Econ 702
Spring 2006
Suggested solutions to Problem Set #1

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Problem 1. *Let $\mathcal{L}$ be a topological vector space, X a convex and open subset of $\mathcal{L}$ and a mapping $u : X \to \mathbb{R}$. Prove that concavity implies continuity.*

Suggested solution

Notice that we need openness of X for the proof to go through. If the set we are dealing with is closed, the same theorem applies to the interior of the set.

Let $x \in X$, and $x_k \to x$, with $x_k \in X$ for all k. If X is open, there is $r > 0$, such that $B_r(x) \subset X$. Pick $\alpha \in (0, r)$ and let $A \subset B_r(x)$, where $A = \{z \mid \|z - x\| = \alpha\}$. Since $x_k \to x$, there exists K big enough, such that for all $k > K$, $\|x_k - x\| < \alpha$.

Then, for $k > K$ and some $\theta_k \in (0, 1)$, there is $z_k \in A$ such that $x_k = \theta_k x + (1-\theta_k)z_k$. Since $x_k \to x$ and $\|z_k - x\| = \alpha > 0$ for all k, we know that $\theta_k \to 1$.

By concavity of u

$$u(x_k) = u(\theta_k x + (1 - \theta_k)z_k) \geq \theta_k u(x) + (1 - \theta_k)u(z_k)$$

taking limits (remember that $\theta_k \to 1$)

$$\liminf_{k\to\infty} u(x_k) \geq u(x) \tag{1}$$

On the other hand, there is also $w_k \in A$ and $\lambda_k \in (0, 1)$ such that $x = \lambda_k x_k + (1 - \lambda_k)w_k$ for all $k > K$. Using the concavity of u again

$$u(x) = u(\lambda_k x_k + (1 - \lambda_k)w_k) \geq \lambda_k u(x_k) + (1 - \lambda_k)u(w_k)$$

taking limits

$$u(x) \geq \limsup_{k\to\infty} u(x_k) \tag{2}$$

Finally, (1) and (2) imply that $\lim_{k\to\infty} u(x_k) = u(x)$, i.e., continuity of f on X □

Problem 2. *Let $p, \ell \in \mathbb{R}^N$. Prove that $p(\ell) = \sum_{n=1}^{N} p_n \ell_n$ is a continuous linear function.*

Suggested solution

First, let's check linearity. Let $a, b \in \mathbb{R}$ and $\ell^1, \ell^2 \in \mathbb{R}^N$. Then

$$\begin{aligned} p\left(a\ell^1 + b\ell^2\right) &= \sum_{i=1}^{N} p_n \left(a\ell_n^1 + b\ell_n^2\right) \\ &= \sum_{i=1}^{N} \left(p_n a\ell_n^1 + p_n b\ell_n^2\right) \\ &= a\sum_{i=1}^{N} p_n\ell_n^1 + b\sum_{i=1}^{N} p_n\ell_n^2 \\ &= ap\left(\ell^1\right) + bp\left(\ell^2\right) \end{aligned}$$

Now, for continuity, we need a sequence ℓ^k converging to an arbitrary $\ell \in \mathbb{R}^N$. So

$$p\left(\ell^k\right) = \sum_{n=1}^{N} p_n \ell_n^k \qquad \forall k$$

Since $\ell^k \to \ell$, we know that $p_n \ell_n^k \to p_n \ell_n$ for all k and $n = 1, ...N$. Since we are dealing with a finite sum, it must be true that

$$\sum_{n=1}^{N} p_n \ell_n^k \to \sum_{n=1}^{N} p_n \ell_n$$

hence, $p\left(\ell^k\right) \to p\left(\ell\right)$ as $\ell^k \to \ell$.

Now, let's see what happens when $N \to \infty$. The proof goes through only if we have $p, \ell \in \mathcal{L}$ (the space of infinite and bounded sequences). Moreover, we need $\sum_{n=1}^{\infty} |p_i| < \infty$.[1]

Linearity is proven just like in the finite case. For continuity, we'll use Theorem 15.1 of Stockey and Lucas (page 446) which states that $p(\ell)$ is continuous iff it is bounded:

$$|p(\ell)| = |\sum_{n=1}^{\infty} p_n \ell_n| \leq \sum_{n=1}^{\infty} |p_n \ell_n| \leq \sup_n |\ell_n| \sum_{n=1}^{\infty} |p_n| < \infty$$

Since $p(\ell)$ is bounded, then it's continuous.

Problem 3. *If X, Y are convex sets, prove that $X \cap Y$ is also convex.*

Suggested solution

Pick z^1 and z^2 from $X \cap Y$. That means $z^1, z^1 \in X \cap Y$.

Now, let $\theta \in (0, 1)$ such that $z^\theta \equiv \theta z^1 + (1-\theta) z^2$. Since both X, Y are convex, $z^\theta \in X$ and $z^\theta \in Y$. So $z^\theta \in X \cap Y$ $\square$

Problem 4. *Consider the (Solow) growth model as presented in Class:*

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u\left[c_t\right]$$

subject to

$$c_t + k_{t+1} = f\left(k_t\right)$$

$$k_0 \quad \textit{given}$$

Show that (i) a solution to the problem exists, (ii) is unique and (iii) the solution is pareto optimal. In order to do this, you need to check the following conditions[2] (in order to apply existence and uniqueness theorems, as well as the basic welfare theorems):

- *Start by defining the commodity space ($\mathcal{L}$), the consumption possibility set (X) and the production possibility set (Y) in a suitable form (This was already done in class).*
- *Show that X and Y are closed and convex*
- *Show that Y has an interior point*

[1] In other words, restriction on prices at infinity

[2] These are NOT all the neccesary conditions. We're not checking the conditions on the objective function for now.

- *Show that the set of feasible allocations ($X \cap Y$) is compact*

Note that you need to assume certain properties of f in order to prove the above.[3]

Suggested solution Q.4

Let's start by defining the commodity space, the consumption possibility set and the production possibility set. As in class, the commodity space is defined as

$$\mathcal{L} = \{\{x_{1t}, x_{2t}, x_{3t}\}_{t=0}^{\infty} \mid \sup_{i,t} < \infty\}$$

or in other words, the space of infinite and bounded real sequences.
The Consumption possibility set was defined as

$$\begin{aligned} X = \{x \in \mathcal{L} \mid \exists \{c_t, k_{t+1}\}_{t=o}^{\infty} \geq 0 \quad & s.t. \\ x_{1t} = c_t + k_{t+1} \quad & \forall t \\ x_{2t} \in [-k_t, 0] \quad & \forall t \\ x_{3t} \in [-1, 0] \quad & \forall t \\ k_0 \quad & given\} \end{aligned}$$

And the production possibility set as

$$Y = \prod_t Y_t$$

where

$$Y_t = \{(y_{1t}, y_{2t}, y_{3t}) \in \mathbb{R} \mid y_{1t} \leq f(-y_{2t}, -y_{3t})\}$$

Now, let's assume the following:

- $f(\cdot)$ is increasing (in both arguments), continuous, concave, homogenous of degree one and bounded
- $f(0, x_{3t}) = f(x_{2t}, 0) = f(0, 0) = 0, \forall t$

X closed
We'll use the sequential definition of closed sets throughout. Pick a converging sequence $x^k \in X, \forall k$. Now, assume that X is not closed, i.e., $lim_{k \to \infty} x^k = x \notin X$. Then, for some k big enough, there must be some $\{c_t^k, k_{t+1}^k\}_{t=0}^{\infty}$ such that

$$x_{1t}^k \neq c_t^k + k_{t+1}^k, \forall t$$
$$or$$
$$x_{2t}^k \notin [-k_{2t}^k, 0]$$
$$or$$
$$x_{3t}^k \notin [-1, 0]$$

Which contradicts $x^k \in X, \forall k$.

Y closed
Analogous proof. Just use f continuous.

X convex
Pick $x^1, x^2 \in X$ and $\theta \in (0, 1)$. Define $z^\theta = \theta z^1 + (1 - \theta) z^2$. By our definition of X, we know that for all t

[3] This is what Victor meant when he said 'use backward engineering'.

$$x_{1t}^1 = c_t^1 + k_{t+1}^1$$
$$x_{1t}^2 = c_t^2 + k_{t+1}^2$$

which imply

$$x_{1t}^\theta = c_t^\theta + k_{t+1}^\theta$$

Also, $x_{2t}^1 \in [-k_{2t}^1, 0]$ and $x_{2t}^2 \in [-k_{2t}^2, 0]$ imply

$$x_{2t}^\theta \in [-k_{2t}^\theta, 0]$$

Finally, $x_{3t}^1 \in [-1, 0]$ and $x_{2t}^2 \in [-1, 0]$ imply

$$x_{3t}^\theta \in [-1, 0]$$

Since every point in X is defined by the same k_0, $x^\theta = \theta x^1 + (1-\theta)x^2 \in X$.

Y convex

Let $y^1, y^2 \in Y_t, \forall t$ and $\theta \in (0,1)$. By our definition of Y_t, for each t

$$y_{1t}^1 \leq f(-y_{2t}^1, -y_{3t}^1)$$
$$y_{1t}^2 \leq f(-y_{2t}^2, -y_{3t}^2)$$

Taking the convex combination of the two equations

$$y_{1t}^\theta \leq \theta f(-y_{2t}^1, -y_{3t}^1) + (1-\theta) f(-y_{2t}^2, -y_{3t}^2)$$

By concavity and homogeneity of degree 1 of f

$$\theta f(-y_{2t}^1, -y_{3t}^1) + (1-\theta) f(-y_{2t}^2, -y_{3t}^2) \leq f(-y_{2t}^\theta, -y_{3t}^\theta)$$

Then,

$$y_{1t}^\theta \leq f(-y_{2t}^\theta, -y_{3t}^\theta)$$

For all t. Hence, Y is convex.

Y has an interior point

Pick a triplet $\{y_{1t}, y_{2t}, y_{3t}\}$ and $\epsilon > 0$ such that $y_{1t} \geq 0$, $y_{2t} \leq 0$, $y_{3t} \leq 0$ and

$$y_{1t} = f(-y_{2t}, -y_{3t}) - \epsilon \qquad \forall t$$

By construction, $y' = \{y_{1t} + \epsilon, y_{2t}, y_{3t}\}_{t=0}^\infty \in Y$. Clearly, any point $y'' = \{y_{1t} + \varepsilon, y_{2t}, y_{3t}\}_{t=0}^\infty$ with $\varepsilon \in (0, \epsilon)$, also belongs in Y. Moreover,

$$\sup \|y' - y''\| = \epsilon - \varepsilon > 0$$

Thus, we have constructed a point in Y, disturbed it by ε and seen that the resulting points remained in Y. Hence, the set has an interior point.

$X \cap Y$ ***Compact***

First, lets define $\mathcal{Z} = X \cap Y$

$$\begin{aligned} \mathcal{Z} = \{z \in \mathcal{L} \mid \exists \{c_t, k_{t+1}\}_{t=o}^\infty \geq 0 \quad & s.t. \\ z_{1t} \leq f(-z_{2t}, -z_{3t}) \quad & \forall t \\ z_{1t} = c_t + k_{t+1} \quad & \forall t \\ -z_{2t} \in [-k_t, 0] \quad & \forall t \\ -z_{3t} \in [-1, 0] \quad & \forall t \\ k_0 \quad & given\} \end{aligned}$$

By the sequential definition of compactness, if we have a sequence $z^n \in \mathcal{Z}, \forall n$ and construct a converging subsequence to a point in $\mathcal{Z}$, then we are done. For all n and t, we have

- $z_{3t}^n \in [-1, 0]$
- $c_t^n \geq 0$, then $k_{t+1}^k \leq f(-z_{2t}^n, -z_{3t}^n) \leq f(-z_{2t}^n, 1) \leq M < \infty$

$$\Rightarrow z_{2t}^n \in [-M, 0]$$

- Since f is bounded (by M), then $z_{1t} \in [0, M]$
- Hence, $z_t^n \in \mathcal{B} \equiv [0, M] \times [-M, 0] \times [-1, 0]$

By Bolzano-Weirstrass' theorem, $\mathcal{B}$ has a concentration point. Then, for any sequence $\{z^n\}$ we can construct a subsequence $\{z^{n_k}\}$ such that $z^{n_k} \to z \in \mathcal{Z}$.