The Design of Teacher Assignment: Theory and Evidence*

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(Job Market Paper)

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Abstract

The reassignment of teachers to schools is a central issue in education policies. In several countries, this assignment is managed by a central administration that faces a key constraint: ensuring that teachers obtain an assignment that they weakly prefer to their current position. To satisfy this constraint a variation on the Deferred Acceptance (DA) mechanism of Gale and Shapley (1962) has been proposed in the literature and used in practice—for example, in the assignment of French teachers to schools. We show that this mechanism fails to be efficient in a strong sense: we can reassign teachers in a way that i) makes them better-off and ii) better fulfills the administration's objectives represented by the priority rankings of the schools. To address this weakness, we characterize the class of mechanisms that do not suffer from this efficiency loss and elicit a set of strategy-proof mechanisms within this class. To empirically assess the extent of potential gains associated with the adoption of our mechanisms, we use a rich dataset on teachers' applications for transfers in France. These empirical results confirm both the poor performance of the modified DA mechanism and the significant improvements that our alternative mechanisms deliver in terms of teachers' mobility and administration's objectives.

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1 Introduction

Teachers are a key determinant of student achievement, and their distribution across schools can have a major impact on achievement gaps between students from different ethnic and/or social backgrounds. Growing concerns that disadvantaged students may have less access to effective teachers have given rise to policies intended to better distribute effective teachers across schools.[1] However, such policies must be implemented with caution, as they might have unexpected effects on teachers' satisfaction and, ultimately, on the overall appeal of the teaching profession.[2] This raises a central question: how can we design a teacher assignment procedure that takes into account both teachers' preferences and the administration's concerns about the distribution of (effective) teachers? This paper introduces a new assignment procedure and assesses its performance.

In many countries, the labor market for teachers is highly regulated by a central administration that is in charge of assigning teachers to schools.[3] Within such systems, teachers submit ranked lists of school preferences, and each school ranks teachers. When policymakers design assignment processes, they must consider several objectives, some of which might be difficult to reconcile. A first intuitive objective is to maximize teachers' satisfaction by implementing all beneficial exchanges of positions across teachers. Yet, policymakers must ensure that such position exchanges do not unintentionally harm some disadvantaged schools. In many countries, teachers' priorities at schools are primarily determined by their experience, which reflects the administration's effort to not assign novice teachers to disadvantaged students. By implementing all beneficial exchanges, one may increase the number of inexperienced teachers in disadvantaged schools, which is a key concern for policymakers. Hence, in this context, one natural objective is to allow teachers to exchange positions only if it does not increase the number of inexperienced teachers in disadvantaged schools.

More generally, the criteria used to define teachers' priorities reflect clear social objectives.[4] A relevant objective is therefore to allow teachers to exchange positions only if schools do not get teachers with lower priorities. We name this requirement *two-sided efficiency*. While

[1]Recent initiatives in the U.S. have intended to measure teacher effectiveness and ensure that disadvantaged students have equal access to effective teachers. See for instance, Teach for America, Teach First in the U.K., and, more generally, Race to the Top, the Teacher Incentive Fund, and the flexibility policy of the Elementary and Secondary Education Act).

[2]Two important issues facing the teaching profession are the increasing shortage of qualified teachers (Corcoran et al., 1994) and the difficulty of retaining new teachers in the profession (Boyd et al., 2005).

[3]This is the case, for example, in France (Terrier, 2014), Germany, Czech Republic (Cechlárová et al., 2015), Italy (Barbieri et al., 2011), Turkey (Dur and Kesten, 2014), Mexico (Pereyra, 2013), Peru, Uruguay (Vegas et al., 2006), and Portugal.

[4]In practice, several other criteria used to determine teacher priorities might also reflect broader social objectives. For instance, in France, spousal reunification and children reunification give a priority bonus to teachers at schools close to where their spouses or children live. Again, one can easily see the social objective motivating these priorities, namely, to allow for position exchanges that are not at the expense of teachers' experience in (possibly disadvantaged) schools, except when an exchange can allow a teacher to join his/her family. This is what our approach will ensure.

this is formally equivalent to an efficiency notion that considers both teachers and schools as welfare-relevant entities, in our context this requirement is mainly instrumental. It allows us to produce teacher assignments that respect the administration's objectives, as reflected by the priority system. In the empirical section (Section 5) of this paper we examine the French teacher assignment system and show that its underlying social objectives are better fulfilled when using assignment schemes that satisfy two-sided efficiency.

The central administration in charge of designing the assignment mechanism faces an additional fundamental constraint. Many teachers already have positions and are willing to be reassigned. In practice, tenured teachers have the right to keep their initial positions if they wish. Thus, the administration has to offer a teacher a position at a school that he or she likes at least as much as the one to which he or she is currently assigned. In other words, the assignment of teachers must be *individually rational*. To fulfill this constraint, a standard approach is to use a variation on the well-known deferred acceptance mechanism (Gale and Shapley, 1962) – DA, for short – to make the assignment individually rational (Compte and Jehiel, 2008; Pereyra, 2013). This variation consists of, first, artificially modifying the school's ordering of teachers such that all teachers initially assigned to a school are moved to the top of that school's ranking. In a second step, we run the DA mechanism using the modified priorities. This mechanism is used to assign teachers to schools in France, to assignment on-campus housing in several places around the world (Guillen and Kesten, 2012), and, in many countries, to assign students to schools when students have the right to stay in their district school.

While, by construction, the modified version of DA is individually rational, a first objective of this paper is to show that the modification loses an important property of DA by failing to be two-sided efficient in that one can find alternative assignments whereby all teachers would get better-off assignments and schools would get teachers to which they assign higher priority. Importantly, this "Pareto improvement" can be achieved while simultaneously improving fairness; i.e., we can reduce the number of teachers who are refused by a school when other teachers with lower rankings by that school are accepted. (We use standard terminology and say that such teachers and schools form a *blocking pair*.)[5] Hence, the modified DA can be improved both in terms of efficiency – beneficial exchanges of positions across teachers – and fairness. This lack of trade-off between these two core notions is in stark contrast with what we know from the college admission and school choice environments and makes clear that the teacher assignment problem is quite different from previously studied settings.

The main goal of this paper is therefore to design mechanisms that do not suffer from the same limitations as the modified version of DA, while keeping the good incentive properties of this mechanism, i.e., strategy-proofness (meaning that teachers have straightforward incentives to report their preferences truthfully).[6] As we will make clear, merely tweaking the DA

[5] Under the (standard) DA, it is well known that one can reassign teachers and make all of them better off, some strictly. However, this will be done at the expense of schools, given that the (standard) DA is in the Core and, hence, two-sided efficient. Here, in stark contrast with the standard DA, we show that, under the modified DA, two-sided efficiency is violated; i.e., both teachers and schools can be made better-off.

[6] Teacher are the only strategic agents in this teacher assignment context.

mechanism is not enough to fulfill this objective. We say that a matching is *two-sided maximal* if (1) compared to the initial assignment, all teachers are better-off and all schools get teachers with higher priorities, and (2) the matching cannot be improved in terms of (2i) (two-sided) efficiency and (2ii) fairness. This requirement is actually quite weak, and two-sided maximal matchings are easily shown to correspond to assignments that are both two-sided efficient and individually rational on both sides of the market.

To characterize two-sided maximal matchings, we provide an algorithm called the block exchange (BE) algorithm. The idea is simple: starting from the initial assignment, if two teachers block one another's schools, we allow these teachers to exchange their schools. Obviously, larger exchanges involving many teachers are possible. As a teacher may be involved in several cycles, the outcome of the BE algorithm depends on the order in which we select the cycles. A first natural result shows that any possible outcome of the BE algorithm is a two-sided maximal matching, and, conversely, any two-sided maximal matching can be achieved with an appropriate selection of cycles.[7] While we obtain a plethora of different possible matchings depending on how we select exchange cycles in the BE algorithm, our main result shows that there are ways to select cycles that make this algorithm strategy-proof for teachers. Such mechanisms are called teacher-optimal BE algorithms (TO-BE). The name emphasizes the fact that TO-BE is a teacher-optimal selection of BE. Our results also give a sense in which adding strategy-proofness drastically reduces the set of possible mechanisms: in the one-to-one environment, there is a unique two-sided maximal mechanism that is strategy-proof.

We provide additional theoretical results in two respects. First, we consider a case in which only teachers are welfare-relevant entities. In this context, we provide a similar characterization to that obtained with the BE algorithm. Depending on how exchange cycles are selected, we once again identify a large class of mechanisms. Although this approach obviously favors teachers, we show that no mechanism in this class is strategy-proof.[8] Second, we consider a large market approach in which preferences and schools' rankings are drawn randomly from a rich class of distributions.[9] We show that when the market size increases, our mechanisms perform quantitatively better than the modified DA in terms of utilitarian efficiency and number

[7]This result is related to Erdil and Ergin (2017), who characterize two-sided Pareto-efficient stable matchings in a two-sided matching framework with indifferences. We discuss the connection in more depth later in the text. In particular, while there is no two-sided Pareto-efficient stable matching mechanism that is strategy-proof for agents on one side of the market (see, Erdil, 2014), we show that, in our environment, some two-sided maximal matching mechanisms are strategy-proof for teachers.

[8]Dur et al. (2015) independently characterize a class of mechanisms similar to the one we obtain when only teachers are welfare-relevant entities. They consider the allocation given by DA in a school choice environment. For each school, the authors define an exogenous set of teachers who are allowed to form a blocking pair with that school. They characterize the allocations that are (one-sided) efficient under this constraint and Pareto dominate the DA assignment. Our class of mechanisms starts from an arbitrary initial assignment, while given their motivation, they are interested only in the class improving on DA, and thus, they begin from the DA allocation. Our main message concerns the non-existence of a strategy-proof selection in our class of mechanisms. When starting from an arbitrary exogenous initial assignment (which excludes starting from the DA allocation), this result is non-trivial.

[9]These markets can involve a large number of agents. For instance, in France, approximately 65,000 tenured teachers ask for an assignment every year.

of blocking pairs. We also identify the potential cost of adopting a strategy-proof mechanism in terms of utilitarian outcomes and number of blocking pairs compared to a first-best approach whereby one could select any two-sided maximal mechanism. Our arguments build on techniques from random graph theory, as in Lee (2014), Che and Tercieux (2015a), and Che and Tercieux (2015b).

Finally, we use a nationwide labor market to empirically estimate the magnitude of gains and trade-offs in a real teacher assignment problem. In France, like in several other countries, the central administration manages teachers' assignment to schools. Exploiting the straightforward incentives of the modified version of DA currently in use, we use teacher-reported preferences to run counterfactuals and quantify our mechanisms' performance. The results confirm that the modified version of DA (DA^* for short) is rarely two-sided maximal. Of the 49 disciplines for which we ran DA^*, 30 of them (representing 95.9% of the teachers) could be simultaneously improved in terms of teachers' welfare, schools' welfare, and fairness. Compared to DA^*, the number of teachers moving from their initial assignment more than doubles under our mechanisms, and the distribution of teachers' ranks (over schools they obtain) stochastically dominates that of DA^*. Regarding fairness, the number of teachers who are not blocking with any school increases by 49%. Finally, contrary to DA^*, under our alternative mechanisms, no region has a position for which the teacher assigned to it has a lower priority than the teacher initially assigned to that position. More importantly, we show that the administration's objectives are better fulfilled when using mechanisms that are two-sided maximal. Indeed, the percentage of unexperienced teachers in disadvantaged regions diminishes under all the mechanisms we suggest, and the number of teachers who get a position closer to their partner increases. These figures are essentially the same for the strategy-proof mechanisms mentioned above, which makes them particularly appealing for practical implementation.

Our work stands at the crossroads of different strands of the literature. Our theoretical setup covers two standard models in matching theory. The first is the *college admission problem*, in which schools have preferences that are taken into account for both efficiency considerations and fairness issues (Gale and Shapley, 1962). Second, our model also embeds the *house allocation problem* (e.g., Shapley and Scarf, 1974, Abdulkadirouglu and Sonmez, 1999, and Sönmez and Ünver, 2010). In this framework, individuals own a house and are willing to exchange their initial assignments. Our paper builds on these two lines of research by incorporating both the initial assignment and the two-sided efficiency criteria. Despite covering important applications, this mixed model remains understudied, though Guillen and Kesten (2012) note that the modified version of the DA mechanism is used to allocate on-campus housing at MIT. Compte and Jehiel (2008) and Pereyra (2013) provide results on the properties of this mechanism. After explaining that fairness and individual rationality are not compatible, they propose weakening the notion of blocking pairs and show that the modified version of DA maximizes fairness under this weakening. By contrast, our work retains the standard definition of blocking pairs and addresses notions of maximal fairness using the usual definition. More importantly, our theoretical and empirical results highlight that maximizing their notion of fairness can have high costs in terms of efficiency and the traditional notion of fairness. Finally, Dur and Unver (2017) introduce a matching model to study two-sided exchange markets, such as tuition or temporary worker exchanges. We discuss the similarities and differences between our papers in more details in the Section 4, after we introduce our setting and algorithms.

2 Teacher Assignment to Schools in France

France, like several other countries, has a highly centralized labor market for teachers. The 400,000 public school teachers in France are civil servants. The French Ministry of Education is responsible for their recruitment, assignment to schools, and salary scale.[10] This gives us the opportunity to use a nationwide labor market to compare the performance of different assignment algorithms. Prior to assignment, the central administration defines teacher priorities using a point system that takes into account three legal priorities: spousal reunification, disability, and having a position in a disadvantaged or violent school. Computing the score also takes into account additional teacher characteristics, including total seniority in teaching, seniority in the current school, time away from the spouse and/or children. This score determines schools' rankings or preferences. Note that we will use the terms "priorities" and "preferences" interchangeably (for the rationale behind this terminology, see the Introduction). The central administration defines the point system, which is well known by all teachers wishing to change schools.[11]

The French Ministry of Education divides French territory into 31 administrative regions, which are called *académies* (see the map in Appendix S.1). We will refer to them as *regions* hereafter. Since 1999, the matching process has comprised two successive phases. First, during a *region assignment* phase, newly tenured teachers and teachers who wish to move to another region submit an ordered list of regions. A matching mechanism (described in the next section) is used to match teachers to regions, using priorities defined by the point system. This phase is managed by the central administration. Then, during the second phase, each region proceeds to *school assignment*. In each region, teachers matched to the region after the first phase and teachers who already have a position but wish to change schools within the same region report their preferences for schools in the region. Matching is completed using the same mechanism as in Phase 1 and a similar priority-defining point system as in Phase 1. The only difference is that teachers are limited in the number of schools that they can rank during this phase, as discussed in the empirical section (Section 5). This phase is locally managed by each regional administration.

[10] Anyone who wishes to become a teacher must pass a competitive examination that is organized once per year by the Ministry of Education. Those who pass this public sector exam will become civil servants, and thus, their salary is completely regulated by a detailed pay scale. Neither schools nor teachers can influence salary or promotions. All teachers with the same number of years of experience and who have passed the same exam earn the same salary. Further details on the recruitment and assignment process are available on the Matching in Practice website: http://www.matching-in-practice.eu/matching-practices-of-teachers-to-schools-france/

[11] An official list of criteria used to compute the point system is available on the government website: http://cache.media.education.gouv.fr/file/42/84/6/annexeI-493_365846.pdf

In 2013, just over 25,000 teachers applied in Phase 1, and in Phase 2 approximately 65,000 teachers submitted lists of preferred within-region schools for reassignment. These figures include all newly tenured teachers who have never been assigned a position and tenured teachers who request a transfer. In practice, the assignment process is decomposed into as many markets as there are subjects taught – 107 – and each market includes different amounts of teachers. Some markets are large, such as Sports (approximately 2,500 teachers), Contemporary Literature (approximately 2,000 teachers), and Mathematics (approximately 2,000 teachers), while others are smaller, such as Thermal Engineering (approximately 60 teachers) or Esthetics (approximately 15 teachers), with a wide range in between. As a teacher teaches only one subject and positions are specific to a subject, the markets can be considered independent from one another.[12]

A lack of mobility has emerged as a concern for the Ministry. In 2013, of the 17,000 tenured teachers requesting a new assignment, only 40.9% had their requests satisfied. Similarly, 29% of the teachers asking to move closer to their families did not obtain a new assignment, many of them for several consecutive years. Due to this lack of mobility, the Mediator of the French Ministry of Education (2015), responsible for resolving conflicts between the Ministry and teachers, receives approximately 700 complaints from primary and secondary school teachers every year related to assignment issues. The mediator states that "the assignment algorithm opens doors to difficult personal situations that can eventually tarnish the quality and the investment of human resources". An additional concern exists. Every year, many teachers who do not obtain reassignment to their desired region decide to resign or request a year off. This leaves some students without teachers and regularly requires regions to hire last-minute replacement staff who are not trained to teach. In the least attractive schools, labeled "priority education," 30% of teachers do not have teaching certification, versus 7.6% in other schools. One of the objectives of this paper is to show that using an alternative mechanism can significantly reduce the current lack of mobility.

3 Basic Definitions and Motivation

Consider a problem in which a finite set of teachers T has to be assigned to a finite set S of schools. Each school s has q_s available seats. Each teacher t has a strict preference relation $\succ_t$ over the set of schools and being unmatched (being unmatched is denoted by $\emptyset$). For any teacher t, we write $s \succeq_t s'$ if and only if $s \succ_t s'$ or $s = s'$. Similarly, each school s has a strict preference relation $\succ_s$ over teachers and being unmatched.[13] For simplicity, we assume that all teachers and schools prefer to be matched rather than being unmatched. A **matching** μ is a mapping from $T \cup S$ into $T \cup S \cup \{\emptyset\}$ such that (i) for each $t \in T$, $\mu(t) \in S \cup \{\emptyset\}$ and for each $s \in S$, $\mu(s) \subset T$ and (ii) $\mu(t) = s$ iff $t \in \mu(s)$. That is, a matching simply specifies the school to which each teacher is assigned or that a teacher is unmatched. It also specifies

[12] In practice, couples from different fields can submit joint applications, which connects the fields. However, we eliminated all couples from our sample. Details are provided in Appendix S.5.

[13] Our results easily extend to the case of weak preferences for schools.

the teachers assigned to each school, if any. We also sometimes use the term "assignment" instead of "matching". Thus far, our environment does not differ from the college admission problem (Gale and Shapley, 1962).

However, in a teacher assignment problem, there is an additional component: teachers have an initial assignment. Let us denote the corresponding matching by μ_0. We assume that $\mu_0(t) \neq \emptyset$ for each teacher t and $|\mu_0(s)| = q_s$ for each school s. Thus, we focus on a pure reassignment process among teachers. We further discuss this assumption in Section 6. All teachers are initially assigned a school (there is no incoming flow of teachers into the market), and there is no available seat at schools (there is no outgoing flow of teachers out of the market). We define a **teacher allocation problem** as a quadruplet $[T, S, \succ, \mathbf{q}]$ where $\succ := (\succ_a)_{a\in S\cup T}$ and $\mathbf{q} := (q_s)_{s\in S}$.

Since we are in a many-to-one setting, one has to define schools' preferences over groups of teachers. We adopt a standard conservative approach here that will only strengthen our main empirical findings. Moreover, our theoretical findings do not depend on this assumption. Consider a school s with q positions to fill and two vectors of size q, say $\mathbf{x} := (t_1, \dots, t_q)$ and $\mathbf{y} := (t'_1, \dots, t'_q)$. Let us assume that each of these vectors is ordered in such a way that for each $k = 1, \dots q-1$, the kth element of vector $\mathbf{x}$ is preferred to its $k+1$th element; we make analogous assumptions for vector $\mathbf{y}$. We say that $\mathbf{x}$ is (weakly) preferred by the school to $\mathbf{y}$ if, for each $k = 1, \dots q$, the kth element of vector $\mathbf{x}$ is (weakly) preferred to the kth element of vector $\mathbf{y}$, i.e. $t_k \succeq_s t'_k$. The preference is strict if at least one of the coordinates is strictly preferred. In the following, when comparing two matchings μ and μ' for a school s, we will abuse notations and note $\mu'(s) \succeq_s \mu(s)$ if $\mu'(s)$ is (weakly) preferred by s to $\mu(s)$. We will say that school s weakly prefers or is weakly better under the matching μ'.[14] Again, we use the terms "preferences" and "priorities" interchangeably even though, in our context, schools' ordering over teachers are priorities given by law. Many of our welfare notions (e.g., two-sided efficiency defined below) will do "as if" these priorities were the schools' true preferences with the motivation that priorities reflect the administration's normative criteria and so are welfare relevant (see the Introduction). Again, Section 5.2.3 will show that the administration's objectives are better fulfilled when using this approach.

We are interested in different efficiency and fairness criteria, depending on whether we regard both teachers and schools or only teachers as welfare-relevant entities. First, we say that a matching μ is **two-sided individually rational** (2-IR) if, for each teacher t, $\mu(t)$ is acceptable to t, i.e., $\mu(t) \succeq_t \mu_0(t)$ and, in addition, for each school s, $\mu(s)$ is acceptable to s, i.e., $\mu(s) \succeq_s \mu_0(s)$.[15] Similarly, a matching is **one-sided individually rational** (1-IR) if each teacher finds his/her assignment acceptable. We say that a matching μ 2-Pareto dominates (resp. 1-Pareto dominates) another matching μ' if all teachers and schools (resp. teachers)

[14]This is a strong notion. However, even with this conservative notion, we can significantly improve on the standard mechanisms. Using such a strong notion only strengthen this result.

[15]Requiring IR on both the teacher and school sides is non standard, but is equivalent to the requirement that the assignment Pareto dominates the initial assignment for both teachers and schools. This is consistent with our motivation for considering both sides of the market as welfare-relevant entities (see the Introduction). Our empirical analysis indeed points out that this notion allows to better fulfill the administration objectives.

are weakly better off – and some strictly better off – under μ than under μ'. A matching is **two-sided Pareto-efficient** (2-PE) if there is no other matching that 2-Pareto dominates it. Similarly, we define **one-sided Pareto-efficient** (1-PE) matchings as assignments for which no alternative matching exists that 1-Pareto dominates it. We say that under matching μ, a teacher t has justified envy for teacher t' if t prefers the assignment of t', i.e., $\mu(t') =: s$, to his own assignment $\mu(t)$ and s prefers t to t'. Using the standard terminology from the literature, we say that (t, s) **blocks** matching μ. A matching μ is **stable** if there is no pair (t, s) blocking μ.[16] We will sometimes say that a matching μ dominates another matching μ' in terms of stability if the set of blocking pairs of μ is included in that of μ'.

Finally, a **matching mechanism** is a function φ that maps problems into matchings. We write $\varphi(\succ)$ for the matching obtained in problem $[T, S, \succ, \mathbf{q}]$. We also write $\varphi_t(\succ)$ for the school that teacher t obtains under matching $\varphi(\succ)$. It is 2-IR/1-IR/1-PE/2-PE/stable if, for each problem, it systematically selects a matching that is 2-IR/1-IR/1-PE/2-PE/stable.

One of the standard matching mechanisms is DA, as proposed by Gale and Shapley (1962). Because we discuss a closely related mechanism, we first recall the definition of DA.

- **Step 1.** Each teacher t applies to his most preferred school. Each school tentatively accepts, up to its capacity, its most preferred teachers among the offers it receives and rejects all other offers.

In general,

- **Step k $\geq$ 1.** Each teacher t who was rejected at step $k-1$ applies to his most preferred school among those to which he has not yet applied. Each school tentatively accepts, up to its capacity, its most favorite teachers among the new offers in the current step and the applicants tentatively selected from the previous step (if any), and rejects all other offers.

The following proposition is well known.

Proposition 1 (Gale and Shapley, 1962) *DA is a stable and 2-PE mechanism.*

While DA is stable and 2-PE, it fails to be 1-IR (and thus 2-IR). This is unavoidable: in general, there is a conflict between individual rationality and stability. The basic intuition is that imposing 1-IR on a mechanism yields situations in which some teacher t may be able to keep his initial assignment $\mu_0(t) =: s$, while school s may perfectly prefer other teachers to t.

[16] Usually, one also adds a *non-wastefulness* condition that requires that there must be no teacher t and school s such that $s \succ_t \mu(t)$ and $|\mu(s)| < q_s$. Since we assumed that for any school s, $|\mu_0(s)| = q_s$ and all teachers and schools are willing to be matched, under any 1-IR or 2-IR matching μ, we have $|\mu(s)| = q_s$. Because our analysis focuses on such matchings, we can omit the non-wastefulness condition.

These other teachers may rank s at the top of their preference relation and hence block with school s. We summarize this discussion in the following observation.[17]

Proposition 2 *There is no mechanism that is both 1-IR and stable. Hence, DA is not 1-IR.*

Because there is a fundamental trade-off between 1-IR and stability, one may wish to find a mechanism that restores individual rationality while retaining DA's other desirable properties, such as its stability and 2-Pareto efficiency, to the greatest extent possible. An approach followed in the literature (see, for instance, Pereyra (2013) or Compte and Jehiel (2008)) and used in practice achieves this balance by artificially modifying the schools' preferences such that each teacher t is ranked, in the (modified) ranking of his initial school $s := \mu_0(t)$, above any teacher $t' \notin \mu_0(s)$. Other than this modification, the schools' preference relations remain unchanged.[18] With this modification in place, DA proceeds, as defined above, using schools' modified preferences. We denote this mechanism as DA*. By construction, this is a 1-IR mechanism. It is used in several real-world situations, including assigning on-campus housing at MIT (Guillen and Kesten, 2012) and, more pertinent to our interests here, assigning teachers to schools in France. Specifically, a *school-proposing deferred acceptance* mechanism is run using the modified priorities and reported preferences. Then, *Stable Improvement Cycles* are executed as defined in Erdil and Ergin (2008). Using Theorem 1 in Erdil and Ergin (2008), we know this process yields the outcome of the *teacher-proposing deferred acceptance* mechanism according to the modified priorities.[19] Hence, the mechanism used to assign French teachers to public schools is equivalent to DA*.

As noted above, this mechanism is 1-IR by construction, and therefore, by Proposition 2, we know it is not stable. Yet, is there a sense in which the violation of stability is minimal? What about efficiency: Is DA* 2-PE? Furthermore, if the answers to these questions are negative, can we find ways to improve upon DA*? The following example will illustrate an important disadvantage of DA* to which we will return in both our theoretical analysis and in our empirical assessment.

[17]This is highlighted in Compte and Jehiel (2008) and Pereyra (2013), who also provide results on the properties of DA*. They note that fairness and individual rationality are not compatible, propose weakening the notion of blocking pairs, and show that the modified version of DA maximizes fairness under this weakening. By contrast, our work retains the standard definition of blocking pairs and addresses notions of maximal fairness using the usual definition. More importantly, our theoretical and empirical results highlight that maximizing their notion of fairness can have high costs in terms of efficiency and the traditional notion of fairness.

[18]Formally, for each school s, a new preference relation $\succ'_s$ is defined such that $t \succ'_s t'$ for each $t \in \mu_0(s)$ and $t' \notin \mu_0(s)$, and for each t, t' not in the school's initial assignment $\mu_0(s)$, we have $t \succ'_s t'$ if and only if $t \succ_s t'$. If $t, t' \in \mu_0(s)$, we assume they are ranked according to $\succ_s$. This is not necessary for the results.

[19]In the French system, teachers' priorities at schools can be coarse. Hence, in practice, the algorithm begins by breaking ties (using teachers' birth dates). Once ties are broken, school-proposing DA is run using the modified priorities *with no ties* and the reported preferences. From this outcome, stable improvement cycles are run, again using the modified (strict) priorities. The outcome is thus equivalent to the teacher-proposing deferred acceptance with the same tie-breaking rule, which, in turn, may be Pareto dominated by a teacher-optimal stable mechanism. Our mechanisms and results can be easily extended to an environment with coarse priorities.

Example 1 *We consider a simple environment with n teachers and n schools with a 1-IR initial assignment μ_0. Let us assume that a teacher t^* is initially assigned to school s^* (i.e., $\mu_0(t^*) = s^*$) and is ranked first by all schools. In addition, school s^* is ranked at the bottom of each teacher's preference relation, including t^*; hence, t^* is willing to move. Under these assumptions, no teacher will move from his initial assignment if we use DA* to assign teachers. To see this, note first that t^* does not move from his initial assignment. Indeed, because DA* is 1-IR, if t^* were to move, then some teacher t would have to take the seat at school s^* (or be unmatched), but since s^* is the worst school for every teacher (and μ_0 is 1-IR), this assignment would violate the individual rationality condition for teacher t, a contradiction. Note that this implies that, under DA* algorithm, t^* applies to every school s (but is eventually rejected). Now, to see that no teacher other than t^* moves, assume on the contrary that $t \neq t^*$ is assigned a school $s \neq \mu_0(t)$. As mentioned above, at some step of DA* algorithm, t^* applies to s. Since t^* is ranked above t in the preference relation of school s (recall that $s \neq \mu_0(t)$), t cannot eventually be matched to school s, a contradiction.*

To recap, under our assumptions, no teacher moves from his initial assignment. Since the initial assignment can perform very poorly in terms of basic criteria such as stability or 2-Pareto efficiency, we can easily imagine the existence of alternative matchings that would make both teachers and schools better off and, thereby, shrink the set of blocking pairs.

The driving force in this example is the existence of a teacher who is ranked at the top of each school's ranking and is initially assigned to the worst school. This is, of course, a stylized example, and one can easily imagine less extreme examples in which a similar phenomenon would occur. The basic idea is that, for DA* to perform poorly, it is enough to have one teacher (a single one is enough) who has a fairly high ranking for a relatively large fraction of the schools, being assigned an unpopular school. Our theoretical analysis and our empirical assessment will show that the described phenomenon is far from being a peculiarity.

Remark 1 *Contrary to what we have in the example, in practice there are open seats at schools. One may argue that high-priority teachers, such as t^*, will succeed in obtaining available seats at schools they desire and that the above phenomenon would thus be considerably weakened. However, in an environment in which teachers' preferences tend to be similar (i.e., are positively correlated), there will be competition to access good schools. These good schools have a limited number of seats available, and one may easily imagine that once these open seats are filled by some of the high-priority teachers, a similar phenomenon as in the example could occur among the remaining teachers. We ran simulations in a rich environment (allowing for correlation in teachers' and schools' preferences, available seats, newcomers, and positive assortment in the initial assignment) that confirm this intuition. The results are reported in Appendix S.2.*

The above example identifies a weakness of DA*: it can be improved upon in terms of both efficiency (on both sides) and the set of blocking pairs (i.e., we can shrink its set of blocking pairs). This is an important difference with the standard college admission DA, which is known to be 2-PE. In DA*, the change in schools' preferences made before running the algorithm to ensure the 1-IR property leads to the failure of the 2-PE property. Thus, we are interested in mechanisms that do not have this type of disadvantage. We also wish to retain the elementary property that our mechanism improves on the initial assignment. This suggests the following definitions.

Definition 1 *A matching μ is **two-sided maximal** if μ is 2-IR*[20] *and there is no other matching μ' such that (1) all teachers and schools are weakly better off and some strictly better off, and (2) the set of blocking pairs under μ' is a subset of that under μ.*

This notion treats both schools and teachers as welfare-relevant entities. This might be surprising, as schools' rankings of teachers (or priorities) are given by law. However, as we argued in the introduction, there are clear social objectives motivating the criteria used to define teacher priorities. This makes it relevant to adopt an approach "as if" schools' priority rankings were schools' preferences. The empirical analysis further proves that this "as if" approach allows us to better fulfill the social objectives behind priorities than approaches focusing only on teachers as welfare-relevant entities.

Accounting for both teachers' and schools' welfare is quite conservative. In particular, some teachers may be unable to leave their positions because no other teacher with a higher priority will be willing to replace them. Accordingly, we examine the cost that our two-sided efficiency requirement imposes on teachers' welfare. When we ignore the school side, we obtain the following natural counterpart.

Definition 2 *A matching is **one-sided maximal** if μ is 1-IR and there is no other matching μ' such that (1) all teachers are weakly better off and some strictly better off, and (2) the set of blocking pairs under μ' is a subset of that under μ.*

Consistent with our previous notions, we say that a mechanism is two-sided (resp. one-sided) maximal if it systematically selects a two-sided (resp. one-sided) maximal matching.

Let us note that, if there is a matching μ' under which all teachers and schools are weakly better off and some strictly better off than under a matching μ, then the set of blocking pairs under μ' is a subset of that under μ. Thus, in the definition of two-sided maximality, requirement (2) can be dropped.[21] This yields the following straightforward equivalent definition.

Proposition 3 *A matching μ is **two-sided maximal** if and only if μ is 2-IR and 2-PE.*

Given Example 1 above, we have the following straightforward proposition.

[20]Recall that the motivation for imposing 2-IR is to ensure that our assignments 2-Pareto-dominate the initial assignment.

[21]However, one can easily check that (2) in the definition of one-sided maximality cannot be dropped.

Proposition 4 *DA^* is not two-sided maximal and, hence, not one-sided maximal. (Thus, DA^* is not 2-PE.)*

Given the above weaknesses of DA*, the obvious goal is to identify the class of mechanisms characterizing two-sided and one-sided maximality and then study the properties of those mechanisms. This is the aim of the next section.

4 Theoretical Analysis

The next two sections identify a class of mechanisms that characterize each notion of maximality defined above (Definitions 1 and 2). Once the characterization results are proved, we analyze the properties of the mechanisms in that class.

4.1 Two-sided maximality

We define a class of mechanisms that characterizes the set of two-sided maximal mechanisms. The mechanism will sequentially clear cycles of an appropriately constructed directed graph in the spirit of Gale's top trading cycle (TTC hereafter), originally introduced in Shapley and Scarf (1974).

4.1.1 The Block Exchange Algorithm

The basic idea behind the mechanisms we define is the following: starting from the initial assignment, if a teacher t has a justified envy toward t' and t' also has a justified envy toward t, then we allow t and t' to trade their initial assignments. This is a pairwise exchange between t and t', but three-way or even larger exchanges could also occur. Once such an exchange has been made, we obtain a new matching and can again search for possible trades. More precisely, our class of mechanisms is induced by the following algorithm, named the Block Exchange (BE):

- **Step** 0 : set $\mu(0) := \mu_0$.
- **Step** $k \geq 1$: Given $\mu(k-1)$, let the teachers and their assignments stand for the vertices of a directed graph where, for each pair of nodes (t, s) and (t', s'), there is an edge $(t, s) \longrightarrow (t', s')$ if and only if teacher t has a justified envy toward teacher t' at s', i.e., t prefers s' to s and s' prefers t to t'. If there is no cycle, then set $\mu(k-1)$ as the outcome of the algorithm. Otherwise, select a cycle in this directed graph. For each edge $(t, s) \longrightarrow (t', s')$ in the cycle, assign teacher t to school s'. Let $\mu(k)$ be the matching so obtained. Go to step $k+1$.

It is easy to verify that this algorithm converges in (finite and) polynomial time.[22] In the above description, we do not specify how the algorithm should select the cycle of the directed graph. Therefore, the above description can be thought to define a class of mechanisms, wherein a mechanism is determined only after we fully specify how to act when confronted with multiple cycles. These selections may be random or dependent on earlier selections. In general, for each preference profile for teachers and schools $\succ$, a possible outcome of BE is a matching that can be obtained by an appropriate selection of cycles in the above procedure. Thus, we consider the following correspondence $BE : \succ \rightrightarrows \mu$ where $BE(\succ)$ stands for the set of all possible outcomes of BE. A selection of the BE algorithm is a mapping $\varphi : \succ \mapsto \mu$ s.t. $\varphi(\succ) \in BE(\succ)$. Obviously, each selection φ of BE defines a mechanism.

As mentioned above, our class of mechanisms shares some similarities with Gale's TTC, but there are two important differences. The first and the more minor difference is that a teacher in a node can point to several nodes and thus, implicitly, to several schools. This is why, contrary to TTC, we have an issue regarding cycle selection and our algorithm does not define a unique mechanism. However, as we will see in the next result, this is necessary for our characterization. Second, and certainly more importantly, our algorithm takes into account welfare on both sides of the market. Indeed, a teacher in a node (t, s) can point to a school in (t', s') only if s' prefers t to its assignment t'. This is what ensures, contrary to TTC, that each time we carry out a cycle, *both* teachers and schools become better off. This has the desirable implication that each time a cycle is cleared, the set of blocking pairs shrinks.

The BE algorithm starts from the initial assignment and then improves on it in terms of teacher and school welfare. More generally, one could start from any matching obtained by running another mechanism φ. Doing so will guarantee that the (modified) BE algorithm will select a matching that dominates that of φ in terms of both teacher and school welfare. This modification of the BE algorithm that takes the composition of BE and φ will be denoted by BE$\circ\varphi$. Given our starting point that DA* performs poorly in terms of teacher and school welfare, we will be particularly interested in BE$\circ$DA*.

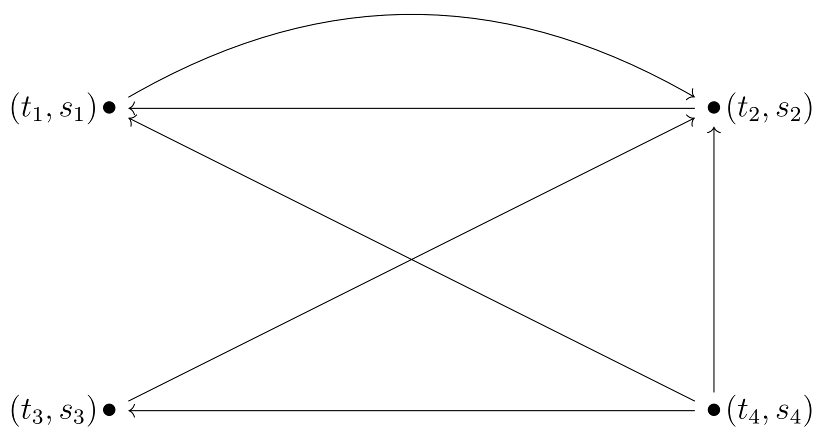
The next example illustrates how the BE algorithm works.

Example 2 *There are 4 teachers $t_1, \ldots, t_4$ and 4 schools $s_1, \ldots, s_4$ with one seat each. The initial matching μ_0 is such that, for $k = 1, \ldots, 4$, $\mu_0(t_k) = s_k$. Preferences are the following:*

$$\begin{array}{llllllllll}
\succ_{t_1}: & s_2 & s_3 & s_1 & s_4 & \qquad \succ_{s_1}: & t_4 & t_2 & t_1 & t_3 \\
\succ_{t_2}: & s_3 & s_1 & s_2 & s_4 & \qquad \succ_{s_2}: & t_4 & t_3 & t_1 & t_2 \\
\succ_{t_3}: & s_1 & s_2 & s_3 & s_4 & \qquad \succ_{s_3}: & t_4 & t_3 & t_2 & t_1 \\
\succ_{t_4}: & s_1 & s_2 & s_3 & s_4 & \qquad \succ_{s_4}: & t_4 & t_1 & t_2 & t_3
\end{array}$$

[22]To see that this algorithm converges in a finite number of steps, observe that, whenever we carry out a cycle, at least one teacher is strictly better off. Hence, in the worst case, one needs $(n-1)n$ steps for this algorithm to end. Because finding a cycle in a directed graph can be solved in polynomial time, the algorithm converges in polynomial time.

This example has a similar feature as Example 1: t_4 is the best teacher and is matched to the worst school. Thus, we know that, in that case, DA* coincides with the initial assignment. We have six blocking pairs: (t_1, s_2), (t_2, s_1), (t_3, s_2), and (t_4, s_k) for $k = 1, 2, 3$. The graph for BE is then the following:



The only cycle in this graph is $(t_1, s_1) \leftrightarrows (t_2, s_2)$, and it can be verified that, once implemented, there are no cycles left in the new matching. Thus, BE matching is given by

$$BE = \begin{pmatrix} t_1 & t_2 & t_3 & t_4 \\ s_2 & s_1 & s_3 & s_4 \end{pmatrix}$$

There are now 4 blocking pairs: (t_3, s_2) and (t_4, s_k) for $k = 1, 2, 3$, and teachers t_1 and t_2 as well as schools s_1 and s_2 are better off.

The next proposition provides a first natural result: the BE algorithm does characterize the set of 2-sided maximal matchings.

Proposition 5 *Fix a preference profile. The set of possible outcomes of the BE algorithm coincides with the set of two-sided maximal matchings.*

The proof uses the following technical lemma:

Lemma 1 *Assume that μ' 2-Pareto dominates μ. Starting from $\mu(0) = \mu$, there is a collection of disjoint cycles in the directed graph associated with the BE algorithm that, once carried out, yields matching μ'.*

Proof. The proof can be found in Appendix A. ∎

We are now in a position to complete the proof of Proposition 5.

Proof of Proposition 5. If μ is an outcome of BE, then it must be two-sided maximal. Indeed, if this were not the case, then by the above lemma, there would exist a cycle, in the directed graph associated with the BE algorithm starting from μ, which contradicts our assumption that μ is an outcome of the BE algorithm. Now, if μ is two-sided maximal, it 2-Pareto dominates the initial assignment μ_0. Hence, appealing again to the above lemma, there is a collection of disjoint cycles in the directed graph associated with the BE algorithm starting from μ_0 that, once carried out, yields the assignment μ. Clearly, once μ is achieved by the BE algorithm, there are no more cycles in the associated graph. ∎

Our result provides a simple and computationally easy procedure to find two-sided maximal matchings. The idea of implementing cycles to achieve two-sided maximality is natural. Erdil and Ergin (2017) already identified a similar algorithm to characterize 2-Pareto-efficient stable matchings in two-sided matching environments where agents may not have strict preferences.[23] Our motivations, however, are very different. They aim at Pareto-improving stable assignments while we are Pareto-improving on an initial assignment. As we will see in the next section, this difference has far reaching consequences once we deal with incentive issues. In particular, in their environment, it is known that no 2-Pareto-efficient stable matching mechanism is strategy-proof even if incentives are restricted to one side of the market as we assume (see, Erdil, 2014). Yet, in our environment, it is possible to find two-sided maximal matchings that are strategy-proof for teachers.

Note that the class of mechanisms defined by the BE algorithm is huge. Indeed, appealing to Proposition 3, this corresponds to the whole class of mechanisms that are both 2-PE and 2-IR. As we will see, by imposing the standard requirement of strategy-proofness, the class of mechanisms shrinks. The next section will identify strategy-proof selections of the BE algorithm.

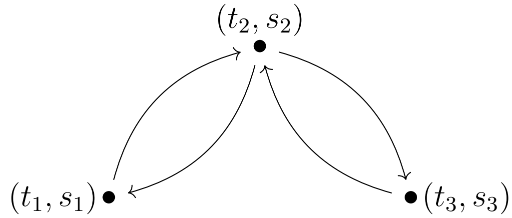
4.1.2 Incentives under Block Exchanges

First, recall that a mechanism φ is **strategy-proof** if, for each profile of preferences $\succ$ and teacher t, $\varphi_t(\succ) \succeq_t \varphi_t(\succ'_t, \succ_{-t})$ for any possible report $\succ'_t$ of teacher t.[24] The following example shows that some selections of the BE algorithm are not strategy-proof.

Example 3 *Consider an environment with three teachers $\{t_1, t_2, t_3\}$ and three schools $\{s_1, s_2, s_3\}$. For each $i = 1, 2, 3$, we assume that teacher t_i is initially assigned to school s_i. Teacher t_1's most preferred school is s_2, and he ranks his initial school s_1 second. Teacher t_2 ranks s_1 first, followed by s_3. Teacher t_3 ranks s_2 first and his initial assignment s_3 second. Finally, we assume that each teacher is ranked in last position by the school to which he is initially assigned. We obtain the following graph for the BE algorithm.*

[23] Technically, Erdil and Ergin (2017) start from a stable matching and, then, run the BE algorithm on top of this matching. As long as agents' preferences are not strict, the BE algorithm may exhibit cycles.

[24] Using standard notation, $\succ_{-t}$ denotes the vector of preference relations $(\succ_{t'})_{t' \neq t}$.



There are two possible cycles that overlap at (t_2, s_2). Consider a selection of the BE algorithm that picks cycle $(t_2, s_2) \leftrightarrows (t_3, s_3)$. In that case, the algorithm ends at the end of step 1, and teacher t_2 is eventually matched to school s_3, his second most preferred school. However, if teacher t_2 lies and claims that he ranks s_3 below his initial assignment, the directed graph associated with the BE algorithm has a single cycle $(t_1, s_1) \leftrightarrows (t_2, s_2)$. In that case, the unique selection of the BE algorithm assigns t_2 to his most preferred school s_1. Hence, t_2 has a profitable deviation under the selection of the BE algorithm considered here.

While this example is simple, an important objection for practical market design purposes is that the manipulation requires that teachers have fairly precise information regarding the preferences in the market (i.e., information about other teachers and schools). While this is true for many mechanisms, there is a sense in which – in some realistic instances – some selections of the BE (or associated) algorithm can be manipulated without requiring considerable information on both preferences in the market and the details of the mechanism. A simple example of a manipulation can be illustrated for BE$\circ$DA*. Indeed, under this mechanism, a teacher who would initially be assigned to a popular school that dislikes him can use the following strategy: report his most preferred school sincerely and then rank the school to which he is initially assigned in second place (even though this may not match his true preferences). If, under DA*, the teacher does not receive his first choice, he will certainly receive his initial assignment. Given that this school is popular and dislikes him, the teacher is likely to be part of a cycle involving his most preferred school under the BE algorithm. Hence, at an intuitive level, this mechanism can be manipulated by teachers who may only have coarse information on preferences in the market.

4.1.3 Strategy-Proof Selection of the Block-Exchange Mechanism

We define a class of mechanisms that are selections of the BE algorithm and are strategy-proof. Before defining the mechanism, we need an additional piece of notation. Given a matching μ, a set of teachers T', a set of school $S' \subseteq S$ and a teacher t, we let $\text{Opp}(t, \mu, T', S') := \{s \in S' \,|t \succeq_s t' \text{ for some } t' \in \mu(s) \cap T'\}$ be the opportunity set of teacher t within schools in S'. Note that for each teacher t, if $\mu_0(t) \in S'$ and $t \in T'$, then $\text{Opp}(t, \mu_0, T', S') \neq \emptyset$ since $\mu_0(t) \in \text{Opp}(t, \mu_0, T', S')$.

Now, for each school $s \in S$, fix an ordering over teachers $f_s : \{1, \ldots, |T|\} \to T$. We denote $f = (f_s)_{s \in S}$ the collection of the orderings, sometimes referred as a collection, one for each school. f is the index for our class of mechanisms.

We define the **Teacher Optimal Block Exchange** (TO-BE) algorithm as follows:

- **Step** 0 : Set $\mu(0) = \mu_0$, $T(0) := T$ and $S(0) := S$.
- **Step** $k \geq 1$: Given $T(k-1)$ and $S(k-1)$, let the teachers in $T(k-1)$ and their assignments stand for the vertices of a directed graph where, for each pair of nodes (t, s) and (t', s'), there is an edge $(t, s) \longrightarrow (t', s')$ if and only if:

  1. teacher t ranks school s' first in his opportunity set $\text{Opp}(t, \mu(k-1), T(k-1), S(k-1))$
  2. teacher t' has a lower priority than teacher t at school s'
  3. teacher t' has the lowest ordering according to $f_{s'}$ among all teachers in school s' who have a lower priority than t at s' (i.e., $f_{s'}(t') \leq f_{s'}(t'')$ for all t'' such that $\mu(k-1)(t'') = s'$ and $t \succeq_{s'} t''$)

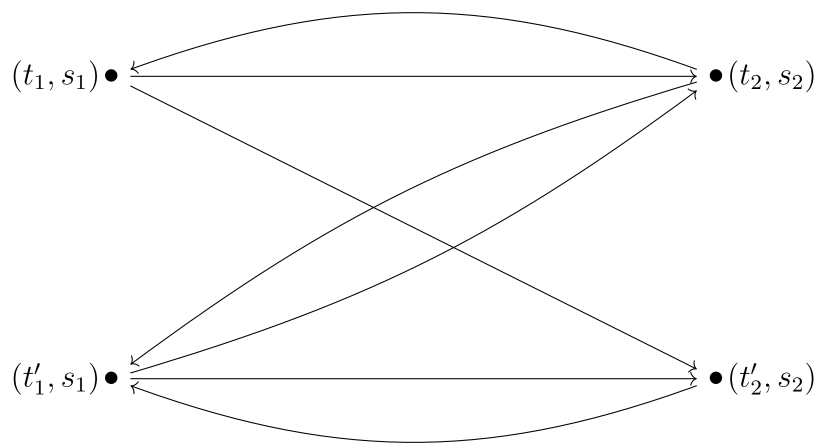
  The obtained directed graph has out-degree one and, as such, at least one cycle; cycles are pairwise disjoint. For each edge $(t, s) \longrightarrow (t', s')$ in a cycle, assign teacher t to school s'. Let $\mu(k)$ be the assignment obtained and $T(k)$ be the set of teachers who are not part of any cycle at the current step. The number of seats of each school is reduced consistently, and we let $S(k)$ be the set of schools with positive remaining capacities. If $T(k)$ is empty, then set $\mu(k)$ as the outcome of the algorithm. Otherwise, go to step $k + 1$.

The following example illustrates how the TO-BE algorithm works.

Example 4 *There are 4 teachers and 2 schools: t_1, t_1' are initially assigned to s_1 while t_2, t_2' are initially assigned to s_2. Preferences are as follows:*

$$\begin{array}{lll} \succ_{t_1}: & s_2 & s_1 \\ \succ_{t_1'}: & s_2 & s_1 \\ \succ_{t_2}: & s_1 & s_2 \\ \succ_{t_2'}: & s_1 & s_2 \end{array} \qquad \begin{array}{lllll} \succ_{s_1}: & t_2 & t_1 & t_2' & t_1' \\ \succ_{s_2}: & t_1 & t_1' & t_2 & t_2' \end{array}$$

With this example, the graph of BE starting at the initial matching is:



Consider the two following cases for a collection of orderings f:

1. *$f_{s_1}(t_1) < f_{s_1}(t'_1)$ and $f_{s_2}(t_2) < f_{s_2}(t'_2)$: then, at the first step of TO-BE$_f$, the implemented cycle is $(t_1, s_1) \leftrightarrows (t_2, s_2)$ and, at the second step, it is $(t'_1, s_1) \leftrightarrows (t'_2, s_2)$. In the final matching, every teacher is assigned his top-ranked school.*

2. *$f_{s_1}(t'_1) < f_{s_1}(t_1)$ and $f_{s_2}(t_2) < f_{s_2}(t'_2)$: then, at the first step of TO-BE$_f$, the implemented cycle is $(t'_1, s_1) \leftrightarrows (t_2, s_2)$. Now, at the second step, since $t_1 \succ_{s_1} t'_2$, there is no cycle left, the algorithm stops, and t_1 and t'_2 stay at their initial school.*

The TO-BE algorithm has some similarities with the (Shapley-Scarf) TTC mechanism. We specify the connection in Appendix B.

Each mechanism in the above-defined class is indexed by its collection of orderings, f, and is denoted TO-BE$_f$. We sometimes omit the collection and simply note TO-BE when there is no risk of confusion. In a one-to-one environment, orderings are irrelevant since there is only one teacher per school.

The next theorem presents two of the main characteristics of the TO-BE mechanism.

Theorem 1 *Fix a collection f. TO-BE$_f$ is strategy-proof and is a selection of the BE algorithm.*

Proof. The proof can be found in Appendix C. ∎

The above result is intuitive given the proximity between TO-BE and TTC. Indeed, let us think of teacher-school pairs involving a school and a teacher initially assigned that school (i.e., of the form $(t, \mu_0(t))$) and apply the usual argument. All teachers matched in the first step of TO-BE are getting their most preferred teacher-school pair (clearly, they are indifferent between two pairs involving the same school) within all pairs $(t, \mu_0(t))$ for which these teachers have a higher priority than t at $\mu_0(t)$. Hence, the only way for teachers matched in the first step to get better-off through a deviation is to get matched to a pair $(t, \mu_0(t))$ for which these teachers would have a lower priority than t at $\mu_0(t)$. However, this is clearly impossible by definition of TO-BE. Hence, these agents matched in the first step have no incentives to misreport their preferences. Under our assumption that the ordering f does not depend on teachers' preferences, by deviating, agents matched in the second step of TO-BE cannot get a seat of a teacher matched in the first step. Then, the argument is similar for these agents matched in the second step and we can proceed inductively.

We show next that if the collection of orderings f is chosen appropriately, TO-BE is optimal for teacher in a well-defined sense. This justifies our terminology for the strategy-proof selection of BE: Teacher-Optimal Block Exchange. We say that a selection φ of the BE algorithm is teacher-optimal if there is no 2-IR matching that 1-Pareto dominates φ.[25] Let $f^* := (f^*_s)_{s\in S}$ be the collection of orderings under which for each school $s \in S$ and each pair of teachers $t, t' \in T$, $f^*_s(t') < f^*_s(t) \Leftrightarrow t' \succ_s t$, i.e. the orderings of the schools follow their priorities. Quite intuitively, the ordering f can match schools' priorities, but it does not have to. f is used to break ties between teachers of a given school and therefore determines which teacher might be pointed at first in an exchange cycle. As such, f might be used as an instrument to achieve certain policy goals, e.g. retain some teachers. For instance, in the first case of Example 4, the schools' orderings match the schools' preferences, and TO-BE produces the teacher-optimal matching. In the second case, however, the schools' orderings differ from the schools' preferences, and TO-BE does not produce the teacher-optimal matching. In that case, however, both schools prefer the resulting assignment compared to the teacher-optimal matching.

We show that, when schools' orderings follow schools' priorities, i.e. the collection is f^*_s, TO-BE$_{f^*}$ is a teacher-optimal mechanism. We further show that TO-BE is not teacher-optimal if a different collection of orderings is used.

Theorem 2 *Let φ be a 2-IR mechanism. TO-BE$_{f^*}$ is not 1-Pareto dominated by φ. Moreover, for any other collection $f \neq f^*$, TO-BE$_f$ is 1-Pareto dominated by some alternative 2-IR mechanism.*

Proof. The proof can be found in Appendix D. ∎

Corollary 1 *TO-BE$_{f^*}$ is a teacher-optimal selection of BE. TO-BE$_f$ with $f \neq f^*$ is not a teacher-optimal selection of BE. In a one-to-one environment, TO-BE$_f$ is a teacher-optimal selection of BE for any f.*[26]

[25] Note that all the results presented above would also go through with a collection $(f_{(t,s)})_{(t,s)\in T\times S}$ of orderings over the teachers which are teacher-school specific. The class of strategy-proof selections of BE would be larger. However, as Theorem 2 will show below, only one collection, that is school-specific, is teacher-optimal.

[26] Recall that, in the one-to-one environment, since there is only one teacher per school, the ordering f is irrelevant.

4.1.4 Characterizations

The class of TO-BE mechanisms we introduce fully characterizes the strategy-proof mechanisms in two cases. First, in a many-to-one environment, we obtain a characterization when each teacher finds a single school acceptable beyond his initial assignment. While this is a very special preference structure, it can be satisfied in several natural environments. In particular, as we will further explain, our field application is close to satisfying this assumption. Second, in a one-to-one setting, we prove that this class of strategy-proof mechanisms reduces to a singleton and coincides with BE's unique strategy-proof selection.[27]

Let us denote by $\mathcal{P}$ the restricted domain of preferences under which each teacher finds acceptable at most one school beyond his initial assignment. A mechanism in this context is a mapping from $\mathcal{P}$ to matchings. In the sequel, we consider an algorithm that gives the outcome set that our collections of TO-BE can achieve. The algorithm follows the same steps as TO-BE but does not refer to any collection f. Formally, along the steps, (t, s) is allowed to point to (t', s') if and only if t ranks s' first in his opportunity set and t' has a lower priority than t at school s'. At any step, cycles may intersect, so that the outcome of this algorithm is not uniquely defined. Here again, we do not specify how the algorithm selects cycles in the graph. Hence, we consider a correspondence TO-BE $:\succ \rightrightarrows \mu$ where TO-BE$(\succ)$ stands for the set of all possible outcomes of this algorithm. A selection of the TO-BE algorithm is then a mapping $\varphi : \succ \mapsto \mu$ s.t. $\varphi(\succ) \in$ TO-BE$(\succ)$.

Theorem 3 *In the restricted domain $\mathcal{P}$, the set of two-sided maximal and strategy-proof mechanisms coincides with all selections of TO-BE.*[28]

Proof. The proof can be found in Appendix E. ∎

The intuition behind this result is simple. In essence, BE lets teachers point to schools with which they can form a blocking pair. TO-BE, however, lets teachers point to their favorite school among those with which they can block. Hence, when each teacher ranks a single school as acceptable, the two mechanisms are very similar. The domain restriction in the above statement is obviously strong, but in the teacher labor market presented in Section 5, teachers are assigned to regions, and most teachers rank only one region acceptable beyond the one to which they are currently assigned (see footnote 47). This result is therefore particularly relevant to the type of field experiment we consider.

We now present characterization results in a one-to-one environment wherein each school is initially assigned one teacher and there is no teacher without an initial school. Note that, in the previous many-to-one setting, the collection of orderings f was used in the procedure leading to preferences $\succ'$ over teacher-school pairs. The orderings were used to define the ranking of pairs (t', s') and (t'', s'), i.e., pairs with two different teachers being assigned the

[27] In a general many-to-one setting, one can exhibit a selection of BE that is strategy-proof but that is not in the class of TO-BE mechanisms.

[28] Note that two-sided maximality implies 1-IR which, under domain restriction $\mathcal{P}$, implies strategy-proofness. Hence, all selections of BE are strategy-proof under domain restriction $\mathcal{P}$.

same school. In a one-to-one framework, such a case does not arise since there is only one seat per school and so the collection f is not needed. The set of TO-BE algorithms is a singleton. We refer to this unique algorithm as simply TO-BE. Theorem 4 below shows that in this setting, TO-BE is the unique selection of BE that is strategy-proof.

Theorem 4 *In a one-to-one environment, TO-BE is the unique selection of the BE algorithm that is strategy-proof.*

Proof. The proof can be found in Appendix F. ∎

This result shows that the teacher assignment problem is structurally similar to the college admission problem. Indeed, in the college admission problem, imposing two-sided efficiency and stability yields a large set of stable mechanisms. Some of these mechanisms favor students whilst others favor colleges. Our characterization of two-sided maximal matchings is similar. We end up with a plethora of possible mechanisms, some favoring teachers and others favoring schools. In the college admission problem, imposing one-sided strategy-proofness produces a unique mechanism is obtained: the stable mechanism that favors students (i.e. DA). Similarly, Theorem 4 shows that, in the one-to-one teacher assignment problem, imposing the same incentive constraints generates a unique mechanism: TO-BE, which favors teachers. While the structure is very similar, the two mechanisms (DA and TO-BE) are very different, as are the underlying arguments.

Before closing this section, we discuss our approach relative to that of Ma (1994) and Dur and Unver (2017). Ma shows that in the Shapley-Scarf economy, TTC is the unique mechanism that is 1-IR, 1-PE and strategy-proof. Intuitively, in a one-to-one setting, Theorem 4 applies to richer environments in which schools have non-trivial preferences that are taken into account when determining welfare. This suggests that Theorem 4 is a generalization of Ma's. Indeed, to see this, note that in the specific situation in which each school ranks its initial assignment at the bottom of its ranking, TO-BE and TTC coincide. In this context, 1-IR and 2-IR are obviously equivalent. In addition, since 1-PE implies 2-PE, the class of mechanisms considered by Ma is a subset of the BE algorithm's selections. Applying Theorem 4 to these selections yields Ma's result. While our argument builds upon that of Ma's, there are a number of crucial differences. As mentioned above, even in the very specific environment in which each school ranks its initial assignment at the bottom of its preference relation, the BE algorithm contains many other mechanisms that include, in particular, all those that are 2-PE but not 1-PE and all 1-PE mechanisms that are sensitive to schools' preferences.[29] In addition, our result applies to settings in which schools' preferences are arbitrary and, thus, to many other types of mechanisms that are not well defined in Ma's environment.

Our paper is also closely related to Dur and Unver (2017), who study two-sided matchings via balanced exchanges, and use tuition and worker exchanges as applications. They propose

[29] That is, 1-PE mechanisms that select two different matchings for two different preference profiles in which teachers' preferences remain unchanged.

an algorithm, called the Two-Sided Top Trading Cycle (2S-TTC), that ensures that imports and exports are balanced. Further, they prove that 2S-TTC is the only mechanism that is balanced-efficient, worker strategy-proof, acceptable, individually rational, and that respects internal priorities. Although our paper pertains to the same two-sided environment with initial assignment, two key features differentiate our analysis. First, the main focus of our paper is the conflict between efficiency and fairness. We show that DA* can be improved upon in these two dimensions, while our alternative mechanism cannot. This focus of our paper is thus independent from Dur and Unver's main purposes. Second, we differentiate ourselves from Dur and Unver (2017)'s setting by taking into account a richer set of schools' priorities. To capture the specificities of worker-exchange programs, Dur and Unver (2017) make assumptions about the preferences of workers and firms. They notably assume that firms do not have strong preferences over acceptable workers, who are all equally desirable for firms. This assumption of coarse preferences over incoming agents is certainly plausible in the environment of temporary worker exchange, as the cost of not being matched with the best candidate is relatively limited. However, when assignments are permanent, firms (or schools) are likely to have finer preferences over applicants. Schools know that teachers' characteristics can vary widely, notably in terms of years of experience, experience teaching in disadvantaged schools, family situation, and so on. These characteristics are all used to define a teacher's priority in a school. Unlike Dur and Unver (2017), we account for schools' finer preferences over incoming teachers. In our context, this is particularly important when the distribution of experienced teachers differs across schools. Schools may want to maintain a balance in their teachers' characteristics and experience. If an experienced teacher wishes to leave, the school may want to replace her with an equally experienced teacher.

We refer to Dur and Unver (2017)'s coarse preferences over acceptable and non-acceptable workers as *group preferences*.[30] In practice, *non-grouped preferences* arise quite frequently. In our dataset, for the ten largest disciplines, 53.2% of the applicants have an "intermediate priority", meaning their priority is strictly higher than internal teachers' lowest priority and strictly lower than internal teachers' highest priority. On average, 91.3% of the regions receive at least one application with intermediate priority.[31]

Incorporating schools' fine preferences has two important consequences. First, with finer preferences, 2S-TTC is not two-sided maximal, and this mechanism could create new blocking pairs due to the possibility to recruit teachers who have lower priority than those leaving. Second, the characterization result in Dur and Unver (2017) relies on the axiom of respect of internal priorities (see Dur and Unver (2017) for a formal definition). In our environment, as we show in Appendix S.4, no mechanism is two-sided maximal, strategy-proof, and respects internal priorities; accordingly, our two approaches radically differ in a many-to-one setting.

[30]Formally, for a school s, its preference $\succ_s$ is a group preference if for any teacher $t' \notin \mu_0(s)$, either: i) $\forall t \in \mu_0(s)$, $t' \succ_s t$ or ii) $\forall t \in \mu_0(s)$, $t \succ_s t'$.

[31]To compute this statistic, for every discipline-by-region combination, we have defined the minimum and the maximum of the internal teachers' priorities. Then, for every applicant teacher, we define her priority as "intermediate" if it is strictly higher than the minimum and strictly lower than the maximum.

4.2 One-sided maximality

We now turn to the characterization of one-sided maximality. As we did for two-sided maximality, we introduce a class of mechanisms with possible outcomes spanning the whole set of one-sided maximal matchings. With two-sided maximality, the underlying criteria targeted by the designer are teacher welfare, school welfare, and the set of blocking pairs. In contrast, with one-sided maximality, the designer only targets teacher welfare and the set of blocking pairs. The basic idea behind the mechanism described in this section is as follows: under the BE algorithm, two teachers can exchange their assignments if and only if they justifiably envy each other. However, one can imagine a pair of teachers t and t' who each desire the other's school – say s and s', respectively – and, while school s does not necessarily rank t' above t, it does rank first t' *among the individuals who desire* s.[32] Similarly, if s' ranks t first among the individuals who desire s', then an exchange between t and t' increases the teachers' welfare *and* shrinks the set of blocking pairs. Hence, based on a similar idea, we will weaken the definition of the pointing behavior in the directed graph defined in BE in such a way that – although schools may become worse off – both teachers' welfare increases and the set of blocking pairs shrinks each time we carry out a cycle. The following algorithm, named one-sided BE (1S-BE for short), accomplishes this weakening, and Theorem 6 below shows how this is the best weakening one can hope to achieve.

- **Step** 0 : set $\mu(0) := \mu_0$.

- **Step** $k \geq 1$: Given $\mu(k-1)$, let the teachers and their assignments stand for the vertices of a directed graph in which, for each pair of nodes (t, s) and (t', s'), there is an edge $(t, s) \longrightarrow (t', s')$ if and only if either (1) teacher t has justified envy toward t' at s' or (2) t desires s' and t is ranked higher by s' than each teacher who also desires s' and does not block with s'.[33, 34] If there is no cycle, then set $\mu(k-1)$ as the outcome of the algorithm. Otherwise, select a cycle in this directed graph. For each edge $(t, s) \longrightarrow (t', s')$ in the cycle, assign teacher t to school s'. Let $\mu(k)$ be the matching so obtained. Go to step $k + 1$.

Here, again, it is easy to verify that this algorithm converges in (finite and) polynomial time. Similar to our process for the BE algorithm, we do not specify how the algorithm should select the cycle of the directed graph, and thus this algorithm defines a class of mechanisms.

[32] Henceforth, given a matching μ, we say that t desires s if $s \succ_t \mu(t)$.

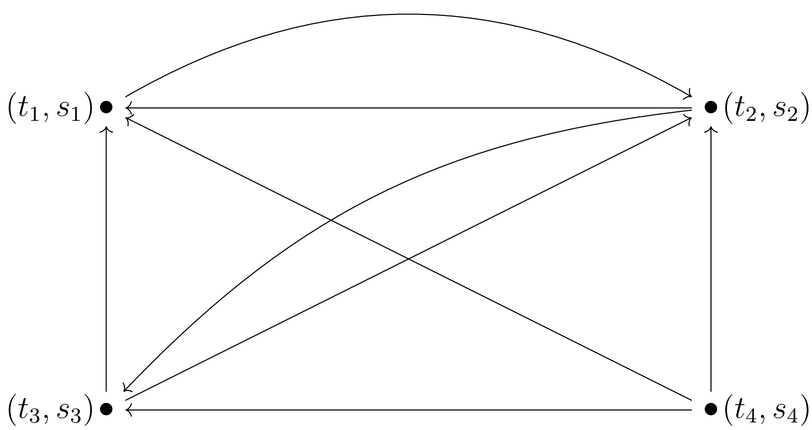
[33] Note that, here, teacher t may block with s' under condition (2). Thus, it is easy to see that, if (1) is satisfied, then (2) is also satisfied. Hence, one could simplify the definition and suppress condition (1). We keep this definition just to have a parallel with the definition of BE.

[34] As already mentioned, Dur et al. (2015) independently define a similar algorithm. For each school s they fix a set of teachers $C(s)$ under which, if $t \in C(s)$ then one can violate the priority of teacher t at school s. They define an algorithm which, for teachers, Pareto-improves on DA while respecting priorities of teachers outside $C(s)$ for each school s. One can show that their algorithm is equivalent to ours when the starting assignment is our initial assignment and $C(s) := \{t' \in T : \exists t \in \mu_0(s) \text{ s.t. } t' \succ_s t\}$ so that the sets can vary with the initial matching and the priorities of the schools.

Each mechanism in this class is a selection from the correspondence between preference profiles and matchings that corresponds to the whole set of possible outcomes that can be achieved by the 1S-BE algorithm.

By construction, starting from $\mu(k-1)$, the directed graph defined above is a supergraph of the directed graph that would have been built under the BE algorithm. Hence, there will be more cycles in our graph and more possibilities for improving teachers' welfare and shrinking the set of blocking pairs. This reflects the fact that we dropped the constraint that schools' welfare must increase along the algorithm, so that more can be achieved in terms of teachers' welfare and the set of blocking pairs. This is illustrated in the following example.

Example 5 *Consider the same market as in Example 2. The graph of 1S-BE contains the edges of the graph of BE but now has two new additional edges. Indeed, t_1 and t_2 both desire s_3 but do not block with it under μ_0, and t_2 is preferred to t_1 at s_3; thus, the node (t_2, s_2) can now point to (t_3, s_3). Since t_3 is the only one who desires s_1 and does not block with it, (t_3, s_3) can point to (t_1, s_1). Therefore, the graph of 1S-BE is as follows:*



Note that now there are two additional cycles: $(t_1, s_1) \to (t_2, s_2) \to (t_3, s_3) \to (t_1, s_1)$ and $(t_1, s_1) \leftrightarrows (t_2, s_2)$. Having implemented the first cycle, it can be verified that there are no cycles left, and thus, the matching given by 1S-BE is[35]

$$\begin{pmatrix} t_1 & t_2 & t_3 & t_4 \\ s_2 & s_3 & s_1 & s_4 \end{pmatrix}$$

Note that there are now only three blocking pairs: (t_4, s_k) for $k = 1, 2, 3$.

Following the notions introduced for the BE algorithm, we will note 1S-BE$\circ\varphi$ for the composition of BE and of a mechanism φ. An outcome of such a (modified) 1S-BE algorithm

[35] Note that even if one wished to select one of the two other cycles, another cycle would lead to the same matching.

selects matchings that dominate that of φ both in terms of teacher welfare and the set of blocking pairs (but not necessarily in terms of school welfare); i.e., all teachers are weakly better off, and the set of blocking pairs is a subset of that of φ. Again, in the sequel, we we will be particularly interested in starting the 1S-BE algorithm from the matching given by DA*, i.e., 1S-BE$\circ$DA*.

We now turn to our characterization result. We note that, while the argument in the proof of Proposition 5 is simple, the proof of the characterization result below is non-trivial.

Proposition 6 *Fix a preference profile. The set of possible outcomes of the 1S-BE algorithm coincides with the set of one-sided maximal matchings.*

PROOF. The proof is relegated to Appendix G. ∎

This result also provides a computationally simple procedure to find one-sided maximal matchings. As for the BE algorithm, we can easily construct selections of the 1S-BE algorithm that are not strategy-proof. Although there are strategy-proof selections of BE, there is no strategy-proof selection for the 1S-BE algorithm.

Theorem 5 *There is no strategy-proof selection of the 1S-BE algorithm.*

PROOF. The proof can be found in Appendix H. ∎

This result highlights an important difference between the classes of two-sided and one-sided maximal mechanisms. In contrast to the graph of BE, 1S-BE can have an edge $(t, s) \longrightarrow (t', s')$ if t desires s' and t is ranked first by s' among teachers who both desire s' and do not block with s'. Because of this condition, a teacher can modify the pointing behavior of others: indeed, if t is ranked first by s' among teachers who both desire s' and do not block with s', then teacher t can change other teachers' set of outgoing edges depending on whether he/she claims to desire s'. The argument for Theorem 5, relies on this additional feature. Consider that, for each possible cycle selection under the 1S-BE algorithm, one teacher can profitably misreport his/her preferences. Two manipulations are used in that case: one is basic and consists of ranking as acceptable an unacceptable school in order to be able, once matched with it, to exchange it for a better one. However, for some cycle selection, another manipulation is needed whereby a teacher ranks as unacceptable an acceptable school in order to expand other teachers' sets of outgoing edges. Again, this new type of manipulation is central to the argument in Theorem 5 and is not available under the BE algorithm.

Lastly, the 1S-BE algorithm shares some similarities – and can indeed be seen as a generalization of – the stable improvement cycle (SIC) algorithm defined by Erdil and Ergin (2008). The SIC algorithm is designed to improve stable outcomes whenever an outcome is not teacher-optimal, as is the case, for instance, with the outcome of the teacher-proposing DA when schools have weak preferences. Starting from a stable outcome, SIC and 1S-BE are the same. However, our mechanism extends the SIC algorithm's properties to cases in which the starting assignment is arbitrary. To illustrate why this is true and why the SIC algorithm does not suit our purposes, consider one of our initial motivations, which is to improve the outcome of DA*. Although both BE$\circ$DA* and 1S-BE$\circ$DA* successfully improve that outcome, the SIC algorithm (starting from the outcome of DA*) does not. Given the individual rationality of DA*, no teacher desires his initial assignment under the matching achieved by DA*. Therefore, the pointing behavior associated with SIC (starting from DA*) remains unchanged if we use the modified schools' preferences used to run DA* as opposed to the true schools' preferences. Under the modified preferences, by definition, DA* yields the teacher-optimal stable matching. Hence, there cannot be any cycle in the graph associated with SIC (again, starting from DA*).

4.3 Large Markets

Thus far, we have provided a stylized example in which DA* performs poorly with regard to the set of teachers moving from their initial position. This lack of movement indicates that this algorithm can be improved in terms teachers' welfare, schools' welfare and the set of blocking pairs. We have also presented a whole class of mechanisms – characterized by the BE algorithm – that does not suffer from such flaws. While this seems to be an improvement over DA*, it remains quite weak. As mentioned above, these results essentially show that DA* is not on the Pareto frontier, while our mechanisms are. These theoretical findings raise a new set of questions concerning both the magnitude of DA*'s underperformance and the performance of the BE algorithm's different selections. In this section, we answer these questions by adopting a large-market approach that allows us to quantify some aspects of the mechanisms' performance when the market grows.

More specifically, this section will answer three questions. First, since lack of movement is a key weakness of DA*, we ask, *Is there more movement under all selections of the BE algorithm compared to DA*?* Second, while all BE algorithm selections are two-sided maximal, as noted in Proposition 3, how do they compare in terms of teacher and school welfare? In particular, we ask, *Based on standard welfare criteria, is there a best selection of BE?* As we will show, there is a best BE selection in terms of welfare on both sides of the market. Third, from our analysis of incentives, we identified natural candidate mechanisms – the TO-BE algorithms – and will next compare them to the best selection of BE. In other words, *Is there a cost of strategy-proofness?* The following large-market analysis helps answer these questions.

For simplicity, we assume we are in a one-to-one setting. However, all our results hold in a many-to-one setting as long as the number of school seat has an upper bound that does not grow with the number n of students or at least does not grow too quickly. We assume that there are K tiers for the schools. More precisely, there is a partition $\{S_k\}_{k=1}^{K}$ of S such that the utility of teacher t for school $s \in S_k$ $(k = 1, ..., K)$ is given by

$$U_t(s) = u_k + \xi_{ts}$$

where $\xi_{ts} \sim U_{[0,1]}$. We assume that $u_1 > u_2 > ... > u_K$. For each $k = 1, .., K$, we denote by x_k

the fraction of schools having common value u_k and further assume that $x_k > 0$.[36]

Distributing preferences in tiers facilitates positive correlation in teachers' preferences. Prior literature has highlighted the positive correlation in preferences. Indeed, by studying teachers' preferences for schools in the US, Boyd et al. (2013) find that teachers demonstrate preferences for schools that are suburban and have a smaller proportion of students in poverty.[37] In addition, although this structure is special, we believe the basic insights extend far beyond this class of distributions in which they were obtained.

For schools' preferences, we assume that

$$V_s(t) = \eta_{ts}$$

where $\eta_{ts} \sim U_{[0,1]}$. The additive separability structure of our utilities and the specific uniform distribution employed are not essential to our argument.[38] In addition, we could assume that school's preferences are drawn in a similar way as students' preferences (allowing tiers); in that case, our results would remain essentially the same. Schools' preferences are based only on an idiosyncratic shock in order to simplify the exposition.[39]

Finally, the initial assignment μ_0 is selected at random among all possible $n!$ matchings, where $n := |T| = |S|$. A random environment is thus characterized by the number of tiers, their size, and common values $[K, \{x_k\}_{k=1}^{K}, \{u_k\}_{k=1}^{K}]$. The maximum normalized sum of teachers' payoffs that can be achieved in this society is $\bar{U}_T := \sum_{k=1}^{K} x_k (u_k + 1)$, which is attained if all teachers are matched to schools with which they enjoy the highest possible idiosyncratic payoff. The maximum normalized sum of schools' payoffs that can be achieved in this society is $\bar{V}_S := 1$, which is attained if all schools are matched to teachers with which they enjoy the highest possible idiosyncratic payoff. Clearly, in our environment, in which preferences are drawn randomly, a mechanism can be seen as a random variable. In the sequel, we let $\varphi(t)$ be the random assignment that teacher t obtains under mechanism φ.

In general, our mechanisms will fail to achieve the maximum sum of utilities on either side. However, a meaningful question is how often this phenomenon occurs when the market increases in size. The following concepts will help to answer this question. We say that a

[36] Assuming full support for the distribution of the u_k does not change the results.

[37] In France, in our dataset, we also observe that some school regions are systematically preferred to others, as measured by the number of teachers ranking these regions first. This shows a clear pattern of tiers: whereas 43 and 57 teachers (out of 10,579) rank the regions of Amiens and Créteil first, respectively, more than 1,000 teachers rank the attractive regions of Paris, Bordeaux, or Rennes as their first choices. The differences observed are likely related to cross-regions differences in the proportion of students from disadvantaged social backgrounds and/or minority students.

[38] We essentially need utilities to be continuous and increasing in both components and the distribution of the idiosyncratic shocks to have full support in a compact interval in $\mathbb{R}$.

[39] More precisely, the only issue when introducing a richer class of schools' preferences is that asymptotic stability and individual rationality become incompatible. However, if we ignore asymptotic stability, all of our results can be extended when allowing the richer class of preferences.

mechanism φ **asymptotically maximizes movement** if, for any random environment,

$$\frac{|\{t \in T \,|\varphi(t) \neq \mu_0(t)\}|}{|T|} \xrightarrow{p} 1.$$

A mechanism φ is **asymptotically teacher-efficient** if, for any random environment,

$$\frac{1}{|T|}\sum_{t\in T} U_t(\varphi(t)) \xrightarrow{p} \bar{U}_T.$$

Similarly, φ is **asymptotically school-efficient** if, for any random environment,

$$\frac{1}{|S|}\sum_{s\in S} V_s(\varphi(s)) \xrightarrow{p} \bar{V}_S.$$

Finally, φ is **asymptotically stable** if, for any random environment and any $\varepsilon > 0$,

$$\frac{|\{(t,s) \in T \times S \,|U_t(s) > U(\varphi(t)) + \varepsilon \text{ and } V_s(t) > V(\varphi(t)) + \varepsilon\}|}{|T \times S|} \xrightarrow{p} 0.$$

The next three results provide some answers to the three questions posed at the beginning of this section. The proofs of these results are relegated to Appendix I (except for Theorem 6).

Theorem 6 *DA* does not maximize movement, and thus is not asymptotically teacher-efficient, asymptotically school-efficient, or asymptotically stable.*

The basic idea behind the above theorem is very similar to the underlying argument in Example 1. Indeed, consider a random environment with two tiers of schools (i.e., $K = 2$) where the second tier corresponds to "bad" schools, while the first corresponds to "good" schools. Formally, we assume that $u_1 > u_2 + 1$, so that, irrespective of the idiosyncratic shocks, a school in tier 1 is always preferred to a school in tier 2. The intuition for the result is as follows. Fix any teacher t initially assigned to a school in the first tier. With non-vanishing probability, if t applies to a school in tier 1 other than his/her initial assignment, some teacher in the second tier will be preferred by that school. Hence, teacher t will be replaced by that teacher. This simple argument implies that, among teachers initially assigned to schools in tier 1, the expected fraction of teachers staying at their initial assignments is bounded away from 0.

Specifically, for each $k = 1, 2$, let T_k denote the set of teachers who are initially assigned to a school in S_k. Consider any teacher $t \in T_1$. Let E_t be the event that for each school $s \in S_1$, there is a teacher $r \in T_2$ s.t. r is ranked above t (according to s's preferences). Note that, for a school s, the probability that t is ranked above each individual in T_2 is the probability that $\{t\} = \arg\max\{\eta_{ts}, \{\eta_{rs}\}_{r\in T_2}\}$. Since $\{\eta_{ts}, \{\eta_{rs}\}_{r\in T_2}\}$ is a collection of iid random variables, for each $r \in T_2$, by symmetry, the probability that the maximum is achieved by t must be the same as the probability that it is achieved by any $r \in T_2$. The probability of $\{t\} = \arg\max\{\eta_{ts}, \{\eta_{rs}\}_{r \in T_2}\}$ must therefore be $\frac{1}{1+|T_2|}$. We can now easily compute the probability of E_t:

$$\begin{aligned}
\Pr(E_t) &= \left(1 - \frac{1}{|T_2|+1}\right)^{|S_1|} = \left(\left(1 - \frac{1}{|T_2|+1}\right)^{|T_2|}\right)^{|S_1|/|T_2|} \\
&\to \left(\frac{1}{e}\right)^{x_1/x_2} \text{ as } n \to \infty.
\end{aligned}$$

Note that, using the same logic as in Example 1, whenever E_t occurs, t cannot move from his/her initial assignment. Indeed, if t applies to some school s, this must be to a school in S_1. By construction, however, each teacher $t \in T_2$ applies to each school in S_1. In particular, the teacher in T_2 being ranked above t by school s applies to s, showing that, eventually, t cannot be matched to s under DA*. Thus, the expected fraction of individuals in T_1 who do not move must be

$$\begin{aligned}
\frac{1}{|T_1|}\mathbb{E}\left[\sum_{t \in T_1} \mathbf{1}_{\{t \text{ does not move}\}}\right] &= \frac{1}{|T_1|}|T_1|\,\mathbb{E}\left[\mathbf{1}_{\{t \text{ does not move}\}}\right] \\
&= \Pr\{t \text{ does not move}\} \\
&\geq \Pr(E_t).
\end{aligned}$$

Thus, the lim inf of the expected fraction of teachers not moving is bounded away from 0. Note that the lower bound computed here can be improved. Indeed, for t not to move, one only needs that, for each school $s \in S_1$ *that t finds acceptable*, there is a teacher $r \in T_2$ s.t. r is ranked above t (according to s's true preferences). In general, simulations suggest that a much larger fraction of teachers are not moving. These simulations also show that the assumption we made above that $u_1 > u_2 + 1$ is not necessary and that the result seems to hold in much broader contexts.[40] Let us now think of the best possible outcome of the BE algorithm. While the way to implement this outcome may not be practical, we consider this a benchmark and want to compare it to what can typically be achieved by mechanisms that can be implemented easily, such as TO-BE.

Theorem 7 *Each selection of BE asymptotically maximizes movement. There is a selection of BE that is asymptotically teacher-efficient, asymptotically school-efficient, and asymptotically stable.*

The intuition for the first part of the result is basic. Assume, toward a contradiction, that the set of teachers not moving under some selection of BE is large. For each pair of teachers t and t' in that set, the probability that t blocks with the initial assignment (and hence the assignment under the given selection) of t' and that, vice versa, t' blocks with the initial assignment of t (and thus gains from the assignment under the given selection) is bounded

[40] Available upon request.

away from 0. Given our assumption that the set of teachers not moving is large, then, with high probability, there will be such a pair of teachers. In other words, there will be a cycle in the graph associated with BE when starting from the assignment given by the selection, that contradicts the definition of a selection.

As for the other part of the theorem, the intuition can be seen as follows: for a given tier k, for any school in S_k and any agent initially matched to such a school in S_k, the probability that they both enjoy high idiosyncratic payoffs when matched with one another is bounded away from 0. Thus, as the market grows, with a probability approaching 1, we appeal to Erdös-Rényi's result on the existence of a (perfect) matching within the set of individuals and schools. However, there are two difficulties here. First, one must ensure the individual rationality of the matching so obtained, which we do by restricting the set of teachers and schools to those having idiosyncratic payoffs for their initial match bounded away from the upper bound. Second, this scenario implicitly assumes that the designer has access to an agent's cardinal utilities. Yet, in this paper, we assume—as is usually the case in practice—that matching mechanisms map ordinal preferences into matchings, and a large part of the proof is devoted to addressing this issue.

As we have already noted, while the BE algorithm treats teachers and schools symmetrically, TO-BE favors teachers at the expense of schools; accordingly, TO-BE is asymptotically teacher-efficient. In addition, TO-BE ensures only that schools are assigned a teacher they weakly prefer over their initial assignment. Hence, each school's assignment under TO-BE is a random draw within the set of teachers it finds acceptable. Given the school's idiosyncratic payoff for its initial assignment $\eta_{\mu_0(s)s}$, the expected payoff for a school s under TO-BE (s) is $\mathbb{E}\left[\eta_{st}\,\middle|\eta_{st} \geq \eta_{\mu_0(s)s}\right] = \frac{1}{2}\left(1+\eta_{\mu_0(s)s}\right)$. Thus, the (unconditional) expected payoff of school s under TO-BE (s) is $\mathbb{E}\left[\frac{1}{2}\left(1+\eta_{\mu_0(s)s}\right)\right] = \frac{3}{4}$, TO-BE cannot therefore be asymptotically school-efficient.

Theorem 8 *TO-BE is asymptotically teacher-efficient. Under TO-BE, the expected payoff of a school is $\frac{3}{4}$, and thus, TO-BE is neither asymptotically school-efficient nor asymptotically stable.*

The heuristic to understand why TO-BE is asymptotically teacher-efficient is as follows: assume it is not. This implies that for some tier, there is a large set of teachers who are obtaining an idiosyncratic payoff bounded away from the upper bound. Now, for each pair of teachers t and t' in that set, intuitively, the probability that t blocks with the assignment of t' and that, vice versa, t' blocks with the assignment of t is bounded away from 0. Hence, given our assumption that the set of teachers not moving is large, intuitively, with high probability, there will be such a pair of teachers. Here again, then, there will be a cycle in the graph associated with BE, when starting from the assignment given by the selection, that contradicts the definition of a selection. While this is an intuitive way of describing the result, there is a difficulty here. If we fix the set of teachers who are obtaining an idiosyncratic payoff bounded away from the upper bound, there are implications for the distribution of preferences. Hence, there is a conditioning issue that intuition provided above does not take into account. This raises an important technical difficulty that we circumvent using random graph arguments in the spirit of those developed by Lee (2014) and, more specifically, Che and Tercieux (2015b).

From the above, we should expect several results from our data analysis. First, DA* should rarely be two-sided maximal, particularly in markets with large numbers of teachers. In addition, the BE algorithm and TO-BE should ensure more movement than DA* and perform better in terms of teachers' welfare. Our data analysis largely confirms these findings. We should also expect TO-BE to perform more poorly than the BE algorithm: TO-BE may exhibit a loss in terms of schools' welfare and blocking pairs relative to the BE algorithm. In terms of schools' welfare and the set of blocking pairs, it is not clear a priori how to compare TO-BE and DA*. Our data analysis will help further discriminate between these mechanisms.

5 Empirical Analysis

The aim of this section is to assess our theoretical findings using a dataset drawn from all public school teacher assignments in France. We provide a brief presentation of the dataset, then run counterfactual scenarios for our mechanisms and measure the extent of the improvements they may yield in terms of school and teacher welfare as well as fairness.

5.1 Data

We use several datasets related to teacher assignment in 2013. For both the first and the second phases of the assignment, these datasets contain four pieces of information: (1) teachers' reported preferences, (2) regions/schools' priorities, (3) each teacher's initial assignment (if any), and (4) regions/schools' vacant positions. This empirical study focuses on the first phase of the assignment primarily because teachers have incentives to report their preferences[41] sincerely, and the preferences we observe are more straightforward to interpret.[42] Several features of the assignment process support this position. First, since the mechanism in use is DA* with no limit on the number of regions teachers can rank, it is a dominant strategy for all agents to be truthful. The fact that strategy-proof mechanisms generate reliable preference data for guiding policy is a common argument in their favor (see, e.g., Abdulkadiroglu et al., 2009, Abdulkadiroglu et al., 2015a).[43] Second, since the ministry manages the assignment process in partnership with teachers' trade unions, teachers are very well informed about the entire process, and trade unions have never advised teachers to strategically rank the school

[41]Given the agents' assessments over schools they may obtain in the second phase, agents have well-defined preferences over regions.

[42]As discussed in Appendix S.5, preferences reported during the second phase of the assignment are more difficult to interpret because of both a binding constraint on the number of schools teachers can rank and the ability to rank larger geographic areas than a school (cities, for instance).

[43]This assumption is sometimes challenged. See, for instance, Fack et al. (2015).

regions.[44] To run our counterfactuals, then, we take agents' reported preferences in Phase 1 as their true preferences.

The sample of teachers used for the analysis takes into account two restrictions. First, the sample is restricted to the 49 subjects that contain more than 10 teachers. Second, to match our theoretical framework, all initially non-matched teachers (newly tenured) and all empty seats in regions are suppressed. Hence, the initial assignment corresponds to a market in which each teacher is initially assigned to a region, and each seat in each region is initially assigned a teacher. The final sample contains 10,579 teachers corresponding to 49 subjects ranging from 6 to 1,753 teachers. We end this section by providing two pieces of information on this market.

Fact 0 *(i) Under the regular DA mechanism, there are at least 1,325 teachers for whom individual rationality is violated; i.e., they are assigned to a region that they consider worse than their initial region.*[45] *(ii) The individually rational mechanism that maximizes movement allows 2,257 teachers to move from their initial assignment.*[46]

This shows that the regular DA mechanism is indeed not individually rational in this market, and the violation of individual rationality is quite strong. The second point shows that there is congestion on this market: if we focus only on individually rational matchings and attempt to ensure as much movement as possible, 21% of teachers will be able to move. We should bear in mind this upper bound when considering the performance of our algorithms and the scope of their improvement.[47]

[44]The mobility process is the main source of teacher unionization in France. Because competition exists among trade unions, they have high incentives to provide teachers with detailed information and tailored help throughout the process. In practice, trade unions help teachers identify the criteria they can use to compute their priorities, they negotiate the number of positions offered in each region with the ministry, and they validate the mobility project submitted by the ministry.

[45]In our dataset, because we only have each teacher's reported preferences up to his initial region, we do not know how teachers rank regions below their initial assignment. However, one can show that, when running DA on these truncated preferences, the number of unassigned teachers is a lower bound on the number of teachers for whom individual rationality is violated when running DA on the full preference lists.

[46]To find such an assignment, we build a bipartite graph with teachers on one side and schools on the other. We then consider the complete bipartite graph, where each edge will be associated with a weight. We assign weight ∞ to edges (t, s), where s is unacceptable to t (i.e., worse than his initial assignment). We assign weight 1 to the edge if t is initially matched to s. Finally, we assign weight 0 to all other edges (i.e., if t finds s strictly better than his initial assignment). The weight of a matching is defined as the sum of weights over all its edges. We use a standard algorithm to find a matching with minimal weight (see Kuhn, 1955 and Munkres, 1957). It is easily verified that such a matching maximizes movement among all individually rational matchings.

[47] The relatively small fraction of teachers able to move is explained primarily by the high proportion of teachers reporting short lists. Indeed, on average, teachers rank 1.64 regions and 75 % of teachers ask for only one region (beyond their initial region). Combined with correlation in preferences, this structurally restricts the possibility of movement in the market.

5.2 Results

The aim of the following empirical analysis is to test our theoretical results. We will therefore focus on three main dimensions: teachers' welfare, regions' welfare, and number of blocking pairs. Since BE and 1S-BE define a class of mechanisms, we randomly select outcomes within this class. For the TO-BE mechanisms parametrized by an ordering over teachers, we randomly select an ordering and thus an outcome in this class. We randomly draw selections for each mechanism ten times. In addition, there are indifferences in priorities of regions over teachers.[48] We use a single tie-breaking rule to break ties in regions' priorities. We randomly draw the tie-breaking rule ten times. The results reported in Tables 1 to 3 for BE, TO-BE, and 1S-BE correspond to average outcomes over one hundred draws – ten random selections of tie-breaking rules times ten random selections of outcomes. The results for DA* correspond to an average over ten iterations of tie-break only. The results for the BE algorithm and the 1S-BE are successively presented in the next section.[49]

5.2.1 Two-sided maximality: BE and TO-BE

How far is DA* from being two-sided maximal?

In the theoretical analysis, we noted that DA* can be improved in three key ways: teachers' welfare, regions' welfare, and fairness. In practice, a first simple way to test whether DA* is two-sided maximal is to run the BE algorithm starting from the matching obtained by DA* and then to observe whether the two matchings obtained with DA* and BE$\circ$DA* differ. If they differ, it means some cycles exist in the graph associated with the BE algorithm starting from DA*, and thus DA* is not two-sided efficient, which is a necessary condition for two-sided maximality. Fact 1 below illustrates that point in our data:

Fact 1 *DA* is not two-sided maximal in 33 out of 49 subjects. These subjects represent 95.9% of the teachers.*[50]

This first fact suggests that the theoretical phenomenon we highlight is not rare. Based on this observation, we now estimate how far DA* is from maximality in terms of the three

[48]Many young teachers use only one criterion – the number of years of experience – to compute their priorities, and thus, they have the same priority in a given region.

[49]As explained in Section 2, regions use multiple criteria to rank teachers (spousal reunification, disability, having a position in a disadvantaged or violent school, total seniority in teaching, seniority in the current school, time away from spouse and/or children, etc.). However, when running our alternative algorithms, we use only the seniority criteria (both total seniority in teaching and seniority in the current school) to determine a teacher's priority in his initial region. Indeed, the other criteria are supposed to help a teacher to leave his current region, so it would not make sense to use these criteria for the region to which he is currently assigned.

[50]Even if BE$\circ$DA* = DA*, it could be the case that DA* is not two-sided maximal. Indeed, in our definition of two-sided maximality, we require that schools must not be harmed relative to the initial allocation. Since DA* may harm schools, it can be two-sided Pareto efficient but still violate two-sided maximality. In 30 subjects, BE$\circ$DA* $\neq$ DA*, and in 3 subjects, BE$\circ$DA* = DA*, but DA* harms the welfare of at least one region relative to the initial assignment. Finally, we note that, if we restrict our attention to the 19 subjects with more than 100 teachers, in only one subject is DA* two-sided maximal.

criteria in which we are interested. Regarding teachers' welfare, Table 1 shows that, on average, BE∘DA* more than doubles the number of teachers who are assigned to a new region relative to DA*: 565 teachers move from their initial allocations under DA* versus 1,488 under BE∘DA*. The same Table also reports the cumulative distribution of the number of teachers who obtain school rank k. While we know from the theory that the distribution of the rank obtained by teachers under BE∘DA* first-order stochastically dominates this same distribution under DA*, the dominance is indeed significant.

There are several possible measures of regions' welfare. Though we focus on one natural approach, detailed below, we also test our results' robustness to the use of alternative approaches, which yield no significant differences in the results. Given a mechanism, we examine the improvement a region obtains (from the initial assignment) in terms of the number of positions improved. More precisely, given a region, we first take the initial assignment and sort it by decreasing order of priorities. We obtain a vector in which the first element/position is the teacher with the highest priority in that region at the initial assignment, the second element/position is the teacher with the second highest, and so forth. Call this vector $\mathbf{x}$. We perform the same operation for this region's assignment with the mechanism under study. Let us call this vector $\mathbf{y}$. Finally, we say that a position k is assigned a teacher with higher (resp., lower) priority if the kth element of vector $\mathbf{y}$ has a higher (resp., lower) priority than the kth element of vector $\mathbf{y}$. Based on this, we compute the percentage of net improvement in positions, i.e., the percentage of positions receiving a teacher with a higher priority minus the percentage of positions being assigned a teacher with a lower priority.

Table 3 reports, for the different mechanisms we run, the cumulative distribution of the percentage of net improvement in positions, i.e., for each percentage x, the proportion of regions having less than x percent of net improvement in positions. Again, we observe that the distribution under BE∘DA* first-order stochastically dominates this same distribution under DA*.

Finally, we compare the performance of DA* and BE∘DA* in terms of fairness. The first row of Table 2 reports that, on average, 2,496.5 teachers are not blocking under DA* and 3,799.4 are not blocking under BE∘DA*, which represents a 52.1% increase in the number of teachers who are not blocking with any region. More generally, we observe that fairness is significantly increased.[51]

Overall, these results show that DA* fails to be two-sided maximal in a large number of cases, and the scope of improvement seems to be very large. Running the BE algorithm from the assignment achieved by DA* would seem to address this issue; however, as mentioned in our theoretical analysis, this mechanism is prone to easy manipulations. Alternatively, we focus our attention on both the BE algorithm that is run directly from the initial assignment (this is referred to as BE(Init) in our tables and graphs) and its strategy-proof selection: the

[51] The number of teachers who are part of a blocking pair is quite high. This is intuitive since the number of teachers moving is low and many teachers stay at their initial allocation, thus possibly creating envy. This can be seen as the cost of imposing the individual rationality constraint.

TO-BE mechanism. In the next section, we evaluate the performance of these two mechanisms in terms of teachers' welfare, regions' welfare, and the number of blocking pairs.

Performance of BE and TO-BE

Before commenting on the results, it is worth briefly discussing the relevance of comparing BE and TO-BE to DA*. We should bear in mind that, for an arbitrary outcome of the BE mechanism, its set of blocking pairs may differ from that of DA*, and similarly, the outcome may not 2-Pareto dominate DA*. Nevertheless, the comparison remains interesting for two reasons. First, we know from the above results that DA* is far from being two-sided maximal, so BE and TO-BE – which are two-sided maximal – can be expected to perform much better. Second, our theoretical results (Theorems 6, 7 and 8) suggest that BE and TO-BE perform better than DA* in large markets.

We first focus our attention on the performance of BE and TO-BE in terms of teachers' welfare. Both mechanisms significantly improve the number of teachers moving relative to DA*: on average, 565 teachers obtain a new assignment under DA*, versus 1,461.5 under BE and 1,373 under TO-BE. In addition,

Fact 2 *The distribution of ranks obtained by teachers under TO-BE first-order stochastically dominates the distribution of BE, which dominates that of DA*.*

Note, however, that there is no 2-Pareto domination between the matchings: some teachers may prefer their assignment under DA* to the one that they obtain under BE or TO-BE.[52]

Regarding stability, BE and TO-BE also perform significantly better than DA*. Table 2 shows that the average number of teachers not being part of a blocking pair increases from 2,496.5 under DA* to 3,731.3 and 3,742.7 under BE and TO-BE, respectively. In addition,

Fact 3 *The distributions of the number of regions teachers can block with under DA*, BE and TO-BE can be ranked stochastically: DA* is dominated by BE, which is dominated by TO-BE.*[53]

Finally, comparing regions' welfare across mechanisms is particularly interesting, as we know that DA* can harm some regions, in contrast to BE and TO-BE. This is confirmed by Table 3. Indeed, under BE and TO-BE, no region has a position for which the teacher assigned to it has a lower priority than the teacher initially assigned to that position.

Fact 4 *The distributions of the percentage of net improvement in positions can be stochastically ordered: the distribution of DA* is dominated by that of TO-BE, which is dominated by that of BE.*

[52] On average, 258.5 teachers obtain a region they rank strictly higher under DA* than under BE (254.5 under TO-BE). Conversely, 1,152.7 teachers strictly prefer their assignment under BE to that under DA*, and the corresponding number is 1,096.7 under TO-BE.

[53] As discussed for teachers' welfare, note that the set of blocking pairs of each matching may differ. Some teachers may block with a region under BE or TO-BE but not under DA*.

TO-BE's lower performance, as compared to BE, in terms of regions' welfare accords with our theoretical predictions regarding the cost of the strategy-proofness imposed by TO-BE (Theorem 7 and 8).

As mentioned earlier, accounting for regions' preferences – through two-sided maximality – is a way to better consider policy-makers' objectives because priorities incorporate several welfare-relevant criteria. As described in Section 2, priorities are defined by a point system. Three main criteria are used to rank a teacher: total seniority in teaching (experience), spousal reunification, and whether the teacher has been teaching for several years in a disadvantaged school.[54] The analysis of regions' welfare presented above pools together all criteria, but we are also interested in the mechanisms' performance for each criterion taken separately. In a sense, the points given for each criterion reflect the policy-maker's trade-offs among the three criteria. For instance, we know that one of the main objectives is to allocate more experienced teachers to disadvantaged regions. But policy-makers would accept a slightly less experienced teacher in such a region if he/she ends up being closer to his/her spouse. This is exactly what the ranking reflects. The lower priority in terms of experience is compensated by a higher priority in terms of spousal reunification. In Section 5.2.3, we decompose the welfare analysis for each criterion, and take a closer look at teachers' distribution across regions under each mechanism.

Overall, these results suggest that BE and TO-BE perform much better than DA^* in terms of teachers' welfare, regions' welfare, and fairness. TO-BE's strong performance is of particular interest due to its incentive properties. These results provide evidence that, although two-sided maximality is a strong requirement, our mechanisms can generate large improvements and distributions that dominate those of DA^*.[55] The next section tests whether we can further improve upon DA^* by relaxing the constraint that no region should be harmed (relative to the initial assignment). To do so, we provide empirical evidence on the performance of 1S-BE, the one-sided maximal algorithm we defined in Section 4.2.

5.2.2 One-sided maximality: 1S-BE

As done previously for BE and TO-BE, we first aim to estimate how far DA^* is from being one-sided maximal. To do so, we compare the matching under DA^* to that under 1S-BE, which we run from DA^*. For a large number of subjects, we have seen that DA^* is not two-sided maximal; thus, it is not one-sided maximal. Because the constraint on regions' welfare is relaxed under 1S-BE relative to BE, the improvements we have found for $BE \circ DA^*$

[54] Additional criteria are used such as disability, being a replacement teacher, or applying for a position after some years off teaching. Fewer teachers use these criteria, so we choose to focus on the three main ones. Time away from the spouse and/or children is used to determine the points attributed for the spousal reunification criterion. Disadvantaged schools are classified as such by the administration. After having taught for five years in a disadvantaged school, teachers get additional points to move anywhere else.

[55] These results are all the more encouraging because they are obtained in a restrictive environment in which teachers rank a very limited number of regions. Even better results could be expected in environments in which agents have longer ranked lists.

in terms of teachers' welfare and blocking pairs can be seen as a lower bound on the potential improvements under 1S-BE∘DA*. Indeed, Table 1 reports that 1S-BE∘DA* yields a threefold increase in the number of teachers moving relative to DA* and increases this figure by 15% compared to BE∘DA*. This suggests that there is still significant potential for improvement when considering one-sided maximality.

Fact 5 *DA* is not one-sided maximal in 31 subjects, and 95.3% of the teachers belong to these subjects.*

In one subject, DA is two-sided maximal but not one-sided maximal.*

We now turn to the results on 1S-BE, starting from the initial allocation (referred to as 1S-BE(Init) in our tables and graphs) to compare its performance with that of the other mechanisms. Regarding teachers' welfare and fairness, the distributions of both the ranks obtained by teachers (Table 1) and the number of teachers blocking (Table 2) under 1S-BE stochastically dominate the distribution of all other algorithms mentioned previously: BE, TO-BE, and DA*.[56]

Finally, regions' welfare is the key difference between two- and one-sided maximality. Even if improvements in teachers' welfare and fairness are large with 1S-BE, we know that this comes at the expense of the regions' welfare.

Fact 6 *Under 1S-BE, in 4.9 % of the regions, the percentage of positions filled by a teacher with a lower priority is higher than the percentage of positions filled by a teacher with a higher priority. This is in contrast with BE or TO-BE, under which, by definition, regions cannot be assigned teachers with a lower priority relative to the initial assignment.*

5.2.3 Administration's objective

We motivated the two-sided efficiency notion by its better ability to fulfill the administration's objectives, as reflected by the three main criteria defining the priority system: i) experience in teaching, ii) spousal reunification, and iii) years of teaching in a disadvantaged school. For instance, under the two-sided efficiency notion, a reassignment of teachers that, ceteris paribus, decreases the number of experienced teachers in disadvantaged schools would not meet the administration's objective to better distribute experienced teachers across schools. We end this empirical section by looking more closely at these three criteria.

The first criterion gives more points to more experienced teachers. Obviously, it is not possible to increase teachers experience in all regions. However, as some regions are more disadvantaged than others, one objective is to control the share of inexperienced teachers in these regions. To define disadvantaged regions, we follow a classification used by the French ministry of education. The regions of Créteil and Versailles are classified as disadvantaged

[56]As explained above, some teachers may prefer their match under DA*; 195 teachers do so under 1S-BE, which is less than the corresponding figure under either BE or TO-BE.

regions because they have high shares of students enrolled in priority education and/or whose parents have no diploma. In addition, every year around 50% of the teachers who request reassignment come from one of these two regions. The upper part of Table 4 reports, for both disadvantaged and non-disadvantaged regions, the percentage of teachers with only one or two years of experience under the current algorithm and the difference under the alternative mechanisms we suggest.[57]

Under the current algorithm, 26.9% of the teachers have only one or two years of experience in non-disadvantaged regions. This rate goes up to 45.6% in disadvantaged regions. A first interesting result is that, under the TO-BE mechanism, this percentage of inexperienced teachers goes down in disadvantaged regions but goes up in non-disadvantaged regions. This fulfills the policy-maker objective to not increase the share of inexperienced teachers in deprived regions. It is also interesting to notice the difference between the TO-BE mechanism and 1S-BE. Given that 1S-BE does not require an exiting teacher to be replaced by a teacher whose priority is higher, we could expect this mechanism not to fulfill the administration's aim to not assign relatively inexperienced teachers to disadvantaged regions. This is what we see in Table 4. The reduction in the share of inexperienced teachers assigned to disadvantaged regions is smaller under 1S-BE than under TO-BE.

It is worth keeping in mind that most of the mobility gain happens in non-disadvantaged regions,[58] while mobility tends to either remain constant or decrease in disadvantaged regions.[59] In order to cancel out these important differences in mobility, we look at two additional statistics: the percentage of inexperienced teachers among the ones *entering* each region and the percentage of inexperienced teachers among the ones *leaving* each region. As expected, these differences are much starker. Among teachers who leave disadvantaged regions, the share of inexperienced teachers is significantly higher under TO-BE than under DA*. This is consistent with the fact that, under TO-BE, any teacher leaving a region must be replaced by a teacher with a higher priority. This makes it more difficult for experienced teachers to move but not inexperienced ones, so that the share of inexperienced teachers among exiting teachers increases. This higher priority rule under TO-BE also justifies why the share of inexperienced teachers among those who enter disadvantaged regions does not go up under TO-BE.

Finally, we look at performance for two additional criteria: spousal reunification and experience in disadvantaged schools. For these criteria, the objectives are to help teachers get closer to their spouse or leave a disadvantaged school. The lower part of Table 4 shows

[57] We distinguish between teachers having only one or two years of experience and more experienced teachers based on evidence that teachers in their first year tend not to perform as well as more experienced teachers (Chetty et al., 2014; Rockoff, 2004).

[58] In Example 1, the disadvantaged school s^* is initially assigned to teacher t^*. Since t^* has the highest priority in all schools, he/she prevents all other teachers from moving from their initial schools under DA* and himself stays in s^*. Under TO-BE, teacher t^* still stays in s^*, but the other teachers can exchange their positions.

[59] Mobility decreases in disadvantaged regions under the BE mechanism, and to a lesser extent under TO-BE, because of the requirement that any teacher leaving the region has to be replaced by a teacher with a higher priority. This requirement does not exist with DA*, and can prevent some teachers from leaving their region if no teacher with a higher priority can replace them.

that the BE, TO-BE, and 1S-BE mechanisms significantly increase the number of teachers moving closer to their spouses. TO-BE performs slightly better than 1S-BE.[60] On the other hand, fewer teachers from disadvantaged schools move under TO-BE and 1S-BE than under DA*.[61] This lower mobility is mainly due to the fact that TO-BE slightly reduces mobility for teachers in disadvantaged regions, and 48% of the teachers who wish to leave a disadvantaged school are located in the disadvantaged regions of Créteil and Versailles.

6 Concluding Remarks

In many countries, a central administration is in charge of assigning teachers to schools. In an attempt to ensure that every teacher is assigned to a school that he or she weakly prefers to his or her current one, several countries have adopted a modified version of the well-known deferred acceptance mechanism (DA) for teacher assignment. In this paper, we show that this mechanism fails to be fair and efficient for both teachers and schools. Ensuring that schools are not "harmed" by teacher reassignments is important as schools' priorities partly reflect a social objective, notably in terms of the experience of teachers assigned to deprived schools. To address the weakness of the modified version of DA, we characterize the class of mechanisms that cannot be improved upon in terms of both efficiency and fairness, and we identify the subclass of strategy-proof mechanisms. We further test and confirm the performance of these alternative mechanisms by showing that, when the market size grows, they perform much better in terms of utilitarian efficiency and fairness. Finally, we use a rich dataset on teachers' applications for transfers in France to measure the relevant gains. As our counterfactual analysis shows, the alternative mechanisms, compared to the modified version of DA, generate significant gains in efficiency and fairness. In particular, the number of teachers moving from their initial assignments more than doubles under our mechanism.

Efficiency vs. Fairness. In designing school choice allocation mechanisms, efficiency and fairness have received considerable attention, yet. Two goals are incompatible (see Roth, 1982, Abdulkadiroglu and Sonmez, 2003 and Abdulkadiroglu et al., 2009). Efficient matching mechanisms, such as the top trading cycle, attain efficiency but fail to be fair. Fair mechanisms, such as the DA algorithm do not guarantee efficiency. This trade-off between efficiency and fairness is well studied, with particular focus on how to attain one objective with the minimum possible sacrifice of the other.[62] The designer's task often boils down to a choice between two

[60] The number of teachers who move closer to their spouses increases from 251.8 under the current algorithm to 513.5 under TO-BE, 547.6 under BE, and 482.5 under 1S-BE.

[61] 110.3 teachers obtain a new assignment under DA* compared to 83.4 under TO-BE and 106.8 under 1S-BE.

[62] For instance, DA selects a fair matching that Pareto dominates all other fair mechanisms for the proposing side (Gale and Shapley, 1962), and the top trading cycle mechanism (Abdulkadiroglu and Sonmez, 2003), which allows agents to sequentially trade their priorities, can be considered efficient with minimal unfairness (Abdulkadiroglu et al., 2015b).

mechanisms: the student-proposing DA mechanism or the top trading cycle mechanism.[63] Yet our work shows that, in the teacher assignment problem, the individually rational version of DA can be improved in terms of both efficiency and fairness. In addition, we identify a class of mechanisms, closely related to the top trading cycle mechanism (TO-BE), as natural alternatives for a designer concerned with these two core notions. Our conclusion's striking contrast with the previous literature confirms that the teacher assignment problem is novel and presents important differences from previously studied contexts.

Vacant positions and newly recruited teachers. In this paper, in order to focus on improving the reallocation of initially assigned teachers, we omitted from our analyses recently graduated teachers (newcomers) and available seats at schools. A separate project complements our current approach by incorporating vacant positions and newly recruited teachers who do not have an initial assignment. France currently faces significant egress of teachers from the least attractive regions (which consequently receive many inexperienced teachers). Under TO-BE this flow is dramatically reduced, because a teacher leaving must be replaced by a teacher with higher priority/experience. As one would naturally expect, experienced teachers' demand for these regions is relatively low, making it more difficult for teachers to leave these regions. While decreasing egress mechanically increases the number of experienced teachers in these regions, it may have several negative outcomes. First, inability to leave may discourage experienced teachers from applying to these regions. Moreover, if newcomers are eventually trapped in such regions, the teaching profession may become less attractive, and the overall quality of teachers may decrease. Overall, we sought to remain agnostic on the degree to which experienced teachers should be retained in these regions. We therefore tailor the current mechanism to ensure that the outgoing flow can be targeted by the decision maker. One option is to leave the number of departing teachers unchanged (compared to DA*).[64] Even under such a conservative approach, the overall mobility of initially assigned teachers can still be increased by 44.9%.

Dynamic environment. One may argue that the teacher assignment problem is a fundamentally dynamic situation, as teachers change positions several times throughout their careers.[65] This is particularly important if one believes that incentives to truthfully report preferences can be affected by this dynamism. Pereyra (2013) defined a dynamic overlapping generation model wherein newly recruited teachers arrive, are assigned to schools, and can later ask for reassignments during a certain number of periods before exiting the market.[66] In this environment, assuming that all newly tenured teachers entering at a given date are less preferred by all schools to the tenured teachers who already have an assignment, he showed

[63] See also Che and Tercieux (2015a) for additional perspectives on this topic.

[64] Our conversation with agents from the Ministry of Education suggests that this is one of their most desired options.

[65] However, in our field experiment, teachers are unlikely to apply many times during their careers. Indeed, most teachers target either the region where their family lives or the region of their home town. Hence, once a desired region is obtained, they are unlikely to reapply in the short or medium term.

[66] In his setting, teachers' preferences are fixed over time while schools' preferences/priorites can evolve.

that DA* is dynamically strategy-proof; i.e., no teacher can ever misreport his/her preferences and obtain a better-school at the current or some later date. If the TO-BE algorithm is properly extended to account for newly recruited teachers and vacant positions, then, in the same setting as in Pereyra (2013), one can show that TO-BE is also dynamically strategy-proof.

Centralized vs. decentralized systems. This study has clear policy implications for countries using a centralized assignment system. However, our work also helps envision what the potential impact of transitioning from a decentralized to a centralized assignment system in other countries. More specifically, we show that adopting the modified version of DA, rather than one of the alternative mechanisms we suggest, would largely underestimate the performance of a centralized system (for instance, in terms of teacher mobility).

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Table 1: Teacher welfare under different mechanisms

Choice	Init	DA*	TO-BE	BE(Init)	BE(DA*)	1S-BE(Init)	1S-BE(DA*)
1	0	422.5 (2.4)	1163.0 (14.5)	1153.5 (17.9)	1169.9 (12.9)	1355.9 (13.5)	1347.8 (11.5)
2	7935	8084.2 (2.2)	8318.8 (9.1)	8287.2 (10.2)	8290.1 (6.8)	8386.1 (11.3)	8370.9 (7.8)
3	9125	9220.5 (0.5)	9361.5 (7.1)	9336.1 (9.9)	9341.7 (7.0)	9399.1 (8.4)	9390.8 (7.1)
4	9743	9796.1 (0.7)	9901.9 (6.2)	9882.7 (6.5)	9884.8 (5.8)	9929.0 (7.9)	9917.7 (6.2)
5	10038	10077.7 (0.5)	10150.3 (5.7)	10137.1 (5.3)	10140.1 (4.3)	10170.5 (5.6)	10162.7 (4.5)
6	10271	10297.0 (0.0)	10341.0 (4.7)	10328.7 (5.0)	10331.0 (3.7)	10354.8 (4.3)	10351.3 (3.7)
7	10366	10383.5 (0.5)	10418.8 (3.6)	10409.5 (3.9)	10408.7 (3.0)	10426.4 (3.5)	10422.7 (3.1)
8	10420	10432.5 (0.5)	10459.3 (3.7)	10450.6 (3.7)	10450.7 (2.7)	10463.5 (3.3)	10461.0 (2.9)
9	10461	10474.5 (0.5)	10493.7 (3.0)	10485.9 (3.4)	10487.5 (2.2)	10495.5 (2.9)	10494.6 (2.4)
>= 10	10579	10579 (0.0)	10579 (0.0)	10579 (0.0)	10579 (0.0)	10579 (0.0)	10579 (0.0)
Nb teachers moving	0	564.7	1373.0	1461.5	1488.2	1732.5	1709.7
Min	0	560.0	1333.0	1416.0	1456.0	1696.0	1677.0
Max	0	568.0	1408.0	1517.0	1513.0	1768.0	1739.0
SD	0	2.7	14.9	17.4	12.5	15.4	12.4

† Notes: This table presents the cumulative distribution of teachers who obtain school rank k under their initial assignment in column 1, under DA* in column 2, TO-BE in column 3, BE(Init) in column 4, BE(DA*) in column 5, 1S-BE(Init) in column 6, and 1S-BE(DA*) in column 7. The data come French teacher assignments to regions in 2013. Simulation standard errors are reported in parentheses.

Table 2: Stability of the matchings obtained with different mechanisms

Nb regions	Init	DA*	TO-BE	BE(Init)	BE(DA*)	1S-BE(Init)	1S-BE(DA*)
0	1980	2496.5 (2.4)	3742.7 (56.8)	3731.3 (55.4)	3799.4 (49.9)	3940.5 (56.9)	4001.2 (54.9)
1	8722	8880.9 (2.0)	9246.9 (16.5)	9215.5 (17.7)	9234.8 (17.7)	9294.2 (17.9)	9306.1 (17.7)
2	9694	9787.8 (0.4)	10004.9 (13.2)	9983.4 (14.4)	9991.9 (12.0)	10026.4 (13.3)	10035.3 (13.3)
3	10096	10149.5 (0.5)	10299.1 (8.2)	10287.5 (9.3)	10292.7 (8.0)	10309.6 (7.9)	10312.3 (7.5)
4	10323	10360.2 (0.6)	10447.0 (5.4)	10438.9 (5.8)	10444.5 (5.3)	10457.6 (5.9)	10459.3 (4.8)
>= 5	10579	10579 (0.0)	10579 (0.0)	10579 (0.0)	10579 (0.0)	10579 (0.0)	10579 (0.0)
Nb of teachers blocking with at least one region							
Mean	.	8082.5	6836.3	6847.7	6779.6	6638.5	6577.8
Min	.	8078.0	6675.0	6703.0	6665.0	6527.0	6453.0
Max	.	8087.0	7003.0	6985.0	6917.0	6809.0	6701.0
SD	.	2.4	56.8	55.4	49.9	56.9	54.9

† Notes: The upper part of this table presents the cumulative distribution of the number of regions with which teachers are blocking. The data are from French teacher assignments to regions in 2013. Column 1 reports the cumulative distribution of the number of regions with which teachers block under their initial assignment. The following columns report the cumulative distribution of the number of regions with which teachers block under DA*, TO-BE, BE(Init), BE(DA*), 1S-BE(Init) and 1S-BE(DA*). Simulation standard errors are reported in parentheses.

Table 3: Regions' welfare under different mechanisms

Net percentage of positions	DA*	TO-BE	BE(Init)	BE(DA*)	1S-BE(Init)	1S-BE(DA*)
-100/-91%	0.18 (0.03)	0 (0)	0 (0)	0.18 (0.03)	0.80 (0.15)	0.71 (0.13)
-90/-71%	0.18 (0.0)	0 (0)	0 (0)	0.18 (0.0)	1.06 (0.21)	0.93 (0.15)
-70/-51%	0.31 (0.03)	0 (0)	0 (0)	0.31 (0.03)	1.74 (0.26)	1.41 (0.21)
-50/-31%	0.57 (0.03)	0 (0)	0 (0)	0.50 (0.04)	3.19 (0.33)	2.57 (0.25)
-30/-1%	1.03 (0.03)	0 (0)	0 (0)	0.93 (0.05)	4.86 (0.39)	4 (0.32)
0%	84.32 (0.06)	72.01 (0.40)	71.18 (0.42)	70.90 (0.31)	72.67 (0.47)	71.81 (0.40)
1/29%	87.95 (0.02)	75.62 (0.44)	74.59 (0.50)	74.73 (0.37)	76.56 (0.52)	76.02 (0.40)
30/49%	91.05 (0.04)	79.11 (0.43)	78.02 (0.55)	77.66 (0.38)	80.01 (0.45)	79.14 (0.46)
50/69%	94.84 (0.03)	85.20 (0.38)	84.31 (0.40)	84.03 (0.38)	85.76 (0.48)	84.94 (0.43)
70/89%	97.40 (0.03)	90.32 (0.36)	89.70 (0.41)	89.04 (0.33)	89.63 (0.43)	88.82 (0.37)
90/100%	100 (0.0)	100 (0.0)	100 (0.0)	100 (0.0)	100 (0.0)	100 (0.0)
% of regions with no priority change	83.22	72.01	71.18	69.83	67.05	67.12
SD	0.08	0.40	0.42	0.32	0.41	0.33

† Note: this table presents the cumulative percentage of regions with a net welfare improvement (relative to their initial assignment). For each of the 49 subjects*31 regions, we compute the number of positions assigned to teachers with higher priority, from which we subtract the number of positions assigned to teachers with lower priority. Then, for each subject*region, the net total is divided by the total number of positions to obtain the percentage of positions with net improvement. Finally, the total number of regions considered is divided by 49×31 to obtain regions' average percentages. For instance, on average, under DA*, 0.18% of the regions have between 91% and 100% of their seats assigned teachers with lower priority (in net terms). Simulation standard errors are reported in parenthesis.

Table 4: Fulfillment of the administration's objectives

	DA*	TO-BE	BE(Init)	1S-BE(Init)
	Share of inexperienced teachers			
Non-disadvantaged regions	26.88 (0.0)	27.50 (0.08)	27.66 (0.09)	27.42 (0.11)
Disadvantaged regions	45.66 (0.0)	44.99 (0.08)	44.82 (0.10)	45.07(0.11)
	Share of inexperienced teachers (among incoming teachers)			
Non-disadvantaged regions	17.57 (0.07)	24.40 (0.65)	26.05 (0.74)	21.14 (0.61)
Disadvantaged regions	28.56 (0.08)	29.00 (1.34)	30.09 (1.19)	30.98 (0.98)
	Share of inexperienced teachers (among exiting teachers)			
Non-disadvantaged regions	18.36 (0.07)	21.87 (0.56)	22.94 (0.63)	19.36 (0.51)
Disadvantaged regions	26.94 (0.07)	48.35 (2.78)	50.73 (2.61)	42.89 (2.68)
Nb spousal reunification	251.8 (1.48)	513.5 (10.11)	547.6 (12.01)	482.5 (10.55)
Nb teacher from disadvantaged school	110.3 (0.78)	83.4 (4.85)	90.4 (5.36)	106.8 (5.08)

† Note: The upper part of this table presents the share of teachers who have only one or two years of experience under the current algorithm (in column 1), TO-BE, BE, and 1S-BE. We successively report statistics for all teachers, for incoming teachers only, and for exiting teachers. The lowest two rows of the table present statistics on the number of teachers who move closer to their spouses and who leave a disadvantaged school. Simulation standard errors are reported in parentheses.

THE DESIGN OF TEACHER ASSIGNMENT: THEORY AND EVIDENCE

Julien Combe, Olivier Tercieux and Camille Terrier

APPENDIX

A Proof of Lemma 1

Consider the directed graph $G := (N, E)$ of BE starting at the matching $\mu(0) = \mu$. For each school s s.t. $\mu'(s) \neq \mu(s)$, consider the sets of teachers $T_s := \{t \in T : t \in \mu(s) \backslash \mu'(s)\}$ and $T'_s := \{t \in T : t \in \mu'(s) \backslash \mu(s)\}$. Since we are in a reassignment setting, we have that $|T_s| = |T'_s| := m_s$ for some $m_s \leq q_s$. Let $\mathbf{x} := (t_1, \ldots, t_m)$ and $\mathbf{x'} := (t'_1, \ldots, t'_m)$ be the two ordered vectors corresponding to T_s and T'_s where each vector orders the teachers in each set in a decreasing order according to $\succ_s$. Since μ' 2-Pareto dominates μ, we have that $\forall k = 1, \ldots, m$, $t'_k \succ_s t_k$.[67] So under G, the graph of BE, we have that $(t'_k, \mu(t'_k))$ points to (t_k, s), i.e., $[(t'_k, \mu(t'_k)), (t_k, s)] \in E$. Let $e^s_k := [(t'_k, \mu(t'_k)), (t_k, s)]$. Consider the subgraph $G' := (N', E')$ where $N' := \{(t, \mu(t)) : \mu(t) \neq \mu'(t)\}$ and $E' := \{e^s_k : s \in S, k = 1, \ldots, m_s\}$. Note that, by construction of G', each node $(t, s) \in N'$ has a unique incoming edge from another node $(t', s') \in E'$ and that $s = \mu'(t')$. If all graph nodes have an in-degree of exactly one, then there exists a collection of disjoint cycles that includes all graph nodes. Indeed, start from any node in N': call it n_1. Then, since n_1 has in-degree one, there is a unique node n_2 s.t. $(n_2, n_1) \in E'$. Applying the same argument for n_2, there is a unique node n_3 s.t. $(n_3, n_2) \in E'$. Iterating the argument, we can identify a cycle in G' involving $L \geq 2$ nodes of N'. Let $A := \{n_1, \ldots, n_L\} \subset N'$ be such nodes. Note that since the nodes of G' have an in-degree of exactly one, there is no node $n \in N' \backslash A$ that points to a node in A since otherwise, it would imply that a node in A has an in-degree of at least two. So if one deletes nodes in A together with their edges, the resulting subgraph of G' will still have edges with an in-degree of exactly one, and we can iterate the argument to find a new cycle in this subgraph, and so on. This process will lead to a collection of disjoint cycles in G' that involve all the nodes in N'. To conclude the proof, note that all these disjoint cycles are actual cycles of G, the original graph of the BE algorithm starting at μ, and that each teacher matched using these cycles is assigned his school under μ'. By implementing these cycles under the BE algorithm, then, one indeed goes from μ to μ'.

B Relationship between TO-BE and TTC

The TO-BE algorithm has some similarities with the (Shapley-Scarf) TTC mechanism. TTC is well-defined only in a one-to-one environment (i.e., when $q_s = 1$ for each school s). But even in this environment TTC and TO-BE are different. Indeed, TTC operates in the same

[67] While the statement is fairly intuitive, we provide the formal argument in Appendix S.3.

manner as the above algorithm but does not refer to schools' preferences/priorities: an edge $(t, s) \longrightarrow (t', s')$ is added if and only if teacher t ranks school s' first within the set of all remaining schools (i.e., at step k, those are the schools in $S(k-1)$). The other additional feature is that TO-BE is defined in a many-to-one setting while TTC is defined in a one-to-one environment.

Yet, TTC and TO-BE can be formally related. Starting from our many-to-one environment, one can construct a one-to-one environment in which the outcome of TTC will give us the outcome of TO-BE. To see this, fix a preference profile and a collection f and let us build this one-to-one environment. We assume that each teacher t is "endowed" with $(t, \mu_0(t))$, which we refer to as an object. Each teacher t has preferences over possible objects in $\{(t', \mu_0(t'))\}_{t' \in T}$ that are given as follows. Teacher t finds unacceptable any object (t', s') for which $t' \succ_{s'} t$ while any other object is acceptable. For any pair of objects (t', s') and (t'', s'') that are acceptable to t, we consider two cases. First, if $s' \neq s''$ then (t', s') is preferred to (t'', s'') if and only if $s' \succ_t s''$. Second, if $s' = s''$ then (t', s') is preferred to (t'', s'') if and only if $f_{s'}(t') < f_{s'}(t'')$. In the sequel, if $\succ_t$ denotes the original preferences of teacher t, we let $\succ'_t$ be his preferences in this modified environment. In this one-to-one environment with strict preferences, standard TTC is well-defined, and it is easily checked that teacher t is matched to an object (t', s') under TTC for some teacher $t' \in \mu_0(s')$ if and only if teacher t is matched to s' under TO-BE$_f$ in our original environment.

We summarize this discussion in the following Lemma:

Lemma 2 *Fix a preference profile $\succ$ and a collection f. TO-BE$_f(\succ)(t) = s'$ if and only if TTC$(\succ')(t) = (t', s')$ for some $t' \in \mu_0(s')$ where the preference relation $\succ'$ is constructed according to the procedure described above.*

C Proof of Theorem 1.

Proof. Given the connection with TTC, a mechanism known to be strategy-proof, pointed out in Lemma 2, it follows that TO-BE$_f$ is likewise strategy-proof. Indeed, TO-BE$_f(\succ)[t]$, the school obtained by teacher t under TO-BE$_f$, corresponds to the element in the second dimension of TTC$(\succ')[t] = (t', s)$. Similarly, when teacher t misreports his preferences to $\tilde{\succ}_t$, he obtains school TO-BE$_f(\tilde{\succ}_t, \succ_{-t})[t]$ under TO-BE$_f$ corresponding to the element in the second dimension of TTC$(\tilde{\succ}'_t, \succ_{-t})[t] = (t'', s')$. By strategy-proofness of TTC we obtain that TTC$(\succ')[t] = (t', s) \succeq'_t$ TTC$(\tilde{\succ}'_t, \succ_{-t})[t] = (t'', s')$. By definition of $\succ'$ this implies that TO-BE$_f(\succ)[t] = s \succeq_t s' =$ TO-BE$_f(\tilde{\succ}_t, \succ_{-t})[t]$.

Now, we show that TO-BE$_f$ is a selection of BE. Assume by contradiction that TO-BE$_f$ is not a selection of BE. Appealing to Proposition 5, this implies that TO-BE$_f$ is not two-sided maximal. By construction, TO-BE$_f$ is 2-IR so we obtain that TO-BE$_f$ is 2-Pareto dominated at some preference profile $\succ$ by an alternative assignment, say μ. Now, let us consider the one-to-one environment described above under which the outcome of TTC corresponds to the outcome of TO-BE$_f$ in our original many-to-one environment. Recall that we denote the modified preference profile in the one-to-one environment by $\succ'$. We claim that, in this modified environment, TTC is 1-Pareto-dominated by some matching μ' which contradicts the well-known 1-Pareto efficiency of TTC. In the sequel, we build μ'.

For each school s, we define $T^{out}(s)$ as the set of teachers who exit school s when we move from TO-BE$_f(\succ)$ to the assignment given by μ, i.e., $T^{out}(s) :=$TO-BE$_f(\succ)[s]\backslash\mu(s)$. We similarly denote $T^{in}(s)$ for the set of teachers entering into school s, i.e., $T^{in}(s) := \mu(s)\backslash$TO-BE$_f(\succ)[s]$. Clearly, for each school s, we must have that $|T^{out}(s)| = |T^{in}(s)|$. Without loss of generality, we denote $T^{out}(s) = \{t_1, ..., t_k\}$ where teacher t_ℓ has the ℓ-th highest preference ranking / priority in $T^{out}(s)$. We similarly denote $T^{in}(s) = \{t'_1, ..., t'_k\}$ where teacher t'_ℓ has the ℓ-th highest preference ranking / priority in $T^{in}(s)$. We then build assignment μ' as follows. Pick any teacher t. If $\mu(t) =$ TO-BE$_f(\succ)[t]$, then let $\mu'(t) =$ TTC$(\succ')[t]$. Trivially, $\mu'(t) \succeq'_t$ TTC$(\succ')[t]$. Now, if $\mu(t) \neq$ TO-BE$_f(\succ)[t]$, then t must belong to $T^{in}(s)$ where $s := \mu(t)$. Assume that t has the ℓ-th highest priority in $T^{in}(s)$. We set $\mu'(t)$ to TTC$(\succ')[t']$ where t' has the ℓ-th highest priority in $T^{out}(s)$.[68] Now, let us show that $\mu'(t) \succ'_t$ TTC$(\succ')[t]$. Because we let TTC$(\succ')[t'] =: (t'', s)$, in the sequel, we would like to show that $(t'', s) \succ'_t$ TTC$(\succ')[t]$. Since $s \succ_t$ TO-BE$_f(\succ)[t]$, we only have to show that (t'', s) is acceptable to t under $\succ'_t$. Since μ 2-Pareto dominates TO-BE$_f(\succ)$, $t \succ_s t'$. In addition, by definition, TTC$(\succ')[t']$ is acceptable to t' under $\succ'_{t'}$ and so, by construction of $\succ'_{t'}$, we must have that $t' \succeq_s t''$. We conclude that $t \succ_s t''$ and so (t'', s) is acceptable to t under preference profile $\succ'$. This proves that μ' 1-Pareto dominates TTC$(\succ')$, thereby yielding our contradiction. ∎

D Proof of Theorem 2.

For the first part of the proof of Theorem 2, we proceed by contradiction and fix a preference profile $\succ$ under which TO-BE$_{f^*}(\succ)$ is 1-Pareto dominated by another 2-IR matching, say μ. Consider the one-to-one environment described in Section 4.1.2, under which the outcome of TTC corresponds to that of TO-BE$_{f^*}$ in the original many-to-one environment (see Lemma 2). The modified teacher preference profile in this environment is $\succ'$. We will show that, in this modified environment, TTC is 1-Pareto dominated by some matching μ', which contradicts the well-known 1-Pareto efficiency of TTC. Let us build μ'.

For each school s, we let $T_0^{out}(s) := \mu_0(s)\backslash\mu(s)$ and $T_0^{in}(s) := \mu(s)\backslash\mu_0(s)$.[69] Remember that for each school s we must have $|T_0^{in}(s)| = |T_0^{out}(s)| := m$. Without loss of generality, let $T_0^{out}(s) = \{t_1, \dots, t_m\}$ where teacher t_ℓ has the ℓ-th highest priority in $T_0^{out}(s)$. Similarly, let $T_0^{in}(s) = \{t'_1, \dots, t'_m\}$. Note that, since μ is 2-IR and so 2-Pareto dominates μ_0, we have that $t'_\ell \succ_s t_\ell$ for all $\ell = 1, \dots, m$. Finally, let $T_0(s) := \mu_0(s) \cap \mu(s)$. We define a first matching μ'_1 as follows.

Fix a teacher t. If $t := t'_\ell \in T_0^{in}(s)$ for some school s, then let $\mu'_1(t) := (t_\ell, s)$. If t does

[68] One can easily check that μ' is a well-defined matching.

[69] Note that the construction is different from the proof of Theorem 1 since here, we consider the move from μ_0 to μ and not the one from TO-BE$_{f^*}(\succ)$ to μ.

not appear in any set $T_0^{in}(s)$ for all schools $s \in S$, then it implies that the teacher belongs to $T_0(s)$ for some school s. In that case, let $\mu_1'(t) = (t, s)$. Note that, under the latter case, we have that $\mu_0(t) = \mu(t) \succeq_t \text{TO-BE}_{f^*}(\succ)[t] \succeq_t \mu_0(t)$ so that $\mu_0(t) = \text{TO-BE}_{f^*}(\succ)[t] = \mu(t)$ and, by definition of $\text{TTC}(\succ')$, we have that $\text{TTC}(\succ')[t] = (t, s)$.

For a teacher t, if $s := \mu(t) \succ_t \text{TO-BE}_{f^*}(\succ)[t] := \tilde{s}$, then $t_\ell' := t \in T_0^{in}(s)$ and $\mu_1'(t) = (t_\ell, s)$. Let $(\tilde{t}, \tilde{s}) := \text{TTC}(\succ')[t]$. Since $s \succ_t \tilde{s}$ and $t = t_\ell' \succ_s t_\ell$, we indeed have that (t_ℓ, s) is acceptable for t under $\succ_t'$ and $(t_\ell, s) \succ_t (\tilde{t}, \tilde{s})$.

If $t \in T_0(s)$ for some s, then, as described above, we trivially have $\mu_1'(t) = \text{TTC}(\succ')[t] = (t, s)$.

Now, assume that teacher t is s.t. $\mu(t) = \text{TO-BE}_{f^*}(\succ)[t] =: s \succ_t \mu_0(t)$ so that $t := t_\ell' \in T_0^{in}(s)$. So $\mu_1'(t_\ell') = (t_\ell, s)$ and $\text{TTC}(\succ')[t] := (t_k, s)$ for some k. By construction of $\text{TTC}(\succ')$, we have that $t \succ_s t_k$. We also know that $t = t_\ell' \succ_s t_\ell$ so that (t_ℓ, s) and (t_k, s) are both acceptable under $\succ_t'$. If $(t_\ell, s) \succeq_t' (t_k, s)$ then $\mu_1'(t) \succeq_t' \text{TTC}(\succ')[t]$.[70]

Assume that $(t_k, s) \succ_t' (t_\ell, s)$. We will build a matching μ_2' s.t. all teachers who preferred their assignments under μ_1' to those under $\text{TTC}(\succ')$ continue to do so and teacher t will weakly prefer his assignment under μ_2' to the one under $\text{TTC}(\succ')$. Let $k_1 := k$, by construction of the profile $\succ'$, since $(t_k, s) \succ_t' (t_\ell, s)$ then $f_s^*(t_{k_1}) < f_s^*(t_\ell)$. By definition of f_s^*, it means that $t_{k_1} \succ_s t_\ell$, so that $k_1 < \ell$. Let $O_{<\ell} := \{(t_1, s), \ldots, (t_{\ell-1}, s)\}$, $O_{\geq \ell} := \{(t_\ell, s), \ldots, (t_m, s)\}$, and $\tilde{O} := O_{\geq \ell} \cup O_{<\ell}$. We build a bipartite digraph (N, E) where $N := T_0^{in}(s) \cup \tilde{O}$. Let $t_\ell' := n_1 \in T_0^{in}(s)$ and let him point to $n_1' := (t_{k_1}, s) \in \tilde{O}$. Now, let n_1' point to $n_2 := t_{k_1}'$ where t_{k_1}' the teacher assigned to (t_{k_1}, s) under μ_1'. Now, there are three cases:

1. If $s = \mu(t_{k_1}') \neq \text{TO-BE}_{f^*}(\succ)[t_{k_1}']$, then let $n_2 = t_{k_1}'$ point to $n_2' := (t_\ell, s)$. Since $k_1 < \ell$, we have that $t_{k_1}' \succ_s t_\ell' \succ_s t_\ell$, so that (t_ℓ, s) is acceptable under $\succ_{t_{k_1}'}'$ and so $(t_\ell, s) \succ_{t_{k_1}'}'$ $\text{TTC}(\succ')[t_{k_1}']$.
2. If $s = \mu(t_{k_1}') = \text{TO-BE}_{f^*}(\succ)[t_{k_1}']$ and $(t_\ell, s) \succ_{t_{k_1}'}' \text{TTC}(\succ')[t_{k_1}'] =: (t_{k_2}, s)$, then let $n_2 = t_{k_1}'$ point to $n_2' := (t_\ell, s)$.
3. If $s = \mu(t_{k_1}') = \text{TO-BE}_{f^*}(\succ)[t_{k_1}']$ and $\text{TTC}(\succ')[t_{k_1}'] =: (t_{k_2}, s) \succ_{t_{k_1}'}' (t_\ell, s)$, then let $n_2 = t_{k_1}'$ point to (t_{k_2}, s). Since (t_ℓ, s) is acceptable at $\succ_{t_{k_1}'}'$, we must have that $k_2 < \ell$ so that $(t_{k_2}, s) \in O_{<\ell}$. Let $n_2' := (t_{k_2}, s)$ and $n_3 := t_{k_2}'$, where t_{k_2}' is the teacher assigned to (t_{k_2}, s) under μ_1', and let the former point to the latter.

If Case 3 holds, if one starts with t_{k_2}' rather than t_{k_1}' and since $k_2 < \ell$, then it is possible to use the same argument as above. If Case 3 still happens, then one can iterate until the pointing reaches (t_ℓ, s). Note that it does reach (t_ℓ, s) only once it reaches Case 1 or 2 and that, by

[70] Remember that we are trying to build a matching μ' of objects that Pareto dominates at $\succ'$ the matching given by $\text{TTC}(\succ')$.

finiteness of the set $O_{<\ell}$, Case 3 cannot holds indefinitely.[71] Let $p-1$ be the number of iterations needed before reaching Case 1 or 2. Once it does, we have p nodes $n_1, n'_1, \dots, n_p, n'_p$ where, for $i = 1, \dots, p$, $n_i = t'_{k_{i-1}}$ points to $n'_i = (t_{k_i}, s)$ (where t'_{k_0} stands for $t = t'_\ell$) and the latter points to $n_{i+1} = t'_{k_i}$. Note that $n'_p = (t_\ell, s)$. Let μ'_2 be the same matching as μ'_1 except that, for $i = 1, \dots, p$, the teacher t'_{k_i} in node n_i is assigned the object (t_{k_i}, s).[72] Note that, by construction, for each teacher $t'_{k_{i-1}}$ with $i = 1, \dots, p-1$, we have that $\mu'_2(t'_{k_{i-1}}) = \text{TTC}(\succ')[t'_{k_{i-1}}]$ and that, for teacher $t'_{k_{p-1}}$, we have $\mu'_2(t'_{k_{p-1}}) = (t_\ell, s) \succ'_{t'_{k_{p-1}}} \text{TTC}(\succ')[t'_{k_{p-1}}]$.

Now, we can iterate the entire above argument with μ'_2 if there is still a teacher t and a school s s.t. $t := t'_\ell \in T_0^{in}(s)$, $\mu(t) = s = \text{TO-BE}_{f^*}(\succ)[t]$ but $\text{TTC}(\succ')[t] \succ'_t \mu'_2(t)$. Note that such teacher t cannot be one of those treated above. So at some point, this process will stop and identify a matching μ'_K where for all $t \in T$, $\mu'_K(t) \succeq'_t \text{TTC}(\succ')[t]$. Since by assumption there is at least one teacher t s.t. $\mu(t) \succ_t \text{TO-BE}_{f^*}(\succ)[t]$, then our above construction implies that $\mu'_K(t) \succ_t \text{TTC}(\succ')[t]$. Setting $\mu' = \mu'_K$, we obtain the desired contradiction.

For the second part of the theorem, assume that $f \neq f^*$, then one can construct an instance of preferences wherein the outcome TO-BE_f is 1-Pareto dominated by a 2-IR matching μ. This is enough for our purpose since we can build a 2-IR mechanism that selects μ at this specific instance and coincides with TO-BE_f at any other instance. Since $f \neq f^*$, then $\exists s^* \in S$ and two teachers t_1^*, t_2^* s.t.

- $\mu_0(t_1^*) = \mu_0(t_2^*) = s^*$
- $t_1^* \succ_{s^*} t_2^*$
- $f_{s^*}(t_1^*) > f_{s^*}(t_2^*)$

Then, let t, t' be two additional teachers initially assigned to respectively s and s' so that these two schools have one seat and s^* has two seats.[73] Let the preferences be:

$\succ_{s^*}$:	t	t_1^*	t'	t_2^*
$\succ_s$:	t_2^*	t		
$\succ_{s'}$:	t_1^*	t'		
$\succ_{t_1^*}$:	s'	s^*		
$\succ_{t_2^*}$:	s	s^*		
$\succ_t$:	s^*	s		
$\succ_{t'}$:	s^*	s'		

[71] Note that it is not possible for the pointing defined in Case 3 to cycle. Indeed, take the first iteration with μ'_1, in which each object points to the assigned teacher assigned under μ'_1, and each teacher in the Case 3 points to his assigned object under μ'_1. If the pointing does cycle, it means that a teacher, say for instance t_{k_4}, points to an object that a previous iteration of the Case 3 used, say for instance (t_{k_2}, s). In that case, it would mean that the object (t_{k_2}, s) was assigned to both teacher t'_{k_2} and teacher t'_{k_4} under μ'_1, a contradiction.

[72] This indeed defines a matching since μ'_2 just reassign the objects corresponding to seats inside the same school s.

[73] One needs at least two additional teachers into two different schools. If there are more teachers and more schools, one can trivially set their preferences s.t. they rank their initial school first so that the exchanges below are the only possible ones.

At the first step of TO-BE$_{f^*}(\succ)$, the only cycle is $(t,s) \to (t_2^*, s^*) \to (t,s)$. Once implemented, (t', s') cannot point to (t_1^*, s^*) since $t_1^* \succ_{s^*} t'$ so the final matching matches t and t_1^* to s^*, t_2^* to s and t' to s'. But in matching t' to s^* and t_1^* to s', we would obtain a 2-IR matching μ that 1-Pareto dominates TO-BE$_{f^*}(\succ)$. Note, however, that, in doing so, school s^* would be worse off compared to under TO-BE$_{f^*}$. Indeed, its assignment under TO-BE$_{f^*}(\succ)$ is $\{t, t_1^*\}$, while the one under μ is $\{t, t'\}$ and $t_1^* \succ_{s^*} t'$.

E Proof of Theorem 3

The proof of Theorem 3 directly follows from the proposition below.

Proposition 7 *Consider any two-sided maximal (and strategy-proof) mechanism φ. Fix any profile $\succ$ which lies in $\mathcal{P}$. We must have $\varphi(\succ) \in$ TO-BE($\succ$).*

PROOF. Consider the graph in the first step of TO-BE($\succ$). We claim that there is a cycle of TO-BE($\succ$) such that, under $\varphi(\succ)$, any teacher who is part of the cycle gets assigned the object he points to. First, note that if there is a self-cycle in this step, i.e., a node (t,s) pointing to itself, then, by 2-IR of φ, $\varphi(\succ)$ must assign t to s. So let us assume that there is no self-cycle in the graph. Pick an arbitrary cycle denoted $(t_1, s_1), (t_2, s_2), ..., (t_K, s_K)$.

We claim that in this cycle, there must be a node (t_k, s_k) such that $\varphi(\succ)[t_k] \neq s_k$. Indeed, if this was not the case, we would have that for all $k = 1, ..., K$, $\varphi(\succ)[t_k] = s_k$. But then $\varphi(\succ)$ is not two-sided maximal since we can assign each t_k to s_{k+1}. Noticing that this gives a 2-IR assignment, the assignment we obtain 2-Pareto dominates $\varphi(\succ)$, a contradiction with the assumption that φ is a selection of BE and, hence, two-sided maximal.

So there must be a node (t_k, s_k) such that $\varphi(\succ)[t_k] \neq s_k$, wlog, let us assume that $k = 1$. The following simple lemma shows that there must be a cycle of TO-BE($\succ$) such that, under $\varphi(\succ)$, any teacher who is part of the cycle gets assigned the object he points to.

Lemma 3 *If there exists t_1 such that $\varphi(\succ)[t_1] \neq \mu_0(t_1)$ then there exists a cycle $(t_1, s_1), (t_2, s_2), ..., (t_K, s_K)$ in the graph such that $\varphi(\succ)[t_k] = s_{k+1}$ for any $k = 1, ..., K$.*

PROOF. Assume that there exists t_1 such that $\varphi(\succ)[t_1] \neq \mu_0(t_1)$. Because $\succ$ lies in $\mathcal{P}$, $\varphi(\succ)[t_1] =: s_2$ where s_2 is t_1's top choice. In addition, since, under $\varphi(\succ)$, one seat of school s_2 is taken by t_1, there must be a teacher t_2 such that $\mu_0(t_2) = s_2$ and $\varphi(\succ)[t_2] \neq s_2$. In addition, because $\varphi(\succ)$ is two-sided maximal, this teacher can be chosen so that $t_1 \succ_{s_2} t_2$. More specifically, we pick t_2 the teacher with the highest priority at s among all those who have a lower priority than t_1 at school s. By definition of the graph, (t_1, s_1) points to (t_2, s_2). Now, since $\varphi(\succ)[t_2] \neq s_2 = \mu_0(t_2)$, we can iterate the reasoning to induce a path $(t_1, s_1), (t_2, s_2), ...$ in the graph such that $\varphi(\succ)[t_k] = s_{k+1}$ for any $k \geq 1$. Since the graph is finite, this path will cycle at some point. ∎

Now, consider the new graph obtained after we removed the teachers who are part of this cycle and the seats they point to in the cycle. The exact same reasoning holds here. Hence, we can iterate the reasoning until we exhaust the market. We obtain a sequence of cycles selected in the graphs associated with TO-BE which, once implemented, yields the assignment given by $\varphi(\succ)$. This shows that $\varphi(\succ) \in$ TO-BE$(\succ)$. ∎

F Proof of Theorem 4

We want to prove the following proposition. Remember that, for this result, we assume that we are in a one-to-one setting so that the set of TO-BE mechanisms is a singleton. We refer to the unique mechanism of this class as simply TO-BE.

Proposition 8 *Let φ be any selection of BE. If $\varphi \neq$ TO-BE then φ is not strategy-proof.*

Lemma 4 *Let φ be any selection of BE. Fix any profile of preferences $\succ$ and assume that $\varphi(\succ) \neq$ TO-BE$(\succ)$. Let x be the outcome of TO-BE$(\succ)$ and let y be that of $\varphi(\succ)$. There exists t s.t. $x(t) \succ_t y(t) \succ_t \mu_0(t)$.*

Proof. Let $T(x, y)$ be the set of teachers for which $x(t) \succ_t y(t) \succeq_t \mu_0(t)$. We know that x is not 1-Pareto dominated by y (by Corollary 1), and since y is individually rational and $x \neq y$, we must have $T(x, y) \neq \emptyset$. Proceed by contradiction and assume that, for all $t \in T(x, y)$, we have $y(t) = \mu_0(t)$. Let $B := T \backslash T(x, y)$. Note that for any $t \in B$, $y(t)$ is a school initially assigned to some teacher in B. In addition, by definition, for all $t \in B$, $y(t) \succeq_t x(t)$. If there was no teacher $t \in B$ for which $y(t) \succ_t x(t)$, then we would have the following situation: y would select the initial allocation for all $t \in T(x, y)$ and would be identical to x for all $t \notin T(x, y)$. Given that $x \neq y$, we must have $x(t) \neq y(t) = \mu_0(t)$ for some $t \in T(x, y)$. Since x is individually rational, we have $x(t) \succ_t y(t) = \mu_0(t)$ for those $t \in T(x, y)$. Hence, x 1-Pareto-dominates y. However, all schools are also better off under x than under y. Indeed, for each school s s.t. $y(s) \notin T(x, y)$, $y(s) = x(s)$ and for each school s s.t. $y(s) \in T(x, y)$, because x is individually rational on both sides, $x(s) \succeq_s y(s) = \mu_0(s)$ with a strict inequality for s satisfying $x(s) \neq y(s)$ (and this s must exist since $x \neq y$). Thus, x is individually rational on both sides and 2-Pareto-dominates y, which is not possible, given that y is an outcome of BE.

To recap, we have that, for any $t \in B$, $y(t)$ is a school initially assigned to some teacher in B and for all $t \in B$, $y(t) \succeq_t x(t)$ with a strict inequality for some $t \in B$. In addition, since y is the outcome of $\varphi(\succ)$ and φ 2-Pareto-dominates the initial allocation μ_0, we must have that, for all schools s, $y(s) \succeq_s \mu_0(s)$. Hence, B is a two-sided blocking coalition for x, which is a contradiction since x must be a point in the two–sided Core. ∎

Proof of Proposition 8. We start from a profile of preferences $\succ$ under which $\varphi(\succ) \neq$ TO-BE$(\succ)$ which must exist because of our assumption that $\varphi \neq$ TO-BE. Given our profile of preferences $\succ$, we let the profile of preferences $\succ'$ be defined as follows. For any t, any school s other than TO-BE$(\succ)[t]$ are ranked as unacceptable for t under $\succ'$. We must have TO-BE$(\succ)$ = TO-BE$(\succ')$. Now, we are in a position to prove the following lemma.

Lemma 5 *TO-BE*$(\succ') = \varphi(\succ')$.

Proof. Suppose $x :=$ TO-BE$(\succ') \neq \varphi(\succ') =: y$. By the above lemma, there exists t s.t. $x(t) \succ'_t y(t) \succ'_t \mu_0(t)$, which yields a contradiction, by construction of $\succ'_t$. ∎

Note that TO-BE also satisfies the following property: for any profile of preferences $\succ$, for any teacher t, TO-BE$(\succ)(t)$ = TO-BE$(\succ_{-t}, \succ'_t)(t)$. This will be used in the following lemma.

Lemma 6 *If φ is strategy-proof, then TO-BE$(\succ_Z, \succ'_{-Z}) = \varphi(\succ_Z, \succ'_{-Z})$ for any $Z \subseteq T$.*

Proof. Assume φ is strategy-proof. The proof is, by induction, on the size of Z. For $|Z| = 0$, the result is given by the previous lemma. Now, the induction hypothesis is that TO-BE$(\succ_Z, \succ'_{-Z}) = \varphi(\succ_Z, \succ'_{-Z})$ for any subset Z with $|Z| = k$. Proceed by contradiction and suppose that there is Z s.t. $|Z| = k+1$ for which $x :=$ TO-BE$(\succ_Z, \succ'_{-Z}) \neq \varphi(\succ_Z, \succ'_{-Z}) =: y$. By the first lemma above, there exists t s.t. TO-BE$(\succ_Z, \succ'_{-Z})(t) \rhd_t \varphi(\succ_Z, \succ'_{-Z})(t) \rhd_t \mu_0(t)$ where $\rhd_t = \succ'_t$ if $t \notin Z$ while $\rhd_t = \succ_t$ otherwise. If $t \notin Z$, then there is a straightforward contradiction since, under $\succ'_t$, there is a single school ranked above $\mu_0(t)$ for teacher t. Now, assume that $t \in Z$. By the property noted just before the lemma statement, we must have TO-BE$(\succ_{Z \setminus \{t\}}, \succ'_{-Z}, \succ'_t)(t)$ = TO-BE$(\succ_Z, \succ'_{-Z})(t)$ and, by our induction hypothesis, $\varphi(\succ_{Z \setminus \{t\}}, \succ'_{-Z}, \succ'_t)(t)$ = TO-BE$(\succ_{Z \setminus \{t\}}, \succ'_{-Z}, \succ'_t)(t)$. Thus, we obtain $\varphi(\succ_{Z \setminus \{t\}}, \succ'_{-Z}, \succ'_t)(t)$ = TO-BE$(\succ_{Z \setminus \{t\}}, \succ'_{-Z}, \succ'_t)(t)$ = TO-BE$(\succ_Z, \succ'_{-Z})(t) \succ_t \varphi(\succ_Z, \succ'_{-Z})(t)$, which is a contradiction with the assumption that φ is strategy-proof (indeed, at $(\succ_Z, \succ'_{-Z})$, teacher $t \in Z$ has an incentive to report $\succ'_t$ instead of $\succ_t$). ∎

Taking $Z = T$ in the above lemma, given that $\varphi(\succ) \neq$ TO-BE$(\succ)$, we obtain the following corollary, which completes the proof of our proposition.

Corollary 2 *φ is not strategy-proof.*

■

G Proof of Proposition 6

In the sequel, we prove our characterization result of one-sided maximal matchings given in Theorem 6. Our proof is divided into two parts. We start by showing that any outcome of the 1S-BE algorithm is a one-sided maximal matching (Section G.1):

Proposition 9 *If μ is an outcome of the 1S-BE algorithm then μ is one-sided maximal.*

Then, we move to the proof that any one-sided maximal matching corresponds to a possible outcome of the 1S-BE algorithm (Section G.2):

Proposition 10 *If μ is one-sided maximal then μ is an outcome of the 1S-BE algorithm.*

G.1 Proof of Proposition 9

Before moving to the proof we introduce a new notation. Given matching μ, we denote $\mathcal{B}_\mu$ for the set of blocking pairs of μ.

In the sequel, we fix two matchings μ and μ' such that μ' Pareto-dominates μ for teachers and $\mathcal{B}_{\mu'} \subseteq \mathcal{B}_\mu$. We show below that starting from μ, the graph associated with the 1S-BE algorithm must have a cycle. Hence, any outcome of 1S-BE must be one-sided maximal, as claimed in Proposition 9.

To give the intuition of each step of the proof, which uses a lot of graphical arguments, we will use an example to illustrate each part. This example involves 6 teachers, $t_1, t_1', t_2, t_2', t_3, t_4$ and 4 schools s_1, s_2, s_3, s_4. In the example, matchings μ and μ' are as follows:

$$\mu = \begin{pmatrix} t_1 & t_1' & t_2 & t_2' & t_3 & t_4 \\ s_1 & s_1 & s_2 & s_2 & s_3 & s_4 \end{pmatrix}$$
$$\mu' = \begin{pmatrix} t_1 & t_1' & t_2 & t_2' & t_3 & t_4 \\ s_2 & s_4 & s_3 & s_1 & s_1 & s_2 \end{pmatrix}$$

As in Lemma 1, we can exhibit "cycles of exchanges"that can be used to go from μ to μ' in the proposition. It is worth noting that in the many-to-one environment, these cycles of exchanges are not uniquely defined. Indeed, if for a given selection of cycles of exchanges, there are two nodes that involve the same school, then this cycle can be decomposed into two cycles of exchanges. Figure 1 illustrates this simple fact: in the left part of the figure, starting from μ, there is a a cycle of exchanges that, once implemented leads to μ'. It is easy to see that we can decompose this cycle into two smaller cycles of exchanges, shown in the right part of the figure, that also lead to μ' once implemented.

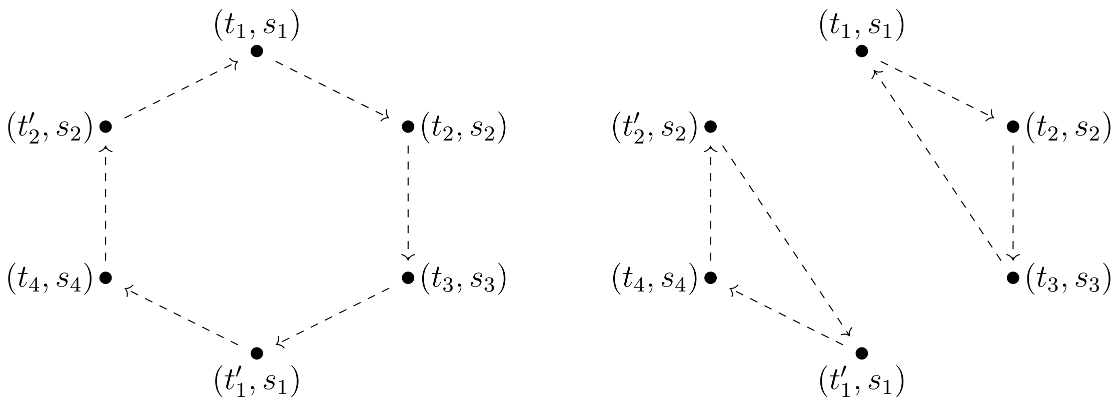
So for the rest of the proof, we fix (a collection of) exchange cycles that takes us from μ to μ' once implemented. To fix ideas, in the example, we consider the one on the left part of Figure 1. In Lemma 1, these cycles of exchanges were actual cycles in the graph associated with BE. However, when considering the graph associated with 1S-BE, this is no longer the case: the cycles of exchanges are not necessarily cycles of the graph associated with 1S-BE. Before moving to the first lemma, we note that all nodes that are not part of cycles of exchanges are those where the teacher of that node has the same allocation between μ and μ'. In the following, the "nodes of the cycles of exchanges"will be all the nodes (t, s) s.t $\mu(t) \neq \mu'(t)$. We will say that a node (t, s) 1S-BE-points to another node (t', s') if (t, s) points toward (t', s') in the graph associated with the 1S-BE algorithm (starting from μ).

Lemma 7 *Fix a node (t, s) of the cycles of exchanges. Then:*

1. *either its predecessor according to the the cycles of exchanges 1S-BE-points toward (t, s);*
2. *or there is a node (t', s') in the cycles of exchanges that such that t' does not block with s under μ, $s \succ_{t'} s'$ and t' has the highest priority among those who desire s but do not block with it under μ. And so (t', s') 1S-BE-points toward (t, s).*

Figure 1: Two equivalent cycles of exchanges in many-to-one.

a) Cycles of exhanges 1 b) Cycles of exhanges 2



Before moving to the proof, let us illustrate this lemma in the example. Assume that all nodes except (t_3, s_3) are 1S-BE-pointed by their predecessors in the cycle of exchanges. According to Lemma 7, there must be a node (t', s') in the cycle of exchanges that 1S-BE-points toward (t_3, s_3). In the graph of Figure 2, this node is assumed to be (t_4, s_4). The dashed edge from (t_2, s_2) to (t_3, s_3) is here to show that this is not an edge of the 1S-BE graph but rather is an edge corresponding to the exchange cycle.

Proof. Call (t'', s'') the predecessor of node (t, s) in the cycles of exchanges so that $s'' := \mu(t'')$ and $s := \mu'(t'')$. Because μ' Pareto-dominates for teachers μ, we know that $s \succ_{t''} s''$ so that t'' desires s under μ. Assume that (t'', s'') does not 1S-BE-point to (t, s). This means that t'' does not block with s under μ and that there is another teacher t' who does not block with s and has the highest priority among those who desire s and do not block with it. Thus, (t', s') (where $s' := \mu(t')$) 1S-BE points toward (t, s). It remains to show that (t', s') is part of the cycles of exchanges. If this was not the case, it would mean that $\mu(t') = \mu'(t') = s'$. Let us recap. We have that t' does not block with s under μ. In addition, by definition of t', we must have that $t' \succ_s t''$ (since t'' does not block with s under μ and desires s). In addition, t' desires s under μ, and so $\mu(t') = \mu'(t')$ implies that t' also desires s under μ'. Hence, because $t'' \in \mu'(s)$, we obtain that t' blocks with s under μ'. This contradicts our assumption that $\mathcal{B}_{\mu'} \subseteq \mathcal{B}_{\mu}$. ∎

Lemma 7 allows us to identify a subgraph (N', E_1) of the 1S-BE graph starting from μ such that N' are the nodes of the cycles of exchanges and the set of edges E_1 is built as follows. We start from $E_1 = \emptyset$ and add the following edges: for each node (t, s) in the cycles of exchange, if its predecessor $(\tilde{t}, \tilde{s})$ under the cycles of exchanges 1S-BE-points to (t, s) then $((\tilde{t}, \tilde{s}), (t, s))$ is added to E_1. If, on the contrary, $(\tilde{t}, \tilde{s})$ does no not 1S-BE-point to (t, s), then we pick the node (t', s') in the cycles of exchanges, identified in the second condition of Lemma 7, that 1S-BE-points toward (t, s) and we add $((\tilde{t}, \tilde{s}), (t, s))$ to E_1. Note that, by construction, each node in N' has a unique in-going edge in (N', E_1). In the example, this subgraph (N', E_1) is given by the right graph of Figure 2 (the solid arrows). Note that this graph admits a cycle: $(t_3, s_3) \rightarrow (t'_1, s_1) \rightarrow (t_4, s_4) \rightarrow (t_3, s_3)$. This is a simple property of digraphs with in-degree one:

Lemma 8 *Fix a finite digraph (N, E) such that each node has in-degree one. There is a cycle in this graph.*

Proof. Fix a node n_1 in the graph (N, E). Because it has in-degree one, we can let n_2 be the unique node pointing to n_1. Again, from n_2 we can let n_3 be the unique node pointing to n_2. Because there are a finite number of nodes in the graph, this process must cycle at some point. ∎

As the example illustrates, applying this lemma to (N', E_1) leads to the following corollary:

Corollary 3 *There is a cycle in the graph associated with 1S-BE starting from μ.*

We are now in a position to prove Proposition 9.

Completion of the proof of Proposition 9. Let μ be an outcome of the 1S-BE algorithm. Proceed by contradiction and assume that μ is not one-sided maximal. Thus, there must be a matching μ' such that μ' Pareto-dominates μ for teachers and $\mathcal{B}_{\mu'} \subseteq \mathcal{B}_{\mu}$. Corollary 3 implies that there must be a cycle in the graph associated with 1S-BE starting from μ, contradicting the fact that μ is an outcome of 1S-BE. ∎

G.2 Proof of Proposition 10

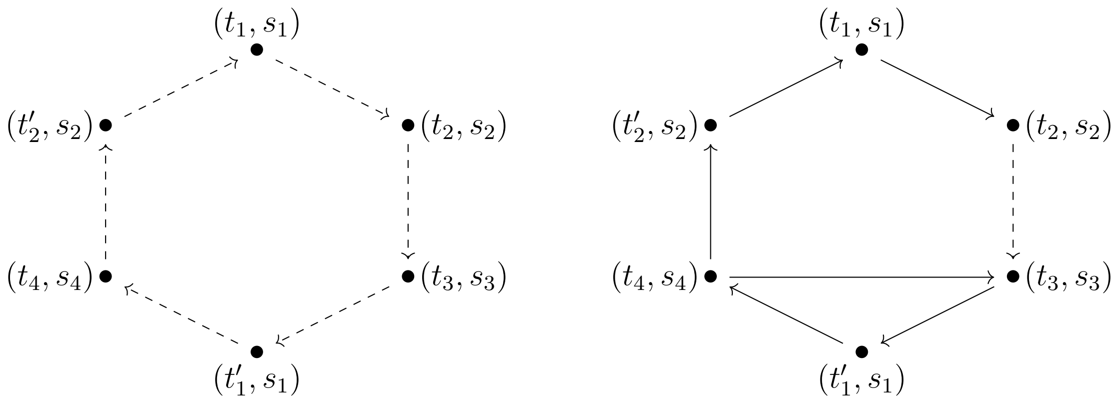
In the sequel, we fix a one-sided maximal matching μ'. We let μ be a matching such that μ' Pareto-dominates for teachers μ and satisfies $\mathcal{B}_{\mu'} \subseteq \mathcal{B}_{\mu}$. We claim there is a cycle in the graph associated with 1S-BE starting from μ which, once implemented, leads to a matching $\tilde{\mu}$ such that μ' Pareto-dominates $\tilde{\mu}$ for teachers and satisfies $\mathcal{B}_{\mu'} \subseteq \mathcal{B}_{\tilde{\mu}}$. Note that this implies Proposition 10. Indeed, because, by definition, μ' Pareto-dominates μ_0 and $\mathcal{B}_{\mu'} \subseteq \mathcal{B}_{\mu_0}$, we must have a cycle in the graph associated with 1S-BE starting from μ_0, which, once implemented, yields to a matching say $\tilde{\mu}_1$ such that μ' Pareto-dominates $\tilde{\mu}_1$ for teachers and satisfies $\mathcal{B}_{\mu'} \subseteq \mathcal{B}_{\tilde{\mu}_1}$. Now, we can iterate the reasoning, and we again see that there is a cycle in the graph associated with 1S-BE starting from $\tilde{\mu}_1$, which, once implemented, yields to a matching say $\tilde{\mu}_2$ such that μ' Pareto-dominates $\tilde{\mu}_2$ for teachers and satisfies $\mathcal{B}_{\mu'} \subseteq \mathcal{B}_{\tilde{\mu}_2}$. We can pursue this reasoning: at some point, because the environment is finite, we must reach matching μ', as we intended to show.

We start by proving a lemma that will be useful in the subsequent arguments. Starting from any matching μ, in the graph associated to the 1S-BE algorithm, if a node points to another node involving school s then it also points to all other nodes involving school s.

Figure 2: Cycles of exchanges and (N', E_1).

a) Cycle of exhanges

b) (N', E_1)



Lemma 9 *Let (t, s) be a node in the graph associated with the 1S-BE algorithm, starting from a matching μ. If (t, s) points to (t', s') then (t, s) points to (t'', s') for all $t'' \in \mu(s')$.*

Proof. Let us assume that (t, s) points to (t', s') and consider any node (t'', s'). Let us first consider the case where $t \succ_{s'} t''$. Given that (t, s) points to (t', s'), we must have that t desires s'. Hence, t has justified envy toward t'' and so, by definition of 1S-BE, (t, s) must point to (t'', s'). Now, consider the other case in which $t'' \succ_{s'} t$. We have to check that t is preferred by s' to each teacher who desires s' and does not block with it. If $t \succ_{s'} t'$, then for any teacher $\tilde{t}$ who desires s' and does not block with s', we have that $t' \succ_{s'} \tilde{t}$ which implies $t \succ_{s'} \tilde{t}$ so that t is preferred by s' to those who desire s' and do not block with it. Hence, (t, s) must point to (t'', s'). Now, if $t' \succ_{s'} t$, because we know that (t, s) points to (t', s'), t must be preferred by s' to those who desire s' and do not block with it so that (t, s) must also point to (t'', s'). ∎

In the sequel, as in the proof of Proposition 9, we fix (a collection of) cycles of exchanges which takes us from μ to μ' once implemented. We consider the digraph (N', E_1) as built in Section G after Lemma 7. Consider a cycle C_1 in this graph (which exists by Lemma 8). Let μ_1 be the matching obtained once the cycle C_1 is implemented. In the example introduced in Section G, this matching would be:

$$\mu_1 = \begin{pmatrix} t_1 & t'_1 & t_2 & t'_2 & t_3 & t_4 \\ s_1 & s_4 & s_2 & s_2 & s_1 & s_3 \end{pmatrix}$$

We first show the following lemma.

Lemma 10 *μ' Pareto-dominates μ_1 for teachers.*

Proof. Fix a teacher t. If the node (t,s) to which t belongs is not part of the cycles of exchanges, we know t does not move from μ to μ' and so (t,s) is not in the cycle C_1. Hence, $\mu(t) = \mu_1(t) = \mu'(t)$. So assume that (t,s) is part of the cycles of exchanges and let $s := \mu(t)$ and $s' := \mu'(t)$ with $s \neq s'$. There are three possible cases:

- *Case 1:* $s = \mu_1(t) \neq s'$. Because μ' Pareto-dominates μ for teachers, we have that $\mu'(t) = s' \succeq_t \mu_1(t) = \mu(t) = s$.
- *Case 2:* $s \neq \mu_1(t) = s'$. In such a case, we trivially have $\mu'(t) \succeq_t \mu_1(t)$.
- *Case 3:* $s \neq \mu_1(t) := s_1 \neq s'$. By construction of the graph (N', E_1) when we implement cycle C_1, we know that there is a unique edge $\big((t,s),(t_1,s_1)\big)$ in C_1 and that (t,s) is not the predecessor of (t_1,s_1) under the cycles of exchanges, since otherwise, t would be matched to s' under μ_1, which is not the case by assumption. Hence, by construction of (N', E_1), the predecessor of (t_1,s_1) under the cycles of exchanges, say (t'', s''), does not 1S-BE point to (t_1,s_1) and, in addition, t does not block with s_1 under μ, $s_1 \succ_t s$, and t has the highest priority among those who desire s_1 but do not block with it under μ and 1S-BE-points to (t_1,s_1). Because (t'',s'') does not 1S-BE point to (t_1,s_1), we know that t'' does not block with s_1. While because (t'',s'') points to (t_1,s_1) under the cycles of exchange, we must have that t'' desires s_1. Thus, we conclude that $t \succ_{s_1} t''$.

  Now, proceed by contradiction and assume that $(\mu_1(t) =)s_1 \succ_t s'(= \mu'(t))$. Because $t'' \in \mu'(s_1)$ (recall that (t'',s'') is the predecessor of (t_1,s_1) under the cycles of exchange) and $t \succ_{s_1} t''$, we have that t blocks with s_1 under μ' i.e. $(t,s_1) \in \mathcal{B}_{\mu'}$. But, as already claimed, $(t,s_1) \notin \mathcal{B}_{\mu}$. This contradicts that $\mathcal{B}_{\mu'} \subseteq \mathcal{B}_{\mu}$. Thus, we must have $\mu'(t) \succeq_t \mu_1(t)$.[74]

So we have shown that $\forall t$, $\mu'(t) \succeq_t \mu_1(t)$. ∎

If we were sure that it is always the case that $\mathcal{B}_{\mu'} \subseteq \mathcal{B}_{\mu_1}$, the proof would be completed. Unfortunately, even if this is true in the one-to-one environment, it may not be true in the many-to-one case. To give an intuition, assume that in the example we have $t_1 \succ_{s_1} t_3 \succ_{s_1} t_2 \succ_{s_1} t'_2 \succ_{s_1} t'_1$ and $s_1 \succ_{t_2} s_3 \succ_{t_2} s_2$. So t_2 blocks with s_1 under both μ and μ'. But after implementing cycle C_1, we see that t_2 does not block with s_1 anymore. Indeed, the only teacher for whom t_2 feels justified envy under μ is t'_1. But t'_1 is replaced by t_3 once C_1 is implemented, and t_1 has a higher priority than t_2 at s_1 (while t_1 stays matched to s_1). We

[74] Case 3 in Lemma 10 can be illustrated in the example. The node (t,s) would be (t_4,s_4) in the right graph of Figure 2. t_4 is matched to s_3 under μ_1 but is matched to s_2 under μ'. Under C_1 (i.e., $(t_3,s_3) \to (t'_1,s_1) \to (t_4,s_4) \to (t_3,s_3)$), node (t_4,s_4) points to (t_3,s_3) while (t_2,s_2) does not 1S-BE-point to (t_3,s_3). Because (t_2,s_2) points to (t_3,s_3) in the cycle of exchanges, it means that $t_2 \in \mu'(s_3)$ so that if t_4 preferred s_3 to his match under μ', s_2, it would imply that t_4 blocks with s_3 under μ' while he does not under μ, and so this would yield the contradiction.

will show that if this arises, we can find another subgraph of the 1S-BE graph starting from μ, call it (N', E_2), still with in-degree one for each node so that there is a cycle C_2 in this subgraph and the matching μ_2 obtained with this cycle keeps the blocking pair (t_2, s_1).

Lemma 11 *Assume there is $(t_1, s_1) \in \mathcal{B}_{\mu'}$ but not in $\mathcal{B}_{\mu_1}$. Then there is a teacher t_1^* with $s_1^* := \mu(t_1^*)$ s.t $t_1 \succ_{s_1} t_1^*$, and (t_1^*, s_1^*) is part of the cycles of exchanges and points to all nodes of the form (t, s_1) in the graph associated with 1S-BE starting from μ.*

Before moving to the proof, let us illustrate the lemma in the example. As explained above, the "problem" in the graph (N', E_1) (which yields to $(t_2, s_1) \notin \mathcal{B}_{\mu_1}$) is that the node pointing to (t_1', s_1) is (t_3, s_3) and that $t_3 \succ_{s_1} t_2$. But we have teacher t_2' who is less preferred by s_1 than t_2, is not matched to s_1 under μ, but is under μ'. In addition, since we assumed that node (t_2', s_2) is 1S-BE-pointing to (t_1, s_1) under μ, we can use Lemma 9 to be sure it is also pointing to (t_1', s_1) so that t_1^* in the above lemma would be t_2' in the example. The argument in the proof below shows that this construction can be made in general.

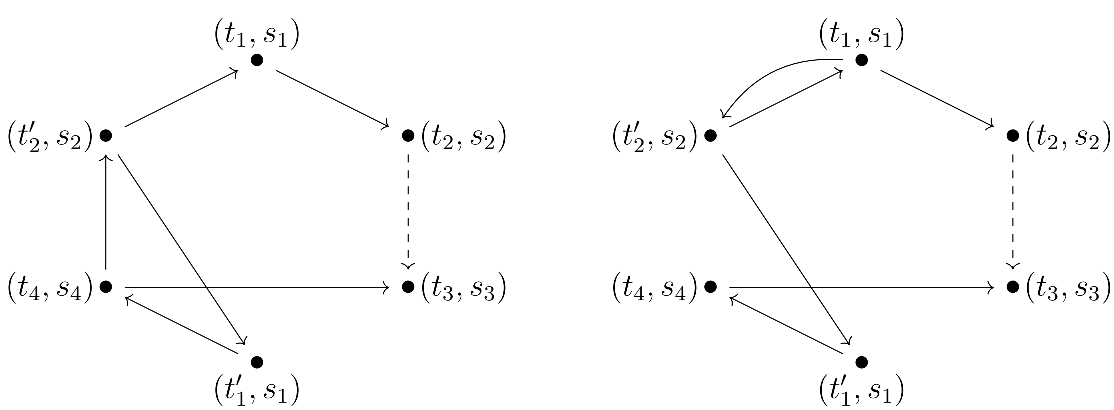
Proof. Note first that because t_1 desires s_1 under μ and μ', t_1 must also desire s_1 under μ_1 because, by Lemma 10, $\mu'(t_1) \succeq_{t_1} \mu_1(t_1)$. Now, because t_1 blocks with s_1 under μ, it means that there is $t \in \mu(s_1)$ s.t $t_1 \succ_{s_1} t$. Fix one such teacher t. Since, by assumption, $(t_1, s_1) \notin \mathcal{B}_{\mu_1}$, it means that t is not matched to s_1 under μ_1 and so, when implementing C_1, t has been replaced by a teacher t' such that $t' \succ_{s_1} t_1$ since t_1 does not block with s_1 under μ_1 but desires s_1 under μ_1. Since t_1 blocks with s_1 under μ' it means that there is a teacher $t_1' \in \mu'(s_1)$ s.t $t_1 \succ_{s_1} t_1'$, let $s_1' := \mu(t_1')$. Note first that (t_1', s_1') is part of the cycles of exchanges. To see this, observe that if it was not the case then we would have that $t_1' \in \mu(s_1)$, but because (t_1, s_1) does not block μ_1, $t_1' \notin \mu_1(s_1)$. Because $\mu_1(t_1) \succ_{t_1} \mu(t_1) = s_1$ and, by Lemma 10, $\mu'(t_1) \succeq_{t_1} \mu_1(t_1)$, we conclude that t_1 cannot be matched to s_1 under μ', a contradiction. Hence, if the node (t_1', s_1') 1S-BE-points to (t, s_1) then we can set $t_1^* := t_1'$ and $s_1^* := s_1'$, and the argument is complete using Lemma 9. Now consider the case in which node (t_1', s_1') does not 1S-BE-point to (t, s_1). We already know that (t_1', s_1') is part of the cycles of exchanges, so let $(\tilde{t}, s_1)$ be its successor under these cycles of exchanges (s_1 has to be part of this node since $t_1' \in \mu'(s_1)$). If (t_1', s_1') was 1S-BE-pointing to $(\tilde{t}, s_1)$, then by Lemma 9 it would also point to (t, s_1), a contradiction. So node (t_1', s_1') does not 1S-BE-point to its successor under the cycles of exchanges, i.e., $(\tilde{t}, s_1)$. Thus, we have that t_1' does not block with s_1 under μ (if he were to block with s_1, (t_1', s_1') would be 1S-BE-pointing to some node which includes school s_1 and so toward $(\tilde{t}, s_1)$, a contradiction) and, by condition 2 of Lemma 7, there is a teacher t_1'', whose node is part of the cycles of exchanges, who does not block with s_1 under μ, desires s_1, and has the highest priority among those who do not block with s_1 under μ and desire it. In particular, the node $(t_1'', \mu(t_1''))$ 1S-BE-points to $(\tilde{t}, s_1)$, and so by Lemma 9 points also to (t, s_1). Since t_1'' does not block with s_1 under μ but t_1 does, it means that $t_1 \succ_{s_1} t_1''$, so we can set $t_1^* := t_1''$ and $s_1^* := \mu(t_1'')$. Here again, we can use Lemma 9 to make sure that (t_1^*, s_1^*) indeed points to all the nodes (t, s_1) under the graph of 1S-BE. ∎

Coming back to our example, we can modify the graph (N', E_1) by deleting the edge $\big((t_3, s_3), (t'_1, s_1)\big)$ and replacing it with $\big((t'_2, s_2), (t'_1, s_1)\big)$. In doing so, we obtain a new subgraph of 1S-BE wherein each node still has in-degree one and so still has a cycle. But, by constructing this new graph, the matching once the new cycle is implemented keeps (t_2, s_1) as a blocking pair. This is illustrated in the left graph in Figure 3, and the new cycle is now $(t'_2, s_2) \to (t'_1, s_1) \to (t_4, s_4) \to (t'_2, s_2)$. The general procedure is given below.

Let us assume there is a node (t_1, s_1) such that it is in $\mathcal{B}_{\mu'}$ but not in $\mathcal{B}_{\mu_1}$. Fix a teacher $t \in \mu(s_1)$ such that $t_1 \succ_{s_1} t$. We know that t must leave s_1 under μ_1 (because (t_1, s_1) does not block μ_1) and is replaced by a teacher t' such that $t' \succ_{s_1} t_1$. Since the teacher t_1^* identified in Lemma 11 satisfies $t_1 \succ_{s_1} t_1^*$, we have that $t' \neq t_1^*$. So under the graph (N', E_1), because t' replaces t when at s_1 when we implement C_1, we must have that $\big((t', \mu(t')), (t, s_1)\big) \in E_1$, and because each node has in-degree one, $\big((t_1^*, s_1^*), (t, s_1)\big) \notin E_1$. But that node is an edge in the graph of 1S-BE by construction. We therefore define a new graph (N', E_2) where E_2 corresponds to E_1, all edges of the form $\big((t', \mu(t')), (t, s_1)\big)$ with $t_1 \succ_{s_1} t$ have been replaced by $\big((t_1^*, s_1^*), (t, s_1)\big)$, and (t_1^*, s_1^*) is as in Lemma 11. So (N', E_2) is still a subgraph of the 1S-BE graph starting from μ, and all the nodes in N' still have in-degree one so that, using Lemma 8, we have a cycle C_2. We let μ_2 be the matching obtained once C_2 is implemented.

Figure 3: Graphs of (N', E_2) and (N', E_3).

a) (N', E_2). b) (N', E_3).



We obtain:

Lemma 12 *We have that:*

1. *μ' Pareto-dominates μ_2 for teachers.*

2. $(t_1, s_1) \in \mathcal{B}_{\mu_2}$.

Proof. For part 1, fix a teacher t and let $s := \mu(t)$. Without loss, assume that (t, s) is part of the cycles of exchanges. First note that the only new edges (i.e. those in $E_2 \backslash E_1$) are those of the form $\big((t_1^*, s_1^*), (\tilde{t}, s_1)\big)$ for $\tilde{t} \in \mu(s_1)$ such that $t_1 \succ_{s_1} \tilde{t}$. So if the edge that matched t under C_2 is an old one (i.e. belongs to E_1), the same argument as in Lemma 10 can be used. Assume the edge that matched t is of the form $\big((t_1^*, s_1^*), (\tilde{t}, s_1)\big)$, so that $t = t_1^*$ and $\mu_2(t) = s_1$. Using the same notations as in Lemma 11, there are two cases to consider:

- *Case 1:* $t_1^* = t_1'$. In that case, we know that $t_1' \in \mu'(s_1)$ and so, trivially, that $s_1 = \mu'(t_1^*) \succeq_{t_1^*} \mu_2(t_1^*) = s_1$.
- *Case 2:* $t_1^* = t_1''$. If $\mu'(t_1'') = s_1$ then trivially, $\mu'(t_1'') \succeq_{t_1''} \mu_2(t_1'') = s_1$. Assume that $\mu'(t_1'') \neq \mu_2(t_1'')$ and toward a contradiction that, $\mu_2(t_1'') = s_1 \succ_{t_1''} \mu'(t_1'')$. By the proof of Lemma 11, we know that t_1'' does not block with s_1 under μ, and since $\mathcal{B}_{\mu'} \subseteq \mathcal{B}_\mu$, he does not block with s_1 under μ' either. Again, using the proof of Lemma 11, we know that because, by assumption, (t_1', s_1') does not 1S-BE-point to (t, s_1), t_1' therefore does not block with s_1 under μ. In addition, since $t_1' \in \mu'(s_1)$, we must have t_1' desires s_1 Thus, because, by construction of t_1'', teacher t_1'' has the highest priority among those who do not block with s_1 under μ and desire s_1, we must have $t_1'' \succ_{s_1} t_1'$. Because $t_1' \in \mu'(s_1)$ and by assumption t_1'' desires s_1 under μ', we obtain that $(t_1'', s_1) \in \mathcal{B}_{\mu'}$, which yields a contradiction since, again by construction of t_1'', we must have $(t_1'', s_1) \notin \mathcal{B}_\mu$.

For part 2 assume that $(t_1, s_1) \notin \mathcal{B}_{\mu_2}$. Since $(t_1, s_1) \in \mathcal{B}_{\mu'}$, we have that $s_1 \succ_{t_1} \mu'(t_1)$. In addition, by Lemma 10, $\mu'(t_1) \succeq_{t_1} \mu_2(t_1) \succeq_{t_1} \mu(t_1)$ and so $s_1 \succ_{t_1} \mu_2(t_1)$. Then because $(t_1, s_1) \notin \mathcal{B}_{\mu_2}$, we must have that all teachers t s.t $t \in \mu(s_1)$ and $t_1 \succ_{s_1} t$ are not matched to s_1 anymore under μ_2, i.e., once cycle C_2 is implemented. But under (N', E_2) the only incoming edge for a node (t, s_1) with $t_1 \succ_{s_1} t$ is $\big((t_1^*, s_1^*), (t, s_1)\big)$, and, since $t_1 \succ_{s_1} t_1^*$, it contradicts that $(t_1, s_1) \notin \mathcal{B}_{\mu_2}$ since t_1 feels justified envy toward t_1^* under μ_2. ∎

As for μ_1, if we were sure that $\mathcal{B}_{\mu'} \subseteq \mathcal{B}_{\mu_2}$, the proof would be completed. However, as for μ_1, this may not be the case. For instance, in the example, if we assume that $t_2 \succ_{s_2} t_4 \succ_{s_2} t_3 \succ_{s_2} t_1 \succ_{s_2} t_2'$ and $s_2 \succ_{t_3} s_1 \succ_{t_3} s_3$, we have that $(t_3, s_2) \in \mathcal{B}_{\mu'} \subset \mathcal{B}_\mu$. Then, when we implement the cycle C_2 given in the left graph of Figure 3, we can see that we delete the blocking pair (t_3, s_2) and so $(t_3, s_2) \notin \mathcal{B}_{\mu_2}$. With this observation in mind, the idea now is to define a new graph, as we did when we constructed (N', E_2) from (N', E_1), in order to be sure that this is a subgraph of 1S-BE and that it contains a cycle C_3, which, once implemented, yields a matching that keeps the desired blocking pairs.

For the general case, assume there is a pair (t_2, s_2) s.t $(t_2, s_2) \in \mathcal{B}_{\mu'} \subseteq \mathcal{B}_\mu$ but $(t_2, s_2) \notin \mathcal{B}_{\mu_2}$. In that case, we can apply exactly the same argument as in Lemma 11 and exhibit a teacher t_2^* such that $t_2 \succ_{s_2} t_2^*$ and (t_2^*, s_2^*) 1S-BE-points to all the nodes of the form (t, s_2) under the graph of 1S-BE starting at μ. However, when $s_2 = s_1$ if $t_2^* \succ_{s_1} t_1^*$ then we reset t_2^* to be t_1^*.

Then, we define a new graph (N', E_3) with E_3 where E_3 corresponds to E_2 where all edges of the form $\big((t', \mu(t')), (t, s')\big)$ with $t_2 \succ_{s_2} t$ are replaced by $\big((t_2^*, s_2^*), (t, s_1)\big)$ where (t_2^*, s_2^*) as in the above paragraph. Here again, (N', E_3) is indeed a subgraph of the graph associated with 1S-BE starting from μ, and each node still has in-degree one. Applying Lemma 8, we get the existence of a cycle C_3 which, once implemented, yields to a matching μ_3. In the example, t_2^* would be t_1 and (N', E_3) is shown in the right graph of Figure 3.

It is easy to see that we can mimic the proof of Lemma 12 in order to obtain the following lemma.

Lemma 13 *We have that:*

1. *μ' Pareto-dominates μ_3 for teachers.*
2. *$\{(t_1, s_1), (t_2, s_2)\} \subset \mathcal{B}_{\mu_3}$.*

In the example, the unique t_2^* is t_1 and the graph of (N', E_3) is given in the right graph of Figure 3. In that case, the cycle C_3 is $(t_1, s_1) \leftrightarrows (t_2', s_2)$. Note that, once C_3 is implemented, we indeed have $\{(t_2, s_1), (t_3, s_2)\} \subset \mathcal{B}_{\mu_3}$ and $\mathcal{B}_{\mu'} \subseteq \mathcal{B}_{\mu_3}$ so that we have found the desired matching.

Of course, in full generality, it is possible to have a pair (t_3, s_3) satisfying $(t_3, s_3) \in \mathcal{B}_{\mu'} \subseteq \mathcal{B}_\mu$ while $(t_3, s_3) \notin \mathcal{B}_{\mu_3}$. In order to prove the desired result – namely that there is a cycle in the graph associated with 1S-BE starting from μ which, once implemented, leads to a matching $\tilde{\mu}$ such that μ' Pareto-dominates $\tilde{\mu}$ for teachers and satisfies $\mathcal{B}_{\mu'} \subseteq \mathcal{B}_{\tilde{\mu}}$ – we would continue to apply the same logic. Because we have a finite environment, at some point we must find a matching $\tilde{\mu}$ with the desired property.

H Proof of Theorem 5

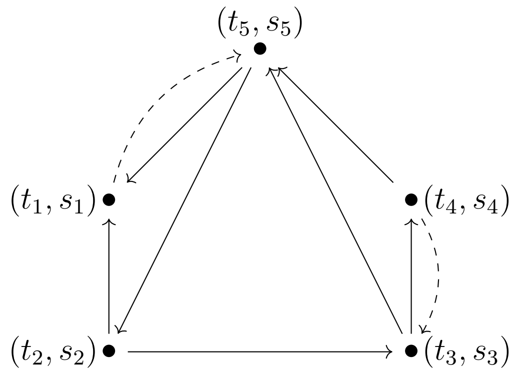
In order to prove this result, we exhibit an instance where, irrespective of which (sequence of) cycle(s) one selects in the graphs associated with 1S-BE , one teacher will gain by misreporting his preferences. Assume that there are five teachers $t_1, \ldots, t_5$ and five schools $s_1, \ldots, s_5$. Teachers' and schools' preferences are given as follows:

$$
\begin{array}{llllllllll}
\succ_{t_1}: & s_5 & s_1 & & \quad \succ_{s_1}: & t_5 & t_2 & t_1 & & \\
\succ_{t_2}: & s_1 & s_3 & s_2 & \quad \succ_{s_2}: & t_5 & t_2 & & & \\
\succ_{t_3}: & s_4 & s_5 & s_3 & \quad \succ_{s_3}: & t_3 & t_2 & t_4 & & \\
\succ_{t_4}: & s_5 & s_3 & s_4 & \quad \succ_{s_4}: & t_3 & t_4 & & & \\
\succ_{t_5}: & s_2 & s_1 & s_5 & \quad \succ_{s_5}: & t_4 & t_2 & t_5 & t_3 & t_1
\end{array}
$$

We let $\succ := (\succ_{t_1}, \ldots, \succ_{t_5})$. The initial assignment is given by:

$$\mu_0 = \left(\begin{array}{ccccc} t_1 & t_2 & t_3 & t_4 & t_5 \\ s_1 & s_2 & s_3 & s_4 & s_5 \end{array} \right)$$

Starting from the initial assignment, the solid arrows in the graph below correspond to the graph associated with 1S-BE.



We add dashed arrows from one node to another if the teacher in the origin of the arrow prefers the school in the pointed node. These arrows are not actual arrows of the graph associated with 1S-BE and therefore cannot be used to select a cycle. These arrows only facilitate understanding of the argument.

When $\succ$ is submitted, there are two possible choices of cycles in the graph:

- A large cycle given by $(t_2, s_2) \to (t_3, s_3) \to (t_4, s_4) \to (t_5, s_5) \to (t_2, s_2)$. Denote this cycle by $\bar{C}$.
- A small cycle given by $(t_2, s_2) \to (t_3, s_3) \to (t_5, s_5) \to (t_2, s_2)$. Denote this cycle by $\underline{C}$.

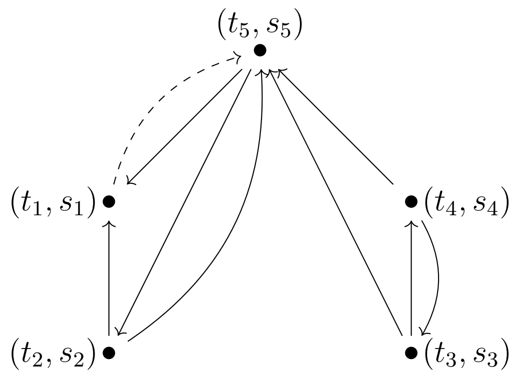
We decompose the analysis for these two cases.

<u>*Case A: Under $\succ$, $\bar{C}$ is selected:*</u>

Once this cycle is cleared, there are no cycles left in the graph associated with 1S-BE, and the final matching of 1S-BE is given by

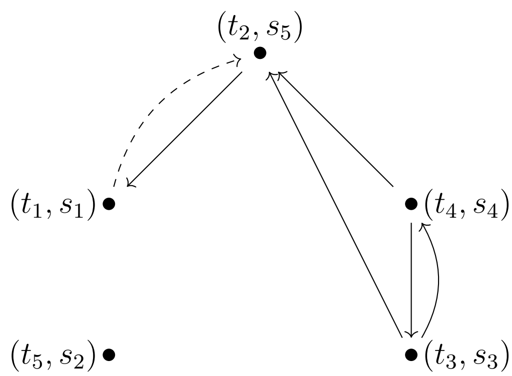
$$\bar{\mu} = \begin{pmatrix} t_1 & t_2 & t_3 & t_4 & t_5 \\ s_1 & s_3 & s_4 & s_5 & s_2 \end{pmatrix}$$

Now, assume that teacher t_2 reports the following preference relation $\succ'_{t_2}$: s_1, s_5, s_2, while others report according to $\succ$. Under this profile, starting from the initial assignment, the graph associated with 1S-BE is



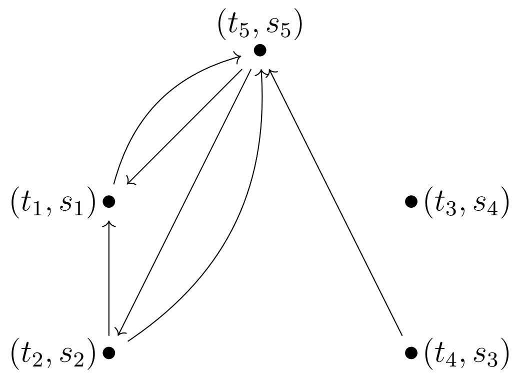
Now, there are two possible cycle choices.

Case A.1: The cycle chosen is $(t_2, s_2) \leftrightarrows (t_5, s_5)$. Once carried out, the graph associated with 1S-BE starting from the new matching is



Clearly, there is a unique cycle $(t_4, s_4) \leftrightarrows (t_3, s_3)$. Consider the new matching once this cycle is implemented. Teacher t_3 obtains his most favorite school. Hence, in the graph associated with 1S-BE starting from the new matching, node (t_1, s_1) will now point to node (t_2, s_5). In this graph, the only cycle is $(t_2, s_5) \leftrightarrows (t_1, s_1)$; therefore, t_2 is eventually matched to school s_1. Hence, t_2 obtains his most preferred school under $\succ_{t_2}$, and we exhibit a profitable misreport.

Case A.2: The cycle chosen is $(t_4, s_4) \leftrightarrows (t_3, s_3)$. Once carried out, the graph associated with 1S-BE starting from the new matching is



In this graph, there are three possible cycle choices:

1. $(t_2, s_2) \to (t_1, s_1) \to (t_5, s_5) \to (t_2, s_2)$: in that case, t_2 is matched to s_1 and so, again, we identified a profitable misreport.

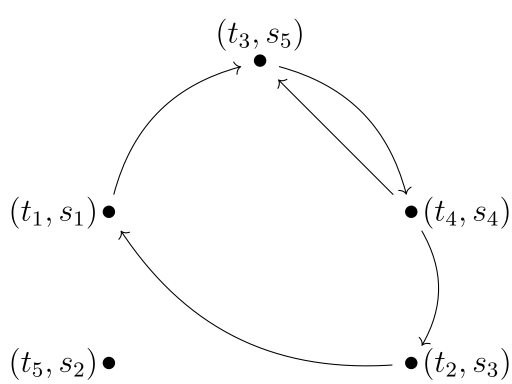
2. $(t_2, s_2) \leftrightarrows (t_5, s_5)$: Once cleared, the only cycle that is left is $(t_1, s_1) \leftrightarrows (t_2, s_5)$; therefore, t_2 will be matched to s_1, leading to a successful manipulation.

3. $(t_1, s_1) \leftrightarrows (t_5, s_5)$: Once cleared, since t_5 prefers s_2 to s_1, there is a unique cycle left: $(t_5, s_1) \leftrightarrows (t_2, s_2)$. Once again, the manipulation of t_2 is successful.

Thus, we have shown that, when cycle $\overline{C}$ is selected under the profile $\succ$, teacher t_2 has a profitable misreport irrespective of the possible selections of cycles performed after t_2's deviation. Let us now move to the other case.

Case B: Under $\succ$, $\underline{C}$ is selected:

Once this cycle is carried out, the graph associated with 1S-BE starting from the new matching is



There are two possible cycle choices.

Case B.1: Choose $(t_3, s_5) \leftrightarrows (t_4, s_4)$. Then, the matching obtained is the same as the one obtained when we selected cycle $\bar{C}$. Therefore, we can come back to *Case A*, and we know that t_2 has a successful misreport.

Case B.2: Choose $(t_1, s_1) \to (t_3, s_5) \to (t_4, s_4) \to (t_2, s_3) \to (t_1, s_1)$. In this case, each teacher but teacher t_4 gets his most preferred school. Hence, there are no more cycles in the new graph associated with 1S-BE. In particular, teacher t_4 is matched to school s_3. Now, assume that t_4 submits the following preferences: $\succ'_{t_4}$: s_5, s_4. The graph associated with 1S-BE starting from the initial assignment is the same as the one under truthful reports (note that, although these are not the arrows of the graph of 1S-BE, the dashed arrow from (t_4, s_4) disappears). Therefore, again, we are left with a choice between cycle $\bar{C}$ and $\underline{C}$.

1. If we carry out $\underline{C}$, the graph starting from the new matching will be given by the graph just above, except that (t_4, s_4) no longer points to (t_2, s_3). Hence, we can pick only cycle $(t_3, s_5) \leftrightarrows (t_4, s_4)$; therefore, t_4 obtains his best school, and we identify a profitable misreport for teacher t_4.

2. If we select $\bar{C}$, we already know that we end up with matching $\bar{\mu}$, as defined above. Therefore, here again, t_4 obtains his best school s_5 and the manipulation is also a success.

To sum up, we have shown that, for each possible cycle selection under 1S-BE, there is a teacher who has a profitable misreport. Thus, no selection of the 1S-BE algorithm is strategy-proof, as we intended to show.

I Proof of Theorem 7 and 8

I.1 Preliminaries in random graph

In the sequel, we will exploit two standard results in random graph theory that are stated in this section. It is thus worth introducing the relevant model of random graph. A **graph** $G(n)$ consists of n vertices, V, and edges $E \subseteq V \times V$ across V. A **bipartite graph** $G_b(n)$ consists of $2n$ vertices $V_1 \cup V_2$ (each of equal size) and edges $E \subset V_1 \times V_2$ across V_1 and V_2 (with no possible edges within vertices in each side). Random (bipartite) graphs can be seen as random variables over the space of (bipartite) graphs. We will see two asymptotic properties of random graphs: one based on the notion of perfect matchings, the other on that of independent sets.

A **perfect matching** of $G_b(n)$ is a subset E' of E such that each node in $V_1 \cup V_2$ is contained in a single edge of E'.

Lemma 14 (Erdös-Rényi) *Fix $p \in (0,1)$. Consider a random graph that selects a graph $G_b(n)$ with the following procedure. Each pair $(v_1, v_2) \in V_1 \times V_2$ is linked by an edge with probability p independently (of edges created for all other pairs). The probability that there is a perfect matching in a realization of this random graph tends to 1 as $n \to \infty$.*

The second important technical result is about so-called independent sets. An **independent set** of $G(n)$ is $\bar{V} \subseteq V$ such that for any $(v_1, v_2) \in \bar{V} \times \bar{V}$, (v_1, v_2) is not in E.

Lemma 15 (Grimmett and McDiarmid, 1975) *Fix $p \in (0,1)$. Consider a random graph that selects a graph $G(n)$ with the following procedure. Each pair $(v_1, v_2) \in V \times V$ is linked by an edge with probability p independently (of edges created for all other pairs). Then,*

$$\Pr\left\{\exists \textit{ an independent set } \bar{V} \textit{ such that } |\bar{V}| \geq \frac{2\log n}{\log \frac{1}{1-p}}\right\} \to 0 \textit{ as } n \to \infty.$$

I.2 Proof of Theorem 7

In the sequel, we fix μ_0 and let T_k be $\mu_0(S_k)$, where μ_0 is the initial allocation. We will prove the following result, which implies the first part of Theorem 7.

Proposition 11 *Consider any selection φ of the BE-algorithm. Fix any k. Let $\bar{T}_k := \{t \in T_k \,|\, \varphi(t) \neq \mu_0(t)\}$. We have*

$$\frac{|\bar{T}_k|}{|T_k|} \xrightarrow{p} 1.$$

Proof of Proposition 11. Fix an arbitrary k and fix $\varepsilon > 0$. We define a random graph with $\{(t, \mu_0(t))\}_{t \in T_k}$ as the set of vertices. An edge between $(t, \mu_0(t))$ and $(t', \mu_0(t'))$ is added if and only if $\xi_{t\mu_0(t')} > 1-\varepsilon$ and $\xi_{t'\mu_0(t)} > 1-\varepsilon$ and $\eta_{t'\mu_0(t)} > 1-\varepsilon$ and $\eta_{t\mu_0(t')} > 1-\varepsilon$. Then, in the random graph, each edge between $(t, \mu_0(t))$ and $(t', \mu_0(t'))$ is added independently with probability $\varepsilon^4 \in (0,1)$. Then, let

$$\hat{T}_k := \{t \in T_k \,\big|\, \varphi(t) = \mu_0(t) \text{ and } U_t(\mu_0(t)) \leq u_k + 1 - \varepsilon \text{ and } V_{\mu_0(t)}(t) \leq 1 - \varepsilon\}$$

It must be that $\{(t, \mu_0(t))\}_{t \in \hat{T}_k}$ is an independent set, or else if there is an edge $(t, \mu_0(t)), (t', \mu_0(t'))$ where $t, t' \in \hat{T}_k$ for some realization of the random graph, then

$$U_t(\mu_0(t')) > u_k + 1 - \varepsilon \geq U_t(\mu_0(t)) = U_t(\varphi(t)) \text{ and } V_{\mu_0(t')}(t) > 1 - \varepsilon \geq V_{\mu_0(t')}(t') = V_{\mu_0(t')}(\varphi(\mu_0(t')))$$

and similarly,

$$U_{t'}(\mu_0(t)) > u_k + 1 - \varepsilon \geq U_{t'}(\mu_0(t')) = U_{t'}(\varphi(t')) \text{ and } V_{\mu_0(t)}(t') > 1 - \varepsilon \geq V_{\mu_0(t)}(t) = V_{\mu_0(t)}(\varphi(\mu_0(t))).$$

Put another way, both $(t, \mu_0(t'))$ and $(t', \mu_0(t))$ block φ. Since, by definition, under φ, t is assigned $\mu_0(t)$ and t' is assigned $\mu_0(t')$, there are still cycles in the graph associated with BE when starting from the assignment given by φ, which contradicts the fact that φ is a selection of BE.

Now, we can use Lemma 15 to obtain that $\Pr\left\{|\hat{T}_k| \geq \frac{2\log(|T_k|)}{\log\frac{1}{1-p}}\right\} \to 0$ as $n \to \infty$ and thus, $\frac{|\hat{T}_k|}{|T_k|} \xrightarrow{p} 0$ as $n \to \infty$. Setting $\tilde{T}_k := \{t \in T_k \,|U_t(\mu_0(t)) \leq u_k + 1 - \varepsilon \text{ and } V_{\mu_0(t)}(t) \leq 1 - \varepsilon\}$, we have

$$\frac{|\hat{T}_k|}{|T_k|} = \frac{|\bar{T}_k^c \cap \tilde{T}_k|}{|T_k|} = \frac{|\bar{T}_k^c \backslash \tilde{T}_k^c|}{|T_k|} \geq \frac{|\bar{T}_k^c|}{|T_k|} - \frac{|\tilde{T}_k^c|}{|T_k|}.$$

We know that, for the left hand-side above, $\frac{|\hat{T}_k|}{|T_k|} \xrightarrow{p} 0$ as $n \to \infty$. By the law of large numbers, $\frac{|\tilde{T}_k^c|}{|T_k|} \xrightarrow{p} 1 - (1-\varepsilon)^2$, which can be made arbitrarily close to 0, given that $\varepsilon > 0$ is arbitrary. Hence, we obtain that $\frac{|\bar{T}_k^c|}{|T_k|} \xrightarrow{p} 0$ as $n \to \infty$, as we intended to prove.

Let us now move to the other part of Theorem 7. We aim to show that there is a selection of BE that is asymptotically teacher-efficient, asymptotically school-efficient, and asymptotically stable. Note that, in our environment, asymptotic school-efficiency implies asymptotic stability. Hence, the following proposition is sufficient for this purpose.

Proposition 12 *There is a mechanism φ that is a selection of the BE algorithm such that, for any k and any $\varepsilon > 0$, we have*

$$\frac{|\bar{T}_k|}{|T_k|} \xrightarrow{p} 1 \text{ and } \frac{|\bar{S}_k|}{|S_k|} \xrightarrow{p} 1$$

where $\bar{T}_k := \{t \in T_k \,|U_t(\varphi(t)) \geq u_k + 1 - \varepsilon\}$ and $\bar{S}_k := \{s \in S_k \,|V_s(\varphi(s)) \geq 1 - \varepsilon\}$.

Proof of Proposition 12. Fix $\varepsilon > 0$. We show that there exists a 2-IR mechanism ψ s.t. for each $k = 1, ..., K$, it matches each teacher $t \in T_k$ to a school in S_k and for each $\delta > 0$:

$$\Pr\left\{\frac{\left|\left\{t \in T_k \,|\xi_{t\psi(t)} \geq 1 - \varepsilon\right\}\right|}{|T_k|} > 1 - \delta\right\} \to 1$$

and

$$\Pr\left\{\frac{\left|\left\{s \in S_k \,|\eta_{\psi(s)s} \geq 1 - \varepsilon\right\}\right|}{|S_k|} > 1 - \delta\right\} \to 1$$

as $n \to \infty$ where we recall that $T_k := \mu_0(S_k)$. This turns out to be enough for our purposes. Indeed, consider the matching mechanism given by $\varphi := \text{BE} \circ \psi$ (i.e., the mechanism that runs BE on top of the assignment found by mechanism ψ). Since ψ is 2-IR, so is φ. Hence, by construction, this must be a selection of BE that satisfies

$$\frac{|\bar{T}_k|}{|T_k|} \xrightarrow{p} 1 \text{ and } \frac{|\bar{S}_k|}{|S_k|} \xrightarrow{p} 1$$

as $n \to \infty$.

Fix $k = 1, ..., K$. Fix $\varepsilon_0 \in (0, \varepsilon)$. Further assume that ε_0 is small enough so that $(1-\varepsilon_0)^2 > 1-\delta$. Consider the set of pairs $(t,s) \in T_k \times S_k$ such that $s = \mu_0(t)$ and either t ranks s within its $\varepsilon_0|S_k|$ most favorite schools in S_k or s ranks t within his $\varepsilon_0|T_k|$ most favorite teachers in T_k. We eliminate these pairs from $T_k \times S_k$. Observing that the remaining set is a product set, we denote it by $T_k^0 \times S_k^0$. Note that, for each pair $(t,s) \in T_k \times S_k$ such that $s = \mu_0(t)$, there is a probability $(1-\varepsilon_0)^2$ that both t ranks s outside his $\varepsilon_0|S_k|$ most favorite schools in S_k and s ranks t outside its $\varepsilon_0|T_k|$ most favorite teachers in T_k. Let us call this event E_{ts}. For each such (t,s) where $s = \mu_0(t)$, we denote $\mathbf{1}_{ts}$ for the indicator function, which takes a value 1 if the event E_{ts} is true and 0 otherwise. Hence, $|T_k^0| = \sum_{(t,s)\in T_k \times S_k : s=\mu_0(t)} \mathbf{1}_{ts}$. Thus, $|T_k^0|(=|S_k^0|)$ follows a Binomial distribution $\text{Bin}(|T_k|,(1-\varepsilon_0)^2)$. By the law of large numbers, $\frac{|T_k^0|}{|T_k|} \xrightarrow{p} (1-\varepsilon_0)^2$, which, by assumption, is strictly greater than $1-\delta$. This proves that

$$\Pr\left\{\frac{|T_k^0|}{|T_k|} \geq 1-\delta\right\} \to 1$$

and

$$\Pr\left\{\frac{|S_k^0|}{|S_k|} \geq 1-\delta\right\} \to 1.$$

In the sequel, we condition w.r.t. a realization of the random set $T_k^0 \times S_k^0$ assuming that both $\frac{|T_k^0|}{|T_k|}$ and $\frac{|S_k^0|}{|S_k|}$ are greater than $1-\delta$. Now, fix $\varepsilon_0' > 0$ and note that, conditional on this, each teacher $t \in T_k^0$ draws randomly[75] in S_k^0 his $\varepsilon_0'|S_k^0|$ most favorite schools in S_k^0. Similarly, each school $s \in S_k^0$ draws randomly in T_k^0 its $\varepsilon_0'|T_k^0|$ most favorite teachers in T_k^0. We build a random bipartite graph on $T_k^0 \cup S_k^0$ where the edge $(t,s) \in T_k^0 \times S_k^0$ is added if and only if t ranks s within his $\varepsilon_0'|S_k^0|$ most favorite schools in S_k^0 and, similarly, s ranks t within its $\varepsilon_0'\,|T_k^0|$ most favorite teachers in T_k^0. This random bipartite graph can be seen as a mapping from the set of ordinal preferences into the set of bipartite graph $G_b(|T_k^0|)$. We denote this random graph by $\tilde{G}_b$. While Lemma 14 does not apply directly to this type of random graph, we will claim below that this random graph has a perfect matching with probability approaching one as the market grows. Before stating and proving this result, we must define the following lemma.

Lemma 16 *With probability approaching one, for any teacher $t \in T_k^0$, any school $s \in S_k^0$ with which $\xi_{ts} \geq 1 - \frac{\varepsilon_0'}{2}$ must be within his $\varepsilon_0'|S_k^0|$ most favorite schools in S_k^0. Similarly, with probability approaching one, for any school $s \in S_k^0$, any teacher $t \in T_k^0$, with whom $\eta_{ts} \geq 1 - \frac{\varepsilon_0'}{2}$ must be within its $\varepsilon_0'|S_k^0|$ most favorite teachers in T_k^0.*

Proof. We prove the first part of the statement, and the other part follows the same argument. Fix $t \in T_k^0$ and let E_t be the event that any school $s \in S_k^0$ with which $\xi_{ts} \geq 1 - \frac{\varepsilon_0'}{2}$

[75] In the following, by randomly, we mean uniformly i.i.d.

must be within his $\varepsilon_0'|S_k^0|$ most favorite schools in S_k^0. Let $X_t := \sum_{s \in S_k^0} \mathbf{1}_{\left\{\xi_{ts} \geq 1 - \frac{\varepsilon_0'}{2}\right\}}$ be the number of schools in S_k^0 with which teacher t enjoys an idiosyncratic payoff greater than $1 - \frac{\varepsilon_0'}{2}$. Observe that X_t follows a Binomial distribution $B(|S_k^0|, \frac{\varepsilon_0'}{2})$ (recall that ξ_{ts} follows a uniform distribution with support $[0,1]$) and that $X_t \leq \varepsilon_0'|S_k^0|$ implies that E_t is true. Hence, we have to prove that $\Pr\{\exists t \in T_k^0 : X_t > \varepsilon_0'|S_k^0|\} \to 0$ as $n \to \infty$. In the sequel, we let Y_t be a Binomial distribution $B(|S_k^0|, 1 - \frac{\varepsilon_0'}{2})$, and thus we have

$$\begin{aligned}\Pr\left\{\exists t \in T_k^0 : X_t > \varepsilon_0'|S_k^0|\right\} &\leq |T_k^0| \Pr\left\{X_t > \varepsilon_0'|S_k^0|\right\} \\ &= |T_k^0| \Pr\left\{Y_t \leq (1-\varepsilon_0')|S_k^0|\right\} \\ &\leq |T_k^0| \exp\left\{-2|S_k^0|(\frac{\varepsilon_0'}{2})^2\right\} \to 0\end{aligned}$$

as $n \to \infty$, where the first inequality is by the union bound and the last one uses Hoeffding inequality. The limit result uses the fact that under our conditioning event, $|T_k^0| = |S_k^0| \geq (1-\delta)|S_k| \to \infty$. ∎

We now move to our statement on the existence of perfect matching in $\tilde{G}_b$.

Lemma 17 *With probability going to* 1 *as* $n \to \infty$*, the realization of* $\tilde{G}_b$ *has a perfect matching.*

Proof. In our random environment, the state space, Ω, can be considered as the set of all possible profiles of idiosyncratic shocks for teachers and schools, i.e., the space of all $\{\{\xi_{ts}\}_{ts}, \{\eta_{ts}\}_{ts}\}$. We denote by ω a typical element of that set. Let E be the event under which, for each $(t,s) \in T_k^0 \times S_k^0 : \xi_{ts} \geq 1 - \frac{\varepsilon_0'}{2}$ and $\eta_{ts} \geq 1 - \frac{\varepsilon_0'}{2}$ imply that both t ranks s within his $\varepsilon_0'|S_k^0|$ most favorite schools in S_k^0 and s ranks t within its $\varepsilon_0'|T_k^0|$ most favorite teachers in T_k^0. By Lemma 16, $\Pr(E) \to 1$. Now, let us build the following random graph on $T_k^0 \cup S_k^0$ where, this time, the edge $(t,s) \in T_k^0 \times S_k^0$ is added if and only if $\xi_{ts} \geq 1 - \frac{\varepsilon_0'}{2}$ and $\eta_{ts} \geq 1 - \frac{\varepsilon_0'}{2}$. Let us call this graph $\tilde{G}_b'$. Therefore, this time, $\tilde{G}_b'$ can be viewed as a mapping from the set of cardinal preferences to the set of bipartite graph $G_b(|T_k^0|)$. Let F be the event that the realization of $\tilde{G}_b'$ has perfect matching. By Lemma 14, $\Pr(F) \to 1$. By definition, $E \cap F \subset \Omega$. Let us consider the set of all possible profiles of teachers and schools' ordinal preferences $\succ$ induced by states $E \cap F$, and let us denote this set by $\mathcal{P}$. Clearly, $\Pr(\mathcal{P}) \geq \Pr(E \cap F) \to 1$. Now, for each profile of preferences $\succ$ in $\mathcal{P}$, let $\tilde{G}_b(\succ)$ be the graph corresponding to $\tilde{G}_b$ when $\succ$ is the profile of realized preferences. We claim that, for any $\succ$ in $\mathcal{P}$, $\tilde{G}_b(\succ)$ has a perfect matching. Indeed, let $\omega \in E \cap F$ be one state that induces $\succ$ (this is well defined by the construction of $\mathcal{P}$). Because $\omega \in F$, the realization of $\tilde{G}_b'$ at profile ω has a perfect matching. In addition, because $\omega \in E$, the realization of $\tilde{G}_b'$ at profile ω is a subgraph of $\tilde{G}_b(\succ)$. We conclude that $\tilde{G}_b(\succ)$ has a perfect matching. Combining this result with the observation that $\Pr(\mathcal{P}) \to 1$, we get

$$\Pr\left\{\exists \text{ a perfect matching in } \tilde{G}_b\right\} \to 1$$

as $n \to \infty$, as claimed. ∎

Now, we build the mechanism ψ as follows. For each realization of ordinal preferences (for each $k = 1, ..., K$), we build a graph on $T_k^0 \cup S_k^0$ as defined above, i.e., where the edge $(t, s) \in T_k^0 \times S_k^0$ is added if and only if t ranks s within his $\varepsilon_0' |S_k^0|$ most favorite schools in S_k^0 and, similarly, s ranks t within its $\varepsilon_0' |T_k^0|$ most favorite teachers in T_k^0. If there is perfect matching, then under ψ, teachers in T_k^0 are matched according to this perfect matching, while teachers in $T_k \backslash T_k^0$ remain at their initial assignments. If there is no perfect matching, then under ψ, all teachers in T_k remain at their initial assignments. Assuming that $\varepsilon_0' + \delta < \varepsilon_0$, we obtain that the mechanism built in that way is 2-IR.[76] To see this, consider a teacher t who is not matched to his initial school. This means that t is matched to a school s given by a perfect matching of the random bipartite graph. By construction, this means that t ranks s within his $\varepsilon_0' |S_k^0|$ most favorite schools in S_k^0. Hence, s is within his $\varepsilon_0' |S_k^0| + \delta |S_k|$ most favorite schools in S_k. Since $\varepsilon_0' |S_k^0| + \delta |S_k| \leq (\varepsilon_0' + \delta) |S_k| < \varepsilon_0 |S_k|$ and because $t \in T_k^0$ implies that $\mu_0(t)$ is not within t's $\varepsilon_0 |S_k|$ most favorite schools in S_k, we obtain that s is preferred by t to his initial assignment. Since a similar reasoning holds for schools, we obtain that ψ is 2-IR.

As we have shown, with probability approaching one, our bipartite graph actually has a perfect matching. Obviously, this perfect matching ensures that all teachers in T_k^0 and all schools in S_k^0 are matched to a partner within their $\varepsilon_0' |S_k^0|$ favorites. This holds for any realization of the random set $T_k^0 \times S_k^0$ such that $\frac{|T_k^0|}{|T_k|}$ and $\frac{|S_k^0|}{|S_k|}$ are greater than $1 - \delta$. Thus, it holds conditional on the random sets $\frac{|T_k^0|}{|T_k|}$ and $\frac{|S_k^0|}{|S_k|}$ being greater than $1 - \delta$. Hence, this perfect matching ensures that all teachers in T_k^0 and all schools in S_k^0 are matched to a partner within their $(\varepsilon_0' + \delta) |S_k|$ favorites in S_k and T_k, respectively. Hence, under our conditioning event that the random sets $\frac{|T_k^0|}{|T_k|}$ and $\frac{|S_k^0|}{|S_k|}$ are greater than $1 - \delta$,

$$\Pr \left\{ \frac{|\{t \in T_k \,|\psi(t) \text{ is within the } (\varepsilon_0' + \delta) |S_k| \text{ most favorite school in } S_k \}|}{|T_k|} > 1 - \delta \right\} \to 1$$

and

$$\Pr \left\{ \frac{|\{s \in S_k \,|\psi(s) \text{ is within the } (\varepsilon_0' + \delta) |S_k| \text{ most favorite teacher in } T_k \}|}{|S_k|} > 1 - \delta \right\} \to 1$$

Given that the conditioning event has a probability approaching 1 as $n \to \infty$, this is even true without conditioning.

Now, without loss of generality, let us assume that δ is small enough so that $\varepsilon_0' + \delta < \varepsilon$. It remains to show that these $(\varepsilon_0' + \delta) |S_k|$ favorite partners in S_k (resp. T_k) yield an idiosyncratic payoff greater than $1 - \varepsilon$. The following lemma completes the argument.

[76] This is without loss of generality because, if $\Pr \left\{ \frac{|\{t \in T_k | \xi_{t\psi(t)} \geq 1 - \varepsilon \}|}{|T_k|} > 1 - \delta \right\} \to 1$ then, $\Pr \left\{ \frac{|\{t \in T_k | \xi_{t\psi(t)} \geq 1 - \varepsilon \}|}{|T_k|} > 1 - \delta' \right\} \to 1$ for any $\delta' > \delta$.

Lemma 18 *With probability approaching* 1 *as* $n \to \infty$*, the* $(\varepsilon_0' + \delta)\,|S_k|$ *most favorite schools of each teacher in* T_k *yield an idiosyncratic payoff higher than* $1-\varepsilon$ *and the* $(\varepsilon_0' + \delta)\,|T_k|$ *most favorite teachers of each school in* S_k *yield an idiosyncratic payoff higher than* $1-\varepsilon$*.*

Proof. We show that with probability going to 1 as $n \to \infty$, the $(\varepsilon_0' + \delta)\,|S_k|$ most favorite schools of each teacher in T_k yield an idiosyncratic payoff higher than $1-\varepsilon$. The other part of the statement is proved in the same way. For each $t \in T_k$, let Z_t be the number of schools s in S_k for which $\xi_{ts} \geq 1-\varepsilon$. Note that if $Z_t > (\varepsilon_0' + \delta)\ |S_k|$ then t's $(\varepsilon_0' + \delta)\ |S_k|$ first schools in S_k must yield an idiosyncratic payoff higher than $1-\varepsilon$. Thus, it is enough to show that

$$\Pr\{\exists t \in T_k \text{ with } Z_t \leq (\varepsilon_0' + \delta)\ |S_k|\} \to 0$$

as $n \to \infty$. Observe that Z_t follows a binomial distribution $B(|S_k|, \varepsilon)$ (recall that ξ_{ts} follows a uniform distribution with support $[0,1]$). Hence,

$$\begin{aligned}
\Pr\{\exists t \in T_k \text{ with } Z_t \leq (\varepsilon_0' + \delta)\ |S_k|\} \leq & \sum_{t \in T_k} \Pr\{Z_t \leq (\varepsilon_0' + \delta)\ |S_k|\} \\
= & \ |T_k| \Pr\{Z_t \leq (\varepsilon_0' + \delta)\ |S_k|\} \\
\leq & \ |T_k| \frac{1}{2} \exp\left(-2 \frac{(|S_k|\varepsilon - (\varepsilon_0' + \delta)\,|S_k|)^2}{|S_k|}\right) \\
= & \ \frac{|T_k|}{2 \exp\left(2\left(\varepsilon - (\varepsilon_0' + \delta)\right)^2 |S_k|\right)} \to 0
\end{aligned}$$

where the first inequality is by the union bound, while the second equality is by Hoeffding's inequality. ∎

I.3 Proof of Theorem 8

Recall that T_k stands for $\mu_0(S_k)$, where μ_0 is the initial allocation. We will prove the following result.

Proposition 13 *Fix any* k *and any* $\varepsilon > 0$*. Let* $\bar{T}_k := \{t \in T_k \,|U_t(\text{TO-BE}(t)) \geq u_k + 1 - \varepsilon\}$*. We have*

$$\frac{|\bar{T}_k|}{|T_k|} \xrightarrow{p} 1.$$

Proof of Proposition 13. Recall that TO-BE is in the two-sided core. In particular, this implies that there is no pair of teachers t and t' so that $\mu_0(t') \succeq_t$ TO-BE(t), $\mu_0(t) \succeq_{t'}$ TO-BE(t') (with a strict preference for either t or t'), $t' \succeq_{\mu_0(t)} t$ and $t \succeq_{\mu_0(t')} t'$. Fix an arbitrary k and let E be the event that the fraction of schools $s \in S_k$ s.t. $\eta_{\mu_0(s)s} \leq 1-\delta$ is greater than $1-2\delta$ where $\delta \in (0,1)$. By the law of large numbers, we have

$$\frac{1}{|S_k|} \sum_{s \in S_k} \mathbf{1}_{\{\eta_{\mu_0(s)s} \leq 1-\delta\}} \xrightarrow{p} 1-\delta.$$

Thus, $\Pr(E) \to 1$. Let $T_k^0 := \{t \in T_k \,|\eta_{t\mu_0(t)} \leq 1-\delta\}$.

In the sequel, we condition on event E and fix a realization of $\left\{\eta_{\mu_0(s)s}\right\}_{s\in S}$ compatible with E. Observe that T_k^0 is non-random once this has been fixed and that, conditional on these, individuals' preferences are still drawn according to the same distribution (as in the unconditional case) and for $t \neq \mu_0(s)$, η_{ts} is also still drawn according to the same distribution. We further observe that, because event E holds, $\frac{|T_k^0|}{|T_k|} \geq 1-2\delta$ and hence $|T_k^0|$ approaches infinity as $n \to \infty$. We define a random graph with $\{(t,\mu_0(t))\}_{t\in T_k^0}$ as the set of vertices. An edge between $(t,\mu_0(t))$ and $(t',\mu_0(t'))$ is added if and only if $\xi_{t\mu_0(t')} > 1-\varepsilon$ and $\xi_{t'\mu_0(t)} > 1-\varepsilon$ and $\eta_{t'\mu_0(t)} \geq \eta_{t\mu_0(t)}$ and $\eta_{t\mu_0(t')} \geq \eta_{t'\mu_0(t')}$. Then, in the random graph, each edge between $(t,\mu_0(t))$ and $(t',\mu_0(t'))$ is added independently with a probability of at least $\varepsilon^2\delta^2 \in (0,1)$. Now, let $\bar{T}_k^0 := \{t \in T_k^0 \,|U_t(\text{TO-BE}(t)) \leq u_k + 1 - \varepsilon\}$. It must be that $\bar{T}_k^0$ is an independent set, or else, if there is an edge $(t,t') \in \bar{T}_k^0 \times \bar{T}_k^0$ for some realization of the random graph, then

$$U_t(\mu_0(t')) > u_k + 1 - \varepsilon \geq U_t(\text{TO-BE}(t)) \text{ and } U_{t'}(\mu_0(t)) > u_k + 1 - \varepsilon \geq U_{t'}(\text{TO-BE}(t')).$$

In addition, $V_{\mu_0(t)}(t') = \eta_{t'\mu_0(t)} \geq \eta_{t\mu_0(t)} = V_{\mu_0(t)}(t)$ and $V_{\mu_0(t')}(t) = \eta_{t\mu_0(t')} \geq \eta_{t'\mu_0(t')} = V_{\mu_0(t')}(t')$ and therefore TO-BE is blocked by a coalition of size two, a contradiction. Now, we can use Lemma 15 to obtain that $\Pr\left\{|\bar{T}_k^0| \geq \frac{2\log(|T_k|)}{\log\frac{1}{1-p}}\right\} \to 0$ as $n \to \infty$ and thus $\frac{|\bar{T}_k^0|}{|T_k^0|} \xrightarrow{p} 0$ as $n \to \infty$. Now, since $\bar{T}_k^c = \bar{T}_k^0 \cup \{t \in T_k \backslash T_k^0 \,|U_t(\text{TO-BE}(t)) \leq u_k + 1 - \varepsilon\}$, we must have

$$\frac{|\bar{T}_k^c|}{|T_k|} \leq \frac{|\bar{T}_k^0| + |T_k \backslash T_k^0|}{|T_k|} \leq \frac{|\bar{T}_k^0|}{|T_k|} + 2\delta$$

Hence, given that $\frac{|\bar{T}_k^0|}{|T_k^0|} \xrightarrow{p} 0$, we must have that, with probability approaching 1 as n approaches infinity, $\frac{|\bar{T}_k^c|}{|T_k|} \leq 3\delta$ and so $\frac{|\bar{T}_k|}{|T_k|} \geq 1-3\delta$.

To recap, given event E and any realization of $\left\{\eta_{\mu_0(s)s}\right\}_{s\in S}$, we have $\frac{|\bar{T}_k|}{|T_k|} \geq 1-3\delta$ with probability approaching 1 as $n \to \infty$. Since the realization of $\left\{\eta_{\mu_0(s)s}\right\}_{s\in S}$ is arbitrary, we obtain that, given event E, $\frac{|\bar{T}_k|}{|T_k|} \geq 1-3\delta$ with probability approaching 1 as $n \to \infty$. Since $\Pr(E) \to 1$ as $n \to \infty$, we obtain that $\frac{|\bar{T}_k|}{|T_k|} \geq 1-3\delta$ with probability approaching 1 as $n \to \infty$. Since $\delta > 0$ is arbitrarily small, we obtain $\frac{|\bar{T}_k|}{|T_k|} \xrightarrow{p} 1$ as $n \to \infty$, as claimed.

Remark 2 *The statement is related to that of Che and Tercieux (2015b) in Theorem 1. However, since TO-BE is not Pareto-efficient, their proof/argument does not apply.*

Remark 3 *The argument relies on the fact that TO-BE is not blocked by any coalition of size* 2*. Hence, the result applies beyond the TO-BE mechanism and applies to any mechanism that cannot be blocked by any coalition of size* 2*.*