PIER Working Paper 12-012

"Dynamic Quality Signaling with Moral Hazard"

by

Francesc Dilmé

http://ssrn.com/abstract=2034393

Dynamic Quality Signaling
with Moral Hazard[*]

Francesc Dilmé[†]

March 19, 2012

Abstract

Asymmetric information is an important source of inefficiency when assets (like firms) are transacted. The two main sources of this asymmetry are unobserved idiosyncratic characteristics of the asset (for example, quality) and unobserved idiosyncratic choices (actions done by the current owners). We introduce moral hazard in a dynamic signaling model where heterogeneous sellers exert effort to affect the distribution of a stochastic signal (for example sales or profits) of their firms. Buyers observe the signal history and make price offers to the sellers. High-quality sellers try to separate themselves from the less quality ones in order to receive high price offers, while the latter try to pool with the first group to avoid receiving a low price. We characterize the competitive equilibria of the model, and we propose an adaptation of existing refinements to the incorporation of moral hazard in dynamic signaling that implies uniqueness of equilibria. We find that similar individual characteristics across types of sellers make everyone worse off, since competition increases signaling waste. Also, due to the new intensive margin (effort), non-trivial signaling will take place even when the cost of signaling is large. In particular cases, we find analytical solutions, that allow transparent comparative statics analysis. The model can be applied to education where grades depend not only on the students' skills, but also on their effort.

Dynamic Signaling, Dynamic Moral Hazard, Endogenous Effort.

JEL Classification: D82, D83, C73, J24.

---

1   Introduction

We develop a model of dynamic signaling with moral hazard. Buyers learn about the type (i.e., quality) of the asset through a noisy signal that takes place over time. At each moment, the signal depends on the effort exerted by the owner of the asset, which is unobserved by buyers. The cost of providing effort is type-dependent, and effort is more costly for the owners of low-quality assets. The model features equilibria where buyers use the history of realizations of the signal to update their beliefs about the type of the asset and make price offers. The signaling process takes place until either the seller accepts an offer or voluntarily leaves the market by keeping the asset for himself.

The difference in the cost of signaling allows for partial separation in equilibrium. The signal can be used by the high-quality sellers to separate themselves from the low-quality sellers and potentially get a higher price. Nevertheless, since low-quality sellers can mimic the signal process of high-quality sellers, they will try to pool with them, in order to avoid getting low price offers.

Our model tries to capture the fact that, for some non-homogeneous assets (like firms), the owner (potential seller) of the asset has private information about its quality. Potential buyers observe some signals of the quality of the asset, like sales, profits, etc. Nevertheless, the owner may put some (unobservable) effort into altering the signal distribution in order to convince the buyers that the quality of the asset is high. The cost of signaling an asset as being of "high" quality is likely to be lower if the actual quality of the asset is high. The buyers, knowing the incentives of the sellers but not the effort exerted by them, should use the signal to correctly infer the quality.

To our knowledge this is the first model in the literature of dynamic signaling that incorporates moral hazard. Although signaling and moral hazard have been studied separately in dynamic models (see the literature review below), their interaction has not been previously analyzed. We analyze the two main sources of inefficiency in signaling with hidden actions: the non-observability of the type (idiosyncratic characteristics) and the non-observability of effort (idiosyncratic effort choice). The model predicts that when idiosyncratic characteristics are similar across types (similar cost of effort) the inefficiency due to the non-observability of effort is higher. The reason is that when types are similar, it is more difficult for high-quality sellers to separate themselves from low-quality sellers. In equilibrium, both types increase the effort put into signaling, so the signal is less informative and there is a high signaling waste. Furthermore, we compare our results with the case where effort is perfectly observable, where we can sometimes get full efficiency. This difference highlights the loss in efficiency due to the incentive compatibility restriction given by the moral hazard.

Like most signaling models, our model exhibits a large multiplicity of equilibria. We adapt one of the standard refinements in this literature (D1 criterion of Cho and Kreps (1987)) to dynamic signaling with moral hazard. Under this refinement we have existence and uniqueness of equilibria in wide regions of the parameter space. These equilibria are characterized by pooling regions (regions where all asset owners immediately accept the equilibrium price offers) and partially separating regions (regions where signaling takes place and there is no trade). In some of these equilibria all high-quality assets are sold, even when the outside option of the sellers is high.

Our model can also be applied to education. Indeed, high-level education is by nature a dynamic process where information (signals) is progressively realized over time. Grades, prizes and test results stochastically depend on individual characteristics (type), such as innate skills, and individual choices, such as effort. Students, knowing their type and past history, decide how much effort they exert to affect the new signals to come. On the other side of the market, firms use the observable signals to infer information about the productivity of each student and use it to make wage offers. If the (utility) cost of effort is correlated with innate skills such as productivity, different types of students would exert different levels of effort. Therefore, the signal history can be used to infer choices, correlated with individual characteristics.

The organization of the paper is as follows. After this introduction, we review the related literature. In Section 2 we discuss our base model, where there is no fixed cost per unit of time. In Section 3 we analyze the robustness of the previous results when we introduce a fixed cost per unit of time. Section 4 concludes. An appendix contains the proofs of all lemmas and propositions of the previous sections.

1.1 Literature Review

Our model is closely related to the literature on preemptive offers, which provides a rationale for why unproductive education may last for long periods of time. Indeed, as Weiss (1983) pointed out, in Spence's (1973) model, beliefs about the type after the first day of class should degenerate toward high productivity, so firms can make offers at this point and obtain (part of) the reduction in the worker's educational costs. Nöldeke and van Damme (1990) assume that workers have different educational cost rates and receive public offers by firms. In their model, delaying the acceptance of offers signals low educational costs and therefore high productivity type. Swinkels (1999) introduces the possibility of private offers, and Hörner and Vieille (2009) make similar arguments in an adverse selection environment. In these models the signals of the workers are given by the rejection of (public or private) offers. Our model focuses on the stochastic nature of the signaling process, as well as on the importance of the intensive margin given by the effort put

into signaling. We find that even when education is not productive and firms can make offers at any moment in time, there is a delay in the expected time of accepting an offer.

There also exists a literature on dynamic signaling where a (non-random) signal is accompanied by a (random) grade, which is correlated with the agents' type. Daley and Green (2009) present a static version of this model, and are able to characterize the corresponding signaling equilibria. Kremer and Skrzypacz (2007) and Daley and Green (2011) introduce dynamic stochastic signaling models. They focus on trade and adverse selection, not on moral hazard. Indeed, in these models the only information that is unknown to the firm is the workers' type (and not the effort put into signaling). We will see that having an extra intensive margin (effort) introduces a moral hazard problem with new interesting economic tradeoffs.

Although our model contains dynamic moral hazard, the modeling assumptions and the questions asked are very different from other models in this literature. Indeed, most of the dynamic moral hazard literature (see Chapter 10 of Bolton and Dewatripont (2005) for a survey) focuses on the design of contracts between agent(s) and principal(s) that optimize the hidden actions chosen by agents. In our model, the effort of the sellers is only valuable for signaling reasons (quality remains unchanged), so buyers do not benefit from it. Furthermore, competition among buyers and the inability of the seller to commit will greatly reduce the set of possible contracts (in our model, equilibria).

Finally, our paper is partially related to the literature on reputations. Indeed, our model has one agent with an unobservable type who performs unobservable actions to pool/separate himself with/from other types. In this literature, inaugurated by the seminal works of Kreps and Wilson (1982) and Milgrom and Roberts (1982), the model closest to ours is in Faingold and Sannikov (2007), set in continuous time. Our model focuses on the buyers' optimal price offer strategy (prices are in general given in reputation models). Furthermore, the nature of our problem (where the only purpose of the dynamic process is signaling) makes the model here much more tractable, which allows rich comparative statics and general equilibrium analysis.

2 Base Model

We begin with the simplest version of our model, where there is no fixed cost per unit of time. This will allow us to get analytical results for the effort and value functions and characterize the resulting equilibria. The next section will be devoted to analyzing the robustness of the results obtained in this section to the introduction of a fixed cost per unit of time.

There is one (potential) seller who owns an asset. The quality (type) of the asset may be either low (then we call the seller an L-seller) or high quality (then we call him an H-seller). Buyers share a common prior $p_0 \in [0, 1]$ about the asset's quality being high.[1]

Time is continuous, $t \in \mathbb{R}_+$. For each seller, there is a continuous-time stochastic process X on a probability space $(\Omega, \mathcal{F}, \mu)$, whose distribution is determined by an effort process provided by the seller. If the effort process that a seller exerts is $(e_t)_{t \in \mathbb{R}_+}$, with $e_t \in \mathbb{R}_+ \; \forall t \in \mathbb{R}_+$, then the signal satisfies the following stochastic equation

$$\mathrm{d}X_t = e_t \, \mathrm{d}t + \sigma \, \mathrm{d}B_t \; ,$$

where B_t is the standard one-dimensional Brownian motion. The cost of effort is type-dependent, given by $c_\theta(\cdot)$. We assume that $c_\theta(0) = 0$, $c'_\theta(\cdot) \geq 0$ and $c''_\theta(\cdot) \geq 0$, for all $\theta \in \{H, L\}$.

We assume $c_H(e) < c_L(e) \; \forall e \in \mathbb{R}_{++}$, that is, the H-seller increases the drift of X at a lower cost than the L-seller. The difference in the cost of signaling implies that the signal may be used to separate sellers. Indeed, the H-seller may try to distinguish himself from the L-seller by increasing the drift of X. Since complete pooling (that is, both types choosing the same effort) may be too costly for the L-seller, the signal may have information about the type. Nevertheless, given the stochastic nature of the signal and the fact that the volatility is independent of the effort, the information about the type will progressively arrive over time. In this section we will use the following specification for the cost of effort:

$$c_\theta(e) \equiv A_\theta \, e^\alpha \quad \forall \theta \in \{L, H\} \; ,$$

with $\alpha > 1$, $\alpha \neq 2$, and $A_H < A_L$. In the next section we will introduce a fixed cost c_0 to the previous cost function.

Buyer Competition

There is a continuum of identical competitive buyers. The value of an asset with a given quality is common across all buyers. The value of an H-asset to a buyer is Π, while the value of an L-asset to a buyer is 0. The seller can leave the market at any moment, and take an outside option that provides him a present value of $0 \leq \Pi_0 < \Pi$.[2] Although it is not important for most of our results,

---

[1] Past performance of the asset, market conditions, etc, may be used by buyers to form priors about the quality of the asset. In the educational interpretation of our model, students have idiosyncratic observable characteristics such as race, sex, city/neighborhood, etc.

[2] This outside option can be interpreted as keeping the ownership of the asset. In the context of dynamic signaling in education, it can be thought of as home production or an additional competitive sector in the economy where the productivity of the workers is Π_0 independent of the type.

we will assume that the outside option is type-independent. This will make the arguments more intuitive and the algebra simpler.[3]

At each moment of time, two identical buyers meet the seller and make him a private price offer. These buyers are short-lived. We denote the maximum of these 2 offers and Π_0 the offer (stochastic) process P_t, and assume that is such that $P_t \in [\Pi_0, \Pi]$ for all histories. We assume that it is a right-continuous process.[4] No transfers take place during the signaling process. If the seller accepts an offer P_t at some time t, the process ends, and the buyer makes a lump-sum payment P_t to the seller.

A strategy for a seller of type $\theta \in \{L, H\}$ is a stochastic process for the effort[5] $(e_{\theta,t})_t$ and a (stochastic) rejection policy $(r_{\theta,t}(\cdot))_t$. We will focus on pure strategies on the decision to accept an offer. This assumption facilitates our analysis and avoids tedious technicalities that unnecessarily complicate the model. Then, given $\omega \in \Omega$ and $t \in \mathbb{R}_+$, $r_{\theta,t}$ is a function from the possible offers $[\Pi_0, \Pi]$ to $\{0, 1\}$, where 0 means rejecting the offer and 1 accepting it. We assume that, fixing $\omega \in \Omega$, $r_{\theta,t}^{-1}(1)$ is a right-upper-hemi-continuous correspondence a.s. This implies that given some offer process P_t and $r_{\theta,t}(\cdot)$, $r_{\theta,t}(P_t)$ is right continuous in histories where $r_{\theta,t}(P_t) = 0$.

Fix an offer process P and a strategy $(e_\theta, r_\theta(\cdot))$ for θ-sellers, for $\theta \in \{L, H\}$. Let $\tau_{\theta,t} \equiv \inf\{s \geq t | r_{\theta,s}(P_t) = 1\}$ be the first time that an offer is accepted after t. The payoff for θ-sellers is composed by the flow cost of providing effort and the lump-sum payoff when the game stops. Since, by right-continuity of P_t, we have $\lim_{s \downarrow \tau_{\theta,t}} P_s = P_{\tau_{\theta,t}}$ a.s., the payoff for θ-sellers has the following expression:

$$V_{\theta,t} = \sup_{(e_\theta, r_\theta)} \mathbb{E}_t\left[-\int_t^{\tau_{\theta,t}} c_\theta(e_{\theta,s})\,\mathrm{d}s + P_{\tau_{\theta,t}} \,\Big|\, e_\theta, r_\theta\right].$$

It is easy to see that since P is right-continuous, this is a right-continuous process. Furthermore, note that $V_{L,t} \leq V_{H,t}$. Indeed, the H-seller has the option of mimicking the strategy of the L-seller. Since in this case the signal will have the same distribution, the expected price offer will be the same. Nevertheless, by assumption, the total cost of signaling will be lower.

Note that a history for a single buyer is only given by the public signal history X^t, so we assume that P is measurable with respect to X. Furthermore, the future payoff of the sellers

---

[3]One could interpret the outside option Π_0 as the value of the physical capital of the firm, which is observed by both the buyer and the seller. If the firm is not sold, the seller sells the physical capital at a given price. The type is then interpreted as the non-phyisical capital of the firm, such as name, reputation, etc, whose value is unknown to the buyer.

[4]Right-continuity ensures that information has a well-defined first time of arrival.

[5]We assume that e_θ is such that the signaling process X and the total cost are well defined. Therefore, we assume its Ito integral is well defined. To see the technical details for this, see Oksendal (2003), chapter 3.

at time t only depends on the future path of P, which is a function of X^t. Therefore, we can restrict ourselves to both the seller and the buyer using public strategies, that is, strategies that are measurable with respect to X.

> *Remark* 2.1. Since there is no time discounting and no fixed cost of time, the seller's payoff is affected only by the expected price when the asset is sold (the term $\mathbb{E}_t[P_{\tau_\theta,t}]$) and the total cost of effort. Therefore, at any moment in time, the seller's tradeoff will be to exert high effort and increase the expected price offer or to exert low effort and lower the expected price offer. The channel by which high effort is translated to high expected prices is the signal, by which will be used by the buyers to update their beliefs about the seller.

Beliefs Process

Buyers use the signaling history to update their beliefs about the type of the seller. The payoff to a buyer who makes a price offer is given by the probability of this offer being accepted and, conditional on being accepted, the asset valuation minus the price. So, we need to characterize the beliefs of the buyers after each history in order to determine their strategy.

Buyers will use the (continuous time) Bayes' rule to update their beliefs. Assume that buyers believe that sellers are following a strategy $(e_\theta, r_\theta)_{\theta \in \{L,H\}}$, and that the offer process is P. Then, for histories where $r_{\theta,t}(P_t) = 0$ for all $\theta \in \{L, H\}$,[6] beliefs are updated according to the standard continuous-time Bayesian updating:

$$\mathrm{d}p_t = p_t \left(1 - p_t\right) s_t \,\mathrm{d}\bar{Z}_t \ , \tag{2.1}$$

where $s_t \equiv \frac{e_{H,t} - e_{L,t}}{\sigma}$ and

$$\mathrm{d}\bar{Z}_t \equiv \frac{1}{\sigma}(\mathrm{d}X_t - p_t \, e_{H,t} \,\mathrm{d}t - \left(1 - p_t\right) e_{L,t} \,\mathrm{d}t) \ .$$

Fix a strategy profile $(e_\theta, r_\theta)_{\theta \in \{L,H\}}$, and assume that buyers believe that it is used by the sellers. Consider a seller who exerts an effort process $(\tilde{e}_t)_t$. Then, for histories where $r_{L,t}(P_t) =$

---

[6]Since $r_{\theta,t}(P_t)$ is right continuous when $r_{\theta,t}(P_t) = 0$ for all $\theta \in \{L, H\}$, we can use standard belief updating, so beliefs move continuously.

$r_{H,t}(P_t) = 0$, the drift of p, $\tilde{\mu}$, and the volatility of p, $\tilde{\sigma}$, take the following form:

$$\tilde{\mu}(\tilde{e}_t, p_t, e_t) = \frac{(1 - p_t)\, p_t\, (e_{H,t} - e_{L,t})\, (\tilde{e}_t - p_t\, e_{H,t} - (1 - p_t)\, e_{L,t})}{\sigma^2} \, , \qquad (2.2)$$

$$\tilde{\sigma}(p_t, e_t) = \frac{(1 - p_t)\, p_t\, (e_{H,t} - e_{L,t})}{\sigma} \, , \qquad (2.3)$$

where e_t refers to the vector $(e_{L,t}, e_{H,t})$. Note that when $e_{L,t} = e_{H,t}$ both the drift and the volatility are 0, independent of the effort choice $\tilde{e}_t$. This is an important feature of our model that differs from the standard dynamic signaling models. Indeed, if buyers believe that the signal is uninformative, then the seller cannot change the beliefs of the buyers.

We finally define what beliefs processes are consistent with the strategies of the seller:

Definition 2.1. A beliefs process p is *consistent* with a strategy profile $(e_\theta, r_\theta)_{\theta \in \{L, H\}}$ and a price offer process P if:

- In all histories X^t where no offer is accepted, p_t follows the Bayesian update process (2.1).

- If in some history $r_{L,t}(P_t) = 1$ and $r_{H,t}(P_t) = 0$ (resp. $r_{L,t}(P_t) = 0$ and $r_{H,t}(P_t) = 1$), then beliefs jump to 1 (resp. to 0), that is, $\lim_{s \downarrow t} p_s = 1$.

2.1 Competitive Equilibria

In this section we define our concept of competitive equilibrium.

Note that after given any history it is optimal for θ-sellers to accept a price offer P if it is higher than or equal to the value of continuing to signal, $V_{\theta,t}$, so $r_{\theta,t}(P) = 1$ if $P > V_{\theta,t}$, for all $\theta \in \{L, H\}$. Since, by upper-hemi-continuity, $r_{\theta,t}(\cdot)^{-1}(1)$ is closed, we have that $r_{\theta,t}(P) = 1$ also when $P = V_{\theta,t}$. Let $\hat{p}_t(P)$ be the beliefs of the buyers about the type of the seller if he accepts an offer P being type H, at time t. Then, in equilibrium, it must be the case that $\hat{p}_t(P)$ is given by Bayes' update:

$$\hat{p}_t(P) = \begin{cases} 0 & \text{for } V_{L,t} \leq P < V_{H,t}, \\ p_t & \text{for } P \geq V_{H,t}, \end{cases} \qquad (2.4)$$

where we used that $V_{L,t} \leq V_{H,t}$. Since we are assuming that buyers share beliefs about the seller and behave competitively, their profits will be 0. Then, in equilibrium, we will assume that buyers offer $p_t \Pi$ when when $V_{H,t} \geq p_t \Pi$ and less than $V_{L,t}$ otherwise.

Let's define the competitive equilibrium as usual, that is, making each agent (the seller and the buyers) behave optimally and consistently with everyone else's strategy:

8

Definition 2.2. A *competitive equilibrium* is, for each prior $p_0 \in [0,1]$, a strategy profile and value function of the seller $(V_\theta^{p_0}, (r_\theta^{p_0}, e_\theta^{p_0}))_{\theta \in \{L,H\}}$, an price offer process P^{p_0} and a beliefs process p^{p_0} such that:

1. The seller's strategy profile is a public perfect Bayesian equilibrium given P^{p_0}.

2. At any moment buyers offer $p_t^{p_0} \Pi$ if $p_t^{p_0} \Pi \geq V_{H,t}^{p_0}$ and less than $V_{L,t}^{p_0}$ otherwise.

3. Beliefs satisfy $p_0^{p_0} = p_0$ and are consistent given $(r_\theta^{p_0}, e_\theta^{p_0})_{\theta \in \{L,H\}}$ and the price offer process P^{p_0}.

As is common in settings where the only payoff-relevant variable for the uninformed part of the market is the type, we restrict ourselves to Markovian strategies and Markovian equilibria with beliefs as the state variable:

Definition 2.3. A *Markovian competitive equilibrium* is a competitive equilibrium where both the seller and the buyers follow Markov strategies, the state variable being the beliefs about the type of the seller p_t. In particular, the strategies of the sellers, $(r_\theta, e_\theta)_{\theta \in \{L,H\}}$ and the offer process P are functions of p.

From now on we will focus on Markovian competitive equilibria, and therefore we will call them simply equilibria. The following lemma establishes an important property of Markovian equilibria

Lemma 2.1. *In any (Markovian competitive) equilibrium there exists an (maybe empty) open region $R \subseteq (0,1)$, called rejection region, such that $r_\theta(R) = \{0\}$ and $r_\theta((0,1)\backslash R) = \{1\}$ for all $\theta \in \{L, H\}$, where $r_\theta(p) \equiv r_\theta(p)(P(p))$.*

Proof. The proof is in the appendix on page 31. □

We will say that a competitive equilibrium is *(partially) separating* if $R \neq \emptyset$, and *pooling* otherwise. When the equilibrium is separating, if the seller enters the market with prior $p_0 \in R$, he is partially separated, that is, the accepted price offer will depend on the signal history. If, instead, $p_0 \notin R$, the seller is offered $\max\{p_0 Y, \Pi_0\}$ and he accepts immediately. Therefore, in pooling equilibria, the seller sells at the same price, independent of the quality of his asset.

2.2 The Seller's Problem

We first focus on the problem that the seller faces. As we will see, given the property of the competitive equilibria established in Lemma 2.1, this will be easy for a given rejection region.

In Section 2.3 we will introduce a new refinement and will devote the section to analyzing the competitive equilibria that passes it.

Using the results from Lemma 2.1, and given a Markovian competitive equilibrium with rejection region R and an initial prior $p_0 \in R$, we define the following limits

$$\underline{p} \equiv \sup \left\{ p \leq p_0 \middle| p \notin R \right\}, \tag{2.5}$$

$$\bar{p} \equiv \inf \left\{ p \geq p_0 \middle| p \notin R \right\}. \tag{2.6}$$

Let's also define $\underline{P} \equiv \max\{\underline{p}\,\Pi, \Pi_0\}$ and $\overline{P} \equiv \max\{\bar{p}\,\Pi, \Pi_0\}$. Note that since R is open, $\underline{p} < p < \bar{p}$. Then, if the initial prior lies in the region $(\underline{p}, \bar{p})$, since p_t moves continuously inside R, the process will stop when p_t reaches either $\underline{p}$ or $\bar{p}$ (where the seller will accept the corresponding price offer).

If there is a competitive equilibrium with rejection region R and effort $(e_\theta)_{\theta \in \{L,H\}}$ it is easy to verify that there is another competitive equilibrium with the same rejection region, with $P(R) = \{\Pi_0\}$ and effort given by

$$\tilde{e}_\theta(p) = \begin{cases} e_\theta(p) & \text{if } p \in R, \\ 0 & \text{if } p \notin R, \end{cases}$$

for all $\theta \in \{L, H\}$. Furthermore, for each interval of the form $(\underline{p}, \bar{p}) \subseteq R$ as defined before (note that R is the union of a finite or countable set of disjoint open intervals), there is a competitive equilibrium with rejection region $(\underline{p}, \bar{p})$ and with the same effort in $(\underline{p}, \bar{p})$. Therefore, we will first focus on equilibria where $R \equiv (\underline{p}, \bar{p})$, which will be called *interval equilibria*.

One of the novel features of our model with respect to other dynamic signaling models is the incorporation of unobservable effort. This generates the usual fixed point problem between the beliefs of the buyers about the strategies of the sellers and the strategies that best respond to these beliefs. In order to be able to solve this fixed point problem, we need the following assumption to hold:

Assumption 1. $e_L(\cdot), e_H(\cdot) \in \mathcal{C}^1(R)$, that is, the beliefs of the buyers about the effort choice of the seller are smooth in the rejection region.

Note that the previous assumption restricts the set of possible equilibria, but not the space of best responses by the seller. Therefore, using the standard stochastic control tools, we will guess that $V_\theta \in \mathcal{C}^2(R) \cap \mathcal{C}^0[0, 1]$ and we will solve the Hamilton-Jacobi-Bellman (HJB) equation. Then, a standard verification theorem (for example, Theorem 11.2.2 in Oksendal (2003)) will ensure that the strategy found is optimal and $V_\theta(\cdot)$ is the corresponding value function.

Remark 2.2. In continuous-time signaling models without moral hazard it is not necessary to assume smooth conditions on equilibrium objects such as policy functions. Indeed, solutions are found assuming smoothness and, using verification theorems, showing that the guess found is an actual solution. In a signaling model with moral hazard this is no longer the case. Even in versions of this model in discrete time, the equilibrium effort is the result of a fixed point problem. For given a (Markovian) price process $P : [0,1] \rightarrow [\Pi_0, \Pi]$ and some strategy $(r_\theta, e_\theta)_{\theta \in \{L,H\}}$ (interpreted as the "beliefs of the buyers" about the strategy of the seller), we can use standard dynamic programming results to show that there is a unique best response $(\hat{r}_\theta, \hat{e}_\theta)_{\theta \in \{L,H\}}$ by the seller. Nevertheless, few things can be said about the beliefs of the buyers $(r_\theta, e_\theta)_{\theta \in \{L,H\}}$ such that $\hat{r}_\theta(p) = r_\theta(p)$ and $\hat{e}_\theta(p) = e_\theta(p)$ for all p.

Under Assumption 1 we will be able to find explicit solutions for the continuous-time model. Furthermore, numerical simulations show that discrete-time versions of this model converge to the continuous time solutions found in this paper when the length of the interval gets small. The key asymptotic feature of the limit that makes it tractable is that asymptotically p moves continuously in R, so the solution of the fixed point problem can be obtained from local conditions (and some boundary conditions), instead of the global conditions required in the discrete time versions of the model.

Consider an interval equilibrium with rejection region $R = (\underline{p}, \bar{p})$, with strategy profile for the seller $(e_\theta, r_\theta)_{\theta \in \{L,H\}}$. Consider a prior $p_0 \in R$. As we mentioned before, we assume that $V_\theta \in \mathcal{C}^2(R) \cup \mathcal{C}^0[0,1]$, and standard verification theorems will ensure that the solution found will be the actual solution of the problem. Then the HJB equation for a seller of type $\theta \in \{L, H\}$ is given by

$$0 = \max_{\hat{e}_\theta} \left(- c_\theta(\hat{e}_\theta) + V_\theta'(p)\, \tilde{\mu}(\hat{e}_\theta, p, e(p)) + \tfrac{1}{2}\, \tilde{\sigma}(p, e(p))^2\, V_\theta''(p) \right), \qquad (2.7)$$

with boundary conditions $V_\theta(\bar{p}) = \overline{P}$ and $V_\theta(\underline{p}) = \underline{P}$. We differentiate (2.7) with respect to $\hat{e}_\theta$ to get the first order condition (FOC). We get

$$A_\theta\, \alpha\, \hat{e}_\theta^{\alpha-1} = \frac{V_\theta'(p)\,(1-p)\,p\,(e_H(p) - e_L(p))}{\sigma^2}. \qquad (2.8)$$

As we see, after imposing $\hat{e}_\theta = e_\theta(p)$ for all $\theta \in \{L, H\}$, Assumption 1 is a necessary condition for $V_\theta \in \mathcal{C}^2(R)$.

The following lemma establishes the functional form of the policy functions

Lemma 2.2. *The unique policy functions $(e_L^*(\cdot), e_H^*(\cdot))$ that solve the system of HJB equations (2.7) (one for each type $\theta \in \{L, H\}$) and are consistent with the beliefs of the buyers,[7] have the following functional form:*

$$e_H^*(p) = C_H \left(\frac{1-p}{p}\right)^{\frac{1}{\alpha}} \quad and \quad e_L^*(p) = C_L \left(\frac{p}{1-p}\right)^{\frac{1}{\alpha}}, \tag{2.9}$$

where C_L and C_H are positive constants uniquely determined by the boundary conditions on the value functions.

Proof. The proof is in the appendix on page 31. $\qquad\square$

Notice that $e_H^*(\cdot)$ is decreasing and $e_L^*(\cdot)$ is increasing. The intuition as to why that is the case is as follows. As p gets close to the truth, p moves slowly, and therefore it is less worthy to exert effort. Indeed, for any $e_H(\cdot)$ and $e_L(\cdot)$ (not necessarily equilibrium) we have

$$\left|\frac{\mu_p(e_H(p), p, e(p))}{\mu_p(e_L(p), p, e(p))}\right| = \frac{1-p}{p} .$$

Therefore, for example, when p is close to 1, the drift of the beliefs conditional on being of type H (and therefore playing $e_H(\cdot)$) is close to 0. Furthermore, we will see in Section 2.4 that the probability of reaching $\bar{p}$ is convex for the L-seller and concave for the H-seller (see Figure 4 (a)). Hence, the L-seller's expected gain from increasing the buyers' beliefs is higher the higher are the beliefs, so the L-seller have more incentives to exert high effort when p is high. The reverse is true for the H-seller.

Using the FOC (2.8) and the policy functions (2.9), we can find an integral expression for $V_\theta(\cdot)$

$$V_\theta(p) = \underline{P} + \int_{\underline{p}}^{p} \frac{A_\theta \, \alpha \, \sigma^2 \, e_\theta(q)^{\alpha-1}}{(1-q) \, q \, (e_H(q) - e_L(q))} \, \mathrm{d}q . \tag{2.10}$$

The boundary condition $V_\theta(\bar{p}) = \overline{P}$ determines the value for C_L and C_H. Therefore, using the expressions for the policy functions (2.9) we have a system of two equations and two unknowns. Unfortunately, it is not possible to find an analytical expression for C_L and C_H.

A system of two equations involving integrals with two unknowns may be slow to solve numerically. Nevertheless, we can compute C_L and C_H solving sequentially, using first that $V_H(\bar{p}) = V_L(\bar{p}) = \overline{P}$, so

$$0 = \frac{\alpha \, \sigma^2 \, A_H}{C_H^{2-\alpha}} \int_{\underline{p}}^{\bar{p}} \frac{\left(\frac{1-q}{q}\right)^{\frac{\alpha-1}{\alpha}} - \frac{A_L}{A_H} \left(\frac{C_L}{C_H}\right)^{\alpha-1} \left(\frac{1-q}{q}\right)^{-\frac{\alpha-1}{\alpha}}}{(1-q) \, q \left(\left(\frac{1-q}{q}\right)^{\frac{1}{\alpha}} - \frac{C_L}{C_H} \left(\frac{1-q}{q}\right)^{-\frac{1}{\alpha}}\right)} \, \mathrm{d}q . \tag{2.11}$$

---

[7]That is, $e_\theta^*(p) = e_\theta(p)$ for all $p \in (\underline{p}, \bar{p})$ and $\theta \in \{L, H\}$.

Since $C_H \neq 0$, the previous expression is an equation for $\frac{C_L}{C_H}$. Note that $\frac{C_L}{C_H}$ is only a function of $\underline{p}$, $\bar{p}$, $\frac{A_L}{A_H}$ and α. Once we know the value of $\frac{C_L}{C_H}$, C_H can be obtained as follows:[8]

$$\overline{P} = V_H(\bar{p}) = \underline{P} + \frac{\alpha\,\sigma^2\,A_H}{C_H^{2-\alpha}} \underbrace{\int_{\underline{p}}^{\bar{p}} \frac{\left(\frac{1-q}{q}\right)^{\frac{\alpha-1}{\alpha}}}{(1-q)\,q\left(\left(\frac{1-q}{q}\right)^{\frac{1}{\alpha}} - \frac{C_L}{C_H}\left(\frac{1-q}{q}\right)^{-\frac{1}{\alpha}}\right)}\,\mathrm{d}q}_{\equiv K_H} \; . \tag{2.12}$$

Note that, as well as $\frac{C_L}{C_H}$, K_H defined above is only a function of $\underline{p}$, $\bar{p}$, $\frac{A_L}{A_H}$ and α. Finally, we get C_L using $C_L = \frac{C_L}{C_H} C_H$.

We can use the equation for (2.10) and the definition of K_H to get the following expression for V_H:[9]

$$V_H(p) = \underline{P} + \frac{\overline{P} - \underline{P}}{K_H}\int_{\underline{p}}^{p} \frac{\left(\frac{1-q}{q}\right)^{\frac{\alpha-1}{\alpha}}}{(1-q)\,q\left(\left(\frac{1-q}{q}\right)^{\frac{1}{\alpha}} - \frac{C_L}{C_H}\left(\frac{1-q}{q}\right)^{-\frac{1}{\alpha}}\right)}\,\mathrm{d}q \; . \tag{2.13}$$

The previous expression implies the remarkable fact that $V_H(\cdot)$ (and the same can be proven for $V_L(\cdot)$) is independent of an important parameter of the signaling process, the volatility σ.[10] Nevertheless, the usual intuition says that when σ is small, it is easier for the H-seller to signal himself, and therefore it should be cheaper to increase the beliefs of the buyers.

One could think that the non-dependence of the value functions on the volatility is driven by "competition" between types of sellers. Indeed, fixing $\frac{A_L}{A_H}$, lower σ provides both types of sellers with higher incentives to increase the effort, so the increase in efficiency because of a more accurate signal can be counterbalanced by more inefficiency coming from the incentives of players to pool (by the L-seller) or separate (by the H-seller) with/from the other type. Nevertheless we will see that even in the case where $A_L = \infty$ (and therefore $e_L(p) = 0 \; \forall p$), $V_H(\cdot)$ is independent of σ. Therefore, even in the extreme case where only the high-quality seller can make the effort to separate himself from the low quality seller, the cost of changing the beliefs of the buyers is independent of the accuracy of the signal.

---

[8]Note that, given $\frac{C_L}{C_H}$, (2.12) is an equation for C_H only if $\alpha \neq 2$. The reason is that when $\alpha = 2$, both the HJB equation (2.7) and the FOC (2.8) are homogeneous in $(e_H(\cdot), e_L(\cdot))$. This implies that the optimal effort choice is linear in the effort beliefs by firms (Figure 2), so there is no equilibrium with $e_\theta(\cdot) \neq 0$ for some $\theta \in \{L, H\}$.

[9]Note that if $\overline{P} = \underline{P} = \Pi_0 \geq \bar{p}\Pi$, then $V_H(p) = \Pi_0$ for all $p \in (\underline{p}, \bar{p})$. Using (2.10) we see that this implies $e_H(p) = 0$ for $p \in (\underline{p}, \bar{p})$. The same argument can be used to see that in this case $e_L(p) = 0$ for $p \in (\underline{p}, \bar{p})$. Since there is no updating of beliefs in $R = (\underline{p}, \bar{p})$ and $r_\theta(R) = \{0\}$, we have $V_H(p) = V_L(p) = 0 < p\Pi$ for all $p \in (\underline{p}, \bar{p})$, what is a contradiction. Therefore $\bar{p} > \frac{\Pi_0}{\Pi}$ and $\overline{P} > \underline{P}$.

[10]Note that V_H is also independent of the levels of A_H and A_L, since $\frac{C_H}{C_L}$ only depends on A_H and A_L through $\frac{A_L}{A_H}$.

Although the seller acts in isolation (there is no strategic interaction between types of sellers), there is an indirect competition between types of sellers. Indeed, all (types of) sellers try to increase the drift of X in order to increase the beliefs of the buyers about them and get higher price offers. The seller "succeeds" if he reaches $\bar{p}$ and "fails" if reaches $\underline{p}$ instead. Since the informativeness of the signal depends on the beliefs of the buyers about the action of both types of sellers, the effort it takes for the seller to succeed depends not only on his individual characteristics (i.e., cost of effort) but also on the other type's characteristics. The following proposition establishes that more similar costs across types make both types worse off, what we call the "*competition effect*":

Proposition 2.1. *Fix α, $\underline{p}$ and $\bar{p}$. Consider two pairs $A_L, A_H, \tilde{A}_L, \tilde{A}_H \in \mathbb{R}_{++}$ satisfying $\frac{A_L}{A_H} < \frac{\tilde{A}_L}{\tilde{A}_H}$. Let V_θ and $\tilde{V}_\theta$ for each $\theta \in \{L, H\}$ be the corresponding equilibrium value functions. Then, for any $p \in (\underline{p}, \bar{p})$ and $\theta \in \{L, H\}$, $V_\theta(p) < \tilde{V}_\theta(p)$.*

Proof. The proof is in the appendix on page 32. $\square$

The intuition behind the previous proposition is the following. The more similar are the types of sellers, the more "competition" one can expect among them. Indeed, when A_L is close to A_H (lower ratio $\frac{A_L}{A_H}$), it is easier for the L-seller to mimic the H-seller, so there will be more signaling waste for the same level of separation. Not only does the H-seller benefit from a higher A_L (and therefore "handicapping" the L-seller), but the L-seller is also better off, since, in equilibrium, the L-seller exerts lower effort. This result is in contrast to some models in races, like Cao (2010), where it is optimal to handicap the advantaged player.

Remark 2.3. In many signaling models, the binding incentive constraint for the most efficient separating equilibrium is the non-mimicking condition for the low-quality seller (see, for example, Riley (1979)). This makes the previous result intuitive, since making mimicking more difficult for the L-seller reduces the inefficiency for all types of sellers. In our model the L-seller still mimics the H-seller by not taking the outside option and waiting for p to reach the boundary of the rejection region. Indeed, below we will consider the $A_L = \infty$ (and $e_L = 0$) case, so the L-seller cannot put any effort into signaling. Nevertheless, since the signal is stochastic, in the model, none of the equilibria will be completely separating, so some inefficiency will persist.

No Effort for L-sellers ($A_L = \infty$)

The remaining sections of this paper will focus on the case where $A_L = \infty$ or, equivalently, the L-seller cannot make any effort, so $e_L(\cdot) \equiv 0$.[11] As we will see, under this assumption we can obtain analytical forms for most of the relevant functions of our basic model (value functions, expected stopping times, profits functions, etc). Furthermore, numerical simulations show that all the relevant conclusions that the model offers under this assumption apply to the general case.[12]

Since now there is no FOC for the L-seller, we have to recalculate the policy function for the H-seller. Doing this we find that the policy functions have the same form as in (2.9), now with $C_L = 0$ and

$$C_H = \left(\frac{(\alpha - 2)(\overline{P} - \underline{P})}{A_H \, \alpha^2 \, \sigma^2 \left(h(\underline{p}) - h(\bar{p}) \right)} \right)^{\frac{1}{\alpha - 2}} \tag{2.14}$$

where

$$h(p) \equiv \left(\frac{p}{1 - p} \right)^{\frac{2 - \alpha}{\alpha}} . \tag{2.15}$$

Note that when $\alpha < 2$, h is increasing, $h(0) = 0$ and $\lim_{p \to 1} h(p) = \infty$. The reverse is true when $\alpha > 2$. The value functions $V_H(\cdot)$ and $V_L(\cdot)$ take the following form

$$V_H(p; \underline{p}, \bar{p}, \underline{P}, \overline{P}) = \frac{h(p) - h(\underline{p})}{h(\bar{p}) - h(\underline{p})} (\overline{P} - \underline{P}) + \underline{P} , \tag{2.16}$$

$$V_L(p; \underline{p}, \bar{p}, \underline{P}, \overline{P}) = \frac{(p - \underline{p})(1 - \bar{p})}{(1 - p)(\bar{p} - \underline{p})} (\overline{P} - \underline{P}) + \underline{P} . \tag{2.17}$$

Note that V_L does not depend on α. The reason is that, as we will see in the next section, the probability of reaching $\bar{p}$ from p is a function of just $\underline{p}$ and $\bar{p}$. Since there is no signaling cost for the L-seller, V_L is just $\underline{P}$ plus the probability of reaching $\bar{p}$ (given in (2.19)) multiplied by $\overline{P} - \underline{P}$.

As mentioned before for the case where $A_L < \infty$, note that neither V_H nor V_L depends on σ^2. To understand this, we will provide some intuition for the case $\alpha > 2$ (the other case is analogous).

---

[11]Our setting is equivalent to assuming a type dependent efficiency-of-signaling function $f_\theta(v) = \frac{v^{1/\alpha}}{A_\theta}$, satisfying $f_\theta(0) = 0$, $f'_\theta > 0$ and $f''_\theta < 0$, for $\theta \in \{L, H\}$. Then, given a cost choice v, the drift of X is $f_\theta(v)$, with $f_H(v) > f_L(v)$ for all $v > 0$. The limit $A_L \to \infty$ corresponds to the case where the L-owner is completely inefficient, that is, unable to change the drift.

[12]The assumption that one of the agents is "handicapped" is common in the reputations literature. Indeed, in many models in this literature there is a behavioral type, that plays an action independently of the history. Here, we allow the L-seller to act strategically through the decision of accepting or not accepting the offer.
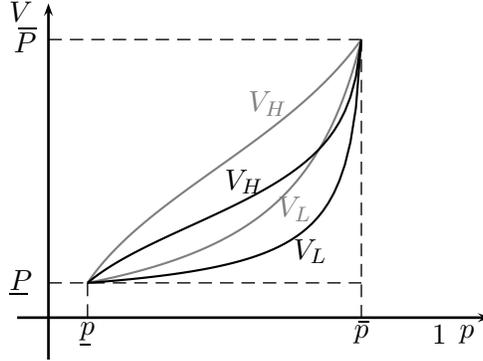
Figure 1: Value functions for $A_L \neq \infty$ and $A_L = \infty$, keeping the other parameters the same. Gray lines correspond to $A_L = \infty$.

If σ decreases, the equilibrium effort of the H-seller increases for all p. Therefore, both the speed of learning (defined as change in p per unit of time, $\tilde{\mu}(\cdot)$) and the effort per unit of time increase. We can then calculate the change in p per unit of effort in the following way:

$$\frac{\tilde{\mu}(e_H(p), p, e_H(p))}{A_H\, e_H(p)^\alpha} = \frac{(1-p)^2\, p\, e_H(p)^2}{\sigma^2\, A_H\, e_H(p)^\alpha} = \frac{\alpha\,(1-p)}{V_H'(p)} \ .$$

As we see, if on the LHS the increase in the numerator is equal to the increase in the denominator, $V_H(\cdot)$ remains unchanged.

Remark 2.4. The reader may wonder why C_H (and therefore $e_H(p)$ for all p) is an increasing function of σ when $\alpha < 2$. Using the FOC (2.8) for $\theta = H$, the optimal effort choice $\hat{e}_H$ is decreasing in σ, holding fixed the beliefs of the buyers (e_L and e_H on the RHS of the equation) and $V_H(p)$ (we know that it is independent of σ). It seems natural that when the signal becomes less accurate, the H-seller would tend to put less effort into signaling his type. This seems to be reinforced in equilibrium, since beliefs of the sellers about $\hat{e}_H$ would have to be consistently lower, lowering further the RHS of (2.8).

Nevertheless, the previous intuition is only valid when $\alpha > 2$. Indeed, when $\alpha < 2$, the previous tatonnement heuristics (a decrease in the beliefs about the effort leads to a decrease in the effort choice that in turn leads to a decrease in the beliefs about the effort) makes the effort choice decrease to 0, which is inconsistent with the assumption $\hat{e}_H(p) > 0$ for $p \in (\underline{p}, \bar{p})$.[13] If, instead, we focus on the non-zero equilibrium-consistent choices, we

---

[13]Note that if $e_H(p) = 0$ for $p > 0$ there is no updating of beliefs, so p remains constant. This implies that it is better for the seller to take the price offer (that is at least $p\,\Pi > 0$) than waiting forever and getting 0. Therefore, in the rejection region, it is always the case that the effort is strictly positive.
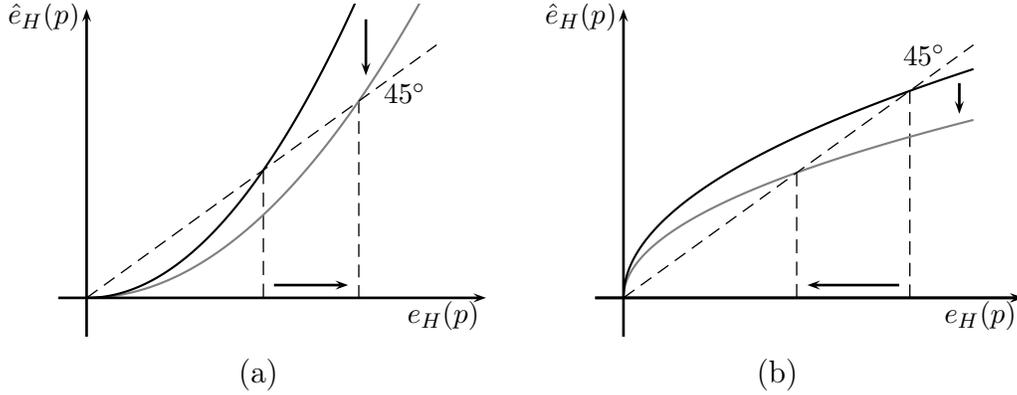
Figure 2: Optimal choice of e given as a function of $\hat{e}_H(p)$ in (2.8), for a given $p \in (\underline{p}, \bar{p})$, where the black line is for σ_1 and the gray line for σ_2, with $\sigma_1 < \sigma_2$. (a) is for $\alpha < 2$ and (b) is for $\alpha > 2$.

see that the action is increasing in the choice. This can be seen in Figure 2, where we see that the effect of increasing σ on the relationship between the beliefs of the buyers about the effort and the optimal response to these beliefs by the H-seller is similar for all α (that is, for the same beliefs $e_H(p)$, the effort choice $\hat{e}_H$ is lower). Nevertheless, the new resulting equilibrium is higher for $\alpha < 2$ and lower for $\alpha > 2$.[14]

2.3 Equilibrium Characterization

In this section we will define a version of the D1 criterion (see Cho and Kreps (1987)) and characterize the equilibria satisfying it. As we will see, the existence of competitive equilibria depends on whether α is higher or lower than 2. In Section 3 we will see that these differences are robust to introducing a fixed cost per unit of time.

The D1 criterion

In our model, given a strategy profile $(e_\theta, r_\theta(\cdot))_{\theta \in \{L,H\}}$ and an offer process P_t, a public history X^t is off the path of play if there exist two times $t_L, t_H < t$ (possibly the same) satisfying $r_{\theta,t_\theta}(P_{t_\theta}) = 1$

---

[14]The intuition is that when σ is large, the H-seller needs more effort to get the same level of separation. In equilibrium, this higher effort makes the signal more informative than when σ is small, which is more efficient when $\alpha < 2$. As we mentioned before, these two effects cancel each other, leaving $V_H(\cdot)$ unchanged.

at X^{t_θ} for all $\theta \in \{L, H\}$. That is, only histories where both types of sellers should have accepted an offer before are off the equilibrium path.

When a deviation by the seller is observed, our definition of consistent beliefs is silent on the value of beliefs immediately afterwards. In particular, lots of equilibria can be generated using the "beliefs threat"; that is, if in a given history everyone should be accepting an offer but someone does not, p and $\hat{p}$ are 0 thereafter, so everyone accepts it. In our model there is an additional "beliefs threat" due to moral hazard. This is given by the fact that firms may believe that, after a given history, the effort of both players is 0 thereafter, and therefore the signal becomes useless.

We adapt Daley and Green's (2009) version of D1 (originally defined by Cho and Kreps (1987)) to allow for dynamic signaling and moral hazard. Instead of defining the criterion in terms of sets of beliefs, we define it using sets of continuation plays, using the fact that in a perfect public Bayesian equilibrium they are new equilibria. While keeping the spirit of the criterion defined for static models, this adaptation allows its use in a dynamic model like ours.[15]

After a deviation is observed, we assume that a new competitive equilibrium will be played afterwards, where the initial beliefs take a new value. The following criterion imposes the constraint that beliefs have to move toward the type that is more likely to be better off deviating:

Definition 2.4. For any $P \in [\Pi_0, \Pi]$, let $\mathcal{E}_\theta(P, p_0)$ be the set of competitive equilibria such that $V_{\theta,0}^{p_0} > P$ and the initial prior is p_0, for $\theta \in \{L, H\}$. A competitive equilibrium $((e_\theta, r_\theta)_{\theta \in \{L,H\}}, P, p)$ satisfies the D1 criterion if, for all t and p_0,

$$
\left.
\begin{array}{l}
r_{H,t}^{p_0}(P_t^{p_0}) = 1 \quad \text{and} \\
\cup_{p_0'} \{(p_0', \mathcal{E}_L(P_t^{p_0}, p_0'))\} \subset \cup_{p_0'} \{(p_0', \mathcal{E}_H(P_t^{p_0}, p_0'))\}
\end{array}
\right\}
\;\Rightarrow\;
\lim_{t' \searrow t} \mathbb{E}_t[p_{t'}^{p_0}] > p_t^{p_0} \;,
$$

where $\subset$ is strict inclusion.

Intuitively, once an unexpected rejection is observed, a perfect Bayesian equilibrium implies that the continuation play should be optimal given some new beliefs. Beliefs should move toward the type that is more likely to reject the offer; that is, the type such that there are more continuation equilibria where he has strict incentives to reject it. Indeed, in our model, in any continuation path, V_H is weakly bigger than V_L. Therefore, an equilibrium satisfies the D1 criterion if for all offers accepted in this equilibrium, there is no equilibrium and history where, if made, the decision to take it would differ a across types.

---

[15]Note that for the model in Daley and Green (2011) it is enough to impose *belief monotonicity*, that is, after an unexpected rejection beliefs do not decrease. The reason is that, in their model, the signal is always informative, since it is type-dependent and exogenous. The signal in our model, since it is endogenous through a hidden effort, is informative only if the efforts of types of sellers differ.

Finally, let us state the following result that will be useful to characterize equilibria satisfying D1:

Lemma 2.3. *Define $\mathcal{R}$ as the set of the rejection regions of all competitive equilibria. If $R \in \mathcal{R}$ is such that $\tilde{R} \subseteq R$ for all $\tilde{R} \in \mathcal{R}$, then any competitive equilibrium with rejection region R satisfies the D1 criterion. If, instead, there is some $\tilde{R} \in \mathcal{R}$ such that $(\tilde{R} \backslash R) \cap (\frac{\Pi_0}{\Pi}, 1) \neq \emptyset$, then the competitive equilibrium with rejection region R does not satisfy the D1 criterion.*

Proof. The proof is in the appendix on page 33. □

For $\alpha < 2$

The following proposition characterizes the unique equilibrium that satisfies the D1 criterion, for $\alpha < 2$:

Proposition 2.2. *When $\alpha < 2$, there is a unique[16] competitive equilibrium that satisfies D1. Its rejection region is $R = (0, \bar{p}_*)$, with*

$$\bar{p}_* \equiv \frac{\alpha - 1}{\alpha} \left(1 + \sqrt{1 + \frac{(2 - \alpha)\,\alpha\,\Pi_0}{(\alpha - 1)^2\,\Pi}} \right) \quad \in \left[\frac{2\,(\alpha - 1)}{\alpha}, 1 \right] . \tag{2.18}$$

Proof. The proof is in the appendix on page 33. □

Figure 3 (a) plots the unique equilibrium described in the previous proposition for different values of Π_0. Note that as Π_0 gets close to Π, the region where signaling takes place (the rejection region) increases. This may seem counterintuitive, since when Π_0 is close to Π the gains from signaling are low because the most that the seller can gain from signaling is $\Pi - \Pi_0$. Nevertheless, as we can see in (2.14), when the boundary payoffs are close, the equilibrium effort is high when $\alpha < 2$. Proposition 2.4 will show that this makes signaling more efficient in this case.

It may seem surprising that an H-asset with initial prior $p_0 \in R$ is sold at $\bar{p}_*$ with probability 1. Indeed, when a signal is random and there is no event that happens with positive probability under one type's strategy and with 0 probability under the other type's strategy, complete information is hardly achievable. One could think that the result is driven by the fact that there is no fixed cost of time, so beliefs could get arbitrarily precise in very long time periods. Nevertheless, as we

---

[16]By unique we mean that the on-the-path-of-playing effort policies, rejection policy functions and accepted offers are uniquely defined.
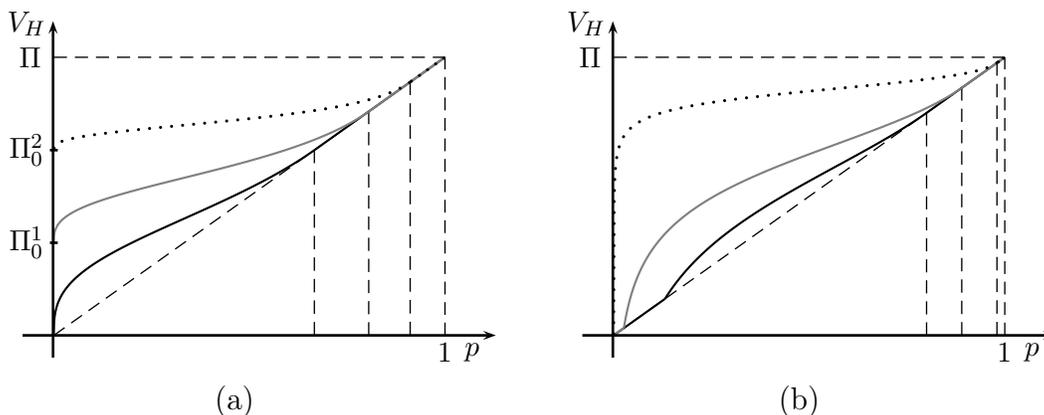
Figure 3: In (a), $V_H(\cdot)$ for equilibria satisfying the D1 criterion for different values of Π_0, when $\alpha < 2$. In (b), $V_H(\cdot)$ for different equilibria when $\alpha > 2$ and $\Pi_0 = 0$.

will see in Section 2.4, the expected time of the seller in the process is bounded. Section 3.1 will show the robustness of this result when we include a fixed cost.

In discrete-time versions of this model, buyers are never perfectly convinced in finite time about the type of the seller. Nevertheless, numerical simulations show that the competitive equilibrium that satisfies D1 is such that the lower bound of R is small. For each length of the period $\Delta > 0$, the rejection region of the D1 equilibrium takes the form $R(\Delta) = (\underline{p}(\Delta), \bar{p}(\Delta))$, and is such that $\lim_{\Delta \to 0} \underline{p}(\Delta) = 0$. Therefore, asymptotically, the type I error tends to 0. Furthermore, since the effort is large around $\underline{p}(\Delta)$ when it is small, beliefs are updated very quickly, and $\underline{p}(\Delta)$ can be reached in a relatively short time.

For $\alpha > 2$

The previous existence and uniqueness result does not carry forward when $\alpha > 2$. As the following proposition states, there is no competitive equilibrium that satisfies the D1 criterion:

Proposition 2.3. *If $\alpha > 2$, no competitive equilibrium satisfies the D1 criterion. In particular, there is a function $\underline{p}_*(\bar{p})$ satisfying $\lim_{\bar{p} \to 1} \underline{p}_*(\bar{p}) = 0$ such that $(\underline{p}_*(\bar{p}), \bar{p})$ is the rejection region of a competitive equilibrium for $\bar{p} < 1$ large enough, but there is no equilibrium with rejection region $(0, 1)$.*

Proof. The proof is in the appendix on page 34. $\qquad\square$

20

We will provide some intuition about this result below, comparing it with the observable effort case. This result is not robust to including a cost per unit of time (i.e., $c_\theta(0) > 0$) or discounting when $\alpha > 2$. Since effort becomes arbitrarily small, the expected time to reach the boundary increases as $\underline{p}$ goes to 0. Therefore, if there is a fixed time cost, the total payoff is negative. In Section 3.2 we will study the analogous result when $c_\theta(0) > 0$.

Observable Effort

In order to understand the previous results (especially for the case where $\alpha > 2$) we consider a variation of our model where the effort made by the seller is observable. Note that in our model we have two sources of inefficiency: the non-observability of the type and the non-observability of the effort. Then, let's see how the model changes when we eliminate one of these two sources.

To make the argument simpler, we still focus on the case $A_L = \infty$. To allow the L-seller to mimic the H-seller,[17] in this section we assume that they can (pretend to) make an observable effort at 0 cost, but that effort leaves the drift of X unchanged. The H-seller, instead, if he makes an observable effort e, they incur a cost $c_H(e)$, but the drift of X is e as before.[18]

Proposition 2.4. *Fix a rejection region R and assume effort is observable. Fix a positive policy function $e_H(\cdot) \in C^1(R)$ and let $V_H(p, e_H)$ be the corresponding value function of the H-seller at p. Then*

1. *If $\alpha < 2$, $V_H(p, \lambda e_H) > V_H(p, e_H)$ for all $\lambda > 1$ and $p \in R$.*

2. *If $\alpha > 2$, $V_H(p, \gamma e_H) > V_H(p, e_H)$ for all $\gamma < 1$ and $p \in R$.*

In the limit $\lambda \to \infty$ and $\gamma \to 0$, signaling waste disappears.

Proof. The proof is in the appendix on page 34. $\square$

Therefore, given a rejection region, signaling is more efficient when effort is high in the case $\alpha < 2$, and the reverse is true when $\alpha > 2$. In particular, when $\alpha > 2$ the equilibrium effort given in (2.14) is small for $\underline{p}$ low and $\bar{p}$ high, and therefore signaling becomes more efficient. Even though it takes more time to reach $\bar{p}$, the cost of effort per unit of time is lower, and the total cost

---

[17]Since effort is observable and $e_L(\cdot) \equiv 0$ when $A_L = \infty$, any observation of effort higher than 0 would lead to perfect knowledge of the type.

[18]This computation is related to Stackelberg action, often used in the reputations literature. The Stackelberg action of a player is the action that he would choose if, at the beginning of the game, he could publicly commit to do at it each period (without considering the incentive constraints in each period's game). Here, we allow for a full strategy, not just a single action.

decreases. In the $\alpha < 2$ case, instead, when $\bar{p}$ is too high, the expected time for p to reach $\bar{p}$ does not decrease, as we will see in the next section (the effort is high, but p moves slowly when it is close to 1). So, when $\alpha < 2$, higher accuracy of the signal does not compensate for the increase in the cost of effort.

2.4   Arrival Probabilities and Expected Finishing Times

To provide some intuition about the previous results, let's compute the arrival probabilities and expected finishing times.

Arrival Probabilities

Fix a rejection region $R \equiv (\underline{p}, \bar{p})$. Let $\pi_\theta(p)$ denote the probability of reaching $\bar{p}$ conditional on the seller's type being $\theta \in \{L, H\}$ and the beliefs of the buyers being $p \in (\underline{p}, \bar{p})$ when the seller follows the equilibrium strategy. Since buyers correctly update beliefs, we can use the Bayes' rule to get expressions with $\pi_H(\cdot)$ and $\pi_L(\cdot)$. These are given by

$$\bar{p} = \frac{p\,\pi_H(p)}{p\,\pi_H(p) + (1-p)\,\pi_L(p)},$$

$$\underline{p} = \frac{p\,(1 - \pi_H(p))}{p\,(1 - \pi_H(p)) + (1-p)\,(1 - \pi_L(p))}.$$

The solutions of the previous equations are the following:

$$\pi_L(p) = \frac{(p - \underline{p})\,(1 - \bar{p})}{(1 - p)\,(\bar{p} - \underline{p})} \quad \text{and} \quad \pi_H(p) = \frac{(p - \underline{p})\,\bar{p}}{p\,(\bar{p} - \underline{p})}. \tag{2.19}$$

Note that this is consistent with the expression (2.17) for $V_L(\cdot)$. Indeed, since there is no cost per unit of time and no discount, $V_L(p) = \underline{P}\,(1 - \pi_L(p)) + \overline{P}\,\pi_L(p)$.

Consider the null hypothesis that the seller is type H. Then, $1 - \pi_H(\cdot)$ is the type I error probability, that is, the probability that the null hypothesis is true but the seller ends up taking the outside option. Figure 4 (a) shows that when $\underline{p} \to 0$, type I error disappears. Indeed, $\lim_{\underline{p} \to 0} \pi_H(p) = 1$ for all $p > 0$.

Expected Arrival Time for $\alpha < 2$

Fix a competitive equilibrium with rejection region $R \equiv (\underline{p}, \bar{p})$. Given a seller with type $\theta \in \{L, H\}$ and prior p_0, let $\tau_\theta(p_0)$ denote the offer-accepting stopping time. Let $T_\theta(p)$ be the expected time
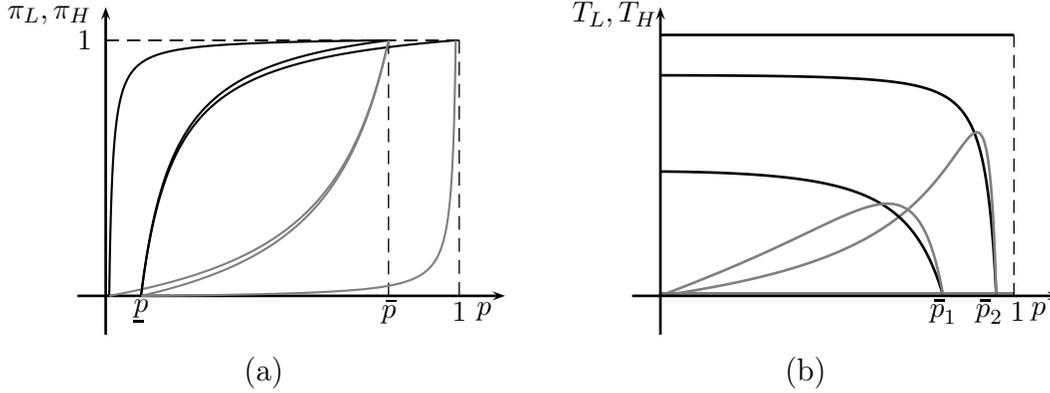
Figure 4: Probability of reaching $\bar{p}$ in (a), and expected arrival times in (b). Gray and black lines correspond to L-sellers and H-sellers, respectively.

before an offer is accepted, that is

$$T_\theta(p_0) \equiv \mathbb{E}\big[\, \tau_\theta \,\big|\, p_0, e_t = e_\theta(p_t)\,\big] = \mathbb{E}\Big[\int_0^{\tau_\theta} \mathrm{d}s \,\Big|\, p_0, e_t = e_\theta(p_t)\Big] \ .$$

Therefore, $T_\theta(\cdot)$ can be thought as the value function for a flow payoff of 1 while the project is active and 0 when it stops. Hence, $T_\theta(\cdot)$ satisfies the following HJB equation:

$$0 = 1 + \tilde{\mu}(e_\theta(p), p, e(p))\, T_\theta'(p) + \tfrac{1}{2}\,\tilde{\sigma}(p, e(p))^2\, T_\theta''(p) \ ,$$

with boundary conditions $T_\theta(\underline{p}) = T_\theta(\bar{p}) = 0$. The previous equation can be analytically solved. We focus on the limiting case $\underline{p} \to 0$, since this is the relevant case for the competitive equilibrium satisfying D1. After some amount of algebra, T_H and T_L can be expressed in the following way

$$T_H(p|\underline{p} = 0) \;=\; \frac{1}{2+\alpha}\left(\frac{(2-\alpha)\,(\bar{p}\,\Pi - \Pi_0)}{\alpha^\alpha\, A_H\, \sigma^\alpha}\right)^{\frac{2}{2-\alpha}}\left(1 - \frac{h(p)^{\frac{2}{2-\alpha}}}{h(\bar{p})^{\frac{2}{2-\alpha}}}\right) ,$$

$$T_L(p|\underline{p} = 0) \;=\; \frac{1}{2-\alpha}\left(\frac{(2-\alpha)\,(\bar{p}\,\Pi - \Pi_0)}{\alpha^\alpha\, A_H\, \sigma^\alpha}\right)^{\frac{2}{2-\alpha}}\left(\frac{(1-\bar{p})\,p}{\bar{p}\,(1-p)} - \frac{h(p)^{\frac{2}{2-\alpha}}}{h(\bar{p})^{\frac{2}{2-\alpha}}}\right) ,$$

where $h(\cdot)$ is defined at (2.15). Figure 4 (b) plots these functions for different values of $\bar{p}$. We see that $T_H(0) \neq 0$. Even though for each $\underline{p} > 0$ we have $T_H(\underline{p}) = 0$, we have $\lim_{p\to 0} T_H(p) > 0$ for all $p > 0$. The rationale, as we explained before, is that $\lim_{\underline{p}\to 0} e_H(\underline{p}) = \infty$, so in the limit the unbounded effort around 0 generates a "wall" in the beliefs.[19]

---

[19]The fact that there is pointwise convergence when $\underline{p} \to 0$ both for $V_\theta(\cdot)$ and $T_\theta(\cdot)$ reinforces the conjecture (that is verified numerically) that the equilibrium described in Proposition 2.2 is the limit of equilibria in a sequence of discrete-time versions of our model. Even though the fact that $\underline{p} = 0$ can be reached in finite time is only true in the continuous-time model, it is asymptotically true in the sequence of equilibria.

3 Fixed Cost of Signaling

In this section we check the robustness of our previous results when we introduce a fixed cost of time. As we will see, the qualitative results when $\alpha < 2$ are essentially unchanged. The results when $\alpha > 2$ change and, instead, a unique D1 equilibrium will always exist.

Now there is a flow cost $c_0 > 0$ of being in the market. For simplicity we still restrict ourselves to the case where $A_L = \infty$, that is, when the L-seller does not make an effort to signal himself. Repeating the same procedure we used in Section 2.2 to get the expression for e_H, we find the following expression:

$$e_H(p) = \left(\frac{C_1 (1-p)}{(2-\alpha)p} + \frac{2 c_0}{A_H (\alpha - 2)} \right)^{1/\alpha}, \tag{3.1}$$

where C_1 is a constant to be determined.

Unfortunately, now there is no closed form for $V_H(p)$. This complicates our analysis, since it is difficult to know the properties of $\underline{p}$ and $\bar{p}$ for $c_0 > 0$. Then, it is impossible to verify whether the equilibrium condition $V_L(p) \geq \Pi_0$ for all p is satisfied.[20] Indeed, to verify this condition we need to know $e_H(\cdot)$, $\underline{p}$ and $\bar{p}$, and then solve for the HJB equation of the L-seller. We instead assume that c_0 is a fixed cost that H-sellers incur if they exert a positive effort, so their total cost is

$$c_H(e) = \begin{cases} c_0 + A_H e^\alpha & \text{if } e > 0, \\ 0 & \text{if } e = 0. \end{cases}$$

The cost (and the effort) for L-sellers remains equal to 0.[21] Under this assumption it is always the case that $V_L(p) \geq \Pi_0$ for all p, for the same reason as in the $c_0 = 0$ case. This implies that Lemma 2.1 still applies in this case.

Before characterizing equilibria, we will establish a lemma about the smoothness of the value function. Note that in our model we cannot apply the usual theorems about smooth pasting conditions for the value functions. In our model many interval equilibria like the one described in

---

[20]Note that if there is a fixed cost it could be optimal for the L-seller at some beliefs to take the outside option rather than waiting to get a higher price offer. This was not the case when $c_0 = 0$, even when $A_L < \infty$. Indeed, since when there is no cost to waiting, if beliefs are not in the rejection region, the L-seller will receive an offer higher than the current price offer with positive probability. Nevertheless, if waiting is costly, it may not be optimal to stay in the market until p hits a boundary of R.

[21]We can interpret this fixed cost of providing an extra effort to increase sales as an opportunity cost of the time devoted to this. In the education setting, this may be regarded as the cost of attending class (opportunity cost in salaries, for example). L-workers, instead, could already be enjoying their outside option, by just taking the exams.

Section 2.2 can be constructed by imposing 0 effort outside the rejection region, without requiring any smooth pasting condition. Nevertheless, the D1 criterion allows us to restrict ourselves to smooth value functions:

Lemma 3.1. *(smooth pasting condition) Assume that there exists an equilibrium satisfying D1 with value function $V_H(\cdot)$ for the H-seller. Then $V_H(p) \in \mathcal{C}^1(0,1)$.*

Proof. The proof is in the appendix on page 35. $\qquad\square$

3.1 Case $\alpha < 2$

As we mentioned before, the results for $c_0 = 0$ when $\alpha < 2$ are qualitatively the same when $c_0 > 0$. The intuitive reason is that, in the equilibrium described in Section 2.3, the H-seller's equilibrium effort is bounded away from zero in the rejection region, $\inf_{p \in R} c_H(e_H(p)) > 0$. Therefore, since the (effort) cost per unit of time is bounded away from 0, including an extra fixed cost should not qualitatively change the results.

The following proposition establishes the existence of equilibria satisfying D1 for all values of c_0 and Π_0:

Proposition 3.1. *For each pair (c_0, Π_0), there exists a unique $\bar{p}_* \in (\frac{\Pi_0}{\Pi}, 1)$ such that $(0, \bar{p}_*)$ is the rejection region of the unique equilibrium satisfying D1.*

Proof. The proof is in the appendix on page 36. $\qquad\square$

It may be surprising that the type I error is 0 even when c_0 is arbitrarily high. We see that when $\Pi_0 > 0$, the rejection region contains $(0, \frac{\Pi_0}{\Pi})$, so signaling takes place in a potentially large region of the beliefs space. In models where the drift is exogenous, the size of the region where the signaling takes place becomes arbitrarily small or disappears when the fixed cost rises. Nevertheless, when the effort is endogenous, the equilibrium effort may be bigger when the fixed cost is high. In this case, the quality of the signal is high when the cost is high and therefore accelerates the process, which compensates for the high cost per unit of time.

3.2 Case $\alpha > 2$

As we saw in Section 2.3, no equilibrium satisfies the D1 criterion when $\alpha > 2$ and there is no discounting and no cost per period. This result is extremely fragile to the second of these assumptions. Let's see what the implications are of introducing a fixed cost per unit of time.

$\Pi_0 = 0$ case

Let's first consider the case where the value of the outside option is 0. The following proposition establishes the conditions for the existence of equilibria in this case:

Proposition 3.2. *Assume $\alpha > 2$ and $\Pi_0 = 0$. Then, there exists $\bar{c}_0 \equiv K \left(\frac{\Pi A_H^{-2/\alpha}}{\alpha \sigma^2} \right)^{\frac{\alpha}{\alpha-2}}$, where $K > 0$ is a constant that depends only on α, such that*

1. *For $c_0 < \bar{c}_0$ there exist some $0 < \underline{p}_*(c_0) < \bar{p}_*(c_0) < 1$ such that $(\underline{p}_*(c_0), \bar{p}_*(c_0))$ is the rejection region of the unique equilibrium that satisfies D1.*

2. *For $c_0 \geq \bar{c}_0$ the only competitive equilibrium is complete pooling and therefore satisfies D1.*

Proof. The proof is in the appendix on page 37. $\qquad\qquad\square$

The intuition for the non-existence of equilibria is clear. A higher cost per unit of time requires a higher effort for the signaling to be informative enough (and then learning fast enough) to make signaling worthwhile. When $\alpha < 2$, we saw that increasing the effort compensates for the increase in cost per unit of time enough to maintain the existence of equilibria. Nevertheless, for $\alpha > 2$, the higher convexity of the cost function makes providing high effort too costly to make signaling worthwhile.

Note that when there is high noise (σ big), the signal is less precise, so signaling becomes less valuable, and the existence constraint gets tighter (i.e. $\bar{c}_0$ gets smaller). The same happens when signaling is more costly (A_H high.) Finally, when Π is higher, separation becomes more profitable, so the existence constraint is relaxed.

Figure 5 plots $\underline{p}$ and $\bar{p}$ of the unique D1 equilibrium as a function of c_0, for Y and σ^2 fixed. We see that for c_0 small $\underline{p}$ is close to 0 and $\bar{p}$ is close to 1. Then, we recover the result established in the Section 2.3 such that, when $c_0 = 0$, we can get equilibria very close to full separation. When c_0 increases, the region of partial separation shrinks, disappearing when c_0 reaches $\bar{c}_0$.

$\Pi_0 > 0$ case

Consider now the case where the outside option is higher than the value of the L-asset to the buyers, i.e. $\Pi_0 > 0$. The following proposition establishes the existence of equilibria in this case, and introduces equilibria where the rejection region is composed of two open intervals instead of one:
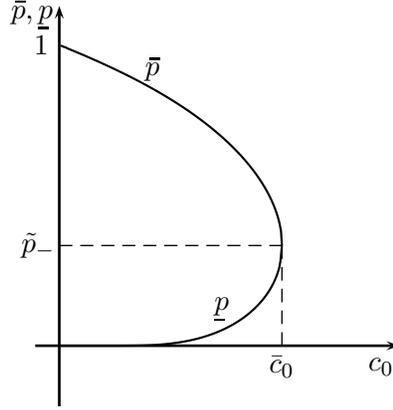
Figure 5: $\underline{p}$ and $\bar{p}$ as a function of c_0 the $\Pi_0 = 0$ case ($\tilde{p}_-$ is defined in the proof of Proposition 3.2.)

Proposition 3.3. *Assume $\alpha > 2$ and $\Pi_0 > 0$. Then there is a unique separating competitive equilibrium that satisfies the D1 criterion. Furthermore, if $c_0 < \bar{c}_0$ ($\bar{c}_0$ defined in Proposition 3.2) there exists some $\bar{\bar{\Pi}}_0(c_0)$ such that, for $0 < \Pi_0 \leq \bar{\bar{\Pi}}_0(c_0)$, the rejection region for the equilibrium that satisfies the D1 criterion is the union of two disjoint open intervals.*

Proof. The proof is in the appendix on page 39. ☐

In Figure 6 (a) we see that when c_0 is low, $\underline{p}$ is close to 0 and $\bar{p}$ is close to 1, that is, the equilibrium is close to full separation, as we obtained in the $c_0 = 0$ case. As c_0 gets large, both curves get closer to $\frac{\Pi_0}{\Pi}$. Indeed, when the fixed cost increases signaling is very costly, so it only takes place in a small region around $\frac{\Pi_0}{\Pi}$. Therefore, when c_0 is large, for most initial beliefs p_0 the seller is either immediately hired (if $p \geq \bar{p}$) or takes the outside option (if $p \geq \underline{p}$), independent of his type. Only in a small neighborhood of the indifference region (when $p_0 \Pi \simeq \Pi_0$) is there signaling and some separation.

In Figure 6 (b) we plot $\underline{p}$ and $\bar{p}$ as a function of $\frac{\Pi_0}{\Pi}$, for two different values of c_0. Again we see that the higher is the cost, the smaller is the region where signaling takes place. Furthermore, we see that when Π_0 is small, the seller accepts an offer immediately for most of the initial priors. The reverse is true when Π_0 is close to Π.

3.3 Observable Effort

As we did in Section 2.3 we now analyze the case when the effort is observable. Note that the result obtained in Proposition 2.4 for $\alpha < 2$ remains the same. Indeed, when the effort is very
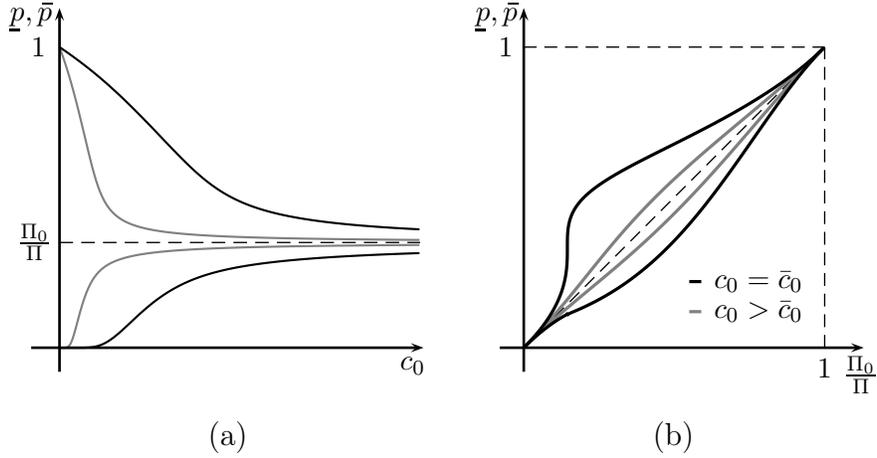
Figure 6: In (a), $\underline{p}$ and $\bar{p}$ as a function of c_0 for a fixed Π_0, and for two different values for σ (the gray line corresponds to the low value of σ). In (b), $\underline{p}$ and $\bar{p}$ as a function of $\frac{\Pi_0}{\Pi}$, for different values of c_0.

large the (expected) signaling time is very small and so is the total fixed cost. Therefore, the seller has further incentives to increase effort.

The result obtained in Proposition 2.4 changes when $c_0 > 0$ and $\alpha > 2$. The reason is that if effort is very low, the (expected) signaling time is very large and therefore so is the total time cost. The following proposition establishes the analogous result for $c_0 > 0$:

Proposition 3.4. *When effort is observable, $c_0 > 0$ and $\alpha > 2$, the optimal effort choice by H-sellers is a constant effort equal to*

$$ e_H(p) = e_*^{OE} \equiv \left(\frac{2\,c_0}{A_H\,(\alpha - 2)} \right)^{\frac{1}{\alpha}}. $$

Proof. The proof is in the appendix on page 41. □

The intuition of the previous result comes from the envelope theorem. When we change p, the change in the payoff of the flow value of the seller (the maximand in expression (2.7)) can be decomposed into a direct change in p and a change in $e_H(p)$. Since the seller now fully internalizes the change in $e_H(p)$, the envelope theorem tells us that only the direct effect takes place, and therefore $e_H(\cdot)$ remains constant.

The previous proposition allows us to recover the case where the drift is constant but type dependent (e_*^{OE} for H-sellers and 0 for L-sellers) and there is a cost per unit of time. Indeed,

models in the literature where the drift depends on the type (but not on the effort) can be reinterpreted as optimal behavior when the effort (but not the type) is observable.

4   Conclusions

We fully characterize the equilibria of a model with dynamic signaling and moral hazard. By introducing a new intensive margin (the effort), the model provides insights into how the interaction between different sources of asymmetric information affects the signal dynamics.

Our model allows us to compare the strength of the two main causes of inefficiency in signaling in dynamic signaling with hidden actions: the non-observability of the type and the non-observability of the effort. We have seen that, the more similar are the types of sellers (that is, the more similar are their cost functions) the lower is their payoff (competition effect). Indeed, if different types of assets are similar, the incentive for the low type sellers to pool with the high types makes the signal less informative and more wasteful. When effort is observable, we saw that full efficiency can be restored when there is no fixed cost of time. When there is a fixed cost of time and the effort curve is convex enough, instead, the optimal effort is constant, which endogenously provides a rationale for some assumptions made in other models in the literature.

Adapting existing refinements to moral hazard in dynamic signaling, we have existence and uniqueness of equilibria in most of the parameter configurations. Hence, we can investigate how the asymmetry of information on the type and the effort affects the misallocation of assets and the signaling waste. The competitive equilibria of our model exhibit a non-rejection region in the space of beliefs, where no asset is traded, and a pooling region, where assets are immediately traded. We see that if the cost function has a low degree of convexity, the signaling region may be wide even if the fixed cost of signaling or the outside option are high. Even with a stochastic signal, all high-quality assets are sold (that is, no type I error takes place) for some parameter values. Although this is not a property of discrete-time versions of this model, numerical simulations show that it is asymptotically satisfied in sequences of equilibria of discrete versions of our model.

When there is no fixed cost of time, the value function of the seller does not depend on the accuracy of the process. This is a property that makes the model suitable for empirical work, since in the prices are independent of the underlying signaling process, which is unobserved by the econometrician. The volatility is still identified by the expected stopping times. If, instead, the fixed time cost is small but positive, this property will still be valid at the first order.

Future research will be devoted to generalizing the results to allow low types to exert effort when there is a fixed cost of time and to introduce additional types. Introducing the usefulness of

the signal (such as productive education) may also introduce new tradeoffs, since the uninformed side of the market will value effort nor only as separation device.

References

Bolton, P., and M. Dewatripont, 2005, *Contract Theory*, vol. 1 of *MIT Press Books*. The MIT Press.

Cao, D. V., 2010, "Racing Under Uncertainty: A Boundary Value Problem Approach," working paper.

Cho, I.-K., and D. M. Kreps, 1987, "Signaling Games and Stable Equilibria," *The Quarterly Journal of Economics*, 102(2), 179–221.

Daley, B., and B. Green, 2009, "Market Signaling with Grades," working paper.

———, 2011, "Waiting for News in the Market for Lemons," *Econometrica, forthcoming*.

Faingold, E., and Y. Sannikov, 2007, "Reputation Effects and Equilibrium Degeneracy in Continuous-Time Games," Cowles Foundation Discussion Papers 1624, Cowles Foundation for Research in Economics, Yale University.

Farber, H. S., and R. Gibbons, 1996, "Learning and Wage Dynamics," *The Quarterly Journal of Economics*, 111(4), 1007–47.

Hörner, J., and N. Vieille, 2009, "Public vs. Private Offers in the Market for Lemons," *Econometrica*, 77(1), 29–69.

Kremer, I., and A. Skrzypacz, 2007, "Dynamic signaling and market breakdown," *Journal of Economic Theory*, 133(1), 58–82.

Kreps, D. M., and R. Wilson, 1982, "Reputation and imperfect information," *Journal of Economic Theory*, 27(2), 253–279.

Milgrom, P., and J. Roberts, 1982, "Predation, reputation, and entry deterrence," *Journal of Economic Theory*, 27(2), 280 – 312.

Noldeke, G., and E. van Damme, 1990, "Signalling in a Dynamic Labour Market," *Review of Economic Studies*, 57(1), 1–23.

Øksendal, B., 2003, *Stochastic differential equations: an introduction with applications.* Springer, 6 edn.

Riley, J. G., 1979, "Informational Equilibrium," *Econometrica*, 47(2), 331–59.

Spence, A. M., 1973, "Job Market Signaling," *The Quarterly Journal of Economics*, 87(3), 355–74.

Swinkels, J. M., 1999, "Education Signalling with Preemptive Offers," *Review of Economic Studies*, 66(4), 949–70.

Weiss, A., 1983, "A Sorting-cum-Learning Model of Education," *Journal of Political Economy*, 91(3), 420–42.

5 Appendix

Proof of Lemma 2.1 (page 9)

Proof. Consider beliefs $p_t \in (0,1)$ are such that $r_L(p_t) \neq r_H(p_t)$. Note that, since $V_H \geq V_L$ and $\{P|r_\theta(p)(P) = 1\}$ is closed, if an offer is accepted by the H-seller, then it is accepted also by the L-seller. Suppose that $r_L(p) = 1$ and $r_H(p) = 0$. At beliefs p, the L-seller receives a payoff of Π_0 (since they are the only ones dropping out) and the H-seller receives a payoff of Π (since beliefs jump to 1 afterwards). Since there is no cost to the L-seller of mimicking the H-seller, he can deviate and get a payoff of Π instead of Π_0, which contradicts optimal behavior. Therefore, in equilibrium, $r_{L,t} = r_{H,t}$ whenever $p_t \in (0,1)$.

Now let's show that R is open. Consider $p \in (0,1)$ such that $r_\theta(p) = 0$ for all $\theta \in \{L, H\}$. By right hemi-continuity of $r_{\theta,t}$ when $r_{\theta,t}(P_t) = 0$ there is continuous beliefs updating around p. Therefore, there exists a neighborhood of p such that where no offer is accepted.

$\square$

Proof of Lemma 2.2 (page 12)

Proof. We impose the equilibrium condition $\hat{e}_\theta(p) \equiv e_\theta(p)$ in the FOC (2.8). Using (2.8) for $\theta = H$ and the corresponding HJB equation (2.7) we find the following expression for $e'_L(p)$:

$$e'_L(p) = \frac{\alpha\, e_L(p) + (2 - \alpha)\, e_H(p)}{\alpha\, p\, (1 - p)} + \frac{e'_H(p)\, ((2 - \alpha)\, e_H(p) + (\alpha - 1)\, e_L(p))}{e_H(p)}\ .$$

Using the same equations for $\theta = L$ and the previous expression for $e'_L(p)$ we obtain the following expression that $e_L(\cdot)$ and $e_H(\cdot)$ must satisfy:

$$0 = -\frac{(e_H(p) - e_L(p))^2 \, (e_H(p) + (1-p)\,p\,\alpha\,e'_H(p))}{2\,e_H(p)\,e_L(p)^{2-\alpha}} \ .$$

Note that since $e_H(p) \neq e_L(p)$ for $p \in R$, the previous equation is a first-order ordinary differential equation for $e_H(p)$. This allows us to find the policy functions, which are exactly those given in (2.9), where C_L and C_H are constants to be determined by the boundary conditions on $V_\theta(\cdot)$.

Now let's prove that C_L and C_H exist and are uniquely determined by the boundary conditions. Define $c \equiv \frac{C_L}{C_H}$. Given a value for c, it is trivial to see that C_H and C_L are unique from equation (2.12). So, we will focus on the existence and uniqueness of c. It is easy to see that, when $c = 0$, the RHS of the expression (2.11) is positive. Note that the upper limit on c is $\bar{c} \equiv \left(\frac{\bar{p}}{1-\bar{p}}\right)^{-2/\alpha}$, that is, the value that makes the denominator 0 (and $e_H(\bar{p}) = e_L(\bar{p})$). Using basic calculus it is easy to see that the RHS expression (2.11) tends to $-\infty$ when $c \to \bar{c}$. Since it is clearly continuous for $c \in (0, \bar{c})$, we have that a solution for c exists.

To show the uniqueness we use the change of variables $q \to m \equiv \frac{q}{1-q}\,c^{-2/\alpha}$. After some amount of algebra, we have that equation (2.11) for c is equivalent to

$$0 = f(c) \equiv \int_{m_-(c)}^{m_+(c)} \frac{\frac{A_H}{A_L}\,m^{2/\alpha - 2} - 1}{1 - m^{2/\alpha}}\,\mathrm{d}m \ ,$$

where $m_-(c) \equiv \frac{p}{1-p}\,c^{-2/\alpha}$ and $m_+(c) \equiv \frac{\bar{p}}{1-\bar{p}}\,c^{-2/\alpha}$. If we differentiate the previous expression with respect to c we get

$$f'(c) = -\frac{2}{\alpha\,c}\left(\frac{\frac{A_H}{A_L}\,m_+^{2/\alpha-2} - 1}{1 - m_+^{2/\alpha}}\,m_+ - \frac{\frac{A_H}{A_L}\,m_-^{2/\alpha-2} - 1}{1 - m_-^{2/\alpha}}\,m_-\right) < 0 \ ,$$

where $m_- \equiv m_-(c)$ and $m_+ \equiv m_+(c)$. The previous equation is clearly signed because $m_- < m_+ < 1$. Hence, since f is monotone, it has at most one 0. Therefore, the solution is unique. $\square$

Proof of Proposition 2.1 (page 14)

Proof. We will only do the proof for the case $\theta = H$, since the case $\theta = L$ is analogous. Note that the RHS of (2.11) is decreasing in $\frac{A_L}{A_H}$, and as can be seen in the proof of Lemma 2.2, decreasing in c. This implies that c is increasing in $\frac{A_L}{A_H}$. We use equation (2.13) to get the following expression:

$$\frac{\mathrm{d}(\log(V'_H(p)))}{\mathrm{d}c} = \frac{\left(\frac{p}{1-p}\right)^{2/\alpha}}{1 - c\left(\frac{p}{1-p}\right)^{2/\alpha}} - \frac{\mathrm{d}(\log(K_H))}{\mathrm{d}c} \ .$$

32

Note that the RHS of the previous expression is clearly increasing in p, and it is easy to see that it is continuous in c. Also, the integral of V'_H between $\underline{p}$ and $\bar{p}$ is $\overline{P} - \underline{P}$, independent of the value of $\frac{A_L}{A_H}$. Hence, there exists $\tilde{p} \in (\underline{p}, \bar{p})$ such that $\frac{dV'(\tilde{p})}{dc} = 0$, so $\frac{d(\log(V'_H(\tilde{p})))}{dc} = 0$. Since the RHS of the previous expression is monotonically increasing in p, this implies that $\frac{d(\log(V'_H(p)))}{dc} < 0$ for $p < \tilde{p}$ and $\frac{d(\log(V'_H(p)))}{dc} > 0$ for $p > \tilde{p}$. Hence, $\frac{d(V'_H(p))}{dc} < 0$ for $p < \tilde{p}$ and $\frac{d(V'_H(p))}{dc} > 0$ for $p > \tilde{p}$. This implies that if $c_1 < c_2$, then $V_H(p; c_1) > V_H(p; c_2)$ for all $p \in (\underline{p}, \bar{p})$. $\qquad\square$

Proof of Lemma 2.3 (page 19)

Proof. Note first that in any competitive equilibrium, if $V_\theta(p) < V_\theta(\tilde{p})$ for some $p, \tilde{p} \in [0, 1]$ and $\theta \in \{L, H\}$, then $p < \tilde{p}$. This is true because $V_\theta \in \mathcal{C}^0[0, 1]$, $V_\theta(p) = \max\{\Pi_0, p\,\Pi\}$ for $p \notin R$ and $V'_\theta(p) > 0$ for $p \in R$, as we see in (2.10), for all $\theta \in \{L, H\}$.

Assume that a competitive equilibrium with strategy profile $((e_\theta, r_\theta)_{\theta \in \{L,H\}}, P)$ and with rejection region R exists and is such that $\tilde{R} \subseteq R$ for all $\tilde{R} \in \mathcal{R}$. So, for any rejection region $\tilde{R} \in \mathcal{R}$ and equilibrium, if $\tilde{V}_H(\tilde{p}) > p\,\Pi$ for some $\tilde{p}$ and $p \notin R$ (so $p \notin \tilde{R}$), then it is the case that $\tilde{p} > p$, so $\tilde{V}_L(\tilde{p}) > p\,\Pi$ (by the first part of this lemma). Therefore, $\mathcal{E}_L(\max\{\Pi_0, p\,\Pi\}, \tilde{p}) = \mathcal{E}_H(\max\{\Pi_0, p\,\Pi\}, \tilde{p})$ for all p and $\tilde{p}$. Hence, this implies that the D1 criterion imposes no restriction on the beliefs following a deviation at $p \notin R$. In particular, beliefs may remain constant (using $e_L(p) = e_H(p) = 0$), so there are no incentives to deviate. Therefore, an equilibrium with rejection region R satisfies the D1 criterion.

Now assume that there is a competitive equilibrium with rejection region R satisfying the D1 criterion. Assume that there is some competitive equilibrium with rejection region $\tilde{R}$ and there is some $\tilde{p} \in \tilde{R} \setminus R$ such that $\tilde{p}\,\Pi > \Pi_0$. Note that since $\tilde{p} \in \tilde{R}$, $\tilde{p} < 1$. Define $\underline{\tilde{p}}$ and $\bar{\tilde{p}}$ as in (2.5) and (2.6) (with $\tilde{p}$ and $\tilde{R}$ instead of p and R). Since $\underline{\tilde{p}} < \bar{\tilde{p}}$ and $\tilde{V}_H(p) > \tilde{V}_L(p)$ for all $p \in (\underline{\tilde{p}}, \bar{\tilde{p}})$ (this can be seen from the explicit functional forms (2.10)) there exists some $\tilde{p}' \in (\underline{\tilde{p}}, \bar{\tilde{p}})$ such that $\tilde{V}_L(\tilde{p}') < \tilde{p}\,\Pi < \tilde{V}_H(\tilde{p}')$. That implies $\mathcal{E}_L(\max\{\Pi_0, p\,\Pi\}, \tilde{p}) \subset \mathcal{E}_H(\max\{\Pi_0, p\,\Pi\}, \tilde{p})$ ($\subset$ being strict inclusion), so the D1 criterion imposes that after a rejection at $\tilde{p}$, beliefs must jump to 1. So, all types have incentives to not take the offer $\tilde{p}\,\Pi$ at $\tilde{p}$, since, by rejecting it, they get Π instead of $\tilde{p}\,\Pi$. This contradicts the existence of a competitive equilibrium with rejection region R (such that $\tilde{p} \notin R$) satisfying the D1 criterion. $\qquad\square$

Proof of Proposition 2.2 (page 19)

Proof. The proof is the same as the one for Proposition 3.1. In Proposition 3.1 the result is proven for $c_0 > 0$, but the argument still applies for $c_0 = 0$. The particular value of $\bar{p}_*$ is obtained by

solving the equation $\frac{\partial}{\partial p}V_H(\bar{p}_*; 0, \bar{p}_*, \Pi_0, \bar{p}_* \Pi) = \Pi$ (using the definition (2.16).) $\qquad\square$

Proof of Proposition 2.3 (page 20)

Proof. We will prove this proposition by finding an explicit sequence of interval equilibria where the corresponding sequence of rejection regions, $R_n = (\underline{p}_n, \bar{p}_n)$, tends to $(0, 1)$. We will argue that there is no equilibrium with $R = (0, 1)$. Therefore, by Lemma 2.3, no equilibrium will satisfy the D1 criterion.

For any given $\bar{p} \in (\frac{\Pi_0}{\Pi}, 1)$ and $p \in (0, \bar{p})$, let's define the following function:

$$V_H(p; \bar{p}) \equiv \bar{p}\,\Pi\left(1 - \frac{\alpha\,(1 - \bar{p})}{\alpha - 2}\,\frac{h(p) - h(\bar{p})}{h(\bar{p})}\right).$$

Note that $V_H(\bar{p}; \bar{p}) = \bar{p}\,\Pi$ and $\frac{\partial}{\partial p}V_H(\bar{p}; \bar{p}) = \Pi$. Since, for $\alpha > 2$, $\lim_{p \to 0} h(p) = \infty$, we have that $\lim_{p \to 0} V_H(p; \bar{p}) = -\infty$. It is easy to show that $\frac{\partial^2 V_H(p; \bar{p})}{\partial p^2} > 0$ when $\bar{p} > \frac{\alpha - 1}{\alpha}$. Therefore, there exists a function $\underline{p}_* : (\frac{\alpha - 1}{\alpha}, 1) \to (0, 1)$ such that $\underline{p}_*(\bar{p}) < \bar{p}$, $V_H(\underline{p}_*(\bar{p}), \bar{p}) = \max\{\underline{p}_*(\bar{p})\,\Pi, \Pi_0\}$ and $V_H(p, \bar{p}) > \max\{p\,\Pi, \Pi_0\}$ for all $p \in (\underline{p}_*(\bar{p}), \bar{p})$. It is easy to verify that $\underline{p}_*(\bar{p})$ also satisfies

$$V_H(p; \bar{p}) = V_H(p; \underline{p}_*(\bar{p}), \bar{p}, \max\{\underline{p}_*(\bar{p})\,\Pi, \Pi_0\}, \bar{p}\,\Pi).$$

Hence, for each $\bar{p} \in (\frac{\alpha - 1}{\alpha}, 1)$ there is an interval equilibrium with rejection region $(\underline{p}_*(\bar{p}), \bar{p})$. Furthermore, it is easy to see that $V_H(p; \bar{p})$ is increasing in $\bar{p}$ when $\bar{p} \in (\frac{\alpha - 1}{\alpha}, 1)$, so $\underline{p}_*(\bar{p})$ is decreasing. It can be easily shown that $\lim_{\bar{p} \to 1} V(p; \bar{p}) = \Pi$ for all p, so $\lim_{\bar{p} \to 1} \underline{p}_*(\bar{p}) = 0$. So, any increasing sequence $(\bar{p}_n)_n$ with $\lim_{n \to \infty} \bar{p}_n = 1$ generates the desired sequence of equilibria. Figure 3 (b) shows $V_H(\cdot)$ for some equilibria of a sequence like this, being the black line first, the gray line second and the dotted line third.

Assume $R = (0, 1)$. In this case, using (2.14), the corresponding policy function for the H-seller is $e_H(p) = 0$ for all $p \in (0, 1)$. Nevertheless, in this case there is no updating of beliefs, so $V_H(p) = 0$ for all $p \in (0, 1)$, which is clearly a contradiction. Proposition 2.4 will shed light on why low effort is more efficient when $\alpha > 2$. $\qquad\square$

Proof of Proposition 2.4 (page 21)

Proof. Fix a $p_0 \in R$ and define $\bar{p}$ and $\underline{p}$ as in (2.5) and (2.6). Fix $e_H \in \mathcal{C}^1(\underline{p}, \bar{p})$ positive. Assume $\alpha < 2$ (the case $\alpha > 2$ is analogous). The equation for the value function for the H-seller exerting effort e_H is given, in $(\underline{p}, \bar{p})$ by the following HJB equation

$$0 = -A_H\,e_H(p)^\alpha + \frac{(p - 1)^2\,p\,(p\,V_H''(p) + 2\,V_H'(p))}{2\,\sigma^2}\,e_H(p)^2, \tag{5.1}$$

and boundary conditions $V_H(\underline{p}) = \underline{P}$ and $V_H(\bar{p}) = \overline{P}$. Let $V_H(p, e_H(\cdot))$ be its solution. Let's consider the following decomposition: $V_H(p, e_H) \equiv V_h(p) + V_t(p, e_H)$. We assume that $p\,V_h''(p) + 2\,V_h'(p) = 0$, and we impose $V_h(\underline{p}) = \underline{P}$ and $V_h(\bar{p}) = \overline{P}$. This leads to

$$V_h(p) = \underline{P} + \frac{(p - \underline{p})\,\bar{p}}{(\bar{p} - \underline{p})\,p}\,(\overline{P} - \underline{P})\ .$$

This is exactly the expected payoff when the signaling waste is 0 (note that the homogeneous equation is "as if" $A_H = 0$), which coincides with the expected accepted price offer conditional on being type H (we can see this using the formula (2.19)). Note that $V_t(p; e_H)$ must satisfy (5.1) and $V_t(\underline{p}; e_H) = V_t(\bar{p}; e_H) = 0$. Consider $\lambda > 1$. Then, it is the case that

$$V_t(p, \lambda\,e_H) = \lambda^{\alpha-2}\,V_t(p, e_H) < V_t(p, e_H)\ .$$

This is true because both $V_t(p, \lambda\,e_H)$ and $V_t(p, \lambda\,e_H)$ satisfy the same equations and boundary conditions (equal to 0 at the boundary). Therefore, increasing the effort by a factor $\lambda > 1$, the absolute value of $V_t(p, \lambda\,e)$ is reduced by a factor $\lambda^{\alpha-2} < 1$. Finally, note that $V_t(p, \lambda\,e) < 0$ $\forall p \in (\underline{p}, \bar{p})$. Indeed, it is the solution of a boundary problem with negative flow payoff and with 0-value at the boundary. So, by increasing the effort we increase V_H, we make it asymptotically equal to V_h, that is, signaling waste asymptotically disappears. $\square$

Proof of Lemma 3.1 (page 25)

Proof. Assume that an equilibrium satisfying D1 exists and let R be its rejection region. Suppose $p_0 \in R$ and define $\underline{p}$ and $\bar{p}$ as in (2.5) and (2.6). Note that by Lemma 2.3 there is no equilibrium such that its rejection region contains $\bar{p}$. We assume $V_H(\underline{p}) = \underline{p}\,\Pi$ and $V_H(\bar{p}) = \bar{p}\,\Pi$ (the other possible case, when $V_H(\underline{p}) = \Pi_0$, is proved analogously). Then, from the FOC (2.8) and the form of the policy function (3.1), there exists some constant C_1 such that

$$V_H(p) \equiv V_H(p, \bar{p}) = \bar{p}\,\Pi - \int_p^{\bar{p}} \frac{A_H\,\alpha\,\sigma^2}{(1-q)\,q}\left(\frac{C_1\,(1-q)}{(2-\alpha)\,q} - \frac{2\,c_0}{(2-\alpha)\,A_H}\right)^{\frac{\alpha-2}{\alpha}}\,\mathrm{d}q\ . \tag{5.2}$$

For $\bar{p}' \in (0, 1)$, define $\underline{p}_*(\bar{p}') \equiv \sup\{p < \bar{p}' | V_H(p, \bar{p}') \leq p\,\Pi\}$. Note that $\underline{p} = \underline{p}_*(\bar{p})$.

Note that since $V_H(p) > p\,\Pi$ for all $p \in R$ and $V_H(\cdot) \in \mathcal{C}^1(R)$, we have $\lim_{p \downarrow \underline{p}} V_H'(p) \geq \Pi$ and $\lim_{p \uparrow \bar{p}} V_H'(p) \leq \Pi$. We need to show that when the equilibrium satisfies the D1 criterion, these weak inequalities are equalities, instead.

Assume first $\lim_{p \downarrow \underline{p}} V_H'(p) > \Pi$ and $\lim_{p \uparrow \bar{p}} V_H'(p) < \Pi$. Note that $\frac{\partial V_H(p,\bar{p})}{\partial \bar{p}} = \Pi - V_H'(\bar{p}) > 0$. Therefore, $\underline{p}_*(\bar{p})$ is decreasing, and since $\lim_{p \downarrow \underline{p}} V_H'(p) > \Pi$ exists in a neighborhood of $\bar{p}$. So, since

$V_H(p,\bar{p})$ is increasing in $\bar{p}$, for $\varepsilon > 0$ small enough $(\underline{p}_*(\bar{p}+\varepsilon), \bar{p}+\varepsilon) \ni \bar{p}$ is the rejection region of some equilibrium. This contradicts the assumption that $\bar{p}$ is not in the rejection region of any equilibrium.

Now consider the case $V_H'(\underline{p}) > \Pi$ and $V_H'(\bar{p}) = \Pi$ (a similar argument can be used when $V_H'(\underline{p}) = \Pi$ and $V_H'(\bar{p}) < \Pi$). It is easy to see that now $V_H(p) = \overline{V}_H(p,\bar{p})$ where

$$\overline{V}_H(p,\bar{p}) \equiv \bar{p}\,\Pi - \int_{\underline{p}}^{\bar{p}} \frac{\Pi\,\bar{p}\,q^{\frac{2}{\alpha}-2}\,(1-\bar{p})^{2/\alpha}\,(1-q)^{-1}}{\left(2\,(2-\alpha)^{-1}\,k\,(\bar{p}-q)((1-\bar{p})\,\bar{p})^{\frac{\alpha}{2-\alpha}} + (1-q)\,\bar{p}\right)^{\frac{2-\alpha}{\alpha}}}\,\mathrm{d}q \qquad (5.3)$$

where $k \equiv c_0 \left(\frac{\Pi\,A^{-2/\alpha}}{\alpha\,\sigma^2}\right)^{\frac{\alpha}{2-\alpha}}$. Simple algebra shows that

$$\frac{\partial^2 \overline{V}_H(\bar{p},\bar{p})}{\partial \bar{p}^2} > 0 \quad \Leftrightarrow \quad \frac{\partial \overline{V}_H(p,\bar{p})}{\partial \bar{p}} > 0$$
$$\Leftrightarrow \quad k - (\alpha\,(1-\bar{p}) - 1)\,((1-\bar{p})\,\bar{p})^{\frac{\alpha}{\alpha-2}} > 0\ .$$

The first condition is a necessary condition for $(\underline{p},\bar{p})$ to be an equilibrium when $\frac{\partial}{\partial p}\overline{V}_H(p,\bar{p}) = \Pi$. Indeed, since $V_H(p,\bar{p}) > p\,\Pi$ for $p \in (\underline{p},\bar{p})$ and $\overline{V}_H(\bar{p},\bar{p}) = \bar{p}\,\Pi$, $\overline{V}_H(\cdot,\bar{p})$ must be convex at $p = \bar{p}$. Using simple algebra we find that when $\alpha < 2$, there exists a unique $\bar{p}^\dagger$ such that $\frac{\partial^2 \overline{V}_H(p,\bar{p})}{\partial p^2} > 0$ iff $\bar{p} > \bar{p}^\dagger$. For $\alpha > 2$, as we will see in the proof of Proposition 3.2, $\frac{\partial^2 \overline{V}_H(p,\bar{p})}{\partial p^2} < 0$ in a (maybe empty) interval contained in $(0,\frac{\alpha-1}{\alpha})$ that contains $\tilde{p}_-$ (defined in (5.4)). Therefore, since by assumption $V_H(\underline{p}) = \underline{p}\,\Pi$, it must be the case that $\bar{p} > \tilde{p}_-$, so $\frac{\partial^2 \overline{V}_H(\bar{p}',\bar{p}')}{\partial p^2} > 0$ for $\bar{p}' > \bar{p}$.

For $\alpha < 2$, the value function (5.3) is well defined for all $p \in (0,1)$. In this case, $\bar{p}$ can be increased to $\bar{p}+\varepsilon$, for $\varepsilon > 0$ small, such that $\underline{p}_*(\bar{p}+\varepsilon)$ exists, and satisfies $\frac{\partial}{\partial p}V_H(p,\bar{p}+\varepsilon) > \Pi$. Since $V_H(p,\bar{p}+\varepsilon) > V_H(p,\bar{p})$ for all p, $\underline{p}_*(\bar{p}+\varepsilon) < \underline{p}$. This, by a similar argument as before, contradicts the assumption that $\bar{p}$ does not belong to the rejection region of any competitive equilibrium.

When $\alpha > 2$, the term inside the parenthesis of the denominator of (5.3) may not be well defined. It is easy to see that it is well defined for $p \geq \bar{p}$. In particular, given $\bar{p}$, either the denominator is well defined for all p or there exists some function $0 < \tilde{p}_0(\bar{p}) < \bar{p}$ such that it is not well defined for $p < \tilde{p}_0(\bar{p})$ and well defined otherwise. Furthermore, if $\tilde{p}_0(\bar{p})$ exists, it is continuous in $\bar{p}$ and $\lim_{p \to \tilde{p}_0(\bar{p})} \frac{\partial}{\partial p}V_H(p,\bar{p}) = 0$. Since, by assumption, $\frac{\partial}{\partial p}\overline{V}(\underline{p},\bar{p}) >$, then $\underline{p} > \tilde{p}_0(\bar{p})$ if $\tilde{p}_0(\bar{p})$ exists. Now, using the same argument as in the case where $\alpha < 2$, $\bar{p}$ can be increased by $\varepsilon > 0$ small such that $(\underline{p}_*(\bar{p}+\varepsilon), \bar{p}+\varepsilon)$ is the rejection region of an equilibrium. This contradicts our initial assumption. $\qquad\square$

Proof of Proposition 3.1 (page 25)

Proof. We will prove this proposition by explicitly constructing the equilibrium.

Define $\overline{V}_H(\cdot,\cdot)$ and $\bar{p}^\dagger$ as in the proof of Lemma 3.1. Since $\lim_{\bar{p}\to 0} \overline{V}_H(0,\bar{p}) = 0$ and $\overline{V}_H(p,\bar{p})$ is decreasing in $\bar{p}$ when $\bar{p} < \bar{p}^\dagger$, we have that $\overline{V}_H(0,\bar{p}^\dagger) < 0$. Furthermore, by simple visual inspection we see that $\lim_{\bar{p}\to 1} \overline{V}_H(0,\bar{p}) = \Pi$. Therefore, by continuity and since $\overline{V}_H(p,\bar{p})$ is increasing in $\bar{p}$ when $\bar{p} > \bar{p}^\dagger$, for each Π_0 there exists a unique $\bar{p}_*(\Pi_0) \in (\bar{p}^\dagger, 1)$ such that $\overline{V}_H(0,\bar{p}_*(\Pi_0)) = \Pi_0$.

Let's show that there is an equilibrium with rejection region $R = (0, \bar{p}_*(\Pi_0))$. Let's denote $V_{H*}(p) \equiv \overline{V}_H(p, \bar{p}_*(\Pi_0))$. Then, since the boundary conditions are satisfied, we only need to show that $V_{H*}(p) \geq \max\{p\,\Pi, \Pi_0\}$ for all $p \in (0, \bar{p}_*(\Pi_0))$. Since $V'_{H*}(p) > 0$ and $V_{H*}(0) = \Pi_0$, we only have to verify that $V_{H*}(p) \geq p\,\Pi$ for all $p \in (0, \bar{p}_*(\Pi_0))$. First, taking derivatives in the expression (5.3) we have that

$$V''_{H*}(\bar{p}_*(\Pi_0)) > 0 \quad \Leftrightarrow \quad \bar{p}_*(\Pi_0) > \bar{p}^\dagger .$$

Second, let's find the solutions of the equation $V'_{H*}(p) = \Pi$ other than $p = \bar{p}_*(\Pi_0)$. Simple algebra transforms this equation into finding the zeros of $f(\cdot)$, where

$$f(p) \equiv \frac{2\,k\,(\bar{p}-p)}{2-\alpha} - (1-\bar{p})\,p^{\frac{\alpha}{\alpha-2}+1}\,(1-p)^{\frac{\alpha}{\alpha-2}} + (1-p)\,(1-\bar{p})^{\frac{\alpha}{\alpha-2}}\,\bar{p}^{\frac{\alpha}{\alpha-2}+1} ,$$

where $\bar{p} \equiv \bar{p}_*(\Pi_0)$. Let's show that it has at most one solution lower than $\bar{p}$. In order for the previous equation to have more than one solution in $(0, \bar{p}_*(\Pi_0))$, the second derivative must have at least one zero in $(0, \bar{p}_*(\Pi_0))$. Nevertheless, if we take the second derivative it is easy to see that it does not have any zero in $(0, \bar{p}_*(\Pi_0))$ for $\alpha < 2$. Then, since $V'_{H*}(0) = \infty$, it must be the case that $V_{H*}(p) > p\,\Pi$ for all $p \in (0, \bar{p}_*(\Pi_0))$. Indeed, if it was not the case, there must exist $\tilde{p}'_1, \tilde{p}'_2 \in (0, \bar{p}_*(\Pi_0))$ such that $V_{H*}(p) < 0$ for $p \in (\tilde{p}'_1, \tilde{p}'_2)$ and

$$V_{H*}(\tilde{p}'_1) = \tilde{p}'_1\,\Pi , \quad V_{H*}(\tilde{p}'_2) = \tilde{p}'_2\,\Pi , \quad V'_{H*}(\tilde{p}'_1) \leq \Pi \quad \text{and} \quad V'_{H*}(\tilde{p}'_2) \geq \Pi .$$

Continuity of $V'_{H*}(\cdot)$ implies that there exist $\tilde{p}_1, \tilde{p}_2, \tilde{p}_3 \in (0, \bar{p}_*(\Pi_0))$ such that $V'_{H*}(\tilde{p}_1) = V'_{H*}(\tilde{p}_2) = V'_{H*}(\tilde{p}_3) = \Pi$ and $0 < \tilde{p}_1 < \tilde{p}'_1 < \tilde{p}_2 < \tilde{p}'_2 < \tilde{p}_3 < \bar{p}_*(\Pi_0)$. But this contradicts the fact that $f(\cdot)$ only has one zero in $(0, \bar{p}_*(\Pi_0))$. So, there is an equilibrium with rejection region $R = (0, \bar{p}_*(\Pi_0))$.

Using a similar argument as in the proof of Lemma 3.1, we can argue that if there is an equilibrium with rejection region R' and $\sup R' > \sup R$, there must exist another equilibrium with rejection region $\tilde{R}'$ such that $\sup R'' > \sup R'$, satisfying the smooth pasting condition. Nevertheless, as we have just seen, the equilibrium defined is the only one that satisfies them. $\square$

Proof of Proposition 3.2 (page 26)

Proof. Note that in all equilibria with $R \neq \emptyset$ there must be a $\tilde{p} \in R$ such that $V'_H(\tilde{p}) = \Pi$ and $V''_H(\tilde{p}) < 0$. Indeed, consider a $p \in R$ and define $\underline{p}$ and $\bar{p}$ as in (2.5) and (2.6). Note that equilibrium

conditions require $V_H(p) > p\,\Pi$ for $p \in (\underline{p}, \bar{p})$, $\lim_{p \downarrow \underline{p}} V'_H(p) \geq \Pi$ and $\lim_{p \uparrow \bar{p}} V'_H(p) \leq \Pi$. So, since $V'_H(\cdot) \in \mathcal{C}^1(\underline{p}, \bar{p})$, using standard calculus we know that there must exist at least one $\tilde{p}$ such that $V'_H(\tilde{p}) = \Pi$ and $V''_H(\tilde{p}) < 0$.

Assume that an equilibrium exists and consider $\tilde{p}$ satisfying the previous conditions. Then, given the form of the policy function (3.1), there must exist some $\tilde{v}$ such that the value function $V_H(p) \equiv V_H(p; \tilde{p}, \tilde{v})$ takes the following form:

$$V_H(p; \tilde{p}, \tilde{v}) \equiv \tilde{v} + \int_{\tilde{p}}^{p} \frac{\Pi\,\tilde{p}\,q^{\frac{2}{\alpha}-2}\,(1-\tilde{p})^{2/\alpha}\,(1-q)^{-1}}{\left(2\,(2-\alpha)^{-1}\,k\,(\tilde{p}-q)\,((1-\tilde{p})\,\tilde{p})^{\frac{\alpha}{2-\alpha}} + (1-q)\,\tilde{p}\right)^{-\frac{2-\alpha}{\alpha}}}\,\mathrm{d}q\ ,$$

where $k \equiv c_0 \left(\frac{\Pi\,A^{-2/\alpha}}{\alpha\,\sigma^2}\right)^{\frac{\alpha}{2-\alpha}}$ as in the proof of Lemma 3.1. Note that $V_H(\tilde{p}; \tilde{p}, \tilde{v}) = \tilde{v}$ and $\frac{\partial}{\partial p} V_H(\tilde{p}; \tilde{p}, \tilde{v}) = \Pi$. If we twice differentiate it with respect to p we find

$$\frac{\partial^2}{\partial p^2} V_H(\tilde{p}; \tilde{p}, \tilde{v}) = \frac{2\,\Pi\,((1-\tilde{p})\,\tilde{p})^{\frac{-\alpha}{\alpha-2}}}{\alpha\,\tilde{p}\,(1-\tilde{p})} \left(k - (\alpha\,(1-\tilde{p})-1)\,((1-\tilde{p})\,\tilde{p})^{\frac{\alpha}{\alpha-2}}\right)\ .$$

Note that the first term in the RHS of the expression is clearly positive. The second term in the RHS is k when $\tilde{p} = 0$, $\tilde{p} = \frac{\alpha-1}{\alpha}$ and $\tilde{p} = 1$. If we differentiate this term, we see that it is strictly convex in the region $(0, \frac{\alpha-1}{\alpha})$ and concave otherwise. Therefore, the minimum of the second term of the RHS of the previous expression is in the region $(0, \frac{\alpha-1}{\alpha})$, and it can be shown that it is reached at

$$\tilde{p}_- \equiv \frac{1}{2 + \sqrt{\frac{\alpha-2}{\alpha-1}}}\ . \tag{5.4}$$

Therefore, using the definition of k, a necessary condition for $V_H(p; \tilde{p}, \tilde{v})$ to be concave at $\tilde{p}$ is that

$$c_0 < \bar{c}_0 \equiv \left(\frac{\Pi\,A_H^{-2/\alpha}}{\alpha\,\sigma^2}\right)^{\frac{\alpha}{\alpha-2}} (\alpha\,(1-\tilde{p}_-)-1)\,((1-\tilde{p}_-)\,\tilde{p}_-)^{\frac{\alpha}{\alpha-2}}\ .$$

So, the previous condition is necessary for the existence of $\tilde{p} \in R$ satisfying $V'_H(\tilde{p}) = \Pi$ and $V''_H(\tilde{p}) < 0$, that itself is a necessary condition for the existence of equilibria. Then, $c_0 < \bar{c}_0$ is a necessary condition for the existence of equilibria.

Let's show that $c_0 < \bar{c}_0$ is also a sufficient condition for the existence of equilibria. Assume $c_0 < \bar{c}_0$, so $\tilde{p}$ exists such that $\frac{\partial}{\partial p} V_H(\tilde{p}; \tilde{p}, \tilde{v}) = \Pi$ and $\frac{\partial^2}{\partial p^2} V_H(\tilde{p}; \tilde{p}, \tilde{v}) < 0$. If we make $\tilde{v}$ higher (close enough to $\tilde{p}\,\Pi$), standard calculus guarantees that there exist $\underline{p} < \tilde{p}$ and $\bar{p} > \tilde{p}$ such that $V_H(\underline{p}; \tilde{p}, \tilde{v}) = \underline{p}\,\Pi$, $V_H(\bar{p}; \tilde{p}, \tilde{v}) = \bar{p}\,\Pi$ and $V_H(p; \tilde{p}, \tilde{v}) > p\,\Pi$ for all $p \in (\underline{p}, \bar{p})$. Since $R = (\underline{p}, \bar{p})$ satisfying the previous conditions is the rejection region of an equilibrium, $c_0 < \bar{c}_0$ is a sufficient condition for an equilibrium to exist.

Note that $V_H(p; \tilde{p}, \tilde{v})$ is well defined as long as the term inside the parenthesis of the denominator is non-negative. It is easy to verify that it is non-negative if p is in the neighborhood of $\tilde{p}$, so the previous argument is valid. Note also that if $\tilde{p}$ is large, the term in the denominator is not well defined for low q. Since the exponent of this term is negative when $\alpha > 2$, this corresponds to the derivative of V_H with respect to p being 0.

Assume that $c_0 < \bar{c}_0$, fix an equilibrium and $\tilde{p}$ satisfying the previous properties. Let's define $\tilde{v}_- \equiv \inf\{\tilde{v} | V_H(p; \tilde{p}, \tilde{v}) > p\,\Pi \ \ \forall p < \tilde{p}\}$ and $\tilde{v}_+ \equiv \inf\{\tilde{v} | V_H(p; \tilde{p}, \tilde{v}) > p\,\Pi \ \ \forall p > \tilde{p}\}$. Note that since $\lim_{p \to 1} V_H(p; \tilde{p}, \tilde{v}) = \infty$, we have $\tilde{p}\,\Pi < \tilde{v}_+ < \infty$. Assume $\tilde{v}_+ \leq \tilde{v}_-$ (the other case is analogous). By continuity, there exists some $\bar{p}$ such that $V_H(\bar{p}; \tilde{p}, \tilde{v}_+) = \bar{p}\,\Pi$. Note that $\bar{p}$ is unique. Indeed, by the previous argument $\frac{\partial^2}{\partial p^2} V_H(p; \tilde{p}, \tilde{v}_+)$ has two zeros when $c_0 < \bar{c}_0$ (one lower than $\tilde{p}$ and one higher than $\tilde{p}$), and $\bar{p}$ must be higher than the higher zero, so $\frac{\partial^2}{\partial p^2} V_H(\bar{p}; \tilde{p}, \tilde{v}_+) > 0$.

Since $V_H(p; \tilde{p}, \tilde{v}) \in \mathcal{C}^1(\tilde{p}, 1)$, it must be the case that $\frac{\partial}{\partial p} V_H(\bar{p}; \tilde{p}, \tilde{v}_+) = \Pi$, and therefore $V_H(\bar{p}; \tilde{p}, \tilde{v}_+) = \overline{V}_H(p, \bar{p})$, where $\overline{V}_H$ is defined in (5.3). Recall that $\overline{V}_H(p, \bar{p})$ is increasing and continuous in $\bar{p}$. Furthermore, by assumption (since $\tilde{v}_- > \tilde{v}_+$), there exists some $p < \tilde{p}$ such that $\overline{V}_H(p, \bar{p}) = p$. Define $\bar{p}_* = \inf\{\bar{p} | \overline{V}_H(p, \bar{p}) > p\,\Pi \ \ \forall p < \bar{p}\}$. Using standard calculus, it is easy to prove that there exists some $\underline{p}_* < \bar{p}_*$ such that $\overline{V}_H(\underline{p}_*, \bar{p}_*) = \underline{p}_*\,\Pi$ and $\frac{\partial}{\partial p} \overline{V}_H(\underline{p}_*, \bar{p}_*) = \Pi$.

Finally, to show that the equilibrium found satisfies D1 we can use a similar argument as in the proof of Lemma 3.1. Indeed, it is easy to see that for all competitive equilibria, there must exist another equilibrium with a bigger rejection region satisfying the smooth pasting condition. Nevertheless, the fact that $\overline{V}_H(p, \bar{p})$ is increasing in $\bar{p}$ ensures that there is no equilibrium satisfying the smooth pasting condition other than the one we just defined. $\square$

Proof of Proposition 3.3 (page 27)

Proof. Let's fix $\underline{p} \in (0, \frac{\Pi_0}{Y})$ and define

$$\underline{V}_H(p, \underline{p}) \equiv \Pi_0 + \int_{\underline{p}}^p \frac{2^{\frac{1}{\alpha}} \alpha \sigma^2 A_H^{2/\alpha} c_0^{\frac{\alpha-2}{\alpha}} (q - \underline{p})^{\frac{\alpha-2}{\alpha}}}{(\alpha - 2)^{1-\frac{2}{\alpha}} (1 - \underline{p})^{1-\frac{2}{\alpha}} (1 - q) q^{2-\frac{2}{\alpha}}} \, dq \ . \tag{5.5}$$

It is easy to see that this is the value function corresponding to $C_1 = \frac{2\,p\,c_0}{A_H\,(1-\underline{p})}$ in (5.2), using $\underline{p}$ as the integration limit instead of $\bar{p}$ and changing $\bar{p}\,\Pi$ by Π_0 in the front of the expression. Note that $\underline{V}_H(\underline{p}, \underline{p}) = \Pi_0$ and $\frac{\partial}{\partial p} \underline{V}_H(\underline{p}, \underline{p}) = 0$. Note also that $\frac{\partial^2}{\partial p^2} \underline{V}_H(p, \underline{p}) > 0$ when $p > \underline{p}$ is close to $\underline{p}$. Therefore, if we choose $\underline{p}$ close enough to $\frac{\Pi_0}{\Pi}$, it is easy to show that there exists some $\bar{p} > \frac{\Pi_0}{\Pi}$ such that $\underline{V}_H(\bar{p}, \underline{p}) = \max\{\Pi_0, \bar{p}\,\Pi\}$ and $\underline{V}_H(p, \underline{p}) > p\,\Pi$ for all $p \in (\underline{p}, \bar{p})$. Therefore, a competitive equilibrium (with rejection region $R = (\underline{p}, \bar{p})$) exists.

Using simple algebra it is easy to show that $\frac{\partial}{\partial p}\underline{V}_H(p,\underline{p}) < 0$. Furthermore, we see that $\lim_{p\to 0}\underline{V}_H(p,\underline{p}) = \infty$ for all $p > 0$ and $\lim_{p\to 1}\underline{V}_H(p,\underline{p}) = \infty$ for all $\underline{p} > 0$. Also, twice differentiating (5.5), we see that $\frac{\partial^2}{\partial p^2}\underline{V}_H(p,\underline{p})$ has at most 2 zeros. Therefore, there exist one and at most two pairs of values $(\underline{p}_1,\bar{p}_1)$ and $(\underline{p}_2,\bar{p}_2)$, with $\bar{p}_1 < \bar{p}_2$, such that $\underline{V}_H(\bar{p}_i,\underline{p}_i) = \bar{p}_i\,\Pi$, $\frac{\partial}{\partial p}\underline{V}_H(\bar{p}_i,\underline{p}_i) = \Pi$ and $\frac{\partial^2}{\partial p^2}\underline{V}_H(\bar{p}_i,\underline{p}_i) > 0$ for $i \in \{1,2\}$. Note also that if $(\underline{p}_i,\bar{p}_i)$ exists for some $i \in \{1,2\}$, then $\underline{V}_H(p,\underline{p}_i) = \overline{V}_H(p,\bar{p}_i)$, where $\overline{V}_H$ is defined in (5.3).

Note that two pairs $\{(\underline{p}_i,\bar{p}_i)\}_{i\in\{1,2\}}$ with the previous properties exist only if $c_0 \le \bar{c}_0$, where $\bar{c}_0$ is defined in Proposition 3.2. Indeed, assume otherwise, that is, two pairs exist and $c_0 > \bar{c}_0$. Then, since $\overline{V}_H(p,\bar{p})$ is increasing in $\bar{p}$, is the case that $\overline{V}_H(\bar{p}_j,\bar{p}_i) \le \bar{p}_j\,\Pi$ for some i and j such that $i,j \in \{1,2\}$, $i \ne j$. Assume it is true for $i = 2$ (the other case is analogous). Since $\frac{\partial^2}{\partial p^2}\overline{V}_H(p,\bar{p}_2) > 0$, there exists some $\underline{p} \in [\bar{p}_1,\bar{p}_2)$ such that $\overline{V}_H(\underline{p},\bar{p}_2) = \underline{p}\,\Pi$ and $\overline{V}_H(p,\bar{p}_2) > p\,\Pi$ for all $p \in (\underline{p},\bar{p}_2)$. Therefore, there exists a competitive equilibrium with rejection region $(\underline{p},\bar{p}_2)$, which contradicts Proposition 3.2. Furthermore, it is easy to show that if two pairs exist (and therefore $c_0 < \bar{c}_0$), it must be the case that $\bar{p}_1 < \tilde{p}_- < \bar{p}_2$, where $\tilde{p}_-$ is defined in (5.4).

Then, we have the following cases:

1. If $c_0 > \bar{c}_0$, only one pair exists (assume it is $i \in \{1,2\}$). Note that, by Proposition 3.2, no competitive equilibrium with rejection region $(\underline{p}_0,\bar{p}_0)$ exists such that $V_H(\underline{p}_0) = \underline{p}_0\,\Pi$ and $V_H(\bar{p}_0) = \bar{p}_0\,\Pi$. Therefore, the only equilibrium satisfying the smooth pasting condition is $(\underline{p}_i,\bar{p}_i)$.

2. If $c_0 < \bar{c}_0$ then, by Proposition 3.2, there is a unique competitive equilibrium satisfying D1 when $\Pi_0 = 0$. Denote its rejection region by $(\underline{p}_0,\bar{p}_0)$. We have two cases:

   - If $\bar{p}_2 > \bar{p}_0$ then $(\underline{p}_2,\bar{p}_2)$ is the rejection region of a competitive equilibrium. Indeed, since the boundary conditions are satisfied, we only need to verify that $\underline{V}_H(p,\underline{p}_2) > \max\{\Pi_0, p\,\Pi\}$ for all $p \in (\underline{p}_2,\bar{p}_2)$. Since $\frac{\partial}{\partial p}\underline{V}_H(p,\underline{p}_2) > 0$ for all $p > \underline{p}_2$, we only need to verify that $\underline{V}_H(p,\underline{p}_2) > p\,\Pi$ for all $p \in (\underline{p}_2,\bar{p}_2)$. Assume otherwise, that is, assume there is some $\tilde{p} \in (\underline{p}_2,\bar{p}_2)$ such that $\underline{V}_H(\tilde{p},\underline{p}_2) \le \tilde{p}\,\Pi$. Since $\underline{V}_H(p,\underline{p}_2) = \overline{V}_H(p,\bar{p}_2)$ and $\overline{V}_H(p,\bar{p})$ is increasing in $\bar{p}$, a similar argument as the one used in the proof of Proposition 3.2 shows that there must an equilibrium with rejection region $(\underline{p}_3,\bar{p}_3)$, with $\bar{p}_3 > \bar{p}_2$, such that $\frac{\partial}{\partial p}\overline{V}_H(\underline{p}_3,\bar{p}_3) = \Pi$. As we see saw in the proof of Proposition 3.2, this is unique, which implies that $\bar{p}_3 = \bar{p}_0$. Nevertheless, we have $\bar{p}_0 = \bar{p}_3 > \bar{p}_2$, which is a contradiction. Also, since $\underline{V}(p,\underline{p})$ is decreasing in $\underline{p}$, we have $(\underline{p}_1,\bar{p}_1) \subset (\underline{p}_2,\bar{p}_2)$.

   - If $\bar{p}_2 \le \bar{p}_0$, it must be the case that $\bar{p}_1 \le \underline{p}_0$. Indeed, otherwise there exists some $\tilde{p} \in (\underline{p}_1, \frac{\Pi_0}{\Pi})$ such that $\overline{V}_H(\tilde{p},\bar{p}_0) = \Pi_0$ and $\overline{V}_H(p,\bar{p}_0) > \max\{\Pi_0, p\,\Pi\}$ for $p \in (\tilde{p},\underline{p}_0)$.

Since $\overline{V}_H(p, \bar{p})$ is increasing in p, there exists some $\bar{p}_4 = \inf\{\bar{p} > \bar{p}_0 | \overline{V}_H(\tilde{p}_0(\bar{p}), \bar{p}) > \Pi_0\}$, where $\tilde{p}_0(\bar{p})$ is defined as in the proof of Lemma 3.1. It is easy to show that $(\tilde{p}_0(\bar{p}_4), \bar{p}_4)$ satisfies the same conditions as $(\tilde{p}_2, \bar{p}_2)$, and since there are only two pairs that satisfy those conditions and $\bar{p}_4 \geq \bar{p}_0 > \tilde{p}_- > \bar{p}_1$, it must be the case that $\bar{p}_4 = \bar{p}_2$. This contradicts our assumption, since $\bar{p}_2 = \bar{p}_4 > \bar{p}_0 \geq \bar{p}_2$. Furthermore, if $\bar{p}_2 < \bar{p}_0$, there is some $\tilde{p} \in (\underline{p}_2, \bar{p}_2)$ such that $\underline{V}(\tilde{p}, \underline{p}_2) < \tilde{p}\,\Pi$. The reason is that $\underline{V}(\cdot, \underline{p}_2) = \overline{V}(\cdot, \bar{p}_2)$, $\underline{p}_2 < \underline{p}_0 < \bar{p}_2$, $\overline{V}(\underline{p}_0, \bar{p}_0) = \underline{p}_0\,\Pi$ and $\overline{V}(p, \bar{p})$ is increasing in $\bar{p}$. Finally, if $\bar{p}_2 = \bar{p}_0$ then $\bar{p}_1 = \underline{p}_0$ and $\underline{p}_1 = \underline{p}_2$. Therefore, in this case, the equilibrium that satisfies the smooth pasting condition with biggest rejection region equal to the union of the (disjoint) intervals $(\underline{p}_0, \bar{p}_0)$ and $(\underline{p}_1, \bar{p}_1)$.

In each case it can be shown that the corresponding equilibrium satisfies D1 using a similar argument as in the proof of Lemma 3.1. Indeed, it is easy to see that if any of them does not satisfy D1, there must exist an equilibrium with a bigger rejection region satisfying the smooth pasting condition. Nevertheless, as we have shown, each of the proposed equilibria is the one with the biggest rejection region among all equilibria satisfying the smooth pasting condition. $\qquad\square$

Proof of Proposition 3.4 (page 28)

Proof. The problem of maximizing the value function of the H-seller can be written as a regular stochastic control problem, since now there is no incentive constraint:

$$0 = \max_{e_H(p)} \left(-c_0 - A_H\,e_H(p)^\alpha + \frac{(p-1)^2\,p\,(p\,V_H''(p) + 2\,V_H'(p))}{2\,\sigma^2}\,e_H(p)^2 \right).$$

The First Order Condition of the previous equation is

$$0 = -\alpha\,A_H\,e_H(p)^{\alpha-1} + \frac{(p-1)^2\,p\,(p\,V_H''(p) + 2\,V_H'(p))}{2\,\sigma^2}\,2\,e_H(p).$$

Note that since $\alpha > 2$ the Second Order Condition is satisfied. Using the previous two equations to solve for $e_H(p)$ it is easy to verify that the statement of the proposition is true (note that the terms of both equations involving p are identical). $\qquad\square$