Continuous Time Stochastic Processes

Jesús Fernández-Villaverde

University of Pennsylvania

November 9, 2013
Filtration

- Fix a probability space \((\Omega, \mathcal{F}, P)\).

- Define \(t \in [0, \infty) = \mathbb{R}_+\).

- Filtration: a family \(\mathcal{F} = \{\mathcal{F}_t : t \geq 0\}\) of increasing \(\sigma\)-algebras contained in \(\mathcal{F}\):

\[
\mathcal{F}_s \subseteq \mathcal{F}_t \text{ for } \forall s \leq t \text{ and } \mathcal{F}_t \subseteq \mathcal{F}
\]

- Clearly, \(\mathcal{F}_\infty = \mathcal{F}\) is the smallest \(\sigma\)-algebras containing \(\forall \mathcal{F}_t\).

- \((\Omega, \mathcal{F}, P)\): filtered probability space.
Continuous-time stochastic process: a mapping \( x : [0, \infty) \times \Omega \rightarrow \mathbb{R} \) which is measurable with respect to \( \mathcal{B}_+ \times \mathcal{F} \) (where \( \mathcal{B}_+ \) are the Borel sets of \( \mathbb{R}_+ \)).

A stochastic process is adapted to \( \mathcal{F} \) if \( x(t, \omega) \) is \( \mathcal{F}_t \)-measurable \( \forall t \).

\( x(t, \cdot) \): \( \mathcal{F}_t \)-measurable function of \( \omega \).

\( x(\cdot, \omega) \): realization, trajectory, or sample path of the process.

Continuous stochastic process: \( x(\cdot, \omega) \in C[0, \infty), \) a.e. \( \omega \in \Omega \).
Brownian Motions: Definition

- A Wiener processes (or Brownian motion) is a stochastic process $W$ having:
  1. continuous sample paths.
  2. independent increments.
  3. $W(t) \sim \mathcal{N}(0, t)$, $\forall t$.

Basic Result

If a stochastic process $\{X(t), t \geq 0\}$ has continuous sample paths with stationary, independent, and i.i.d. increments, then it is a Wiener process.

Differential:

$$dW = \lim_{dt \downarrow 0} (W(t + dt) - W(t))$$
Properties of Differentials

- Moments:
  1. \[ \mathbb{E} [dW] = 0. \]
  2. \[ \mathbb{E} [(dW)^2] = dt. \]

- Also, as \( dt \to 0 \) (we skip the proof):
  1. \( dW \sim o \left( \sqrt{dt} \right) \).
  2. \( (dW)^2 \to \mathbb{E} [(dW)^2] = dt. \)

- Note that, while \( W(t) \) has a continuous path, it is not differentiable:
  \[ \frac{dW}{dt} = o \left( \frac{\sqrt{dt}}{dt} \right) \to \infty \text{ as } dt \to 0. \]
Diffusions I

A Brownian motion with drift:

\[ dX(t) = \mu dt + \sigma dW(t) \quad \text{with} \quad X(0) = x_0 \]

More generally, we have a diffusion:

\[ dX(t) = \mu(t, x) \, dt + \sigma(t, x) \, dW(t) \quad \forall t, \forall \omega \]

Properties:

1. \( \mathbb{E}_t [dX] = \mu(t, x) \, dt. \)
2. \( \text{var}_t [dX] = \sigma^2(t, x) \, dt. \)
Diffusions II

- Diffusion are important in arbitrage-free asset pricing. Aït-Sahalia (2006).

- Particularly useful cases are:
  1. Geometric Brownian motion
     \[ dX = \mu X dt + \sigma X dW \]
  2. Ornstein-Uhlenbeck process
     \[ dX = -\theta (X - \mu) \, dt + \sigma (t, x) \, X dW \]
Functions of Stochastic Processes I

- Let $F(t, x)$ be a function that is at least once differentiable in $t$ and twice in $x$.
- We approximate the total differential of $F(t, X(t, \omega))$ by a Taylor expansion:

$$dF = F_t dt + F_x dX + \frac{1}{2} F_{tt} (dt)^2 + \frac{1}{2} F_{xx} (dX)^2 + F_{xt} dt (dX) + ...$$

- We substitute in:

$$dF = F_t dt + F_x [\mu dt + \sigma dW] + \frac{1}{2} F_{tt} (dt)^2 + \frac{1}{2} F_{xx} [\mu^2 (dt)^2 + 2\mu \sigma dt dW + \sigma^2 (dW)^2] + F_{xt} dt (\mu dt + \sigma dW) + H.O.T...$$
Itô’s lemma: we can drop the terms that have order higher than $dt$ or $(dW)^2$ and use $(dW)^2 \rightarrow dt$:

$$dF = F_t dt + F_x [\mu dt + \sigma dW] + \frac{1}{2} \sigma^2 F_{xx} (dW)^2$$

$$= \left( F_t + \mu F_x + \frac{1}{2} \sigma^2 F_{xx} \right) dt + \sigma F_x dW$$

Since $\mathbb{E}[dW] = 0$,

$$\mathbb{E}[dF] = \left[ F_t + \mu F_x + \frac{1}{2} \sigma^2 F_{xx} \right] dt$$

$$Var[dF] = \mathbb{E}[dF - \mathbb{E}[dF]]^2 = \sigma^2 F_x^2 dt$$
Particular case \( F(t, x) = e^{-rt} f(x) \):

\[
\mathbb{E}[dF] = \left[ -rf + \mu f' + \frac{1}{2} \sigma^2 f'' \right] e^{-rt} dt
\]

and when \( r = 0 \) (that is, \( F(t, x) = f(x) \), an often relevant case in economics)

\[
\mathbb{E}[dF] = \left[ \mu f' + \frac{1}{2} \sigma^2 f'' \right] dt
\]