1. Bubbly Liquidity

Consider an overlapping generations economy in which one household is born at each period \( t, t = 0, 1, 2, \ldots \). The household lives for three periods: \( t, t+1, \) and \( t+2 \). We will call the household in her first period of life “young”, in her second “middle-age”, and in her last “old.” The household consumes the only good in the economy when she is old, \( c^t_{t+2} \), and orders consumption allocations according to a linear utility function:

\[
u(c^t_{t+2}) = c^t_{t+2}.
\]

Also, at \( t = 0 \), there are one initial old (\( -2 \)) and one initial middle-age (\( -1 \)) households alive.

The household is born with an amount \( A^t \) of the good, a wealth which she can invest when young in three different assets:

1. Lucas trees: the household can rent, for one period, Lucas trees owned by an agent outside the economy. In exchange for a rental fee per-tree \( p^l_t \) paid at time \( t \), the household will get the fruit of the tree: 1 unit of the good in time \( t+1 \), also per-tree. Let us call \( A^l_t \) the amount of wealth invested in renting Lucas trees.

2. Securities issued by the middle-age household. The securities \( s^s_t \) with price \( p^s_t \) pay 1 unit of good at time \( t+1 \). Let us call \( A^s_t \) the amount of wealth invested in securities.

3. A bubble: a bubble is an asset with price \( b^b_t \) that does not yield any dividend and that is purchased only because it will have a positive value \( b^b_{t+1} \geq 0 \) in the next period. Let us call \( A^b_t \) the amount of wealth invested in the bubble.

Therefore:

\[
A = A^l_t + A^s_t + A^b_t.
\]

A middle-age household born at time \( t \) can invest \( i^t_{t+1} \) units of good in period \( t+1 \) in a project that delivers \( \rho^t_1 i^t_{t+1} \) units of good in period \( t+2 \). To finance \( i^t_{t+1} \), the household can use the current value of the portfolio invested in period \( t \) plus an amount of securities \( s^s_{t+1} \) pledged against the payoffs from the project. Because of financial frictions, the household can only pledge a total amount of \( \rho^t_0 i^t_{t+1} \) to pay those securities. We assume that \( \rho^t_1 > \rho^t_0 > 0 \).

When old, the household receives the return of the project \( \rho^t_1 i^t_{t+1} \), pays off \( s^s_{t+1} \), and consumes the remanent \( c^t_{t+2} \).

The interest rate between period \( t \) and period \( t+1 \) is \( 1 + r^t_{t+1} \). While \( 1 + r^t_{t+1} \) is endogenous, you can assume that in all equilibria, \( \rho^t_1 > 1 + r^t_{t+1} > \rho^t_0 \) for all \( t \).

There is a total of \( l > 0 \) Lucas trees in the economy. The initial old invested at scale \( i^o_{-1} \) and issued \( s^o_{-1} \) securities. The initial middle-age rented \( l \) Lucas trees and owns \( b^o_0 \) and \( s^o_{-1} \). All agents behave competitively with respect to prices and all markets clear.

1. Show that a middle-age household will invest in the production technology the maximum amount she can finance. Once you have shown this, you can use the result for the next questions.

2. Find \( p^l_t, p^s_t \), and the relation between \( b^o_{t+1} \) and \( b^t_i \), all as a function of \( r^t_{t+1} \).
3. Use the results in 2. to describe the non-arbitrage condition existing between the three assets that a young household can invest in. Once you have shown this, you can use the result for the next questions.

4. Define a sequential markets equilibrium for this economy.

5. Characterize the sequential markets equilibrium. In particular you need to:

   1. From the problem of the young household, find an expression for \( i_t \) that depends on \( A, l, b_t \), and \( r_{t+1} \) (hint: think about this equation as an asset demand equation).
   2. From the problem of the middle-age household, find an expression for \( i_t \) that depends on \( i_{t-1}, l, b_t \), and \( r_{t+1} \) (hint: think about this equation as an asset supply equation).
   3. Use the non-arbitrage condition for the bubble found in 2.
   4. Argue that the asset demand equation, the asset supply equation, and the non-arbitrage condition for the bubble fully describe the dynamic behavior of the economy.

6. Assume that we are in an equilibrium where the bubble asset is not valued: \( b_0 = 0 \). Find the steady state of the economy.

7. Bonus question: show that when \( b_0 = 0 \), the economy converges monotonically to the steady state you just found.