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On Preservation of Increasing Hazard Rate Under Convolution

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Abstract

The increasing hazard rate (IHR) property of distributions of asymmetric information parameters play a critical role in characterizing a separating Perfect Bayesian–Nash Equilibria in screening problems. This paper studies sufficient conditions on these distributions for IHR to be preserved under convolution. When different sources of asymmetric information aggregate into a single scalar, these preservation results prove very useful in designing alternative optimal mechanisms. The paper proves that if the distributions of all convoluting parameters are IHR the resulting distribution is also IHR. This result does not necessarily requires that the corresponding densities have to be log–concave. JEL: C00, D42, D82.

Keywords: Convolution, Increasing Hazard Rate, Log–Concavity.

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1   Introduction

In a recent work, Biais, Martimort, and Rochet (2000), BMR hereafter, develop a common agency model in which agents' types have two dimensions that lie on the real line and define a single dimensional aggregate:

$$\theta_0 = \theta_1 + \theta_2, \tag{1}$$

Thus, BMR face two alternative models of screening: either accounting for each source of asymmetry of information separately, *i.e.*, using $F_i(\theta_i)$, $i = 1, 2$, or targeting the aggregate directly using the convolution distribution:

$$F_0(\theta_0) = \int\limits_{\Theta_j} F_i(\theta_0 - \theta_j) dF_j(\theta_j). \tag{2}$$

BMR choose to make the necessary assumptions on the distribution of θ_0 in order to characterize an equilibrium in nonlinear schedules that depends exclusively on θ_0, thus reducing the dimensionality of the screening problem. Theorem 1 of BMR argues that such procedure is not restrictive because log–concavity is preserved under convolution. They claim that the convolution of a log–concave density $f_1(\theta_1)$, with any *arbitrary* density $f_2(\theta_2)$ leads to a probability distribution $F_0(\theta_0)$, and survival function $1 - F_0(\theta_0)$ that are both log–concave for the convolution defined by equations $(1)-(2)$. Their only requirement is that $f_1(\theta_1)$ and the *arbitrary* density $f_2(\theta_2)$ are defined on a bounded support.

Figures 1 and 2 cast some doubt on the validity of Theorem 1 of BMR. In these figures, the first row presents the probability density functions $f_i(\theta_i)$, $i = 0, 1, 2$. The second row pictures the ratio $f_i'(\theta_i)/f_i(\theta_i)$ to analyze the log–concavity of the density functions, while the third row shows the ratio $f_i(\theta_i)/F_i(\theta_i)$ to analyze the log–concavity of the corresponding distribution functions. Finally, the bottom row describes the behavior of the hazard rate, $r_i(\theta_i)$.

Figure 1 shows the convolution of a uniform distribution defined on the unit interval, with a beta distribution with parameters $p = 0.4$ and $q = 0.5$, also defined on the unit interval. The third column represents the distribution of their convolution defined on the $[0, 2]$ interval.[1] As it is well known, the uniform is a log–concave distribution with increasing hazard rate. Thus, both the ratio $f_1'(\theta_1)/f_1(\theta_1)$ and $f_1(\theta_1)/F_1(\theta_1)$ are non–increasing functions to be consistent with log–concavity. The beta distribution –defined on a bounded support as required by BMR–, may or may not be log–concave depending on the values of the indexing parameters p and q. If these parameters are sufficiently small as in the present case, the ratio $f_2'(\theta_2)/f_2(\theta_2)$ becomes increasing (log–convex density),

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[1] Pham and Turkkam (1994) study this type of convolution. A general reference is Johnson, Kotz, and Balakrisnahan (1995, §25.8). To obtain the convolution density function, the range of integration was divided in 10,000 intervals. For each one of these intervals, the convolution was computed using a 40–points Gauss–Legendre quadrature.

the distribution function also fails to be log–concave, and the hazard rate includes regions where it decreases. The consequence for the convolution distribution is that small values of p and q make the density of θ_2 sufficiently log–convex to turn the convolution density sufficiently peaked, so that there is a nontrivial region in the neighborhood of the mode of θ_0 where $f_0'(\theta_0)/f_0(\theta_0)$ becomes increasing, thus violating the log–concavity of $f_0(\theta)$. Although there might be log–concave distribution functions whose densities are not log–concave, this is not the case for the present example either because the ratio $f_0(\theta_0)/F_0(\theta_0)$ in the figure also rejects such hypothesis around the mode of θ_0 and in a neighborhood of its lower bound (not shown in the figure due to scale issues). Finally, the bottom figure of the third column clearly shows that the hazard rate is not increasing for the whole support of θ_0, thus contradicting the presumed log–concavity of the convolution survival function $1 - F_0(\theta_0)$.

Figure 2 removes the restriction of bounded supports for the distributions of θ_1 and θ_2. The first column presents a standard lognormal distribution, and the second column shows a standard normal distribution. The third column is the lognormal–normal convolution. This case has some appeal for economic modeling since the normally distributed variable may represent an error of measurement in the appraisal of each individual's own type which in addition, due to some economic reason, might be restricted to take only positive values in many models.[2] It is well known that the lognormal density is not log–concave and that it is characterized by a decreasing hazard rate as θ_1 increases [Sweet (1990)]. Consequently, and as I will show in the next section, the convolution density function cannot be log–concave. However, the lognormal distribution is log–concave [Bagnoli and Bergstrom (1989)], a property that is preserved under convolution since the normal distribution is also log–concave. The normal density is log–concave, and thus by Proposition 1 later in Section 2, its distribution and survival functions are also log–concave, and therefore IHR. Finally, as it can easily be confirmed looking at the third column of Figure 2, the convolution of a log–concave density as the normal, and an *arbitrary* distribution such as the lognormal, does not ensure that the convolution distribution is IHR in the case of unbounded supports either.

The proof of Theorem 1 of BMR ignores that the convolution is a commutative operation,[3] and thus, the distribution $f_2(\theta_2)$ (in my notation) plays no role; it just smears the effect of the endowment shock θ_2 on the support of θ_0 according to the rule of the distribution of asset values $f_1(\theta_1)$. Intuitively, the same result should be obtained by spreading the effect of θ_1 according to the distribution of θ_2. Therefore, the proof should be true if $f_1(\theta_1)$ is replaced by $f_2(\theta_2)$ and *vice versa*, but in such a case, $f_2(\theta_2)$ cannot

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[2] Hawkins (1991) studies in detail the basic properties of this convolution. For a general overview, see Johnson, Kotz, and Balakrisnahan (1995, §14.8). Romberg integration was used to compute the convolution density with a minimum of 10,000 divisions of the initial range $[-25, 75]$. Convergence required an error of integration smaller than 10^{-8} to define the final range of integration.

[3] The characteristic function of $F_0(\theta_0)$ is the product of the Fourier transforms of the distributions of its components [Hirschman and Widder (1955, §2.5); Karlin (1968, §7.1-7.3)]

be any *arbitrary* density defined on a bounded support, but rather a *log–concave* density function without a necessarily bounded support.

There are two alternatives to overcome this difficulty. One is to assume that both density functions are log–concave. As log–concavity is preserved under convolution, the distribution $F_0(\theta_0)$ has the desired properties and the rest of results of BMR will remain correct. Section 2 presents these preservation results taking the log–concavity of the density functions as starting point of the analysis. However, what BMR need, as well as in many other screening problems, is that the distribution $F_0(\theta_0)$ is IHR to ensure the existence of a separating Perfect Bayesian–Nash Equilibria. Thus, Section 3 only assumes that the distributions $F_1(\theta_1)$ and $F_2(\theta_2)$ are IHR to show that the convolution distribution $F_0(\theta_0)$ is IHR. The approach of Section 3 is less restrictive because it does not exclude those IHR distributions whose densities are not log–concave and that will be excluded under the approach of BMR or that of Section 2. Finally, Section 4 concludes.

2 Preservation of Log–Concavity

This section proves that the convolution of log–concave densities is also log–concave. It also proves that any random variable whose density functions is log–concave is also characterized by a log–concave distribution and survival function, and therefore is IHR. I start by presenting the minimal mathematical tools needed to prove these preservation results.

ASSUMPTION 1: The random variable θ_i, $i = 1, 2$, has a continuously differentiable probability density function $f_i(\theta_i) \geq 0$ on $\Theta_i = [\underline{\theta}_i, \overline{\theta}_i] \subseteq \Re$, such that the cumulative distribution function given by:

$$F_i(\theta_i) = \int_{\underline{\theta}_i}^{\theta} f_i(z)dz, \tag{3}$$

is absolutely continuous.

Log–concavity is a smoothness property common to many distributions. The following is a formal definition for continuously differentiable probability density functions.

DEFINITION 1: A probability distribution function $F_i(\theta_i)$ is log–concave if:

$$\frac{\partial^2 \log[f_i(\theta_i)]}{\partial \theta_i^2} = \frac{\partial}{\partial \theta_i}\left[\frac{f_i'(\theta)}{f_i(\theta)}\right] \leq 0 \quad \text{on} \quad \Theta_i. \tag{4}$$

The IHR property keeps a close relation with the log–concavity of $f_i(\theta_i)$. Actually, it is equivalent to the log–concavity of the corresponding survival function $\overline{F}_i(\theta_i) = 1 - F_i(\theta_i)$. The IHR property is defined as follows.

DEFINITION 2: If a univariate random variable θ_i has density $f_i(\theta_i)$ and distribution function $F_i(\theta_i)$, then the ratio:

$$r_i(\theta_i) = \frac{f_i(\theta_i)}{1 - F_i(\theta_i)} \qquad \text{on} \quad \{\theta_i \in \Theta_i : F_i(\theta_i) < 1\}, \tag{5}$$

is called the hazard rate of either θ_i or $F_i(\theta_i)$. A univariate random variable θ_i or its cumulative distribution function $F_i(\theta_i)$ are said to be increasing hazard rate if $r_i'(\theta_i) \geq 0$ on $\{\theta_i \in \Theta_i : F_i(\theta_i) < 1\}$.

In order to prove that log–concavity is preserved under convolution, I need to introduce the set of *Pólya frequency functions*. The major practical significance of *Pólya frequency functions* is that their smoothness properties are preserved under convolution, and more importantly, that a class of *Pólya frequency functions* coincides with the set of log–concave functions.

DEFINITION 3: A function $g(z)$ is a *Pólya frequency function* of order n (PF_n) if $\forall x_1 < x_2 < \cdots < x_m$, $x_i \in X \subseteq \Re$; and $\forall y_1 < y_2 < \cdots < y_m$, $y_i \in Y \subseteq \Re$; and all $1 \leq n \leq m$:

$$\begin{vmatrix} g(x_1 - y_1) & g(x_1 - y_2) & \cdots & g(x_1 - y_n) \\ g(x_2 - y_1) & g(x_2 - y_2) & \cdots & g(x_2 - y_n) \\ \vdots & \vdots & \ddots & \vdots \\ g(x_n - y_1) & g(x_n - y_2) & \cdots & g(x_n - y_n) \end{vmatrix} \geq 0. \tag{6}$$

The following lemma is the basis of the preservation results.

LEMMA 1: *Let $f_1(\theta_1)$ and $f_2(\theta_2)$ be PF_n, and θ_1 and θ_2 be stochastically independent, then the convolution:*

$$f_0(\theta_0) = \int_{\Theta_2} f_1(\theta_0 - \theta_2) f_2(\theta_2) d\theta_2 = \int_{\Theta_1} f_1(\theta_1) f_2(\theta_0 - \theta_1) d\theta_1, \tag{7}$$

is also PF_n.

PROOF: Without loss of generality, let $n = 2$. By definition of PF_2, the convolution $f_0(x - y)$ has to be such that $\forall x_1, x_2 \in X \subseteq \Re$ and $\forall y_1, y_2 \in Y \subseteq \Re$, such that $x_1 < x_2$ and $y_1 < y_2$, the following condition holds:

$$\begin{vmatrix} f_0(x_1 - y_1) & f_0(x_1 - y_2) \\ f_0(x_2 - y_1) & f_0(x_2 - y_2) \end{vmatrix} = \begin{vmatrix} \int f_1(x_1 - z) f_2(z - y_1) dz & \int f_1(x_1 - z) f_2(z - y_2) dz \\ \int f_1(x_2 - z) f_2(z - y_1) dz & \int f_1(x_2 - z) f_2(z - y_2) dz \end{vmatrix}$$

$$= \int \int_{z_1 < z_2} \begin{vmatrix} f_1(x_1 - z_1) & f_1(x_1 - z_2) \\ f_1(x_2 - z_1) & f_1(x_2 - z_2) \end{vmatrix} \cdot \begin{vmatrix} f_2(z_1 - y_1) & f_2(z_2 - y_1) \\ f_2(z_1 - y_2) & f_2(z_2 - y_2) \end{vmatrix} dz_1 dz_2 \geq 0, \tag{8}$$

where the last inequality is the *Basic Composition Formula* that relates compositions of *totally positive functions*.[4] From here the proof is straightforward since the first determinant in the double integral is positive as $f_1(x-y)$ is PF_2 and the second determinant is also positive as $f_2(x-y)$ is PF_2. ∎

An immediate consequence of the application of the *Basic Composition Formula* is the following result that will be used later in this section.

COROLLARY 1: *If $f_1(\theta_1)$ is PF_m and $f_2(\theta_2)$ is PF_n, then $f_0(\theta_0)$, the probability density function defined by convolution (7), is $PF_{\min(m,n)}$.*

I now need to link *Pólya frequency functions* with log–concavity. The previous mathematical results of this section have shown that the smoothness properties of *Pólya frequency functions* of the same order are preserved under convolution. While reliability properties such as IHR keep a close relation with the log–concavity of the probability density functions, the preservation of such smoothness condition is easily ensured if we focus on the family of *Pólya frequency functions*. The remaining results of this section rely on the equivalence between log–concave and a class of *Pólya frequency functions*. The following Lemma establishes such equivalence.

LEMMA 2: *A continuously differentiable function $g(z)$ is PF_2 if and only if $g(z) > 0$ $\forall z \in \Re$ and $g(z)$ is log–concave on $\Re$.*

PROOF: Since $g(z) > 0$ $\forall z \in \Re$, it follows from Definition 1 that a continuously differentiable function $g(z)$ is log–concave if and only if it is monotone decreasing in $\Re$. Next, without loss of generality, assume $x_1 < x_2$ and $0 = y_1 < y_2 = \Delta$. Then, from the definition of PF_2 in equation (6) the following inequalities hold:

$$\begin{vmatrix} g(x_1) & g(x_1 - \Delta) \\ g(x_2) & g(x_2 - \Delta) \end{vmatrix} = \Delta \cdot \begin{vmatrix} \dfrac{g(x_1) - g(x_1 - \Delta)}{\Delta} & g(x_1 - \Delta) \\ \dfrac{g(x_2) - g(x_2 - \Delta)}{\Delta} & g(x_2 - \Delta) \end{vmatrix} \geq 0. \qquad (9a)$$

Since $\Delta > 0$, we can take limits in the latter determinant to obtain:

$$\lim_{\Delta \to 0} \begin{vmatrix} \dfrac{g(x_1) - g(x_1 - \Delta)}{\Delta} & g(x_1 - \Delta) \\ \dfrac{g(x_2) - g(x_2 - \Delta)}{\Delta} & g(x_2 - \Delta) \end{vmatrix} = \begin{vmatrix} g'(x_1) & g(x_1) \\ g'(x_2) & g(x_2) \end{vmatrix} \geq 0, \qquad (9b)$$

leading to:

$$\frac{g'(x_1)}{g(x_1)} \geq \frac{g'(x_2)}{g(x_2)}, \qquad (9c)$$

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[4] The *Basic Composition Formula* is the continuous version of the Binet–Cauchy formula that expresses any minor of order k of the product of two rectangular matrices as the product of all possible minors of order k [Gantmacher (1958, §1.1)]. The proof of this intermediate result is sketched in Karlin (1968, §1.2).

which, given $g(z) > 0$, proves that $\forall z \in \Re$, $g'(z)/g(z)$ is monotone decreasing in $\Re$, and therefore log–concave. ∎

I can prove now the main results of this section. By imposing the log–concavity assumption on $f_1(\theta_1)$ and $f_2(\theta_2)$, I identify a wide class of distributions with nice properties for economic modeling, and further ensure that $f_0(\theta_0)$ also share those properties. These results are summarized in the following Proposition and Corollary.

PROPOSITION 1: *If the probability density function $f_i(\theta_i)$ is continuously differentiable and log–concave, all the following properties are equivalent:*
   (a) $F_i(\theta_i)$ is log–concave,
   (b) $\overline{F}_i(\theta_i) = 1 - F_i(\theta_i)$ is log–concave,
   (c) $F_i(\theta_i)$ is IHR in θ_i on $\{\theta_i \in \Theta_i : F_i(\theta_i) < 1\}$.

PROOF: In order to prove parts (a) and (b) let first study the properties of the function $\delta : \Re \to \{0, 1\}$ defined as follows:

$$\delta(x - y) = \begin{cases} 0 & \text{if} \quad x < y \\ 1 & \text{otherwise} \end{cases} \tag{10}$$

It is straightforward to show that $\delta(x - y)$ is PF_2 by direct application of Definition 3. It follows that $\hat{\delta}(x - y) = 1 - \delta(x - y)$ is also PF_2. By Lemma 1, $\hat{\gamma}(\theta_i)$, the convolution of $\hat{\delta}(x - \theta_i)$ and $f_i(\theta_i)$ is PF_2. Hence:

$$\hat{\gamma}(\theta_i) = \int_{\Re} \hat{\delta}(x - \theta_i) f_i(\theta_i) d\theta_i = \int_{-\infty}^{x} f_i(\theta_i) d\theta_i = F_i(\theta_i = x), \tag{11}$$

because $\hat{\delta}(x - \theta_i) = 1$ only if $x < \theta_i$, and therefore the cumulative distribution function $F_i(\theta_i)$ is PF_2. Similarly, $\gamma(\theta_i)$ the convolution of $\delta(x - \theta_i)$ and $f_i(\theta_i)$ is also PF_2, which in this case implies that:

$$\gamma(\theta_i) = \int_{\Re} \delta(x - \theta_i) f_i(\theta_i) d\theta_i = \int_{x}^{\infty} f_i(\theta_i) d\theta_i = \overline{F}_i(\theta_i = x), \tag{12}$$

because $\delta(x - \theta_i) = 1$ only if $x \geq \theta_i$, and the survival function $1 - F_i(\theta_i)$ is also PF_2. Finally, to prove part (c), note that by Definition 2, it follows that the hazard rate is $r_i(\theta_i) = -\overline{F}'_i(\theta_i)/\overline{F}_i(\theta_i)$ on $\{\theta_i \in \Theta_i : F_i(\theta_i) < 1\}$, which has to be increasing in θ_i because by part (b) of this Proposition, $\overline{F}_i(\theta_i)$ is log–concave, and according to Definition 1, this implies that the quotient $\overline{F}'_i(\theta_i)/\overline{F}_i(\theta_i)$ is decreasing in θ_i. ∎

The following Corollary shows that all the above properties are preserved under convolution, and thus, assuming that the density functions of each type component is log–concave suffices for all distributions involved to be well behaved.

COROLLARY 2: *If the probability density functions $f_i(\theta_i)$, $i = 1, 2$, are continuously differentiable and log–concave, and θ_1 and θ_2 are stochastically independent, then:*

(a) *$f_0(\theta_0)$ is continuously differentiable and log–concave,*
(b) *$F_0(\theta_0)$ is log–concave,*
(c) *$\overline{F}_0(\theta_i) = 1 - F_0(\theta_i)$ is log–concave,*
(d) *$F_0(\theta_0)$ is IHR in θ_0 on $\{\theta_0 \in \Theta_0 : F_0(\theta_0) < 1\}$.*

PROOF: By Lemma 2, $f_1(\theta_1)$ and $f_1(\theta_2)$ are both PF_2. Thus, Lemma 1 ensures that $f_0(\theta_0)$ is also PF_2. Part (a) results from applying Lemma 2 again to the convolution density function $f_0(\theta_0)$. Since the premises of Proposition 1 are now fulfilled by $f_0(\theta_0)$, parts (b)–(d) follow straightforwardly from its application. ∎

These results can be used to comment Theorem 1 of BMR more rigourously. Any single dimensional density function is, by Definition 3, at least PF_1. This is the case of all convoluting distributions of Figures 1 and 2. However, by Lemma 2, only log–concave densities are PF_2. Thus, the uniform in Figure 1 and the normal density in Figure 2 are PF_2. As shown by Corollary 1, the convolution of *Pólya frequency functions* of different order is also a *Pólya frequency function* of order equal to the lower order of the convoluting distributions. Therefore, the convolutions of Figures 1 and 2 are necessarily PF_1 which, while still well defined as densities, lack the log–concavity property, a sufficient condition to prove that the convolution distribution and survival functions are both log–concave.

3   Preservation of the IHR Property

It could still be argued that Theorem 1 of BMR does not make any inference about the log–concavity of the convolution density function but only about the distribution and survival function of the convolution. However, both examples in Figure 1 and 2 show that there are regions in Θ_0 where log–concavity of $F_0(\theta_0)$ and/or the IHR property fails to hold.

In many screening problems, as in BMR or in nonlinear pricing, the critical assumption to ensure the existence of a separating equilibria is the IHR property of the distribution of types instead of the more restrictive assumption of log–concavity of the corresponding density functions. The results of Section 2 imply that if all density functions of type components are log–concave, then the screening problems could also be solved when demands are stochastic or have more than one source of asymmetric information, but only at the cost of reducing the set of distributions that could be used for modeling these problems. Taking the approach of Section 2, we will not be able to use IHR distribution functions whose density functions are not log–concave.

Fortunately, this is not a additional restriction of models with multiple type components because, as Proposition 2 shows, IHR is preserved under convolution regardless of the log–concavity of the respective density functions.[5]

PROPOSITION 2: *If $F_1(\theta_1)$ and $F_2(\theta_2)$ are IHR, then their convolution $F_0(\theta_0)$ defined in equation (2) is also IHR.*

PROOF: Since parts (b) and (c) of Proposition 1 are equivalent, I only have to prove that the survival function of the convolution distribution is log–concave, *i.e.*, for $x_1 < x_2$ and $y_1 < y_2$:

$$
\begin{vmatrix}
1 - F_0(x_1 - y_1) & 1 - F_0(x_1 - y_2) \\
1 - F_0(x_2 - y_1) & 1 - F_0(x_2 - y_2)
\end{vmatrix} =
$$

$$
\begin{vmatrix}
\int [1 - F_1(x_1 - z)] f_2(z - y_1) dz & \int [1 - F_1(x_1 - z)] f_2(z - y_2) dz \\
\int [1 - F_1(x_2 - z)] f_2(z - y_1) dz & \int [1 - F_1(x_2 - z)] f_2(z - y_2) dz
\end{vmatrix} =
$$

$$
\begin{vmatrix}
\int [1 - F_1(x_1 - z)] f_2(z - y_1) dz & \int f_1(x_1 - z)[1 - F_2(z - y_2)] dz \\
\int [1 - F_1(x_2 - z)] f_2(z - y_1) dz & \int f_1(x_2 - z)[1 - F_2(z - y_2)] dz
\end{vmatrix} =
$$

$$
\int\limits_{z_1 < z_2} \int
\begin{vmatrix}
1 - F_1(x_1 - z_1) & f_1(x_1 - z_2) \\
1 - F_1(x_2 - z_1) & f_1(x_2 - z_2)
\end{vmatrix} \cdot
\begin{vmatrix}
f_2(z_1 - y_1) & 1 - F_2(z_2 - y_1) \\
f_2(z_1 - y_2) & 1 - F_2(z_2 - y_2)
\end{vmatrix} dz_1 dz_2 \geq 0. \quad (13)
$$

The second determinant just states the survival function $1 - F_0(\cdot)$ in terms of the distributions $F_1(\cdot)$ and $F_2(\cdot)$. The third determinant integrates the expressions in the second column of the second determinant by parts using the convolution identity:

$$
\int F_1(x - z) f_2(z - y) dz = \int f_1(x - z) F_2(z - y) dz, \quad (14)
$$

while the double integral of the product of determinants in (13) is again the *Basic Composition Formula*. For the last expression of equation (13) to be positive, and thus to ensure that the distribution $F_0(\cdot)$ is IHR, both determinants must have the same sign. Assuming without loss of generality that $0 = z_1 < z_2 = \Delta$, the IHR property of the convoluting distributions imply that these determinants are positive. To prove this statement observe that:

$$
[1 - F_1(x_1)] f_1(x_2 - \Delta) - [1 - F_1(x_2)] f_1(x_1 - \Delta) \geq 0, \quad (15)
$$

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[5] Observe that Proposition 2 does not exclude the possibility that the convolution of distributions is IHR even when at least one of the convoluting distributions is not IHR. See Karlin (1968, §3.8.C) for an example.

implies:

$$\frac{f_1(x_2 - \Delta)}{1 - F_1(x_2 - \Delta)} \cdot \frac{1 - F_1(x_2 - \Delta)}{1 - F_1(x_2)} \geq \frac{f_1(x_1 - \Delta)}{1 - F_1(x_1 - \Delta)} \cdot \frac{1 - F_1(x_1 - \Delta)}{1 - F_1(x_1)}. \tag{16}$$

But since $\Delta > 0$ and $x_1 < x_2$, lets compare the first ratios on each side of this inequality:

$$\frac{f_1(x_2 - \Delta)}{1 - F_1(x_2 - \Delta)} \geq \frac{f_1(x_1 - \Delta)}{1 - F_1(x_1 - \Delta)}, \tag{17}$$

which is just the hypothesis that $F_1(\cdot)$ is IHR. Similarly, comparing the other ratios of inequality (16):

$$\frac{1 - F_1(x_2 - \Delta)}{1 - F_1(x_2)} \geq \frac{1 - F_1(x_1 - \Delta)}{1 - F_1(x_1)}, \tag{18}$$

which is equivalent to:

$$\begin{vmatrix} 1 - F_1(x_1) & 1 - F_1(x_1 - \Delta) \\ 1 - F_1(x_2) & 1 - F_1(x_2 - \Delta) \end{vmatrix} \geq 0, \tag{19}$$

that is the condition for the survival function $1 - F_1(\cdot)$ to be log–concave, which holds by assumption as $F_1(\cdot)$ is IHR. A similar argument proves that if $F_2(\cdot)$ is IHR, the second determinant in the last inequality of (13) must also be positive. Thus, $F_0(\cdot)$ is IHR. ∎

4   Concluding Remarks

Models of multidimensional screening can sometimes just define a single aggregate type that embodies the effects of all different sources of asymmetric information. This opens the possibility of screening agents using the distributional information on either the aggregate or its type components. The present paper has shown that such procedure is feasible and that separating Perfect Bayesian–Nash equilibria can still be characterized in both frameworks for a wide set of distribution functions.

There are two main results in this paper. First, log–concavity is preserved under convolution. This implies that that if all density functions of type components are log–concave, then the density function of the aggregate type is also log–concave. It then follows that the corresponding distribution an survival functions are also log–concave, and therefore IHR. Second, I have shown that IHR is preserved under convolution, which ensures that the set of distributions that might be used to model screening problems is not restricted to those whose density functions are log–concave. Miravete (2001a) makes use of Proposition 2 in the framework of optional nonlinear pricing and test empirically the implications of such model using data from a tariff experiment in local telephone service.
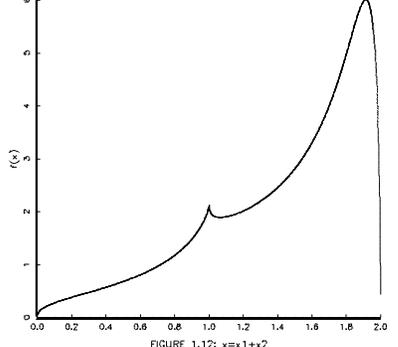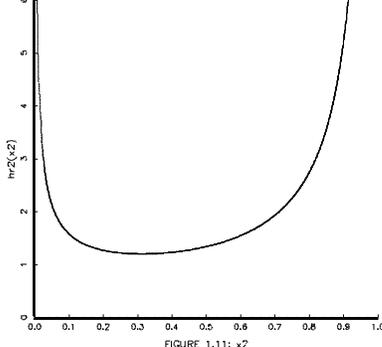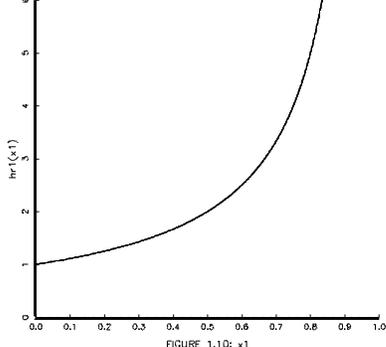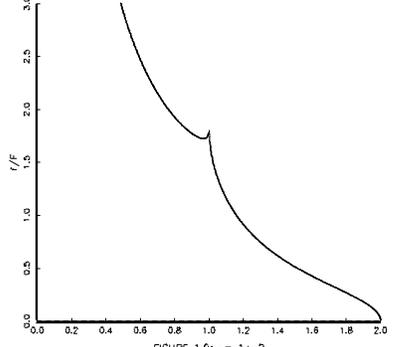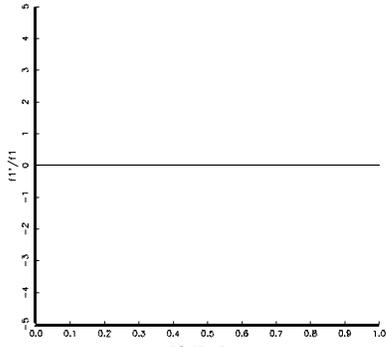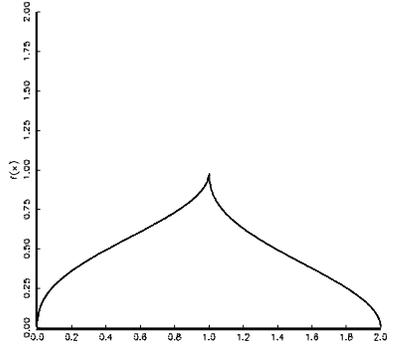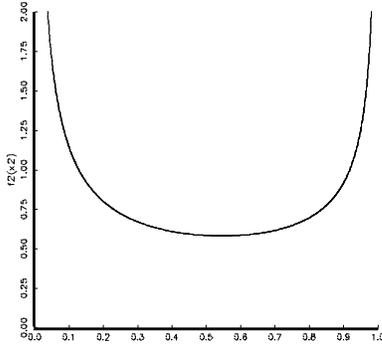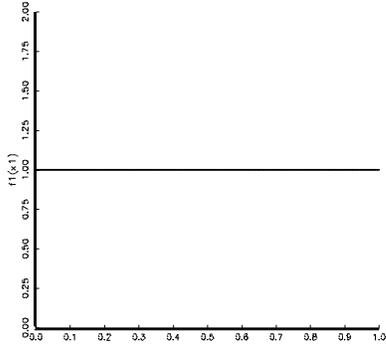
Pólya frequency functions and preservation of log–concavity under convolution are just particular cases of the preservation of regularity conditions of *totally positive functions*

under composition. Miravete (2001b) explores whether the commonly observed preference for bundling solutions when types are multidimensional can be explained by the properties of convolution distributions relative to the original type component distributions. In that work it is shown that preservation results of *totally positive functions* can also be applied to models of voting as well as to multidimensional moral hazard problems.

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Figure 1. Uniform–Beta Convolution

FIGURE 1.1: x1

FIGURE 1.2: x2

FIGURE 1.3: x=x1+x2

FIGURE 1.7: x1

FIGURE 1.8: x2

FIGURE 1.9: x=x1+x2

FIGURE 1.7: x1

FIGURE 1.8: x2

FIGURE 1.9: x=x1+x2

FIGURE 1.10: x1

FIGURE 1.11: x2

FIGURE 1.12: x=x1+x2
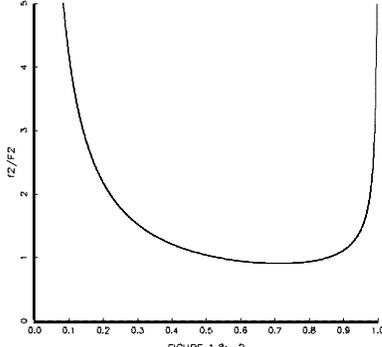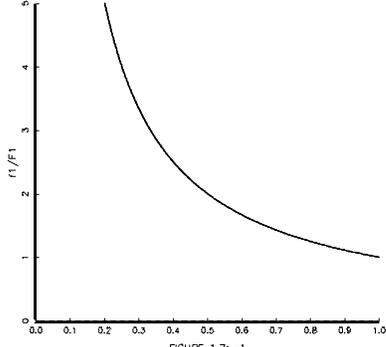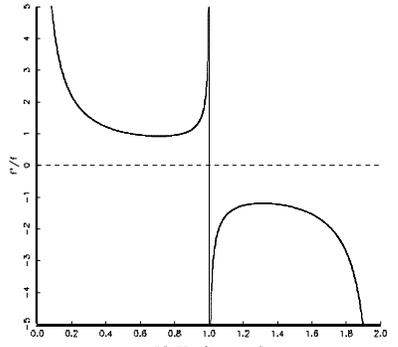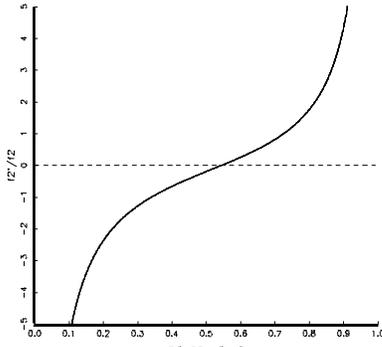
Figure 2. Lognormal–Normal Convolution

FIGURE 1.1: x1

FIGURE 1.2: x2

FIGURE 1.3: x=x1+x2

FIGURE 1.7: x1

FIGURE 1.8: x2

FIGURE 1.9: x=x1+x2

FIGURE 1.7: x1

FIGURE 1.8: x2

FIGURE 1.9: x=x1+x2

FIGURE 1.10: x1

FIGURE 1.11: x2

FIGURE 1.12: x=x1+x2