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The Maximum Efficient Equilibrium Payoff in the Repeated Prisoners' Dilemma*

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Abstract

We describe the maximum efficient subgame perfect equilibrium payoff for a player in the repeated Prisoners' Dilemma, as a function of the discount factor. For discount factors above a critical level, every efficient, feasible, individually rational payoff profile can be sustained. For an open and dense subset of discount factors below the critical value, the maximum efficient payoff is not an equilibrium payoff. When a player cannot achieve this payoff, the unique equilibrium outcome achieving the best efficient equilibrium payoff for a player is eventually cyclic. There is an uncountable number of discount factors below the critical level such that the maximum efficient payoff is an equilibrium payoff.

1. Introduction

While the discounted repeated Prisoners' dilemma is one of the most intensively studied games, little is known about the set of subgame perfect equilibrium payoffs for a wide range of discount factors. It is known that for low values of the discount factor, only the minmax payoff vector is an equilibrium payoff vector, while the

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rationality constraint. Denote this value for player i by v_i^*.

For other discount factors, not all individually rational and efficient payoffs are equilibrium payoffs. For an open and dense subset of discount factors below the critical value,[3] v_i^* is not an equilibrium payoff for player i. On the other hand, the set of discount factors below the critical value for which the maximum equilibrium payoff for player i is v_i^* is uncountable. Thus, the maximum equilibrium payoff does not exhibit monotonicity with respect to the discount factor.

If the discount factor is such that v_i^* is not an equilibrium payoff for player i, the (unique) best efficient equilibrium outcome for player i is eventually cyclic: after some finite history, play follows a cycle. We also show that the best efficient equilibrium payoff is sometimes different from the maximum of *all* equilibrium payoffs for a player (the remark at the end of Section 3). On the other hand, when v_i^* is an equilibrium payoff for player i, various types of outcomes are consistent with being the best equilibrium, among which are acyclic outcomes.

2. Preliminary analysis

We study the Prisoners' Dilemma $g : \{C, D\}^2 \to \Re^2$, where g is described in the following bimatrix:

		Player 2 C	Player 2 D
Player 1	C	$1,1$	$-1,2$
	D	$2,-1$	$0,0$

While we have chosen to work with a particular version of the Prisoners' Dilemma for clarity, our results hold for any Prisoners' Dilemma.[4] The set of individually rational and feasible payoffs is denoted V^*. Our interest lies in equilibrium payoffs on the Pareto boundary of this set. Without loss of generality, we restrict attention to the boundary $B = \left\{(v_1, v_2) : v_1 = \frac{3}{2} - \frac{v_2}{2},\ v_2 \in [0,1]\right\}$ (see Figure 1).

We base our analysis on self-generation (Abreu, Pearce, and Stacchetti [2]). A pair (α, w), where α is a (possibly mixed) action profile and $w : A \to V^*$ is a specification of continuation payoffs, is *admissible* if α is a Nash equilibrium of the game with payoffs $(1-\delta)\, g(a) + \delta w(a)$. A payoff vector v is *decomposable* with respect to an action profile α and continuation values w if the pair (α, w) is admissible and has value v.

[3] We do not know if the set of such discount factors has full measure.

[4] Since we focus on the maximum efficient payoff from a player's (say 1) point of view, the other player always plays C. As a consequence the only relevant payoffs are those from CC and DC.

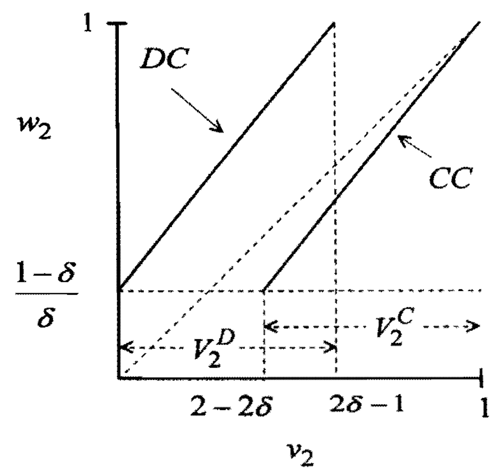

Figure 2: Self-generating sets for $\delta \geq \frac{3}{4}$. This is drawn for $\delta = \frac{4}{5}$. Any v_2 can be decomposed into a current action profile and continuation value w_2.

Thus, $V_2^C = [2 - 2\delta, 1]$.

Denote by V_2^D the set of payoffs for player 2 that can be decomposed using DC and a payoff $w_2 \in [(1-\delta)/\delta,\ 1]$:

$$v_2 \in V_2^D \iff \exists w_2 \in \ [(1-\delta)/\delta,\ 1] \quad \text{s.t. } v_2 = (1-\delta)\, g_2(DC) + \delta w_2 = \delta w_2 - (1-\delta)\,. \tag{2}$$

Thus, $V_2^D = [0, 2\delta - 1]$.

Note that for $\delta < \frac{1}{2}$, V_2^C and V_2^D are both empty and so any action profile in which player 2 plays C is not admissible. In fact, it is easy to show that for $\delta < \frac{1}{2}$, the only equilibrium payoff is $(0, 0)$. Moreover, for $\delta \geq 1/2$, grim trigger is an equilibrium. Thus, the best symmetric equilibrium payoff is $(0, 0)$ for $\delta < 1/2$ and $(1, 1)$ for $\delta \geq 1/2$.

If $V_2^C \cup V_2^D = [0, 1]$ (which is implied by $\delta \geq \frac{3}{4}$), on the other hand, every payoff on the segment $\left\{(v_1, v_2) : v_1 = \frac{3}{2} - \frac{v_2}{2},\ v_2 \in [0, 1]\right\}$ can be supported as an equilibrium payoff in the first period. This is illustrated in Figure 2. These last two observations imply our first result. For $\delta \geq 1/2$, let $\bar{v}_1(\delta)$ be the maximum of player 1's payoff in any equilibrium with payoffs on B, given discount factor δ.

Lemma 1. *For $\delta < \frac{1}{2}$, $(0, 0)$ is the only subgame perfect equilibrium payoff. For $\delta \geq \frac{3}{4}$, $\bar{v}_1(\delta) = \frac{3}{2}$.*

Proof. Let $\{v_2^t\}_{t=0}^{\infty}$ and π be generated by v. Iteratively applying (5) yields

$$v_2^t = (1-\delta)\sum_{\tau=t}^{T-1} \delta^{\tau-t} g_2(\pi_\tau) + \delta^{T-t} v_2^T \tag{6}$$

for any $t \geq 0$ and any $T > t$. Since the sequence $\{v_2^t\}_{t=1}^{\infty}$ generated by v is bounded, taking $T \to \infty$ in (6) gives

$$v_2^t = h_2^t(\pi; \delta) \tag{7}$$

for any $t \geq 0$.

Suppose the sequence $\{v_2^t\}_{t=0}^{\infty}$ generated by $v \in [0,1]$ under δ satisfies $v_2^t \geq \frac{1-\delta}{\delta}$ for all $t \geq 1$. Consider the strategy profile in which π is played on the path and any deviation is punished by the Nash reversion. Then $h_2^t(\pi, \delta) = v_2^t \geq \frac{1-\delta}{\delta}$ for all $t \geq 1$ ensures that player 2 has no incentive to deviate from the path. Player 1 also has no profitable deviation because her continuation payoff from any period is greater than player 2's continuation payoff, which proves that π is an equilibrium outcome and so $\left(\frac{3-v}{2}, v\right)$ is an equilibrium payoff. ■

Next we show that the path generated by $v \in [0,1]$ is the unique equilibrium path that achieves the equilibrium payoff $\left(\frac{3-v}{2}, v\right)$ when $\delta < \frac{3}{4}$.

Lemma 3. *Fix $\delta < \frac{3}{4}$ and let π be the path generated by $v \in [0,1]$. If a pure outcome path $\mu \neq \pi$ achieves $\left(\frac{3-v}{2}, v\right)$, then $h_2^{T+1}(\mu; \delta) < \frac{1-\delta}{\delta}$ where T is the smallest $t \geq 0$ such that $\mu_t \neq \pi_t$.*

Proof. For any $t \geq 1$, we have

$$v = h_2(\mu; \delta) = (1-\delta)\sum_{\tau=0}^{t-1} \delta^{\tau} g_2(\mu_\tau) + \delta^t h_2^t(\mu; \delta) \tag{8}$$

By the definition of T, if $T \geq 1$, (6) and (8) imply that $h_2^t(\mu; \delta) = v_2^t$ for any $t \leq T$. If $T = 0$, we trivially have $h_2^T(\mu; \delta) = v_2^T = v$.

If $\pi_T = CC$, then $2\delta - 1 < v_2^T = h_2^T(\mu; \delta)$. Since $\mu_T = DC$,

$$h_2^{T+1}(\mu; \delta) = \frac{1}{\delta} h_2^T(\mu; \delta) + \frac{1-\delta}{\delta} > 1. \tag{9}$$

However, since μ achieves $\left(\frac{3-v}{2}, v\right)$ and therefore consists of CC and DC only, we must have $h_2^{T+1}(\mu; \delta) \leq 1$, a contradiction.

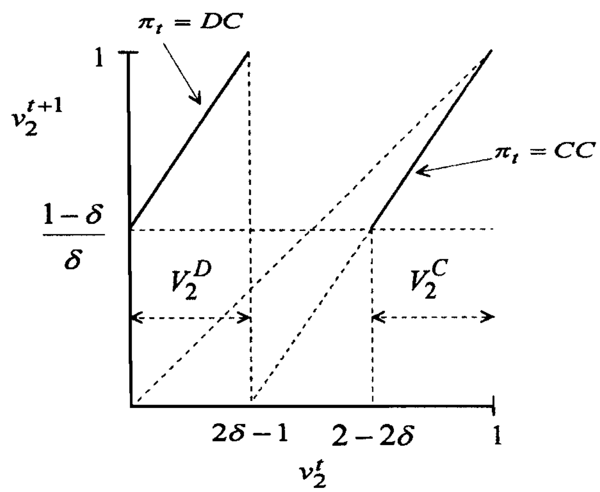

Figure 3: The dynamic for $\delta \in \left(\frac{1}{2}, \frac{1}{\sqrt{2}}\right)$. This is drawn for $\delta = \frac{2}{3}$.

$(2\delta - 1, 1)$ is an efficient equilibrium payoff for player 2, $\left(\frac{3-v}{2}, v\right)$ is again not an equilibrium outcome. Therefore, for all $\delta \in (\frac{1}{2}, \frac{1}{\sqrt{2}})$, $\bar{v}_1(\delta) = 2 - \delta$.

It is also straightforward to show that $\bar{v}_1(\frac{1}{\sqrt{2}}) = \frac{3}{2}$, because the path generated by 0 is $(0, \sqrt{2} - 1, 1, 1, \dots)$, with associated outcome path DC, DC, CC^∞. Therefore $(\frac{3}{2}, 0)$ is an equilibrium payoff when $\delta = 1/\sqrt{2}$ (see Figure 4). Moreover, for $\delta = 1/\sqrt{2}$, there are a countable number of equilibrium payoff vectors in B: any path of the form $(DC)^x$, $(CC)^t$, DC, $(CC)^\infty$, where $x \in \{0, 1\}$ and t is a nonnegative integer, is an equilibrium outcome path. The path DC, CC, CC, DC, $(CC)^\infty$ is illustrated in Figure 4.

We have thus proved the following proposition.

Proposition 2. *Suppose $\delta \geq \frac{1}{2}$. For $\delta \notin (\frac{1}{\sqrt{2}}, \frac{3}{4})$, the maximum efficient equilibrium payoff for player i is*

$$\bar{v}_i(\delta) = \begin{cases} 2 - \delta, & \text{if } \delta \in [\frac{1}{2}, \frac{1}{\sqrt{2}}), \text{ and} \\ \frac{3}{2}, & \text{if } \delta = \frac{1}{\sqrt{2}} \text{ or } \delta \geq \frac{3}{4}. \end{cases}$$

Things are more complicated when $\delta \in (\frac{1}{\sqrt{2}}, \frac{3}{4})$, because there are too many equilibria to describe explicitly. This multiplicity is due to the ability of the

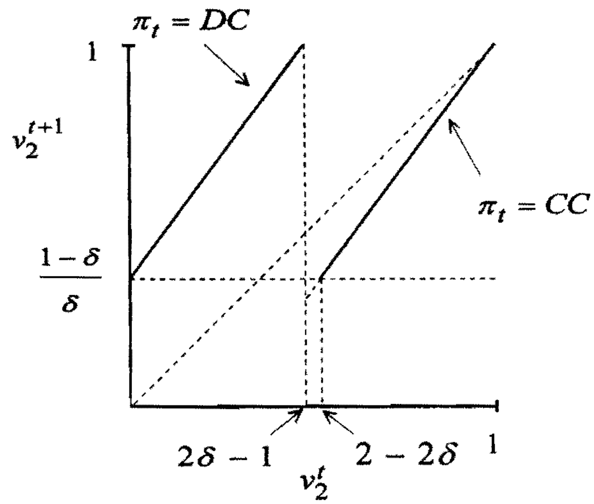

Figure 5: The dynamic for $\delta \in \left(\frac{1}{\sqrt{2}}, \frac{3}{4}\right)$. This is drawn for $\delta = 0.74$.

where v_1^a is player 1's continuation value after action a. Moreover, on B, we have $v_2^a = 3 - 2v_1^a$, so that

$$v_2^D = v_2^C + \frac{2(1-\delta)}{\delta} \geq \frac{3(1-\delta)}{\delta}.$$

For $\delta \geq \frac{3}{4}$, every payoff in B can be achieved in a pure strategy equilibrium, and so mixing is redundant. For $\delta < \frac{3}{4}$, the above inequality implies $v_2^D > 1$, which is impossible if the continuation values are to lie in B.

3. The Set of Nonwonderful Discount Factors

Now we consider $\delta \in (\frac{1}{\sqrt{2}}, \frac{3}{4})$. We do not attempt to derive the whole set of equilibrium payoff vectors in B explicitly. Rather, we describe the equilibrium in B that maximizes player 1's payoff for any nonwonderful δ. We also show that the set of nonwonderful δ is open.

We start with some preliminary results.

Lemma 4. *Suppose $\{v_2^t\}_{t=0}^{\infty}$ and $\{\pi_t\}_{t=0}^{\infty}$ are generated by 0 under some $\delta \in (\frac{1}{\sqrt{2}}, \frac{3}{4})$. Define $T(\delta) \equiv \max\{T : v_2^t \geq (1-\delta)/\delta,\ t = 1, \ldots, T\}$. Then, $T(\delta) \geq 3$ and the first four periods of the outcome path are given by DC, DC, CC, and CC. Moreover, for $1 \leq t \leq T(\delta)$, if $\pi_t = DC$ then $\pi_{t+1} = \pi_{t+2} = CC$.*

where the first inequality follows from $g_2(\mu_T) \leq 1$, and the second from $\delta > 1/\sqrt{2}$.

Let $\pi' = \{\pi'_t\}_{t=0}^{\infty}$ be the path generated by $w = -\frac{(1-\delta)}{\delta^{T+1}} \sum_{t=0}^{T} \delta^t g_2(\mu_t)$. Define the path ρ as

$$\rho_t = \begin{cases} \mu_t, & \text{if } t \leq T, \\ \pi'_{t-T-1}, & \text{if } t > T. \end{cases}$$

Note that, by construction, $h_2(\rho;\delta) = 0$, and that (11) and (12) imply $h_2^t(\rho;\delta) \geq \frac{1-\delta}{\delta}$ for all $t \in \{1, 2, \cdots, T\}$.

Suppose that δ is wonderful, and let π be the wonderful equilibrium outcome path. Then, by Lemma 3, π is generated by 0. Observe that the actions in the first T periods of π and ρ coincide. [If not, Lemma 3 implies that there exists $t \in \{1, 2, ..., T\}$ such that $h_2^t(\rho;\delta) < \frac{1-\delta}{\delta}$, which is a contradiction.] Hence, $h_2^T(\pi;\delta) = h_2^T(\rho;\delta)$, and from (12), $h_2^T(\pi;\delta) = h_2^T(\rho;\delta) > 2\delta - 1$. By (4), we then have $\pi_T = CC$. Moreover, $h_2^T(\rho;\delta) = (1-\delta)\, g_2(\mu_T) + \delta w < (1-\delta)(g_2(\mu_T) + 1)$ (since (13) implies $w < \frac{(1-\delta)}{\delta}$). The inequality $h_2^T(\rho;\delta) > 2\delta - 1$ then requires $\mu_T = CC$. But now π and ρ also agree in period T, and so $h_2^{T+1}(\pi;\delta) = h_2^{T+1}(\rho;\delta) = w < \frac{1-\delta}{\delta}$, contradicting the assertion that π is an equilibrium outcome path. Consequently, δ is not wonderful. ■

Lemma 6. *Suppose* $\{\hat{v}_2^t\}_{t=1}^{\infty}$ *and* $\hat{\pi}$ *are generated by* 0 *under* $\hat{\delta} \in [\frac{1}{\sqrt{2}}, \frac{3}{4})$.

1. *If*

$$f_k(\hat{\pi};\delta) = \sum_{\tau=0}^{k} \delta^\tau g_2(\hat{\pi}_\tau),$$

then $\partial f_k(\hat{\pi};\delta)/\partial\delta > 0$ *for all* $\delta \in (\frac{1}{\sqrt{2}}, \frac{3}{4}]$ *and any* k *such that* $2 \leq k \leq T(\hat{\delta})$.

2. *Suppose* $T(\hat{\delta}) = \infty$. *Then* $h_2(\hat{\pi};\delta) < 0$ *for all* $\delta \in (\frac{1}{\sqrt{2}}, \hat{\delta})$ *and* $h_2(\hat{\pi};\delta) > 0$ *for all* $\delta \in (\hat{\delta}, \frac{3}{4}]$.

Proof. Since

$$\frac{\partial f_k(\hat{\pi};\delta)}{\partial\delta} = \sum_{\tau=1}^{k} \tau\delta^{\tau-1} g_2(\hat{\pi}_\tau).$$

we just need to prove $\sum_{\tau=1}^{k} \tau\delta^{\tau-1} g_2(\hat{\pi}_\tau) > 0$. For $\ell \geq 1$, define $s_\ell = \sum_{\tau=\ell}^{k} \tau\delta^{\tau-1} g_2(\hat{\pi}_\tau)$, and recall from Lemma 4 that $\hat{\pi}_1 = DC$ and $\hat{\pi}_2 = CC$. For $k = 2$, $s_1 = -1 + 2\delta > 0$. For $k \geq 3$,

$$s_1 = -1 + 2\delta + s_3 > s_3, \tag{15}$$

We are now ready to state and prove the main result of this section: if no wonderful equilibrium exists, only cyclic behavior, namely, a path eventually ending in a cycle, forms the best efficient equilibrium outcome for a player. In other words, other behavior is only consistent with the best efficient equilibrium if it is wonderful. One virtue of the result is that we are able to provide the best efficient equilibrium explicitly and to describe the range of discount factors for which the same path continues to be the best efficient equilibrium.

Definition 2. *A path $\pi = \{\pi_t\}_{t=0}^{\infty}$ is* eventually n-cyclic, *where n is a positive integer, if*

1. *there exists $T \geq 1$ such that for all $s > T$ and $t = kn + s$ for some integer k, $\pi_t = \pi_s$, and*

2. *the above property does not hold for any $n' < n$.*

Proposition 3. *If $\delta_0 \in (\frac{1}{\sqrt{2}}, \frac{3}{4})$ is not wonderful, then the best efficient equilibrium outcome for player 1 under δ_0 is a path π^* that is eventually n-cyclic with $n \neq 1$. Moreover, there exists a half-open interval $[\delta_1, \delta_2)$ containing δ_0 such that*

1. *for any $\delta' \in [\delta_1, \delta_2)$, π^* is the best efficient equilibrium outcome for player 1, being wonderful if and only if $\delta' = \delta_1$; and*

2. *δ_2 is wonderful, and the corresponding wonderful equilibrium outcome is eventually 1-cyclic.*

Proof. Let π and $\{v_2^t\}_{t=0}^{\infty}$ be the path and the sequence generated by 0 under δ_0. Since δ_0 is not wonderful, $T(\delta_0) < \infty$. From Lemma 4, $T(\delta_0) \geq 3$. We write T for $T(\delta_0)$ in this proof.

Definition of δ_1 and δ_2:

As in the proof of Lemma 5, we have at $\delta = \delta_0$ the following three inequalities:

$$-\sum_{t=0}^{k} \delta^t g_2(\pi_t) \geq \delta^k \tag{16}$$

for any $k \in \{0, 1, 2, \ldots, T-2\}$,

$$-\sum_{t=0}^{T-1} \delta^t g_2(\pi_t) > \frac{\delta^T(2\delta - 1)}{1 - \delta} \tag{17}$$

where the first equality follows from $\delta = \frac{3}{4}$, the second from $\pi_T = CC$ (because $v_2^T > 2\delta - 1$, recall the first paragraph of the proof of Lemma 5), and the last inequality from (18). Thus, $\delta_2 \in (\delta_0, \frac{3}{4})$ is also well-defined. The above argument also shows that (17) holds for all $\delta \in (\delta_1, \delta_2)$.

No $\delta \in (\delta_1, \delta_2)$ is wonderful:

Recall that (17) and (18) hold for all $\delta \in (\delta_1, \delta_2)$. Thus, if (16) holds at all $k \in \{0, 1, 2, \cdots, T-2\}$ for any $\delta \in (\delta_1, \delta_2)$, Lemma 5 implies the desired result. Lemma 4 implies that (16) always holds for $k = 0$ and 1. Since Lemma 6 applies to the left hand side of (16) for any $k \in \{2, \ldots, T-2\}$, it suffices to show that (16) is true for all $k \in \{2, \ldots, T-2\}$ at δ_2 (note that the right hand side of (16) is always increasing).

Suppose, then, that (16) does not hold for some $k \in \{2, \ldots, T-2\}$ at δ_2. This implies the existence of $\delta_3 \in [\delta_0, \delta_2)$ such that (16) is true for all $k \in \{2, \ldots, T-2\}$ at δ_3, with an equality for some $\hat{k} \in \{2, \ldots, T-2\}$. Since (17) and (18) hold at δ_3, δ_3 is not wonderful by Lemma 5. Now consider the path $\hat{\rho}$ that starts with DC and then cycles through $\{\pi_t\}_{t=1}^{\hat{k}}$. We have

$$h_2(\hat{\rho}; \delta_3) = -(1 - \delta_3) + \frac{1 - \delta_3}{1 - \delta_3^{\hat{k}}} \sum_{\tau=1}^{\hat{k}} \delta_3^\tau g_2(\pi_\tau), \tag{21}$$

because $\hat{\rho}_0 = DC$. Since (16) holds at $\hat{k}$ with equality, (21) implies $h_2(\hat{\rho}; \delta_3) = 0$. Furthermore, since (16) holds at any $k \leq \hat{k}$, it follows that

$$h_2^k(\hat{\rho}; \delta_3) = -\frac{1 - \delta_3}{\delta_3^k} \sum_{\tau=0}^{k-1} \delta_3^\tau g_2(\pi_\tau) \geq \frac{1 - \delta_3}{\delta_3}$$

for any $k = 2, ..., \hat{k}$. Since for any $t > \hat{k}$, $h_2^t(\hat{\rho}; \delta_3) = h_2^k(\hat{\rho}; \delta_3)$ for some $k \leq \hat{k}$, $\hat{\rho}$ is a wonderful equilibrium path, a contradiction. Thus, (16) holds for all $k \in \{2, \cdots, T-2\}$ at δ_2, and therefore at any $\delta \in (\delta_1, \delta_2)$. Hence no discount factor $\delta \in (\delta_1, \delta_2)$ is wonderful.

Definition of π^*: $\pi_t^* = \pi_t$ if $t \leq T$, and for $t = kT + s$, for positive integers k and $1 \leq s \leq T$, $\pi_t^* = \pi_s$. Thus, π^* is an eventually T-cyclic path starting with DC and then cycling through $\{\pi_t\}_{t=1}^T$.

The path π^* is an equilibrium path for any $\delta \in [\delta_1, \delta_2]$, being wonderful at δ_1:

Since $\pi_0^* = DC$, we have

$$h_2(\pi^*; \delta) = -(1 - \delta) + \frac{1 - \delta}{1 - \delta^T} \sum_{\tau=1}^{T} \delta^\tau g_2(\pi_\tau).$$

In other words, the best equilibrium outcome has no "frills." Proposition 3 also shows what type of equilibrium dominates the original best equilibrium when it ceases to be best at δ_2. The equilibrium plays the same as the original one until the very last period of the first phase of the cycle, then switches to DC followed by CC forever. Therefore, the equilibrium is eventually 1-cyclic and, more importantly, wonderful.

Remark. We have so far considered the best efficient equilibrium for player 1, i.e., the equilibrium which gives the greatest payoff to player 1 among equilibrium payoffs on the Pareto frontier of the feasible payoff set. We should emphasize that the best efficient equilibrium payoff is sometimes different from the maximum of *all* equilibrium payoffs for player 1.

To illustrate this possibility, fix a nonwonderful discount factor δ_0 and consider the half-open interval $[\delta_1, \delta_2)$ presented in Proposition 3. Let ρ be the wonderful equilibrium path for δ_2, which is eventually 1-cyclic. This path is not an equilibrium path for $\delta < \delta_2$ but in a neighborhood of δ_2 (Lemma 6). Intuitively, the problem is that in period T, the path ρ requires DC, leading to $h_2(\rho; \delta) < 0$. In contrast, the best efficient equilibrium outcome path specifies $\pi_T = CC$. Now modify ρ by replacing CC in a distant future period with CD. The modified path, denoted ρ', results in a greater payoff for player 2. So, if we consider $\delta < \delta_2$ sufficiently close to δ_2, ρ' gives player 2 more than 0 and player 1 almost $\frac{3}{2}$. It is easy to see that ρ' is indeed an equilibrium path, which gives player 1 a greater payoff than the eventually cyclic efficient equilibrium path we consider in Proposition 3.

The above argument suggests that the full characterization of the best equilibrium for player 1 is significantly more complicated when we remove the restriction to efficient paths. Nonetheless, we conjecture that the optimality of cyclic behavior is a general phenomenon.

4. Denseness

One message of the analysis in the previous section is that the set of nonwonderful discount factors is open. The purpose of this section is to show that it is also a dense subset of $(\frac{1}{\sqrt{2}}, \frac{3}{4})$. With this result, our previous finding that the best efficient equilibrium behavior is eventually cyclic is shown to be pervasive in the discount factor. We should note here that we do not know the Lebesgue measure of the set of nonwonderful discount factors.[8]

[8] Recall that there are open and dense subsets of $[0, 1]$ of arbitrarily small measure, for example complements of generalized Cantor sets (see Royden [5, Problem 14.b, page 64]).

Choose an element of the neighborhood $\delta < \delta_T$ so that (12) holds at δ, and that

$$-\sum_{t=0}^{T'-1} \delta^t g_2(\pi_t^T) < 2\delta^{T'}. \tag{27}$$

This is possible because $2\delta_T - 1 < 2(1-\delta_T)$ and Lemma 6 applies. Lemma 6 also guarantees that 25 continues to hold. Consider a finite path $\mu = \{\mu_t\}_{t=1}^{T'}$ defined as: $\mu_t = \pi_t^T$ for any $t \neq T'$, and $\mu_{T'} = CC$. Then, (27) implies

$$-\sum_{t=0}^{T'} \delta^t g_2(\mu_t) < \delta^{T'}. \tag{28}$$

By Lemma 5, (25), (12) and (28) imply that δ is not wonderful. Since the neighborhood of δ' is arbitrarily chosen, the set of nonwonderful discount factors is dense. ■

5. The Set of Wonderful Discount Factors

So far, we have limited our attention to nonwonderful discount factors, and described the best equilibrium outcome paths for nonwonderful discount factors. Now we turn to the set of wonderful discount factors and the properties of corresponding wonderful equilibria. We start by classifying behavior consistent with wonderful equilibria.

Unlike the case of nonwonderful discount factors, where we can derive a certain type of behavior as the best equilibrium path, wonderful equilibria exhibit much more diversity. For example, a wonderful equilibrium may be eventually 1-cyclic, like the one we have seen at δ_2 of Proposition 3. Or it may be eventually n-cyclic like the one we have observed at δ_1 of Proposition 3. However, wonderful equilibrium paths need not be of the type considered there, that is, an eventually n-cyclic path with no frills. An *eventually n-cyclic path with a frill*, denoted by $\pi = (DC, \mu, \rho^\infty)$, where μ is a finite path, ρ is a different finite path, and ρ^∞ is the infinite repetition of ρ, could be a wonderful equilibrium.

To make things far more complicated, there is another type of wonderful equilibrium path, which never converges to any cycle. Consider the following path ρ:

$$\rho_t = \begin{cases} DC, & \text{if } t = 0, 1 \text{ or } t = 100^k \text{ for some integer } k, \\ CC, & \text{otherwise.} \end{cases}$$

path. Note that π^T assigns DC to periods 0 and 1, and any period written as $mT+1$ for a positive integer m.

Let $s = \{s_t\}_{t=1}^{\infty}$ be a sequence of natural numbers. Associated with s, define the set of natural numbers

$$Z(s) = \{m : m = \sum_{i=1}^{t} s_i \text{ for some } t\}.$$

We also define the path $\pi(s)$ as

$$\pi_t(s) = \begin{cases} DC, & \text{if } t = 0, 1 \text{ or } t = mT+1 \text{ for } m \in Z(s), \\ CC, & \text{otherwise.} \end{cases}$$

It is immediate that $h_2^t(\pi(s); \delta) > h_2^t(\pi^T; \delta)$ for any $t \geq 1$. Therefore $\pi(s)$ is an equilibrium path. Moreover, we obtain $h_2^t(\pi(s); \delta) \geq h_2^1(\pi(s); \delta)$ for any $t \geq 1$, because in the continuation path from period $t > 1$, DC's are located more distantly and more sparsely than in the continuation path from period 1. Thus Lemma 7 applies, and we have a wonderful discount factor $\delta(s) \in (\frac{1}{\sqrt{2}}, \delta)$ at which $\pi(s)$ is the wonderful equilibrium path. Therefore, $\delta(s) \in W \cap (\frac{1}{\sqrt{2}}, \delta)$.

Note that if we choose a different sequence s', $Z(s') \neq Z(s)$ and therefore $\pi(s') \neq \pi(s)$. Consequently, $\delta(s') \neq \delta(s)$. Since the set of all sequences of natural numbers has the power of continuum, $W \cap (\frac{1}{\sqrt{2}}, \delta)$ has at least the same power, hence there are uncountably infinite elements in $W \cap (\frac{1}{\sqrt{2}}, \delta)$ for any $\delta \in (\frac{1}{\sqrt{2}}, \frac{3}{4})$.
■

6. Monotonicity

Our analysis in the previous sections has shown that the maximum efficient equilibrium payoff, $\bar{v}_1(\delta)$ is *not* monotonic with respect to δ, in the region $(0, \frac{3}{4})$. The analysis has also demonstrated that the set of all efficient equilibrium payoffs given δ does not exhibit monotonicity with respect to δ.[9]

However, while we do not have monotonicity of the maximum equilibrium payoff or the equilibrium payoff set, we do have monotonicity of efficient equilibrium *paths* with respect to δ, for $\delta < \frac{3}{4}$. Indeed, this monotonicity is a nice aspect of efficient equilibrium; for the set of (not necessarily efficient) equilibrium outcomes, it does not increase monotonically as δ increases.

[9]This observation does not contradict Abreu, Pearce, and Stacchetti [2, Theorem 6], which proves monotonicity of equilibrium payoff sets, since they assume the public signal is distributed on a subset of a finite Euclidean space, and that the distribution function has a density. In our context, this is equivalent to requiring the presence of a public correlating device.

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