Econ 702, Spring 2006
Problem set 11
Due Tuesday April 11th

Suggested Solutions (prepared by Se Kyu Choi)

Problem 1 *Consider the problem of optimal unemployment insurance by a social planner when the effort of the worker is observable. Explicitly, the problem of the Planner is*

$$\Psi(V) = \min_{c,V',a} c + \frac{1}{1+r}(1-p(a))\Psi(V')$$

subject to

$$V = u(c) - a + \beta\{p(a)V^e + (1-p(a))V'\}$$

where $V^e = \frac{u(w)}{1-\beta}$ *is the value of being employed at wage* w*. Also, define*

$$V^u = \max_a u(b) - a + \beta\{p(a)V^e + (1-p(a))V^u\}$$

as the value of being unemployed. For simplicity, assume $b = 0$*.*

1. *If* $V > V^u$*, show that* $\Psi(V) > 0$*.*
2. *Setting up the problem of the planner as a Lagrangian (with multiplier* θ *for the constraint) show carefully that the envelope condition implies*

$$\Psi'(V) = \theta$$

3. *Prove that* $\Psi'(V) > 0$*.*
4. *We saw in class that in an interior solution, the optimum unemployment insurance has* $V = V'$*. Show that this implies that* $c = c'$ *and* $a = a'$*.*

Suggested Solution

1. First, consider the first order condition when the agent is on her own:

$$-1 + \beta p(a)(V^e - V^u) = 0$$

From this condition, we get an optimal level of effort a^*_{aut}. Clearly, $\Psi(V^u) = 0$; if the planner promises V^u, the cheapest way to solve the problem is to set consumption to zero, effort level to a^*_{aut} and promise V^u. With a consumption of stream of zero, the cost of the insurance is zero also (we don't have to give anything to the agent). In a simple analogy, it's like promising you to give you the 10 dollar bill you already have in your wallet. The cost for me of providing such insurance is zero, since (if I can contract on your effort, i.e., tell you what to do) I can make you reach your wallet and get the bill.

Now, if the planner promises $V > V^u$, the optimal choice of $\{c^*(V), a^*(V), V'^*(V)\}$ must be different. More precisely, if $c^*(V) > 0$ the cost of the insurance plan is positive, hence $\Psi(V) > 0$. How to show this? a simple way is to impose inada conditions on the utility function

$$\lim_{c \to 0} u_c(c) = \infty$$

In this way, to provide a promise of $V > V^u$ we need to provide $c^*(V) > 0$.[1]

2. The problem (including constraints and setting $\beta = (1+r)^{-1}$) is

$$\Psi(V) = \min_{c,a,V'} \{c + \beta [1 - p(a)] \Psi(V') + \theta [V - u(c) + a - \beta \{p(a) V^e + (1 - p(a)) V'\}]\}$$

The FOCs are

$$\begin{aligned} c &: \quad 1 - \theta u_c = 0 \\ a &: \quad \Psi(V) = \theta \left[\frac{1}{\beta p(a)} - (V^e - V') \right] \\ V' &: \quad \Psi'(V') = \theta \end{aligned}$$

The envelope condition comes from including $\{c^*(V), a^*(V), V'^*(V)\}$ as the optimizers of the problem and taking the derivative of $\Psi(V)$ with respect to V:

$$\begin{aligned} \Psi'(V) &= \frac{\partial c}{\partial V}[1 - \theta u_c] + \frac{\partial a}{\partial V}\left[-\beta p(a) \Psi(V) + \theta - \beta p(a) \left(V^e - V^{'}\right)\right] \\ &\quad + \frac{\partial V'}{\partial V}[-\Psi'(V') + \theta] + \theta \end{aligned}$$

By the FOCs, all terms in brackets are equal to zero, which leaves us with

$$\Psi'(V) = \theta$$

3. Proving $\Psi'(V) > 0$ is the same as proving $\theta > 0$, i.e., that the constraint is always binding.[2] Suppose not, that is, in an optimal solution $\{c^*(V), a^*(V), V'^*(V)\}$ to the minimization problem, we have that the constraint is not binding

$$V < u(c^*) - a^* + \beta \{p(a) V^e + (1 - p(a)) V'^*\}$$

But then the planner is not optimizing, since consumption can still be lowered further, maintaining the same promise V'^*. Hence, a contradiction. □

4. In class we saw that the FOC with respect to V' and the envelope condition together implied that $V = V'$. Taking this into the FOC for effort

$$\Psi(V) = \theta \left[\frac{1}{\beta p(a)} - (V^e - V') \right]$$

we see that if $V = V'$, and assuming $p(a)$ strictly concave, we get a unique solution a^* which will repeat itself each period. Further, if the promise value is constant, the promise keeping constraint binds at the same point: then, θ is also constant. From the FOC for consumption

$$u_c = \frac{1}{\theta}$$

[1] In the following questions, we'll get a more straightforward way of proving this.

[2] Note that the original constraint is

$$V \leq u(c) - a + \beta [p(a) V^e - (1 - p(a)) V']$$

in other words, $\{c^*(V), a^*(V), V'^*(V)\}$ must provide at least utility V.

we get a unique consumption value that will be constant over time (the assumption on u is that it's strictly concave).[3]

Problem 2 *Now consider the same problem, but when effort is not observable.*

1. *Setup the problem as a Lagrangian and carefully derive the first order conditions as well as the envelope condition*
2. *Sketch a proof for Ψ being a convex function*
3. *Show that if Ψ is convex, the optimum unemployment insurance has V decreasing which implies that c is also decreasing.*

Suggested Solution

1. The problem with unobservable effort is described by an additional constraint (incentive compatibility constraint)

$$\Psi(V) = \min_{c,a,V'} \left\{ \begin{array}{c} c + \beta[1-p(a)]\Psi(V') + \theta[V - u(c) + a - \beta\{p(a)V^e + (1-p(a))V'\}] \\ +\eta[1 - \beta p'(a)(V^e - V^u)] \end{array} \right\}$$

The FOCs with respect to c, a and V' are respectively

$$\begin{aligned} u_c &= 1/\theta \\ \Psi(V) &= \theta\left[\frac{1}{\beta p(a)} - (V^e - V')\right] - \eta\frac{p''(a)}{p'(a)}(V^e - V') \\ \Psi'(V') &= \theta - \eta\frac{p'(a)}{1-p'(a)} \end{aligned}$$

The envelope condition is again

$$\Psi'(V) = \theta$$

2. A simple proof of Ψ being convex (using differentiability) is to set the following equality (using the Envelope condition along with the FOC for consumption)

$$\Psi'(V) = \frac{1}{u_c(c(V))}$$

Given that c is increasing in V and that u_c is decreasing (by concavity of u) we see that

$$\Psi''(V) = -\frac{u_{cc}}{u_c}\frac{\partial c}{\partial V} > 0$$

[3] Now, we can prove question 1 directly: θ is the multiplier for the promise keeping constraint. Obviously, the value of relaxing the constraint is higher the higher the level of current value V, i.e.,

$$V^1 > V^2 \Rightarrow \theta(V^1) > \theta(V^2)$$

Now, from the FOC for consumption we see that there is a positive relation between θ and c (given concavity of u, u_c is decreasing). All this together, imply that

$$\frac{\partial c^*}{\partial V} > 0$$

Hence, if $V > V^u$, and $c(V^u) = 0$ (as showed previously) the cost of the insurance plan is positive since we have to increase promised consumption.

i.e., $\Psi(V)$ is convex. The intuition for this result comes from the fact that the utility function is concave: the agent values additional consumption in a decreasing way, so its increasingly costly for the planner to provide that consumption (if V is to be increased).

3. Lets compare the FOC with respect to V' and the envelope condition

$$\begin{aligned} FOC\{V'\} &: \quad \Psi'(V') = \theta - \eta \frac{p'(a)}{1 - p'(a)} \\ EC &: \quad \Psi'(V) = \theta \end{aligned}$$

As before, we can argue that θ is strictly positive. The term $\eta \frac{p'(a)}{1-p(a)}$ is also strictly positive: η is the multiplier for the incentive compatibility constraint which always binds, since it's the FOC of the maximization problem that the agent performs on her own. From this, we have that

$$\Psi(V) > \Psi(V')$$

and by strict convexity of Ψ, we get that

$$V > V'$$

or in other words, the promise to the agent is decreasing. For consumption, the FOC implies that

$$u_c(c) = 1/\theta$$

and

$$u_c(c') = 1/\left(\theta - \eta \frac{p'(a)}{1 - p'(a)}\right)$$

Hence, by concavity of u, we get $c > c'$ i.e., consumption must be also decreasing.