1 Introduction

1.1 What is an equilibrium?

An equilibrium is a statement about what the outcome of an economy is. Tells us **what happens in an economy,** and by an ecomomy we mean a well defined environment in terms of primitives such as preferences and technology.

Then an equilibrium is a particular mapping from the environment (preference, technology, information, market structure) to allocations where,

- 1. Agents maximize
  2. Agents' actions are compatible.
- One of the important questions is, given the environment what type of equilibria we should look at. The economist doesn't have the right to choose what happens, but free to define the environment.
- For the theory to be able to predict precisely what is going to happen in a well defined environment, the outcome we define as the equilibrium needs to exist and must be unique. For this reason uniqueness is property that we want the equilibrium to have. We also know with certain assumptions that will be covered we can ensure the existence and uniqueness of an equilibrium outcome.
- By now, you have learned how to solve Social planner's problem (SPP) of neoclassical growth model with representative agent (RA-NGM), using dynamic programming. Also we know that solution to SPP, if it exists, is Pareto Optimal (PO). The solution of SPP can be interpreted as the allocation to be chosen if the God exists and has control over everything (by definition!) and is benevolent (maybe by definition). In other words, the solution does not predict what is going to happen in an environment but what will prevail as the wish of the social planner. As we will see, under certain conditions we are able define the 'right' environment such that the equilibrium outcome coincides with coincides with the solution to SPP.
- Other good things for solution to SPP is that, in RA-NGM with certain assumptions, we know that (i) it exists and (ii) it's unique.
- Besides, we have two welfare theorems (FBWT, SBWT) from Dave's class. If we carefully define the environment, those two theorems guarantee (loosely) that (i) under certain conditions, Arrow-Debrew Competitive Equilibrium (ADE, or Walrasian equilibrium or valuation equilibrium) is PO, and (ii) also under certain conditions, we can construct an ADE from a PO allocation.
- Using those elements, we can argue that ADE exists and is unique, and we just need to solve SPP to derive the allocation of ADE, which is much easier task than solve a monster named ADE.
- But we have another problem: The market assumed in ADE is not palatable to us in the sense that it is far from what we see in the world. So, next, we look at an equilibrium with sequential markets (Sequential Market Equilibrium, SME). Surprisingly, we can show that, for our basic RA-NGM, the allocation in SME and the allocation of ADE turn out to be the same, which let us conclude that even the allocation of the equilibrium with sequential markets can be analyzed using the allocation of SPP.
- Lastly, we will learn that equilibrium with sequential markets with recursive form (Recursive Competitive Equilibrium, RCE) gives the same allocation as in SME, meaning we can solve the problem using our best friend = Dynamic Programming.
- (Of course, these nice properties are available for limited class of models. We need to directly solve the equilibrium, instead of solving SPP, for large class of interesting models. We will see that Dynamic Programming method is also very useful for this purpose. We will see some examples later in the course.)

- In this class, we will go over some of the 'popular' notions of equilibrium in dynamic macroeconomics that might be different from a static Walrasian equilibrium covered last semester. Neoclassical growth model (NGM), which is the workhorse of modern macroeconomics will be our departure point.

1.2 Growth Model

1.2.1 Technology

- Represented by production function:

$$f : R_+^2 \to R_+ \quad \text{such that } y_t = f(k_t, n_t) \tag{1}$$

- We assume (i) Constant Returns to Scale (CRS, or homogeneous of degree one, meaning $f(\lambda k, \lambda n) = \lambda f(k, n)$), (ii) strictly increasing in both arguments, and ((iii) INADA condition, if necessary)

1.2.2 Preference

- We assume infinitely-lived representative agent (RA).[1]
- We assume that preference of RA is (i) time-separable (with constant discount factor $\beta < 1$), (ii) strictly increasing in both consumption (iii) strictly concave
- Our assumptions let us use the utility function of the following form:

$$\sum_{t=0}^{\infty} \beta^t u(c_t) \tag{2}$$

- Initial capital stock k_0 is given.

  With these in hand the SP problem is,

$$\max_{\{c_t, n_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{3}$$

subject to[2]

$$k_{t+1} + c_t = f(k_t, n_t) + (1-\delta)k_t \tag{4}$$

$$c_t, k_{t+1} \geq 0 \tag{5}$$

$$n_t \in [0,1] \tag{6}$$

$$n_t + l_t = 1 \tag{7}$$

$$k_0 \text{ is given} \tag{8}$$

[1]For now, let's treat the economy as if there were only one agent in the economy. We might interpret it as the result of normalization (so the number of population is 1) of the economy with FINITE number of identical (sharing the same technology, preference, and allocation) agents. If we proceed to the economy with mass of zero measure agents, things will be not so trivial because changing allocation of one agent does not change the aggregate amount of resources in the economy (since, by assumption, measure of an agent is zero), but let's forget it for now.

[2]We can also define f (the production function) as including depreciation of capital. In the 1st class, Victor actually took this approach, but I modified the notation to make notation consistent across classes.

1.2.3 Property 1: Existence

- Need to show (i) maximand is continuous function and (ii) constraint set is compact (closedness and boundedness).
- Not go into details but be aware that commodity space is infinite dimensional space (so exactly the same argument as in 701 where commodity space is finite dimensional space is not valid here). In particular, need to define commodity space as a topological linear space with sup-norm. For those interested, the argument involves the product topology and compactness of products of compact vector spaces

1.2.4 Property 2: Uniqueness

- Need (i) convex constraint set, and (ii) strictly concave function, which through our assumptions satisfied.

1.2.5 Property 3: Pareto Optimality

- Trivial (if assume finite number of agents).
- We know how to solve SPP of RA-NGM.
- But what we want to know is equilibrium (price and allocation).
- If we can apply welfare theorems to the allocation of SPP, we can claim that "God's will realizes" and can analyze allocation of SPP instead of directly looking at an equilibrium allocation.

  Note that we want the INADA conditions everywhere so that we have interior solutions. INADA conditions make sure that the nonnegativity constraints are irrelevant. This is so that we can use the First Order Conditions and not deal with the KT conditions. Since leisure is not in the utility function, we don't have to worry about it. Agent doesn't care about it so he will work as much as he can, therefore it must be that $l_t = 1$.

Rewriting the above problem with full deprication assumption:

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(f(k_t) - k_{t+1}) \tag{9}$$

$$k_0 \ \ given \tag{10}$$

Because of the INADA conditions we know that the solution is interior. So if $\{k_{t+1}^*\}$ is a solution then it satisfies the first order conditions,

$$-\beta^t u_c(f(k_t) - k_{t+1}) + \beta^{t+1} f_k(k_t + 1) u_c(f(k_{t+1}) - k_{t+2}) = 0 \tag{11}$$

But notice that these conditions are not sufficient. The above is a second order difference equation. It has two degrees of freedom. Therefore this equation is not enough to find the solution. There can be many sequences that satisfy it. We need to more conditions to pin down the right solution: The initial condition k_0 and the transversality condition.

1.3 From SPP to Valuation Equilibrium

- In order to use the argument above, we formalize the environment of RA-NGM in the way such that we can apply welfare theorems. By using (i) existence of solution to SPP, (ii) uniqueness of solution of SPP, and (iii) welfare theorems, we can claim that ADE (i) exists, (ii) is unique, (iii) and PO. However, market arrangement of ADE is not palatable to us in the sense that set of markets that are open in the ADE is NOT close to the markets in our real world. In other words, there is notion of time in ADE: all the trades are made before the history begins and there is no more choices after the history begins. So we would like to proceed to the equilibrium concept that allows continuously open markets, which is SME and we will look at it closely later.

1.3.1 Valuation Equilibrium

- In macroeconomics, we are interested in infinite- dimensional commodity spaces. We want to look at the relationship between competitive equilibrium and Pareto optimality in models with infinite-dimensional spaces. You looked at competitive equilibrium and Pareto optimality in 701, but the proofs of the FBWT and SBWT were done in the context of finite-dimensional commodity spaces. Here we want to show that the welfare theorems hold for economies with infinite dimensional spaces. To do this, we introduce the equilibrium concept 'valuation equilibrium'.
  Before defining valuation equilibrium, we first need to define the environment, unlike the social planner problem, which is a problem of allocation, in a AD world we will have exchange among agents. This requires definition of markets in which the relevant commodities to be defined are traded.:

1. $\mathcal{L}$, Commodity space:
   $\mathcal{L}$ is a topological vector space.

   Definition 1 (Vector Space) *A vector space is a space where the operations addition and scalar multiplication are defined, and where the space is closed under these two operations. i.e. If we take two sequences $a = \{a_i\} \in \mathcal{L}$ and $b = \{b_i\} \in \mathcal{L}$, it must be that $a + b \in \mathcal{L}$. And if we take $k \in \mathcal{R}^+, k > 0$, it must be that $a \in \mathcal{L} \Rightarrow d = ka \in \mathcal{L} \;\; \forall\, k > 0$.*

   Definition 2 (Topological Vector Space) *A topological vector space is a vector space which is endowed with a topology such that the maps $(x, y) \to x + y$ and $(\lambda, x) \to \lambda x$ are continuous. So we have to show the continuity of the vector operations addition and scalar multiplication.*

2. $X \subset \mathcal{L}$, Consumption Possibility Set:
   Specification of the 'things' that people could do (that are feasible to them). X contains every (individually) technologically feasible consumption point.

3. $U : X \to \mathcal{R}$, Specifies the preference ordering.

4. Y, Production possibility set.

What is an allocation in this environment? An allocation is a pair (x, y). On the other hand, a <u>feasible</u> allocation is (x, y) such that x=y (agents' actions need to be compatible). What are the commodites we need to make tradable in this environment? Output, labor services, capital services. So lets define the commodity space.

$$\mathcal{L} = \{\{s_t\}^{t=0,\infty} = \{s_{it}\}_{i=1,2,3}^{t=0,\infty},\;\; s_{it} \in R : \sup_t |s_t| < \infty\;\}$$

so our commodity space will be the set of bounded sequences in sup norm. Interested reader can refer to Stokey and Lucas (1989) for the reasons behind the choice of this particular space. Next is the definition of consumption possibility set X with full depriciation $\delta = 1$.

$$X(k_0) = \{x \in \mathcal{L} = l_\infty^3 : \exists \{c_t, k_{t+1}\}_{t=0}^\infty \geq 0 \text{ such that}$$

$$\begin{aligned} k_{t+1} + c_t &= x_{1t} + (1-\delta)k_t \qquad \forall t && (12)\\ x_{2t} &\leq k_t \qquad \forall t && (13)\\ x_{3t} &\in [0,1] \qquad \forall t\} \end{aligned}$$

- Interpretation is that x_{1t}=received goods at period t, x_{3t}=labor supply at period t, x_{2t}=capital service at period t. $k_{t+1} + c_t = x_{1t} + (1-\delta)k_t$ comes from real accounting. Note: capital and capital service are not the same thing. Think of the difference between a house and to rent a house.

- Production possibility set Y.

  Firm's problem is relatively simple as firm do not have intertemporal decision. Firms just rent production factors and produce period by period.

  $Y = \Pi_{t=0}^{\infty}\widehat{Y}$:

$$Y = \Pi_{t=0}^{\infty}\widehat{Y}_t : \widehat{Y}_t = \{y_{1t} \geq 0, y_{2t}, y_{3t} \leq 0 : y_{1t} \leq f(-y_{2t}, -y_{3t})\} \tag{14}$$

  Interpretation is that y_1=production at period t, y_{3t}=labor input at period t, y_{2t}=capital input at period t.

- Then defining preferences over this space $U : X \to R$.

$$U(x) = \sum_{t=0}^{\infty} \beta^t u(c_t(x)) \tag{15}$$

  c_t is unique given x because each x implies a sequence $\{c_t, k_{t+1}\}_{t=0}^{\infty}$. If $x_{3t} = k_t$, $c_t = x_{1t} + (1-\delta)x_{2t} - x_{2t+1}$.

- And the budget constraint of the agent

$$v(x) \leq 0$$

  where the v is the valuation function, i.e. gives the value of a particular commodity bundle. Note that this formulation is a little different than the usual dot product representation of prices that we are used to. The reason behind this is the fact that it is not always possible to represent the prices as a vector due to the nature of infinete dimensional spaces as we will see below.

Prices We want the price of a good at time t in terms of good at time 0. But that's not what v^* tells us. v^* is the cost or value of a commodity point. It is an arbitrary continuous linear function that is defined on our commodity space; it may not always be possible to find a sequence of prices to represent it with. Now we will go over the conditions under which this can be done. This is the Prescott-Lucas Theorem. This theorem tells us under which conditions the valuation function, or the pricing scheme, can be represented as an inner product for the infinite time horizon case.

Theorem 3 *(based on Prescott and Lucas 1972) If, in addition to the conditions to SBWT, $\beta < 1$ and u is bounded, then $\exists \hat{p}$ such that $(x^*, y^*, \hat{p})$ is a Quasi-Equilibrium and*

$$\hat{p}(x) = \sum_{t=0}^{\infty} \sum_{i=1}^{3} \hat{p}_{it} x_{it} \tag{16}$$

i.e. price system has an inner product representation.

Theorem 4 *(based on Prescott and Lucas 1972) If, in addition to the conditions to SBWT, $\beta < 1$ and u is bounded, then $\exists \hat{p}$ such that $(x^*, y^*, \hat{p})$ is a QE and*

$$\hat{p}(x) = \sum_{t=0}^{\infty} \sum_{i=1}^{3} \hat{p}_{it} x_{it} \tag{17}$$

i.e. price system has an inner product representations.

The result above is a special case of the more general theorem proved by Prescott and Lucas (1972). Before stating the theorem, let's define some notations. Let L^n be the subspace of L such that, for $x \in L^n$, $x = ((x_{11,}x_{21,x31}), (x_{12}, x_{22}, x_{32}), (x_{13}, x_{23}, x_{33}), ..., (x_{1n-1}, x_{2n-1}, x_{3n-1}), (0,0,0), (0,0,0), ...)$, i.e. $x_{it} = 0$ for $t \geq n$. Also Let x^n denote the projection of $x \in L$ on L^n.

Now we are ready to state the theorem in a more general form.

Theorem 5 *(Prescott and Lucas 1972) If (i) X is convex, (ii) preference is convex (these two conditions are same as those in the SBWT), (iii) for every n, $x^n \in X$ and $y^n \in Y$, (iv) if $x, x' \in X$ and $U(x) > U(x')$, then there exists and integer N such that, for $\forall n \geq N$, $U(x^n) > U(x')$, then, for a QE (x^*, y^*, p^*) with non-satiation point x^*, there exists $\hat{p}$ such that (1) $\hat{p}(x) = \lim_{n\to\infty} p(x^n)$ for a $p \in Dual(L)$, and (2) $(x^*, y^*, \hat{p})$ is a QE.*

Remark 6 *The results of the theorem allows us to consider the price system of a QE as the limit of a price system of the finite commodity space and thus represent price system of a QE by inner product representations. Intuitively, the additional two conditions of the theorem ((iii) and (iv)) tell that (iii) truncated consumption or production allocation is also feasible, and (iv) truncation of the sufficiently "future" consumption does not change the preference relationship.*

With the Prescott-Lucas Theorem, from now on we can use the inner product representation of prices (assuming the conditions for this theorem hold). So the SBWT told us that for the PO allocation, we can get prices that will support it as an ADE. The Prescott-Lucas Theorem told us that these prices can be written as an inner product. Next we define the Arrow-Debreu equilibrium in its general form (i.e. with a valuation function).

Definition 7 (Arrow-Debreu/Valuation Equilibrium) *Valuation equilibrium is a feasible allocation (x^*, y^*) and a valuation function v^* such that,*

1. *x^* solves the consumer's problem.*

2. *y^* solves the firm's problem.*

3. *$x^* = y^*$ markets clear.*

With this definition in hand, next go back to our original aim of constructing this environment that is using welfare theorems to show the existence and uniquenes of competitive equilibrium.

1.4 Welfare Theorems (Linking SSP and AD)

Theorem 8 (First Basic Welfare Theorem) *Suppose that for all $x \in X$ there exists a sequence $\{x_n\}_{n=0}^{\infty}$ inX converging to x with $u(x_n) \geq u(x)$ for all n (local nonsatiation). If an allocation (x^*, y^*) and a continous linear functional ν constitute a competitive equilibrium, then the allocation (x^*, y^*) is Pareto optimal.*

Theorem 9 (Second Basic Welfare Theorem) *If (i) X is convex, (ii) preference is convex (for $\forall x, x' \in X$, if $x' < x$, then $x' < (1-\theta)x' + \theta x$ for any $\theta \in (0,1)$), (iii) $U(x)$ is continuous, (iv) Y is convex, (v) Y has an interior point, then with any PO allocation (x^*, y^*) such that x^* is not a saturation point, there exists a continuous linear functional ν^* such that (x^*, y^*, ν^*) is a Quasi-Equilibrium with transfers((a) for $x \in X$ which $U(x) \geq U(x^*)$ implies $\nu^*(x) \geq \nu^*(\nu^*)$ and (b) $y \in Y$ implies $\nu^*(y) \leq \nu^*(y^*)$)*

Note that an additional assumption we are making for SBWT to go through in infinitely dimensional spaces is that Y has an interior point i.e.

$$\exists \bar{y} \in Y,\ B \subset Y,\ B \textit{ open and } \bar{y} \in B$$

Also that the SBWT states that under certain conditions listed above, we can find prices to support any Pareto optimal allocation as a quasi equilibrium with transfers. Transfers are not relevant in our case since we are working in an representative agent environment with identical households. Taking care of the transfers still leaves us with Quasi-Equilibrium so SBWT by itself it does not say anything about the existence of Arrow-Debreu equlibrim. The following lemma takes care of this.

Lemma 10 *If, for (x^*, y^*, ν^*) in the theorem above, the budget set has cheaper point than x^* ($\exists x \in X$ such that $\nu(x) < \nu(x^*)$), then (x^*, y^*, ν^*) is a ADE.*

With the SBWT, we established that there exists a ν that will support our PO allocation as a competitive equilibrium. What's the problem with this approach? It is that SBWT only tells us that such a ν exists, it doesn't tell us what it is. Assuming the prices has inner product representations, we will next show how to construct them using the necessary FOCs of the households and firms. Before doing that lets have a brief rewiev of where are we coming from.

Remember, our main purpose is to be able to apply the welfare theorems to the most commonly used models in macroeconomics where we have an infinite-dimensional commodity space. Until now, we set up an environment (Arrow-Debreu economy) (which consisted of the commodity space, consumption possibility set, production possibility set, and preferences) with infinite-dimensional commodity space and we stated that under certain conditions the Welfare Theorems hold in this environment. Now we will map the growth model into the environment that we talked about until here, and show that in the context of the growth model the assumptions we need for the Welfare Theorems are satisfied. Then we can conclude that any competitive equilibrium allocation is Pareto optimal and moreover we can support a PO allocation with some prices as a competitive equilibrium. This result is very important in macroeconomics. It helps us in solving for the equilibria. With the FBWT and SBWT, we can just solve for the PO allocations and then get the prices. This makes life much easier.

What's next? Constructing the prices.

1.5 The Growth Model

Houshold Problem:

$$\max_{\{c_t, l_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t(x)) \tag{18}$$

subject to

$$\sum_t (p_{1t}x_{1t} + p_{2t}x_{2t} + p_{3t}x_{3t}) = 0 \tag{19}$$

Look at the First Order Conditions with respect to x_{1t} and x_{1t+1} and using the relationships between k, c and x we can get (with full depricaton),,

$$\frac{p_{1t}^*}{p_{1t+1}^*} = \frac{u_c(x_{1t}^* - k_{t+1}^*)}{\beta u_c(x_{1t+1}^* - k_{t+2}^*)} = F_k(k_t^*, n_t^*) \tag{20}$$

Also using the firms problem we can get,

$$\frac{p_{2t}^*}{p_{1t}^*} = F_k(k_t^*, n_t^*) \tag{21}$$

$$\frac{p_{3t}^*}{p_{1t}^*} = F_n(k_t^*, n_t^*) \tag{22}$$

where the variables with asterix are the optimal allocation from the solution to the SPP.

Note that in a AD world all the trades are made at period 0, and after the history starts, all that agents can do is to follow what was promised (full commitment is assumed). But this is an unrealistic market arrangement. To see the point this more clearly, imagine the decision of an agent who is going to be born in period t. At period 0, although the agents is not born yet, the agent also joins the market at period 0. At period 0, she trades (by solving the consumer's problem above), and she goes to limbo from period 0 (after trade) until period t-1, and she is born in period t. As we want the market arrangement of the model to be comparable to the one in the real world, this unrealistic assumption on market arrangement is not desirable. That is the motivation to consider Sequential Market Equilibrium (SME), where markets are open every period next.

1.6 Sequence of Marktes Equlibrium

Note that two things are important in the setup of this new environment: (i) there are infinitely many markets in SME (because markets are open every period), which means that there are infinitely many budget constraints to be considered, (ii) an allocation in SME has to give as much utility as in ADE to agents in order to be PO. Otherwise, agents will choose to trade in AD markets, meaning SME doesn't work. Remember that we cannot force agents to do certain things.

Also note that there are many ways of arranging markets so that the equilibrium allocation is equivalent to that in ADE. We'll see two of them. Note that if the number of markets open is TOO FEW, we cannot achieve the allocation in the ADE (incomplete market). To the contrary, if the number of markets are TOO MANY, we can close some of the markets and still achieve the ADE allocation in this market arrangement. Also it means that there are many ways to achieve ADE allocation because some of the market instruments are redundant and can be substituted by others. If the number of markets are not TOO FEW nor TOO MANY, we call it JUST RIGHT. The fact that these two arrangements have to be equal at the end of the day requires us to introduce a new tradable commodity that will utilize households to move resources in time in both directions, loans.

With sequential markets, people have capital k_t, and rent it to the firm at rental R_t. People have time 1 and rent it to firm at wage w_t. They also consume c_t and save k_{t+1}. Agents can also borrow and lending one period loan l_{t+1} at price q_t. Then the budget constraint at time t is

$$k_tR_t + w_t + l_t = c_t + k_{t+1} + q_tl_{t+1} \tag{23}$$

Next thing we notice is, we can close the market of loans without changing the resulting allocation. This is because we need someone to lend you loans in order that you borrow loans, but there is only one agents in the economy. But surprisingly, we will see that even though there is no trade in certain markets in equilibrium, we can solve for prices in those markets, because prices are determined even though there is no trade in equilibrium, and agents do not care if actually trade occurs or not because they just look at prices in the market (having market means agents do not care about the rest of the world but the prices in the market). Using this technique, we can determine prices of all market instruments even though they are redundant in equilibrium. This is the virtue of Lucas Tree Model and this is the fundamental for all finance literature (actually, we can price any kinds of financial instruments in this way. we will see this soon.). Writing down the problem of the household in SM world,

Consumer's Problem in SME can be written as follows:

$$\max_{\{c_t,k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{24}$$

subject to

$$k_tR_t + w_t + l_t = c_t + k_{t+1} + q_tl_{t+1} \quad \forall t \tag{25}$$

$$k_0 \text{ is given} \tag{26}$$

and the firm solves the following static problem,

$$\max_{k_t,n_t} f(k_t,1) - R_tk_t - w_tn_t$$

and with the well defined environments and behavioral assumptions we are ready to define equlibrium in this world.

Definition A sequence of markets equilibrium is prices and allocation $\{w_t^*, r_t^*, q_t^*, c_t^*, k_{t+1}^*, 0, n_t^*\}$ such that

- Households maximize utility,

$$\begin{aligned} \{c_t^*, k_{t+1}^*, l_t^* &= 0\}_{t=0,\infty} \in \arg\max_{\{c_t,k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \\ s.t.\ k_tR_t + w_t + l_t &= c_t + k_{t+1} + q_tl_{t+1} \end{aligned}$$

- Producers maximaze profits,

$$\{k_t, n_t\} \in \arg\max_{k_t, n_t} f(k_t, 1) - R_t k_t - w_t n_t$$

- Feasibility and Markets clear,

$$\begin{aligned} c_t + k_{t+1} &= f(k_t, 1) \\ l_{t+1} &= 0 \end{aligned}$$

So far we have worked with two different form of market arrangements, AD and SME, and defined equilibrium in both of these worlds. We have also seen that the problem defined in the sequential market setup is at least as hard, if not more, as the AD setup to solve. The way we went around this in the AD setup was the use of welfare theorems by making sure a unique PO solution to SPP existed, which coincided with the solution to AD problem, furthermore we were able to support this allocation with appropriate prices as the equilibrium outcome of the AD problem. If we can establish the equivalence between these two equilibrium concepts than we can figure out the euqilibrium outcome in the SME world without having to solve the sequential problem. This brings us to the following propositions.

Proposition 11 *Let the allocation* $\{x^*_{it}, y^*_{it}\}^{\infty}_{t=0,i=1,2,3}$ *and a valuation function (assuming it has an inner product representation)* $\widehat{p}(x) = \sum\limits_{t=0}^{\infty} \sum\limits_{i=1}^{3} p^*_{it} x_{it}$ *form an AD equilibrium. Then there exists an allocation* $\{c^*_t,$ $k^*_{t+1}, n^*_t, l^*_{t+1}\}^{\infty}_{t=0}$ *, a sequence of prices* $\{R^*_t,\ w^*_t\}^{\infty}_{t=0}$ *that can be constructed such that it is an SME.*

The way to prove this proposition is first to establish an implication from AD budget set and technology constraints to SME budget set and technology constraints. Once we do so, we can show that given our constructed prices, our constructed allocations satisfy the sufficient FOCs of the SM problem and thus form an SME and this is done throuhgly in the homework solutions. The opposite implication is stated in the following proposition.

Proposition 12 *Let the allocation* $\{c^*_t,\ k^*_{t+1}, n^*_t, l^*_{t+1}\}^{\infty}_{t=0}$ *and a sequence of prices* $\{R^*_t,\ w^*_t\}^{\infty}_{t=0}$ *form an SME equilibrium. Then there exists an allocation* $\{x^*_{it}, y^*_{it}\}^{\infty}_{t=0,i=1,2,3}$*, and a valuation function (assuming it has an inner product representation)* $\widehat{p}(x) = \sum\limits_{t=0}^{\infty} \sum\limits_{i=1}^{3} p^*_{it} x_{it}$ *to be constructed such that it is an ADE.*

A similar line of logic can be followed to prove this proposition as well with one additional condition. We know by the definition of the AD equilibrium, the valuation function must be linear and continuous. Linearity is trivial to show in this case with our implied valuation function. The other important property continiuty seems to be harder to show but we know that in the infinite dimensional spaces, boundedness is a sufficient condition for continuity and we can exploit this fact. Once we do so, we have our AD equilibrium constructed from SME objects.

Why we did not go from SPP to SME directly? Because Welfare Theorems are available only between SPP and ADE, though what we want is to derive equivalence between SPP allocation and SME (or RCE) allocation. For some particular environments, as the equivalence result between SPP allocation and RCE allocation is available, we can exploit the result and can argue directly that some RCE allocation is indeed PO. Next we introduce uncertainty into our deterministic world. The most conventional way of doing it is introducing productivity shocks to technology.

2 Stochastic Growth Model

2.1 Markov Process

In this course, we will concentrate on Markov productivity shock. Considering shock is really a pain, so we want to use less painful one. Markov shock is a stochastic process with the following properties:

1. there are FINITE number of possible states for each time. More intuitively, no matter what happened before, tomorrow will be represented by one of a finite set.
2. what only matters for the realization tomorrow is today's state. More intuitively, no matter what kind of history we have, the only thing you need to predict realization of shock tomorrow is TODAY's realization.

More formally, for each period, suppose either z^1 or z^2 happens [3]. Denote z_t is the state of today and Z_t is a set of possible state today, i.e. $z_t \in Z_t = \{z^1, z^2\}$ for all t. Since the shock follow Markov process, the state of tomorrow will only depend on today's state. So let's write the probability that z^j will happen tomorrow, conditional on today's state being z^i as $\Gamma_{ij} = prob[z_{t+1} = z^j | z_t = z^i]$. Since Γ_{ij} is a probability, we know that

$$\sum_j \Gamma_{ij} = 1 \qquad for\ \forall i \tag{27}$$

Notice that 2-state Markov process is summarized by 6 numbers: z^1, z^2, Γ_{11}, Γ_{12}, Γ_{21}, Γ_{22}.

The great beauty of using Markov process is we can use the explicit expression of probability of future events, instead of using ambigious operator called expectation, which very often people don't know what it means when they use.

2.1.1 Representation of History

- Let's concentrate on 2-state Markov process. In each period, state of the economy is $z_t \in Z_t = \{z^1, z^2\}$.
- Denote the history of events up to t (which of $\{z^1, z^2\}$ happened from period 0 to t, respectively) by $h_t = \{z_1, z_2, ..., z_t\} \in H_t = Z_0 \times Z_1 \times ... \times Z_t$.
- In particular, $H_0 = \emptyset$, $H_1 = \{z^1, z^2\}$, $H_2 = \{(z^1, z^1), (z^1, z^2), (z^2, z^1), (z^2, z^2)\}$.
- Note that even if the state today is the same, past history might be different. By recording history of event, we can distinguish the two histories with the same realization today but different realizations in the past (think that the current situation might be "you do not have a girl friend", but we will distinguish the history where "you had a girl friend 10 years ago" and the one where you didn't (tell me if it is not an appropriate example...).)
- Let $\Pi(h_t)$ be the unconditional probability that the particular history h_t does occur. By using the Markov transition probability defined in the previous subsection, it's easy to show that (i) $\Pi(h_0) = 1$, (ii) for $h_t = (z^1, z^1)$, $\Pi(h_t) = \Gamma_{11}$ (iii) for $h_t = (z^1, z^2, z^1, z^2)$, $\Pi(h_t) = \Gamma_{12}\Gamma_{21}\Gamma_{12}$.

2.2 SPP,ADE, and SME in a Stochastic RA-NGM

2.2.1 Big Picture

- Now we have Nature, who decides the realization of productivity shock every period.
- Social Planner's Problem (the benevolent God's choice) in this world is a state-contingent plan, i,e, optimal consumption and saving (let's forget about labor-leisure choice in this section for simplicity [4]) choice for all possible nodes (imagine the nodes of a game tree. we need to solve optimal consumption and saving for each node in the tree).
- Notice that the number of nodes for which we have to solve for optimal consumption and saving is countable. This feature allows us to use the same argument as the deterministic case to deal with the problem. The only difference is that for deterministic case, the number of nodes is equal to number of periods (which is infinite but countable), but here the number of nodes is equal to the number of date-events (which is also infinite but countable).

[3] Here we restrict our attention to the 2-state Markov process, but increasing the number of states to any finite number does not change anything fundamentally.

[4] Or just assuming the consumers do not value leisure (drop leisure from utility function) is enough to let agents work as much as possible in this world.

- More mathematically, the solution of the problem is the mapping from the set of date-events (which is specified by history) to the set of feasible consumption and saving.

2.3 The SPP and ADE

$$\max_{\{k_{t+1}(h_t),c_t(h_t)\}} \sum_{t=0}^{\infty} \beta^t \sum_{h_t \in H_t} \Pi(h_t) u(c_t(h_t)) \tag{28}$$

subject to

$$z_t f[k_t(h_{t-1}), 1] = c_t(h_t) + k_{t+1}(h_t) \tag{29}$$

$$k_0 \qquad \text{given} \tag{30}$$

Couple of comments:

- Here capital in indexed by the time it is used. k_t is a mapping from h_{t-1} because the amount of capital used today is determined yesterday. Alternatively, you can index capital by the time when the amount is chosen, but the former notation is the tradition and more common so we use the former notation. Anyway it is just a matter of notation.
- An assumption here is leisure is not valued by consumer so time of consumer is inelastically supplied for working.
- Measurability (very loosely) means whether an object is known when agents make their choice. Choice of agents must not depend on an object which agents do not know when they make choices.

Let's denote the solution as $x^* = \{c_t^*(h_t), k_{t+1}^*(h_t)\}$. It's easy to show that (i) the utility function is strictly concave, (ii) the constraint set is convex, (iii) commodity set is same as deterministic case. Using these properties, we can show (i) existence of the solution, (ii) uniqueness of the solution, (iii) FBWT (ADE is PO), (iv) SBWT (PO allocation can be supported as an ADE), (v) price system has a nice inner product representation (Lucas and Prescott (1972))[5], (vi) some equations (derived from FOC of SPP) which characterize the ADE allocation (remember (21), (22), and (**??**)).

A history is given by $h_t = \{z_1, z_2, ..., z_t\} \in H_t = Z^t$, and let the probability of a history be denoted by $\pi(h_t)$.

Preferences are given by

$$u(x) = \sum_t \beta^t \sum_{h_t \in H_t} \pi(h_t) u[c_t(h_t)]$$

.

First connsider an Arrow -Debreu world. The constraint will look like

$$\sum_t \sum_{h_t \in H_t} \sum_j p_t^j(h_t)\ x_t^j(h_t) \le 0, where j = 1,2,3.$$

In a SM setting we need to give to the agent enough tools, so that he can consume different quantities in different states of the world. In other words, we have to make sure that whatever he was able to do in an AD setting, he will also be able to do it the SM setting. To that end, we will introduce the notion of a state contigent claim. For example, $b_t(h_{t-1}, z^i)$ is a claim that the agent bought in period $t-1$, and will pay 1 unit of consumption for sure if state i occurs. In the SM world, the budget constraint will be

[5]Remember the deterministic version of Lucas-Prescott Theorem. For the stochastic model, we need two additional assumptions, corresponding (iii) and (iv) of the deterministic one. Very loosely, we need additionally, that (iii) and (iv) of the deterministic one hold for truncation with respect to certain history when probability of occurrence of the history with truncation is sufficiently small. For more details, see Lucas-Prescott paper or Harris (p62-64).

$$c_t\left(h_t\right)+k_{t+1}\left(h_t\right)+\sum_{z_{t+1}} q_t(h_t,z_{t+1})b_{t+1}\left(h_t,z_{t+1}\right)=k_t\left(h_{t-1}\right)\ R_t\left(h_t\right)+w_t\left(h_t\right)+b_t\left(h_{t-1},z_t\right),$$

where $q_t(h_t,z_{t+1})$ is the price of the state contigent claim that pays 1 in period $t+1$ if state z_{t+1} occurs.

In this model, the agent has two different ways to transfer resources to the future. As we saw in Problem Set 2, there has to be an arbitrage condition connecting the prices of these two assets (capital and state contigent claims). Recall that this condition is

$$\sum_{z_{t+1}} q_t(h_t,z_{t+1})R_{t+1}\left(h_t,z_{t+1}\right)=1$$

The problem of the agent is to

$$\max_{c_t(h_t),k_{t+1}(h_t)} \sum_t \beta^t \sum_{h_t} \pi(h_t)u\left(c_t(h_t)\right)$$

$$s.t:\ c_t(h_t)+k_{t+1}(h_t)+\sum_{z_{t+1}} q_t(h_t,z_{t+1})\ b_{t+1}(h_t,z_{t+1})=w_t(h_t)+k_t(h_{t-1})R_t(h_t)+b_t(h_{t-1},z_t)$$

The Langrangian is given by

$$\mathcal{L}=\sum_t \beta^t \sum_{h_t} \pi(h_t)u\left(c_t(h_t)\right)$$

$$+\sum_t\sum_{h_t} \lambda_t(h_t)\left[c_t(h_t)+k_{t+1}(h_t)+\sum_{z_{t+1}} q_t(h_t,z_{t+1})b_{t+1}(h_t,z_{t+1})-w_t(h_t)-k_t(h_{t-1})R_t(h_t)-b_t(h_{t-1},z_t)\right]$$

You should note that here there is one Langrangian multiplier for every period and for every possible history.

Take the first order conditions

$$\{c_t(h_t)\}:\ \beta^t\pi(h_t)u_c(c_t(h_t))=\lambda_t(h_t) \quad (1)$$
$$\{b_{t+1}(h_t,z^i)\}:\ \lambda_t(h_t)\ q_t(h_t,z^i)=\ \lambda_{t+1}(h_t,z^i) \quad (2)$$

Then, replacing the Langrangian multiplier from (1) into (2) gives

$$\beta^t\pi(h_t)u_c(c_t(h_t))\ q_t(h_t,z^i)=\ \beta^{t+1}\pi(h_t,z^i)u_c(c_{t+1}(h_{t+1}))$$

We will assume that the random shock follows a Markov process. In this case we know that $\pi(h_t,z^i)=\pi(h_t)\Gamma_{ji}$, where Γ_{ji} is the probability of state i occuring in period $t+1$ given that the current state is j. Using this observation in the above expression we get

$$\pi(h_t)u_c(c_t(h_t))\ q_t(h_t,z^i)=\ \beta\pi(h_t)\Gamma_{ji}u_c(c_{t+1}(h_{t+1})) \text{ and so}$$
$$q_t(h_t,z^i)=\beta\Gamma_{ji}\tfrac{u_c(c_{t+1}(h_t,z^i))}{u_c(c_t(h_t))}$$

so we have a formula for the price of the state i contigent claim.

We will now define a Sequential Markets Equilibrium in the stochastic world.

DEFINITION A SME for the stochastic growth model is an allocation $\{c_t^*(h_t),k_{t+1}^*(h_t),b_t^*(h_t)=0\}$ together with prices $\{q_t^*(h_t,z_{t+1}),R_t^*(h_t),w_t^*(h_t)\}$ such that

1) Given prices, $\{c_t^*(h_t),k_{t+1}^*(h_t),b_t^*(h_t)\}$ solve the agent's problem.
2) $w_t^*(h_t)=z_t\ f_n\left(k_t^*(h_{t-1}),1\right)$ and $R_t^*(h_t)=z_t\ f_k\left(k_t^*(h_{t-1}),1\right)$
3) Markets clear.

3 Recursive Competitive Equilibrium

As you have seen in 704, the beauty of the recursive representation lies in the fact that, in a stationary environment, the nature of the problem do not change with passage of time. Unlike the sequential formulation, in which the solution to the problem depends on at what point in time you solve it, the solution to the recursive problem do not depend on time and we do not have to keep track of time.

So what do we keep track of? Everything that matters to the structure of our problem. These are the variables that our agents respond to either directly or indirectly and we call them the STATE VARIABLES. State variables need to satisfy the following criteria:

1. PREDETERMINED: when decisions are made, the state variables are taken as given and cannot be effected by the agent.

2. It must MATTER for decisions of agents: there is no sense of adding irrelevant variables as state variable.

3. It must VARY across time and state: otherwise, we can just take it as a parameter.

One important thing is to be able to distinguish the aggregate and individual state variables. Aggregate state is not affected by individual choice. But aggregate state should be consistent with the individual choice (we will consider the meaning of "consistency" more formally later), because aggregate state represents the aggregated state of individuals. In particular, in our RA-NGM aggregate state turns out to be the same as individual state in equilibrium, but this does not mean that the agent decide the aggregate state or the agent is forced to follow the average behavior, but rather the behavior of the agent turns out to be the aggregate behavior, in equilibrium.

Also note that prices (wages, and rental rates of capital) is determined by aggregate capital, rather than individual capital, and since individual takes aggregate state as given, she also takes prices as given (because they are determined by aggregate state). Again, the aggregate capital turns out to coincide with the individual choice, but it is not because of the agent's choice, rather it is the result of consistency.

One notational note. Victor is going to use a for individual capital and K for aggregate capital, in order to avoid the confusion between K and k. But the problem with aggregate and individual capital is often called as "big-K, small-k" problem, because the difference of aggregate capital and individual capital is crucial. So for our case, the counterpart is "big-K, small-a" problem.

What does the agent has to know in order to solve her problem? Our agent has to know how wealthy she is. She also needs to know the prices but we do not need $\{R, w\}$ directly. Why? Because they are redundant: K is a sufficient statistic to calculate $\{R, w\}$ as they must be equal to marginal products in the firm problem at equilibrium.If we put K as a state variable instead of these prices, we do not need $\{R, w\}$. So are we done? Not yet. As the problem of the HH is formulated below, our agent not only needs to know $\{R, w\}$ but $\{R', w'\}$ as well thus K'. But this is a variable that our agent has no control over and the best she can do is to have a 'belief' about it. These beliefs in our model are parametrized by the G^e function which maps todays state to a unique belief about next period's value. As it is formulated, it is an exogenous parameter, i.e. we can solve this problem for any sort of beliefs under which the problem is well defined. But from the beginning of this course we want to be able to 'predict' the outcome, once we setup our environment as precise as possible. To continue to be able to do so, as we will see later, we will impose an additional constraint on the beliefs and make them an equilibrium object as well, i.e. endogenize them.

Now let's write the representative consumer's problem in the recursive way.

$$V(K, a; G^e) = \max_{c,a'}[u(c) + \beta V(K', a'; G^e)]$$

$$\begin{aligned} s.t: \quad c + a' &= w + Ra \\ w(K) &= F_L(K, 1) \\ R(K) &= 1 + F_K(K, 1) - \delta \\ K' &= G^e(K) \end{aligned}$$

NOTE: For the above problem to be a well-defined one, all the variables in the maximand (in the problem above: $[u(c) + \beta V(K', a'; G^e)]$) have to be either (i) a state variable (so an argument of V(.)), (ii) a choice variable (so appear below max operator), (iii) or defined by a constraint, in order for the problem to be well defined.

Moreover, prices $\{R, w\}$ are functions of aggregate variables, so agents have to take them as given. Note that this is because individual is measure zero, by assumption (so, although we are dealing with representative agent, at the same time we assume that agents are measure zero and have no power to affect aggregate state of the world, hence prices).

Finally, notice that $\{K, w(K), R(K), G^e(K)\}$ are enough to generate all future prices if today's aggregate capital is K.

Definition 13 *A Recursive Competitive Equilibrium with arbitrary expectations $G^e(.)$ is $\{V^*(.), g^*(.), G^*(.), R(.), w(.)\}$ such that*

1) Given $\{G^e(.), R(.), w(.)\}$, $\{V^*(.), g^*(.)\}$ solves the household problem above.

Definition 14 *1. 2) $\{R(.), w(.)\}$ are characterized by the optimal decisions of firms.*

3) $G^*(K; G^e) = g^*(K, K; G^e)$

Some comments on the third condition. This condition is called 'Representative agent condition' and is a specific case of the compatibility condition that any equilibrium must satisfy. It basically means that if a consumer turns out to be average this period (her individual capital stock is K, which is aggregate capital stock), the consumer will choose to be average in the next period (she chooses $G^*(K)$, which is a belief on the aggregate capital stock in the next period if today's aggregate capital stock is K). This condition guarantees that in an equilibrium, individual choice turns out to be consistent with the aggregate law of motion. This is true not because our agent is constrained to do so but because the prices are such that she choses to do so.

Definition 15 : *A Recursive Competitive Equilibrium with rational expectations is a list*

$\{V^*(.), g^*(.), G^*(.), R(.), w(.)\}$ such that:

1) Given $\{G^e(.), R(.), w(.)\}$, $\{V^*(.), g^*(.)\}$ solves the household problem above.
2) $\{R(.), w(.)\}$ are characterized by the optimal decisions of firms.
3) $G^*(K; G) = g^*(K, K; G)$

Consider the following problem, a neoclassical growth model with leisure (from now on we replace G^e with G) .

$$
\begin{aligned}
V(K, a; G) = \max_{c,n,a'} & [u(c, n) + \beta V(K', a'; G)] \\
s.t: \quad & c + a' = nw + Ra \\
& w(K) = F_L(K, 1) \\
R(K) = & 1 + F_K(K, 1) - \delta \\
& K' = G(K)
\end{aligned}
$$

Is this problem consistent (in the sense we analyzed in the previous lecture, i.e, every variable is either a state, or a control, or a function of the states)? The answer is yes. However, this problem is wrong. Why? Because it doesn't take under consideration that prices will now depend on aggregate labor as well as capital. Now K along with G are not enough for the agent to predict prices. The correct formulation is as follows:

$$
\begin{aligned}
V(K, a; G, H) = \max_{c,n,a'} & [u(c, n) + \beta V(K', a'; G, H)] \\
s.t: \quad & c + a' = nw + Ra \\
& w(K) = F_L(K, N) \\
R(K) = & 1 + F_K(K, N) - \delta \\
& K' = G(K) \\
& N = H(K)
\end{aligned}
$$

with solutions be $a' = g(K, a; G, H)$ $n = h(K, a; G, H)$

Definition 16 *Here, a RCE is alist of functions* $\{V^*(.), g^*(.), G^*(.), H^*(.), R(.), w(.)\}$ *such that*

1) Given $\{G^*(.), H^*(.), R(.), w(.)\}$, $\{V^*(.), g^*(.)\}$ solves the household problem above.
2) $\{R(.), w(.)\}$ are characterized by the optimal decisions of firms.
3) Agent is representative: $g^*(K, K; G^*, H^*) = G^*(K)$ and $h^*(K, K; G^*, H^*) = H^*(K)$.

Comment: Why do we need the RCE? Isn't it more easy to work with the Social Planner's Problem? The answer is yes. However, there are lots of cases where the solution to SPP does not coincide with that of the Competitive Economy. The most characteristic example is the presence of an externality (see next Lecture) or a distortionary tax. Moreover, we will also see models where we don't have a representative agent (heterogeneous agents models), and so we don't really know what the SP Problem looks like. In all the above cases the SPP is not helpful anymore, and we have to "attach" the Competitive Equilibrium directly.

The most common way to find and characterize A RCE (given its complexity) is to use computational methods. But we will not cover any of these techniques in 702.

3.1 *Externalities:*

Consider the following problem in which the utility function of the agent is given by $u(c, n, N)$, and satisfies $u_2 < 0$ and $u_3 < 0$.

$$\begin{gathered} V(k, a; G, H) = \max_{c,a',n} [u(c, n, N) + \beta V(k', a'; G, H)] \\ s.t: \ c + a' = nw(k, N) + aR(k, N) \\ k' = G(k) \\ N = H(k) \end{gathered}$$

or equivalently

$$\begin{gathered} V(k, a; G, H) = \max_{a',n} [u(nw(k, N) + aR(k, N) - a', n, N) + \beta V(k', a'; G, H)] \\ s.t: k' = G(k) \\ N = H(k) \\ w(k, N) = F_2(k, N) \\ R(k, N) = F_1(k, N) \end{gathered}$$

It was part of your Homework to show that the allocation implied by the solution to the SPP will not coincide with the one implied by the Competitive solution. In particular we have seen that $N^{SP} < N^{CE}$ (see problem set 3). The reason is that the agent thinks she cannot affect the aggregate labor and thus works more than the optimal level.

3.2 Introducing Goverment

In this lecture we will introduce goverment in the economy. The simplest problem with goverment is the one in which it imposes a lump-sum tax whose revenues are thrown in the ocean. The recursive formulation of this simple problem is as follows:

$$\begin{gathered} V(K, a; G) = \max_{c,a'} [u(c) + \beta V(K', a'; G)] \\ s.t: \quad c + a' = w(K) + aR(K) - T \\ \\ K' = G^e(K) \end{gathered}$$

The Goverment Budget Constraint is

$$T + c + k' = w(K) + KR(K) = 1F_n(K, 1) + KF_K(K, 1) = F(K, 1)$$

This is just the Euler Theorem for homogeneous functions. It's economic interpretation is that payments to factors exhaust the output.

It should be clear that in this setting the tax is not distortionary. It is not an income tax, and so agents' decisions will not be affected. Hence, the easiest way to characterize the RCE here, is to solve the (much easier) Social Planner's problem.

The next problem to consider is that with an income tax. In this case if , for example, the agent works more today, he will get a higher wage income. But this income will be taxed at the constant rate τ. This will clearly affect the agent's decision and it will not be true that the allocations implied by SPP and RCE coincide. The problem is as below:

$$V(K,a;G) = \max_{c,a'}[u(c) + \beta V(K',a';G)]$$
$$s.t: \quad c + a' = w(K)(1-\tau) + a(R(K)-1)(1-\tau) + a$$
$$K' = G^e(K)$$

If we suppose that the goverment also has some public spending (which does not appear in the utility function however) it should be true that

$$\tau(F(K,1) - K) = G$$

You should be careful with the budget constraint above: the tax is only income tax. So only the part $(R(K)-1)\,a$ of the individual's capital will be taxed. You should avoid writting $aR(K)(1-\tau)$ in the budget constraint. It's a serious mistake.

As we said above, we expect the RCE not to coincide with the socially optiaml (as this is given by the solution to the SPP). So let's verify that.

The First Order Condition for this problem is:

$$\{a'\}: \quad -u'(c) + \beta\frac{\partial V(K',a';G)}{\partial a'} = 0 \qquad (1)$$

Suppose that the solution has the form $a' = g(K,a;G,H)$. Then the envelope condition is just

$$\frac{\partial V(K,a;G,H)}{\partial a} = u'(c)\left[1 + (R(K)-1)(1-\tau)\right]$$

and so (1) becomes

$$u'(c) = \beta u'(c')\left[1 + (R(G(K)) - 1)(1-\tau)\right]$$

and this is clearly different than

$$u'(c) = \beta u'(c')\,R(G(K))$$

the condition that we have from the solution to the SPP.

The *next example* has a goverment that gives back the tax revenues in a lump-sum manner. Here the formulation of the problem will be slightly different:

$$V(K,a;G) = \max_{c,a'}[u(c) + \beta V(K',a';G)]$$

$$s.t: \quad c + a' = w(K)(1-\tau) + a(R(K)-1)(1-\tau) + a + T$$

$$K' = G^e(K)$$

and, of course, T will be a function of the aggregate state. So we have to add

$$T = T(K) = \tau(F(K,1) - K) = \tau\left[w(K) + K\ (R(K)-1)\right]$$

In problem Set 3 you were asked to characterize the RCE for this economy, but with an agent that values leisure, so that $u = u(c,n)$.

3.3 Goverment Issuing Debt

The *last example* with goverment that we're looking at has the goverment issuing debt. What are the state variables for this economy?

Aggregate states: B, K

Individual states: a

The Goverment Budget Constraint is given by

$$(1+r_b)B + \bar{G} \;=\; B' + \tau\left[F(k,1) - k\right]$$

The problem of the agent is given by

$$\begin{aligned}
V(k,B,a;G,H) &= \max_{c,a'}\left[u(c) + \beta V(k',B',a';G,H)\right] \\
s.t: \quad c + a' &= w(1-\tau) + a(R-1)(1-\tau) + a \\
k' &= G(k,B) \\
B' &= H(k,B) \\
w &= w(k) = F_2(k,1) \\
R &= R(k) = F_1(k,1) \\
\tau &= \tau(k,B)
\end{aligned}$$

or equivalently

$$\begin{aligned}
V(k,B,a;G,H) &= \max_{a'}\left[u\left(w(1-\tau) + a(R-1)(1-\tau) + a - a'\right) + \beta V(k',B',a';G,H)\right] \\
s.t: k' &= G(k,B) \\
B' &= H(k,B) \\
w &= w(k) = F_2(k,1) \\
R &= R(k) = F_1(k,1) \\
\tau &= \tau(k,B)
\end{aligned}$$

We have shown in problem Set 3 that we need to impose a No-Ponzi condition on the sequence of bonds, for the RCE to make sense. Taking that into account, we can define a RCE for the model with public debt as follows:

DEFINITION: A Recursive Competitive Equilibrium is a list of functions $\{V(k,B,a;G,H),\; g(k,B,a;G,H),\; G(k,B),\; H(k,B), \tau(k,B)\}$ such that:

1) Given $G(k,B),\; H(k,B), \tau(k,B)$ the functions $V(k,B,a;G,H),\; g(k,B,a;G,H)$ solve the agent's maximization problem.

2) The Goverment Budget Constraint is balanced

$$(1+r_b)B + \bar{G} \;= H(k,B) + \tau(k,B)\left[F(k,1) - k\right]$$

3) The following arbitrage condition holds

$$g(k,B,k+B) = \; G(k,B) + H(k,B)$$

4) Representative agent condition

$$g(k,B,k+B) = \; G(k,B) + H(k,B)$$

5) There exist $\underline{B}, \bar{B}$ and $\bar{k}$ such that for every $(k,B) \in \left[0,\bar{k}\right] \times \left[\underline{B},\bar{B}\right]$, $G(k,B) \in \left[0,\bar{k}\right]$ and $H(k,B) \in \left[\underline{B},\bar{B}\right]$.

3.4 Economy with Government Debt

Now assume that the government taxes labor income and issues debt to pay for a constant stream of government expenditures $\bar{G}$. This economy is more complicated and tricky than the previous economy without debt (where the amount of the government expenditure is equal to the tax income).

- When the government issues debt, government have the option to delay taxation.
- Government budget constraint will not be satisfied automatically in defining equilibrium.
- Tax policy, that is represented by a function $\tau(.)$, should depend on state of the economy. In particular, since the government always spends a constant expenditure, (i) the government will retire the debt that was issued before when it has a higher revenue, and , (ii) the government will issue more debt when it has a lower revenue.

The tricky part of the problem is to ensure that the government budget constraint is satisfied in the sense of present value. In other words, we want to rule out the insufficient taxation when debt keeps growing. We call such situation as "snowball effect" or "Ponzi scheme".

3.4.1 Defining RCE

State variables

- Aggregate state variable: K, B. K is the aggregate capital in the economy. B is the government debt stock. Government debt here is one period debt in the form of discount bond. Government sell bond today at price q and promise to repay one unit of good tomorrow.
- Individual state variables: a. a is a total asset holding of the agent.

Representative agent only cares about the value of her asset holding, not the composition of her asset portfolio. So, in defining RCE, we only need one state variable for the asset, not both physical capital holding k and financial asset b.In doing so, one equilibrium condition is implied: physical capital holding k and financial asset b bear the same rate of return. This condition holds because they are perfect substitutes, and by No Arbitrage argument.

3.4.2 Household's problem

$$V(K, B, a) = \max_{c,a'} u(c) + V(K', B', a') \tag{31}$$

subject to

$$\begin{aligned} c + a' &= a + [r(K,B)a + w(K,B)](1 - \tau(K,B)) \\ K' &= H(K,B) \\ B' &= G(K,B) \end{aligned} \tag{32}$$

And the solutions are:

$$a' = g(K, B, a) \tag{33}$$

There are different ways of writing an equilibrium. Some are long and tedious, but here we are using short cut in the following sense. The functional form of w and r are given explicitly by marginal product of labor and marginal product of labor capital minus depreciation. So in the definition of equilibrium, we do not need to write out firm's problem.

Household needs to know B because B will affect future prices. In our problem, law of motion for K, B and future taxes τ depend on B, so future prices are affected by B. Why in this problem household expects K' and B' to evolve according to Φ and Ψ? We set it so and this is true in RCE.

There is no government expenditure in household's problem, because household does not care G, rather G will affect individual problem indirectly through B and τ.

3.4.3 Definition of RCE

Definition 17 *Given a feasible policy $\tau(K,B)$, a RCE is a set of functions $\{V^*, G^*, H^*, g^*, r^*,\}$ such that*

1. *(Household's optimization) Given $\{H^*, G^*\}, \{g^*, V^*\}$ solve the household problem.*
2. *(Consistency)*
$$H^*(K,B) + G^*(K,B) = h^*(K,B,K+B) \tag{34}$$
3. *(No Arbitrage Condition)*
$$r(K,B) = F_K(\Phi^*(K,B),1) - \delta \tag{35}$$
4. *(Government Budget Constraint)*
$$H^*(K,B) = \bar{G} + B(1+r(K,B))(1-\tau(K,B)) - \tau(K,B)[F(K,1) - \delta K] \tag{36}$$
5. *(No Ponzi Scheme Condition) $\exists \underline{B}$ and $\overline{B}$, such that $\forall K \in [0, \overline{\overline{K}}]$, $B \in [\underline{B}, \overline{B}]$*
$$H^*(K,B) \in [\underline{B}, \bar{B}],\ G^*(K,B) \in [\underline{K}, \overline{\overline{K}}] \tag{37}$$

Note that the market clearing condition is implicitly there through Walras Law.

3.5 Extensions to our standard economy

3.5.1 Economy with two type of agents

Assume that in the economy there are two types of agents, called type A and type B (B denotes rich A poor in wealth terms). Measure of the agents of type A and type B are the same . Without loss of generality, we can think of the economy as the one with two agents, both of whom are price takers.

Agents can be different in many ways, including in terms of wealth, preference, ability, etc. We will first look at an economy where agents are different in wealth and efficiency in terms of their labor services. We also assume poor types care . There are measure one population of rich people and measure one population of poor people. For simplicity, we assume there are no shocks and agents do not value leisure.

The state variables are aggregate wealth of both types, K^A and K^B. Why? We know wage and rental only depends on total capital stock $K = K^A + K^B$. But K is not sufficient as aggregate state variables because agents need know tomorrow's price which depends on tomorrow's aggregate capital. To not to carry superscripts on K we can also define a new state varible λ which denotes the share of total wealth held by the agents of type A. The problem of type A agent is,

$$V^A(K,\lambda,a) = \max_{c,a'} \left\{u(c, C^A) + \beta V^A(G(K,\lambda), H(K,\lambda), a')\right\} \tag{38}$$

subject to

$$c + a' = R(K)\,a + e^A w(K) \tag{39}$$

Given,

$$C^A = C^A(K,\lambda)$$

Solutions are:

$$a' = g(K_A, K_B, a)$$

The problem of type B agent is similar with different functional forms. Next we define the RCE

Definition 18 *RCE is a set of functions $\{V^i(.), g^i(.)\}$ and $\{G(.), H(.), C^A(.)\}$, $i \in \{A, B\}$, such that;*

1. Given $\{G(.), H(.), C^A(.)\}, \{V^i(.), g^i(.)\}$ solves the households problem.

(RA condition)

$$
\begin{aligned}
G\left(K, \lambda\right) &= \frac{g^{A}(K, \lambda, 2K\lambda) + g^{B}(K, \lambda, 2K(1-\lambda))}{2} \\
H(K, \lambda) &= \frac{g^{A}(K, \lambda, 2K\lambda)}{g^{A}(K, \lambda, 2K\lambda) + g^{B}(K, \lambda, 2K(1-\lambda))} \\
C^{A}(K, \lambda) &= 2\lambda K R(K, \lambda) + e^{A} w(K, \lambda) - g^{A}(K, \lambda, 2K\lambda)
\end{aligned}
$$

So with these powerful tools in our hand, we are able to define richer and more interesting environments than the RA framework. The important thing when defining equilibrium in these environments is to be consistent.

3.6 Neo-classical firm with a dynamic problem

Our analysis so far have left the firm's static problem lingering in the background and primarily focused on the HH behaviour. This is merely a matter of choice and as we will show below firm's problem can be formulated in a dynamic manner without resulting any substansive changes in our main results. The firm is defined as an entity with a unit of land. The land is not used in the production process. The firm makes the dynamic investment decision and owns the capital and households owns the shares of the firm. Then the problem of our household and firm is,

$$
\begin{aligned}
V(K, a) &= \max_{a', c} \{U(c) + \beta V[G(K), a']\} \qquad (40) \\
s.t. \quad c + a' &= R(K)a
\end{aligned}
$$

$$
\Omega(K, k) = \max_{k'} \{F(k, 1) - k' + q(G(K))\Omega(G(K), k')\} \qquad (41)
$$

with solutions;

$$
\begin{aligned}
a' &= g(K, a) \\
k' &= h(K, k)
\end{aligned}
$$

The way to proceed in defining the recursive equilibrium and characterizing it is similar only a bit more tedious.

First lets write down the functional equations that implicitly defines the RCE.

The FOC and the EC for the firm is as follows,

$$
\begin{aligned}
FOC(k') &: -1 + q(K')\Omega_2(K', k') = 0 \qquad (42) \\
EC &: \Omega_2(K', k') = F_k(k', 1) \qquad (43)
\end{aligned}
$$

and the Euler Equation for the firm becomes;

$$
1 = q(K')F_k(k', 1) \qquad \text{(EE FIRM)}
$$

and using the RA condition;

$$
G(K) = h(K, K) = k' \qquad (44)
$$

we get ;

$$
1 = q(G(K))F_k(G(K), 1) \qquad (45)
$$

For the HH, the FOC and EC are;

$$
\begin{aligned}
FOC(a') &: -U_c(R(K)a - a') + \beta V_2[G(K), a'] = 0 \qquad (46) \\
EC &: V_2[G(K), a'] = R(K')U_c(R(K')a' - a'') \qquad (47)
\end{aligned}
$$

and HH Euler Equation is;

$$
U_c(R(K)a - a') = \beta R(K')U_c(R(K')a' - a'') \qquad \text{(EE HH)}
$$

with the relevant RA condition;

$$\begin{aligned} a &= \Omega(K, K) & (48) \\ a' &= h[K, q(K)\Omega(K, K)] = \Omega(G(K), G(K)) & (49) \end{aligned}$$

Definition 19 *A RCE is a list of functions $\{V, g, \Omega, h, q, G\}$ such that,*

1. *Given $\{q, G\}$, $\{V, g, \Omega, h\}$ solves the HH and firm problems*
2. *Representative agent conditions*

$$\begin{aligned} G(K) &= h(K, K) \\ \Omega(G(K), G(K)) &= g(G(K), \Omega(K, K)) \end{aligned}$$

Next we look at the simplest possible model that can explain great deal in asset pricing.

4 Lucas Tree Model (Lucas 1978)

4.1 The Model

Suppose there is a tree which produces random amount of fruits every period. We can think of these fruits as dividends and use d_t to denote the stochastic process of fruits production. Further, assume d_t follows Markov process. Formally:

$$d_t \sim \Gamma(d_{t+1} = d_i \mid d_t = d_j) = \Gamma_{ji} \tag{50}$$

Let h_t be the history of realization of shocks, i.e., $h_t = (d_0, d_1, ..., d_t)$. Probability that certain history h_t occurs is $\pi(h_t)$.

Household in the economy consumes the only good, which is fruit. With usual assumption on preference retained, consumers maximize:

$$\sum_t \beta^t \sum_{h_t \in H_t} \pi(h_t) u(c_t) \tag{51}$$

Since we assume representative agent in the economy, and there is no storage technology, in an equilibrium, the representative household eats all the dividends every period. So the lifetime utility of the household will be:

$$\sum_t \beta^t \sum_{h_t \in H_t} \pi(h_t) u(d_t) \tag{52}$$

Now suppose that the household is given some STUFF at period 0 and there exists a market to trade fruits. It's trivial to guess that the equilibrium allocation will be an autarky (almost by definition), but the key thing is to find the price which can support the equilibrium allocation of autarky.

Define the household's problem.

$$\max_{\{c(h_t)\}_{t=0}^{\infty}} \sum_t \beta^t \sum_{h_t \in H_t} \pi(h_t) u(c_t(h_t)) \tag{53}$$

subject to

$$\sum_t \beta^t \sum_{h_t \in H_t} p(h_t) c(h_t) = STUFF \tag{54}$$

and

$$p_0 = 1 \tag{55}$$

Note that we are considering the Arrow-Debreu market arrangement, with consumption goods in period 0 as a numeraire.

4.1.1 First Order Condition

Take first order condition of the above maximization problem:

$$FOC \quad c(h_t) \qquad \frac{p(h_t)}{p_0} = p_t(h_t) = \frac{\beta^t \pi(h_t) u'(c(h_t))}{u'(c(h_0))} \tag{56}$$

By combining this FOC with the following equilibrium condition:

$$c(h_t) = d_t \ \forall t, h_t \tag{57}$$

We get the expression for the price of the state contingent claim in the Arrow-Debreu market arrangement.

$$p_t(h_t) = \frac{\beta^t \pi(h_t) u'(d(h_t))}{u'(d(h_0))} \tag{58}$$

4.1.2 Price the tree

Now we can compute the mysterious STUFF which satisfies the budget constraint.
What is the STUFF? STUFF is the sufficient amount to buy fruits in every period in every contingency from time 0 on, measured in period 0 consumption good. We can Imagine that the STUFF is a TREE, which bears fruits.
Tree in this model is a package of a stream of good. In asset pricing,

the price of an asset = value of all the things that the asset entitles you to get.

Therefore, the formula to compute q_t =the price of tree at period 0 is:

$$q_0 = \sum_t \sum_{h_t \in H_t} p_t d_t = \sum_t \sum_{h_t \in H_t} \frac{\beta^t \pi(h_t) u'(d(h_t))}{u'(d(h_0))} d(h_t) \tag{59}$$

4.2 Sequential Market

In sequential market, the household can buy and sell fruits in every period, and the tree (the asset). To consider the trade of the asset, let s_t be share of asset and q_t be the asset price at period t. The budget constraint at every time-event is then:

$$q_t s_{t+1} + c_t = s_t(q_t + d_t) \tag{60}$$

Thus, the consumer's optimization problem turns out to be:

$$\max_{\{c_t(h_t), s_{t+1}(h_t)\}_{t=0}^{\infty}} \sum_t \beta^t \sum_{h_t \in H_t} \pi(h_t) u(c_t(h_t)) \tag{61}$$

subject to

$$q_t(h_t) s_{t+1}(h_t) + c_t(h_t) = s_t(h_{t-1})[q_t(h_t) + d_t] \tag{62}$$

Again, from first order condition, we can derive q_t, which is the price of one tree after history h_t in terms of consumption goods at node h_t. To solve the problem, construct Lagrangian as follows:

$$L: \quad \sum_t \beta^t \sum_{h_t \in H_t} \pi(h_t)[u(c_t(h_t)) - \lambda_t(h_t)\{s_t(h_{t-1})[q_t(h_t) + d_t] - q_t(h_t)s_{t+1}(h_t) + c_t(h_t)\}] \tag{63}$$

Note that there are many ways to write equivalent Lagrangians. In the case above, the sequence of Lagrange multipliers is $\{\beta^t \pi(h_t)\lambda_t\}$. We write it in this way to simplify expressions of the first order conditions. First order conditions are:

$$FOC w.r.t. c_t(h_t) \qquad u'(c_t(h_t)) = \lambda_t(h_t) \tag{64}$$

$$FOC\ w.r.t.\ s_{t+1}(h_t) \qquad \pi(h_t)\lambda_t(h_t)q_t(h_t) = \beta \sum_{h_{t+1}|h_t} \pi(h_t)\lambda_{t+1}(h_{t+1})[q_{t+1}(h_{t+1}) + d_{t+1}(h_{t+1})] \tag{65}$$

Recall, d_t follows a Markov process,

$$\pi(h_{t+1}) = \pi(h_t)\Gamma_{ij} \text{ where } d_t(h_t) = d_i,\ d_{t+1} = d_j \tag{66}$$

so, combine (64) and (65), we get:

$$u'(c_t(h_t))q_t(h_t) = \beta \sum_j \Gamma_{ij} u'(c_{t+1}(h_{t+1}))[q_j + d_j] \tag{67}$$

In equilibrium, $c_t(h_t) = d_t(h_t)$. Let's pick $d_t(h_t) = d_i$, then,

$$u'(d_i)q_i = \beta \sum_j \Gamma_{ij} u'(d_j)[q_j + d_j] \tag{68}$$

From this equation, we can see that (i) the price of asset is also Markovian, and (ii) the marginal utility today is equal to marginal utility tomorrow weighted by prices at each node. Looking at the recursive version of the same problem with denoting discrete state variable as subscripts,

$$V_i(s) = \max_{s',c}\ u(c) + \beta \sum_{d'} \Gamma_{ij} V_j(s')$$

$$s.t.\ \ c + s\prime q_i = s[q_i + d_i]$$

In equilibrium, the solution has to be such that c=d and s$'$ = 1. Impose these on the FOC and get the prices that induce the agent to choose that particular allocation. Then the FOC for a particular state i would imply,

$$q_i = \beta \sum_j \Gamma_{ij} \frac{u'(d_j)}{u'(d_i)}[q_j + d_j] \tag{69}$$

A closer look to these conditions reveals they form a system of linear equations in prices. In order to solve for the prices of q_i, we need to solve the system of equations that consists of (**??**) for each i.

$$\begin{bmatrix} q_1 \\ .. \\ .. \\ q_I \end{bmatrix} = \begin{bmatrix} \beta\Gamma_{11}\frac{u'(d_1)}{u'(d_1)} & \beta\Gamma_{12}\frac{u'(d_2)}{u'(d_1)} & .. & \beta\Gamma_{1J}\frac{u'(d_J)}{u'(d_1)} \\ .. & \beta\Gamma_{22}\frac{u'(d_2)}{u'(d_2)} & .. & .. \\ .. & .. & .. & .. \\ \beta\Gamma_{I1}\frac{u'(d_1)}{u'(d_I)} & .. & .. & \beta\Gamma_{IJ}\frac{u'(d_J)}{u'(d_I)} \end{bmatrix} \left[\begin{bmatrix} q_1 \\ .. \\ .. \\ q_I \end{bmatrix} + \begin{bmatrix} d_1 \\ .. \\ .. \\ d_I \end{bmatrix} \right]$$

Let q=$\begin{bmatrix} q_1 \\ .. \\ .. \\ q_I \end{bmatrix}$ and d=$\begin{bmatrix} d_1 \\ .. \\ .. \\ d_I \end{bmatrix}$ and A=$\begin{bmatrix} \beta\Gamma_{11}\frac{u'(d_1)}{u'(d_1)} & \beta\Gamma_{12}\frac{u'(d_2)}{u'(d_1)} & .. & \beta\Gamma_{1J}\frac{u'(d_J)}{u'(d_1)} \\ .. & \beta\Gamma_{22}\frac{u'(d_2)}{u'(d_2)} & .. & .. \\ .. & .. & .. & .. \\ \beta\Gamma_{I1}\frac{u'(d_1)}{u'(d_I)} & .. & .. & \beta\Gamma_{IJ}\frac{u'(d_J)}{u'(d_I)} \end{bmatrix}$

and let b=Ad
we have,

$$q = Aq + b$$

so that,

$$q = (I - A)^{-1}b$$

Now, suppose that the dividend process is not Markovian. We can still get price of tree in terms of h_t good as follows:

$$q_t(h_t) = \frac{\sum_{\tau=t+1}^{\infty}\sum_{h_\tau|h_t} p(h_\tau)d_t(h_\tau)}{p(h_t)} \tag{70}$$

or

$$p(h_t)q_t(h_t) = \sum_{\tau=t+1}^{\infty}\sum_{h_\tau|h_t} p(h_\tau)d_t(h_\tau) = \sum_{h_{t+1}|h_t} p(h_{t+1})[d_{t+1}(h_{t+1}) + q_{t+1}(h_{t+1})] \tag{71}$$

Now that we are able to price any asset we would like, we can utilize it to look into one of the most popular puzzles in economic literature.

4.3 Equity premium puzzle

This puzzle basically says that standard representative agent neoclassical growth model with CRRA utility function with "normal" parameter values fails to explain the huge difference between risky stock returns and riskless bond in US. For example, Dimson, Marsh, and Staunton (2002) reported that the average annual real returns of equity (over 1900-2000) is 6.7%, while the average annual returns of risk-free[6] T-bill over the same period is 0.9%. So the risk premium is around 6% annually. Of course, equity premium puzzle depends on many assumptions, as I listed above, so there are many other assumptions which might cause the problem. But if we change only σ to match this high equity premium, it is known that we need $\sigma = 20 - 50$. In other words, people have to be very very risk averse to hold T-bills regardless of the huge difference in average return. What is the average rate of return in our model economy? To be able to pin that down, we need to know the stationary distribution of the shocks to our economy (more to be discussed about stationary distributions later in the course but for now it basically gives the relative frequency of each state occuring in long run). Given the stationary distribution μ the average rate of return on bonds is given by

$$r^b = \sum_j \mu_j r_j^b$$

and to pin down r_j^b we can use our simple Lucas tree model to price the bond. We know a bond is a asset that gives a unit of consumption good for sure regardless of the state of the economy where as the return on shares is state dependent and usually pays out good when the times are good and marginal utility consumption is low and vice versa. Using the following budget constraint, with state contingent assets b, with the usual problem,

$$c + s\prime q_i + \sum_i p_{ij}b_i' = s[q_i + d_i]$$

one can show the price of the asset that pays a unit of consumption for sure next period if state j is realized,

$$p_{ij} = \beta\Gamma_{ij}\frac{U_c(j)}{U_c(i)}$$

and the price of bond and the return on it is,

$$\begin{aligned} p_i^b &= \sum_j p_{ij} \\ r_j^b &= \frac{1}{q_j^b} - 1 \end{aligned}$$

The average rate of return on shares in our model is,

$$\sum_j \mu_j \sum_i \Gamma_{ji}\frac{q_i + d_i}{q_j} - 1 = r^{sm}$$

[6] We ignore the inflation risk here. If we consider the inflation risk, T-bill is also risky unless it is inflation adjusted (and it is the case).

and it is the discrepancy between differences of these rates in the model and in real world that is in the core of this puzzle

4.4 Pricing an Arbitrary Asset

Because in a complete market any asset can be reproduced by buying and selling contingent claims at every node, we can use this model as a powerful asset pricing formula. For example, discount bond is a promise to pay one unit of good tomorrow no matter what happens. To reproduce bond, it suffices to buy one unit of state contingent claim at every node in the next period. Therefore, at h_t, the price of bond is:

$$p^b(h_t) = \frac{\sum_{h_{t+1}|h_t} p(h_{t+1})}{p(h_t)} \tag{72}$$

Consols is a promise to pay one unit of good forever from now on. Thus its price is:

$$p^{consol}(h_t) = \frac{\sum_{\tau=t}^{\infty} \sum_{h_\tau|h_t} p(h_\tau)}{p(h_t)} \tag{73}$$

Consider a one period call option, which is a right to buy one share of a tree at the fixed price (exercise price) $\bar{q}$. The price of this option is:

$$p^{o,\bar{q}}(h_t) = \frac{\sum_{h_{t+1}|h_t} p(h_{t+1})[q(h_{t+1}) - \bar{q}]1_{[q(h_{t+1})-\bar{q}]>0}}{p(h_t)} \tag{74}$$

where 1 is an indicator function (see the note of the next class).

4.4.1 Two Period Option

To see that we can price any kinds of assets or options using this principle, let's price two periods option. Option here is the RIGHT to buy a consumption goods at a negotiated price. When we talk about multiple period options, we have to be aware the difference between American and European option. American option can be exercised AT ANY TIME before its maturity. On the contrary, European option can be exercised ONLY AT ITS MATURITY. But the principle to price them is same. By the way, notice that American option is always more expensive than its European counterpart, because American option contains more options to its holders.

Here let's price two period American and European options at a node h_t. As a set up, assume that the set of the possible aggregate shock contains two elements. Start from h_t, possible nodes in the next periods are h_{t+1}^1 and h_{t+1}^2. In the two period ahead, there are four possible nodes, h_{t+2}^1, h_{t+2}^2, h_{t+2}^3, h_{t+2}^4, where h_{t+2}^1, and h_{t+2}^2 can be reached only from h_{t+1}^1.

Firstly, remember the price of an one period option at the node $h_{t+1} : p^o(h_t)$ with negotiated price $\bar{q}$. This is:

$$p^{o1}(h_t) = \sum_{h_{t+1}|h_t} [q(h_{t+1}) - \bar{q}]1_{[q(h_{t+1})-\bar{q}]>0} \frac{p(h_{t+1})}{p(h_t)} \tag{75}$$

where $1_{[\text{expression}]}$ is an indicator function that takes value of 1 if the [expression] is true and 0 if false, and $p(h_t)$ is the price of consumption goods at node h_t. You can also use χ for an indicator function.

Price of an European option (option which can be exercised ONLY in the two period ahead), which is just the natural extension of this one period option, is as follows:

$$p^{o2}(h_t) = \sum_{h_{t+2}|h_t} [q(h_{t+2}) - \bar{q}]1_{[q(h_{t+2})-\bar{q}]>0} \frac{p(h_{t+2})}{p(h_t)} \tag{76}$$

Price of an American option is a little bit more tricky:

$$p^{oa2}(h_t) = \sum_{h_{t+1}|h_t} \max\left\{p^{o1}(h_{t+1}), [q(h_{t+1}) - \bar{q}]\right\} \frac{p(h_{t+1})}{p(h_t)} \tag{77}$$

In the period t+1, a holder can either (i) exercise the option (and then the option expires), or (ii) keep the option to the next period (in this case, the option is exactly the same as the one period option bought in the period t+1).

4.5 Final Remark

In this fashion, we can price any kinds of assets or options. For example, you can easily price future transaction[7]. This is basically finance guys are doing during while their life. They are just solving the price, without solving the allocation (because of RA assumption, we do not need to solve the asset portfolio of agents, which are the same in equilibrium).

5 Economy with Heterogeneous Agents

5.1 Introduction

So far, in environments we anaylyzed, the type of agents does not change over time. In this case, especially, if the number of type of agents is small, as the example we did with only two different types of agents, it's easy to keep track of all the types, and so is to define an equilibrium .From now on, we will consider the economies with (i) many agents who are very different among themselves at a given time (crosssection), and (ii) change their types over time.

Since agents might trade each other, we need to keep track of the aggregate state of the world . There are two ways to do it. One is "Spanish Interior Minister way". People in the economy are given identification number and you record the types of agents according to the number. But this way is not efficient, because the id number does not tell the properties of agents: we use the id numbers just to keep track of individuals. So we take the second way. We are not going to keep track of agents by id numbers given to each agent but we use MEASURE. To further proceed, we need some knowledge on thew measure theory, so let's study it briefly, and after that we will see how measure theory is useful for our purpose.

5.2 Introduction to Measure Theory

5.2.1 Intuition

Measure theory can be understood nicely by comparing to weight. Measure is useful in literally measuring a mass in a mathematically consistent way, which is similar to the way of weighting a mass. Therefore, intuitively the following properties are expected to be satisfied by measure:

1. $measure(nothing) = 0$
2. if $A \cap B = \emptyset$, $measure(A + B) = measure(A) + measure(B)$

These properties are intuitive with weight. The weight of nothing is zero. If a body is 200 pounds, and you chop off a hand from the body and put the hand and the rest of the body together on the scales, they must weight 200 pounds. Now consider an economy with many agents. The measure of nobody in the economy is zero. If a measure of the total population is normalized to one, and you take away the rich people from the population and measure the sum of rich people and the rest of the population, they must have measure one.

[7]Future transaction is a contract to buy or sell a goods in a negotiated period at a negotiated price. The difference from option is that you MUST perform the transaction, no matter whether you want to do or not. Naturally, option contract is more expensive, as you are given an option not to exercise.

5.2.2 Definitions

Definition 20 *For a set A, $\mathcal{A}$ is a set of subsets of A.*

Definition 21 *σ-algebra $\mathcal{A}$ is a set of subsets of A, with the following properties:*

1. *$A, \emptyset \in \mathcal{A}$*
2. *$B \in \mathcal{A} \Rightarrow B^c \in \mathcal{A}$ (closed in complementarity)*
3. *for $\{B_i\}_{i=1,2...}$, $B_i \in \mathcal{A} \Rightarrow [\cap_i B_i] \in \mathcal{A}$ (closed in countable intersections)*

The intuition of the property 2 of σ-algebra is as follows. If we chop off a hand from a body, and if the hand is an element of $\mathcal{A}$, the rest of the body is also an element of $\mathcal{A}$. Soon we will define measure as a function from σ-algebra to a real number Then the property of σ-algebra implies that if we can measure the chopped hand, we can measure also the rest of the body.

Examples of σ-algebra are the follows:

1. Everything (all the possible subsets of a set A)
2. $\{\emptyset, A\}$
3. $\{\emptyset, A, A_{1/2}, A_{2/2}\}$ where $A_{1/2}$ means the lower half of A (imagine A as an closed interval on $\mathcal{R}$).
4. $\{\emptyset, A, A_{1/4}, A_{2/4}, A_{3/4}, A_{4/4}, A^c_{1/4}, A^c_{2/4}, A^c_{3/4}, A^c_{4/4}, A_{1,2/4}, A_{1,2/4}, A_{1,3/4}, A_{1,4/4}, A^c_{1,2/4}, A^c_{1,3/4}, A^c_{1,4/4}\}$

Remark 22 *A convention is (i) use small letters for elements, (ii) use capital letters for sets, (iii) use "fancy" letters for set of subsets.*

Look at the examples of 3 and 4. Imagine you are given $a \in A$. If the only information we can get with respect to a is whether a is included in an element of $\mathcal{A}$ or not, it is true that we have richer information on a with σ-algebra 4 than 3 because, with 4, we can know a is included in which of $A_{1/4}, A_{2/4}, A_{3/4}, A_{4/4}$, where with 3, we only know a is included in which of $A_{1/2}, A_{2/2}$. In this sense, σ-algebra is similar to the notion of information.

Definition 23 *A measure is a function $x : \mathcal{A} \to \mathcal{R}_+$ such that*

1. *$x(\emptyset) = 0$*
2. *if $B_1, B_2 \in \mathcal{A}$ and $B_1 \cap B_2 = \emptyset \Rightarrow x(B_1 \cup B_2) = x(B_1) + x(B_2)$ (finite additivity)*
3. *if $\{B_i\}_{i=1}^{\infty} \in \mathcal{A}$ and $B_i \cap B_j = \emptyset$ for all $i \neq j \Rightarrow x(\cup_i B_i) = \sum_i x(B_i)$ (countable additivity)*

In English, countable additivity means that measure of the union of countable disjoint sets is the sum of the measure of these sets.

Definition 24 *Borel-σ-algebra is (roughly) a σ-algebra which is generated by a family of open sets.*

Remember the discussion on the information. Since Borel-σ-algebra contains all the subsets generated by intervals, you can recognize any subset of a set, using Borel-σ-algebra. In other words, Borel-σ-algebra corresponds to the complete information.

You might find that σ-algebra is similar to topology. Topology is also a set of subsets, but its elements are open intervals and it does not satisfy closedness in complementarity (complement of an element is not an element of topology). Very roughly, the difference implies that topologies are useful in dealing with continuity and σ-algebra is useful in dealing with measure.

Definition 25 *Probability (measure) is a measure such that $x(A) = 1$*

6 INDUSTRY EQUILIBRIUM

6.1 Introduction

Consider a firm who uses labor to produce the output good and has productivity $s \in S$. The production technology of the firm is given by $y = sf(n)$. The firm chooses how much labor to employ given the wage, w, and given its productivity,s. The firm's problem is,

$$\max_n psf(n) - wn$$

The solution n^* solves,

$$psf'(n) = w$$

Then the two period profit of the firm is,

$$\pi_2 = [p^* sf(n^*) - wn^*] \left[1 + \frac{1}{1+r}\right]$$

Now suppose that the firm will only operate next period with probability $(1-\delta)$. With probability δ it will die. In that case, the two period profit of the firm is,

$$\pi_2 = [p^* sf(n^*) - wn^*] \left[1 + \frac{1-\delta}{1+r}\right]$$

Now consider the infinite periods profit of the firm,

$$\begin{aligned}\pi_\infty &= [p^* sf(n^*) - wn^*] \sum_{t=0}^{\infty} \left(\frac{1-\delta}{1+r}\right)^t \\ &= [sf(n^*) - wn^*] \left(\frac{1+r}{r+\delta}\right)\end{aligned}$$

The zero profit condition is that the profit from entry is equal to the cost of entry, denoted by c_e. This condition says that there are no further incentives to enter the industry:

$$c_e = \pi_\infty$$

Define $x : \mathcal{S} \to R$ as the measure of firms, where $\mathcal{S}$ is the $\sigma - algebra$ defined on the set S

DEFINITION

An industry equilibrium is a set $\{p^*, y^*, n^*, x^*(s)\}$, such that

1) $p^* = p(y^*)$ (demand is satisfied)

2) $y^* = x^*(s, p^*)\ \ s\ f(n^*)$ (feasibility)

3) Firms optimize: $n^* \in \arg\max_n \ \ p^* sf(n) - w^* n$

4) Zero profit condition: $c_e = \pi_\infty$

6.2 Tweak No1

Suppose that each firm has to pay a cost of entry c_e, and the productivity shock is drawn from the distribution $\gamma(s)$. Once the firm draws s it keeps it forever.

DEFINITION:

An industry equilibrium is a set $\{p^*, y^*, n^*, x^*(s)\}$, such that

1) $p^* = p(y^*)$ (demand is satisfied)

2) $y^* = \int_S \ s\ f(n^*(s))\, dx^*$ (feasibility)

3) Firms optimize: $n^* \in \arg\max_n \;\; p^* s f(n) - w^* n$

4) Zero profit condition: $c_e = \int_S \pi(s)\, d\gamma(s)$

Note that here, the distribution of firms completely reflects the distribution from which they draw their productivity shocks, $\gamma(s)$. This is because what types of firms remain or what types of firms exit is not an issue since there is exogenous entry and exit. For example, if exit was endogenous we would expect the 'bad' firms to exit and the better ones to stay, and therefore the type distribution of incumbent firms would be different than the initial distribution $\gamma(s)$. But in our case, the distribution of incumbents and the initial type distribution are identical.

So this model is not interesting because it has no economics. The next version of the model that we study is:

6.3 Tweak No 2 (Changing productivity)

Here s is drawn from $\gamma(s)$ as before, but after the initial shock is obtained, $s' \sim \Gamma_{ss'}$. We will assume that Γ satisfies First Order Stochastic Dominance. This means that

$$For\ s_1, s_2 \in S, \;\; s_1 < s_2 \Rightarrow \int_{\tilde{s}}^{\bar{s}} \Gamma(s_1, s)\, ds \leq \int_{\tilde{s}}^{\bar{s}} \Gamma(s_2, s)\, ds$$

The entry and exit decisions are still exogenous. Incumbent firms die at rate δ. The following condition is needed for stationarity:

$$x^*(B) = \int_S \int_S (1-\delta)\Gamma_{ss'} 1_{s' \in B}\, dx^*(s)\, ds' + \int_S 1_{s' \in B} d\gamma(s'),$$
where B is a Borel set over the state space.

6.4 Tweak No 3 (Endogenous Exit Decision of Firms)

Now at each period firms make entry and exit decisions. What is a sufficient mechanism to get firms to quit? Having fixed costs. A fixed cost c_f must be paid every period by incumbent firms. Each period, incumbent firms decide to stay or exit.

The value of a firm with current productivity shock s is givne by:

$$\pi(s) = \max \;\; \left[\max_n \; p^* s f(n) - wn - c^f + \tfrac{1}{1+r} \int_S \Gamma_{ss'} \pi(s') ds', \;\; 0\right]$$

As we saw in class the decision of the firm (under the crucial assumption of FOSD of the transition matrix Γ) will be characterized by a threshold. There exists a $s^* \in S$, such that if $s < s^*$ the firm quits, abd if $s \geq s^*$ the firm stays in the industry.

The formula for a stationary distribution (see also Problem Set 6) is the folloing:

$$x^*(B) = \int_S \left(\int_{s^*}^{\bar{s}} \Gamma_{ss'}\, dx^*(s) \right) 1_{s' \in B}\, ds' + \left(\int_{\underline{s}}^{s^*} dx^* \right) \int_S 1_{s' \in B} d\gamma(s)$$

Note that $\left(\int_{\underline{s}}^{s^*} dx^* \right)$ is the number of firms that exit the market, and which will be equal to the number of firms that enter (in equilibrium). Of course, these firms will draw their productivity shock from the distribution γ, and this is why the integration in the second term is with respect to that distribution instead of x^*.

The market clearing condition is given by

$$y^D(p^*) = \int_S s f(n^*) dx^*(s) + \left(\int_{\underline{s}}^{s^*} dx^* \right) \int_S s f(n^*(s)) d\gamma(s)$$

The zero profit condition:

$$c_f = \int \pi(s) d\gamma(s)$$

The interpretation is that the firm that is out of the market should have expected profits equal to the fixed cost, because if it enters the firm has to pay that cost at least for one period (if it enters and the shock it obtains is really bad, the firm can walk out in the second period).

6.5 Employment protection with Firing Costs

Note that in this problem the labor force of the last period is a state variable for the firm. This means that the state space will also be different. The new state space is given by $X = S \times N$. N is the set of the possible values of labor force. For convenience assume that it is bounded, i.e, $N = \left[0, \bar{N}\right]$, where $\bar{N} < \infty$.

Assuming that there is a cost of firing equal to a per worker, the profit function is given by

$$\Pi\left(s, n^{-1}\right) = \max\left\{-a\ n^{-1},\ \max_{n}\left[p\ s\ f(n) - w\ n -\ a\left(n^{-1} - n\right)\ \{n^{-1} > n\}\ + \tfrac{1}{1+\ r}\ \sum_{s'} \Gamma_{s\ s'} \Pi\left(s', n\right)\right]\right\}$$

In Problem Set 6 you had to find a formula for the stationary distribution in this model. Maintaining the assumption of FOSD, we saw that this formula will be given by:

$$x^*(B) = \int_S \int_N \left[\int_{s^*(n^{-1})}^{\bar{s}} \int_N \Gamma_{s\ s'} dx^*\left(s, n^{-1}\right)\right] \{\left(s', n\left(s, n^{-1}\right)\right) \in B\}\ ds' dn +$$
$$+ \left(\int_{\underline{s}}^{s^*(n^{-1})} \int_N dx^*\left(s, n^{-1}\right)\right) \int_S \{(s', n(s, 0)) \in B\}\ d\gamma(s'),$$

where $B \in \mathcal{X}$, the set of subsets of the (new) state space,

$\int_{\underline{s}}^{s^*(n^{-1})} \int_N dx^*\left(s, n^{-1}\right)$ is the number of firms that quit (or- in equilibrium- enter the market),

and $\int_S \{(s', n(s, 0)) \in B\}\ d\gamma(s')$ is the probability measure that a new firm will end up in B (that's why $n^{-1} = 0$)

7 Problem of Fishermen (Once again)

Imagine a Archipelago that has a continuum of islands (instead of the pig farmers we used in class). There is a fisherman on each island. The fishermen get an endowment e each period which follows a Markov process with transition $\Gamma_{ee'}$ and,

s$\in \{e^1, ..., e^{n_e}\}$

The fishermen cannot swim. There is a storage technology such that, if the fishermen store q units of fish today, they get 1 unit of fish tomorrow. (e,a) is the type of a fisherman and the set consisting of all possible such pairs is,

$$E \times A = \{e^1, e^2, ..., e^n\} \times [0, \overline{a}]$$

Let $\mathcal{A}$ be the set of Borel sets on SxA. And define a probability measure x on $\mathcal{A}$,

$$x : \mathcal{A} \to [0, 1]$$

The fisherman's problem is:

$$V(e,a) = \max_{c,a' \geq 0} u(c) + \beta \sum_{e'} \Gamma_{ee'} V(e',a')$$

subject to

$$c + qa\prime = e + a$$

$$c \succeq 0 \text{ and a' } \in [0, \overline{a}]$$

With the decision rule $a' = g(e,a)$ and the transition matrix for the endowment process $\Gamma_{ee'}$
The First Order Conditions are,

$$u_c(e + a - qa\prime) = \frac{\beta}{q} \sum_{s'} \Gamma_{ee'} u_c(e\prime + a\prime - q' a'')$$

You'll notice that a$'$ $\in [0, \overline{a}]$ is already one of the constraints of the above maximization problem. But now rather than just imposing such a constraint, we will find a natural reason that savings should have a lower bound and we will consider a condition that ensures an upper bound for savings.

For the lower bound, we assume that there is no technology which allows negative amount of saving and this sounds natural since storing a negative amount of fish does not make much sense. So savings has a lower bound because Mother Nature says so.

Here, the fisherman has the risk of getting a very bad shock tomorrow. So the fisherman would save just in case he has this bad shock; he would want to store some fish today in order to insure himself against getting very small number of fish tomorrow so he is not hungry in case that happens. In this case we need to think more about how to put an upper bound on savings, because with uncertainty even if $\beta < q$, the fisherman is willing to save due to gains from insurance. The kind of savings to protect oneself from risk in the future in the absence of state contingent commodity markets which can be used to insure against any contingency to make sure consumption is constant across states, which is usually called precautionary savings. In order to ensure an upper bound for savings, we need to bound the gains from insurance somehow. The way to do this is to impose the condition on the utility function that its negative curvature (keeping in mind that the utility function is concave) is diminishing as wealth increases. This means that wealthier agents are less risk-averse. Formally, that u' is convex. The wealthier the agent is, the smaller the variance of his endowment next period proportional to his wealth so he doesn't want to save if he is very wealthy. This is simply because of the fact that the wealth is not subject to any uncertainty but income is thus as the income wealth ratio rise, the overall uncertainty the agent faces diminishes.

So in the economy with uncertainty, in order to have an upper bound on savings, we need the first derivative of the utility function to be convex so that the following Jensen's Inequality holds:

$$\frac{\beta}{q} \int \Gamma_{ss'} u_c(c') > \frac{\beta}{q} u_c(\int \Gamma_{ss'} c')$$

Theorem 26 *If $\beta < q$ and $u\prime$is convex then $\exists \overline{a}$ such that $a_0 < \overline{a}$, g(s,a)$< \overline{a} \quad \forall s$.*

Now consider the case of lower bound. Suppose we let the fisherman borrow and lend to each other but not store any fish, how can we make sure that our agents always has the capability to pay back what they owe. What would be the endogenous lower bound to ensure this? Such a condition would make sure that in the worst case scenario our agent should be able pay the interest rate on its debt and roll over the same amount (the lowest possible amount). Thus, letting the lower bound be $\underline{a}$ and the lowest possible shock be $\underline{e}$ then,

$$\begin{aligned} 0 + q\underline{a} &= \underline{a} + \underline{e} \\ \underline{a} &= \frac{\underline{e}}{q-1} \end{aligned}$$

This is called the solvency constraint rather than a borrowing constraint. Note that when we let the fishermen to get into lending contracts with each other, we need a consistency condition to make sure agents actions are compatible with each other. Here the price will be endogenously determined will ensure the agents hold just the right amount of assets.

The stationary equilibrium of such an economy is defined as,

Definition 27 *A stationary equilibrium for an Huggett(1993) economy is a set* $\{q^*,\ x^*(q^*),\ Q(s,a,B;q^*),\ g(s,a;q^*)\}$ *such that*

1. *(Agent Optimization) Given* q^**,* $g(s,a;q^*)$ *solves the agent's problem.*

2. *(Consistency)* $Q(s,a,B;q^*)$ *is a transition matrix associated with* $\Gamma_{ss'}$ *and* $g(s,a;q^*)$*.*

3. *(Stationarity)* x^* *is the unique stationary distribution associated with* $Q(s,a,B;q^*)$*.*

4. *(Market clear)*
$$\int adx^*(q^*) = 0$$

7.1 Transition analysis in an Heteregenous Agent Economy

Now suppose instead of a natural technology (q) we have a production function of the Cobb-Douglas form that is CRTS. What is sufficient variable to pin down the prices now? We know $\frac{K}{L}$ is sufficent, so there is a particular capital labor ratio that pins down the stationary equilibrium prices. The problem we have is,

$$V\left(e,a;\frac{K}{L}\right) = \max_{c\geq 0, a\in[0,\overline{a}]} u\left(c\right) + \beta\sum_{s'}\Gamma_{ss'}V\left(s',a';\frac{K}{L}\right) \tag{78}$$

subject to

$$c + a' = a\left(1 + r\left(\frac{K}{L}\right)\right) + w\left(\frac{K}{L}\right)e$$

The optimal solution is

$$a' = g\left(e,a;\frac{K}{L}\right)$$

In equilibrium,

$$(\frac{K}{L}) = \frac{\int adx^*\left(\frac{K}{L}\right)}{\int edx^*\left(\frac{K}{L}\right)} \tag{79}$$

where,

$$L = \int ed\gamma^*$$

and γ^* is the stationary distribution of efficiency labor units e. Now suppose we would like to do some policy analysis in this economy (such as a change in tax on capital income), which would require us to do analysis off the stationary equilibrium. Once we leave the stationarity, we know the distribution of x becomes a state variable. This is because of the fact that the law of motion for K/L depends on the wealth distribution and so does the prices. We know that dealing with an object like distribution as a state variable is extremely demanding. The way to circumvent that problem is to make prices independent of distribution by either with an assumption on technology (linear) or directly assuming a small country framework where the prices are given exogenously.

8 Growth

In our analysis so far, we have used Neo-classical Growth Model as our benchmark model and built on it for the analysis of more interesting economic questions. One peculiar characteristic of our benchmark model, unlike its name suggested, was lack of growth. Many interesting questions in economics are related to the cross-country differences of growth rates and we will cover some models that will allow for growth so that we will be able to attempt to answer such questions.

8.1 Exogenous growth

What does it take for an economy to grow? Before answering that question, we know in our standard NGM there is basically two ways of growth, one in which everything grows, which is not necessarily a per-capita growth, and the other is per-capita growth. We will be focusing on per-capita growth. The title exogenous growth refers to the structure of models in which growth rate is determined exogenously, and is not an outcome of the model. First and the simplest one of these is one in which the determinant of the growth rate is population growth. (Note that these notes are written to compliment the notes of Per Krusell which can be downloaded from the course website)

8.1.1 Growth with population

Suppose the population of our economy grows at rate γ and we have the classical CRTS technology in capital and labor inputs.

$$\begin{aligned} Y_t &= AF(K_t, N_t) \\ N_t &= N_0 * \gamma^t \end{aligned} \tag{80}$$

Note that our economy is no longer stationary but as we will see, within the exogenous growth framework we can make these economies look like stationary ones by re-normalizing the variables. Thus, at the end of the day it will only be a mathematical twist on our standard growth model. Once we do that, we will be looking for the counterpart of a steady state that we have in our stationary economies, the Balanced Growth Path, in which all the variables will be growing at constant rates but not necessarily equal. Back to our population growth model, we know

$$AF(K, N) = A[KF_k(K, N) + NF_N(K, N)] \tag{81}$$

Question is, if N is growing at rate γ, can this economy have a balanced growth path. Can we construct one? We know by CRTS property F_K and F_N are homogenous of degree zero. If we assume capital stock grows at rate γ as well, then prices stay constant and per-capita variables are constant and output grow at the same rate. So we get growth on a balanced growth path without per-capita growth. One question is how do we model population growth in our representative agent model. One way is to assume there is a constant proportion of immigration to our economy from outside but this has to assume the immigrants are identical to our existing agents in our economy, which is a bid problematic. The other way could be to assume growing dynasties which preserves the representative agent nature of our economy. If we do so, the problem of the social planner becomes,

$$\begin{aligned} & \max \sum_{t=0}^{\infty} \beta^t N_t U(\frac{C_t}{N_t}) \\ st \quad C_t + K_{t+1} &= AF(K_t, N_t) + (1-\delta)K_t \end{aligned} \tag{82}$$

To transform the budget set to per capita terms, divide all terms by N_t and to make the environment stationary by dividing all the variables by γ^t and assume $N_0 = 1$, we get,

$$\begin{aligned} & \max \sum_{t=0}^{\infty} (\beta\gamma)^t U(\widehat{c}_t) \\ st \quad \widehat{c}_t + \gamma \widehat{k}_{t+1} &= AF(\widehat{k}_t, 1) + (1-\delta)\widehat{k}_t \end{aligned} \tag{83}$$

So how is this transformed model any different than our NGM? By the discount factor, the agents in this economy with growth discounts the future less but everything else is identical to NGM of course with the exception of this economy growing at a constant rate.

Now suppose we have a 'labor augmenting' productivity growth with constant population normalized to one, i.e. have the following CRTS technology,

$$Y_t = AF(K_t, \gamma^t N_t) \tag{84}$$

$$AF(K_t, \gamma^t N_t) = A[K_t F_k(K_t, \gamma^t N_0) + \gamma^t N_0 F_N(K_t, \gamma^t N_0)] \tag{85}$$

Can we have an BGP? The problem is,

$$\begin{aligned} & \max \sum_{t=0}^{\infty} \beta^t U(C_t) \qquad (86) \\ st \quad C_t + K_{t+1} &= AF(K_t, \gamma^t N_0) + (1-\delta)K_t \end{aligned}$$

and since we have a population of one, these variables are already per-capita terms. For stationarity, we have to normalize the variables to 'per productivity' units, by dividing all by γ^t. Then the problem becomes,

$$\begin{aligned} & \max \sum_{t=0}^{\infty} \beta^t U(\gamma^t \widehat{c}_t) \qquad (87) \\ st \quad \widehat{c}_t + \gamma \widehat{k}_{t+1} &= AF(\widehat{k}_t, 1) + (1-\delta)\widehat{k}_t \end{aligned}$$

Suppose we have a CRRA preferences, then the question is how can we represent the preferences as a function of $\widehat{c}_t$ only. Writing the CRRA,

$$\sum_{t=0}^{\infty} \beta^t \frac{(\gamma^t \widehat{c}_t)^{1-\sigma} - 1}{1-\sigma} = \sum_{t=0}^{\infty} (\beta(\gamma^{1-\sigma}))^t \frac{\widehat{c}_t^{\,1-\sigma} - 1}{1-\sigma} \qquad (88)$$

and the problem becomes,

$$\begin{aligned} & \max \sum_{t=0}^{\infty} (\beta(\gamma^{1-\sigma}))^t \frac{\widehat{c}_t^{\,1-\sigma} - 1}{1-\sigma} \qquad (89) \\ st \quad \widehat{c}_t + \gamma \widehat{k}_{t+1} &= AF(\widehat{k}_t, 1) + (1-\delta)\widehat{k}_t \end{aligned}$$

and once again it is exact same problem as the NGM with a different discount factor. Note that the existence of a solution to this problem depends on $\beta(\gamma^{1-\sigma})$.In this set-up we get per-capita growth at rate γ. Also note that CRRA is the only functional form for preferences that is compatible with BGP. This is because as per-capita output grows, for consumption to grow at a constant rate, our agent has to face the same trade-off at each period.

Now suppose we have the TFP growing at rate γ with a CRTS Cobb-Douglas technology

$$\begin{aligned} Y_t &= A_t F(k_t, 1) \\ \frac{A_{t+1}}{A_t} &= \gamma \end{aligned}$$

What would be the growth rate of this economy? We can show that like the previous cases the growth rate of the economy is the growth rate for the productivity of labor, which is $\gamma^{\frac{1}{1-\alpha}}$ in this case.

8.2 Endogenous Growth

So far in the models we covered growth rate has been determined exogenously. Next we will look to models in which the growth rate is chosen by the model itself. We do know for a fixed amount of labor, the curvature of our technology limits the growth due to diminishing marginal return on capital and with depreciation there is an upper limit on capital accumulation. So if our economy is to experience sustainable growth for a long period of time, we either give up the curvature assumption on our technology or we have to be able to shift our production function up. Given a fixed amount of labor, this shift is possible either by an increasing TFP parameter or increasing labor productivity, . The simplest of such models where we can see t is the AK model, where the technology is linear in capital stock so that diminishing marginal return on capital does not set in.

8.2.1 AK Model

We have the usual social planner's problem with linear technology and full depreciation,

$$\max \sum_{t=0}^{\infty} \beta^t U(C_t) \tag{90}$$
$$st \;\; C_t + K_{t+1} \;=\; AK_t$$

and the FOCs

$$(c_t) \;\; : \;\; \beta^t U_c(.) = \lambda_t \tag{91}$$
$$(k_{t+1}) \;\; : \;\; \lambda_t = \lambda_{t+1} \tag{92}$$

together implies the Euler equation,

$$U_c(c_t) = A\beta U_c(c_{t+1}) \tag{93}$$

and on the BGP with consumption growing at rate γ with CRRA utility we get,

$$\gamma = (A\beta)^{1/\sigma} \tag{94}$$

and the growth rate is determined by the model parameters endogenously. Note that capital also grows at rate γ and the fate of the economy is determined by pre-determined parameters of the model. The capital stock will diverge to infinity if $(A\beta)^{1/\sigma} > 1$ or the economy is destined to vanish if $(A\beta)^{1/\sigma} < 1$. Also note that there is no transitional dynamics in this model (we loose the state variables in the euler equation after substituting for the balanced growth rate relation) and conditional on γ,asymptotically all economies are same regardless of the initial capital level. If we de-centralize this economy we know wages will be zero since labor has no use and gross rental rate of capital will be fixed at the A. This is at odds with what we observe in real world. We would rather like to have a model that allows for both transitional dynamics, labor and growth at the same time. Allowing for labor implies that we need a variable that proxies the increasing productivity of labor endogenously and be reproducible in terms of output, such that we are able to shift our production function continually in the output-capital space without hitting a natural bound.

8.2.2 Human Capital and Growth

One way of doing this is, introducing the variable 'Human Capital' as an input of production, to proxy continuously and endogenously increasing labor efficiency. We have two ways of modelling the human capital, one way is to see it very much like physical capital, in the sense output has to be invested to increase the existing stock of human capital. That is the Lucas' approach, in which you can think of investing in education by building more schools as a way to increase the existing human capital stock. The alternative way would be to reserve a part of the leisure time for increasing the human capital stock, which can be thought of studying harder to get better in a fraction of the leisure time. Unfortunately, the second approach puts limit on the rate human capital can grow and might fail to generate sustainable endogenous growth. Next, we look at the Lucas' human capital model.

Lucas' Human Capital Model We have an Cobb-Douglas technology with CRTS and human capital (H) as an input of production instead of labor and the laws of motion for the inputs,

$$F(H, K) \;=\; AK^{\alpha} H^{1-\alpha} \tag{95}$$
$$K' \;=\; i_k + (1 - \delta_k)K \tag{96}$$
$$H' \;=\; i_h + (1 - \delta_h)H \tag{97}$$

Now that there is no limit to the accumulation of human capital and sustainable growth on a BGP is feasible. Furthermore, an analysis of the characterization of the balanced growth path will indicate that this model indeed has transitional dynamics, so unlike the AK model if economy starts out of this optimal growth path economy can adjust and converge to it by responding to prices in a de-centralized setting. If we model the law of motion for human capital as,

$$H' = (1 - N) + (1 - \delta_h)H \tag{98}$$

where $(1 - N)$ is the time devoted to accumulating human capital, say by studying harder, we see there is a natural limit to the growth of human capital and such an economy might not have a BGP. The key ingredient of endogenous growth with labor is then the reproducibility of the human capital without such a limit.

8.2.3 Growth through Externalities (Romer)

We have seen in the AK model the growth rate is determined solely by model primitives and endogenized but still it is not a directly or indirectly determined by the agents' choices in our model. In Lucas' human capital model, the growth rate is determined by the choice of agents, specifically by the optimal ratio of human and physical capital. The source of growth in Lucas' model is reproducibility of human capital. In the next model, Romer introduces the notion of externality generated by the aggregate capital stock to go through the problem of diminishing marginal returns to aggregate capital. In this model, the source of growth will be the aggregate capital accumulation, which is possible with a linear aggregate technology in capital as we saw in the AK model. The firms in our model will not be aware of this externality and will have the usual CRTS technology and observe the source of growth coming from the TFP parameter. As usual with externalities, the equilibrium outcome will not be optimal. Each firm has the following technology,

$$y_t = AK_t^{1-\alpha} k_t^{\alpha} n_t^{1-\alpha} \tag{99}$$

but since the firms are not aware of the positive externality they are facing they are solving the problem with the following technology.

$$\begin{aligned} y_t &= \overline{A}_t k_t^{\alpha} n_t^{1-\alpha} && (100) \\ where\ \overline{A}_t &= A_t K_t^{1-\alpha} && (101) \end{aligned}$$

We can see the social planner in fact is solving an AK model in per-capita terms. So does the de-centralized version of this economy have a BGP and if it does how would it look like? Assuming CRRA preferences without leisure and , we can derive the BGP condition and pin down the growth rate from the euler equation of a typical household,

$$1 = \beta\gamma^{-\sigma}(1 + r) \tag{102}$$

where $\gamma = \dfrac{c_{t+1}}{c_t}$ is the growth rate at the balanced path as usual and r=MP$_k$. So to find out the marginal product of capital for the firm we differentiate the technology w.r.t. k_t,

$$1 + r_t = \alpha AK_t^{1-\alpha} k_t^{\alpha-1} n_t^{1-\alpha} + (1 - \delta) \tag{103}$$

and since the prices are determined by aggregate state variables $K_t = k_t$ gives,

$$A\alpha - \delta = r \tag{104}$$

and substituting this into the euler equation we get the growth rate of consumption.

$$[(A\alpha - \delta + 1)\beta]^{\frac{1}{\sigma}} = \gamma \tag{105}$$

Solving the AK problem the SP faces we can verify the optimal growth rate for consumption is,

$$[(A - \delta + 1)\beta]^{\frac{1}{\sigma}} = \gamma^{sp}. \tag{106}$$

The important properties of the decentralized model are,

1. It is sub-optimal due to firms' unawareness of the externality they are facing and thus have lower growth rate.

2. Once again, the rental rate do not depend on the capital stock (due to the linear technology in aggregate the state variable capital stock drops out of the euler equation) and there is no transitional dynamics generated by the model.

To sum up what we have done so far, we have started with models that had exogenous growth and saw that we can make these models look an behave like our NGM after appropriate transformation. Then we went on to look at the models that generate growth endogenously and saw that a prerequisite for growth in these models is linearity of the technology in reproducible factors. We looked at the simple AK model, where the technology is linear in capital stock and analyzed the BGP of such an economy. Then we looked at Lucas' human capital model, in which we had two forms of capital, human and physical, both of which are reproducible in terms of output. Then we analyzed the model by Romer, which again has linearity in the reproducible factor at the aggregate level (capital stock), but firms were facing the CRTS technology with diminishing marginal return on capital and not aware of the positive externality they face. Next we will see another model by Romer with monopolistic competition and a R&D sector which can generate endogenous growth.

8.2.4 Monopolistic Competition, Endogenous Growth and R&D

Romer's monopolistic competition model has three production sectors, the final goods production, intermediate goods production and R&D i.e. variety production. Our usual TFP parameter in production function will represent the 'variety' in production inputs and as we will see the growth of varieties through research and development firms will make sure a balanced growth path is sustainable. The production function in this economy is,

$$Y_t = L_{1t}^{\alpha} \int_0^{A_t} x_t(i)^{1-\alpha} di \tag{107}$$

where $x_t(i)$ is the type i intermediate good and there is a measure A_t of different intermediate goods and L_{1t} is the amount of labor allocated to the final good production. The production function exhibits CRTS. The intermediate goods are produced with the following linear technology,

$$\int_0^{A_t} \eta x_t(i) di = K_t \tag{108}$$

Now suppose the variety of goods grows at rate γ, $A_{t+1} = \gamma A_t$, is long run sustainable growth possible? The answer to this question will depend whether our final goods production technology is linear in growing terms. We do know by the curvature of the technology, optimality implies equal amount of each variety will be used in production, $x_t(i) = x_t$,then we have,

$$A_t \eta x_t = K_t \tag{109}$$

and our output at this equal variety becomes,

$$Y_t = L_{1t}^{\alpha} A_t {x_t}^{1-\alpha} \tag{110}$$

then substituting for x_t we have,

$$Y_t = \frac{L_{1t}^{\alpha}}{\eta^{1-\alpha}} A_t^{\alpha} K_t^{1-\alpha} \tag{111}$$

thus if both A_t and K_t are growing at rate γ, then production function is linear in growing terms and long run balanced growth is feasible. Note that this model becomes very similar to our previous exogenous labor productivity growth under these assumptions. The purpose of this model is to determine γ endogenously. What will be the source of growth, where does γ come from? As we will see, there will be incentives for R&D firms to produce new 'varieties' because there will be a demand for it. These new varieties will be patented to intermediate good production firms, where a patent will mean exclusive rights to produce that intermediate good. So we will have monopolistic competition in the intermediate goods production. Now suppose the law of motion for 'varieties', which is the technology in R&D sector is given by,

$$A_{t+1} = (1 + L_{2t}\zeta)A_t \tag{112}$$

where L_{2t} is the labor employed in R&D sector. Note that this is not a regular law of motion in the sense every new variety produced helps the production of further new varieties.such that there is a positive externality to variety production. Also assume leisure is not valued and we have aggragate feasibility condition for labor as,

$$L_{2t} + L_{1t} = 1 \tag{113}$$

As a homework, we have calculated the BG rate of SP version of this economy, now we will de-centralize this economy and characterize the equlibrium growth rate.and see that it is sub-optimal. The period t problem of a firm in the competitive final good production sector is,

$$\max_{x_t(i), L_{1t}} \{L_{1t}^{\alpha} \int_0^{A_t} x_t(i)^{1-\alpha} di - w_t L_{1t} - \int_0^{A_t} q_t(i) x_t(i) di\} \tag{114}$$

and since we have CRTS with perfect competition we have zero profit with following FOCs,

$$\begin{aligned} w_t &= \alpha L_{1t}^{\alpha-1} \int_0^{A_t} x_t(i)^{1-\alpha} di & (115) \\ q_t(i) &= (1-\alpha) L_{1t}^{\alpha} x_t(i)^{-\alpha} & (116) \end{aligned}$$

notice that the inverse demand function for good of variety i is,

$$\left(\frac{q_t(i)}{(1-\alpha)L_{1t}^{\alpha}}\right)^{\frac{-1}{\alpha}} = x_t(i) \tag{117}$$

The intermediate goods industry will be monopolistic competition, in which there is only one firm, that is one patent holder, producing each variety. Each firm takes the demand of its variety and prices as given, and solves the following problem each period,

$$\begin{aligned} \Pi_t(i) &= \max_{x_t(i), K_t(i)} \{q_t(i)x_t(i) - R_t K_t(i)\} & (118) \\ s.t. \quad x_t(i) &= \frac{K_t(i)}{\eta} \end{aligned}$$

plugging in the inverse demand function and the technology constraint, the FOC is,

$$(1-\alpha)^2 x_t(i)^{-\alpha} L_{1t} = R_t \eta \tag{119}$$

and because of the symmetry we mentioned $x_t(i) = x_t = \frac{K_t}{\eta A_t}$ we can write this FOC as,

$$(1-\alpha)^2 (\frac{K_t}{\eta A_t})^{-\alpha} L_{1t} = R_t \eta \tag{120}$$

i.e. the rental price of capital is not equal to it's marginal product and there is opportunuties for positive profit. But also remember there is a fixed cost of entering to this industry, namely the price paid for the patent. Then as we will see the relation between the two will be one of our equilibrium conditions. Now lets look at the problem of R&D firms,

$$\max_{A_{t+1}, L_{2t}} \{p_t^P (A_{t+1} - A_t) - w_t L_{2t}\} \tag{121}$$

$$s.t. A_{t+1} = (1 + L_{2t}\zeta)A_t$$

where p_t^P is the patent of the price. Free entry is assumed thus there will be zero profit in equilibrium. Notice also the R&D firm is solving a static problem without realizing the positive externality this period's decision creates on next periods production. As we will see, this and the monopoly power of the patent owners will be the sources of sub-optimality in decentralized solution.The FOC is,

$$p_t^P = \frac{w_t}{\zeta A_t} \tag{122}$$

where wage is determined in the final goods market and given this price equilibrium quantity will come from the deman function. As we mentioned before, one equilibrium condition will be that at any point in time, total profit a patent generates will be equal to price of it such that there will also be zero profit in the intermediate goods market.

$$p_t^P = \sum_{\tau=t}^{\infty} \frac{\Pi_t(i)}{(1+r)^{\tau-t}} \tag{123}$$

These conditions with constant growht equations for the growing variables is sufficient to characterize the equilibrium growth rate of this economy.

8.2.5 MONOPOLISTIC COMPETITION ENDOGENOUS GROWTH AND R&D(Repeated)

Romer's monopolistic competition model has three production sectors, the final goods production, intermediate goods production and R&D i.e. variety production. Our usual TFP parameter in production function will represent the 'variety' in production inputs and as we will see the growth of varieties through research and development firms will make sure a balanced growth path is sustainable. The production function in this economy is,

$$Y_t = L_{1t}^{\alpha} \int_0^{A_t} x_t(i)^{1-\alpha} di$$

where $x_t(i)$ is the type i intermediate good and there is a measure A_t of different intermediate goods and L_{1t} is the amount of labor allocated to the final good production. The production function exhibits CRTS. The intermediate goods are produced with the following linear technology,

$$\int_0^{A_t} \eta x_t(i) di = K_t$$

Now suppose the variety of goods grows at rate γ, $A_{t+1} = \gamma A_t$, is long run sustainable growth possible? The answer to this question will depend whether our final goods production technology is linear in growing terms. We do know by the curvature of the technology, optimality implies equal amount of each variety will be used in production, $x_t(i) = x_t$,then we have,

$$A_t \eta x_t = K_t$$

and our output at this equal variety becomes,

$$Y_t = L_{1t}^{\alpha} A_t {x_t}^{1-\alpha}$$

then substituting for x_t we have,

$$Y_t = \frac{L_{1t}^{\alpha}}{\eta^{1-\alpha}} A_t^{\alpha} K_t^{1-\alpha}$$

thus if both A_t and K_t are growing at rate γ, then production function is linear in growing terms and long run balanced growth is feasible. Note that this model becomes very similar to our previous exogenous labor productivity growth under these assumptions. The purpose of this model is to determine γ endogenously. What will be the source of growth, where does γ come from? As we will see, there will be incentives for R&D firms to produce new 'varieties' because there will be a demand for it. These new varieties will be patented to intermediate good production firms, where a patent will mean exclusive rights to produce that intermediate good. So we will have monopolistic competition in the intermediate goods production. Now suppose the law of motion for 'varieties', which is the technology in R&D sector is given by,

$$A_{t+1} = (1 + L_{2t}\zeta)A_t$$

where L_{2t} is the labor employed in R&D sector. Note that this is not a regular law of motion in the sense every new variety produced helps the production of further new varieties.such that there is a positive externality to variety production. Also assume leisure is not valued and we have aggragate feasibility condition for labor as,

$$L_{2t} + L_{1t} = 1$$

As a homework, we have calculated the BG rate of SP version of this economy, now we will decentralize this economy and characterize the equlibrium growth rate.and see that it is sub-optimal. The period t problem of a firm in the competitive final good production sector is,

$$\max_{x_t(i),L_{1t}}\{L_{1t}^{\alpha}\int_0^{A_t} x_t(i)^{1-\alpha}di - w_tL_{1t} - \int_0^{A_t} q_t(i)x_t(i)di\}$$

and since we have CRTS with perfect competition we have zero profit with following FOCs,

$$w_t = \alpha L_{1t}^{\alpha-1}\int_0^{A_t} x_t(i)^{1-\alpha}di$$

$$\text{and} \quad q_t(i) = (1-\alpha)L_{1t}^{\alpha}x_t(i)^{-\alpha}$$

Notice that the inverse demand function for good of variety i is,

$$\left(\frac{q_t(i)}{(1-\alpha)L_{1t}^{\alpha}}\right)^{\frac{-1}{\alpha}} = x_t(i)$$

The intermediate goods industry will operate under monopolistic competition. There is only one firm, that is one patent holder, producing each variety. Each firm takes the demand of its variety and prices as given, and solves the following problem each period,

$$\Pi_t(i) = \max_{x_t(i),K_t(i)}\{q_t(i)x_t(i) - R_tK_t(i)\}$$

$$s.t. \quad x_t(i) = \frac{K_t(i)}{\eta}$$

plugging in the inverse demand function and the technology constraint, the FOC is,

$$(1-\alpha)^2x_t(i)^{-\alpha}L_{1t} = R_t\eta$$

and because of the symmetry we mentioned $x_t(i) = x_t = \frac{K_t}{\eta A_t}$ we can write this FOC as,

$$(1-\alpha)^2(\frac{K_t}{\eta A_t})^{-\alpha}L_{1t} = R_t\eta$$

i.e. the rental price of capital is not equal to it's marginal product and there is opportunuties for positive profit. But also remember there is a fixed cost of entering to this industry, namely the price paid for the patent. Then as we will see the relation between the two will be one of our equilibrium conditions. Now lets look at the problem of R&D firms,

$$\max_{A_{t+1},L_{2t}}\{p_t^P(A_{t+1} - A_t) - w_tL_{2t}\}$$

$$s.t. A_{t+1} = (1 + L_{2t}\zeta)A_t$$

where p_t^P is the patent of the price. Free entry is assumed thus there will be zero profit in equilibrium. Notice also the R&D firm is solving a static problem without realizing the positive externality this period's decision creates on next periods production. As we will see, this and the monopoly power of the patent owners will be the sources of sub-optimality in decentralized solution.The FOC is,

$$p_t^P = \frac{w_t}{\zeta A_t}$$

where wage is determined in the final goods market and given this price equilibrium quantity will come from the deman function. As we mentioned before, one equilibrium condition will be that at any point in time, total profit a patent generates will be equal to price of it such that there will also be zero profit in the intermediate goods market.

$$p_t^P = \sum_{\tau=t}^{\infty} \frac{\Pi_t(i)}{(1+r)^{\tau-t}}$$

These conditions with constant growht equations for the growing variables is sufficient to characterize the equilibrium growth rate of this economy. For more details see Problem Set 9.

9 Optimal unemployment insurance

9.1 Observable effort

Consider an economy where the probability of finding a job $p(a)$ is a function of effort $a \in [0,1]$. And we assume that once the agent gets a job, she will have wage w for ever. Thus, the individual problem is

$$\max_{a_t} E \sum_t \beta^t \left[u(c_t) - a_t\right]$$

There are two cases: when the agent has got a job, she will pay no effort and enjoy w for ever. The life long utility is

$$V^E = \sum_t \beta^t u(w) = \frac{u(w)}{1-\beta} \tag{124}$$

When the agent is still unemployed, she will have nothing to consumer. Her problem is

$$V^u = \max_a \left\{u(0) - a + \beta\left[p(a) V^E + (1 - p(a) V^{u\prime})\right]\right\} \tag{125}$$

If the optimal solution of a is interior, $a \in (0,1)$, then the first order condition gives

$$-1 + \beta p'(a)\left(V^E - V^u\right) = 0 \tag{126}$$

And since the V^u is stationary,

$$V^u = \max_a \left\{u(0) - a + \beta\left[p(a) V^E + (1 - p(a) V^u)\right]\right\} \tag{127}$$

Solving (126)(127) gives the optimal a and V^u. Another way is to successively substitute a and obtain solution because (**??**) defines a contraction mapping operator. We can fix V_0^u, then solve (127) to get $a(V_0^u)$ and obtain V_1^u. Keeping going until $V_n^u = V_{n+1}^u$. In a word, optimal effort level a^* solves (127) with $V^u = V^{u\prime}$.

The probability of finding a job $p(a)$ is called hazard rate. If agents did not find a job with effort level a^*, next period, she will still execute the same effort level a^*. Why? Because the duration of unemployment is not state variable in agent's problem. (If agents do not have enough realization about the difficulty of getting a job. With learning, their effort a will increase as they revise their assessment of the difficulty. But such revision of belief is not in this model.)

Now suppose resource is given to people who is unemployed to relive her suffering by a benevolent planner. This planner has to decide the minimal cost of warranting agent a utility level V: $c(V)$. To warrant utility level V, the planner tells the agent how much to consume, how much effort to exert and how much utility she will get if she stay unemployed next period. Obviously, the cost function $c(V)$ is increasing in V.

The cost minimization problem of the planner can be written in the following recursive problem:

$$c(V) = \min_{c,a,V^u} c + [1 - p(a)] \frac{1}{1+r} c(V^u) \tag{128}$$

subject to

$$V = u(c) - a + \beta \left[p(a) V^E + (1 - p(a)) V^u\right] \tag{129}$$

To solve the problem, construct Lagragian function

$$\mathcal{L} = c + [1 - p(a)] \beta c(V^u) + \theta \left[V - u(c) + a - \beta \left[p(a) V^E + (1 - p(a)) V^u\right]\right]$$

FOC: (c)

$$\theta = \frac{1}{u_c} \tag{130}$$

(a)

$$c(V^u) = \theta \left[\frac{1}{\beta p'(a)} - \left(V^E - V^u\right)\right] \tag{131}$$

(V^u)

$$c'(V^u) = \theta \tag{132}$$

Envelope condition

$$c'(V) = \theta \tag{133}$$

We will work on some implication of these conditions:

1. Compare (131) and (126), we can see that the substitution between consumption and effort is different from the one in agent's problem without unemployment insurance. This is because the cost of effort is higher for work that it is from the viewpoint of planner.

2. (132) tells us that the marginal cost of warranting an extra unit of utility tomorrow is θ.,provided that tomorrow V^u is optimally chosen when today's promise is V. And (133) tells us that the marginal cost of warranting an extra unit of V today is θ.

3. Given that c is strictly concave, $V = V^u$.

4. Regardless of unemployment duration, $V = V^u$. So, effort required the the planner is the same over time. Hazard rate is still constant.

Next , we will study the case when effort is not observable. Planner can only choose consumption and V^u. Effort level is chosen optimally by worker and it is unobservable.

Now suppose there is a social planner who will warrant utility level V for unemployed agent, where V summarize all the past information. The cost minimization problem is

$$c(V) = \min_{c,a,V^u} c + [1 - p(a)] \beta c(V^u) \tag{134}$$

subject to

$$V = u(c) - a + \beta \left[p(a) V^E + (1 - p(a)) V^u\right] \tag{135}$$

FOC: (c)

$$\theta = \frac{1}{u_c} \tag{136}$$

(a)

$$c(V^u) = \theta \left[\frac{1}{\beta p'(a)} - \left(V^E - V^u\right)\right] \tag{137}$$

(V^u)

$$c'(V^u) = \theta \tag{138}$$

Envelope condition

$$c'(V) = \theta \tag{139}$$

The optimal promise for tomorrow is $V^u(V)$. Now, let's work out the property of $V^u(.)$.

Lemma 28 *If $V > V^A$, then $c(V) > 0$, where V^A is the utility for unemployed agent when they are in autarky..*

The intuition for the lemma is that if the planner promises the agent something more than what agent can achieve by herself, it will cost the planner something because the planner cannot do anything more than what people can do on their own.

Lemma 29 *Lagrangian multiplier $\theta > 0$.*

The second lemma tells us that if the planner promise more, she has to pay more.

In the problem without unemployment insurance, (**??**) implies that

$$\frac{1}{\beta p'(a)} = V^E - V^u$$

In the planner's problem, since $c(V) > 0$, (137) implies that the effort level chosen by the planner are different from agent's choice in autarky. The reason is that effort does not cost that much in planner's thought.

9.2 Unobservable effort

When a is not observable, planner can only choose c and V^u. And households choose a optimally. Now it becomes a principle-agent problem. We will solve the problem backward.

If given c and V^u, the agent will solve

$$\max_a u(c) - a + \beta \left[p(a) V^E + (1 - p(a)) V^u\right] \tag{140}$$

FOC is

$$\left[p'(a)\beta\right]^{-1} = V^E - V^u \tag{141}$$

This FOC gives an implicit function of a as a function of V^u: $a = g(V^u)$. (Because c and a are separate in the utility function, a is not a function of c).

Then, the planner solve her cost minimization problem, in which the optimality condition is also one constraint.

$$c(V) = \min_{c,a,V^u} c + [1 - p(a)] \beta c(V^u)$$

subject to

$$\begin{aligned} V &= u(c) - a + \beta \left[p(a) V^E + (1 - p(a)) V^u\right] && (142) \\ 1 &= \left[p'(a)\beta\right]\left[V^E - V^u\right] && (143) \end{aligned}$$

Lagragian is

$$\begin{aligned} &c + [1 - p(a)] \beta c(V^u) + \theta \left[V - u(c) + a - \beta \left[p(a) V^E + (1 - p(a)) V^u\right]\right] \\ &+\eta \left[1 - \left[p'(a)\beta\right]\left[V^E - V^u\right]\right] \end{aligned}$$

FOC: (c)

$$\theta^{-1} = u_c$$

(a)

$$c(V^u) = \theta \left[\frac{1}{\beta p'(a)} - \left(V^E - V^u\right)\right] - \eta \frac{p''(a)}{p'(a)} \left(V^E - V^u\right) \tag{144}$$

(V^u)

$$c'(V^u) = \theta - \eta \frac{p'(a)}{1 - p(a)} \tag{145}$$

Envelope condition

$$c'(V) = \theta \tag{146}$$

Again, (145) tells the marginal cost to warrant additional amount of delayed promise. (146) gives the marginal cost to increase today's utility. The Lagrangian multiplier associated with constraint (143) is positive, $\eta > 0$, which means that the constraint is binding. So,

$$\eta \frac{p'(a)}{1 - p(a)} > 0$$

Therefore, we have

$$c'(V^u) < c'(V) \Rightarrow V^u < V$$

from the strict concavity of $c(.)$. The delayed promised utility decreases over time.

Let $\theta^u = \theta - \eta \frac{p'(a)}{1-p(a)}$, then $\theta^u < \theta$,which tells us about the consumption path. Consumption decreases over time because $\theta^{-1} = u_c$.

Overall, we get the following model implications: optimal unemployment insurance says that longer unemployment period the agent stays, the less insurance she will be insured for. In this way, the planner induces the higher effort level. Although you cannot let people do what is optimal, such behavior can be achieved by giving out less consumption and promised utility over time. This model implies that time-varying unemployment insurance plan is optimal, under which the replacement rate θ goes down over time.

10 Models with Lack of Commitment

10.1 One sided lack of commitment

We will study a model with one-said lack of commitment. This is an endowment economy (no production). There is no storage technology. Consider the village of fisherladies, where young granddaughters receive $y_s \in \{y_1, y_2, ..., y_S\}$ every period. y is iid. The probability that a certain y_s realized is Π_s. h_t is a history of shocks up to period t, i.e. $h_t = \{y_0,\, y_1,\, y_2,\, ...,\, y_t\}$.

First, if the granddaughter stays autarky, she will enjoy total utility,

$$V_{AUT} = \sum_{t=0}^{\infty} \beta^t \sum_s \Pi_s u(y_s) = \frac{\sum_s \Pi_s u(y_s)}{1 - \beta}$$

Note that here V^A is the utility of the young lady before endowment shock is realized.

Now we assume that the grandmother offers a contract to the granddaughter, which transfer resources and provide insurance to her. Grandmother can commit. But the young granddaughter may leave grandmother and break her word. Thus, this model is one-sided commitment model: an agent can walk away from a contract but the other cannot. Therefore, the contract should be always in the interest of granddaughter for her to stay.

We define a contract $f_t : H_t \to c \in [0, \tau]$. We will see next class that incentives compatibility constraint requires that at each node of history H_t, the contract should guarantee a utility which is higher than that in autarky.

Notice that the problem is different from Lucas tree model because of the shock realization timing. In Lucas tree model, shock is state variable because action takes place after shock is realized. Thus, action is indexed by shock. Here action is chosen before shock realization. Therefore, shock is not a state variable and action is state contingent.

In Lucas tree model, $V(s) = \max_c u(c) + \beta \sum_{s'} \Pi_{ss'} V(s')$. Here, if we write the problem recursively, it is $V = \max_{c_s} \sum_s \Pi_s u(c_s) + \beta V$.

Remember, the grandmother will make a deal with her granddaughter. They sign a contract to specify what to do in each state. $h_t \in H_t$. Contract is thus a mapping $f_t(h_t) \to c(h_t)$. With this contract, granddaughter gives y_t to the grandmother and receives $c_t = f_t(h_{t-1}, y_t)$. But if the granddaughter decided not to observe the contract, she consumes y_t this period and cannot enter a contract in the future, i.e. she has to live in autarky in the future.

For grandmother to keep granddaughter around her, the contract has to be of interest to granddaughter because although grandmother keeps her promise, granddaughter does not. There are two possible outcome if this contract is broken. One is that granddaughter goes away with current and future endowment. The other is that they renegotiate. We ignore the second possibility as no renegotiation is allowed. But we need deal with the possibility that the granddaughter says no to the contract and steps away.

The first best outcome is to warrant a constant consumption c_t to granddaughter who is risk averse. But because of the one-side lack of commitment, the first best is not achievable. The contract should always be attractive to granddaughter, otherwise, when she gets lucky with high endowment y_s, she will feel like to leave. So, this is a dynamic contract problem which the grandmother will solve in order to induce good behavior from granddaughter. The contract is dynamic because the nature keeps moving.

We say the contract $f_t(h_t)$ is incentive compatible or satisfies participation constraint if for all h_t,

$$u(f_t(h_t)) + \sum_{\tau=1}^{\infty} \beta^{\tau} \sum_s \Pi_s u(f_{t+\tau}(h_{t+\tau})) \geq u(y_s(h_t)) + \beta V^A \tag{147}$$

The left hand side is utility guaranteed in the contract. And the right hand side is the utility that granddaughter can get by herself. The participation constraint is not binding if y_s is low. And when y_s is high, PC is binding.

10.1.1 Problem of the grandmother

In this model, problem of the grandmother is to find an optimal contract that maximizes the value of such a contract of warranting V to her. We define the problem using recursive formula. Firstly, let's define the value of contract to grandmother if she promised V to her granddaughter by $\Omega(V)$. $\Omega(V)$ can be defined recursively as the following:

$$\Omega(V) = \max_{\{c_s, \omega_s\}_{s=1}^{S}} \sum_s \Pi_s [(y_s - c_s) + \beta \Omega(\omega_s)] \tag{148}$$

subject to

$$u(c_s) + \beta \omega_s \geq u(y_s) + \beta V^A \quad \forall s \tag{149}$$

$$\sum_s \Pi_s [u(c_s) + \beta \omega_s] \geq V \tag{150}$$

Notice that there are $1 + S$ constraints. The choice variables c_s, ω_s are state-contingent where ω_s is the promised utility committed to granddaughter in each state. In the objective function, $\sum_s \Pi_s (y_s - c_s)$ is the expected value of net transfer.

There are two sets of constraints. (149) is PC and (150) is promise keeping constraint.

The First Order Conditions to the grandmother's problem are:

(c_s)

$$\Pi_s = (\lambda_s + \mu \Pi_s) u'(c_s) \tag{151}$$

(ω_s)

$$-\Pi_s \Omega'(\omega_s) = \mu \Pi_s + \lambda_s \tag{152}$$

(μ)

$$\sum_s \Pi_s [u(c_s) + \beta \omega_s] = V \tag{153}$$

(λ)

$$u(c_s) + \beta \omega_s \geq u(y_s) + \beta V^A \tag{154}$$

In addition, Envelope Theorem tells that:

$$\Omega'(v) = -\mu \tag{155}$$

Interpreting the first order conditions:

1. (151) tells that in an optimal choice of c_s, the benefit of increasing one unit of c equals the cost of doing so. The benefit comes from two parts: first is $\mu\Pi_s u'(c_s)$ as increasing consumption helps grandmother to fulfill her promise and the second part is $\lambda_s u'(c_s)$ since increase in consumption helps alleviated the participation constraint. And the cost is the probability of state s occurs.

2. (152) equates the cost of increasing one unit of promised utility and the benefit. The cost to grandmother is $-\Pi_s\Omega'(\omega_s)$ and the benefit is $\mu\Pi_s + \lambda_s$ which helps grandmother deliver promise and alleviate participation constraint.

How about the contract value $\Omega(V)$. First, $\Omega(V)$ can be positive or negative.

Claim 30 *(1) There exits V such that $\Omega(V) > 0$*[8].

What's the largest V we will be concerned with? When PC will be binding for sure. If PC binds for the best endowment shock y_S, then PC holds for all the shock y_s. When granddaughter gets the best shock y_S, the best autarky value is then

$$V_{AM} = u(y_S) + \beta V_A$$

And the cheapest way to guarantee V_{AM} is to give constant consumption $\overline{c_S}$, such that

$$V_{AM} = \frac{u(c_S)}{1-\beta}$$

From this case, we can see that because of lack of commitment, the grandmother will have to give more consumption in some states. While when there is no lack of commitment, strict concavity of $u(.)$ implies that constant stream of consumption beats any $\{c_t\}$ that have the same present value, as there is no PC.

10.1.2 Characterizing the Optimal Contract

We will characterize the optimal contract by considering the two cases: (i) $\lambda_s > 0$ and (ii) $\lambda_s = 0$.

Firstly, if $\lambda_s = 0$, we have the following equations from FOC and EC:

$$\Omega'(\omega_s) = -\mu \tag{156}$$

$$\Omega'(V) = -\mu \tag{157}$$

Therefore, for s where PC is not binding,

$$V = \omega_s$$

c_s is the same for all s. For all s such that the Participation Constraint is not binding, the grandmother offers the same consumption and promised future value.

Let's consider the second case, where $\lambda_s > 0$. In this case, the equations that characterize the optimal contract are:

$$u'(c_s) = \frac{-1}{\Omega'(\omega_s)} \tag{158}$$

$$u(c_s) + \beta\omega_s = u(y_s) + \beta V^A \tag{159}$$

Note that this is a system of two equations with two unknowns (c_s and ω_s). So these two equations characterize the optimal contract in case $\lambda_s > 0$. In addition, we can find the following properties by carefully observing the equations:

1. The equations don't depend on V. Therefore, if a Participation Constraint is binding, promised value does not matter for the optimal contract.

[8]When $P(V)$ is positive, it shows that there is gains from trade.

2. From the first order condition with respect to ω_s, $\Omega'(\omega_s) = \Omega'(v) - \frac{\lambda_s}{\Pi_s}$, where $\frac{\lambda_s}{\Pi_s}$ is positive. Besides, we know that Ω is concave. This means that $v < \omega_s$. In words, if a Participation Constraint is binding, the moneylender promises more than before for future.

Combining all the results we have got, we can characterize the optimal contract as follows:

1. Let's fix V_0. We can find a $y_s(V_0)$, where for $\forall y_s \leq y_s(V_0)$, the participation constraint is not binding. And vice versa.

2. The optimal contract that the moneylender offers to an agent is the following:

   If $y_t \leq y_s(v_0)$, the moneylender gives $(v_0, c(v_0))$. Both of them are the same as in the previous period. In other words, the moneylender offers the agent the same insurance scheme as before.

   If $y_t > y_s(v_0)$, the moneylender gives $(v_1, c(y_s))$, where $v_1 > v_0$ and c doesn't depend on v_0. In other words, the moneylender promises larger value to the agent to keep her around.

   So the path of consumption and promised value for an agent is increasing with steps.

10.2 Two sided lack of commitment

10.2.1 The Model

- Two brothers, A and B, and neither of them has access to a commitment technology. In other words, the two can sign a contract, but either of them can walk away if he does not feel like observing it.
- This is an endowment economy (no production) and there is no storage technology. Endowment is represented by $(\mathrm{y}_s^A, y_s^B) \in Y \times Y$, where y_s^i is the endowment of brother i. s=(y_s^A, y_s^B) follows a Markov process with transition matrix $\Gamma_{ss'}$.

10.2.2 First Best Allocation

We will derive the first best allocation by solving the social planner's problem:

$$\max_{\{c_i(h_t)\}_{\forall h_t, \forall i}} \lambda^A \sum_{t=0}^{\infty} \beta^t \sum_{h_t} \Pi(h_t) u(c^A(h_t)) + \lambda^B \sum_{t=0}^{\infty} \beta^t \sum_{h_t} \Pi(h_t) u(c^B(h_t))$$

subject to the resource constraint:

$$\sum_i c^i(h_t) - y^i(h_t) = 0 \qquad \forall h_t \quad \text{w/ multiplier } \gamma(h_t)$$

The First Order Conditions are:

$$\begin{aligned} FOC(c^A(h_t)) &: \quad \lambda^A \beta^t \Pi(h_t) u'(c^A(h_t)) - \gamma(h_t) = 0 \\ FOC(c^B(h_t)) &: \quad \lambda^B \beta^t \Pi(h_t) u'(c^B(h_t)) - \gamma(h_t) = 0 \end{aligned}$$

Combining these two yields:

$$\frac{\lambda^A}{\lambda^B} = \frac{u'(c^A(h_t))}{u'(c^B(h_t))}$$

The first best allocation will not be achieved if there is no access to a commitment technology. Therefore, the next thing we should do is look at the problem the planner is faced with in the case of lack of commitment. Due to lack of commitment, the planner needs to make sure that at each point in time and in every state of the world, h_t, both brothers prefer what they receive to autarky. Now we will construct the problem of the planner adding these participation constraints to his problem.

10.2.3 Constrained Optimal Allocation

The planner's problem is:

$$\max_{c^A(h_t),c^B(h_t)} \lambda^A \sum_{t=0}^{\infty} \beta^t \sum_{h_t} \Pi(h_t)u(c^A(h_t)) + \lambda^B \sum_{t=0}^{\infty} \beta^t \sum_{h_t} \Pi(h_t)u(c^B(h_t))$$

$$\sum_i c^i(h_t) - y^i(h_t) = 0 \qquad \forall h_t \quad \text{w/ multiplier } \gamma(h_t)$$

$$\sum_{r=t}^{\infty} \beta^{r-t} \sum_{h_r} \Pi(h_r|h_t)u(c^i(h_r)) \geq \Omega_i(h_t) \qquad \forall h_t, \forall i \quad \text{w/ multiplier } \mu_i(h_t)$$

where $\Omega_i(h_t) = \sum_{r=0}^{\infty} \beta^{r-t} \sum_{h_r} \Pi(h_r|h_t)u(y_i(h_t))$ (the autarky value)

- How many times does c$^A(h_{17})$ appear in this problem? Once in the objective function, once in the feasibility constraint, and it appears in the participation constraint from period 0 to period 16.
- We know that the feasibility constraint is always binding so that $\gamma(h_t) > 0 \ \ \forall h_t$. On the other hand the same is not true for the participation constraint.
- Both participations cannot be binding but both can be nonbinding.
- Define M$_i(h_{-1}) = \lambda^i$
  and M$_i(h_t) = \mu_i(h_t) + M_i(h_{t-1})$
  (We will use these definitions for the recursive representation of the problem in the next class)

10.2.4 Recursive Representation of the Constrained SPP

We want to transform this problem into the recursive, because it would be easier to solve the optimal allocation with a computer. Now we will show how to transform the sequential problem with the participation constraints into its recursive representation.

Before we do this transformation, first recall the Lagrangian associated with the sequential representation of the social planner's problem:

$$\begin{aligned}
&\lambda^A \sum_{t=0}^{\infty} \beta^t \sum_{h_t} \Pi(h_t)u(c^A(h_t)) + \lambda^B \sum_{t=0}^{\infty} \beta^t \sum_{h_t} \Pi(h_t)u(c^B(h_t)) \\
&+ \sum_{t=0}^{\infty} \beta^t \sum_{h_t \in H_t} \Pi(h_t) \sum_{i=1}^{2} \mu_i(h_t) \left[\sum_{r=t}^{\infty} \beta^{r-t} \sum_{h_r} \Pi(h_r|h_t)u(c^i(h_r)) - \Omega_i(h_t) \right] \\
&+ \sum_{t=0}^{\infty} \sum_{h_t \in H_t} \gamma(h_t) \left[\sum_{i=1}^{2} c_i(h_t) - \sum_{i=1}^{2} y_i(h_t) \right]
\end{aligned}$$

Note that here the Lagrangian multiplier associated witih the participation constraint for brother i after history h$_t$ is $\beta^t \Pi(h_t)\mu_i(h_t)$.

Now we will use the definitions from the previous class (for M$_i(h_t)$) to rewrite the above Lagrangian in a more simple form,

Collect terms and rewrite,

$$\begin{aligned}
&\sum_{t=0}^{\infty} \beta^t \sum_{h_t} \Pi(h_t) \sum_i \left\{ \lambda^i u(c^i(h_t) + \mu_i(h_t) \left[\sum_{r=t}^{\infty} \beta^{r-t} \sum_{h_r} \Pi(h_r|h_t)u(c^i(h_r)) - \Omega_i(h_t) \right] \right\} \\
&+ \sum_{t=0}^{\infty} \sum_{h_t \in H_t} \gamma(h_t) \left[\sum_{i=1}^{2} c_i(h_t) - \sum_{i=1}^{2} y_i(h_t) \right]
\end{aligned}$$

Note that, $\sum_{r=t}^{\infty}\beta^{r-t}\sum_{h_r}\Pi(h_r|h_t)u(c^i(h_r))-\Omega_i(h_t) = u(c^i(h_t))+\sum_{r=t+1}^{\infty}\beta^{r-t}\sum_{h_r}\Pi(h_r|h_t)u(c^i(h_r))-\Omega_i(h_t)$,

and that $\Pi(h_r|h_t)\Pi(h_t) = \Pi(h_r)$ so using these, rewrite as,

$$\sum_{t=0}^{\infty}\beta^t\sum_{h_t}\Pi(h_t)\sum_{i}\left\{\lambda^i u(c^i(h_t) + \mu_i(h_t)u(c^i(h_t)\right\}$$
$$+\sum_{t=0}^{\infty}\sum_{h_r}\sum_{i}\mu_i(h_t)\left[\sum_{r=t+1}^{\infty}\beta^r\sum_{h_r}\Pi(h_r)u(c^i(h_r)) - \Omega_i(h_t)\right]$$
$$+\sum_{t=0}^{\infty}\sum_{h_t\in H_t}\gamma(h_t)\left[\sum_{i=1}^{2}c_i(h_t) - \sum_{i=1}^{2}y_i(h_t)\right]$$

Collect the terms of $u(c^i(h_r)$,

$$\sum_{t=0}^{\infty}\beta^t\sum_{h_t}\Pi(h_t)\sum_{i}\left\{\left[\lambda^i + \sum_{r=0}^{t-1}\mu_i(h_r)\right]u(c^i(h_t) + \mu_i(h_t)\left[u(c^i(h_t) - \Omega_i(h_t)\right]\right\}$$
$$+\sum_{t=0}^{\infty}\sum_{h_t\in H_t}\gamma(h_t)\left[\sum_{i=1}^{2}c_i(h_t) - \sum_{i=1}^{2}y_i(h_t)\right]$$

Introduce the variable $\mathrm{M}_i(h_t)$ and define it recursively as,

$$\begin{aligned} M_i(h_t) &= M_i(h_{t-1}) + \mu_i(h_t) \\ M_i(h_{-1}) &= \lambda^i \end{aligned}$$

where $\mathrm{M}_i(h_t)$ denotes the Pareto weight plus the cumulative sum of the Lagrange multipliers on the participation constraints at all periods from 1 to t.

So rewrite the Lagrangian once again as,

$$\sum_{t=0}^{\infty}\beta^t\sum_{h_t}\Pi(h_t)\sum_{i}\left\{M_i(h_{t-1})u(c^i(h_t) + \mu_i(h_t)\left[u(c^i(h_t) - \Omega_i(h_t)\right]\right\}$$
$$+\sum_{t=0}^{\infty}\sum_{h_t\in H_t}\gamma(h_t)\left[\sum_{i=1}^{2}c_i(h_t) - \sum_{i=1}^{2}y_i(h_t)\right]$$

Now we are ready to take the First Order Conditions:

$$\frac{u'(c^A(h_t))}{u'(c^B(h_t))} = \frac{M_A(h_{t-1}) + \mu_A(h_t)}{M_B(h_{t-1}) + \mu_B(h_t)}$$

$$\left[\sum_{r=t}^{\infty}\beta^{r-t}\sum_{h_r}\frac{\Pi(h_r)}{\Pi(h_t)}u(c^i(h_r)) - \Omega_i(h_t)\right]\mu_i(h_t) = 0$$

$$\sum_{i=1}^{2}c_i(h_t) - \sum_{i=1}^{2}y_i(h_t) = 0$$

10.2.5 Recursive Formulation

Our goal is make the problem recursive, which is very nice when we work with computer. To do this, we need to find a set of state variables which is sufficient to describe the state of the world. We are going to use x as a state variable.So the state variables are the endowment: $y = (y^A, y^B)$ and weight to brother 2: x. Define the value function as follows:

$$V = \{(V_0, V_A, V_B) \text{ such that } V_i : X \times Y \to \mathcal{R},\ i = 1, 2, V_0(x, y) = V_A(x, y) + xV_B(x, y)\}$$

What we are going find is the fixed point of the following operator (operation is defined later):

$$T(V) = \{T_0(V), T_1(V), T_2(V)\}$$

Firstly, we will ignore the participation constraints and solve the problem:

$$\max_{c_A, c_B} u(c^A(y,x)) + xu(c^B(y,x)) + \beta \sum_{y'} \Gamma_{yy'} V_0(y', x)$$

subject to

$$c^A + c^B = y^A + y^B$$

First Order Conditions yield:

$$\frac{u'(c_A)}{u'(c_B)} = x$$

Second, we will check the participation constraints. There are two possibilities here:

1. Participation constraint is not binding for either 1 or 2. Then set $x(h_t) = x(h_{t-1})$. In addition,

$$\begin{aligned} V_0^N(y,x) &= V_0(y,x) \\ V_i^N(y,x) &= u(c^i(y,x)) + \beta \sum_{y'} \Gamma_{yy'} V_i(y', x) \end{aligned}$$

2. Participation constraint is not satisfied for one of the brothers (say A).

This means that agent A is getting too little. Therefore, in order for the planner to match the outside opportunity that A has, he needs to change x so that he guarantees person A the utility from going away. We need to solve the following system of equations in this case:

$$\begin{aligned} c^A + c^B &= y^A + y^B \\ u(c^A) + \beta \sum_{y'} \Gamma_{yy'} V_A(y', x) &= u(y_A) + \beta \sum_{y'} \Gamma_{yy'} \Omega_A(y') \\ x' &= \frac{u'(c_A)}{u'(c_B)} \end{aligned}$$

This is a system of three equations and three unknowns. Denote the solution to this problem by,

$$\begin{gathered} c^A(y,x) \\ c^B(y,x) \\ x'(y,x) \end{gathered}$$

So that,

$$\begin{aligned} V_0^N(y,x) &= V_A^N(y,x) + xV_B^N(y,x) \\ V_i^N(y,x) &= u(c^i(y,x)) + \beta \sum_{y'} \Gamma_{yy'} V_i(y', x'(y,x)) \end{aligned}$$

Thus we have obtained T(V)=V^N. And the next thing we need to do is find V* such that T(V*) = V^*.

Final question with this model is "how to implement this allocation?" or "Is there any equilibrium that supports this allocation?". The answer is yes. How? Think of this model as a repeated game. And define the strategy as follows: keep accepting the contract characterized here until the other guy walks away. If the other guy walks away, go to autarky forever. We can construct a Nash equilibrium by assigning this strategy to both of the brothers.

10.3 Variations of NGM

As you will see as you go along, our benchmark Neo-classical Growth model has hardtime replicating certain aspects of data and not usable for certain questions of interest such as monetary policy. In practice lots of variations and extensions of the benchmark model is utilized depending on the question being worked on. We will cover some of these extensions here.

10.3.1 Adjustment costs in investment

Behaviour of investment in our model seems to be somewhat smoother than that is observed in data. Our benchmark model produces too smooth investment (less volatile than data which seems to happen in spurts) relative to the output variability. One way to think about a cause of this observed fact is possibility of adjustment costs in investment which introduces some inertia to investment behaviour of firms. One way to introduce, suppose the law of motion for capital is,

$$k_{t+1} = (1-\delta)k_t + \varphi(x_t, k_t)$$

where x_t is investment. If we assume φ is CRTS then,

$$k_{t+1} = (1-\delta)k_t + x_t\varphi(\frac{k_t}{x_t})$$

and $\varphi' > 0$, $\varphi(\delta) = 1$ i.e. the adjustment cost is zero at the steady state level of investment. Another way would be to introduce a concave transformation technology between consumption and investment goods instead of the linear technology in the benchmark model. The problem with this approach is it is in odds with data.

10.3.2 Variable capital utilization

Benchmark model assumes in equilibrium existing capital is fully utilized. An extension is variable capital utilization would generate a smoother interest rate behaviour relative to benchmark model which seems to be generate excess volatility in the interest rate. Formally, the production function and the law of motion of capital becomes,

$$\begin{aligned} y_t &= z_t(i_t k_t)^\alpha n_t^{1-\alpha} \\ k_{t+1} &= (1-\delta(i))k_t + x_t \end{aligned}$$

where i is the intensity of capital usage.

10.3.3 Habits in consumption

Another aspect of the benchmark model is relatively more volatile consumption than data. One way to model consumption which might generate less volatility is introduction of habits in consumption. There are several ways to do this. 'Hangover effect' would be a utility function in which utility depends negatively on the previous period's consumption. A more subtle way to model is accumulation of habits over time, in which habit stock enters negatively to the utility function and current stock of habits and consumption has a positive marginal effect on habit accumulation,

$$\begin{aligned} & U(c_t, h_t) \\ h_t &= \varphi(c_{t-1}, h_{t-1}) \end{aligned}$$

One natural way to introduce consumption inertia would be through introduction of durables which enters the utility as a stock variable and provide consumption services over time.

All these variations had 'own' consumption as the driving force of habits. Another way would be 'external habits' or 'keeping up/back with the Jonesses', in which utility depends on current or past aggragate consumption level.

10.3.4 Introducing Money

The simplest way to introduce money is basically putting in the utility function in an ad hoc fashion. An indirect way of doing it is imposing a cash in advance constraint to force the households to hold money. Both of these supplies a framework which allows analysis of monetary policy on real variables. Of course any possibility of non-neutrality of money requires some nominal rigidity in some part of the model.

11 Overlaping Genarations Models

So far we have been utilizing the NGM with inifinite horizon with no demographic details. This was not because we did not have the tools to consider a finite horizon. Many interesting questions in macroeconmics should be approached within a framework where these demographic details matter. Suppose that agents can live up to period I. Agents are born with zero asset and can save. The rate of return from saving is assumed to be $[R]$. The wage rate per efficiency unit is w. Agents have a limited amount of time (normalized to one) and can allocate the time to either (i) work or (ii) enjoy leisure. Efficiency units which an agent can supply by working for a unit time changes as the agent grows older. This is captured by ε_i, where i is the age of the agent. The problem of the agent is as follows:

$$\max_{\{a_{i+1},c_t,n_i\}_{i=1}^I} \sum_{i=1}^{I} \beta_i(u_i(c_i)+\varphi(1-n_i)) \tag{160}$$

subject to

$$\begin{aligned} a_1 &= 0 && (161)\\ a_{i+1}+c_i &= a_i[R]+w\varepsilon_i[1-n_i] && (162) \end{aligned}$$

Notice that β_i is not β^i. Time discount factor can be different according to age. Maybe young agents discount future more (NOW is the important time for the young) and adult agents discount future less (considering the future more than kids). Different β_i can capture these.

11.1 Labor Earnings

What is a good theory on ε? If we look at the average wage per hour at the different age ($w\varepsilon_i$), the wage per hour increases with age, peaks at around 40, and slowly decreases until the retirement. Since w is assumed to be same for all agents, we need a theory that explains the difference in ε to replicate the hump shape of the average wage profile. What kind of theory do we have? There are two ways, in general:

1. Take $\{\varepsilon_i\}$ as exogenous; i.e., assuming that the young agents are useless because they are young.
2. Human capital theory. Assume that the difference in capital stock between the young agents and the old agents yields the difference in ε. There are three branches:
   (a) Learning-by-doing: assume that agents accumulate human capital (ε) by working. Agents learn something which enhances their human capital stock while they are working. Imagine an interns of doctor. The young doctors learn how to do operations by actually working at hospitals. This idea is represented by:
   $$\varepsilon_{i+1}=\varphi_i(\varepsilon_i,n_i)$$
   where n_i is hours worked of agents of age i. φ is indexed by i because learning ability can be different depending on age.
   (b) Learning-by-not-doing: assume that agents accumulate human capital by actually learning (which is different from working or enjoying leisure). This idea is represented by:
   $$\varepsilon_{i+1}=\varphi_i(\varepsilon_i,l_i)$$
   where l_i is the time spent on learning, which is different from working or enjoying leisure. Agents allocate their time in learning to accumulate human capital.

(c) Education: the difference from learning models above is that most of education is acquired in the early stage of life. Keane and Ken Wolpin (REStat1994)[9] showed that 90% of people's fate is determined before age 16, by using structurally estimated model of the career choice.

11.2 Constructing Recursive Problem (1): Stationary Equilibrium

Let's define a stationary equilibrium in the recursive way for a model of learning by not doing. Stationary equilibrium means that the prices: r and w do not change over time. Firstly, let's define the problem of an agent of age i,conditional on $\mathcal{K}$, which is a capital labor ratio. Since r and w are functions of $\mathcal{K}$, we only need to record $\mathcal{K}$ instead of keeping track of prices. Individual agent's problem is:

$$V_i(a;\mathcal{K}) = \max_{c,n,a'} \{u(c, 1-n) + \beta V_{i+1}(a';\mathcal{K})\} \tag{163}$$

subject to

$$\begin{aligned} c + a' &= a[R(\mathcal{K})] + w(\mathcal{K}) n \varepsilon_i && (164) \\ \varepsilon_{i+1} &= \varphi_i(\varepsilon_i, l_i) && (165) \\ n &\in [0,1] && (166) \\ V_{I+1} &= 0 && (167) \\ a_1 &= 0 && (168) \end{aligned}$$

Solution of the problem is sequences $\{a_{i+1}, c_i, n_i\}_{i=1}^{I}$ Now we are ready to define a stationary equilibrium.

Definition 31 *A stationary equilibrium is a set of allocations $\{a^*_{i+1}, c^*_i, n^*_i\}_{i=1}^{I}$, a pair of prices r^* and w^*, and $\mathcal{K}$ such that:*

1. *(Agents' optimization) Given prices r^* and w^*, $\{a^*_{i+1}, c^*_i, n^*_i\}_{i=1}^{I}$, solves the optimization problem of agents.*
2. *(Firm's optimization) Prices r^* and w^* are determined competitively.*
3. *(Consistency)*

$$\frac{\sum_{i=1}^{I} a^*_i}{\sum_{i=1}^{I} \varepsilon_i n^*_i} = \mathcal{K}$$

11.3 Constructing Recursive Problem (2): Non-Stationary Equilibrium

Now consider a non-stationary version of this economy where there is a constant population growth and technology shocks. Note that the constant growth assumption is convinient since it ensures the demographic structure of the economy is stable over time.

$$\mu_i = (\frac{1}{1+g_p})^i$$

Let's define an equilibrium which is not restricted to stationary one. Prices can change over time. First define the vector of assets, indexed by age as, $\overrightarrow{A} = \{A_i\}_{i=2}^{I}$, then we know,

$$K = \sum_i \mu_i A_i$$

The recursive formulation of the agent's problem is as follows:

$$V_i(z, a, \overrightarrow{A}; G, H) = \max_{c,n,a'} \left\{ u(c, 1-n) + \beta E\{V_{i+1}(z, a', \overrightarrow{A}'; G, H) \mid z\} \right\} \tag{169}$$

[9]Keane , M., and Wolpin, K. (1994), "The Solution and Estimation of Discrete Choice Dynamic Programming Models by Simulation: Monte Carlo Evidence,' Review of Economics and Statistics, 76-4, 648-672.

subject to

$$
\begin{aligned}
c + a' &= a[R(z, \overrightarrow{A})] + w(z, \overrightarrow{A})n && (170)\\
n &\in [0,1] && (171)\\
V_{I+1} &= 0 && (172)\\
a_1 &= 0 && (173)\\
K_0 &= 0 && (174)\\
\overrightarrow{A}' &= G(z, \overrightarrow{A}) && (175)\\
N &= H(z, \overrightarrow{A}) && (176)
\end{aligned}
$$

The solution of this problem is

$$
\begin{aligned}
a_i' &= g_i(a, z, \overrightarrow{A}; G, H) && (177)\\
c_i &= c_i(a, z, \overrightarrow{A}; G, H) && (178)\\
n_i &= h_i(a, z, \overrightarrow{A}; G, H) && (179)
\end{aligned}
$$

Now we are ready to define a nonstationary recursive competitive equilibrium.

Definition 32 *A nonstationary equilibrium is a set of functions* $\{V_i^*(.),\ g_i^*(.),\ c_i^*(.),\ n_i^*(.)\}_{i=1}^{I}$, $G^*(z, \overrightarrow{A})$, $H^*(z, \overrightarrow{A})$, $r^*(z, \overrightarrow{A})$, $w^*(z, \overrightarrow{A})$ *such that:*

1. *(Agent's optimization) Given* $G^*(z, \overrightarrow{A})$, $H^*(z, \overrightarrow{A})$, $r^*(\overrightarrow{A}, z)$ $w^*(\overrightarrow{A}, z)$,$\{V_i^*(.),\ g_i^*(.),\ c_i^*(.),\ n_i^*(.)\}_{i=1}^{I}$ *solves the agents' problem.*

2. *(Firm's optimization)* $r^*(\overrightarrow{A}, z)$ *and* $w^*(\overrightarrow{A}, z)$ *are determined competitively.*[10]

3. *(Consistency)*

$$
\begin{aligned}
H^*(z, \overrightarrow{A}) &= \sum_{i=1}^{I} h_i^*(z, A_i, \overrightarrow{A}; G^*, H^*)\mu_i\\
G_i^*(z, \overrightarrow{A}) &= g_i^*(z, A_i, \overrightarrow{A}; G^*, H^*) \qquad \forall i
\end{aligned}
$$

[10] In case of Cobb Douglas production function, $r^*(K, N) = \alpha \quad \frac{\sum_{i=1}^{I} K_i}{N}^{\alpha-1}$ and $w^*(K, N) = (1-\alpha) \quad \frac{\sum_{i=1}^{I} K_i}{N}^{\alpha}$.