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"Inference of Bidders' Risk Attitudes in Ascending Auctions with Endogenous Entry"
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by

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Inference of Bidders' Risk Attitudes in Ascending Auctions with Endogenous Entry*

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Abstract

Bidders' risk attitudes have important implications for sellers seeking to maximize expected revenues. In ascending auctions, auction theory predicts bid distributions in Bayesian Nash equilibrium does not convey any information about bidders' risk preference. We propose a new approach for inference of bidders' risk attitudes when they make endogenous participation decisions. Our approach is based on the idea that bidders' risk premium – the difference between *ex ante* expected profits from entry and the certainty equivalent – required for entry into the auction is strictly positive if and only if bidders are risk averse. We show bidders' expected profits from entry into auctions is nonparametrically recoverable, if a researcher observes the distribution of transaction prices, bidders' entry decisions and some noisy measures of entry costs. We propose a nonparametric test which attains the correct level asymptotically under the null of risk-neutrality, and is consistent under fixed alternatives. We provide Monte Carlo evidence of the finite sample performance of the test. We also establish identification of risk attitudes in more general auction models, where in the entry stage bidders receive signals that are correlated with private values to be drawn in the bidding stage.

Keywords: Ascending auctions, Risk attitudes, Endogenous entry, Nonparametric Test, Bootstrap

JEL Classification Codes: D44, C12, C14

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1 Introduction

We propose a nonparametric test to infer bidders' risk attitudes in auctions with endogenous entry of potential bidders. In these auctions, potential bidders observe some (possibly idiosyncratic) entry costs, such as bid preparation/submission costs or information acquisition costs that need to be incurred before learning about private values, and decide whether to pay the costs to be active in the bidding stage. In any Bayesian Nash Equilibrium (BNE), bidders make rational entry decisions by comparing expected utility from entry with that from staying out, based on their knowledge of entry costs or preliminary signals of private values to be realized in the subsequent bidding stage.

Inference of bidders' risk attitudes have important implications for sellers' choice of revenue-maximizing auction format. When participation of bidders is exogenously given and fixed, the Revenue Equivalence Theorem states that expected revenues from first-price and ascending auctions are the same if bidders are risk-neutral with symmetric, independent private values (IPV). On the other hand, Matthews (1987) showed that, if bidders are risk-averse in such models, then first-price auctions yield higher expected revenues than ascending auctions.

Bidders' risk attitudes also affect revenue rankings among symmetric IPV auctions when participation decisions are endogenous. For risk-neutral bidders, Levin and Smith (1994) implied any given entry cost induces the same entry probabilities in first-price auctions (with entrants observing the number of other entrants) and in ascending auctions. Thus the Revenue Equivalence Theorem implies expected revenues must be the same from both first-price and ascending formats under endogenous entry. On the other hand, Smith and Levin (1996) established the revenue ranking of first-price over ascending auctions under endogenous entry for risk-averse bidders, except for the case with decreasing absolute risk aversions (DARA).[1]

While some earlier papers had studied the identification and estimation of bidders' risk attitudes in first-price auctions (e.g. Bajari and Hortascu (2006), Campo, Guerre, Perrigne and Vuong (2007) and Guerre, Perrigne and Vuong (2009)), inference of risk attitudes in ascending auctions remains an open question. Athey and Haile (2007) pointed out bidders' risk attitudes cannot be identified from bids *alone* in ascending auctions where participation is given exogenously. This is because bidding one's true values is a weakly dominant

[1] Even in the case with DARA, first-price format can yield higher expected revenues than ascending formats when entry costs are low enough. To see this, consider a simple case where entry costs are low enough so that the difference between entry probabilities in first-price and ascending auctions are sufficiently small. In such a case, these two probabilities are both close to 1 and only differ by some $\varepsilon > 0$. By Matthews (1987), conditioning on any given number of entrants, ascending auctions have smaller expected revenues than first-price auctions. With difference between the two entry probabilities ε being small enough, such a revenue ranking result will be preserved.

strategy in ascending auctions, regardless of bidders' risk attitudes. Thus, bidders with various risk attitudes could generate the same distribution of bids in Bayesian Nash equilibria. Consequently, the distribution of bids from entrants is not sufficient for inferring bidders' risk attitudes. Furthermore, we show in Section 4 that risk attitudes cannot be recovered from transaction prices and entry decisions of a given set of potential bidders, when nothing is known about entry costs. It follows that some knowledge of entry costs is necessary for recovering risk attitudes.

We propose a non-parametric test for bidders' risk attitudes when researchers observe the transaction prices and bidders' entry decisions in ascending auctions. Our approach only requires data to contain some noisy measures of bidders' entry costs. This is motivated by the fact that entry costs are often measurable (at least up to some noises) in applications even when risk attitudes are unknown. For example, entry costs may consist of bid preparation costs (such as mailing costs), admission fees or other information acquisition expenses, which are usually observed with noises in data.

The main insight for our test can be illustrated using the mixed-strategy entry model (which is analogous to that considered in Levin and Smith (1994) for first-price auctions). In the entry stage, all potential bidders observe some common entry cost and decide whether to pay the cost and enter an ascending auction in the bidding stage. In a Bayesian Nash equilibrium, potential bidders' participation in the auction will be in mixed strategies with the mixing probability determined to ensure that a bidder's expected utility from entry equals that from staying out. Hence bidders' risk attitudes can be identified by comparing the expected profits from entry and the certainty equivalent. As long as the expectation of entry costs can be identified from data, the distribution of transaction prices and entry decisions alone can be used to make such a comparison. Building on this intuition, we show that identification of risk attitudes can also be achieved in a related model where bidders' entry costs are idiosyncratic and entry decisions follow a pure-strategy. Perhaps more interestingly, we extend the idea to recover risk attitudes in more general models where private values in the bidding stage are affiliated with preliminary signals observed by potential bidders in the entry stage.

We apply the analog principle to construct a non-parametric test statistic, using data on transaction prices and entry decisions as well as estimates of the mean of entry costs. We characterize the limiting distribution of this statistic, and propose a bootstrap test that attains correct asymptotic level and is consistent under any fixed alternative of risk-aversion or risk-loving. We provide evidence for its decent finite sample performance through Monte Carlo simulations.

The remainder of the paper is structured as follows. In Section 2 we discuss the related literature; in Section 3 we present two basic models of auction entry and bidding; in Section 4 we describe the theoretical result underlying our test for bidders' risk attitudes under the two basic models; in Section 5 we propose the test statistic and derive its asymptotic distribution; in Section 6 we present Monte Carlo evidence for the small sample performance of our test statistics; in Section 7 we extend our test to an auction model with selective entry; and in Section 8 we conclude. The proofs are collected in the appendices.

2 Related Literature

This paper fits in and contributes to two branches of the literature on structural analyses of auction data. Some earlier papers analyzed the equilibrium and its empirical implications in auctions with endogenous entry and risk-*neutral* bidders. These include Levin and Smith (1994), Li (2000), Ye (2007) and Li and Zheng (2009). Marmer, Shneyerov and Xu (2011) study a model of first-price auctions between risk-neutral bidders with selective entry, and discuss testable implications of various nested entry models. Roberts and Sweeting (2010) estimated a model of ascending auctions with selective entry and risk-neutral bidders, assuming identification is attained for a model with parametrized structure.

Other papers studied the identification and estimation of bidders' utility functions along with the distribution of private values in first-price auctions without endogenous entry. Campo, Guerre, Perrigne and Vuong (2009) showed how to estimate a semiparametric model of *first*-price auctions with risk-averse bidders when the identification of a parametric utility function is assumed. Bajari and Hortascu (2007) used exogenous variations in the number of bidders in first-price auctions to semi-parametrically estimate the utility function while leaving the distribution of bidders' private values unrestricted. Guerre, Perrigne and Vuong (2009) used exogenous variations in the number of potential bidders to non-parametrically identify bidders' utility functions along with the distribution of private values in first-price auctions. Lu and Perrigne (2008) considered a context where data contain bids from both first-price and ascending auctions that involve bidders with the same underlying utility function and the distribution of private values. Their idea is to first use bids from ascending auctions to recover the distribution of private values, and then use bids from first-price auctions to recover the utility function.

Our work in this paper contributes to these two branches of empirical auction literature by studying a model which endogenizes bidders' entry decisions and relaxes the risk-neutrality assumption at the same time. To the best of our knowledge, our paper is the first effort to non-parametrically infer bidders' risk attitudes in ascending auctions with endogenous entry. Levin and Smith (1996) presented some results on the revenue ranking of auction formats in terms of seller revenues when auctions are known to involve risk-averse bidders who make endogenous entry decisions. Their focus is not on the identification of bidders' risk attitudes.

Ackerberg, Hirano and Shahriar (2011) studied a class of e-Bay auctions where a typical online ascending auction is combined with an option of paying the buy-out price posted by the seller in order to purchase the object immediately. They showed how to identify the bidders' utility functions and the distribution of private values using exogenous variations in the buy-out prices and other auction characteristics. The format of auctions they consider is qualitatively different from the one we consider in this paper, which is a standard ascending format with endogenous entry. We do not embark on a full identification of the utility function in this paper, and therefore require fewer sources of exogenous variations to perform the test. (With exogenous variations in entry costs, identification of the utility function may be possible in our model as well.) Our approach does not rely on variations in entry costs, for our test can be performed for any given level of entry costs. Another difference is that, we also goes beyond identification and propose a method of robust inference. We propose a non-parametric test statistic, derive its limiting distribution, and present evidence for good performance in finite samples.

Our paper fits in a category of empirical auction literature on nonparametric tests of the empirical implications/predictions of auction theory. Earlier works in this category included tests of bidders' rationality in first-price auctions with common values in Hendricks, Pinkse and Porter (2003), tests for presence of interdependent values in Haile, Hong and Shum (2004), and test for affiliations between bidders' private values in Li and Zhang (2010) and Jun, Pinkse and Wan (2010).[2]

3 Ascending Auctions with Endogenous Entry

Consider an empirical context where researchers observe data from a large number of independent single-unit ascending auctions. Each of these auctions involve N potential bidders who have symmetric independent private values and make endogenous entry decisions. In the entry stage, each potential bidder decides whether to incur an entry cost K_i so as to become active. Following their entry, active bidders see their private values V_i, and then compete in an open out-cry (ascending) auction in the bidding stage. A binding reserve price r may be implemented in an auction, and is observed by all potential bidders in the entry stage. In each auction, private values and entry costs are independent draws from some distribution $F(V_1, ., V_N, K_1, ., K_N)$, which is common knowledge among all potential bidders *before* making entry decision. Upon entry, each bidder may or may not be aware of the total number of active entrants (denoted A). All bidders in data share the same bounded von Neumann-Morgenstein utility function $u : \mathbb{R}_+ \to \mathbb{R}$ with $u' > 0$ and the sign of u'' is the same over $\mathbb{R}_+$. A winner who has value V_i and pays a price at P_i receives a payoff of $u(V_i - P_i - K_i)$.

Similar to Li and Zheng (2009), we consider two related entry models that differ in whether entry costs K_i vary across potential bidders. With a slight abuse of notation, we

[2] See Athey and Haile (2007) and Hendricks and Porter (2007) for recent surveys.

use N and A to denote respectively both the number and the set of potential and active bidders. Let F_{ξ_1,ξ_2}, $F_{\xi_2|\xi_1}$ denote respectively the joint and conditional distributions of generic random vectors ξ_1, ξ_2. We use upper cases to denote random variables and lower cases to denote their realizations. For notational simplicity, we drop the reference to N when there is no ambiguity.

Model A (Identical Entry Costs). *In each auction, bidders share the same costs ($K_i = K$ for all i) in the entry stage. Private values V_i are independently and identically distributed over bounded support $[\underline{v}, \overline{v}]$ given K. Across auctions, entry costs are drawn independently from the same distribution F_K over $[\underline{k}, \overline{k}]$. That is,* $\Pr(V_1 \leq v_1, ., V_N \leq v_N | K = k) = \prod_{i \in N} F_{V|k}(v_i)$.

Model B (Heterogeneous Entry Costs). *Idiosyncratic entry costs K_i are privately known to bidder i, and are i.i.d. draws from some continuous, increasing distribution F_K over $[\underline{k}, \overline{k}]$. Private values are independent from entry costs, and are i.i.d. draws from some distribution F_V. (That is, the joint distribution of private values and entry costs is $\Pi_{i \in N} F_K(k_i) F_V(v_i)$.)*

In Models A and B, we assume F_V is continuous, atomless and increasing over $[\underline{v}, \overline{v}]$. We focus on cases where bidders' private values are symmetrically distributed in Sections 4-5. In both models, in Bayesian Nash equilibrium (BNE), each entrant i in bidding stages follows a dominant strategy to drop out at his true value V_i if $A \geq 2$. When $A = 1$ in the bidding stage, the lone entrant wins and pays the reserve price r. Yet entry strategies in BNE differ across these two models. In Model A, there is no private information available to potential bidders in the entry stage. Players adopt mixed strategies and make independent decisions to enter with certain probabilities. In Model B, potential bidders have private information about their own entry costs, and we focus on pure strategies where potential bidders decide to enter if and only if their private entry costs are lower than certain cutoffs. We characterize the BNE in both models for the rest of this section.

Let A_{-i} denote the set of *active* entrants that bidder i competes with if he decides to enter. Let r be a binding reserve price ($r > \underline{v}$). Define $P_i \equiv \max_{j \in A_{-i}} \{\max\{V_j, r\}\}$ as i's payment if he enters and wins while other bidders in A_{-i} follow weakly dominant bidding strategies. If $A_{-i} = \varnothing$, then define $P_i \equiv r$. Then i's profit under dominant strategies is $(V_i - P_i)_+ - k$, where $(.)_+ \equiv \max\{., 0\}$. Let $\omega^A(k; \boldsymbol{\lambda}_{-i})$ denote the expected utility for a bidder i in Model A conditional on paying entry cost k and on potential competitors entering with probabilities $\boldsymbol{\lambda}_{-i} \equiv \{\lambda_j\}_{j \in N \backslash \{i\}}$. Under assumptions of Model A,

$$\omega^A(k; \boldsymbol{\lambda}_{-i}) \equiv u(-k) F_{V|k}(r) + \int_r^{\overline{v}} h(v, k; \boldsymbol{\lambda}_{-i}) dF_{V|k}(v),$$

where for all $v > r$,

$$\begin{aligned} h(v,k;\boldsymbol{\lambda}_{-i}) &\equiv u(v-r-k)F_{P_i}(r|k,\boldsymbol{\lambda}_{-i}) + \int_r^v u(v-p-k)dF_{P_i}(p|k,\boldsymbol{\lambda}_{-i}) \\ &\quad +u(-k)[1-F_{Pi}(v|k,\boldsymbol{\lambda}_{-i})] \end{aligned} \tag{1}$$

with $F_{P_i}(.|k,\boldsymbol{\lambda}_{-i})$ being the distributions of P_i in Model A when $K = k$, and i's potential competitors enter with probabilities $\boldsymbol{\lambda}_{-i}$.

Similarly, in Model B, let $\omega^B(k_i;\mathbf{k}_{-i})$ denote the expected utility for a bidder i conditional on paying k_i to enter and potential competitors entering when their costs are lower than $\mathbf{k}_{-i} \equiv \{k_j\}_{j\neq i}$. Let $F_{P_i}(.|\mathbf{k}_{-i})$ denote the distribution of P_i conditional on i's competitors entering when their private entry costs are lower than $\mathbf{k}_{-i}$. Under assumptions of Model B,

$$\omega^B(k_i;\mathbf{k}_{-i}) = u(-k_i)F_V(r) + \int_r^{\overline{v}} \tilde{h}(v,k_i;\mathbf{k}_{-i})dF_V(v)$$

where $\tilde{h}(v,k_i;\mathbf{k}_{-i})$ is defined by replacing k and $F_{Pi}(.|k,\boldsymbol{\lambda}_{-i})$ in (1) respectively with k_i and $F_{Pi}(.|\mathbf{k}_{-i})$ in Model B. Due to symmetry in distributions of private values across bidders in both models, both $F_{Pi}(.|k,\boldsymbol{\lambda}_{-i})$ in Model A and $F_{P_i}(.|\mathbf{k}_{-i})$ in Model B do not change with the bidder identity i. Consequently, ω^A, ω^B are also independent from the bidder identity i.

Given our specification of Model A, ω^A is decreasing in $\boldsymbol{\lambda}_{-i}$ for any given k. This is due to the following two observations: First, the distribution of active entrants competing with a bidder i is stochastically increasing in $\boldsymbol{\lambda}_{-i}$. (A higher $\boldsymbol{\lambda}_{-i}$ leads to a higher probability of competing with a greater number of rivals.) Second, the distribution of $u((V_i - P_i)_+ - k)$ conditional on entry is stochastically decreasing in the number of active competitors. By similar reasoning, we can show in Model B that ω^B is decreasing in k_i and $\mathbf{k}_{-i}$. Using these properties, entry strategies in Bayesian Nash equilibrium is characterized in the following lemma. Its proof is included in Appendix A.

Lemma 1 *(a) Suppose the common entry cost k is such that $\omega^A(k;(1,.,1)) < u(0) < \omega^A(k;(0,.,0))$ in Model A. Then there is a unique symmetric BNE in which all bidders enter with probability λ_k^*, where λ_k^* is the unique solution to $\omega^A(k;(\lambda_k^*,.,\lambda_k^*)) = u(0)$. (b) Suppose $\omega^B(\overline{k};(\overline{k},.,\overline{k})) < u(0) < \omega^B(\underline{k};(\underline{k},.,\underline{k}))$ in Model B. Then exists a unique symmetric BNE in which all bidders enter iff $k_i \leq k^*$ where k^* is the unique solution to $\omega^B(k^*;(k^*,.,k^*)) = u(0)$.*

In Model A, whenever $\omega^A(k;(0,.,0)) \leq u(0)$ (or $\omega^A(k;(1,.,1)) \geq u(0)$), the equilibrium entry probabilities must be 0 (or 1 respectively). Thus the assumption $\omega^A(k;(1,.,1)) < u(0) < \omega^A(k;(0,.,0))$ can be tested as long as entry decisions are observed in data. Likewise, entry strategies in Model B will be characterized by $\underline{k}$ (or $\overline{k}$ respectively) if $u(0) > \omega^B(\underline{k};(\underline{k},.,\underline{k}))$ (or $u(0) < \omega^B(\overline{k};(\overline{k},.,\overline{k}))$). The equilibrium entry processes in Models A and

B are both non-selective, in the sense that the potential bidders' entry decisions are not based on any informational variables that are correlated with private values to be drawn in the bidding stage.

4 Identification of Bidders' Risk Attitudes

In this section, we assume the number of potential bidders is fixed at N and known to the researcher. We start by assuming that researchers have complete knowledge of entry costs. In Model A, this means researchers observe entry costs. In Model B, this means the researcher knows the distribution of entry costs F_K. In practice, the auctioneer sometimes charges a fixed admission fee to entrants (such as art auctions). This fits in Model A provided the admission fees are observed in data.

Our identification argument builds on the simple intuition that the certainty equivalent for risk-averse bidders must be strictly smaller than the expected profits from entry. The difference between these two can be recovered from bidders' entry decisions and the distribution of transaction prices alone. In Section 4.2, we generalize this argument to allow for the case when the distribution of entry costs is only imperfectly known to the researcher.

4.1 Complete knowledge of entry costs

Let λ_k^* denote entry probabilities in symmetric BNE of Model A when the common entry cost is k; and let k^* denote the cutoff that characterizes the entry strategies in symmetric BNE of Model B. Let $\pi^A(\lambda_k^*; k)$ denote expected profits for a bidder i if he enters in Model A, conditional on the entry cost k and that each of his potential competitors also enters with probability λ_k^*. That is, $\pi^A(k) \equiv E[(V_i - P_i)_+ - k|\ K = k]$. [We suppress "$\lambda_j = \lambda_k^*\ \forall j \neq i$" in the event conditioned on in order to simplify notations]. Similarly, let $\pi^B(k^*)$ denote i's expected profits from entry in Model B, given his idiosyncratic cost k^*, and that competitors enter if and only if their costs are lower than k^*. That is, $\pi^B(k^*) \equiv E[(V_i - P_i)_+ - k^*|\ k^*]$, where "$k^*$" denotes the event "$A_{-i} = \{j \neq i : k_j \leq k^*\}$". Both π^A and π^B are independent from bidder identities due to the symmetry in private value distributions.

Lemma 2 *(a) Suppose k is such that $0 < \lambda_k^* < 1$ in Model A. Then $\pi^A(k) = 0$ iff bidders are risk-neutral, and $\pi^A(k) > 0$ (or < 0) iff bidders are risk-averse (or respectively, risk-loving). (b) Suppose $\underline{k} < k^* < \overline{k}$ in Model B. Then $\pi^B(k^*) = 0$ iff bidders are risk-neutral, and $\pi^B(k^*) > 0$ (or < 0) iff bidders are risk-averse (or respectively, risk-loving).*

Proof. Consider Model A. By Lemma 1, in any symmetric BNE, bidders decide to enter with probability λ_k^* with $\omega^A(k; \lambda_k^*) \equiv E[u((V_i - P_i)_+ - k)|k$, "$\lambda_j = \lambda_k^*\ \forall j \neq i$"$] = u(0)$. Thus, 0 is the certainty equivalent associated with u and the distribution of $(V_i - P_i)_+ - k$ given entry cost k and private value distribution $F_{V|k}$. Therefore, $\pi^A(k) > 0$ if $u'' < 0$ (bidders are risk-averse). Likewise, it can be shown that $\pi^A(k) = 0$ (or, respectively, $\pi^A(k) < 0$) if bidders are risk-neutral (or risk-loving). Similar arguments proves part (b) for Model B. $\square$

Lemma 2 suggests bidders' risk attitudes can be identified in Model A if $\pi^A(k)$ can be recovered at least for some k with $0 < \lambda_k^* < 1$. Likewise, risk attitudes can be identified in Model B with $\underline{k} < k^* < \overline{k}$, provided $\pi^B(k^*)$ can be constructed from entry decisions and the distribution of transaction prices. Proposition 1 states this can be done when k is observed in data. Let $F_{V^{(s:m)}|k}$ denotes the s-th smallest out of m independent draws from $F_{V|k}$ for all $1 \leq s \leq m$.

Proposition 1 *(a) For any entry cost k such that $0 < \lambda_k^* < 1$ in Model A, $\pi^A(k)$ is identified from bidders' entry decisions and the distribution of transaction prices, provided the entry cost k is observed in data. (b) If $\underline{k} < k^* < \overline{k}$ in Model B, then $\pi^B(k^*)$ is identified from entry decisions and the distribution of transaction prices, provided the entry cost distribution F_K is known.*

Proof. Proof of (a). By definition, $\pi^A(k) = E[(V_i - P_i)_+ - K|\ K = k]$. Conditional on k, entry decisions are independent across bidders, and jointly independent from private values in Model A. Furthermore, private values are i.i.d. across bidders given k. Hence, once conditional on k *and* A_{-i} (the number of active competitors for i in the bidding stage), (V_i, P_i) are independent from mixed strategies adopted by potential competitors. Using the Law of Iterated Expectations,

$$E[(V_i - P_i)_+|k] = \textstyle\sum_{a=0}^{N-1} E[(V_i - P_i)_+|k,\ A_{-i} = a] \Pr(A_{-i} = a|k). \tag{2}$$

With common cost k and entry decisions observed from data, λ_k^* is directly identified as the probability that a bidder enters under cost k. Consequently, $\Pr(A_{-i} = a|k)$ is identified as a binomial distribution with parameters $N-1$ and λ_k^*. Conditional on entering with cost k, private values are independent draws from $F_{V|k}$. Let $1\{.\}$ denote the indicator function. By the Law of Iterated Expectations, $E[(V_i - P_i)_+|k,\ A_{-i} = a]$ is

$$\begin{aligned} & E[(V_i - P_i)1\{V_i > P_i > r\}|k,\ A_{-i} = a] + E[(V_i - r)1\{V_i > r\}1\{P_i = r\}|k,\ A_{-i} = a] \\ = & \int_r^{\bar{v}} \left(\int_v^{\bar{v}} (s - v)\, dF_{V|k}(s) \right) dF_{V|k}(v)^a + F_{V|k}(r)^a \int_r^{\bar{v}} (v - r)\, dF_{V|k}(v) \end{aligned} \tag{3}$$

for all $A_{-i} = a \geq 1$, due to independence between V_i and P_i given k. Applying integration by parts to the first term in (3), we have

$$E[(V_i - P_i)_+|A_{-i} = a, k] = \int_r^{\bar{v}} F_{V|k}(v)^a - F_{V|k}(v)^{a+1} dv$$

for $a \geq 1$. Besides, $E[(V_i - P_i)_+|\ A_{-i} = 0, k] = E[(V_i - r)_+|k] = \int_r^{\bar{v}} \left[1 - F_{V|k}(v)\right] dv$. Since k is observed in data, $\pi^A(k)$ can be recovered as long as $F_{V|k}(v)$ is identified for $v \geq r$.

Let W denote transaction prices observed in data. If no entrants bid above r, then define $W < r$. The symmetric IPV assumption implies for any $m \geq 2$, $\Pr(W < r|A = m, k) = \Pr(V^{(m:m)} < r|k) = F_{V|k}(r)^m$ and $\Pr(W = r|A = m, k) = mF_{V|k}(r)^{m-1}[1 - F_{V|k}(r)]$. Hence for any $m \geq 2$ and $t \geq r$,

$$
\begin{aligned}
& \Pr(W \leq t|A = m, k) = \Pr(W < r|m, k) + \Pr(W = r|m, k) + \Pr(r < W \leq t|m, k) \\
= \quad & F_{V|k}(t)^m + mF_{V|k}(t)^{m-1}[1 - F_{V|k}(t)] = F_{V^{(m-1:m)}|k}(t).
\end{aligned}
$$

For any $m \geq 2$, define $\phi_m(t) \equiv t^m + mt^{m-1}(1-t)$ so that $F_{V^{(m-1:m)}|k}(t) = \phi_m(F_{V|k}(t))$. Since $\phi_m(t)$ is one-to-one for any $m \geq 2$ over $t \in [0, 1]$, $F_{V|k}(t)$ is (over-)identified for each $t \geq r$ from the distributions of W conditional on k and $A = m$. Proof of part (b) uses similar arguments and is included in the appendix. $\square$

Remark 1 If bids from those who lose (i.e. prices at which they drop out) are observed in data, the distribution of the other order statistics $V^{(s:m)}$ with $s \leq m - 2$ can also be used for identifying $\pi^A(k)$ and $\pi^B(k^*)$. This is because a one-to-one mappings between $F_{V|k}$ and the distributions of these smaller order statistics in Model A still exists. Likewise for Model B. Such an over-identification can be exploited to improve efficiency in estimation.

Remark 2 In practice, auction data may be "truncated" in the sense that only those involving at least one entrants are observed. In such a case, a positive entry probability λ_k^* in Model A is (over-)identified using ratios of truncated distributions

$$\frac{\Pr(A=n|A\geq 1,k)}{\Pr(A=n+1|A\geq 1,k)} = \frac{C_n^N \left(\lambda_k^*\right)^n (1-\lambda_k^*)^{N-n}}{C_{n+1}^N \left(\lambda_k^*\right)^{n+1} (1-\lambda_k^*)^{N-n-1}} = \frac{(n+1)(1-\lambda_k^*)}{(N-n)\lambda_k^*}$$

for all $n \leq N-1$. Similar arguments show $F_K(k^*)$ is identified from truncated data in Model B. Same arguments in Proposition 1 show $\pi^A(k)$ and $\pi^B(k^*)$ can be recovered.

4.2 Imperfect observation of entry costs

We now consider cases where researchers only have imperfect knowledge about entry costs. First, consider an extension of Model A where entry costs K common to potential bidders vary across auctions, but the researcher only observes noisy measures of costs $\tilde{K} = K + \epsilon$ with $E(\epsilon) = 0$. Then $\mu_K \equiv E(K) = E(\tilde{K})$ is directly identifiable from data. In such a case, the test for bidders' risk attitudes can still be conducted, provided entry costs vary independently from bidders' values.

Corollary 1 *Let $[\underline{k}, \overline{k}]$ denote the support of entry costs K in Model A. (a) Suppose $0 < \lambda_k^* < 1$ for all $k \in [\underline{k}, \overline{k}]$. Then $E[\pi^A(K)] = 0$ when bidders are risk-neutral, and $E[\pi^A(K)] > 0$ (or < 0) when bidders are risk-averse (or respectively, risk-loving). (b) If K is independent from $(V_i)_{i \in N}$, then $E[\pi^A(K)]$ is identified from bidders' entry decisions, the distribution of transaction prices and noisy cost measures $\tilde{K}$.*

Proof. Part (a). By Proposition 1 and the support condition in part (a), $\pi^A(k) = 0$ for all $k \in [\underline{k}, \overline{k}]$ if bidders are risk-neutral, and $\pi^A(k) > 0$ (or $\pi^A(k) < 0$) for all k if bidders are risk-averse (or, respectively, risk-loving). Integrating out k using F_K proves (a).

Part (b). That $K \perp (V_i)_{i \in N}$ implies, given $A_{-i} = a$, the vector of order statistics $(V^{(s:a+1)})_{s \leq a+1}$ is independent from K. Thus $\varphi(a) \equiv E[(V_i - P_i)_+ | A_{-i} = a, k]$ does not depend on k for all $a \geq 0$ (recall $P_i \equiv r$ when $A_{-i} = \varnothing$). By (2), $E[\pi^A(K)]$ is:

$$\int_{\underline{k}}^{\overline{k}} -k + \textstyle\sum_{a=0}^{N-1} \varphi(a) \Pr(A_{-i} = a|k) dF_K(k) = \textstyle\sum_{a=0}^{N-1} \varphi(a) \Pr(A_{-i} = a) - \mu_K. \tag{4}$$

To identify $\Pr(A_{-i} = a)$ (or $\int_{\underline{k}}^{\overline{k}} \Pr(A_{-i} = a|k) dF_K$), note that given any k and N, A_{-i} is binomial $(N-1, \lambda_k^*)$ while A is binomial (N, λ_k^*). By construction,

$$\Pr(A_{-i} = a|k) = \tfrac{N-a}{N} \Pr(A = a|k) + \tfrac{a+1}{N} \Pr(A = a+1|k) \tag{5}$$

for all k and $a \leq N-1$. Integrating out k on both sides of (5) implies $\Pr(A_{-i} = a) = \frac{N-a}{N} \Pr(A = a) + \frac{a+1}{N} \Pr(A = a+1)$. Since the unconditional distribution of A is directly identified, so is the distribution of A_{-i}. As for μ_K, it is identified as $E(\tilde{K})$. $\square$

Observing some noisy measures of entry costs is sufficient but not necessary for identifying risk attitudes in Model A. As suggested by the proof of Corollary 1, only μ_K needs to be known (or recoverable from data).

There is an alternative approach for recovering the unconditional distribution of A_{-i}. Because entry decisions in data are rationalized by a symmetric BNE, the distribution of A_{-i} can be identified as the distribution of the number of entrants from a random subset of $N-1$ potential bidders. Such subsets can be formed by removing a randomly-selected bidder i from the set of N potential bidders. We adopt this alternative approach while constructing the test statistic.

Now consider a special case of Model B where researchers have imperfect knowledge of entry costs. Suppose the data contain noisy measure of idiosyncratic costs $K_i' = K_i + \epsilon_i$. The measurement errors ϵ_i's are independently drawn from the same distribution F_ϵ, which is independent from K_i. Even with K_i now unobservable, expected profits from entry $E[(V_i - P_i)_+ |\ k^*]$ is recoverable as in Proposition 1. Thus, bidders' risk attitudes can be identified as long as k^* can be recovered using the distribution of K_i'.

If the distribution F_ϵ is known to researchers and has a non-vanishing characteristic function, then the characteristic function of K_i is recovered as $E(e^{itK_i'})/E(e^{it\epsilon_i})$. With the equilibrium entry probability identified from data, the equilibrium cutoff k^* can be recovered by inverting F_{K_i} at the entry probability. Even with F_ϵ unknown to researchers, the cutoff k^* can be recovered provided researchers have multiple noisy measures of idiosyncratic costs K_i. Suppose for each K_i, data contain two noisy measures $(K_{i,1}', K_{i,2}') \equiv (K_i + \epsilon_{i,1}, K_i + \epsilon_{i,2})$, where $(K_i, \epsilon_{i,1}, \epsilon_{i,2})$ are mutually independent with non-vanishing characteristic functions, and the mean or some quantile of either $\epsilon_{i,1}$ or $\epsilon_{i,2}$ is known.[3] The Kotlarski Theorem (see Kotlarski (1967) and Rao (1992)) then applies to identify the marginal distributions of K_i and $\epsilon_{i,1}$. Inverting the distribution of K_i at the entry probability then gives k^*.

4.3 Further discussions about entry costs

Our test for bidders' risk attitudes requires that the researcher have at least some partial knowledge about entry costs (such as its expectation). We now elaborate on theoretical and empirical justifications of such an assumption. We also discuss possibilities for constructing a test without such knowledge by exploiting the exogenous variations in the number of potential bidders.

First, with the number of potential bidders fixed and researchers having no information about entry costs, it is impossible to nonparametrically infer bidders' risk attitudes from observed entry decisions and transaction prices alone. To see this, consider a simplified version of Model A' where all auctions in data have the same fixed entry cost k, which is not reported in data. The distribution of private values is still identified from the distribution of transaction prices and the number of entrants; and the equilibrium entry probability is also identified from entry decisions. Nonetheless, even in this simplified case, the bidders' utility $u(.)$ and the fixed cost k cannot be jointly identified. To see this, suppose bidders are risk-neutral with utility function $u(.)$ and $\lambda_k^* \in (0,1)$. Then one can always replace u with a slightly concave $\tilde{u} \neq u$, the observed entry behaviors (λ^*) would still be rationalized by some level of entry cost $\tilde{k} \neq k$ that equals the expected utility from entry to that from certainty equivalent (i.e. $E[\tilde{u}((V_i - P_i)_+ - \tilde{k})|\lambda^*, k] = u(0)$). Thus, our hope for inferring risk attitudes from entry and bidding behaviors (when the number of potential bidders is fixed) must utilize at least some partial knowledge of entry costs. To our knowledge, our work in this paper is the first attempt to exploit possible information from entry costs to recover risk preferences.

Second, depending on the empirical application considered, entry costs may well be measurable (at least up to random noises) through additional surveys or data collection work. For example, in timber auctions held by US Forest Service, the entry costs for potential bidders (i.e. that millers and loggers located in a nearby geographic region) consists largely of information acquisition costs. These costs are the prices for conducting a "cruise" on forest tracts in order to plot the distribution of the diameter and height of trees, etc. (See Athey, Levin and Seira (2011) for details.) Such private cruises are institutionalized and standard practices on the market. Thus it is plausible that their costs vary little across potential bidders, and can be learned through surveys or market observations up to random measurement

[3] The two measurement errors $\epsilon_{i,1}$, $\epsilon_{i,2}$ could be drawn from different marginal distributions, in which case the identification would require researchers to know how to distinguish $K'_{i,1}$ and $K'_{i,2}$ in data.

errors. Thus the assumption of having an unbiased estimator for the common entry costs seems plausible in such contexts. Other well-known examples where this assumption might be expected to hold include the high-way procurement auctions considered in Li and Zheng (2009) and Krasnokutskaya and Seim (2011).

If the number of potential bidders can be expected to vary exogenously in data (in the sense that marginal distribution of V_i is invariant to N), then there is hope for constructing a non-parametric test for bidders' risk attitudes from ascending auctions with IPV without relying on any information about entry costs. To see this, consider the scenario where the data contains two sets of auctions, with each observed to involve either N or $N' > N$ potential bidders respectively. The entry cost k is not observed in data, but known to remain the same across all auctions. Then by the same arguments above, the null of risk neutrality implies $E[(V_i - P_i)_+|\lambda^*_{N,k}, N] = k$ and $E[(V_i - P_i)_+|\lambda^*_{N',k}, N'] = k$, where $\lambda^*_{N,k}$ denotes the equilibrium entry probability when entry cost is k and the number of potential bidders is N. Thus the null yields a testable implication

$$E[(V_i - P_i)_+|\lambda^*_{N,k}, N] = E[(V_i - P_i)_+|\lambda^*_{N',k}, N'], \tag{6}$$

where both sides are identifiable from data and do not require observations of entry costs. However, the main challenge is to derive a formal description of the power of a test under the alternatives (of risk-aversion or risk-loving). We conjecture (6) should fail in general when the null of risk-neutrality is false. Nevertheless, it is not clear how the direction of inequality would be related to the types of alternatives (i.e. aversion or loving). We leave this for future research.

5 Inference of Bidders' Risk Attitudes

We now construct a test for bidders' risk attitudes using the analog principle. We focus on the extended version of Model A, where entry costs K vary across auctions independently from $(V_i)_{i\in N}$. Researchers only get to observe noisy measures $\tilde{K} = K + \epsilon$, where $\epsilon \perp (K, (V_i)_{i\in N})$ and $E(\epsilon) = 0$. Furthermore, $0 < \lambda^*_k < 1$ for all $k \in [\underline{k}, \overline{k}]$. Hereinafter we refer to these assumptions as the *Conditions of Model A'*. Independence of ϵ from $(K, (V_i)_{i\in N})$ is not necessary for our inference procedure in Section 5.2 to be valid. Nonetheless it simplifies derivation of the limiting distribution of our test statistic. To simplify exposition, we fix the number of potential bidders N, and drop it from the notations for observable distributions in data when there is no ambiguity.

5.1 Asymptotic property of the test statistic

Recall our goal is to draw a conclusion as to which of the following three competing hypotheses is supported by data:

$$
\begin{aligned}
H_0 &: \text{ Bidders are risk-neutral} \Leftrightarrow \ \tau_0 = 0; \\
H_A &: \text{ Bidders are risk-averse} \Leftrightarrow \ \tau_0 > 0; \text{ and} \\
H_L &: \text{ Bidders are risk-loving} \Leftrightarrow \ \tau_0 < 0.
\end{aligned}
$$

where $\tau_0 \equiv E[\pi^A(K)]$. Our data contain T independent auctions, each of which is indexed by t, involves N potential bidders and has at least one active entrants. Let A_t denote the number of entrants in auction t. By definition, $W_t < r$ if and only if there is no transaction. On the other hand, when there is transaction ($W_t \geq r$), the transaction price in auction t is $W_t = \max\{r, V^{(A_t-1:A_t)}\}$ when $A_t \geq 2$ and $W_t = r$ when $A_t = 1$. Our test statistic amounts to estimating the right-hand side of (4) using the analog principle.

Construction of the Test Statistic $\hat{\tau}_T$

Step 1 *For* $m \geq 2$, *calculate the empirical distribution of transaction prices*

$$\hat{F}_{W,m,T}(s) \equiv \tfrac{1}{N}\textstyle\sum_{t\leq N} 1\{W_t \leq s \text{ and } A_t = m\} / \tfrac{1}{N}\textstyle\sum_{t\leq N} 1\{A_t = m\}$$

for any $s \geq r$.

Step 2 *For any* $m \geq 2$, *estimate the distribution of private values at* $s \geq r$ *by* $\hat{F}_{V,m,T}(s) \equiv \phi_m^{-1}(\hat{F}_{W,m,T}(s))$, *where* $\phi_m(t) \equiv t^m + mt^{m-1}(1-t)$. *Aggregate these estimates by* $\hat{F}_{V,T}(s) \equiv \frac{1}{N-1}\sum_{m=2}^{N} \hat{F}_{V,m,T}(s)$ *for any* $s \geq r$.

Step 3 *Estimate* $\varphi(a)$ *for* $a \geq 0$ *using sample analogs:*[4]

$$\hat{\varphi}_T(a) \equiv \int_r^{\bar{v}} \left[\hat{F}_{V,T}(s)\right]^a \left[1 - \hat{F}_{V,T}(s)\right] ds.$$

The integral is calculated using mid-point approximations.

Step 4 *In each auction* t, *select a potential bidder* i *randomly. Let* $\tilde{A}_t$ *denote the number of entrants from the other* $N-1$ *potential bidders (excluding* i*). Estimate* $\rho \equiv (\rho(a))_{0\leq a\leq N-1}$

[4] As shown while proving part (a) of Proposition 1,

$$\varphi(a) = \int_r^{\bar{v}} [F_V(s)]^a [1 - F_V(s)]\, ds$$

for each $a \geq 0$. The distribution of private values $(V_i)_{i\in N}$ are independent from K under Conditions of Model A' in Corollary 1.

by $\hat{\rho}_T \equiv (\hat{\rho}_T(a))_{0 \leq a \leq N-1}$*, where* $\hat{\rho}_T(a) \equiv \frac{1}{T}\sum_{t \leq T} 1\{\tilde{A}_t = a\}$.[5] *Then estimate* μ_K *by* $\hat{\mu}_T \equiv \frac{1}{T}\sum_{t \leq T} \tilde{K}_t$*. Finally, estimate* τ_0 *by*

$$\hat{\tau}_T \equiv \sum_{a=0}^{N-1} \hat{\varphi}_T(a)\hat{\rho}_T(a) - \hat{\mu}_T.$$

We now derive the limiting distribution of $\sqrt{G}\,(\hat{\tau}_T - \tau_0)$. Let "$\rightsquigarrow$" denote weak convergence of stochastic processes in a normed space. (For Euclidean spaces $\mathbb{R}^N$, this is reduced to convergence in distribution, denoted "$\xrightarrow{d}$".) Let $F_{\tilde{V}}$ be a short-hand for the section of F_V over the domain $[r, \bar{v})$ and $\hat{F}_{\tilde{V},T}$ be a short-hand for its estimator as defined in Step 2. Denote the limiting distribution of $\sqrt{T}(\hat{\rho}_T - \rho)$ and $\sqrt{T}(\hat{\mu}_T - \mu_K)$ by $\mathcal{N}_\rho$ and $\mathcal{N}_\mu$ respectively. We characterize the covariance between $\mathcal{N}_\rho$ and $\mathcal{N}_\mu$ in Lemma B4 in the appendix. We also show in Lemma B5 in the appendix that under mild conditions $\sqrt{T}\left(\hat{F}_{\tilde{V},T} - F_{\tilde{V}}\right) \rightsquigarrow \mathbb{G}_V$, where $\mathbb{G}_V$ is a zero-mean Gaussian Process indexed by $[r, \bar{v})$. Lemma B5 characterizes the covariance kernel of $\mathbb{G}_V$ as well as its covariance with $\mathcal{N}_\mu$ and $\mathcal{N}_\rho$. The proof of these results builds on the fact that $\hat{\rho}_T$, $\hat{\mu}_T$ are simple sample averages, while $\hat{F}_{V,T}$ is a known function of sample averages.

Let $\mathcal{S}_{[r,\bar{v})}$ denote the set of functions defined over the domain $[r, \bar{v})$ that are strictly positive, bounded, integrable, right-continuous and have limits from the left. Under the sup-norm, $\mathcal{S}_{[r,\bar{v})}$ is a normed linear space with a non-degenerate interior. Define $\varphi : \mathcal{S}_{[r,\bar{v})} \mapsto \mathbb{R}^N_+$ as $\varphi(F) \equiv (\varphi(a; F))_{a=0}^{N-1}$, where

$$\varphi(a; F) \equiv \int_r^{\bar{v}} F(s)^a - F(s)^{a+1} ds.$$

By definition, $\varphi(\hat{F}_{\tilde{V},T}) = (\hat{\varphi}_T(a))_{a=0}^{N-1} \equiv \hat{\varphi}_T$ and $\varphi(F_{\tilde{V}}) = (\varphi(a; F_{\tilde{V}}))_{a=0}^{N-1} \equiv \varphi$. The mapping φ is Hadamard differentiable at $F_{\tilde{V}}$ tangentially to $\mathcal{S}_{[r,\bar{v})}$ (see Appendix B). For any $h \in \mathcal{S}_{[r,\bar{v})}$, the Hadamard derivative $D_{\varphi,F_{\tilde{V}}} : \mathcal{S}_{[r,\bar{v})} \to \mathbb{R}^N_+$ is

$$D_{\varphi,F_{\tilde{V}}}(h)(a) \equiv \int_r^{\bar{v}} \left[aF_{\tilde{V}}(s)^{a-1} - (a+1)F_{\tilde{V}}(s)^a\right] h(s) ds$$

for $1 \leq a \leq N-1$; and $D_{\varphi,F_{\tilde{V}}}(h)(0) \equiv -\int_r^{\bar{v}} h(s) ds$.

Proposition 2 *Suppose the reserve price* r *is binding and* F_V *is continuously distributed with positive densities over* $[r, \bar{v})$*. Under the Conditions in Model A',*

$$\sqrt{T}(\hat{\tau}_T - \tau_0) \rightsquigarrow \mathcal{N}_\tau$$

where $\mathcal{N}_\tau \equiv \rho D_{\varphi,F_{\tilde{V}}}(\mathbb{G}_V) + \varphi\mathcal{N}_\rho - \mathcal{N}_\mu$ *follows a univariate normal distribution with zero mean.*

[5] An alternative is to estimate $\rho(a)$ by $\frac{1}{T}\sum_{t \leq T} \frac{N-a}{N} 1\{A_t = a\} + \frac{1}{T}\sum_{t \leq T} \frac{a+1}{N} 1\{A_t = a+1\}$. (See proof of Corollary 1 for details.)

The proposition uses the Functional Delta Method (Theorem 3.9.4 in van der Vaart and Wellner 1996), and its proof is included in Appendix B. The proof builds on the fact that both $\hat{\rho}_T, \hat{\mu}_T$ are simple sample averages while $\hat{\varphi}_T$ is a known non-linear functional involving sample averages. This allows us to first apply the Functional Delta Method to show that $\sqrt{T}(\hat{\varphi}_T - \varphi) \rightsquigarrow D_{\varphi,F_{\tilde{V}}}(\mathbb{G}_V)$, with the covariance kernel of $\mathbb{G}_V$ and its covariance with $\mathcal{N}_\rho, \mathcal{N}_\mu$ completely characterized in terms of population distribution in the data-generating process (DGP). The Jacobian of the right-hand side of (4) with respect to (φ, ρ, μ_K) at the true DGP is the 1-by-$(2N+1)$ vector $[\rho, \varphi, -1]$. Thus another application of the multivariate delta method delivers the result. The Gaussian process $\mathbb{G}_V$ and $\mathcal{N}_\rho, \mathcal{N}_\mu$ are Borel-measurable and tight. Besides, by construction the Hadamard derivative $D_{\varphi,F_{\tilde{V}}}$ is a linear mapping over $\mathcal{S}_{[r,\bar{v})}$. Thus the limiting distribution $\rho D_{\varphi,F_{\tilde{V}}}(\mathbb{G}_V) + \varphi\mathcal{N}_\rho - \mathcal{N}_\mu$ is a zero-mean univariate normal (by Lemma 3.9.8. of van der Vaart and Wellner (1996)).

5.2 Bootstrap Inference Procedure

Our goal is to test the null $H_0 : \tau_0 = 0$ against two directional alternatives $H_A : \tau_0 > 0$ (risk-averse) and $H_L : \tau_0 < 0$ (risk-loving). Set the level for our test to be α. In principle, we can estimate the standard derivation of the limiting distribution in Proposition 2 using the analog principle. Then the asymptotic plug-in approach can be applied to estimate the critical value for testing $H_0 : \tau_0 = 0$. In this section, we adopt an alternative approach using bootstrap procedures to test H_0. This avoids the explicitly estimating the standard deviation of the limiting distribution of $\sqrt{T}\,(\hat{\tau}_T - \tau_0)$. Building on results in Proposition 2, we show the bootstrap test is consistent against fixed alternatives (of risk-aversion or risk-loving), and attains the correct level asymptotically.

Bootstrap Procedure for Testing $H_0 : \tau_0 = 0$

Step 1: *Calculate $\hat{\tau}_T$ using the original sample.*

Step 2: *Draw a bootstrap sample with size T from the original sample with replacement. Estimate τ_0 using this bootstrap sample and denote the estimate by $\hat{\tau}_{T,1}$.*

Step 3: *Repeat Step 2 for B times and denote the bootstrap estimates by $\{\hat{\tau}_{T,b}\}_{b \leq B}$. Find the $1-\alpha$ quantile of the empirical distribution of the bootstrap estimates $\{\sqrt{T}|\hat{\tau}_{T,b} - \hat{\tau}_T|\}_{b \leq B}$ (denoted by $\hat{c}_{1-\alpha/2,T}$).*

Step 4: *Do not reject H_0 if $-\hat{c}_{1-\alpha/2,T} \leq \sqrt{T}\hat{\tau}_T \leq \hat{c}_{1-\alpha/2,T}$. Reject the null in favor of H_A (or H_L) if $\sqrt{T}\hat{\tau}_T > \hat{c}_{1-\alpha/2,T}$ (or respectively if $\sqrt{T}\hat{\tau}_T < -\hat{c}_{1-\alpha/2,T}$).*

Proposition 3 establishes consistency and asymptotic validity of the test. Let $\Pr(\hat{\tau}_T \leq . \mid \tau_0)$ denote the distribution of $\hat{\tau}_T$ given true value of τ_0 in the data generating process.

Proposition 3 *Suppose the Conditions in Model A' hold, and data contains T independent auctions, each involving N potential bidders and*

$$\lim_{T\to+\infty} \Pr\left(\sqrt{T}\hat{\tau}_T \geq \hat{c}_{1-\alpha/2,T} | \tau_0 = c\right) = 1 \ \forall c > 0; \tag{7}$$

$$\lim_{T\to+\infty} \Pr\left(\sqrt{T}\hat{\tau}_T \leq -\hat{c}_{1-\alpha/2,T} | \tau_0 = c\right) = 1 \ \forall c < 0; \tag{8}$$

$$\lim_{T\to+\infty} \Pr\left(\sqrt{T}\hat{\tau}_T > \hat{c}_{1-\alpha/2,T} \ or \ \sqrt{T}\hat{\tau}_T < -\hat{c}_{1-\alpha/2,T} | \tau_0 = 0\right) = \alpha. \tag{9}$$

These results are due to the fact that the empirical distribution of $\sqrt{T}\left(\hat{\tau}_{T,b} - \hat{\tau}_T\right)$ calculated from bootstrap samples provides a consistent estimator for the finite sample distribution of $\sqrt{T}\left(\hat{\tau}_T - \tau_0\right)$ under mild conditions. Such conditions are stated in Beran and Ducharme (1991) and verified for our context here in Appendix C. (Here consistency means the distribution of $\sqrt{T}\left(\hat{\tau}_T - \tau_0\right)$ as estimated from bootstrap samples gets uniformly close to the asymptotic distribution of $\sqrt{T}(\hat{\tau}_T - \tau_0)$ as sample size increases.See Horowitz (2000) for a formal definition of bootstrap consistency.) The results then follow from the fact that $\sqrt{T}\tau_0$ is zero under the null but diverges to positive (or negative) infinity under the alternative.

Our bootstrap inference uses an asymptotically non-pivotal statistic $\sqrt{T}\left(\hat{\tau}_T - \tau_0\right)$. One could construct asymptotically pivotal statistics using the pre-pivoting approach. This would help attain asymptotic refinements in the approximation of test statistic distribution relative to first-order asymptotic approximation or bootstrap using asymptotically non-pivotal statistics. This is computationally intensive due to bootstrap iterations and therefore we do not pursue this approach here..

6 Monte Carlo Experiments

This section presents evidence for the performance of our test in simulated finite samples. We consider the following data-generating process (DGP): Each auction involves N potential bidders who face the same entry cost K. Upon entry, bidders draw private values from a uniform distribution with support $[\underline{v}, \overline{v}] \equiv [0, 10]$. The reserve price is $R = 3$. The data report prices paid by the winner and the number of entrants in each auction. When there is no transaction (when all entrants' realized privates values are lower than R), the transaction prices is set to an arbitrary level lower than R. The data only report noisy measures of the entry costs $\tilde{K} = K + \epsilon$ where ϵ is drawn from a uniform distribution $[-\frac{1}{2}, \frac{1}{2}]$. Bidders' von-Neumann-Morgenstern utility is specified as $u(t) \equiv \left(\frac{t+5}{10}\right)^{\gamma}$. That is, bidders are risk-neutral (and respectively, risk-averse or risk-loving) if $\gamma = 1$ (and $\gamma < 1$ or $\gamma > 1$).For a fixed level of entry cost and γ, the entry probability decreases in the number of potential bidders. This confirms that the pattern as stated for the case with risk-neutral bidders in Li and Zheng (2009) is carried over into the general case with non-risk-neutral bidders. Besides, once controlling for the entry costs and the number of potential bidders, the entry probability increases as bidders become more risk-loving (γ increases).

[Insert Figure 1 (a), (b), (c), (d) here.]

To illustrate how performance of the test depends on bidders' risk preferences, we first focus on a simple design with the entry cost fixed at $K = 0.7$ or $K = 0.9$ respectively. We also set $N = 4$ in the data-generating process. We report test performance under various γ in the DGP in Figure 1. For each gridpoints γ between 0.8 and 1.2 (with gridwidths being 0.2), we simulate $S = 300$ data sets, each containing $T = 5,000$ or $T = 10,000$ independent auctions. In every single auction, data contain $\tilde{K} = 0.7 + \epsilon$ where ϵ is uniform on $[-1/2, 1/2]$. For each simulated sample, we calculate the statistic $\hat{\tau}_T$, and then perform the test by drawing $B = 300$ bootstrap samples with replacement from that estimation sample. We experiment with $\alpha = 5\%$ and 10% respectively. The solid curves in Figure 1 show the percentage of these S simulated samples where the test fails to reject the null of risk-neutrality ($H_0 : \gamma = 1$). The dashed (and dotted) curve plots the proportion of these samples in which the test rejects the null in favor of the alternative $H_A : \gamma < 1$ (and $H_L : \gamma > 1$) respectively. Each panel of Figure 2 reports these proportions for a given pair of sample size T and entry costs K.

In all panels of Figure 1, the test attains approximately the targeted level for both $\alpha = 5\%$ and $\alpha = 10\%$, when the null is true in the DGP. All panels show that under the alternative the power of the test increases fairly quickly to 1 as γ moves away from the risk-neutral value 1.[6] A comparison of panels (a) and (c) with panels (b) and (d) suggests an increase in sample size improves test performance both in terms of errors in rejection probabilities under the null, and power under the alternative. For a given sample size T, the test performs better when $K = 0.9$. This can be partly ascribed to the fact that the difference between entry probabilities under the null and the alternatives are slightly more pronounced when $K = 0.9$ than when $K = 0.7$.

Next, we present performance of the test when the DGP contains random entry costs that are observed with noises under the Conditions of Model A' (which are specified at the beginning of Section 5). We let K be drawn from a uniform multinomial distribution with support [0.7, 0.8, 0.9], and $\tilde{K} = K + \epsilon$ as before. We experiment with DGP with $\gamma \in \{1, 0.9, 1.1\}$ and $N \in \{4, 5\}$. For each DGP, we perform our test in $S = 300$ simulated data sets, each containing $T = 5,000$ or $10,000$ independent auctions. As before, for each simulated sample, we conduct our test by drawing $B = 300$ bootstrap samples. The test results are summarized in Table 1.

Table 1(a): Test Performance under Random Costs ($N = 4$)

[6] In the presence of directional alternatives (i.e. H_A and H_L), we define the power of the test as the probability of rejecting the null in favor of the true alternative that underlies the DGP.

	$\alpha = 5\%$	$\alpha = 10\%$	$\alpha = 15\%$
$T = 5,000$			
$\gamma = 1$	[5.00%, 93.67%, 1.33%]	[4.00%, 93.00%, 3.00%]	[5.33%, 89.67%, 5.00%]
$\gamma = 0.9$	[0.00%, 36.00%, 64.00%]	[0.00%, 25.00%, 75.00%]	[0.00%, 20.33%, 79.67%]
$\gamma = 1.1$	[77.00%, 23.00%, 0.00%]	[86.00%, 14.00%, 0.00%]	[90.67%, 9.33%, 0.00%]
$T = 10,000$			
$\gamma = 1$	[3.33%, 95.33%, 1.33%]	[8.67%, 87.67%, 3.67%]	[11.67%, 84.00%, 4.33%]
$\gamma = 0.9$	[0.00%, 9.67%, 90.33%]	[0.00%, 3.00%, 97.00%]	[0.00%, 2.33%, 97.67%]
$\gamma = 1.1$	[95.33%, 4.67%, 0.00%]	[97.33%, 2.67%, 0.00%]	[99.00%, 1.00%, 0.00%]

The rows in Table 1 represent different DGPs and sample sizes, each corresponding to a pair (N, γ). The column heads are targeted levels. We test the null of risk-neutrality (H_0) against two alternatives risk-aversion (H_A) and risk-loving (H_L). For each cell in Table 1, we report (from the left to the right) the proportions of S simulated samples where the test rejects H_0 in favor of H_L, where the test does not reject H_0, and where H_0 is rejected in favor of H_A respectively.

Table 1(a) shows the result for $N = 4$ and $T = 5,000$ or $10,000$. In such cases, our test attains a rejection probability close to the targeted level α when the null of risk-neutrality is true ($\gamma = 1$). When the bidders are not risk-neutral, the test yields reasonably high chances for rejecting the null in favor of the correct alternative. When bidders are not risk-neutral, the percentage of samples in which the null is rejected in favor of an incorrect alternative (also known as the "Type-three" error) is zero across all specifications and sample sizes. Table 1(a) also shows the performance of the test improves as the sample size T increases. For a fixed specification, the probability for rejecting the null when bidders are risk-neutral is closer to the targeted level when $T = 10,000$. There is also a quite substantial increase in the power of the test (i.e. probability for rejecting H_0 in favor of the correct alternative) under both alternatives as T increases.

Table 1(b): Test performance under Random Costs ($N = 5$)

	$\alpha = 5\%$	$\alpha = 10\%$	$\alpha = 15\%$
$T = 5,000$			
$\gamma = 1$	[3.67%, 94.67%, 1.67%]	[6.00%, 91.00%, 3.00%]	[8.00%, 86.33%, 5.67%]
$\gamma = 0.9$	[0.00%, 45.67%, 54.33%]	[0.00%, 32.33%, 67.67%]	[0.00%, 27.00%, 73.00%]
$\gamma = 1.1$	[73.00%, 27.00%, 0.00%]	[80.00%, 20.00%, 0.00%]	[88.00%, 12.00%, 0.00%]
$T = 10,000$			
$\gamma = 1$	[3.67%, 95.67%, 0.67%]	[7.33%, 90.33%, 2.33%]	[10.33%, 84.33%, 5.33%]
$\gamma = 0.9$	[0.00%, 12.67%, 87.33%]	[0.00%, 6.33%, 93.67%]	[0.00%, 3.67%, 96.33%]
$\gamma = 1.1$	[93.67%, 6.33%, 0.00%]	[98.33%, 1.67%, 0.00%]	[98.67%, 1.33%, 0.00%]

Table 1(b) reports similar results for $N = 5$. A comparison between Table 1(a) and 1(b) shows the impact of a larger number of potential bidders on the relative performance of the test is ambiguous, depending on the true DGP. With $\gamma = 1$ in DGP and for a fixed sample size T, the test yields smaller errors in the rejection probability when $N = 5$ in most cases. (The only exception takes place when $T = 10,000$ and $\alpha = 5\%$, in which case errors in rejection probabilities are both small when $N = 4$ and $N = 5$.) By construction, increasing the number of potential bidders affects the test performance through two channels that have offsetting impacts. First, with T fixed, the number of auctions involving a given number of entrants $m \leq N$ may decrease as N gets larger. Hence *for each* m, the empirical price distribution $\hat{F}_{W,m}$ becomes a worse estimator for $F_{W,m}$ in the population as its variance is increased. On the other hand, the number of empirical distributions $\{\hat{F}_{W,m}\}_{m \leq N}$ available for estimating F_V increase as N increases, which might help increase performance of $\hat{F}_{V,T}$ as an estimator for F_V. The improvement of test performance under a larger N when $\gamma = 1$ appears to be evidence that the second impact has dominated the first when the DGP involves risk-neutral bidders only. On the other hand, the impact of a large N on the power of the test when $\gamma \neq 1$ is ambiguous. For example, consider the case with larger sample size $T = 10,000$. When bidders are risk-averse with $\gamma = 0.9$, the power of the test is higher for any given α when $N = 4$. Meanwhile, the power appears to be higher when $N = 5$ if $\gamma = 1.1$ and $\alpha = 10\%$. We conjecture that such patterns are due to combined impacts of the shape of $u(.)$ and the fact that the entry probability decreases as N increases (which in turn implies the distribution of payoffs $(V_i - P_i)_+$ from entry is stochastically decreasing).

7 Extension: Selective Entry with Informative Signals

So far we have considered models where bidders' information in the entry stage is uncorrelated with their private values to be drawn in the bidding stage. In this section, we relax this assumption and consider models where bidders' entry decisions are based on informative signals correlated with private values in the bidding stage. Nonparametric identification of bidders' risk attitudes can be attained in this case, provided data is rich enough to contain auctions with continuous variations in observable entry costs.

The model is specified as follows. In the entry stage, each bidder i receives a preliminary signal S_i and decides whether to become active by paying the entry cost k (which is the same for and known to all bidders). The costs are incurred only for entrants. Upon entry, each bidder sees his true value V_i, and bids in an ascending auction with a reserve price r. The joint distribution $F(S_1, ., S_N, V_1, ., V_N)$ and r are common knowledge among all potential bidders in the entry stage. Each entrant may or may not be aware of the number of active competitors A.

SAF *(i) Preliminary signals and private values* (S_i, V_i) *are i.i.d. across bidders (*$F(S_1, ., S_N,$

$V_1, ., V_N) = \Pi_{i=1}^N F(S_i, V_i)$); *(ii)* (S_i, V_i) *is affiliated for each* i; *(iii) Marginal distributions* F_{S_i} *and* F_{V_i} *are continuous and increasing over bounded supports* $[\underline{s}, \overline{s}]$ *and* $[\underline{v}, \overline{v}]$ *for all* i.

The SAF (which stands for "*symmetry and affiliation*") condition requires that bidders' private information (S_i, V_i) are independently and identically distributed across bidders. It also requires that each bidder's preliminary signal S_i is affiliated with his/her private values to be drawn in the bidding stage. Suppose $r > \underline{v}$. Let $\bar{\omega}_i(s_i, k; s_{-i})$ denote ex ante expected utility for bidder i with signal s_i if potential competitors follow monotone, pure-strategy BNE characterized by cutoffs $s_{-i} \equiv (s_j)_{j \neq i}$. That is, $\overline{\omega}_i(s_i, k; s_{-i}) \equiv E[u((V_i - P_i)_+ - k)|s_i, A_{-i} = \{j : S_j \geq s_j\}]$.

Lemma 3 *Under SAF,* $\overline{\omega}_i(s_i, k; s_{-i})$ *is increasing in* s_i *and non-decreasing in* s_{-i} *for any fixed* k.

Lemma 3 accommodates cases where S_j takes either discrete or continuous values. The proof of Lemma 3 exploits the fact that affiliation exists between S_i and V_i for each bidder i, while each bidder's private information (S_i, V_i) are independent across all i.

Let $\bar{\omega}(s, k)$ be a shorthand for $\bar{\omega}_i(s, k; (s, ., s)) = E[u((V_i - P_i)_+ - k)|S_i = s; A_{-i} = \{j \neq i : S_j \geq s\}]$. Under SAF, $\bar{\omega}(s, k)$ is the same for all i, and increasing in s due to Lemma 3. Using Lemma 3, we can show symmetric pure-strategy BNE in the entry stage must be characterized as in Lemma 4 below. The proof follows from similar arguments for Lemma 1.

Lemma 4 *Under SAF, there exists a unique pure-strategy BNE in the entry stage of auctions with entry costs* k. *In such an equilibrium, each bidder* i *decides to enter if and only if* $s_i \geq s_k^*$, *where* $\bar{\omega}(s_k^*, k) = u(0)$ *if* $\bar{\omega}(\underline{s}, k) \leq u(0) \leq \bar{\omega}(\overline{s}, k)$ *and* $s_k^* = \underline{s}$ *(or, respectively,* $s_k^* = \overline{s}$*) if* $\bar{\omega}(\underline{s}, k) > u(0)$ *(or* $\bar{\omega}(\overline{s}, k) < u(0)$*).*

Let $\bar{\pi}(s, k) \equiv E[(V_i - P_i)_+ - k|S_i = s, A_{-i} = \{j \neq i : S_j \geq s\}]$, which is identical for all i under SAF. Similar to Section 4, Lemma 5 below suggests bidders' risk attitudes can be identified as long as $\bar{\pi}(s_k^*, k)$ can be constructed from the observable distributions of entry decisions and submitted bids for any k that induces non-degenerate entry probabilities in PSBNE.Its proof is similar to that of Lemma 2.

Lemma 5 *Suppose SAF holds and fix any* k *s.t.* $\underline{s} < s_k^* < \overline{s}$. *Then* $\bar{\pi}(s_k^*, k) = 0$ *if bidders are risk-neutral, and* $\bar{\pi}(s_k^*, k) > 0$ *(or* < 0*) if bidders are risk-averse (or respectively, risk-loving).*

That $\underline{s} < s_k^* < \overline{s}$ is obviously testable using the observed individual entry probabilities. The proof follows from the same argument of Lemma 2. The next proposition shows $\bar{\pi}(s_k^*, k)$ can be recovered from observed entry probabilities and the distribution of transaction prices. By definition,

$$\bar{\pi}(s_k^*, k) \equiv E\left[(V_i - P_i)_+ - k|S_i = s_k^*, A_{-i} = \{j \neq i : S_j \geq s_k^*\}\right] \tag{10}$$

Given SAF, it suffices to show the distributions of V_i given $S_i = s_k^*$ and the distribution of P_i given $A_{-i} = \{j \neq i : S_j \geq s_k^*\}$ can be recovered respectively from data.

Proposition 4 *Suppose SAF holds and $(V_i, S_i)_{i \in N}$ is independent from entry costs. Suppose for k, there exists $\varepsilon > 0$ such that $\underline{s} < s_{k'}^* < \overline{s}$ for all $k' \in (k - \varepsilon, k + \varepsilon)$, and entry costs, entry decisions and transaction prices are observed in data. Then $\bar{\pi}(s_k^*, k)$ is identified.*

Proof. Because (V_i, S_i) are independent across bidders, the joint distribution of (P_i, V_i) conditional on $A_{-i} = \{j \neq i : S_j \geq s_k^*\}$ and $S_i = s_k^*$ can be factored as:

$$F_{P_i|A_{-i}=\{j \neq i: S_j \geq s_k^*\}} \cdot F_{V_i|S_i = s_k^*}. \tag{11}$$

Because (V_i, S_i) are identically distributed across i, for any $t \geq r$,

$$\begin{aligned} & \Pr(P_i \leq t | A_{-i} = \{j \neq i : S_j \geq s_k^*\}) \\ = \ & \textstyle\sum_{m=0}^{N-1} \left[F_V\left(t | S_j \geq s_k^*\right)\right]^m \binom{N-1}{m} F_S(s_k^*)^{N-m-1} [1 - F_S(s_k^*)]^m \end{aligned} \tag{12}$$

where F_S, $F_{V|S \geq s_k^*}$ are the marginal distribution of S_i and the conditional distribution of V_i given S_i, and N is the number of potential bidders including i. (The subscript i is suppressed to simplify notations.) By definition, the probability in (12) is 0 for all $t < r$. Note $F_S(s_k^*) = \Pr(i \text{ enters}|k)$ is identified. The next two lemmas show both terms in (11) are identified under the given conditions. Let a denote the realized number of entrants in an auction.

Lemma A2 *Under SAF, for any k such that $\underline{s} < s_k^* < \overline{s}$, $F_V(t|S_i \geq s_k^*)$ is identified for $t \geq r$ from the distribution of W conditional on any k and any number of entrants a (with $a \geq 2$).*

Lemma A3 *Under SAF, $F_V(t|S_i = s_k^*)$ is identified for any $t \geq r$ and any k s.t. $\underline{s} < s_k^* < \overline{s}$, provided $\Pr(W \leq t|k, a)$ and $\Pr(i \text{ enters}|k)$ are observed in an open neighborhood around k for some $a \geq 2$.*

Proofs are included in Appendix A. Lemma **A2** uses the one-to-one mapping between a second-order statistic and the underlying parent distribution. Lemma **A3** builds on the fact that under the independence between private information and entry costs, $\frac{\partial}{\partial K} \Pr(V_i \leq t$ *and* i *enters*$|K)|_{K=k} = -F_V(t|S = s_k^*) \left(\frac{dF_S(s_K^*)}{dK}|_{K=k} \right)$. Thus the product in (11), and consequently $\bar{\pi}(s_k^*)$, is identified. □

We leave the construction of a test statistic and derivation of its asymptotic properties for such a model with selective entry to future research.

8 Concluding Remarks

This paper proposes an approach for robust inference of bidders' risk attitudes in ascending auctions where potential bidders make endogenous entry decisions based on their information of entry costs and private value distributions. We show that risk premium can be nonparametrically recovered from the distribution of transaction prices and entry decisions. A test is proposed and shown to have good finite sample performance under various data-generating processes. We have also extend our results to establish identification of bidders' risk preference in a more general auction model where entry is selective. A direction for future research is to derive tests that exploit exogenous variations in the number of potential bidders and thus do not require knowledge about entry costs (see discussions in Section 4.3).

APPENDIX

A Proofs of Identification Results

The proof of Lemma **1** builds on Lemma A1 below.

Lemma A1 *(a) In Model A, $\omega^A(k;\boldsymbol{\lambda}_{-i})$ is continuous and non-increasing in $\boldsymbol{\lambda}_{-i}$ for all k. (b) In Model B, $\omega^B(k_i;\mathbf{k}_{-i})$ is continuous, decreasing in k_i and non-increasing in $\mathbf{k}_{-i}$.*

Proof of Lemma A1. *Proof of part (a).* Recall that by the Law of Iterated Expectations,

$$\omega^A(k;\boldsymbol{\lambda}_{-i}) = u(-k)F_{V|k}(r) + \int_r^{\bar{v}} h(v,k;\boldsymbol{\lambda}_{-i})dF_{V|k}(v) \tag{A1}$$

where for $v > r$,

$$\begin{aligned} h(v,k;\boldsymbol{\lambda}_{-i}) \equiv & \quad u(v-r-k)F_{P_i}(r|k,\boldsymbol{\lambda}_{-i}) + \int_r^v u(v-p-k)dF_{P_i}(p|k,\boldsymbol{\lambda}_{-i}) \\ + & \quad u(-k)[1-F_{Pi}(v|k,\boldsymbol{\lambda}_{-i})] \end{aligned}$$

and we have used the independence between private values V_i conditional on entry costs. Note for any $t \in [r, \bar{v}]$, the event "$P_i \leq t$" can be represented as

$$\cap_{j\in N\backslash\{i\}}\{\text{"}j \textit{ stays out}\text{" or "}j \textit{ enters} \cap V_j \leq t\text{"}\}.$$

Due to the independence between entry decisions and between private values, $F_{P_i}(t|k,\boldsymbol{\lambda}_{-i}) = \prod_{j\neq i}[1-\lambda_j+\lambda_j F_{V|k}(t)]$. The marginal effect of λ_j on this conditional probability is strictly negative for all $j \neq i$ at $\lambda_j \in [0,1]$ and $t \in [r,\bar{v}]$. (Recall $F_{V|k}(r) > 0$ when r is binding.) This implies $h(v,k;\boldsymbol{\lambda}_{-i})$ is decreasing in $\boldsymbol{\lambda}_{-i}$ for any fixed k. Hence $\omega^A(k;\boldsymbol{\lambda}_{-i})$ is decreasing in $\boldsymbol{\lambda}_{-i}$. The continuity follows from Dominated Convergence Theorem.

Proof of part (b). Due to independence between idiosyncratic costs and private values, as well as the assumption of symmetric IPV,

$$\omega^B(k_i;\mathbf{k}_{-i}) = u(-k_i)F_V(r) + \int_r^{\bar{v}} \tilde{h}(v,k_i;\mathbf{k}_{-i})dF_V(v)$$

where $\tilde{h}(v,k_i;\mathbf{k}_{-i})$ is defined in the same way as $h(v,k;\boldsymbol{\lambda}_{-i})$ in (A1), except with k and $F_{Pi}(.|k,\boldsymbol{\lambda}_{-i})$ from Model A replaced by k_i and $F_{Pi}(.|\mathbf{k}_{-i})$ in Model B. By construction, $\tilde{h}$ is decreasing in k_i for all $(v,\mathbf{k}_{-i})$. Besides,

$$\begin{aligned} F_{P_i}(t|\mathbf{k}_{-i}) &= \Pr(\text{“}j \text{ does not enter” or “}j \text{ enters and } V_j \leq t\text{" } \forall j \neq i|\mathbf{k}_{-i}) \\ &= \Pi_{j\neq i}[1-F_K(k_j)+F_K(k_j)F_V(t)] \end{aligned}$$

which is decreasing in $\mathbf{k}_{-i}$ at $t \geq r$. This implies $\tilde{h}$ is non-increasing in $\mathbf{k}_{-i}$. Consequently, $\omega^B(k_i; \mathbf{k}_{-i})$ is decreasing in k_i and non-increasing in $\mathbf{k}_{-i}$. $\square$

Proof of Lemma 1. Given part (a) of Lemma A1, the proof of equilibrium entry strategies in Model A is similar to the risk-neutral case in Levin and Smith (1994) and omitted. To prove equilibrium entry strategies in Model B, suppose some bidder i deviates and enters when $k_i > k^*$. Then his expected utility, conditioning on all other potential competitors following entry strategies characterized by the cutoffs k^*, will be strictly lower than payoff from staying out (because $\omega^B(k_i; (k^*, ., k^*)) < \omega^B(k^*; (k^*, ., k^*)) = u(0)$). On the other hand, if i stays out while his signal is $k_i < k^*$, he is not maximizing his expected utility conditional on others' strategies, because entry could yield higher expected utility (i.e. $u(0)$ $< \omega^B(k_i; (k^*, ., k^*))$). The uniqueness of the equilibrium follows from the monotonicity of $\omega^B(k; (k, ., k))$ in k. $\square$

Proof of Proposition 1. *Proof of part (b).* By definition,

$$\pi^B(k^*) \equiv E[(V_i - P_i)_+ | k^*] - k^*.$$

With data rationalized by symmetric BNE, the entry probability is identical across all potential bidders and is equal to $F_K(k^*)$. This entry probability is identified as the proportion of potential bidders who enter. With F_K known, the cutoff k^* is identified by inverting F_K at this entry probability. By replacing the event "$\lambda_j = \lambda_k^* \; \forall j \neq i$" in our arguments for part (a) with "$A_{-i} = \{j \neq i : K_j \leq k^*\}$", we can use the Law of Total Probability to write the first term in the definition of $\pi^B(k^*)$ as:

$$\textstyle\sum_{a=0}^{N-1} E[(V_i - P_i)_+ | \; A_{-i} = a] \Pr(A_{-i} = a | \; k^*).$$

By independence between K and $(V_i)_{i \in N}$, we have

$$E[(V_i - P_i)_+ | \; A_{-i} = a] = \int_r^{\bar{v}} F_V^a(v) - F_V^{a+1}(v) dv$$

for $a \geq 1$ as in part (a). Besides, $E[(V_i - P_i)_+ | A_{-i} = 0] = E[(V_i - r)_+] = \int_r^{\bar{v}} 1 - F_V(v) dv$. The distribution of $A_{-i} = \#\{j \neq i : K_j \leq k^*\}$, conditional on all potential bidders enter below the cutoff k^*, can be recovered as a binomial distribution derived from $N-1$ independent trials, each with success probability $F_K(k^*)$. Similar to part (a), $F_V(t)$ is (over-)identified for $t \geq r$ from $\Pr(W \leq t | A = m)$ for all $m \geq 2$. $\square$

Proof of Lemma 3. Let $P_i \equiv \max_{j \in A \setminus \{i\}} \{\max\{V_j, r\}\}$. Under SAF, $F(p_i | v_i, s_i, s_{-i}) = F(p_i | s_{-i})$ and $F(v_i | s_i, s_{-i}) = F(v_i | s_i)$. Thus by the Law of Iterated Expectations,

$$\bar{\omega}_i(s_i, k; s_{-i}) = u(-k) \Pr(V_i \leq r | s_i) + \int_r^{\bar{v}} \bar{h}(v, k; s_{-i}) dF_{V_i|S_i}(v | s_i)$$

where $\bar{h}(v,k;s_{-i}) \equiv u(v-r-k)F_{P_i|S_{-i}}(r|s_{-i})+ \int_r^v u(v-p-k)dF_{P_i|S_{-i}}(p|s_{-i})+ u(-k)[1-F_{P_i|S_{-i}}(v|s_{-i})]$. By the Leibniz Rule,

$$\tfrac{\partial}{\partial v}\bar{h}(v,k;s_{-i}) = u'(v-r-k)F_{P_i|S_{-i}}(r|s_{-i}) + \int_r^v u'(v-p-k)dF_{P_i|S_{-i}}(p|s_{-i}) > 0 \qquad \text{(A2)}$$

Thus $\bar{h}$ is increasing in v for fixed s_{-i} and k. By the affiliation of V_i and S_i for all i, the distribution $F(.|s_i)$ is stochastically increasing in s_i. Hence $\bar{\omega}_i(s_i,k;s_{-i})$ is increasing in s_i given k and s_{-i}. To show $\bar{\omega}_i(s_i,k;s_{-i})$ is non-decreasing in s_{-i} given s_i and k, it suffices to show $F_{P_i|S_{-i}}(p|s_{-i})$ is stochastically non-decreasing in s_{-i} for all $p \geq r$, which would imply $\bar{h}(v,k;s_{-i})$ is non-decreasing in s_{-i} for any $v \in [\underline{v},\bar{v}]$. Note for any $t \in [r,\bar{v}]$, the event "$P_i \leq t$" can be written as

$$\cap_{j\neq i}\{\text{"}S_j < s_j\text{"} \cup \text{"}S_j \geq s_j \cap V_j \leq t\text{"}\}$$

Under SAF, $\Pr(P_i \leq t|s_{-i}) = \prod_{j\neq i}[F_{S_j}(s_j) + \Pr(V_j \leq t, S_j \geq s_j)]$. Also note for all t and any $s'_j > s_j$,

$$F_{S_j}(s_j) + \Pr(V_j \leq t, S_j \geq s_j) \leq F_{S_j}(s'_j) + \Pr(V_j \leq t, S_j \geq s'_j).$$

Hence $F_{P_i|S_{-i}}(t|s_{-i})$ is non-decreasing in s_{-i} for all $t \in [r,\bar{v}]$. $\square$

Lemma A2 *Under SAF, for any k such that $\underline{s} < s_k^* < \bar{s}$, $F_V(t|S_i \geq s_k^*)$ is identified for $t \geq r$ from the distribution of W conditional on any k and any number of entrants a (with $a \geq 2$).*

Proof of Lemma A2. By definition, for any $t \geq r$, $\Pr(W \leq t|k, A=a)$ is identical to the distribution of the second-highest order statistic among a independent draws from the same conditional distribution $F_V(.|S_i \geq s_k^*)$. That is, for any $t \geq r$

$$\Pr(W \leq t|k,a) = \textstyle\sum_{m=a-1}^{a} \binom{n}{m} F_V(t|S_i \geq s_k^*)^m[1 - F_V(t|S_i \geq s_k^*)]^{a-m}$$

Thus for all $a \geq 2$, there exists a one-to-one mapping ϕ_a so that $F_V(t|S_j \geq s_k^*) = \phi_a(\Pr(W \leq t|k,a))$. (Note the mapping ϕ_a does not depend on k as (V_i, S_i) is assumed to be independent from entry costs.) That $\Pr(i \text{ } enters|k) > 0$ for k implies $\Pr(A \geq 2|k) > 0$. Thus $F_V(t|S_i \geq s_k^*)$ is over-identified for $t \geq r$, because the identification arguments above can be applied for any a such that $\Pr(A = a|k) > 0$. $\square$

Lemma A3 *Under SAF, $F_V(t|S_i = s_k^*)$ is identified for any $t \geq r$ and any k s.t. $\underline{s} < s_k^* < \bar{s}$, provided* $\Pr(W \leq t|k,a)$ *and* $\Pr(i \text{ } enters|k)$ *are observed in an open neighborhood around k for some $a \geq 2$.*

Proof of Lemma A3. Given Lemma A2, $F_V(t|S_i \geq s_k^*)$ is identified for any such k using distributions of entry decisions and transaction prices. Hence

$$\begin{aligned}
\Pr(V_i \leq t, S_i \geq s_k^*) &= F_{V|S}(t|S_i \geq s_k^*)\Pr(i\ enters|k) \\
&= \phi_a(\Pr(W \leq t|k,a))\Pr(i\ enters|k) \qquad \text{(A3)}
\end{aligned}$$

is also identified using any a such that $\Pr(A = a|k) > 0$. We consider this joint distribution as known for the rest of the proof. For any $t \geq r$, differentiating this distribution with respect to entry costs at k gives:

$$\begin{aligned}
& \tfrac{\partial}{\partial K}\Pr(V_i \leq t, S_i \geq s_K^*)|_{K=k} \\
= & -\tfrac{\partial}{\partial K}\Pr(V_i \leq t, S_i \leq s_K^*)|_{K=k} = -\left[\tfrac{\partial}{\partial S}\Pr(V_i \leq t, S_i \leq S)|_{S=s_k^*}\right]\left(\tfrac{ds_K^*}{dK}|_{K=k}\right) \\
= & -F_V(t|S = s_k^*)f_S(s_k^*)\left(\tfrac{ds_K^*}{dK}|_{K=k}\right) = -F_V(t|S_i = s_k^*)\left(\tfrac{dF_S(s_K^*)}{dK}|_{K=k}\right),
\end{aligned}$$

where we have used the independence between (V_i, S_i) and entry costs. Hence

$$F_V(t|S_i = s_k^*) = -\frac{\frac{\partial}{\partial K}\Pr(V_i \leq t, i\ enters|K)|_{K=k}}{\frac{d}{dK}\Pr(i\ does\ not\ enter|K)|_{K=k}}$$

because $F_S(s_k^*) = \Pr(i\ does\ not\ enter|k)$ in the pure-strategy BNE and $\frac{d}{dK}F_S(s_K^*) = \frac{d}{dK}\Pr(i$ *does not enter*$|K)$. The denominator is non-zero under the assumption of the proposition. Hence $F_V(t|S_i = s_k^*)$ is identified for $t \geq r$ as long as $\Pr(W \leq t|k,a)$ and $\Pr(i\ enters|k)$ are observed in an open neighborhood around k for some $a \geq 2$. $\square$

B Limiting Distribution of $\sqrt{T}(\hat{\tau}_T - \tau_0)$

Consider Model A' where (i) K vary across auctions independently from $(V_i)_{i\in N}$ over a support $[\underline{k}, \overline{k}]$ such that $\lambda_k \in (0,1)$ for all $k \in [\underline{k}, \overline{k}]$; and (ii) researchers observe $\tilde{K} = K + \epsilon$, while $\epsilon \perp (K, (V_i)_{i\in N})$ with $E(\epsilon) = 0$.

Define three classes of functions over the joint support of $(W, A, \tilde{A})$: $\mathcal{F}_0 \equiv \{1\{W \leq s\}\ for\ r \leq s < \bar{v}\}$; $\mathcal{F}_1 \equiv \{1\{A = m\}\ for\ 2 \leq m \leq N\}$; and $\mathcal{F}_2 \equiv \{1\{\tilde{A} = a\}\ for\ 0 \leq a \leq N-1\}$. It follows from Donsker (1952) that $\mathcal{F}_0$, $\mathcal{F}_1$ and $\mathcal{F}_2$ are all Donsker Classes. By Corollary 9.3.2 in Kosorok (2008), a class formed by taking the pair-wise infimum $\mathcal{F}_3 \equiv \{f_0 \wedge f_1 : f_0 \in \mathcal{F}_0, f_1 \in \mathcal{F}_1\} \equiv \{1\{W \leq s\ and\ A = n\}\ for\ r \leq s < \bar{v}\ and\ 2 \leq n \leq N\}$ is also Donsker. It follows from Theorem 2.1 in Kosorok (2008) that the Donsker property is preserved under finite unions. Thus $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$ is Donsker.

Throughout Appendix B, we use $\mathbb{P}_0$ to denote the true probability measure for $(W, A, \tilde{A})$ in the data-generating process. Let $\mathbb{P}_T$ denote the corresponding empirical measure. For any signed measure $\mathbb{Q}$, let $\mathbb{Q}f \equiv \int f\ d\mathbb{Q}$ (e.g. $\mathbb{P}_0 1\{W \leq s\} = E[1\{W \leq s\}] = \Pr(W \leq s)$). The empirical process $\mathbb{G}_T \equiv \sqrt{T}(\mathbb{P}_T - \mathbb{P}_0)$ (indexed by the Donsker class of functions $\mathcal{F}$)

thus converges weakly to a zero-mean Gaussian Process $\mathbb{G}$, whose covariance kernel is given by $\mathbb{P}_0 hh' - \mathbb{P}_0 h \mathbb{P}_0 h'$ for any pair $h \neq h'$ in $\mathcal{F}$. (We include specific forms of the covariance kernel of $\mathbb{G}$ in the web appendix.)

Lemma B4 below characterizes the weak convergence of estimators for the distribution of transaction prices W. Let $F_{W,m}$ denote the distribution of W given $A = m$. To simplify notations, let $F_{\tilde{W},m}$ denote the section of $F_{W,m}$ over $[r, \bar{v})$ and $\hat{F}_{\tilde{W},m,T}$ denote corresponding estimators for $F_{\tilde{W},m}$ defined in Section 5. Let $\hat{\pi}_T(m) \equiv \mathbb{P}_T 1\{A = m\}$, $\hat{\pi}_T \equiv (\hat{\pi}_T(m))_{m=2}^N$ and π denote its population counterparts (i.e. $\pi(m) \equiv \Pr(A = m)$). Note $\mathbb{P}_0 1\{W \leq s \textit{ and } A = m\}/\pi(m) = \Pr(W \leq s | A = m) \equiv F_{W,m}(s)$. Let $\hat{\rho}_T \equiv (\hat{\rho}_T(a))_{a=0}^{N-1}$ and ρ denote its population counterparts (i.e. $\rho(a) \equiv \Pr(\tilde{A} = a)$).

Let $\{\mathbb{F}_{\tilde{W},m}\}_{m=2}^N$ denote $N-1$ mutually independent, zero-mean Gaussian Processes, each indexed by $[r, \bar{v})$, such that the covariance kernel $(\mathbb{F}_{\tilde{W},m}(s), \mathbb{F}_{\tilde{W},m}(v))$ is given by:

$$\tilde{\mathbf{\Sigma}}_{s,v,m} \equiv \begin{bmatrix} F_{W,m}(s)\,(1 - F_{W,m}(s)) & F_{W,m}(s)\,(1 - F_{W,m}(v)) \\ F_{W,m}(s)\,(1 - F_{W,m}(v)) & F_{W,m}(v)\,(1 - F_{W,m}(v)) \end{bmatrix}$$

for any $s < v$ on $[r, \bar{v})$ and $m \geq 2$. Let $\mathcal{N}_\rho$ be a multivariate normal random vector in $\mathbb{R}^N$ with variance $\rho(a)[1 - \rho(a)]$ on the diagonal, and covariance $-\rho(a)\rho(a')$ off-diagonal.

Lemma B4 *Suppose the Conditions in Model A' (as defined in Section 5) hold. Then*

$$\sqrt{T} \begin{pmatrix} \hat{F}_{\tilde{W},2,T} - F_{\tilde{W},2} \\ \vdots \\ \hat{F}_{\tilde{W},N,T} - F_{\tilde{W},N} \\ \hat{\rho}_T - \rho \\ \hat{\mu}_T - \mu_K \end{pmatrix} \rightsquigarrow \begin{pmatrix} \frac{1}{\sqrt{\pi(2)}} \mathbb{F}_{\tilde{W},2} \\ \vdots \\ \frac{1}{\sqrt{\pi(N)}} \mathbb{F}_{\tilde{W},N} \\ \mathcal{N}_\rho \\ \mathcal{N}_\mu \end{pmatrix} \tag{B4}$$

where $\{\mathbb{F}_{\tilde{W},m}\}_{m=2}^N$ is defined above and $(\mathcal{N}_\rho, \mathcal{N}_\mu)$ is a multivariate normal random vector in $\mathbb{R}^{N+1}$. The covariance between the processes $\{\mathbb{F}_{\tilde{W},m}\}_{m=2}^N$ and $(\mathcal{N}_\rho, \mathcal{N}_\mu)$ is given by

$$\begin{aligned} &Cov\left(\mathbb{F}_{\tilde{W},m}(s), \mathbb{F}_{\tilde{W},m'}(s')\right) = 0,\ Cov(\mathbb{F}_{\tilde{W},m}(s), \mathcal{N}_\rho(a)) = 0, \\ &Var(\mathcal{N}_\rho(a)) = \rho(a)(1 - \rho(a)),\ Cov(\mathcal{N}_\rho(a), \mathcal{N}_\rho(a')) = -\rho(a)\rho(a'), \\ &Cov(\mathbb{F}_{\tilde{W},m}(s), \mathcal{N}_\mu) = \tfrac{1}{\sqrt{\pi(m)}} \left(E[1\{W \leq s, A = m\}K] - F_{W|m}(s) E[1\{A = m\}K]\right), \\ &Cov(\mathcal{N}_\rho(a), \mathcal{N}_\mu) = E[1\{\tilde{A} = a\}K] - \rho(a)\mu_K \text{ and } Var(\mathcal{N}_\mu) = Var(K) + Var(\epsilon), \end{aligned}$$

for all $s \neq s'$ with $s, s' \geq r$, for all $a \neq a'$ in $\{0, ., N-1\}$ and for all $m \neq m'$ in $\{2, ., N\}$.

Proof of Lemma B4. For a set $\mathcal{S}$, let $\mathcal{B}(\mathcal{S})$ denote the space of bounded, real-valued functions with domain $\mathcal{S}$, equipped with the sup-norm. Let κ be a mapping from $\mathcal{B}([r, \bar{v})) \otimes$

$(0,1)^{N-1}\otimes(0,1)^N$ to $\mathcal{B}([r,\bar{v})\otimes\{2,.,N-1\})\otimes(0,1)^N$ such that the resulted function $\kappa(F,\pi,\rho)$ evaluated at any $s\in[r,\bar{v})$, $m\in\{2,.,N-1\}$ and $a\in\{0,.,N-1\}$ is:

$$\kappa(F,\pi,\rho)(s,m,a)\equiv\left(\tfrac{F(s,m)}{\pi(m)},\ \rho(a)\right).$$

With a slight abuse of notation, we also write $\kappa(F,\pi,\rho)(s,m,a)$ as $\kappa(F(s,m),\pi(m),\rho(a))$ below. By construction, $\hat{F}_{\tilde{W},m,T}(s)$ is the first coordinate in $\kappa(\mathbb{P}_T 1\{W\leq s,A=m\},\mathbb{P}_T 1\{A=m\},\mathbb{P}_T 1\{\tilde{A}=a\})$ and $F_{\tilde{W},m}(s)$ is the first coordinate in $\kappa(\mathbb{P}_0 1\{W\leq s,A=m\},\mathbb{P}_0 1\{A=m\},\mathbb{P}_0 1\{\tilde{A}=a\})$ for any $s\geq r$, $m\geq 2$ and $0\leq a\leq N-1$. Because $\mathcal{F}$ is a Donsker Class, the empirical process $\mathbb{G}_T\equiv\sqrt{T}(\mathbb{P}_T-\mathbb{P}_0)$ converges weakly to a tight limiting process $\mathbb{G}$ in $\mathcal{B}(\mathcal{F})$, i.e. the space of uniformly bounded functions defined over $\mathcal{F}$. (See Theorem 2.1 and page 16 in Kosorok (2008).) Besides, under the *Conditions in Model A'*, $\mathbb{P}_0 f>0$ for all $f\in\mathcal{F}_1$ (or equivalently, $\rho(a)>0$ for all $0\leq a\leq N-1$). Therefore κ is Hadamard-differentiable at $\theta_0\equiv(\mathbb{P}_0 f_3,\ \mathbb{P}_0 f_1,\ \mathbb{P}_0 f_2)_{f_3\in\mathcal{F}_3,f_1\in\mathcal{F}_1,f_2\in\mathcal{F}_2}$ (or equivalently, at $(\Pr(W\leq .,A=.),\ \pi(.),\ \rho(.))$) tangentially to the domain of κ. The Hadamard derivative is

$$\kappa'_{\theta_0}(h_1,h_2,h_3)(s,m,a)=\left(\tfrac{h_1(s,m)}{\pi(m)}-\tfrac{h_2(m)\mathbb{P}_0 1\{W\leq s,A=m\}}{\pi(m)^2}\ ,\ h_3(a)\right)$$

for $h_1\in\mathcal{B}([r,\bar{v}))$, $h_2\in(0,1)^{N-1}$, $h_3\in(0,1)^N$. The Functional Delta Method (Theorem 2.8 in Kosorok (2008)) applies to characterize the weak convergence in (B4). The covariance function is thus derived by applying a finite-dimensional, multivariate delta method to the limiting process evaluated at a generic finite set of elements from the index set $[r,\bar{v})\otimes\{2,.,N\}\otimes\{0,.,N-1\}$. The mutual independence between the $N-1$ zero-mean Gaussian processes $\{\mathbb{F}_{\tilde{W},m}\}_{m=2}^N$ and their joint independence from $\mathcal{N}_\rho$ both follow from an application of the Delta Method to these random vectors. Most importantly, independence between $\{\mathbb{F}_{\tilde{W},m}\}_{m=2}^N$ and $\mathcal{N}_\rho$ results from the fact that $\tilde{A}$ is independent from W conditional on A (since K is independent from $(V_i)_{i\in N}$). Finally, the covariance between $\mathcal{N}_\mu$ and the Gaussian Processes $\{\mathbb{F}_{\tilde{W},m}\}_{m=2}^N$ and $\mathcal{N}_\rho$ follows from an application of the Multivariate Delta Method to $\hat{\mu}_T$ and $\mathcal{N}_\rho(a)$ and $\mathbb{F}_{\tilde{W},m}(s)$ for any arbitrarily s,m,a. More details in the derivation are included in the web appendix. $\square$

Next, we characterize the joint limiting behavior of estimators for F_V and ρ. To simplify notations, let $F_{\tilde{V}}$ denote the section of F_V over the support $[r,\bar{v})$, and let $\hat{F}_{\tilde{V},m,T}$ and $\hat{F}_{\tilde{V},T}$ denote the section of $\hat{F}_{V,m,T}$ and $\hat{F}_{V,T}$ over $[r,\bar{v})$. For each m, let $\xi_m(t)\equiv\phi_m^{-1}(t)$ for $t\in[0,1)$. For any $s,v\in[r,\bar{v})$ and $m\geq 2$, let $\mathbf{D}_{0,m}$ denote a 2-by-2 diagonal matrix with diagonal entries being $\xi'_m\left(F_{W,m}(s)\right)$ and $\xi'_m\left(F_{W,m}(v)\right)$. Let $\mathbf{\Sigma}_{s,v}$ be a $(2N-2)$-by-$(2N-2)$ block-diagonal matrix such that the $(m-1)$-th diagonal block is the 2-by-2 matrix $\frac{\mathbf{D}_{0,m}\tilde{\mathbf{\Sigma}}_{s,v,m}\mathbf{D}'_{0,m}}{\pi(m)}$. Define:

$$\underset{\text{2-by-}(2N-2)}{\mathbf{D}_1}\equiv\begin{bmatrix}\frac{1}{N-1} & 0 & \frac{1}{N-1} & 0 & \cdots & \frac{1}{N-1} & 0\\ 0 & \frac{1}{N-1} & 0 & \frac{1}{N-1} & \cdots & 0 & \frac{1}{N-1}\end{bmatrix}.$$

For any $s \geq r$, let $\mathbf{\Sigma}_s$ denote the variance of the limiting distribution of the random vector $\mathbb{G}_T(1\{W \leq s, A = 2\}, 1\{A = 2\}, ..., 1\{W \leq s, A = N\}, 1\{A = N\}, \tilde{K})$ in $\mathbb{R}^{2N-1}$. The specific form of $\mathbf{\Sigma}_s$ is provided in the web appendix along details in the proof of Lemma B4. Define a 2-by-$(2N-1)$ matrix $\mathbf{D}_2$ as:

$$\begin{bmatrix} \frac{1}{N-1} & \cdots & \frac{1}{N-1} & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} \mathbf{D}_\xi \underbrace{\begin{bmatrix} \frac{1}{\pi(2)} & \frac{-p_{s2}}{\pi(2)} & \cdots & 0 & 0 & 0 \\ & & \ddots & & & \\ 0 & 0 & \cdots & \frac{1}{\pi(N)} & \frac{-p_{sN}}{\pi(N)} & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix}}_{N\text{-by-}(2N-1)}$$

where $\mathbf{D}_\xi$ is a N-by-N diagonal matrix with diagonal matrix being $[\xi_2'(F_{W,2}(s)), ., \xi_N'(F_{W,N}(s)), 1]$ and $p_{sm} \equiv \mathbb{P}_0 1\{W \leq s, A = m\}$ for $s \geq r$

Lemma B5 *Suppose $F_{\tilde{V}}$ is continuously distributed with positive densities over $[r, \bar{v})$. Under conditions in Model A',*

$$\sqrt{T}\left(\hat{F}_{\tilde{V},T} - F_{\tilde{V}}\right) \rightsquigarrow \mathbb{G}_V, \tag{B5}$$

where $\mathbb{G}_V$ is a zero-mean Gaussian Process indexed by $[r, \bar{v})$ and is independent from $\mathcal{N}_\rho$. The covariance kernel for $\mathbb{G}_V$ is $(\mathbb{G}_V(s),\mathbb{G}_V(v)) = \mathbf{D}_1\mathbf{\Sigma}_{s,v}\mathbf{D}_1'$ for any $s, v \in [r, \bar{v})$; and $(\mathbb{G}_V(s),\mathcal{N}_\mu)$ is bivariate normal with covariance $\mathbf{D}_2\mathbf{\Sigma}_s\mathbf{D}_2'$ for any $s \in [r, \bar{v})$.

Proof of Lemma B5. Let $\mathbb{D}_\mathcal{S}$ denote the space of bounded functions defined over $\mathcal{S}$ ($\mathbb{D}_\mathcal{S}$ is equipped with the sup-norm). By definition, $\hat{F}_{\tilde{V},m}(s) = \xi_m(\hat{F}_{\tilde{W},m}(s))$ and $F_{\tilde{V}}(s) = \xi_m(F_{\tilde{W},m}(s))$ for all $r \leq s \leq \bar{v}$. Thus ξ_m can also be interpreted as a mapping from $\mathbb{D}_\mathcal{S}$ to $\mathbb{D}_\mathcal{S}$ for each m, and is Hadamard differentiable at $F_{\tilde{W},m}$ tangentially to $\mathbb{D}_\mathcal{S}$. The Hadamard derivative of ξ_m at $F_{\tilde{W},m}$ is given by $\xi_m'(F_{\tilde{W},m}(s))h(s)$. By the Functional Delta Method,

$$\sqrt{T}\begin{pmatrix} \hat{F}_{\tilde{V},2} - F_{\tilde{V}} \\ \vdots \\ \hat{F}_{\tilde{V},N} - F_{\tilde{V}} \end{pmatrix} \rightsquigarrow \begin{pmatrix} \mathbb{F}_{\tilde{V},2} \\ \vdots \\ \mathbb{F}_{\tilde{V},N} \end{pmatrix}$$

where $\{\mathbb{F}_{\tilde{V},m}\}_{m=2}^N$ are $N-1$ mutually independent zero-mean Gaussian Processes, each indexed by $v \in [r, \bar{v})$. These $N-1$ Gaussian Processes are jointly independent from $\mathcal{N}_\rho$. Furthermore, for each $m \in \{2, ., N-1\}$, the covariance function for $\mathbb{F}_{\tilde{V},m}(s), \mathbb{F}_{\tilde{V},m}(v)$ for all $v > s \geq r$ is

$$\mathbf{\Sigma}_{s,v,m} \equiv \mathbf{D}_{0,m}\left[\tilde{\mathbf{\Sigma}}_{s,v,m}/\pi(m)\right]\mathbf{D}_{0,m}'.$$

Since $\hat{F}_{\tilde{V},T} \equiv \frac{1}{N-1}\sum_{m=2}^N \hat{F}_{\tilde{V},m,T}$, another application of the Delta Method shows $\sqrt{T}\left(\hat{F}_{\tilde{V},T} - F_{\tilde{V}}\right)$ converges weakly to $\mathbb{G}_V$, a zero-mean Gaussian Process indexed by $v \in [r, \bar{v})$. The independence of $\mathbb{G}_V$ and $\mathcal{N}_\rho$ follows from independence of $\{\mathbb{F}_{\tilde{V},m}\}_{m=2}^N$ and $\mathcal{N}_\rho$. Again under conditions in Lemma B4, the Delta Method can be applied to show that the covariance kernel

$(\mathbb{G}_V(s), \mathbb{G}_V(v))$ is $\mathbf{D}_1\boldsymbol{\Sigma}_{s,v}\mathbf{D}_1'$ for any $r \leq s < v < \bar{v}$, where $\boldsymbol{\Sigma}_{s,v}$ is a block diagonal matrix defined above (it is a block diagonal matrix because $\{\mathbb{F}_{\tilde{V},m}\}_{m=2}^N$ are mutually independent Gaussian processes). As for the covariance between $\mathbb{G}_V(s)$ and $\mathcal{N}_\mu$ also follows from the Delta Method. (See the web appendix for more details in derivations.) $\square$

Recall that $\mathcal{S}_{[r,\bar{v})}$ denotes the set of functions defined on the support $[r, \bar{v})$ that are strictly positive, bounded, integrable, right-continuous and have limits from the left. Equipped with a sup-norm, $\mathcal{S}_{[r,\bar{v})}$ is normed linear spaces with a non-degenerate interior. With a slight abuse of notation, define $\varphi : \mathcal{S}_{[r,\bar{v})} \mapsto \mathbb{R}_+^N$ as

$$\begin{aligned}\varphi(F) &\equiv (\varphi(a;F))_{a=0}^{N-1}, \text{ where } \varphi(a;F) \equiv \int_r^{\bar{v}} F(s)^a[1-F(s)]ds \\ \hat{\varphi}_T(a) &\equiv \int_r^{\tilde{v}} \hat{F}_{V,T}(s)^a - \hat{F}_{V,T}(s)^{a+1}ds.\end{aligned}$$

By definition, $\hat{\varphi}_T \equiv (\hat{\varphi}_T(a))_{a=0}^{N-1} = \varphi(\hat{F}_{\tilde{V},T})$ and $\varphi \equiv (\varphi(a; F_{\tilde{V}}))_{a=0}^{N-1} = \varphi(F_{\tilde{V}})$. The mapping φ is Hadamard differentiable at F_V^+ tangentially to $\mathcal{S}_{[r,\bar{v})}$. The Hadamard derivative $D_{\varphi,F_{\tilde{V}}}$: $\mathcal{S}_{[r,\bar{v})} \rightarrow \mathbb{R}_+^N$ is

$$D_{\varphi,F_{\tilde{V}}}(h)(a) \equiv \int_r^{\bar{v}} a[F_{\tilde{V}}(s)]^{a-1}h(s) - (a+1)[F_{\tilde{V}}(s)]^a h(s)ds$$

for any $h \in \mathcal{S}_{[r,\bar{v})}$ and $a \geq 1$; and $D_{\varphi,F_{\tilde{V}}}(h)(0) \equiv -\int_r^{\bar{v}} h(s)ds$. To see this, note for $a \geq 1$ and any $t_n \rightarrow 0$ and $h_n \rightarrow h \in \mathcal{S}_{[r,\bar{v})}$ such that $F_{\tilde{V}} + t_n h_n \in \mathcal{S}_{[r,\bar{v})}$ for all n,

$$\begin{aligned}&\frac{\varphi(a; F_{\tilde{V}} + t_n h_n) - \varphi(a; F_{\tilde{V}})}{t_n} \\ \equiv& \int_r^{\bar{v}} \left(\frac{\left[F_{\tilde{V}}(s)+t_n h_n(s)\right]^a - \left[F_{\tilde{V}}(s)\right]^a}{t_n h_n(s)} - \frac{\left[F_{\tilde{V}}(s)+t_n h_n(s)\right]^{a+1} - \left[F_{\tilde{V}}(s)\right]^{a+1}}{t_n h_n(s)} \right) h_n(s)ds \\ =& \int_r^{\bar{v}} \left[aF_{\tilde{V}}(s)^{a-1} - (a+1)F_{\tilde{V}}(s)^a\right] h(s)ds,\end{aligned}$$

where we have used that $S_{[r,\bar{v})}$ is strictly positive over $[r, \bar{v})$. The case with $a = 0$ follows from the same arguments. Define the Jacobian of τ with respect to (φ, ρ, μ_K) evaluated at their true values in the DGP as:

$$[\rho(0), ., \rho(N-1), \varphi(0), ., \varphi(N-1), -1] \equiv [\rho, \varphi, -1].$$

Since φ is Hadamard differentiable at F_V^+ tangentially to $\mathcal{S}_{[r,\bar{v})}$, it follows from Lemma B5 and the Functional Delta Method that

$$\sqrt{T} \begin{pmatrix} \hat{\varphi}_T - \varphi \\ \hat{\rho}_T - \rho \\ \hat{\mu}_T - \mu_K \end{pmatrix} \rightsquigarrow \begin{pmatrix} D_{\varphi,F_{\tilde{V}}}(\mathbb{G}_V) \\ \mathcal{N}_\rho \\ \mathcal{N}_\mu \end{pmatrix} \tag{B6}$$

where $\mathbb{G}_V$ is a zero-mean Gaussian Process defined in Lemma B5 and is independent from $\mathcal{N}_\rho$. The covariance between $\mathcal{N}_\rho$ and $\mathcal{N}_\mu$ is given in Lemma B4, while the covariance between $\mathbb{G}_V$ and $\mathcal{N}_\mu$ is given in Lemma B5. An application of the Multivariate Delta Method shows that under the conditions in Lemma B5,$\sqrt{T}\left(\hat{\tau}_T - \tau_0\right) \rightsquigarrow \mathcal{N}_\tau \equiv \rho D_{\varphi,F_{\tilde{V}}}(\mathbb{G}_V) + \varphi\mathcal{N}_\rho - \mathcal{N}_\mu$. To see that the limiting distribution $\mathcal{N}_\tau$ is univariate normal with zero-mean, note that the Gaussian process $\mathbb{G}_V$ is Borel-measurable and tight (see Example 1.7.3. in van der Vaart and Weller (1996)) and that by construction the Hadamard derivative $D_{\varphi,F_{\tilde{V}}}$ is a linear mapping defined over $\mathcal{S}_{[r,\bar{v})}$. It follows from Lemma 3.9.8. of van der Vaart and Wellner (1996) that $\mathcal{N}_\tau$ is univariate normal with zero mean.

C Proof of Consistency and Asymptotic Validity

Let $c_{1-\alpha/2}$ denote the actual $1-\alpha/2$ quantile of the limiting distribution of $\sqrt{T}(\hat{\tau}_T - \tau_0)$ in the data generating process. Our first step is to establish $\hat{c}_{1-\alpha/2,T} \xrightarrow{p} c_{1-\alpha/2}$ as $T \to +\infty$ by showing the consistency of bootstrap. We proceed by verifying the consistency conditions specified in Beran and Ducharme (1991).

First, as stated in Proposition 2, the limiting distribution of $\sqrt{T}(\hat{\tau}_T - \tau_0)$ is characterized by the joint distribution of W (transaction prices), A (the number of entrants) and $\tilde{K}$ (noisy measures of entry costs). Let P_0 denote the true distribution in DGP. Across auctions in data, $(W_t, A_t, \tilde{K}_t)$ are i.i.d. draws from P_0, and by the Uniform Law of Large Numbers, the empirical distribution of $(W_t, A_t, \tilde{K}_t)$ converges in probability to its population counterpart uniformly over the joint support of $(W, A, \tilde{K})$. Second, Proposition 2 suggests that given any permissible joint distribution of $(W, A, \tilde{K})$ underlying the DGP, the limiting distribution of $\sqrt{T}(\hat{\tau}_T - \tau_0)$ is continuous over $\mathbb{R}$.

Let $G_T(.;P)$ and $G_\infty(.;P)$ denote respectively the finite sample distribution and the limiting distribution of $\sqrt{T}(\hat{\tau}_T - \tau_0)$ when the actual joint distribution of $(W, A, \tilde{K})$ in DGP is given by a generic permissible P. The third and last condition in Beran and Ducharme (1991) to be checked is that for any sequence of permissible distributions P_T that converge to P_0 in sup-norm as $T \to +\infty$, the finite sample distribution of $\sqrt{T}(\hat{\tau}_T - \tau_0)$ under the DGP P_T (denoted by $G_T(.;P_T)$) converges to $G_\infty(.;P_0)$ pointwise on the real line. To see this, note for any $s \in \mathbb{R}$,

$$\left|G_T(s;F_1) - G_\infty(s;F_2)\right| \leq \left|G_T(s;F_1) - G_\infty(s;F_1)\right| + \left|G_\infty(s;F_1) - G_\infty(s;F_2)\right|. \tag{C7}$$

For any s, the first term on the right-hand side of (C7) converges to 0 as $T \to +\infty$ (by the definition of convergence in distribution). Let $\|.\|_\infty$ denote the sup norm. It follows from the continuity of the limiting distribution characterized by Proposition 2 that, for any $s \in \mathbb{R}$ and $\varepsilon > 0$, there exists $\eta > 0$ (possibly depending on s, ε) such that for any F_1, F_2 with $\|F_1 - F_2\|_\infty \leq \eta$, the second term is smaller than ε for T large enough. Now suppose a sequence of distribution P_T converges to P_0 uniformly over $\mathbb{R}$ as $T \to +\infty$. For any $\varepsilon > 0$, we can always send T to be large enough so that $\|P_T - P_0\|_\infty$ is small enough to induce $|G_T(s; P_T) - G_\infty(s; P_0)| \leq \varepsilon$. (To see this, replace F_1, F_2 in (C7) with P_T, P_0 respectively.) Thus conditions for bootstrap consistency in Beran and Ducharme (1991) hold. Hence the bootstrap estimator for the finite sample distribution of $\sqrt{T}(\hat{\tau}_T - \tau_0)$ converges in probability uniformly to the limiting distribution characterized in Proposition 2, *regardless of bidders' risk attitudes in DGP*. (To simplify notations, we suppress dependence of $\mathcal{N}_\tau$ on the joint distribution of $(W, A, \tilde{K})$ in DGP for the rest of the proof.) With the limiting distribution of $\sqrt{T}(\hat{\tau}_T - \tau_0)$ being absolutely continuous with positive densities almost surely over $\mathbb{R}$, the bootstrap estimator for the $1 - \alpha/2$ quantile $\hat{c}_{1-\alpha/2,T}$ converges in probability to the actual $1 - \alpha/2$ quantile of the limiting distribution (denoted by $c_{1-\alpha/2}$) as $T \to +\infty$. Again, such consistency holds regardless of bidders' risk attitudes in DGP.

Suppose in the true DGP, bidders are risk-averse with $\tau_0 = c$ for some $c > 0$. By definition, $\Pr\left(\sqrt{T}\hat{\tau}_T \geq \hat{c}_{1-\alpha/2,T} | \tau_0 = c\right) = \Pr\left(\sqrt{T}(\hat{\tau}_T - \tau_0) - \hat{c}_{1-\alpha/2,T} \geq -\sqrt{T}\tau_0 | \tau_0 = c\right)$. It follows from Proposition 2 that $\sqrt{T}(\hat{\tau}_T - \tau_0)$ converges in distribution to a zero-mean, univariate normal $\mathcal{N}_\tau$. For any $\varepsilon \in (0,1)$, let $c_\varepsilon < +\infty$ denote the ε-quantile of $\mathcal{N}_\tau$. Since $\hat{c}_{1-\alpha/2,T} \xrightarrow{p} c_{1-\alpha/2}$ and $\tau_0 > 0$ under H_A, we have $\lim_{T\to+\infty} \Pr(\hat{c}_{1-\alpha/2,T} < c_\varepsilon + \sqrt{T}\tau_0 | \tau_0 = c) \to 1$ for any $\varepsilon \in (0,1)$. Hence for any $c > 0$,

$$
\begin{aligned}
& \Pr\left(\sqrt{T}\hat{\tau}_T \geq \hat{c}_{1-\alpha/2,T} | \tau_0 = c\right) \geq \Pr\left(\sqrt{T}\hat{\tau}_T \geq \hat{c}_{1-\alpha/2,T} \textit{ and } \hat{c}_{1-\alpha/2,T} < c_\varepsilon + \sqrt{T}\tau_0 | \tau_0 = c\right) \\
> \; & \Pr\left(\sqrt{T}\left(\hat{\tau}_T - \tau_0\right) \geq c_\varepsilon \textit{ and } \hat{c}_{1-\alpha/2,T} < c_\varepsilon + \sqrt{T}\tau_0 | \tau_0 = c\right) \\
\to \; & \Pr\left(\sqrt{T}\left(\hat{\tau}_T - \tau_0\right) \geq c_\varepsilon | \tau_0 = c\right) = 1 - \varepsilon
\end{aligned}
$$

as $T \to +\infty$. This proves the consistency of our test under fixed alternatives of risk-averse bidders ($H_A : \tau_0 = c$ with $c > 0$). Symmetric arguments show

$$
\lim_{T\to+\infty} \Pr\left(\sqrt{T}\hat{\tau}_T \leq -\hat{c}_{1-\alpha/2,T} | \tau_0 = c\right) = 1
$$

for any $c < 0$ (bidders are risk-loving in DGP). If bidders are risk-neutral with $\tau_0 = 0$,

$$
\begin{aligned}
& \Pr\left(-\hat{c}_{1-\alpha/2,T} \leq \sqrt{T}\hat{\tau}_T \leq \hat{c}_{1-\alpha/2,T} | \tau_0 = 0\right) \\
= \; & \Pr\left(\sqrt{T}\left(\hat{\tau}_T - \tau_0\right) + \hat{c}_{1-\alpha/2,T} \geq 0 \textit{ and } \sqrt{T}\left(\hat{\tau}_T - \tau_0\right) - \hat{c}_{1-\alpha/2,T} \leq 0 | \tau_0 = 0\right) \\
\longrightarrow \; & \Pr(-c_{1-\alpha/2} \leq \mathcal{N}_\tau \leq c_{1-\alpha/2}) = 1 - \alpha \text{ as } T \to +\infty,
\end{aligned}
$$

where the second equality follows from that $\sqrt{T}\left(\hat{\tau}_T - \tau_0\right) \xrightarrow{d} \mathcal{N}_\tau$ and $\hat{c}_{1-\alpha/2,T} \xrightarrow{p} c_{1-\alpha/2}$ ($1-\alpha/2$ quantile of the zero-mean normal variable $\mathcal{N}_\tau$) and an application of the Continuous Mapping Theorem.

D Figures

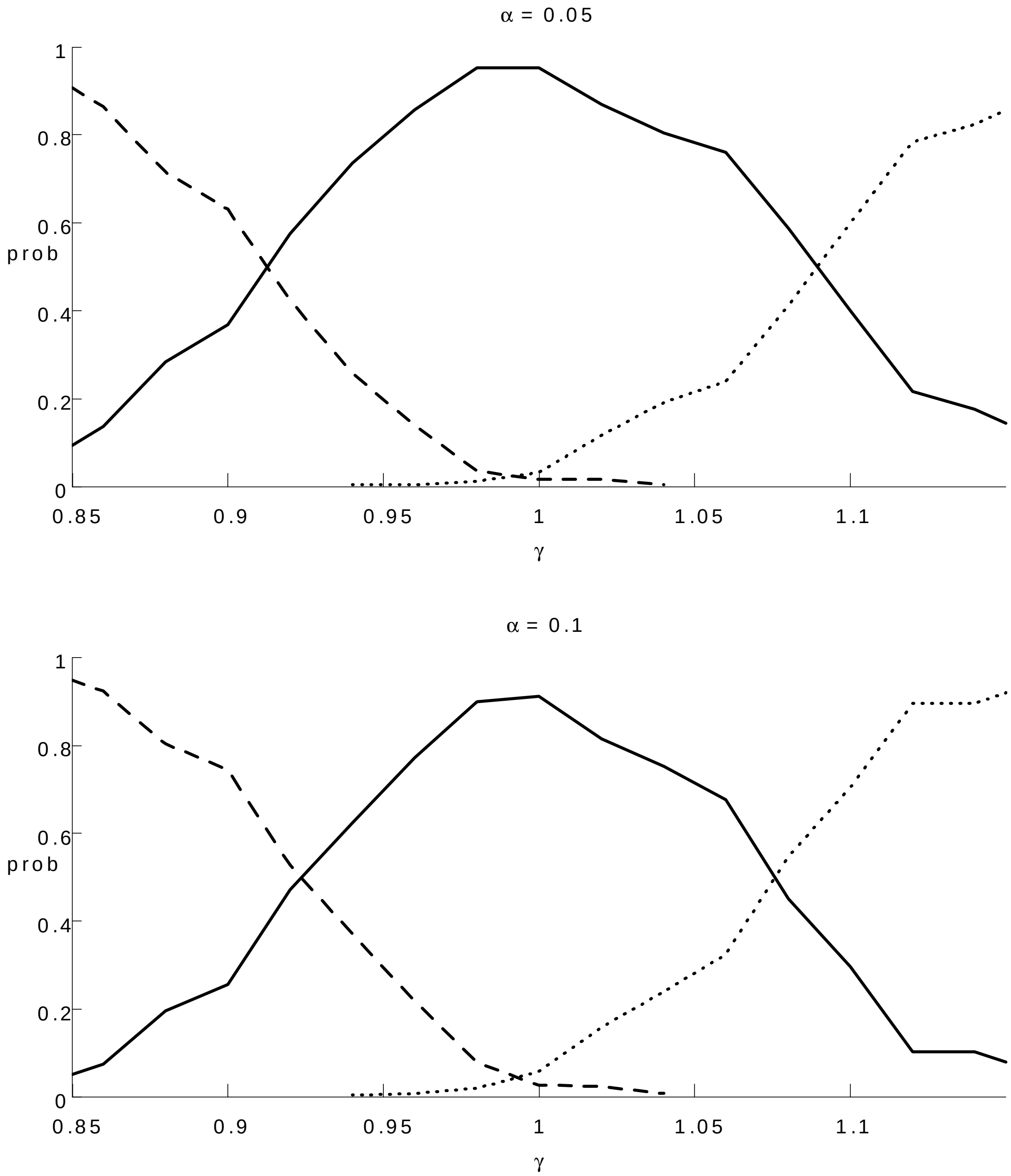


Figure 1 (a): Performance of the test given various risk attitudes in DGP.
($K = 7/10$. $T = 5000$.)

Notes: Solid lines plot the proportion of $S = 300$ simulated samples in which our test fails to reject the null of risk-neutrality. Dashed lines plot the probability that the null is rejected in favor of H_A (risk-aversion with $\gamma < 1$). Dotted lines plot the probability that the null is rejected in favor of H_L (risk-loving with $\gamma > 1$).

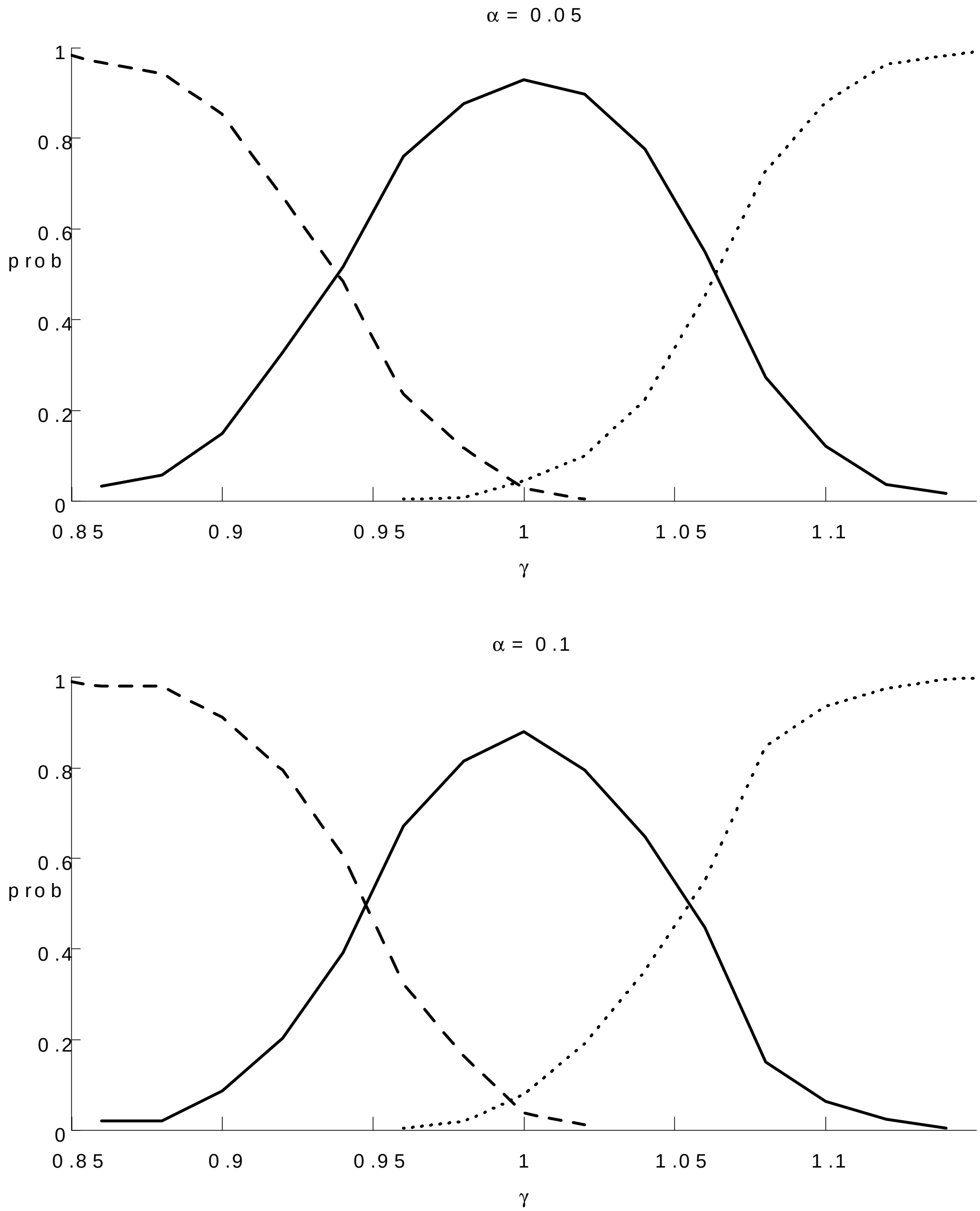
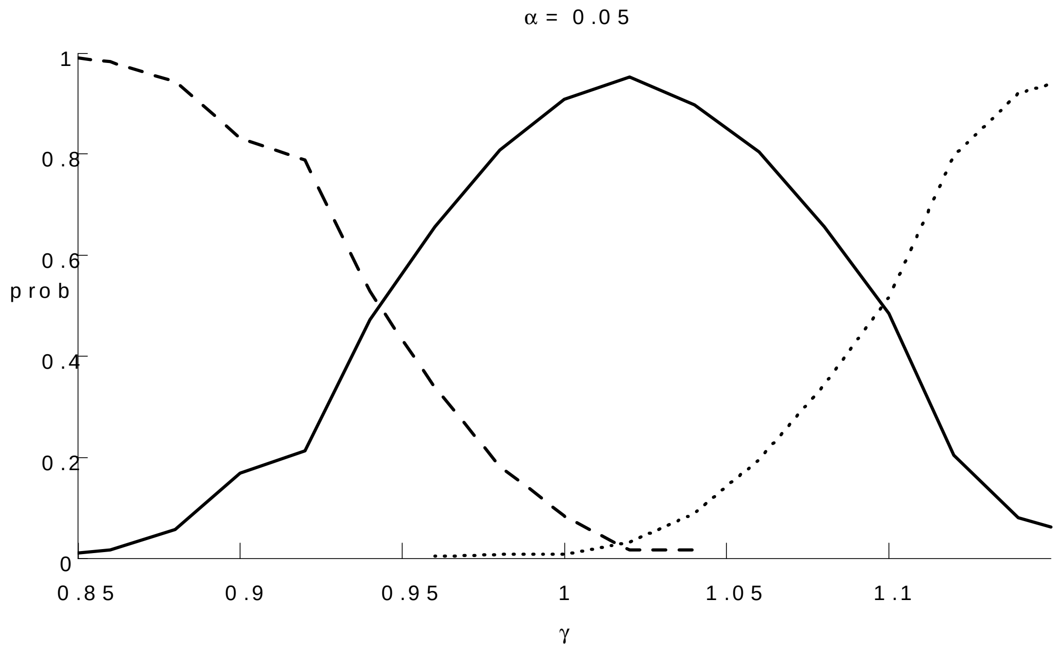


Figure 1 (b): Performance of the test given various risk attitudes in DGP.
$(K = 7/10.\ T = 10000.)$



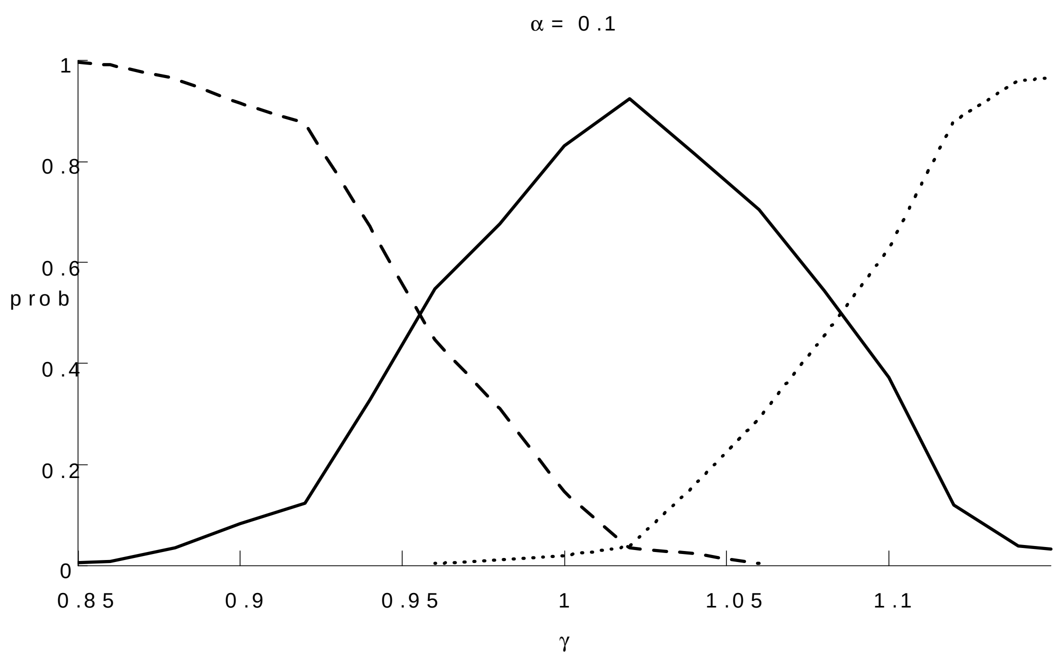
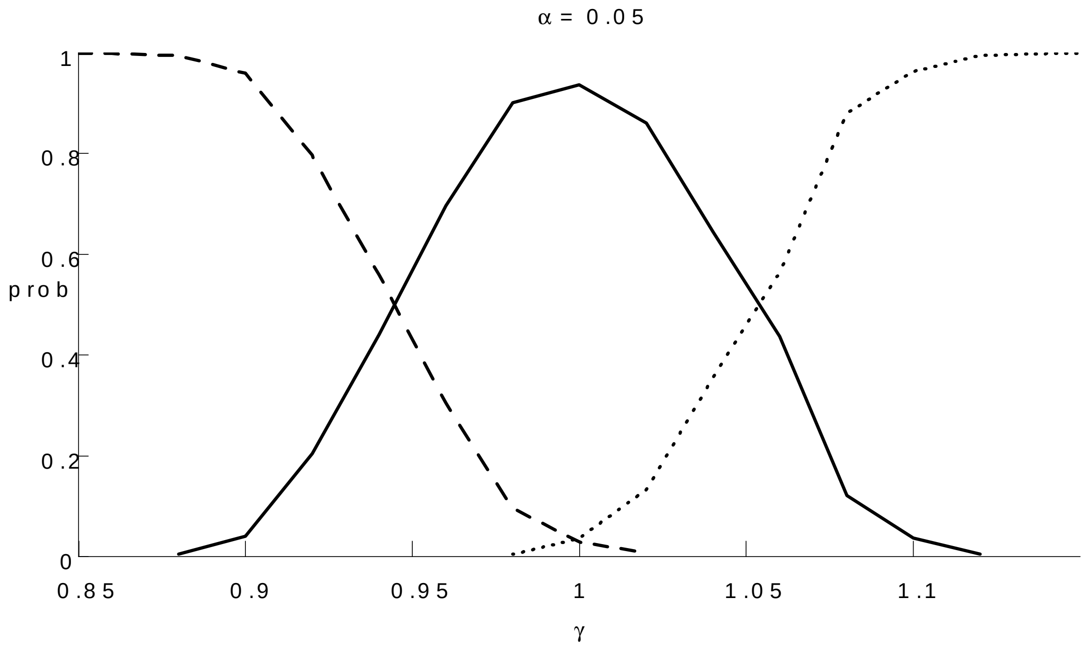


Figure 1 (c): Performance of the test given various risk attitudes in DGP.
$(K = 9/10.\ T = 5000.)$



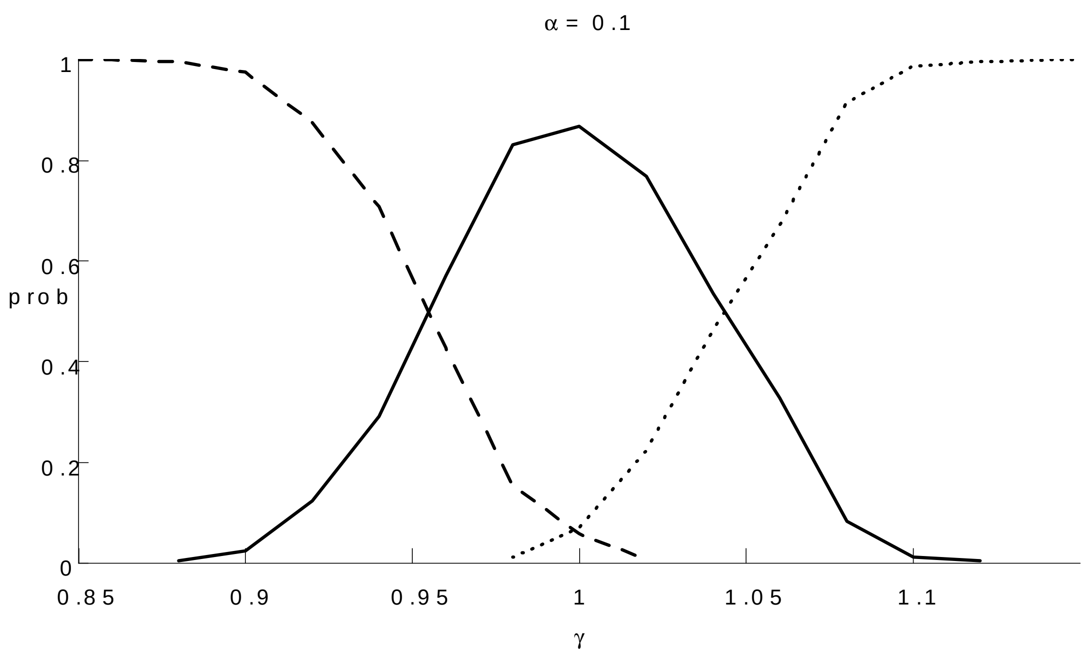


Figure 1 (d): Performance of the test given various risk attitudes in DGP.
($K = 9/10$. $T = 10000$.)

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