March 27th, 2007

Technology Improvements in the Goat Farmers Economy (Transitions-simple)

In the model we were studying, we get a stationary distribution X^* for the assets of all goat farmers. This equilibrium object clearly depends on the 'fundamentals' of the economy, namely $\{\beta, \Gamma, q, u(c)\}$. In general, the problem is defined as

$$\begin{aligned} V(s, a; q) &= \max_{a'} \left\{ u[c] + \beta \sum_{s'} \Gamma_{ss'} V(s', a'; q) \right\} \\ &\quad \text{subject to} \\ c + qa' &= s + a \end{aligned}$$

with solution $y(s, a; q)$

Note that we are showing explicitly the importance of q in the value functions. Now imagine that the economy is in a steady state with some q^0. We would like to know what happens if unexpectedly the cost of storage decreases permanently from q^0 to q^1. In a first step, we can solve the above problem with this new $q^1 < q^0$ and get a new policy function $y(s, a; q^1)$. Applying the updating operator for this new economy

$$T\left[X^*\left(q^0\right), y\left(s, a; q^1\right)\right] \neq X^*\left(q^0\right)$$

Clearly, since a fundamental of the economy has changed, the distribution of the fishermen will move away from $X^*(q^o)$ into a new stationary distribution. This will be true if Γ satisfies the 'AD-AN (American Dream-American Nightmare (Monotone Mixing Condition))' conditions. Then, we know for sure that

$$\lim_{n \to \infty} T^n\left[X\left(q^0\right), y(.)\right] = X^*\left(q^1\right)$$

How do we understand transitions for this economy? Some economist forget about transitions and just compare steady states, which is not correct if we are interested in welfare. The correct way to proceed is to guess some N (number of periods necessary to achieve the new stationary equilibrium) and then compare the sequence of distributions $\{T^n [X(q^0), y(.)]\}_{n=1}^N$ to $X^*(q^0)$. Note that we can do this because we can calculate the decision rules independently from $X^*(.)$.

Welfare Questions

How much would this society be willing to pay for a reduction of q^0 to q^1? We can answer this for a particular household and for the whole society. For a particular household, we have three different values (call them $\mathcal{V}$)

- Utility gain from the switch

$$\mathcal{V}_1 = V\left(s, a; q^1\right) - V\left(s, a; q^0\right)$$

  Note that this is a calculation that we can perform for particular households, defined by particular (s, a).

- Consumption/assets willing to be sacrificed (for each household) in order to have q^1

$$\mathcal{V}_2 = V\left(s, a - z; q^1\right) - V\left(s, a; q^0\right)$$

  where z represents the amount of consumption/assets that the household would sacrifice in order to get the better technology. Note that we don't make the distinction between consumption and assets, since in this economy we have only one good. Also, $\mathcal{V}_1$ is not defined for households with $a = 0$ i.e., no assets. One way of avoiding this problem is to use the next calculation

- Consumption/assets needed to compensate the household if it stays with the old technology

$$\mathcal{V}_3 = V\left(s, a; q^1\right) - V\left(s, a + z; q^0\right)$$

Now, for the whole society, the gains from technical improvements are given by the sum of the individual gains. The formula is given by

$$\mathcal{V}_{soc} = \int z(s,a)\, dX^*\left(q^0\right)$$

where $z(s,a)$ is the one calculated from $\mathcal{V}_3$, for each household (pairs of s,a) and the integration is with respect to $X^*(q^0)$, the distribution of fishermen at the beginning.

Technology Improvements in the Fishermen Economy (Transitions-difficult)

Suppose now that instead of storage technology, we have an aggregate technology of the form

$$Y = A_0 K^\theta N^{1-\theta}$$

where $K = \int a dX^*$ and $L = \int s dX^*$. In this case, if the economy receives an improvement in total factor productivity (i.e., $A_1 > A_0$), answering the welfare questions becomes much harder. Why? simply, because now the measure matters. Recall that in the previous example, the measure of fishermen was exogenous to the calculation of the policy functions. Now, given the new value A_1, we will be moving away from a steady state, which means that the interest rate ($F_K = \partial Y/\partial K$) and the wage ($F_N = \partial Y/\partial N$) are not constant. Hence, we need the entire sequences of prices to get the policy function. The problem is that to get prices, we need the whole distribution (measure) of agents in the economy, which makes this a non-trivial problem. The problem is still solvable with computer-intensive methods and comparing steady states is still wrong.

Growth

In our analysis so far, we have used Neo-classical Growth Model as our benchmark model and built on it for the analysis of more interesting economic questions. One peculiar characteristic of our benchmark model, unlike its name suggested, was lack of growth. Many interesting questions in economics are related to the cross-country differences of growth rates. We will cover some models that will allow for growth so that we will be able to attempt to answer such questions.

Exogenous growth

What does it take for an economy to grow? Before answering that question, we know in our standard NGM there is basically two ways of growth, one in which everything grows, which is not necessarily a per-capita growth, and the other is per-capita growth. We will be focusing on per-capita growth, hence, the next definition is useful

Balanced Growth Path is a situation where all variables of a model grow at a constant rate (not neccesarily at the same rate)

The title exogenous growth refers to the structure of models in which growth rate is determined exogenously, and is not an outcome of the model. First and the simplest one of these is one in which the determinant of the growth rate is population growth.

Growth with population

Suppose the population of our economy grows at rate γ and we have the classical CRTS technology in capital and labor inputs.

$$\begin{aligned} Y_t &= AF(K_t, N_t) \qquad (1) \\ N_t &= N_0 * \gamma^t \end{aligned}$$

Note that our economy is no longer stationary but as we will see, within the exogenous growth framework we can make these economies look like stationary ones by re-normalizing the variables. Thus, at the end of the day it will only be a mathematical twist on our standard growth model. Once we do that, we will be looking for the counterpart of a steady state that we have in our stationary economies, the Balanced Growth Path, in which all the variables will be growing at constant rates but not necessarily equal. Back to our population growth model, we know

$$AF(K, N) = A[KF_k(K, N) + NF_N(K, N)] \qquad (2)$$

Question is, if N is growing at rate γ, can this economy have a balanced growth path. Can we construct one? We know by CRTS property F_K and F_N are homogenous of degree zero. If we assume capital stock grows at rate γ as well, then prices stay constant and per-capita variables are constant and output grow at the same rate. So we get growth on a balanced growth path without per-capita growth. One question is how do we model population growth in our representative agent model. One way is to assume there is a constant proportion of immigration to our economy from outside but this has to assume the immigrants are identical to our existing agents in our economy, which is a bid problematic. The other way could be to assume growing dynasties which preserves the representative agent nature of our economy. If we do so, the problem of the social planner becomes,

$$\max \sum_{t=0}^{\infty} \beta^t N_t U(\frac{C_t}{N_t}) \tag{3}$$
$$st \quad C_t + K_{t+1} = AF(K_t, N_t) + (1-\delta)K_t$$

To transform the budget set to per capita terms, divide all terms by N_t and to make the environment stationary by dividing all the variables by γ^t and assume $N_0 = 1$, we get,

$$\max \sum_{t=0}^{\infty} (\beta\gamma)^t U(\widehat{c}_t) \tag{4}$$
$$st \quad \widehat{c}_t + \gamma\widehat{k}_{t+1} = AF(\widehat{k}_t, 1) + (1-\delta)\widehat{k}_t$$

So how is this transformed model any different than our NGM? By the discount factor, the agents in this economy with growth discounts the future less but everything else is identical to NGM of course with the exception of this economy growing at a constant rate.

March 29th, 2007

Labor-Productivity Growth

Now suppose we have a 'labor augmenting' productivity growth with constant population normalized to one, i.e. have the following CRTS technology,

$$Y_t = AF(K_t, \gamma^t N_t) \tag{5}$$
$$AF(K_t, \gamma^t N_t) = A[K_t F_k(K_t, \gamma^t N_0) + \gamma^t N_0 F_N(K_t, \gamma^t N_0)] \tag{6}$$

Can we have an BGP? The problem is,

$$\begin{aligned} & \max \sum_{t=0}^{\infty} \beta^t U(C_t) \qquad (7) \\ st \quad C_t + K_{t+1} &= AF(K_t, \gamma^t N_0) + (1-\delta) K_t \end{aligned}$$

and since we have a population of one, these variables are already per-capita terms. For stationarity, we have to normalize the variables to 'per productivity' units, by dividing all by γ^t. Then the problem becomes,

$$\begin{aligned} & \max \sum_{t=0}^{\infty} \beta^t U(\gamma^t \widehat{c}_t) \qquad (8) \\ st \quad \widehat{c}_t + \gamma \widehat{k}_{t+1} &= AF(\widehat{k}_t, 1) + (1-\delta) \widehat{k}_t \end{aligned}$$

Suppose we have a CRRA preferences, then the question is how can we represent the preferences as a function of $\widehat{c}_t$ only. Writing the CRRA,

$$\sum_{t=0}^{\infty} \beta^t \frac{(\gamma^t \widehat{c}_t)^{1-\sigma} - 1}{1-\sigma} = \sum_{t=0}^{\infty} (\beta(\gamma^{1-\sigma}))^t \frac{\widehat{c}_t^{\,1-\sigma} - 1}{1-\sigma} \qquad (9)$$

and the problem becomes,

$$\begin{aligned} & \max \sum_{t=0}^{\infty} (\beta(\gamma^{1-\sigma}))^t \frac{\widehat{c}_t^{\,1-\sigma} - 1}{1-\sigma} \qquad (10) \\ st \quad \widehat{c}_t + \gamma \widehat{k}_{t+1} &= AF(\widehat{k}_t, 1) + (1-\delta) \widehat{k}_t \end{aligned}$$

and once again it is exact same problem as the NGM with a different discount factor. Note that the existence of a solution to this problem depends on $\beta(\gamma^{1-\sigma})$.In this set-up we get per-capita growth at rate γ. Also note that CRRA is the only functional form for preferences that is compatible with BGP. This is because as per-capita output grows, for consumption to grow at a constant rate, our agent has to face the same trade-off at each period.

Now suppose we have the TFP growing at rate γ with a CRTS Cobb-Douglas technology

$$\begin{aligned} Y_t &= A_t F(k_t, 1) \\ \frac{A_{t+1}}{A_t} &= \gamma \end{aligned}$$

What would be the growth rate of this economy? We can show that like the previous cases the growth rate of the economy is the growth rate for the productivity of labor, which is $\gamma^{\frac{1}{1-\alpha}}$ in this case.

0.1 Endogenous Growth

So far in the models we covered growth rate has been determined exogenously. Next we will look to models in which the growth rate is chosen by the model itself. We do know for a fixed amount of labor, the curvature of our technology limits the growth due to diminishing marginal return on capital and with depreciation there is an upper limit on (physical) capital accumulation. So if our economy is to experience sustainable growth for a long period of time, we either give up the curvature assumption on our technology or we have to be able to shift our production function up. Given a fixed amount of labor, this shift is possible either by an increasing TFP parameter or increasing labor productivity, . The simplest of such models where we can see that is the AK model, where the technology is linear in capital stock so that diminishing marginal return on capital does not set in.

0.2 AK Model

We have the usual social planner's problem with linear technology and full depreciation,

$$\max \sum_{t=0}^{\infty} \beta^t U(C_t)$$

such that

$$C_t + K_{t+1} = AK_t$$

and the FOCs

$$(c_t) \quad : \quad \beta^t U_c(.) = \lambda_t \qquad (11)$$

$$(k_{t+1}) \quad : \quad \lambda_t = \lambda_{t+1} \qquad (12)$$

together implies the Euler equation,

$$U_c(c_t) = A\beta U_c(c_{t+1}) \tag{13}$$

and on the BGP with consumption growing at rate γ with CRRA utility we get,

$$\gamma = (A\beta)^{1/\sigma} \tag{14}$$

and the growth rate is determined by the model parameters endogenously. Note that capital also grows at rate γ and the fate of the economy is determined by the fundamentals of the model. The capital stock will diverge to infinity if $(A\beta)^{1/\sigma} > 1$ or the economy is destined to vanish if $(A\beta)^{1/\sigma} < 1$. Also note that there is no transitional dynamics in this model (we loose the state variables in the euler equation after substituting for the balanced growth rate relation) and conditional on γ,asymptotically all economies are same regardless of the initial capital level. If we de-centralize this economy we know wages will be zero since labor has no use and gross rental rate of capital will be fixed at A. This is at odds with what we observe in the real world. We would rather like to have a model that allows for both transitional dynamics, labor and growth at the same time. Allowing for labor implies that we need a variable that proxies the increasing productivity of labor endogenously and be reproducible in terms of output, such that we are able to shift our production function continually in the output-capital space without hitting a natural bound.

0.3 Human Capital and Growth

Another way of getting our models to grow "endogenously", is by introducing the variable 'Human Capital' as an input of production. This will proxy continuous and endogenous increasing labor efficiency. We have two ways of modeling human capital:

- one way is to see it very much like physical capital, in the sense output has to be invested to increase the existing stock of human capital. That is the Lucas' approach, in which you can think of investing in education by building more schools as a way to increase the existing human capital stock.

- The alternative way would be to reserve a part of the leisure time for increasing the human capital stock, which can be thought of studying harder to get better in a fraction of the leisure time. Unfortunately, the second approach puts limit on the rate human capital can grow and might fail to generate sustainable endogenous growth. Next, we look at the Lucas' human capital model.

Lucas' Human Capital Model We have an Cobb-Douglas technology with CRTS and human capital (H) as an input of production instead of labor and the laws of motion for the inputs,

$$\begin{aligned} F(H,K) &= AK^{\alpha}H^{1-\alpha} && (15) \\ K' &= i_k + (1-\delta_k)K && (16) \\ H' &= i_h + (1-\delta_h)H && (17) \end{aligned}$$

Now that there is no limit to the accumulation of human capital and sustainable growth on a BGP is feasible. Furthermore, an analysis of the characterization of the balanced growth path will indicate that this model indeed has transitional dynamics, so unlike the AK model if economy starts out of this optimal growth path economy can adjust and converge to it by responding to prices in a de-centralized setting. If we model the law of motion for human capital as,

$$H' = (1-N) + (1-\delta_h)H \tag{18}$$

where $(1-N)$ is the time devoted to accumulating human capital, say by studying harder, we see there is a natural limit to the growth of human capital and such an economy might not have a BGP. The key ingredient of endogenous growth with labor is then the reproducibility of the human capital without such a limit.