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"IS BARGAINING OVER PRICES EFFICIENT?"

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Is Bargaining over Prices Efficient?*

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Abstract

We consider the problem of two agents bargaining over the relative price of two goods they are endowed with. They alternatingly exchange price offers and the utilities are discounted. The recipient of an offer can either accept it and choose the quantities to be traded, or reject and counter-offer a different relative price. We study the set of equilibria as discounting frictions vanish and find that: (1) any generic economy has bargaining equilibria that are inefficient even as discounting frictions vanish; and (2) a bargaining equilibrium converging to a Walrasian outcome exists for some robust types of convergence of the discount factors, but it does not exist for other equally robust convergences. Moreover, in case there exists a bargaining equilibrium converging to a Walrasian outcome, then there is necessarily a multiplicity of them. As a consequence, unlike in Rubinstein's (1982) alternating-offer bargaining, the equilibrium outcome of this set-up is not generically unique and efficient.

Keywords. Alternating-offer bargaining, Bargaining over prices.
JEL. C78.

1 Introduction

In the alternating-offer bargaining game (Rubinstein (1982), Ståhl (1972)), the subgame perfect equilibrium exists, and is unique and efficient. This is, arguably, one of the most important results in economic theory in recent history. In that bargaining game, two players are bargaining over the partition of a pie, say of size 1. The pie will be partitioned only after players reach an agreement. Each player in turn offers a certain partition: player 1 offers the share $1-x$ to the opponent, and keeps x. The opponent can either accept or reject. If she accepts, the game ends and she consumes $1-x$ and the other player consumes x. If she rejects, the same procedure is repeated in the next period, where the opponent can now make a counter-offer x. Discounting frictions make delay costly. The main result in Rubinstein (1982) is that agents will reach an agreement in the first period, and that there only exists one subgame

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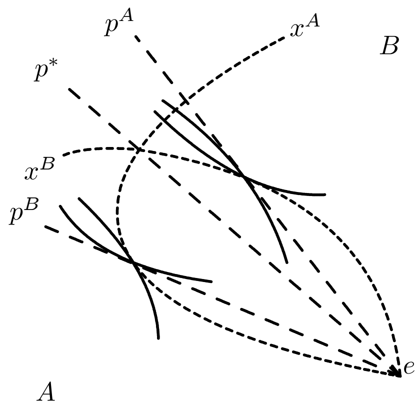
perfect equilibrium division of the surplus. This unique equilibrium is efficient because there is no delay. Moreover, the uniqueness and efficiency result is robust to the use non-stationary strategies (Rubinstein 1982) and general preferences over bundles of goods (Binmore 1987).

In this paper, we consider the following generalization of the standard alternating-offer bargaining game to exchange economies. First, players now hold general preferences over bundles of two goods (say x_1 and x_2) and have a given initial endowment of resources. Since, typically, the initial allocation of resources is not efficient, the agents will be interested in trading goods in order to benefit from the gains from trade and, if possible, attain an efficient one. Obviously, they have conflicting interests about the final allocation of resources and hence, in the absence of a vector of equilibrium prices announced by a Walrasian auctioneer to which they would react as price-takers and that would lead them to an efficient allocation, they will be led to bargain between them in order to improve their welfare with respect to their initial endowments. The second generalization of the alternating-offer bargaining game is to assume that players bargain over prices. The player who makes an offer announces a price p (specifically, a price of good x_1 in terms of good x_2) that he is bound to honor, and the player who receives the price offer p now can accept the offer and choose the quantities to be traded given p, or reject the offer. If an offer is rejected, the roles are switched and the bargaining continues until an agreement is reached.

Intuition may lead to expect that, as the frictions from discounting vanish, bargaining over prices will lead to a Walrasian outcome and hence to efficiency. In effect, consider first a unilateral monopoly in a two-person exchange economy. The monopolist (say agent A) announces a take-it-or-leave-it price offer, and the customer (agent B) chooses the quantity that maximizes her utility given the monopoly price (i.e. chooses an allocation on her offer curve). In this two-stage game, a subgame perfect equilibrium then requires the monopolist to announce a price that maximizes his utility, taking into account agent B will choose an allocation on her offer curve, i.e. the monopolist chooses a price that allows him to attain an indifference curve that is tangent to B's offer curve. Note that a take-it-or-leave-it offer is a particular instance of the alternating-offers protocol with myopia, i.e. extreme discount factors equal to 0: there is no point in rejecting an offer because it is as if there is no second stage in the negotiation and, as a consequence, the proposer has all the market power. As discount factors are made positive, there appears room for true negotiation, and as they converge to 1, the market power becomes evenly distributed between the two agents (given their endowments and preferences). In such negotiation, every agent knows that any outcome, upon acceptance of an offer, will be on the offer curve of the agent accepting the offer. In effect, in Figure 1 below,[1] the take-it-or-leave-it offer p^A of a monopolist A (similarly for p^B when B is the monopolist), corresponds to the extreme case in which the discount factors are equal to 0. It may seem intuitive that as the discount factors approach 1, the extreme imbalance of market power that makes the outcome to be on one offer curve but not on the other vanishes and, therefore, the outcome be an allocation on *both* offer curves x^A and x^B.In other words, as the agents become infinitely patient, the outcome of the bargaining seems to have to be an intersection point of the offer curves, i.e. a Walrasian allocation. Thus, constraining the agents to play stationary strategies in the alternating-price-offers bargaining game points to a Walrasian allocation as a natural candidate equilibrium when discounting frictions disappear.

[1] In Figure 1 the point e represents the initial endowments in this exchange economy, the dotted curves are A's and B's offer curves, and the solid curves are indifference curves of A and B. The price supporting the Walrasian allocation is p^*.

Figure 1



We show nonetheless that the existence of inefficient stationary subgame perfect equilibria (SSPE henceforth) that remain bounded away from efficiency even as the discounting frictions vanish, is a robust phenomenon for this bargaining problem[2] (see Theorem 4 in Section 4). These equilibria are not inefficient because there is delay, but rather because prices are offered that result in one of the agents accepting an offer and choosing an inefficient allocation. This turns out to be an equilibrium because the agent making the offer is indifferent between, on the one hand, consuming an allocation on the opponent's offer curve, and on the other hand, accepting an offer next period and consuming an allocation on his own offer curve. We show that for a generic economy there exist equilibrium prices that correspond to allocations off the contract curve, even as the discount factors approach one. The interesting feature of these inefficient SSPE is that on the stationary equilibrium path, both agents offer different terms of trade (i.e. the price) when called to offer, but accept the other agent's (less favorable) terms of trade when called to accept. Still, both agents are indifferent because for either of the different terms of trade, different quantities are traded. More generally, we show in Theorem 5 in Section 5 that the existence of SSPE that remain bounded away from efficiency is in fact a generic property for vanishing discounting frictions.

Still, the existence of inefficient SSPE in the limit does not *a priori* prevent the existence of other SSPE converging to a Walrasian allocation. We investigate this possibility in Section 6, and we find that either there is no equilibrium converging to a Walrasian allocation as the discounting frictions vanish, or else there is an even number of them. As a result, the uniqueness and efficiency result from the Rubinstein alternating-offer bargaining cannot be replicated when bargaining is over prices. In effect, if a unique equilibrium does exist, then it must be inefficient in the limit as discounting frictions disappear. This is because the equilibrium is unique only if there exists no equilibrium converging to the Walrasian allocation. Finally, unlike the Rubinstein alternating-offer bargaining game, even existence of an SSPE is in general not guaranteed, although generically it is (an example of non-existence is provided in Section 7).

Negotiation over prices seems to be a natural bargaining protocol to consider in a simple set-up with more than one good and perfect information about each other's characteristics. In effect, although what

[2]In the sense that there is a non-empty open set of economies with such SSPE.

really matters to each agent is the actual trade eventually carried out, with common knowledge of each other's preferences this can be equivalently summarized by the terms of trade agreed upon. Moreover, negotiations similar to bargaining over prices in a bilateral monopoly are frequently observed in reality. For example, negotiation over *labor contracts* between employers and unions are typically conducted over wages, i.e. the terms of trade, and not over both wages and employment. Although one can reasonably claim that upon acceptance of a wage offer made by the firm, unions do not choose the number of workers to be employed, nonetheless unions can reasonably well guess the level of employment that a wage rate will lead to, and hence take that into account in making their decision about whether to accept the offer or not. As a matter of fact, firm-union wage-bargaining corresponds rather to a set-up in which two agents bargain over prices making alternating offers while only one of the agents is entitled to choose the quantities to be traded upon an agreement over terms of trade. This problem is more naturally addressed in the framework of axiomatic bargaining and it is considered in Section 8.2 below. There we actually show the existence of bargaining powers leading to the same outcome of the alternating-offers bargain over prices considered here. There is an extensive literature in labor economics (see, among others, Solow and MacDonald (1981), Farber (1986)) documenting the prevalence of such bargaining over wages. Moreover, the early bargaining models pre-dating Nash's axiomatic bargaining solution all consider bargaining over wages in the context of firm-union relations (see Harsanyi (1956)). Those models of axiomatic bargaining point to an outcome on the Pareto frontier. The model we consider in this paper can be interpreted as a formalization of such bargaining over wages by means of a particular extensive form bargaining game (i.e. alternating-offer bargaining), as an alternative to an axiomatic bargaining approach.

The issue addressed in this paper is also addressed in an interesting and appealing paper by Yildiz (2001). In that paper, a unique SSPE of the bargaining problem considered here is shown to converge, under some conditions, towards a Walrasian allocation as the agents become infinitely patient. Our work differs from Yildiz (2001) in two fundamental ways. First, our approach to the issue is different in that we work mostly in the space of allocations of resources, rather than only in the space of profiles of utilities attained by the agents. Second, and most importantly, this approach allows us to obtain the generic existence of SSPE that remain inefficient as the agents' discount factors converge to one. The explanation for the apparent contradiction of our results with those of Yildiz (2001) is that the assumptions under which the convergence to the Walrasian outcome was obtained in Yildiz (2001) turn out to characterize a degenerate set of economies.[3] A perturbation of those economies results in the existence of stationary subgame perfect equilibrium that stay bounded away from efficiency. A detailed account of this fact can be found in the discussion in Section 8.3.

The indeterminacy results reported here refers back to Edgeworth's (1881) conjecture that all allocations in the core are equilibrium outcomes of a two person bargaining problem. He attributes this to the nature of the bilateral monopoly. Like Edgeworth, we find indeterminacy, but unlike Edgeworth, we do not find that all allocations in the core are equilibrium outcomes. On the contrary, there exist equilibrium allocations that are outside the core, and hence Pareto inefficient. Our results seem to confirm that for a Walrasian outcome to obtain with certainty, it is not sufficient to allow a small number of individuals to compete intertemporally.[4] A large number of agents seems to be essential, as established

[3] In the sense of being closed and nowhere dense.

[4] Even if players can bargain over allocations (instead of over terms of trade), the outcome does not coincide with the

in Gale (1986) where with a continuum of randomly matched agents, the Walrasian outcome through pairwise alternating-offer bargaining can be achieved in the limit as agents become impatient.

The remainder of the paper is as follows. In the Section 2 we illustrate, in a simple Cobb-Douglas setup, an example of a SSPE that remains bounded away from efficiency as the discount factors converge to 1. We present the general model in Section 3. We show in Section 4, Theorem 4, a sufficient condition for the existence of an inefficient SSPE, and in Section 5, Theorem 5, we show the generic existence of such SSPE. In Section 6, Theorem 6, we derive conditions for the existence of SSPE converging to a Walrasian allocation. Section 7 provides further examples, and Sections 8 and 9 conclude with a general discussion and some concluding remarks.

2 A Simple Example

In this section we illustrate the existence of a stationary subgame perfect equilibrium (SSPE) of bargaining over prices within a simple Cobb-Douglas setup. In effect, consider an economy with two agents A and B with preferences over bundles of two goods $x = (x_1, x_2)$ that are represented by the Cobb-Douglas utility functions $u^A = \sqrt{x_1^A x_2^A}$ and $u^B = \sqrt{x_1^B x_2^B}$.[5] The total resources are $e = (1, 1)$ and the distribution of initial endowments between A and B is $e^A = (0.9, 0.3)$ and $e^B = (0.1, 0.7)$. Since the initial allocation of resources is not efficient,[6] there are gains from trade.

Consider an extensive form bargaining game that consists of letting one agent i offer terms of trade between the two goods, represented by the vector of prices $(p^i, 1)$, good 2 acting as the numeraire. Given the terms p^i offered by agent i, the other agent $-i$ reacts either announcing a desired quantity traded at the proposed terms of trade (that the proposer is bound to honor), or making a counter-offer of terms of trade, and so on until a trade takes place. As a measure of the costly nature of the bargaining process itself, the utility obtained from the consumption of the two goods decreases by factors $\delta^A, \delta^B \in (0, 1)$ for each iteration in the sequence of offers and counteroffers made until the trade takes place.

Whenever an agent i decides to accept an offer at terms of trade p^{-i} by agent $-i$, individual rationality guarantees that he will choose to demand quantities $x^i(p^{-i})$ on his offer curve (i..e. that maximize his utility given p^{-i}). The resulting instantaneous utilities are $u^{-i}(e - x^i(p^{-i}))$ to the agent $-i$ who makes the offer, and $u^i(x^i(p^{-i}))$ to the agent i that receives it. Our objective is to solve for the SSPE of this bargaining game, denoted by (p^A, p^B). Necessary conditions for an SSPE of this bargaining game are given by[7]

$$\begin{aligned} u^A(x^A(p^B)) &= \delta^A u^A(e - x^B(p^A)) \\ u^B(x^B(p^A)) &= \delta^B u^B(e - x^A(p^B)). \end{aligned} \tag{1}$$

The interpretation of these equations is that, in each subgame, the agent accepting the offer should be no worse off than waiting one period and having his counter-offer accepted. For expositional clarity, let

Walrasian allocation as the discount factor approaches 1 (see Binmore, Rubinstein and Wolinsky (1986) and Binmore (1987)).

[5] As it will become apparent later on, the kind of returns to scale plays no role and hence this particular choice is of no consequence.

[6] In this example, the contract curve is the diagonal of the Edgeworth box, while the initial endowments are off the diagonal.

[7] In the next section, the complete optimization program will be presented and solved.

us first consider the case in which δ^A and δ^B are arbitrarily close to one.[8] In the limit, when δ^A and δ^B become equal to 1, there is a convenient graphical interpretation of equations (1). In effect, the first equation requires that the bundles of agent A resulting from accepting an offer $x^A(p^B)$, and from having an offer accepted $e - x_B(p_A)$, are on the same indifference curve for agent A. The second equation has a similar interpretation for agent B. Moreover, note that individual rationality implies that $x^A(p^B)$ and $x^B(p^A)$ are on the offer curves of A and B respectively.

There are two obvious solutions to the previous system of equations when $\delta_A = \delta_B = 1$, which are (i) the Walrasian allocation x^* (with $x^{*A} = (0.6, 0.6), x^{*B} = (0.4, 0.4)$ and supported by the price $p^A = p^B = 1$), and (ii) the initial endowment allocation e (supported by prices equal to the marginal rates of substitution at this point).[9] Figure 2 below exhibits a third solution this system, with prices

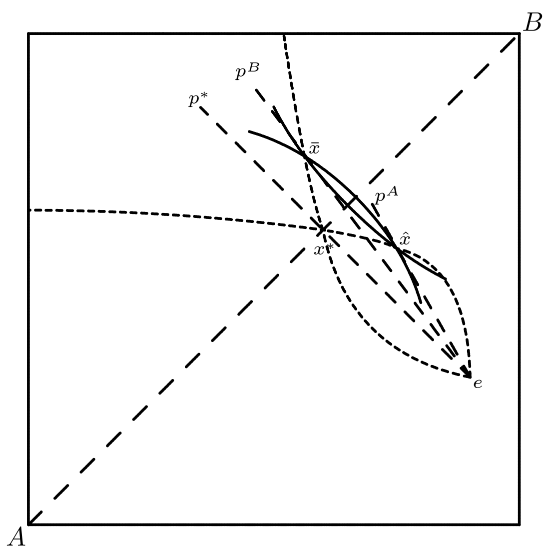
$$(p^A, p^B) = (1.750, 1.333) \tag{2}$$

and two points $\bar{x}, \hat{x}$ corresponding to two allocations on A's and B's offer curve respectively, more specifically

$$\begin{aligned} \bar{x}^A \quad &: \quad (\bar{x}_1^A(p^B), \bar{x}_2^A(p^B)) = (0.5625, 0.75) \\ \hat{x}^A \quad &: \quad (e - \hat{x}_1^B(p^A), e - \hat{x}_2^B(p^A)) = (0.75, 0.5625) \end{aligned} \tag{3}$$

and the complementary bundles for agent B. Note that unlike the Walrasian solution, the latter solution is not Pareto-efficient.

Figure 2



By a simple continuity argument, there exists a solution $(p^A(\delta), p^B(\delta))$ to the system of equations (1) close to the third solution $(1.750, 1.333)$, and hence bounded away from efficiency, for every δ^A, δ^B

[8] Of course, discounting is crucial for the previous equations to be necessary. Without discounting, many other SSPE exist that are no solutions to this system of equations.

[9] Though the initial endowment is a solution to this system of equations, below we show that it will never generate a SSPE for discounting frictions vanishing, because it will never satisfy the conditions for subgame perfection of the equilibrium.

sufficiently close to one. Moreover, as $(\delta^A, \delta^B) \to (1,1)$, there may also exist an even number (possibly zero) of SSPE converging to the Walrasian equilibrium. In Section 7 we return to this example and verify the conditions of existence of these equilibria. For now, this standard Cobb-Douglas example suffices to illustrate the possible inefficiency, even in the limit, of the SSPE of this bargaining game. In the next sections we consider the general set-up and show that the existence of such SSPE inefficient in the limit is actually the typical situation.

3 The Model

Consider an exchange economy with two agents $i \in \{A, B\}$ with standard preferences over nonnegative consumptions of two goods given by the utility functions u^A, u^B satisfying

Assumption 1 *For all $i \in \{A, B\}$, u^i is $\mathbb{R}_+$-valued,*[10] *continuous in $\mathbb{R}^2_+$, differentiable in $\mathbb{R}^2_{++}$, monotone,*[11] *differentiably strictly concave,*[12] *well-behaved at the boundary,*[13] *and such that i's demand is never simultaneously upward-sloped for both goods.*[14]

The agents are endowed with the nonnegative amounts $e^A = (e^A_1, e^A_2)$ and $e^B = (e^B_1, e^B_2)$ of the goods respectively. The total resources of the economy are $e = e^A + e^B$. Let us denote by $x^i = (x^i_1, x^i_2)$ the vector of goods consumed by $i \in \{A, B\}$, and an exchange economy by $\{u^i, e^i\}_{i \in \{A,B\}}$.

In the absence of a Walrasian auctioneer, the agents bargain over the terms of trade letting one agent make and offer of terms of trade by means of a price p of good 1 in terms of good 2, to which the other agent reacts either announcing a desired trade at the proposed terms of trade, or making a counter-offer of terms of trade, and so on until a trade takes place. The cost of the bargaining process itself is captured by the discount of the utility obtained from consumption by a factor $\delta^A, \delta^B = (0, 1)$ for each offer rejected by A and B respectively. Not reaching an agreement amounts to consuming the initial endowments.

In any subgame agent A's best response to an offer p^B from agent B, will in general depend on A's beliefs about B's strategy. We will restrict ourselves to stationary strategies profiles and hence stationary beliefs. Then A's best response to B's offer p^B is

1. to accept B's offer, if the discounted utility A can obtain from making an optimal counter-offer p^A is, if accepted, not more than what A would obtain accepting B's offer immediately, i.e.

[10] For a given set of preferences, there is no reason to assume positive utility. However, in the bargaining game, negative utility would render delay desirable, rather than costly.

[11] In the sense that $Du^i(x) \in \mathbb{R}^2_{++}$ for all $x \in \mathbb{R}^2_{++}$.

[12] As a matter of fact, differentiably strictly quasi-concave (in the sense that $D^2u^i(x)$ is definite negative on the space orthogonal to $Du^i(x)$ for all $x \in \mathbb{R}^2_{++}$) suffices. The strict concavity will only be needed for Theorem 6.

[13] In the sense that every indifference curve going through a point in $\mathbb{R}^2_{++}$ is completely contained in $\mathbb{R}^2_{++}$.

[14] That is to say, there is no vector of prices $(\bar{p}, 1)$ such that $\frac{dx^i_1}{dp}(\bar{p}) > 0$ and $\frac{dx^i_2}{dp^{-1}}(\bar{p}) > 0$. Note that this assumption does allow for backward-bending offer curves, and hence for any good being inferior for some range of prices. It only makes sure that the income effect is not so strong as to offset the substitution effect on the demand for *both* goods simultaneously.

$\delta^A u^A(e - x^B(p^A)) \leq u^A(x^A(p^B))$. This counter-offer p^A is the solution to the problem

$$\max_{p^A} u^A(e - x^B(p^A)) \tag{4}$$
$$u^B(x^B(p^A)) \geq \delta^B u^B(e - x^A(p^B))$$

where the constraint is a necessary condition for B to accept A's counter-offer,

2. to reject B's offer otherwise, and make the optimal counter-offer p^A described above.

Similar conditions characterizes B's optimal behavior.

A Stationary Subgame Perfect Equilibrium (SSPE) of the bargaining problem described above consists of a pair of two prices (p^A, p^B), corresponding to each player's price offer, such that both B and A respectively would accept in any subgame if confronted with these or better offers. If confronted with worse offers, B and A reject and offer p^B and p^A respectively in the next period. Since at an SSPE (p^A, p^B), the offers are accepted, they must satisfy $\delta^A u^A(e - x^B(p^A)) \leq u^A(x^A(p^B))$, and similarly for B. Moreover, sequential rationality requires p^A and p^B to be solutions to the maximization problems above for A and B. The definition of a Stationary Subgame Perfect Equilibrium follows.

Definition 2 *A Stationary Subgame Perfect Equilibrium (SSPE) of the bargaining problem above is a pair p^A, p^B such that*

$$p^A \in \arg\max_{\tilde{p}^A} u^A(e - x^B(\tilde{p}^A)) \tag{5}$$
$$u^B(x^B(\tilde{p}^A)) \geq \delta^B u^B(e - x^A(p^B)),$$

$$p^B \in \arg\max_{\tilde{p}^B} u^B(e - x^A(\tilde{p}^B)) \tag{6}$$
$$u^A(x^A(\tilde{p}^B)) \geq \delta^A u^A(e - x^B(p^A)),$$

and[15]

$$\begin{aligned} \delta^A u^A(e - x^B(p^A)) &\leq u^A(x^A(p^B)) \\ \delta^B u^B(e - x^A(p^B)) &\leq u^B(x^B(p^A)). \end{aligned} \tag{7}$$

The first-order conditions the problem of agent A above

$$\begin{aligned} Du^A(e - x^B(p^A))Dx^B(p^A) - \lambda^A Du^B(x^B(p^A))Dx^B(p^A) &= 0 \\ u^B(x^B(p^A)) - \delta^B u^B(e - x^A(p^B)) &= 0 \end{aligned} \tag{8}$$

characterize its solution, for some $\lambda^A > 0$, and similarly for agent B

$$\begin{aligned} Du^B(e - x^A(p^B))Dx^A(p^B) - \lambda^B Du^A(x^A(p^B))Dx^A(p^B) &= 0 \\ u^A(x^A(p^B)) - \delta^A u^A(e - x^B(p^A)) &= 0 \end{aligned} \tag{9}$$

[15]Note that the conditions (7) are redundant with (5) and (6).

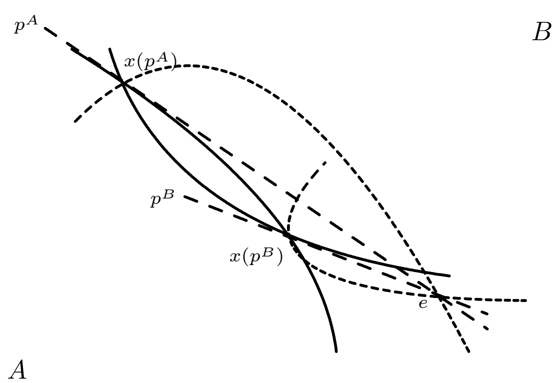
for some $\lambda^B > 0$ also. Therefore, necessary and sufficient[16] conditions for (p^A, p^B) to be an SSPE are

$$\begin{aligned}
Du^A(e - x^B(p^A))Dx^B(p^A) - \lambda^A Du^B(x^B(p^A))Dx^B(p^A) &= 0 \\
Du^B(e - x^A(p^B))Dx^A(p^B) - \lambda^B Du^A(x^A(p^B))Dx^A(p^B) &= 0 \\
u^A(x^A(p^B)) - \delta^A u^A(e - x^B(p^A)) &= 0 \\
u^B(x^B(p^A)) - \delta^B u^B(e - x^A(p^B)) &= 0.
\end{aligned} \tag{10}$$

Note that the last two inequalities (7) in the definition of a SSPE are trivially satisfied by any solution to this system of equations, and therefore need not be added.

The two last equations in (10) have a convenient interpretation when $\delta^A, \delta^B = 1$. In effect (see Figure 3), $x^A(p^B)$ and $x^B(p^A)$ must be points on the offer curves (short-dashed curves) of A and B respectively, and leading to allocations of resources within the Edgeworth box between which both A and B are indifferent (the indifferent curves are represented in solid lines)

Figure 3



The first two equations in (10) can also be interpreted in this figure as follows. Consider, for instance, the second equation. Its equivalent rewriting in (11) below amounts to requiring that the tangent to A's offer curve at $x(p^B)$ — whose direction is given by $Dx^A(p^B)$ — be normal to a positive linear combination of the gradients of u^A and u^B at that point

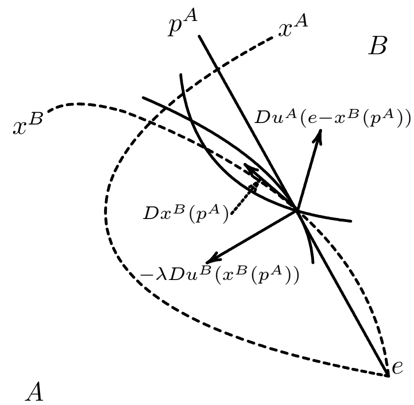
$$\left(Du^B(e - x^A(p^B)) - \lambda^B Du^A(x^A(p^B))\right)Dx^A(p^B) = 0. \tag{11}$$

As illustrated in Figure 4, this is equivalent to saying that the slope of the offer curve of A at $x(p^B)$ is not simultaneously smaller that A's marginal rate of substitution (with its negative sign) at that point, and bigger than B's. In more graphical terms: A's offer curve does not enter at $x(p^B)$ into the lens formed by the agents' indifference curves going through this point.[17] The first equation has a similar interpretation about the behavior of B's offer curve at $x(p^A)$.

[16] By the local nature of these conditions, we will assume that conditions guaranteeing that a local maximum is a global maximum hold.

[17] Obviously, in general the conditions above are not sufficient (see footnote 16) since the equation 1 may be satisfied locally, but the offer curve may still re-enter the lens away from this point. The same caveat of footnote 16 then applies here.

Figure 4



4 A sufficient condition for the existence of SSPE

The conditions for the existence of SSPE of the bargaining problem can be studied in the following way. Instead of looking for two prices p^A and p^B offered by the agents and satisfying (10) for some $\lambda^A, \lambda^B > 0$, we can equivalently look for two allocations $(\bar{x}^A, \bar{x}^B)$ and $(\hat{x}^A, \hat{x}^B)$ playing the roles of $(x^A(p^B), e - x^A(p^B))$ and $(e - x^B(p^A), x^B(p^A))$ respectively, and satisfying the equations (10). The prices p^A and p^B will then be determined unambiguously by the offer curves.

Let us focus first on the last two equations of the system (10). Note that they can equivalently be written in terms of allocations $(\bar{x}^A, \bar{x}^B)$ and $(\hat{x}^A, \hat{x}^B)$ as

$$\begin{aligned} \delta^A u^A(\hat{x}^A) - u^A(\bar{x}^A) &= 0 \\ u^B(e^A + e^B - \hat{x}^A) - \delta^B u^B(e^A + e^B - \bar{x}^A) &= 0 \\ Du^A(\bar{x}^A)(\bar{x}^A - e^A) &= 0 \\ Du^B(e^A + e^B - \hat{x}^A)(e^A - \hat{x}^A) &= 0 \end{aligned} \tag{12}$$

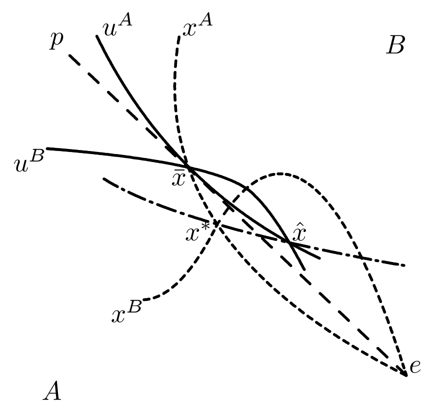
The system of equations (12) consists respectively of the following conditions: (i) $\hat{x}$ gives to A a utility equal to δ^A times the utility that $\bar{x}$ gives him, (ii) $\bar{x}$ gives to B a utility equal to δ^B times the utility that $\hat{x}$ gives him, (iii) $\bar{x}$ is on the offer curve of A, and (iv) $\hat{x}$ is on the offer curve of B, everything written in terms of A's variables, i.e. substituting out the feasibility conditions.

In order to find a solution to this system consider the following modified system instead

$$\begin{aligned} \delta^A u^A(\hat{x}^A) - u^A(\bar{x}^A) &= 0 \\ u^B(e^A + e^B - \hat{x}^A) - \delta^B u^B(e^A + e^B - \bar{x}^A) &= 0 \\ Du^A(\bar{x}^A)(\bar{x}^A - e^A) &= 0 \\ (p, 1)(\bar{x}^A - e^A) &= 0 \end{aligned} \tag{13}$$

where (iv) has been substituted by: (iv') $\bar{x}$ is affordable to A if the price of good 1 in terms of good 2 is p. That is to say, we have substituted a budget constraint to the condition that $\hat{x}$ be on B's offer curve. Intuitively, this modified system (13) defines $\bar{x}^A$ and $\hat{x}^A$ as a differentiable functions of p. As a consequence, $(\hat{x}^A, \hat{x}^B)$ follows a continuous path within the Edgeworth box as p varies (see Figure 5, where the auxiliary curve is represented by the dashed-dotted line).

Figure 5



An intersection of the curve followed by $(\hat{x}^A, \hat{x}^B)$ with the offer curve of B corresponds then to a solution to the original system of equations (12) and hence is a candidate to be an SSPE of the economy. The next lemma establishes the generic existence of inefficient allocations $\bar{x}$ and $\hat{x}$ solution to the original system of equations (12).

Lemma 3 *For any generic*[18] *exchange economy* $\{u^i, e^i\}_{i\in\{A,B\}}$ *satisfying Assumption 1, and discount factors* δ^A, δ^B *close enough to* 1*, there exist inefficient allocations* $\bar{x}, \hat{x}$ *solution to the system of equations* (12).

Proof. Note that, if $\delta^A = \delta^B = 1$, a Walrasian allocation of the economy x^* is a solution to the modified system (13) when p is the corresponding Walrasian equilibrium price p^* for good 1. Given this value for the discount factors, we will be interested in the slope of the curve followed by $\hat{x}$ at the Walrasian equilibrium allocation x^*. In effect, if $\hat{x}$ approaches the Walrasian allocation following a path with slope slightly smaller than that of B's offer curve at x^*, then there will necessarily exist an intersection —distinct from this Walrasian equilibrium and hence bounded away from it— of the path followed by $\hat{x}$ with B's offer curve. By continuity, that intersection will still exist for $\delta^A, \delta^B < 1$ but close enough to 1. Finally, it suffices to note that, for discount factors smaller than 1, $\hat{x}$ cannot be equal to $\bar{x}$, and hence none of them can be efficient allocations.

In order to obtain the slope of the path followed by $\hat{x}$ as p varies at a Walrasian allocation x^*, note that the function that determines A's bundle $(\hat{x}_1^A, \hat{x}_2^A)$ in $\hat{x}$ for each p in the system (13) is the composition of the function $\bar{\xi}^A$ associating $(\bar{x}_1^A, \bar{x}_2^A)$ to each p implicitly defined by

$$\begin{aligned} Du^A(\bar{x}^A)(\bar{x}^A - e^A) &= 0 \\ (p,1)(\bar{x}^A - e^A) &= 0 \end{aligned} \tag{14}$$

[18]With respect to the usual topology in the space of endowments, and the topology of C^1 uniform convergence on compacts in the space of utility functions (actually for any C^n convergence as well).

and the function $\hat{\xi}^A$ associating $(\hat{x}_1^A, \hat{x}_2^A)$ to each $(\bar{x}_1^A, \bar{x}_2^A)$ implicitly defined by

$$\begin{aligned} u^A(\hat{x}^A) - u^A(\bar{x}^A) &= 0 \\ u^B(e^A + e^B - \hat{x}^A) - u^B(e^A + e^B - \bar{x}^A) &= 0. \end{aligned} \tag{15}$$

Note, on the one hand, that the Jacobian of the left-hand side of (14) is

$$\begin{pmatrix} \nabla_1^A(\bar{x}^A) & \nabla_2^A(\bar{x}^A) & 0 \\ p & 1 & \bar{x}_1^A \end{pmatrix} \tag{16}$$

where $\nabla^A(\bar{x}^A)$ denotes the gradient of the offer curve of A at $\bar{x}^A$. For a utility function u^A generic with respect to the topology of C^1 uniform convergence on compacts,[19] the first two columns are always linearly independent, even at a Walrasian equilibrium allocation, and therefore the system (14) defines indeed $(\bar{x}_1^A, \bar{x}_2^A)$ as a function $\bar{\xi}^A$ of p implicitly and

$$\begin{aligned} D\bar{\xi}^A(p) &= -\begin{pmatrix} \nabla_1^A(\bar{x}^A) & \nabla_2^A(\bar{x}^A) \\ p & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \bar{x}_1^A \end{pmatrix} \\ &= -\begin{vmatrix} \nabla_1^A(\bar{x}^A) & \nabla_2^A(\bar{x}^A) \\ p & 1 \end{vmatrix}^{-1} \begin{pmatrix} -\nabla_2^A(\bar{x}^A)\bar{x}_1^A \\ \nabla_1^A(\bar{x}^A)\bar{x}_1^A \end{pmatrix}. \end{aligned} \tag{17}$$

On the other hand, the Jacobian of the left-hand side of (15) is

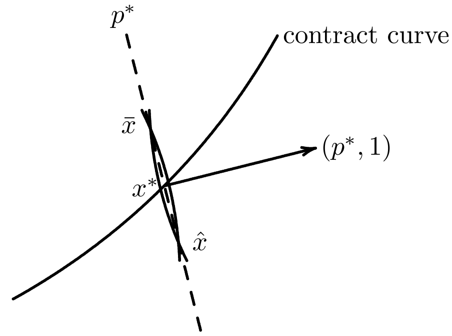
$$\begin{pmatrix} D_1u^A(\hat{x}^A) & D_2u^A(\hat{x}^A) & -D_1u^A(\bar{x}^A) & -D_2u^A(\bar{x}^A) \\ -D_1u^B(\hat{x}^B) & -D_2u^B(\hat{x}^B) & D_1u^B(\bar{x}^B) & D_2u^B(\bar{x}^B) \end{pmatrix} \tag{18}$$

and it drops rank only at efficient allocations, and hence at any Walrasian allocation. As a consequence, the theorem of the implicit function does not apply there. Nonetheless, the two equations (15) clearly define $(\hat{x}_1^A, \hat{x}_2^A)$ as a function of $(\bar{x}_1^A, \bar{x}_2^A)$ since, for strictly convex preferences and any given point $\bar{x}$ of the Edgeworth box, there exists a unique $\hat{x}$ where the two indifference curves going through $\bar{x}$ cross each other again. If $\bar{x}$ happens to be efficient, then $\hat{x}$ actually coincides with $\bar{x}$. This function is continuous and differentiable off the contract curve (there the implicit function theorem applies), but also on the contract curve (where the implicit function theorem cannot be called for). In effect, as $\bar{x}$ departs slightly from an efficient allocation x^* on the contract curve, the lens formed by the two indifference curves going through $\bar{x}$ will cross again at (almost) a point across the contract curve in the direction of

[19] Actually for any such C^n topology also.

the line supporting x^* as a Walrasian equilibrium (see Figure 6).

Figure 6



The linear mapping approximating locally this behavior is

$$D\hat{\xi}^A(x^{*A}) = \begin{pmatrix} p^* & -1 \\ 1 & p^* \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -c^* \end{pmatrix} \begin{pmatrix} p^* & -1 \\ 1 & p^* \end{pmatrix}^{-1} \tag{19}$$

for some $c^* > 0$ that in general will depend on the curvature of A's and B's indifference curves at the Walrasian allocation x^*. In words, $D\hat{\xi}^A(x^{*A})$ consists of the composition of (i) a change to an orthogonal basis containing the price vector $(p^*, 1)$, (ii) a jump across the first axis of that basis, and (iii) the undoing of the change of basis.

Therefore,

$$\begin{aligned} \begin{pmatrix} \frac{d\hat{x}_1^A}{dp}(p^*) \\ \frac{d\hat{x}_2^A}{dp}(p^*) \end{pmatrix} &= D\hat{\xi}^A(x^{*A}) D\bar{\xi}^A(p^*) && (20) \\ &= -\frac{x_1^{*A}}{p^{*2}+1} \begin{vmatrix} \nabla_1^A(x^{*A}) & \nabla_2^A(x^{*A}) \\ p^* & 1 \end{vmatrix}^{-1} \begin{pmatrix} (c^* - p^{*2})\nabla_2^A(x^{*A}) + (1+c^*)p^*\nabla_1^A(x^{*A}) \\ (1 - c^* p^{*2})\nabla_1^A(x^{*A}) - (1+c^*)p^*\nabla_2^A(x^{*A}) \end{pmatrix} \end{aligned}$$

and as a result,

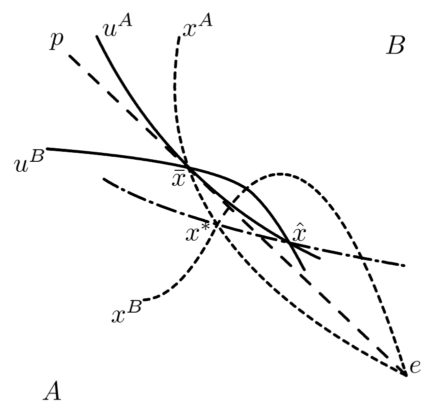
$$\frac{d\hat{x}_2^A}{d\hat{x}_1^A}(x_1^{*A}) = \frac{\frac{d\hat{x}_2^A}{dp}(p^*)}{\frac{d\hat{x}_1^A}{dp}(p^*)} = \frac{(1 - c^* p^{*2})\nabla_1^A(x^{*A}) - (1+c^*)p^*\nabla_2^A(x^{*A})}{(c^* - p^{*2})\nabla_2^A(x^{*A}) + (1+c^*)p^*\nabla_1^A(x^{*A})}. \tag{21}$$

If this slope is distinct from that of B's offer curve at the Walrasian allocation x^*, then there necessarily exists a solution to the system of equations (12) for $\delta^A, \delta^B < 1$ but close to 1. In effect, assume that

$$\frac{d\hat{x}_2^A}{d\hat{x}_1^A}(x_1^{*A}) < -\frac{\nabla_1^B(x^{*B})}{\nabla_2^B(x^{*B})}. \tag{22}$$

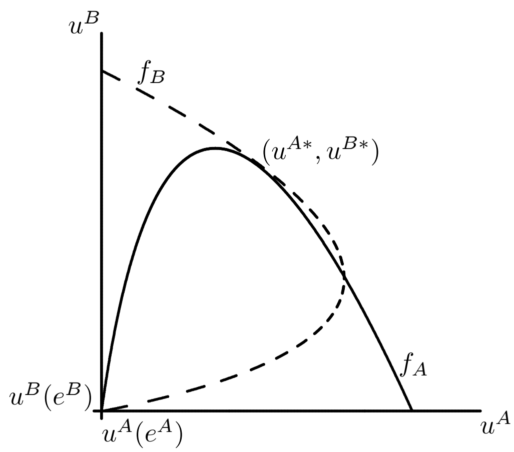
This implies that for any given level of utility u^A on A's offer curve and close to the Walrasian equilibrium one, B's utility is not smaller on B's offer curve than on A's (see Figure 7).

Figure 7



As a consequence, around the profile of utilities $(u^{A*}, u^{B*}) = (u^A(x^{*A}), u^B(x^{*B}))$ corresponding to the Walrasian equilibrium x^*, the profiles of utilities attainable along B's offer curve provide at least as big a utility to B than those attainable along A's offer curve, for any given level of utility for u^A. Figure 8 below represents the utility profiles along each of the offer curves: f_A (the solid line) associates u^A and u^B for those allocations on A's offer curve, and similarly f_B (the dashed line) for allocations on B 's offer curve (a formal definition will be given in Section 6 below).

Figure 8

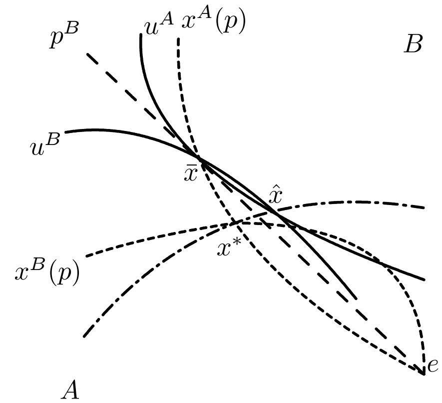


The boundary behavior of these curves of profiles of utilities guarantee the existence of at least one other intersection. On the contrary, assume that

$$\frac{d\hat{x}_2^A}{d\hat{x}_1^A}(x_1^{*A}) > -\frac{\nabla_1^B(x^{*B})}{\nabla_2^B(x^{*B})}. \tag{23}$$

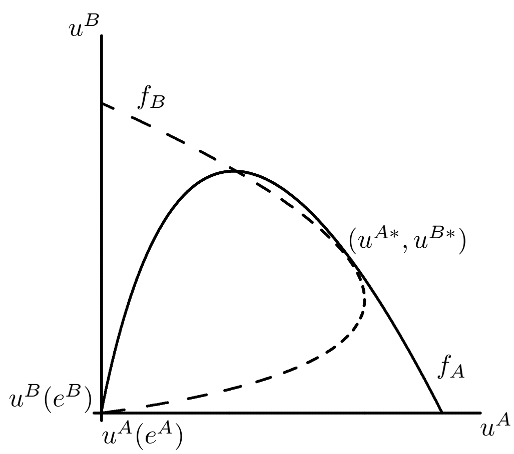
This implies that for any given level of utility u^A on A's offer curve and close to the Walrasian equilibrium one, B's utility is not bigger on B's offer curve than on A's (see Figure 9).

Figure 9



This means that, around this Walrasian equilibrium profile of utilities, those attainable along B's offer curve provide at least as big a utility to A as those attainable along B's offer curve, for any given level of utility for u^A (see Figure 10).

Figure 10



Again, the boundary behavior of these curves of profiles of utilities guarantee the existence of one other intersection.

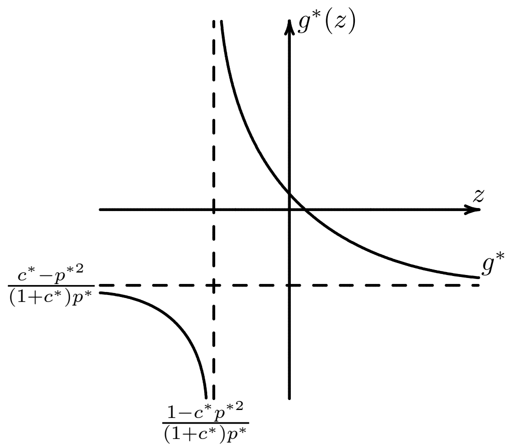
Note that, generically, not all the intersections of f_A and f_B can correspond to Walrasian allocations, since the boundary behavior of these curves would require then that at one of these intersections we have simultaneously a tangency and a crossing of the two curves f_A and f_B. While the tangency of f_A and f_B at profiles of utilities corresponding to Walrasian equilibria is a property intrinsic to such allocations (see Proposition 9 in the Appendix), their crossing is not; quite on the contrary, it is actually a degenerate property. In effect, a crossing of f_A and f_B at a profile of utilities corresponding to a Walrasian equilibrium x^* corresponds to the satisfaction of the equation

$$\frac{(1-c^*p^{*2})\nabla_1^A(x^{*A}) - (1+c^*)p^*\nabla_2^A(x^{*A})}{(c^*-p^{*2})\nabla_2^A(x^{*A}) + (1+c^*)p^*\nabla_1^A(x^{*A})} = -\frac{\nabla_1^B(x^{*B})}{\nabla_2^B(x^{*B})}. \tag{24}$$

This equation imposes a constraint on the partial derivatives of order two of the utility functions at this Walrasian allocation x^* that is degenerate in the space of utility functions with respect to the topology of C^1 uniform convergence on compacts.[20] Equivalently, if preferred, it constrains the profile of slopes of the offer curves of A and B at every Walrasian allocation x^*, $\left(-\frac{\nabla_1^A(x^{*A})}{\nabla_2^A(x^{*A})}, -\frac{\nabla_1^B(x^{*B})}{\nabla_2^B(x^{*B})}\right)$, to be on the graph of the function g^* below, which is clearly a degenerate requirement too (see its graph in Figure 11 below for the case $c^* = 1$ and $1 < p^{*2}$),[21]

$$g^*(z) = \frac{(1+c^*)p^* + (1-c^*p^{*2})z}{(1+c^*)p^*z + (c^* - p^{*2})} \tag{25}$$

Figure 11



Therefore, at least one intersection of f_A and f_B is a transversal crossing, i.e. not corresponding to a Walrasian allocation, and hence corresponding to inefficient allocations $\bar{x}, \hat{x}$. At this solution $\bar{x}, \hat{x}$, $\delta^A = \delta^B = 1$ of (12) the Implicit Function Theorem applies, making $\hat{x}, \bar{x}$ a differentiable function of δ^A, δ^B around $\delta^A, \delta^B = 1$. Therefore, the conclusion follows. ■

Interestingly enough, the following theorem uses the degenerate condition derived in Lemma 3 that would prevent to obtain a solution to the necessary conditions for the existence of a SSPE, to show the

[20] Actually, with respect to any such C^n topology. Interestingly enough, the perturbation need not always be made in the space of utility functions. For instance, in the case of the Cobb-Douglas example introduced in section 2 (for which $c^* = 1$ always), this condition is satisfied only for initial endowments on the anti-diagonal of the Edgeworth box, i.e. in a closed and nowhere dense subset of endowments for the given utility functions.

[21] In effect,

$$-\frac{\nabla_1^B(x^{*B})}{\nabla_2^B(x^{*B})}) = \frac{(1+c^*)p^* + (1-c^*p^{*2})(-\frac{\nabla_1^A(x^{*A})}{\nabla_2^A(x^{*A})})}{(1+c^*)p^*(-\frac{\nabla_1^A(x^{*A})}{\nabla_2^A(x^{*A})}) + (c^* - p^{*2})}.$$

Also

$$\lim_{z\to\infty} g^*(z) = \frac{1-c^*p^{*2}}{(1+c^*)p^*}$$
$$\lim_{z \to \frac{c^*-p^{*2}}{(1+c^*)p^*}} g^*(z) = \infty$$

and, for all $z \in \mathbb{R}$,

$$\frac{dg^*}{dz}(z) = -\frac{(1-c^*p^{*2})(c^*-p^{*2}) + ((1+c^*)p^*)^2}{((1+c^*)p^*z + (c^*-p^{*2}))^2} \neq 0$$

for all $c^*, p^* > 0$.

existence of SSPE whenever there is a Walrasian equilibrium that almost satisfies it.

Theorem 4 *For any generic*[22] *exchange economy* $\{u^i, e^i\}_{i\in\{A,B\}}$ *satisfying Assumption 1 and such that, at every Walrasian equilibrium* (x^*, p^*)*, the profile of slopes of the offer curves* $\left(-\frac{\nabla_1^A(x^{*A})}{\nabla_2^A(x^{*A})}, -\frac{\nabla_1^B(x^{*B})}{\nabla_2^B(x^{*B})}\right)$ *is close enough to the graph of* g^* *such that*

$$g^*(z) = \frac{(1+c^*)p^* + (1-c^*p^{*2})z}{(1+c^*)p^*z + (c^* - p^{*2})}, \tag{26}$$

and for any discount factors δ^A, δ^B *close enough to 1, there exist at least one SSPE of the problem of bargaining over prices. Moreover, that SSPE remains bounded away from efficiency even as* $\delta^A, \delta^B \to 1$.

Proof. In effect, for any utility functions u^A, u^B generic with respect to the topology of C^1 uniform convergence on compacts[23]

$$\frac{(1-c^*p^{*2})\nabla_1^A(x^{*A}) - (1+c^*)p^*\nabla_2^A(x^{*A})}{(c^* - p^{*2})\nabla_2^A(x^{*A}) + (1+c^*)p^*\nabla_1^A(x^{*A})} \neq -\frac{\nabla_1^B(x^{*B})}{\nabla_2^B(x^{*B})}. \tag{27}$$

Then Lemma 3 guarantees the existence of inefficient allocations $\hat{x}, \bar{x}$ that solve the necessary conditions (12) for a SSPE. By continuity, this solution remains for discount factors δ^A, δ^B close enough to 1, and still corresponds to inefficient allocations. This solution corresponds indeed to a SSPE if both curves f_A and f_B of profiles of utilities are downward-sloped at $(u^A(\hat{x}^A), u^B(\hat{x}^B)) = (u^A(\bar{x}^A), u^B(\bar{x}^B))$.[24] Note then that if the slope $\frac{d\hat{x}_2^A}{d\hat{x}_1^A}$ of the path followed by $(\hat{x}_1^A, \hat{x}_2^A)$ in the modified system (13) is not far from the slope of B's offer curve at the Walrasian allocation, $-\frac{\nabla_1^B(x^{*B})}{\nabla_2^B(x^{*B})}$, then the slopes of the two curves of profiles of utilities at $(u^A(\hat{x}^A), u^B(\hat{x}^B)) = (u^A(\bar{x}^A), u^B(\bar{x}^B))$ would be not far from their common negative slope at the Walrasian allocation profile $(u^A(x^{*A}), u^B(x^{*B}))$, and hence they will be negative also. But recall that

$$\frac{d\hat{x}_2^A}{d\hat{x}_1^A}(x_1^{*A}) = \frac{(1-c^*p^{*2})\nabla_1^A(x^{*A}) - (1+c^*)p^*\nabla_2^A(x^{*A})}{(c^* - p^{*2})\nabla_2^A(x^{*A}) + (1+c^*)p^*\nabla_1^A(x^{*A})} \tag{28}$$

also, i.e. $\left(-\frac{\nabla_1^A(x^{*A})}{\nabla_2^A(x^{*A})}, \frac{d\hat{x}_2^A}{d\hat{x}_1^A}(x_1^{*A})\right)$ is on the graph of g^*. Hence $\frac{d\hat{x}_2^A}{d\hat{x}_1^A}(x_1^{*A})$ is not far from $-\frac{\nabla_1^B(x^{*B})}{\nabla_2^B(x^{*B})}$, whenever $\left(-\frac{\nabla_1^A(x^{*A})}{\nabla_2^A(x^{*A})}, -\frac{\nabla_1^B(x^{*B})}{\nabla_2^B(x^{*B})}\right)$, is close to the graph of g^*. As a consequence, $\hat{x}, \bar{x}$ would indeed be the allocations arising from a SSPE (p^A, p^B) according to whether it is A or B who accepts the other agent's offer. ■

[22] With respect to the usual topology in the space of endowments, and the topology of C^1 uniform convergence on compacts in the space of utility functions (actually for any C^n convergence as well).

[23] Actually, with respect to any such C^n topology.

[24] Note that the negative slope at $(u^A(\hat{x}^A), u^B(\hat{x}^B)) = (u^A(\bar{x}^A), u^B(\bar{x}^B))$ is equivalent to satisfying the first two equations of (10). To see this, note that the first two equations require that at a candidate equilibrium allocation, there exists no allocation on the offer curve of either agent that is inside the lens of Pareto-improving allocations, which means precisely that no profile of utilities exists where both are increasing along the offer curve. Thus, should both curves of profiles of utilities not be downward-sloped, then a mutually beneficial counter-offer could be made to the agent whose curve of profiles of utilities along his offer curve is upward-sloped.

5 Generic existence of SSPE that remain bounded away from efficiency

The previous section has shown in Lemma 3 the generic existence of at least one other intersection of the curves f_A and f_B of profiles of utilities along each agent's offer curve not corresponding to a Walrasian outcome and moreover inefficient, in the extreme case in which there is no discounting. This intersection is robust to the introduction of discounting and, under a condition guaranteeing its subgame perfection, corresponds thus to a Stationary Subgame Perfect Equilibrium of the bargaining problem. Still one may wonder whether there are SSPE even when the sufficient condition of Theorem 4 is not satisfied. As a matter of fact, the existence of such equilibria is a quite general phenomenon. In effect, the next theorem establishes the generic existence of multiple SSPE that remain bounded away from efficiency as δ^A, δ^B converge to 1.

Theorem 5 *Within any neighborhood of any generic*[25] *exchange economy* $\{u^i, e^i\}_{i\in\{A,B\}}$ *satisfying Assumption 1, and for any discount factors* δ^A, δ^B *close enough to* 1*, there exists an economy* $\{\widetilde{u}^i, \widetilde{e}^i\}_{i\in\{A,B\}}$ *with multiple SSPE of the bargaining over prices that remain bounded away from efficiency as* $\delta^A, \delta^B \to 1$*. Moreover, the same is true of any economy in some neighborhood of* $\{\widetilde{u}^i, \widetilde{e}^i\}_{i\in\{A,B\}}$.

Proof. Consider a Walrasian equilibrium (p^*, x^*) of the economy. Since

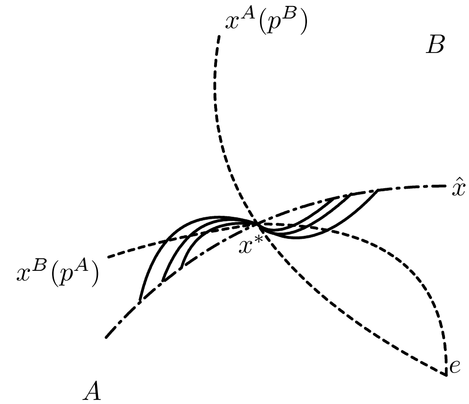
$$\frac{d\hat{x}_2^A}{d\hat{x}_1^A}(x_1^{*A}) = \frac{(1-c^*p^{*2})\nabla_1^A(x^{*A}) - (1+c^*)p^*\nabla_2^A(x^{*A})}{(c^*-p^{*2})\nabla_2^A(x^{*A}) + (1+c^*)p^*\nabla_1^A(x^{*A})}, \tag{29}$$

then the function g^* defined in (25) above expresses $\frac{d\hat{x}_1^A}{d\hat{x}_2^A}(x_1^{*A})$ as an injective function of the slope of A's offer curve at x^* whose range is $\mathbb{R}\backslash\left\{\frac{1-c^*p^{*2}}{(1+c^*)p^*}\right\}$. As a consequence, the slope $\frac{d\hat{x}_2^A}{d\hat{x}_1^A}(x_1^{*A})$ of the path followed by $\hat{x}$ at x^* can be made to take (almost) any value varying adequately the slope of the offer curve of A at x^*. In particular, this implies that there exists an intersection of the curve followed by $\hat{x}$ with B's offer curve for an arbitrarily close economy with respect to the topology of the C^1 uniform convergence on compacts in the space of A's utility function u^A. In effect, consider a sequence of paths $\hat{x}_n$ that differ from the path $\hat{x}$ corresponding to any given offer curve of A only in a compact neighborhood of x^*, and that converge pointwise to $\hat{x}$ while keeping their slopes at x^* smaller than that

[25] With respect to the usual topology in the space of endowments, and the topology of C^1 uniform convergence on compacts in the space of utility functions (actually for any C^n convergence as well).

of B's offer curve (see Figure 12 below).

Figure 12



To each of these paths for $\hat{x}$ there is associated a different offer curve $x_n^A(p^B)$ for A (not depicted in Figure 12 for the sake of readability). The pointwise convergence of these paths $\hat{x}_n$ to $\hat{x}$ guarantees the pointwise convergence within a compact of the associated offer curves $x_n^A(p^B)$ of A to $x^A(p^B)$. Also the (piecewise) monotone and pointwise convergence of $x_n^A(p^B)$ within a compact guarantees that their convergence to $x^A(p^B)$ is uniform indeed. As a consequence, the utility functions u_n^A of A generating these offer curves $x_n^A(p^B)$ converge in the topology of C^1 convergence on compacts towards the utility function u^A that generates the offer curve $x^A(p^B)$ of A.

Note, finally, that it is essential to the argument to be sure that the new intersections between the path followed by $\hat{x}$ and B's offer curve are not new Walrasian equilibria, which cannot be excluded *a priori* since, in perturbing the slope of the path followed by $\hat{x}$, we necessarily are perturbing also that of A's offer curve $x^A(p^B)$ at x^*, and hence new crossings of $x^A(p^B)$ with B's offer curve $x^B(p^A)$ corresponding to new Walrasian allocations might occur. As a matter of fact, it turns out that this is not the case.

In effect, in order to make the path followed by $\hat{x}$ cross again B's offer curve arbitrarily close to a Walrasian allocation x^*, we just need to be able to make $\frac{d\hat{x}_2^A}{d\hat{x}_1^A}(x_1^{*A})$ equal to the slope of B's offer curve at x^*, i.e.

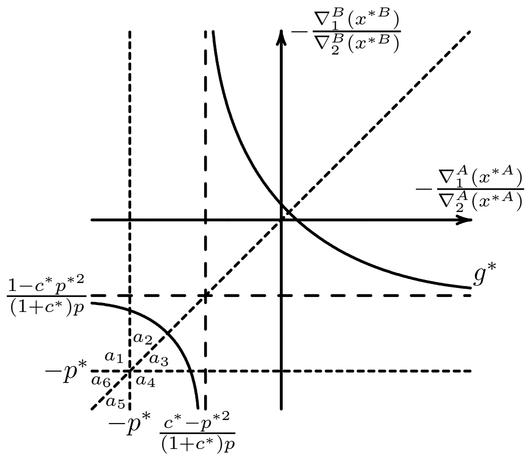
$$\frac{d\hat{x}_2^A}{d\hat{x}_1^A}(x_1^{*A}) = -\frac{\nabla_1^B(x^{*B})}{\nabla_2^B(x^{*B})}. \tag{30}$$

But since

$$\frac{d\hat{x}_2^A}{d\hat{x}_1^A}(p^*) = \frac{(1 - c^*p^{*2})\nabla_1^A(x^{*A}) - (1 + c^*)p^*\nabla_2^A(x^{*A})}{(c^* - p^{*2})\nabla_2^A(x^{*A}) + (1 + c^*)p^*\nabla_1^A(x^{*A})}, \tag{31}$$

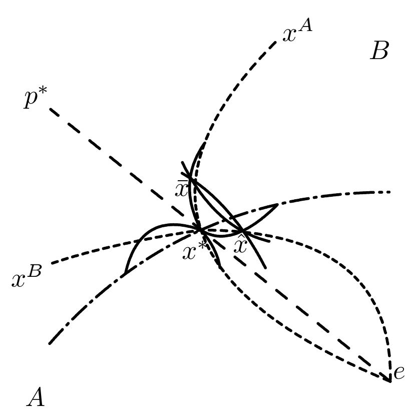
this is achieved making the profile $\left(-\frac{\nabla_1^A(x^{*A})}{\nabla_2^A(x^{*A})}, -\frac{\nabla_1^B(x^{*B})}{\nabla_2^B(x^{*B})}\right)$ of slopes of the offer curves at the Walrasian allocation x^* to be on the graph of the function g^* in (25) above. It is straightforward to check that the graph of g^* can be attained from any profile of slopes for the offer curves without making appear new intersections between them. This can be seen in Figure 13 below for the case $c^* = 1$ and $1 < p^{*2}$. There the whole plane of profiles of slopes of the offer curves of A and B at a Walrasian equilibrium is partitioned in the six areas a_i, $i = 1, \ldots, 6$ within which the number of Walrasian equilibria remains locally constant. The graph of g^* intersects every a_i, except for the areas a_5 and a_6 that correspond to preferences that do not satisfy the Assumption 1 (in particular, they violate the requirement of A's and B's demand not being simultaneously upward-sloped for both goods).[26]

Figure 13



Far enough in the sequence, the marginal rate of substitution of A at the intersection $\hat{x}$ that gives him a higher utility than the Walrasian allocation x^* is close to the relative price supporting the Walrasian allocation, and hence smaller than the slope of B's offer curve at x^*(see Figure 14 below). By continuity, the same will be true for δ^A, δ^B smaller but close enough to 1. This is enough to guarantee that this intersection corresponds to a SSPE. A similar argument applies to the intersection that gives him a lower utility than the Walrasian allocation, and the existence of a multiplicity of SSPE follows. Note that as $\delta^A, \delta^B \to 1$ these SSPE remain bounded away from efficiency. ■

Figure 14



[26]The same holds true for any $c^*, p^* > 0$. In effect, the relevant property is that, since

$$-p^* < \frac{c^* - p^{*2}}{(1+c^*)p^*}$$

and

$$-p^* < \frac{1 - c^* p^{*2}}{(1+c^*)p^*}$$

always, then the asymptotes of g^* (and hence g^* itself) intersect every a_i, but a_5, a_6.

6 SSPE converging to Walrasian allocations

Note now that at every profile of utilities corresponding to a Walrasian allocation, one of the curves f_A and f_B of profiles of utilities along the offer curves of A and B intersects tangently (see Proposition 9 in the Appendix), and generically from below (by Lemma 3), the other curve. As a consequence, any such intersection is not robust to the introduction of discounting. In fact, two different possibilities arise: (i) either the intersection vanishes as at least one of δ^A, δ^B becomes distinct from 1, and therefore there cannot be SSPE converging to that Walrasian outcome as the agents become infinitely patient, or (ii) the tangent intersection bifurcates into two new transversal intersections that correspond to two distinct SSPE (since they would necessarily satisfy the sufficient condition for subgame perfection), that moreover would converge to the Walrasian outcome as the agents become infinitely patient.

It is worth noting that, even in the second case, the efficiency in the limit of the outcome of bargaining over prices does not follow from this convergence towards a Walrasian equilibrium, since the generic existence established in Theorem 5 of other SSPE that remain bounded away from efficiency creates an indeterminacy of the outcome of the bargaining that prevents to guarantee its efficiency. This section addresses in detail the conditions under which each of the two cases above arises.

Consider first the profiles of utilities of each agent along A's and B's offer curve, i.e. respectively

$$\begin{array}{c}(u^A(x^A(p)), u^B(e - x^A(p))) \\ (u^A(e - x^B(p)), u^B(x^B(p)))\end{array} \tag{32}$$

for all $p \in \mathbb{R}_{++}$. Consider now these same profiles but after discounting B's utility in the first, and A's in the second, i.e.

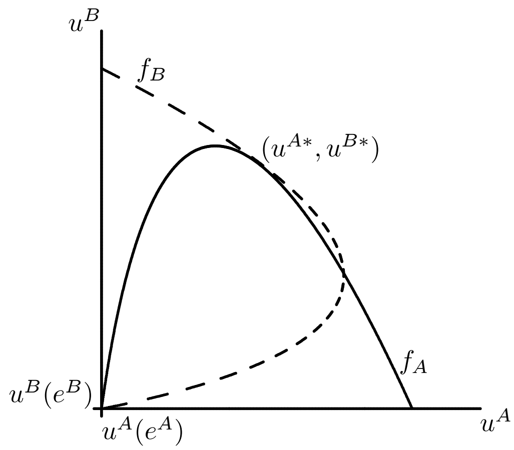
$$\begin{array}{c}(u^A(x^A(p)), \delta^B u^B(e - x^A(p))) \\ (\delta^A u^A(e - x^B(p)), u^B(x^B(p))).\end{array} \tag{33}$$

Note that a point in the intersection of these sets of profiles corresponds to a solution, for some p^A, p^B, to the last two equations of the system (10) of necessary conditions for a SSPE studied in the previous sections. As in the previous section, f_A denotes the function associating $u^B(e - x^A(p))$ to $u^A(x^A(p))$ as $p \in \mathbb{R}_{++}$, and similarly f_B denotes the function associating $u^A(e - x^B(p))$ to $u^B(x^B(p))$ as $p \in \mathbb{R}_{++}$. If p^* is a Walrasian price of the economy, let

$$\begin{array}{rcl}u^{A*} & = & u^A(x^A(p^*)) \\ u^{B*} & = & u^B(x^B(p^*)).\end{array} \tag{34}$$

Therefore, $u^{B*} = f_A(u^{A*})$ and $u^{A*} = f_B(u^{B*})$ (see Figure 15 below).

Figure 15



We are interested in seeing what happens to this intersection (u^{A*}, u^{B*}) of f_A and f_B when δ^A and δ^B depart from 1, more specifically whether an intersection of the graphs of the functions $\delta^A f_A$ and $\delta^B f_B$ (that now would correspond to a SSPE) exists or not around (u^{A*}, u^{B*}) for δ^A, δ^B smaller than but close to 1. The next theorem establishes that actually both cases are possible, depending on how δ^A, δ^B approach 1. It will also be established below that none of these two cases is degenerate, i.e. that the conditions for both the existence and the nonexistence of SSPE converging to a Walrasian equilibrium do not constrain too much the way in which δ^A and δ^B must approach 1.

Theorem 6 *Given a generic*[27] *exchange economy* $\{u^i, e^i\}_{i\in\{A,B\}}$ *satisfying Assumption 1, and a Walrasian equilibrium* (x^*, p^*) *of the economy,*

1. *there exist sequences of discount factors* (δ_n^A, δ_n^B) *converging to* $(1,1)$ *for which there is an even number of SSPE that converge to* (x^*, p^*)*, and*

2. *there also exist sequences of discount factors* $(\delta_n'^A, \delta_n'^B)$ *converging to* $(1,1)$ *for which there is no SSPE converging to* (x^*, p^*)*.*

Proof. Note that the efficiency of the Walrasian allocation implies that f_B is invertible in a neighborhood of the profile of utilities (u^{A*}, u^{B*}) corresponding to the Walrasian outcome x^*, and hence so is $\delta^B f_B$ around u^{A*} for δ^B close enough to 1. Thus we can wonder about whether, for given $\delta^A, \delta^B < 1$, $(\delta^B f_B)^{-1}(u^{A*})$ is smaller or bigger than $\delta^A f_A(u^{A*})$. This is useful for our purposes because

$$(\delta^B f_B)^{-1}(u^{A*}) < \delta^A f_A(u^{A*}) \tag{35}$$

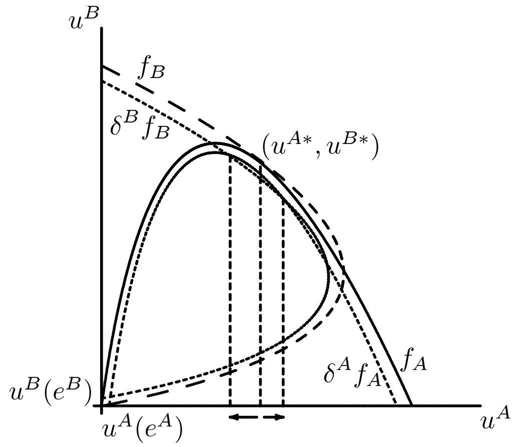
along with

$$f_A(u^A) \leq f_B^{-1}(u^{A*}) \tag{36}$$

[27]With respect to the usual topology in the space of endowments, and the topology of C^1 uniform convergence on compacts in the space of utility functions (actually for any C^n convergence as well).

for every u^A close enough to u^{A*} implies the existence of two other intersections of $\delta^A f_A$ and $\delta^B f_B$ (see Figure 16).

Figure 16



If, on the contrary, equation (35) holds with the opposite inequality (i.e. $(\delta^B f_B)^{-1}(u^{A*}) > \delta^A f_A(u^{A*})$ along with $f_A(u^A) \leq f_B^{-1}(u^{A*})$), then there exists no intersection of $\delta^A f_A$ and $\delta^B f_B$ in some neighborhood of (u^{A*}, u^{B*}) for δ^A, δ^B close enough to 1.[28]

Let us consider the first case. Clearly, since $f_A(u^{A*}) = u^{B*}$, then for any $\delta^A < 1$,

$$\delta^A f_A(u^{A*}) = \delta^A u^{B*}. \tag{37}$$

As for $(\delta^B f_B)^{-1}(u^{A*})$, let $\tilde{f}_B(u^B, \delta^B) = \delta^B f_B(u^B)$. Then

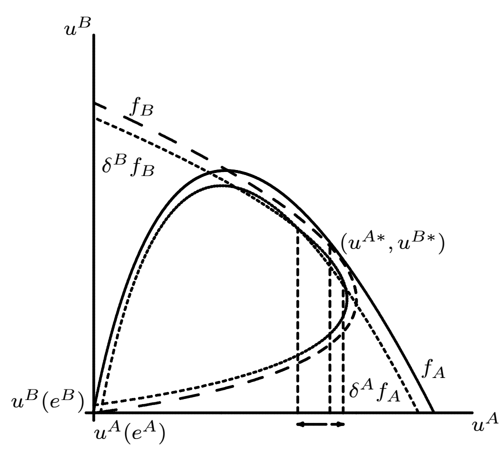
$$\tilde{f}_B(u^{B*}, 1) = f_B(u^{B*}) = u^{A*}. \tag{38}$$

[28]Similarly $(\delta^B f_B)^{-1}(u^{A*}) > \delta^A f_A(u^{A*})$ along with

$$f_A(u^A) \geq f_B^{-1}(u^{A*})$$

guarantees the existence of two other intersections as well (see Figure 17), while $(\delta^B f_B)^{-1}(u^{A*}) < \delta^A f_A(u^{A*})$ along with $f_A(u^A) \geq f_B^{-1}(u^{A*})$ guarantees that there exists no intersection of $\delta^A f_A$ and $\delta^B f_B$ in some neighborhood of (u^{A*}, u^{B*}) for δ^A, δ^B close enough to 1.

Figure 17



Since the linear approximation of $\tilde{f}_B$ at $(u^{B*}, 1)$ is

$$\tilde{f}_B(u^B, \delta^B) \approx \tilde{f}_B(u^{B*}, 1) + D_1\tilde{f}_B(u^{B*}, 1)(u^B - u^{B*}) + D_2\tilde{f}_B(u^{B*}, 1)(\delta^B - 1), \tag{39}$$

then $(\delta^B f_B)^{-1}(u^{A*})$ is the level of utility u^B for B such that

$$u^{A*} \approx u^{A*} + f'_B(u^{B*})(u^B - u^{B*}) + f_B(u^{B*})(\delta^B - 1) \tag{40}$$

i.e.

$$(\delta^B f_B)^{-1}(u^{A*}) \approx u^{B*} + \frac{u^{A*}}{f'_B(u^{B*})}(1 - \delta^B). \tag{41}$$

Therefore

$$(\delta^B f_B)^{-1}(u^{A*}) < \delta^A f_A(u^{A*}) \tag{42}$$

holds for δ^B smaller but close to 1 if, and only if,

$$u^{B*} + \frac{u^{A*}}{f'_B(u^{B*})}(1 - \delta^B) < \delta^A u^{B*} \tag{43}$$

i.e. if, and only if,

$$\frac{u^{B*}}{u^{A*}} < -\frac{1}{f'_B(u^{B*})}\frac{1 - \delta^B}{1 - \delta^A}. \tag{44}$$

Note that the range of values taken by $\frac{1-\delta^B}{1-\delta^A}$ in every neighborhood of $(\delta^A, \delta^B u^{A*}) = (1, 1)$ in $(0, 1)\times(0, 1)$ is $\mathbb{R}_{++}$. Therefore there always exist discount factors δ^A, δ^B arbitrarily close to 1 for which the condition (44) holds, as well as discount factors δ^A, δ^B arbitrarily close to 1 for which the reversed inequality

$$\frac{u^{B*}}{u^{A*}} > -\frac{1}{f'_B(u^{B*})}\frac{1 - \delta^B}{1 - \delta^A} \tag{45}$$

holds. Since, generically, either

$$f_A(u^A) \leq f_B^{-1}(u^{A*}) \tag{46}$$

or

$$f_A(u^A) \geq f_B^{-1}(u^{A*}) \tag{47}$$

holds for all u^A close enough to u^{A*}, the conclusion follows. ■

Clearly, neither $(\delta^B f_B)^{-1}(u^{A*}) < \delta^A f_A(u^{A*})$ nor $(\delta^B f_B)^{-1}(u^{A*}) > \delta^A f_A(u^{A*})$ is a condition that is necessarily satisfied by any given economy, and hence one may wonder whether the sequences of discount factors guaranteeing or preventing the existence of SSPE arbitrarily close to a Walrasian equilibrium of Theorem 6 are too special. It turns out that this is not the case for any of the two possibilities.

In effect, letting $(\bar{u}^{A*}, \bar{u}^{B*})$ denote a profile of utilities on the contract curve, and letting $\frac{d\bar{u}^{B*}}{d\bar{u}^{A*}}$ denote the common slope $\frac{1}{f'_B(u^{B*})} = f'_A(u^{A*})$ of f_A and f_B at (u^{A*}, u^{B*}), the condition

$$\frac{u^{B*}}{u^{A*}} = -\frac{1}{f'_B(u^{B*})}\frac{1 - \delta^B}{1 - \delta^A}. \tag{48}$$

that separates the two cases in Theorem 6 of existence and nonexistence of SSPE converging to a Walrasian equilibrium, can be written equivalently as

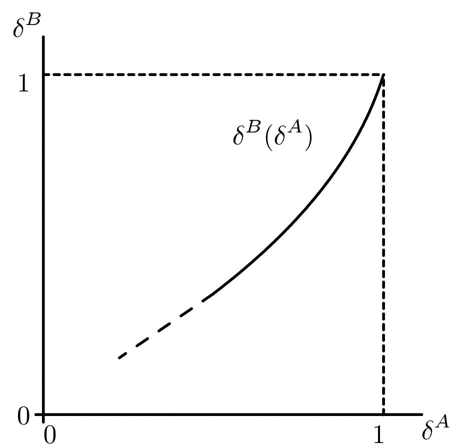
$$\delta^B = 1 + \frac{u^{B*}}{u^{A*}}\frac{d\bar{u}^{A*}}{d\bar{u}^{B*}}(1 - \delta^A), \tag{49}$$

which implies that the slope of this separating curve at $(\delta^A, \delta^B) = (1, 1)$ is

$$\frac{d\delta^B}{d\delta^A} = -\frac{u^{B*}}{u^{A*}} \frac{d\overline{u}^{A*}}{d\overline{u}^{B*}}. \tag{50}$$

As a consequence, $\frac{d\delta^B}{d\delta^A}$ is bounded above and bounded away from 0. In effect, because of the discounting in the bargaining game, utility functions are constrained to be positive. As a result, anywhere interior in the Edgeworth box, $0 < \frac{u^{B*}}{u^{A*}} < \infty$. Similarly, for any concave utility functions $0 < \frac{d\bar{u}^{A*}}{d\bar{u}^{B*}} < \infty$ anywhere on the contract curve and interior in the Edgeworth box. As a result, the curve of (δ^A, δ^B) separating the set of discount factors for which there is no SSPE converging to the Walrasian equilibrium from the set of those discount factors for which an even number of such SSPE exist, approaches $(1, 1)$ with a slope bounded away from the horizontal and vertical slopes of the boundary of $[0, 1] \times [0, 1]$ at $(1, 1)$ (see Figure 18 below). As a consequence, no degenerate convergence is required for either the existence or the nonexistence of SSPE converging to the Walrasian allocation to obtain.

Figure 18



Theorem 6 above provides a link between multiplicity and inefficiency of outcomes of the bargaining problem considered. In effect, from Theorem 6 it follows that for a generic economy either there is no SSPE converging to a Walrasian equilibrium as the agents become infinitely patient, or an even number of them exist. As a consequence, uniqueness of SSPE only obtains if this equilibrium is inefficient even as discounting frictions vanish. Hence the corollaries below follow.[29]

Corollary 7 (*Multiplicity*) *For a generic*[30] *exchange economy* $\{u^i, e^i\}_{i\in\{A,B\}}$ *satisfying Assumption 1, if there is one SSPE of the bargaining over prices that converges to a Walrasian equilibrium as* $\delta^A, \delta^B \to 1$*, then there exists an even number of them.*

Corollary 8 (*Inefficiency*) *For a generic*[30] *exchange economy* $\{u^i, e^i\}_{i\in\{A,B\}}$ *satisfying Assumption 1, if there is a unique SSPE of the bargaining over prices for all* δ^A, δ^B *close enough to* 1*, then it remains bounded away from efficiency as* $\delta^A, \delta^B \to 1$.

[29] Note that Corollary 7 implies Corollary 8, but not the other way around.

[30] With respect to the usual topology in the space of endowments, and the topology of C^1 uniform convergence on compacts in the space of utility functions (actually for any C^n convergence as well).

Finally, note also that the existence of a SSPE of the bargaining problem is not guaranteed in general,[31] even for δ^A, δ^B close to 1. In effect, for an economy where no SSPE converging to a Walrasian equilibrium exists, it may well be the case that the candidate to SSPE that generically exists by Lemma 3 does not satisfy the first two equations (10), i.e. the conditions for subgame perfection, and hence is not a SSPE.

7 Some Further Examples: uniqueness and non-existence

Consider the Cobb-Douglas example of Section 2 again. Both agents have identical utility functions $u^A = \sqrt{x_1^A x_2^A}$ and $u_B = \sqrt{x_1^B x_2^B}$, the total resources are $e = (1,1)$, and now $\delta \to (1,1)$ at a rate $r = \frac{\log \delta^A}{\log \delta^B} = 2$. Then

$$\lim_{\delta \to (1,1)} \frac{1-\delta^B}{1-\delta^A} = \frac{1}{r} = \frac{1}{2}$$

from de l'Hôpital's rule. Now consider an economy with initial endowments $e^A = (0.9, 0.3)$ and $e^B = (0.1, 0.7)$. Then the corresponding Walrasian equilibrium price is $p^* = 1$ and the corresponding equilibrium allocation is $x^{A*} = (0.6, 0.6)$ and $x^{B*} = (0.4, 0.4)$, so that $\frac{u^{B*}}{u^{A*}} = \frac{2}{3}$. It is easily verified that the utility possibility frontier is given by $\overline{u}^B = 1 - \overline{u}^A$ and that as a result, $\left|\frac{d\overline{u}^B}{d\overline{u}^A}\right| = 1$, so that $\lim_{\delta \to (1,1)} \left|\frac{d\overline{u}^B}{d\overline{u}^A}\right| \frac{1-\delta^B}{1-\delta^A} = \frac{1}{2}$, which is strictly smaller than $\frac{u^{B*}}{u^{A*}} = \frac{2}{3}$. As a result, condition (45) is satisfied.

On the other hand, the condition $f_A(u^A) \geq f_B^{-1}(u^{A*})$ is satisfied, since it is equivalent to[32]

$$\frac{(1-p^{*2})\nabla_1^A(x^{*A}) - 2p^*\nabla_2^A(x^{*A})}{(1-p^{*2})\nabla_2^A(x^{*A}) + 2p^*\nabla_1^A(x^{*A})} < -\frac{\nabla_1^B(x^{*B})}{\nabla_2^B(x^{*B})} \tag{51}$$

where the left-hand side of this condition at the Walrasian allocation, with $p^* = 1$, reduces to,

$$-\frac{\nabla_2^A(x^{*A})}{\nabla_1^A(x^{*A})} = -\frac{2x_2^{*A} - e_2^A}{2x_1^{*A} - e_1^A} = -3, \tag{52}$$

which is the inverse of the slope of A's offer curve in the Walrasian equilibrium allocation $(0.6, 0.6)$, and the right hand side of condition (51) is the slope of B's offer curve in the Walrasian allocation

$$-\frac{\nabla_1^B(x^{*B})}{\nabla_2^B(x^{*B})} = -\frac{2x_1^{*B} - e_1^B - 1}{2x_2^{*B} - e_2^B - 1} = -\frac{1}{3}.$$

It follows from Theorem 6 that there is no SSPE converging to the Walrasian equilibrium for this economy.

As for the existence of SSPE that remain bounded away from efficiency, it is sufficient to show that the slope of B's offer curve at $\hat{x}$ in Figure 1 (see section 2) is flatter than the slope of A's indifference curve through the same point, i.e.

$$\frac{2x_1^B - e_1^B - 1}{2x_2^B - e_2^B - 1} < \frac{x_2^A}{x_1^A} \tag{53}$$

where x^A and x^B are evaluated at $\hat{x}$, i.e. $x^A = (0.75, 0.5625)$ and $x^B = (0.25, 0.4375)$. It can be immediately verified that this is satisfied, since $0.727 < 0.75$. As a result, by Theorem 4 there will exist

[31] Although Theorem 5 guarantees its generic existence.

[32] Note that $c^* = 1$ in this example.

an SSPE of this economy converging to this inefficient outcome as δ^A, δ^B converge to 1. Moreover, by Theorem 6 this is the only SSPE of this economy.

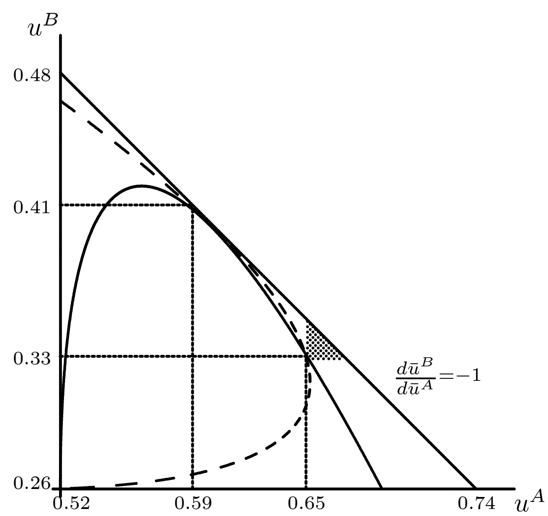
One can repeat the same exercise for an economy with initial endowments $e^A = (E, 0.3)$ and $e^B = (1-E, 0.7)$ where $0.95 < E \leq 1$. It can be easily verified that there is still no SSPE converging to the Walrasian equilibrium. However, now the only candidate to be a SSPE delivered by Lemma 3 ceases to be one. To see this, note that for, say, $E = 0.96$ at the new candidate solution $\hat{x}$, with corresponding allocations $\hat{x}^A = (0.882, 0.578)$ and $\hat{x}^B = (0.118, 0.421)$, the slope of B's offer curve is steeper than of A's indifference curve (more specifically $0.939 > 0.655$). As a result, for this economy, there does not exist any SSPE.

Finally, compare the outcome of the SSPE of the bargaining game over prices with the bargaining equilibrium over allocations. Binmore (1987) shows that for discounting frictions disappearing, bargaining over allocations leads to an allocation on the contract curve and hence efficiency (but in general different from the Walrasian allocation). In the example we considered earlier, the players' outside options are equal to the utility obtained in the initial endowment, $\left(u^A, u^B\right) = (0.52, 0.26)$. Since the total surplus on the contract curve is constant and equal to 1, the net surplus is 0.22 (i.e. 1 minus the sum of the outside options). Then given $\frac{\log \delta^A}{\log \delta^B} = r = 2$, bargaining over allocations converges to

$$\left(0.52 + 0.22\frac{1}{1+r}, 0.26 + 0.22\frac{r}{1+r}\right) = (0.59, 0.41),$$

which is different from the utility profile at the Walrasian allocation $(0.6, 0.4)$. On the other hand, bargaining over prices leads to the unique, inefficient SSPE in this example with payoffs $(0.65, 0.33)$. See Figure 16.

Figure 16



8 Discussion

8.1 The Coase Conjecture and "Renegotiation" of Inefficient Outcomes

Coase (1972) conjectures that the monopolist seller of a durable good will tend to price at marginal cost, absent some mechanism to commit to withholding supply. The underlying idea is that once the monopolist has sold at the monopoly price, he can now lower the price on unsold units, because past units have already been traded. Rational consumers will anticipate the price cuts, and will delay purchases until lower prices are offered. If the monopolist can change prices sufficiently fast, subgame perfection will lead her to set the price equal to the marginal cost (see Gul, Sonnenschein and Wilson (1986)).

A similar criticism applies to bargaining over prices in the exchange of durable goods. If, and when an inefficient equilibrium is obtained, the parties may not be able to commit to terminating the negotiation after trade has taken place. Since the outcome is inefficient, gains from trade remain and both agents may be tempted to resume negotiation. As in the case of the durable goods monopoly, this may lead to different prices being offered on the equilibrium path.

Analyzing a game with such lack of commitment of either party to stop the negotiation is an interesting an promising research venture. A sensible conjecture is that under certain restrictive assumption – e.g. Markov perfect equilibrium, restricting attention to strategies that are only dependent on the allocation where each bargaining round starts – such renegotiation will improve efficiency, and may well achieve an allocation on the contract curve. However, there are good reasons to believe that it may not necessarily lead to the Walrasian allocation on the contract curve because the Walrasian allocation is not in the Pareto set relative to the allocation of the initially inefficient equilibrium.

Unfortunately, the renegotiation argument for price bargaining is subject to the same shortcomings as the Coase conjecture (see Bulow (1982)). Renegotiation only occurs when the good is traded "permanently", i.e. when there is a permanent change of property rights. If this is not the case, then any renegotiation starts at the initial endowment. Examples that have been provided in the literature include leasing contracts. Of course, in the case of negotiation for services rather than goods, as in union-wage bargaining, the trade is considered not permanent.

8.2 Comparison with the Nash Axiomatic Bargaining-over-prices Solution

Does there exist a Nash Bargaining-over-prices problem that obtains the same outcome as the SSPE of the bargaining game over prices? The answer is yes. To see this, consider the Nash bargaining problem where one player (say player A) chooses the quantity of trade, and the Nash bargaining program selects a price, given bargaining power α, that solves the following maximization program

$$p \in \arg\max_{p} \left(u^A\left(x^A(p)\right)\right)^{\alpha}\left(u^B\left(e - x^A(p)\right)\right)^{1-\alpha}$$

where $x^A(p)$ denotes the allocation on A's offer curve given price p. Note that this is a modified version of Nash's static bargaining problem. Here, once p is determined, player A chooses the quantity. As a result, Nash's second axiom – that the solution be Pareto optimal – is not satisfied. However, in this modified version, conditional on optimal behavior in the ensuing subgame by player A (i.e. conditional on the equilibrium being subgame perfect), the bargaining solution is required to be "constrained" Pareto optimal.

The first order condition for this problem is:

$$\begin{aligned}\alpha u^A\left(x^A(p)\right)^{\alpha-1} u^B\left(e - x^A(p)\right)^{1-\alpha} Du^A\left(x^A(p)\right) Dx^A(p)& \\ -(1-\alpha)\, u^A\left(x^A(p)\right)^{\alpha} u^B\left(e - x^A(p)\right)^{-\alpha} Du^B\left(e - x^A(p)\right) Dx^A(p) &= 0\end{aligned}$$

which is satisfied if, and only if,[33]

$$\left[\left(\frac{u^B\left(e-x^A(p)\right)}{u^A\left(x^A(p)\right)}\right)\alpha Du^A\left(x^A(p)\right)-(1-\alpha)\,Du^B\left(e-x^A(p)\right)\right]Dx^A(p)=0.$$

This holds if either the first vector in brackets is the null vector, or both the vector in brackets and $Dx^A(p)$ are non-null but orthogonal.[34] This second case implies[35]

$$\frac{Du^B\left(e-x^A(p)\right)Dx^A(p)}{Du^A\left(x^A(p)\right)Dx^A(p)}=\frac{\alpha}{1-\alpha}\left(\frac{u^B\left(e-x^A(p)\right)}{u^A\left(x^A(p)\right)}\right).$$

Remind that $\left(u^A\left(x^A(p)\right),u^B\left(e-x^A(p)\right)\right)$ is the profile of utilities along A's offer curve f_A, i.e. $u^B\left(e-x^A(p)\right)=f_A\left(u^A\left(x^A(p)\right)\right)$. Then the derivative is of f_A is given by

$$f_A'\left(u^A\left(x^A(p)\right)\right)=-\frac{Du^B\left(e-x^A(p)\right)Dx^A(p)}{Du^A\left(x^A(p)\right)Dx^A(p)}$$

and as a result, the solution p of this Nash bargaining problem has to satisfy

$$f_A'\left(u^A\left(x^A(p)\right)\right)=-\frac{\alpha}{1-\alpha}\left(\frac{u^B\left(e-x^A(p)\right)}{u^A\left(x^A(p)\right)}\right).$$

Note that since $\frac{\alpha}{1-\alpha}$ takes any positive value for some $\alpha\in[0,1]$, any negative slope $f_A'\left(u^A\left(x^A(p)\right)\right)$ of f_A can be made equal to the right-hand side for some $\alpha\in[0,1]$. Therefore any point on the negatively sloped portion of f_A, the constrained Pareto frontier, is a Nash bargaining solution for some choice of α within $[0,1]$.

8.3 More than two goods

It is worth to note that the existence of SSPE that remain bounded away in the limit for a nonempty open set of economies shown above (Theorem 1), does not depend on having just two goods as in the previous set-up, but is rather a completely general property. As a matter of fact, this property follows from the fact that the curves of profiles of utilities f_A and f_B along the agents' offer curves do not cross generically at any profile of utilities (u^{A*},u^{B*}) corresponding to a Walrasian outcome, even though they necessarily intersect tangently there.

In effect, on the one hand, the tangent intersection follows intuitively from the fact that any profile of utilities (u^{A*},u^{B*}) corresponding to a (isolated) Walrasian allocation must be both on f_A and f_B, and on the (smooth) Pareto frontier, while f_A and f_B must be within (the interior of the) set of attainable utilities (at points distinct from (u^{A*},u^{B*})). This tangency implies that the slopes of f_A and f_B at (u^{A*},u^{B*}), i.e. the derivatives $f_A'(u^{A*})$ and $(f_B^{-1})'(u^{A*})$, coincide. On the other hand, assuming a (necessarily non-transversal) crossing of f_A and f_B at (u^{A*},u^{B*}) would require the derivative $(f_B^{-1})'(u^A)$ to be smaller than the derivative $f_A'(u^A)$ at every u^A in a neighborhood of u^{A*} and distinct from u^{A*}.

[33] Note that $\left(\frac{u^B\left(e-x^A(p)\right)}{u^A\left(x^A(p)\right)}\right)^{-\alpha}\neq 0$ because the utility functions take only positive values.

[34] The second vector $Dx^A(p)$ is non-null for any generic utility function u^A.

[35] Generically in utility functions, $Du^A\left(x^A(p)\right)$ and $Dx^A(p)$ are not orthogonal.

As a consequence, the graphs of the derivatives of f_A and f_B^{-1} would necessarily have to intersect at $(u^{A*}, f_A'(u^{A*})) = (u^{A*}, (f_B^{-1})'(u^{A*}))$ without crossing each other, and hence that intersection of the graphs of f_A' and $(f_B^{-1})'$ itself would have to be tangent, i.e. non-transversal, as well.

Now, while the non-transversality of the intersection of f_A and f_B^{-1} at (u^{A*}, u^{B*}) follows, as explained above, from the fact that (u^{A*}, u^{B*}) is the profile of utilities corresponding to a Walrasian allocation, the non-transversality of the intersection of f_A' and $(f_B^{-1})'$ at $(u^{A*}, f_A'(u^{A*})) = (u^{A*}, (f_B^{-1})'(u^{A*}))$ does not follow necessarily from any assumption on the economy or the properties of a Walrasian allocation. As a matter of fact, the non-transversality of the intersection of f_A' and $(f_B^{-1})'$ at $(u^{A*}, f_A'(u^{A*})) = (u^{A*}, (f_B^{-1})'(u^{A*}))$ imposes a constraint on the derivatives *up to the order* 2 of the utility functions u^A and u^B at the Walrasian allocation x^{*A} and x^{*B} respectively (like condition (24) above for the two goods case, where the coordinates of the gradients $\nabla^A(x^{*A})$ and $\nabla^B(x^{*B})$ of A and B's offer curves depend on the derivatives up to the order 2 of the utility functions) that will not be satisfied generically. Now, with f_A and f_B not crossing at (u^{A*}, u^{B*}), their boundary behavior guarantees the existence of at least one crossing that, for the reasons explained above, will generically be transversal and hence interior to the utilities possibility set, i.e. inefficient. If moreover f_A and f_B were close enough to have a crossing at (u^{A*}, u^{B*}) indeed, then this other crossing will necessarily satisfy also the condition for subgame perfection that guarantees it to correspond to a SSPE.

Note that the argument above is independent of the number of goods in the economy and relies only on the properties of the curves f_A and f_B of profiles of utilities attainable along the agents offer curves.

It is easy to see now why the assumptions A3 and A4 made in Yildiz (2001),[36] under which convergence of a unique SSPE towards a unique Walrasian outcome is obtained, are degenerate. In effect, while each of the two assumptions A3 and A4 in Yildiz (2001) are not degenerate on their own, nonetheless the requirement of both of them holding simultaneously amounts to having a non-transversal crossing of f_A and f_B at a Walrasian profile of utilities that, as explained above, is a degenerate property in the space of economies.

9 Conclusion

We have analyzed in this paper a model of alternating-offer bargaining over prices in an exchange economy. Because the only allocations that arise in equilibrium must necessarily be on the offer curve of the agent accepting the offer, and the market power of infinitely patient agents is evenly distributed between both agents, a sensible conjecture about the equilibrium outcome of the bargaining over prices as discounting frictions vanish is an allocation on both offer curves, i.e. a Walrasian allocation.

We have indeed shown that the Walrasian allocation can be the outcome of the bargaining over prices in the limit, as the agents become infinitely patient. In effect, Theorem 6 provides conditions under which such convergence of a stationary subgame perfect equilibrium to the Walrasian equilibrium occurs. Moreover, it establishes that whenever this convergence obtains there is actually an even number of stationary subgame perfect equilibria converging to the same Walrasian equilibrium. As a result,

[36] In words, A3: both monopolistic outcomes are dominated by some allocation attainable along an offer curve; and A4: there is a unique crossing of f_A and f_B within the interval defined by the profiles of utilities attained at the monopolistic outcomes.

convergence only obtains along with the indeterminacy of the outcome. Nevertheless, the convergence of the stationary subgame perfect equilibria of bargaining over prices to Walrasian outcomes is not even guaranteed, and Theorem 6 provides robust conditions under which there is no such convergence.

As a matter of fact, contrarily to what intuition tells us, the convergence of the bargaining over prices to a Walrasian outcome should not be expected. Theorem 4 proves that the existence of stationary subgame perfect equilibria that remain bounded away from efficiency as the agents become infinitely patient is a robust outcome of bargaining over prices. More importantly, Theorem 5 establishes that the subset of economies with such stationary subgame perfect equilibria is actually open and dense, which makes the non-convergence to a Walrasian outcome a generic property of these economies.

Interestingly enough, these results are in stark contrast with the existence of a unique and efficient subgame perfect equilibrium in the Rubinstein (1982) alternating-offer bargaining model.

What can be learned from the results presented in this paper is, in the first place, that bargaining over prices as a procedure for negotiation can be thoroughly questioned. The generic inefficiency of its outcomes suggests that this may not be the right procedure to solve a bilateral monopoly problem. In the second place though, bargaining over prices seems to be pervasive in real life. For instance, union wage bargaining is typically over prices because by law, firms have the right to choose the level of employment. And even without legal restrictions, in similar environments it may not be possible to contract on the quantity on because of the lack of verifiability and enforcement in court. As a result, when bargaining over which contract to write, parties may be restricted to including unit prices. The results derived in this paper suggest that the efficiency of such contracts can be a real concern.

10 Appendix

Proposition 9 *For a generic[37] exchange economy $\{u^i, e^i\}_{i\in\{A,B\}}$ satisfying Assumption 1, the curves f_A and f_B of profiles of utilities along the offer curves of A and B are tangent at every profile of utilities $(u^{A*}u^{B*})$ of a Walrasian allocation x^*.*

Proof. Consider the set of $2+n+1$ equations

$$\begin{aligned} u^A(x^A) - u^A &= 0 \\ u^B(x^B) - u^B &= 0 \\ x^A + x^B - e^A - e^B &= 0 \\ Du^A(x^A)(x^A - e^A) &= 0 \end{aligned}$$

in the $2(1+n)$ variables u^A, u^B, x^B, x^A. Note that the degrees of freedom of the system $n-1$ are positive as long as $n \geq 2$. The Jacobian matrix of the system is

$$\begin{pmatrix} 1 & 0 & 0 & -Du^A(x^A) \\ 0 & 1 & -Du^B(x^B) & 0 \\ 0 & 0 & I_n & I_n \\ 0 & 0 & 0 & \nabla^A(x^A) \end{pmatrix}$$

For u^B to be implicitly defined as a function of u^A, then the submatrix of the columns corresponding to every variable distinct from u^A

$$\begin{pmatrix} 0 & 0 & -Du^A(x^A) \\ 1 & -Du^B(x^B) & 0 \\ 0 & I_n & I_n \\ 0 & 0 & \nabla^A(x^A) \end{pmatrix}$$

The only way in which this submatrix can fail to have a full rank, i.e. the only way in which the null row vector can be a nontrivial linear combination of its rows, is multiplying its second row and the n rows corresponding to the identity matrices I_n by 0, and multiplying the first and last rows by λ and μ respectively such that

$$\lambda Du^A(x^A) = \mu \nabla^A(x^A).$$

This last condition requires the gradients of the utility function u^A and of the function defining A's offer curve to be collinear at x^A. This can be excluded generically (it can happen only countably many times along an offer curve and it is not robust to perturbations), and therefore u^B is indeed implicitly defined as a differentiable function of u^A by the system of equations above.

[37] With respect to the usual topology in the space of endowments, and the topology of C^1 uniform convergence on compacts in the space of utility functions (actually for any C^n convergence as well).

In the case $n = 2$, in order to compute the derivative of u^B with respect to u^A at any point, we need to compute the second entry of

$$-\begin{pmatrix} 0 & 0 & 0 & -D_1u^A(x^A) & -D_2u^A(x^A) \\ 1 & -D_1u^B(x^B) & -D_2u^B(x^B) & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & \nabla_1^A(x^A) & \nabla_2^A(x^A) \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

For that purpose we only need to know the determinant of the matrix above and the cofactor of the entry $(1,1)$. As for the determinant, it is

$$\begin{vmatrix} 0 & 0 & 0 & -D_1u^A(x^A) & -D_2u^A(x^A) \\ 1 & -D_1u^B(x^B) & -D_2u^B(x^B) & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & \nabla_1^A(x^A) & \nabla_2^A(x^A) \end{vmatrix}$$

$$= D_1u^A(x^A)\nabla_2^A(x^A) - D_2u^A(x^A)\nabla_1^A(x^A),$$

and the $(1,1)$ cofactor is

$$\begin{vmatrix} -D_1u^B(x^B) & -D_2u^B(x^B) & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & \nabla_1^A(x^A) & \nabla_2^A(x^A) \end{vmatrix}$$

$$\begin{aligned} &= -D_1u^B(x^B)\begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & \nabla_1^A(x^A) & \nabla_2^A(x^A) \end{vmatrix} + D_2u^B(x^B)\begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & \nabla_1^A(x^A) & \nabla_2^A(x^A) \end{vmatrix} \\ &= D_1u^B(x^B)\nabla_2^A(x^A) - D_2u^B(x^B)\nabla_1^A(x^A). \end{aligned}$$

Therefore

$$\frac{du^B}{du^A} = -\frac{D_1u^B(x^B)\nabla_2^A(x^A) - D_2u^B(x^B)\nabla_1^A(x^A)}{D_1u^A(x^A)\nabla_2^A(x^A) - D_2u^A(x^A)\nabla_1^A(x^A)}.$$

By a similar argument

$$\frac{du^A}{du^B} = -\frac{D_1u^A(x^A)\nabla_2^B(x^B) - D_2u^A(x^A)\nabla_1^B(x^B)}{D_1u^B(x^B)\nabla_2^B(x^B) - D_2u^B(x^B)\nabla_1^B(x^B)}.$$

Finally, since at any Walrasian allocation x^*, $Du^A(x^{A*}) = \lambda Du^B(x^{B*})$ for some $\lambda > 0$, then at x^*

$$
\begin{aligned}
\frac{du^B}{du^A}\frac{du^A}{du^B} &= \frac{D_1 u^B(x^{B*})\nabla_2^A(x^{A*}) - D_2 u^B(x^{B*})\nabla_1^A(x^{A*})}{D_1 u^A(x^{A*})\nabla_2^A(x^{A*}) - D_2 u^A(x^{A*})\nabla_1^A(x^{A*})} \\
&\quad \cdot \frac{D_1 u^A(x^{A*})\nabla_2^B(x^{B*}) - D_2 u^A(x^{A*})\nabla_1^B(x^{B*})}{D_1 u^B(x^{B*})\nabla_2^B(x^{B*}) - D_2 u^B(x^{B*})\nabla_1^B(x^{B*})} \\
&= \frac{D_1 u^B(x^{B*})\nabla_2^A(x^{A*}) - D_2 u^B(x^{B*})\nabla_1^A(x^{A*})}{\lambda(D_1 u^B(x^{B*})\nabla_2^A(x^{A*}) - D_2 u^B(x^{B*})\nabla_1^A(x^{A*}))} \\
&\quad \cdot \frac{\lambda(D_1 u^B(x^{B*})\nabla_2^B(x^{B*}) - D_2 u^B(x^{B*})\nabla_1^B(x^{B*}))}{D_1 u^B(x^{B*})\nabla_2^B(x^{B*}) - D_2 u^B(x^{B*})\nabla_1^B(x^{B*})} \\
&= 1
\end{aligned}
$$

which implies that the graphs of $u^B(u^A)$ and $u^A(u^B)$, i.e. f_A and f_B, are tangent at any profile of utilities corresponding to a Walrasian allocation. Q.E.D. ■

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