Econ 702, Spring 2006
Problem set 9

Suggested Solutions

Problem 1 *Consider the following growth problem*

$$\sum_{t=0}^{\infty} \beta^t N_t u\left[\frac{C_t}{N_t}\right]$$

subject to

$$C_t + K_{t+1} = F\left(K_t, N_t\right)$$

The number of individuals is growing at rate γ*. Set* $N_0 = 1$ *and define* $\hat{x} = \gamma^{-t} x_0$*. Rewrite the growth problem with hats and look for the steady state. At the SS, what is the growth rate of all variables?*

Suggested Solution

If we set $N_0 = 1$ and dividing the budget constraint by γ^t we get

$$\sum_{t=0}^{\infty} \beta^t \gamma^t u\left[\frac{C_t}{\gamma^t}\right]$$

st

$$\frac{C_t}{\gamma^t} + \frac{K_{t+1}}{\gamma^t} = F\left(\frac{K_t}{\gamma^t}, 1\right)$$

The last term comes from the homogeneity of degree one of the production function. Using the definition of 'hats', we get that our problem becomes

$$\sum_{t=0}^{\infty} \tilde{\beta}^t u\left[\hat{c}_t\right]$$

st

$$\hat{c}_t + \gamma \hat{k}_{t+1} = F\left(\hat{k}_t, 1\right)$$

where $\tilde{\beta} = (\beta\gamma)$. Now, this growth problem looks identical to our typical neoclassical growth problem (up to the nuisance parameter γ multiplying $\hat{k}_{t+1}$) , which we have studied extensively and for which we know a steady state exists. Using the FOC of the problem we get that

$$\gamma \frac{u_c\left[\hat{c}_t\right]}{u_c\left[\hat{c}_{t+1}\right]} = \beta\left(1 + r_{t+1}\right)$$

where $(1 + r_{t+1}) = F_k\left(\hat{k}_{t+1}, 1\right)$. Now, in a steady state we have that $\hat{c}_t = \hat{c}_{t+1} = c$ and $\hat{k}_t = \hat{k}_{t+1} = k$, so the FOC becomes

$$\gamma = \beta\left(1 + r\right)$$

Hence, the rate of growth of this economy is given by exogenous parameters and the imposed technology (hence the 'exogenous growth' label to the model). To see more clearly that all variables are growing at the rate γ, just use the fact that in steady state $\hat{x}_t = \hat{x}_{t+1}$, then $\frac{x_t}{\gamma^t} = \frac{x_{t+1}}{\gamma^{t+1}}$ which imply that $\frac{x_{t+1}}{x_t} = \gamma$.

Problem 2 *The FOC of the usual growth problem can be expressed like this*

$$\frac{u_c(c_t)}{u_c(c_{t+1})} = \beta(1 + r_{t+1})$$

We saw in class that with CRRA preferences, we could find an expression for the steady state of the form $1 = \beta\gamma^{-\sigma}(1 + r)$. *Show that if we use the same CRRA preferences but with a minimum level of consumption needed (i.e.,* $\hat{u}(c) = \frac{(c-b)^{1-\sigma}-1}{1-\sigma}$ *with* $b > 0$*) the result doesn't go through.*

Suggested Solution

With CRRA preferences, we have that $u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$ and $u_c(c) = c^{-\sigma}$. Then, the FOC can be written as

$$\left(\frac{c_{t+1}}{c_t}\right)^{\sigma} = \beta(1 + r_{t+1})$$

Where $(1 + r_{t+1})$ is again $F_k(K_{t+1}, 1)$. Guessing that a steady state value for consumption is $\bar{c}$

$$\begin{aligned}\left(\frac{\gamma\bar{c}}{\bar{c}}\right)^{\sigma} &= \beta(1 + r_{t+1}) \\ &\Rightarrow \gamma^{\sigma} = \beta(1 + r) \\ &\Rightarrow 1 = \beta\gamma^{-\sigma}(1 + r)\end{aligned}$$

Now, if $u(c) = \frac{(c-b)^{1-\sigma}-1}{1-\sigma}$ we have that $u_c(c) = (c - b)^{-\sigma}$. In the FOC this implies

$$\left(\frac{c_{t+1} - b}{c_t - b}\right)^{\sigma} = \beta(1 + r_{t+1})$$

from where is not possible to get a closed form solution to γ.

Problem 3 *Show that CRRA preferences have constant elasticity of intertemporal substitution equal to* σ*, which is equal to the coefficient of relative risk aversion.*

Suggested Solution

Given some preferences $u(c)$, where c is part of a sequence of consumption over time, the elasticity of intertemporal substitution is defined as

$$\varepsilon = \left|\frac{\Delta\%\left(\frac{u_c(c)}{u_c(c')}\right)}{\Delta\%\left(\frac{c}{c'}\right)}\right|$$

or in English, the percentage change of the marginal rate of substitution between consumption today and consumption tomorrow, divided by the percentage change in the ratio of consumption today with respect of tomorrow. With CRRA preferences, we have that

$$\frac{u_c(c)}{u_c(c')} = \left(\frac{c'}{c}\right)^{\sigma}$$

Taking logs

$$\begin{aligned}
\log\left(\frac{u_c(c)}{u_c(c')}\right) &= \sigma \log\left(\frac{c'}{c}\right) \\
&= -\sigma \log\left(\frac{c}{c'}\right)
\end{aligned}$$

Since $\Delta\% \approx d\log$

$$\begin{aligned}
\varepsilon &\approx \left|\frac{d\log\left(\frac{u_c(c)}{u_c(c')}\right)}{d\log\left(\frac{c}{c'}\right)}\right| \\
&= \sigma
\end{aligned}$$

On the other hand, the coefficient of relative risk aversion is defined as

$$\Omega \equiv -c\frac{u''(c)}{u'(c)}$$

With CRRA preferences, $u'(c) = c^{-\sigma}$ and $u''(c) = -\sigma c^{-\sigma-1}$. Then

$$\begin{aligned}
\Omega &= -c\frac{-\sigma c^{-\sigma-1}}{c^{-\sigma}} \\
&= \sigma\frac{c}{c} \\
&= \sigma
\end{aligned}$$

Hence, in the CRRA class of utility functions, both the elasticity of intertemporal substitution as well as the coefficient of relative risk aversion are set by the same parameter, σ.

Problem 4 *Imagine an economy where there is a total factor productivity growth, i.e., the production function is*

$$Y_t = A_t F(K_t, N_t)$$

where $A_t = \gamma^t A_0$. Show that this economy exhibits a balanced growth path and calculate the rate at which all variables grow. Is it equal to γ?

Suggested Solution

Set $A_0 = 1$ and $N = 1$ to ease notation. Given a production function that is homogeneous of degree one, the TFP growth implies

$$\begin{aligned}
Y_t &= \gamma^t F(K_t, 1) \\
&= F\left(\gamma^t K_t, \gamma^t\right)
\end{aligned}$$

From the first question, we know that if we can write the production function as $Y_t = F\left(K_t, \tilde{\gamma}^t\right)$ we can solve the problem of balanced growth path, since it's observationally equivalent to a model with increasing labor productivity. Hence, we need some additional properties for the production function. First, let's check with a Cobb-Douglas function

$$\begin{aligned} Y_t &= \gamma^t F\left(K_t, 1\right) \\ &= \gamma^t K_t^{\alpha} 1^{1-\alpha} \\ &= K_t^{\alpha}\left(\gamma^{\frac{1}{1-\alpha}t}\right)^{1-\alpha} \\ &= F\left(K_t, \tilde{\gamma}^t\right) \end{aligned}$$

here $\tilde{\gamma} = \gamma^{\frac{1}{1-\alpha}}$. Then, in addition to homogeneity of degree one, we need our production function to be flexible enough in order to accommodate different types of technological growth. As we've seen, the Cobb-Douglas does the trick, since the problem now reduces to

$$\max \sum_{t=0}^{\infty} \beta^t u\left[C_t\right]$$

st

$$C_t + K_{t+1} = F\left(K_t, \tilde{\gamma}^t\right)$$

which is almost identical to problem 1: here, we need also CRRA preferences in order to construct the

'hat' economy: As you can see, the TFP is growing at rate γ but the economy is growing at a different rate: $\tilde{\gamma} = \gamma^{\frac{1}{1-\alpha}}$.

Similarly, suppose that the technological change is affecting capital:

$$Y_t = F\left(\gamma^t K_t, 1\right)$$

Using a Cobb-Douglas

$$\begin{aligned} Y_t &= \left(\gamma^t K_t\right)^{\alpha} 1^{1-\alpha} \\ &= K_t^{\alpha}\left(\gamma^{\frac{\alpha}{1-\alpha}t}\right)^{1-\alpha} \\ &= F\left(K_t, \check{\gamma}^t\right) \end{aligned}$$

Hence, this economy is growing at rate $\check{\gamma} = \gamma^{\frac{1}{1-\alpha}}$. \

Problem 5 *Imagine a Lucas-tree model with CRRA utility and where the fruit is constant every period ($d_t = 1$ $\forall t$). Price this tree. Now, assume that the fruit starts growing at rate γ. Calculate the price of the new tree. Is it bigger/smaller than before?*

Suggested Solution

Remember the lucas tree model

$$\max_{\{c_t(h_t)\}_{t,h_t}} \sum_{t=0}^{\infty} \beta^t \sum_{h_t} \pi\left(h_t\right) u\left[c_t\left(h_t\right)\right]$$

subject to

$$\sum_{t=0}^{\infty} \beta^t \sum_{h_t} \pi(h_t)\, c_t(h_t) = \sum_{t=0}^{\infty} \beta^t \sum_{h_t} \pi(h_t)\, d_t(h_t)$$

where $c_t(h_t)$ is consumption at node h_t and $d_t(h_t)$ is the fruit at the same node. In class we derived the price of consumption at a particular node h_t in terms of consumption at the beginning of time:

$$p(h_t) = \beta^t \pi(h_t) \frac{u_c(c_t(h_t))}{u_c(c_0)}$$

Given this prices, we can derive the price of any tree

$$\begin{aligned} \mathcal{P} &= \sum_{t=0}^{\infty} \sum_{h_t} p(h_t)\, d_t(h_t) \\ &= \sum_{t=0}^{\infty} \sum_{h_t} \beta^t \pi(h_t) \frac{u_c(d_t(h_t))}{u_c(d_0)} \end{aligned}$$

Now, consider a tree that gives us one fruit every period, no matter what. The price of that tree (call it $\mathcal{P}_c$) is given by

$$\mathcal{P}_c = \sum_{t=0}^{\infty} \beta^t \frac{u_c(1)}{u_c(1)}$$

Note that the tree is deterministic, so we don't need to weight by probabilities. Using CRRA preferences we get

$$\begin{aligned} \mathcal{P}_c &= \sum_{t=0}^{\infty} \beta^t \left(\frac{1}{1}\right)^{\sigma} \\ &= \sum_{t=0}^{\infty} \beta^t \\ &= \frac{1}{1-\beta} \end{aligned}$$

On the other hand, take the tree that gives you one unit of fruit at the beginning of time and then the fruit starts growing at rate γ forever (deterministically). Its price will be given by

$$\begin{aligned} \mathcal{P}_\gamma &= \sum_{t=0}^{\infty} \beta^t \frac{u_c(\gamma^t)}{u_c(1)} \\ &= \sum_{t=0}^{\infty} \beta^t \left(\gamma^{-\sigma}\right)^t \\ &= \frac{1}{1-\beta\gamma^{-\sigma}} \end{aligned}$$

Which tree is more valuable? It all depends on σ.