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On redundant types and Bayesian formulation of incomplete information

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Abstract

A type structure is non-redundant if no two types of a player represent the same hierarchy of beliefs over the given set of basic uncertainties, and it is redundant otherwise. Under a mild necessary and sufficient condition termed separativity, we show that any redundant structure can be identified with a non-redundant structure with an extended space of basic uncertainties. The belief hierarchies induced by the latter structure, when "marginalized," coincide with those induced by the former. We argue that redundant structures can provide different Bayesian equilibrium predictions only because they reflect a richer set of uncertainties entertained by players but unspecified by the analyst. The analyst shall make use of a non-redundant structure, unless he believes that he misspecified the players' space of basic uncertainties. We also consider bounding the extra uncertainties by the action space for Bayesian equilibrium predictions.

JEL classification: C72; D80; D82

Keywords: Incomplete information; Bayesian equilibrium; Redundant types; Separativity

1. Introduction

Players' beliefs and higher order beliefs are often important in an interactive strategic situation. Harsanyi [18] proposed that a probabilistic type structure implicitly describes players' belief hierarchies over a set of payoff-relevant parameters, which enables researchers to extend the concept of strategic equilibrium to games with incomplete information. Mertens and Za-

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s_1	L	R
U	1, 1	0, 0
D	0, 0	1, 1

s_2	L	R
U	0, 0	1, 1
D	1, 1	0, 0

Fig. 1. Coordination under uncertainty: it is commonly believed that Nature chooses s_1 and s_2 with equal probability.

mir [24][1] show that the coherent belief hierarchies can be modeled as types in a *non-redundant* universal type structure, where non-redundancy roughly says that two different types of each player should specify different hierarchies of beliefs over the given set of parameters.

There is a class of type structures that could be used to model a given set of coherent belief hierarchies. Researchers have observed that two different type structures within this class may yield different equilibrium predictions (see Battigalli and Siniscalchi [5], Ely and Peski [16], and Dekel, Fudenberg, and Morris [12,13]). Let us first look at the following example which simplifies the leading example in [16].

Consider the following game (see Fig. 1), where Alice chooses rows $\{U, D\}$, Bob chooses columns $\{L, R\}$, and Nature chooses matrices $\{s_1, s_2\}$.

Write $S = \{s_1, \ s_2\}$. First consider the type structure $M = \langle S; T_a, T_b; m_a, m_b \rangle$: for each player i, take the type set $T_i = \{t_i\}$ as a singleton and $m_i : T_i \to \Delta(S \times T)$ with

$$m_i(t_i)(s_1, t_i, t_{-i}) = m_i(t_i)(s_2, t_i, t_{-i}) = \frac{1}{2}.$$

That is, type t_i of player i assigns equal probability to (s_1, t_i, t_{-i}) and (s_2, t_i, t_{-i}). Thus, it is commonly believed that Nature chooses s_1 and s_2 with equal probability. Any Bayesian equilibrium associated with this type structure yields each player an expected payoff of $\frac{1}{2}$. Next consider the structure $M' = \langle S; T'_a, T'_b; m'_a, m'_b \rangle$: for each player i, take $T'_i = \{u_i, v_i\}$ and $m'_i : T'_i \to \Delta(S \times T')$ with

$$m'_i(u_i)(s_1, u_i, u_{-i}) = m'_i(u_i)(s_2, u_i, v_{-i}) = \frac{1}{2},$$
$$m'_i(v_i)(s_1, v_i, v_{-i}) = m'_i(v_i)(s_2, v_i, u_{-i}) = \frac{1}{2}.$$

In this type structure, each type of player i assigns equal probability to s_1 and s_2. Thus, it is also commonly believed that Nature chooses s_1 and s_2 with equal probability. In this case, there exists a Bayesian equilibrium (σ_a, σ_b) with $\sigma_a(u_a) = U$, $\sigma_a(v_a) = D$, $\sigma_b(u_b) = L$, and $\sigma_b(v_b) = R$. Under this equilibrium, the two players each have an expected payoff of 1.

In both structures, each player has exactly the same hierarchies of beliefs about the matrix Nature chooses. For example, Alice believes that each matrix is chosen with probability $\frac{1}{2}$, she believes that Bob believes this, and she believes that Bob believes that she believes this, and so on ad infinitum. Yet, there is a prediction in the second model that is not a prediction in the first.

This seems peculiar. Players reason within a particular language and this language captures their hierarchies of beliefs. We—the analysts—use a type structure as a model of this language, i.e., as a model of the hierarchies. We then go on to make a prediction based on this model. Yet, the prediction we provide depends on the model we choose.

[1] See also Brandenburger and Dekel [9], Heifetz [19], and Mertens, Sorin, and Zamir [23]. Battigalli and Siniscalchi [4] construct the universal type structure of hierarchies of conditional beliefs, thereby establishing an epistemic framework for dynamic games.

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In the example above, M satisfies the non-redundancy condition, but M' does not because the two types of each player in M' have the same belief hierarchies. *Technically*, the new prediction associated with M' arises from a particular *correlation* between redundant types and payoff-relevant parameters: for instance, "type" u_a of Alice would believe s_1 is chosen by Nature were she to know that Bob's true "type" is u_b. However, this technical explanation in terms of correlation does not resolve the conceptual question. Alice's single hierarchy summarizes all the information that Alice has about Nature's move and Bob's beliefs about it. How, then, can Alice distinguish the two "types" of Bob if she cannot describe their difference using her own language? In effect, Alice cannot even distinguish her own two types.

This observation suggests that the new prediction associated with the redundant structure M' utilizes more information than the players themselves actually have and hence we should restrict our attention to a non-redundant type structure in order to conduct Bayesian equilibrium analysis. However, in conjunction with the Bayesian equilibrium given above, the redundant type structure M' reflects an interesting belief held by Alice: type u_a of Alice assigns equal probability to (i) Nature choosing s_1 and Bob playing L and (ii) Nature choosing s_2 and Bob playing R. The Bayesian equilibrium framework, following the natural problem-solution paradigm, first formulates the problem by modeling the interactive beliefs over the set of payoff-relevant parameters via a type structure, and then solves the problem by imposing the solution concept. But if we restrict our attention to the non-redundant type structure in the problem-formulating stage, as in the example above, we necessarily preclude the "correlated beliefs" in the problem-solving stage.

This seems puzzling: redundant type structures are conceptually problematic but a Bayesian framework based on non-redundant structures excludes certain interesting strategic situations. One natural solution to this puzzle is to account for the redundancy and hence for the predictions based on it. Since the redundancy results from the formulation of the players' language in the Bayesian framework, it should be understood independently of solution concepts. This paper argues that the redundancy reflects some basic uncertainties that the players entertain but are unspecified by the analyst, and hence a redundant type structure corresponds to the non-redundant structure with an extended space of basic uncertainties.

Let us consider the motivating example again. Suppose the payoff-relevant parameters s_1 and s_2 indicate next week's gasoline price: high/low. This price can be nailed down if information from both the demand side and the supply side is known, but information from one side alone is insufficient. If Alice and Bob have information only from the supply side, then we could use M to model this situation. However, if Alice has private information about the demand side and Bob has private information from the supply side, then M', instead of M, is a reasonable type structure to apply to this situation because the players can deduce the true price if they pool their information. This is to say, if the analyst knows only the payoff structures—he is unaware of (or unable to specify) some other variables that the players know, e.g., the demand and supply, but he is aware of his unawareness (or misspecification)—then a redundant type structure is a "safe" modeling choice: the players "reason" within a redundant structure *as if* they were reasoning about some parameters unknown to the analyst. In other words, the analyst should not make use of a redundant structure unless he is not sure of the players' space of basic uncertainties.

This observation might be relevant in practice. It is perhaps sufficient for an auction designer to specify only the payoff structures for an *anonymous online* bidding game, but other variables that are hard to specify or easy to neglect could be strategically important in an *auction house*:

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for example, the appearance of a bidder in a dress with particular colors may impact the auction results.[2]

This paper presents and proves the following results. With one mild condition that we term *separativity* (which is both necessary and sufficient), we show in Theorem 1 that any redundant type structure can be identified with some non-redundant structure with an expanded set of underlying uncertainties and that the hierarchies generated by the latter, when "marginalized" onto the partial space of underlying uncertainties, coincide with the hierarchies generated by the former. In Theorem 2, we further show that the two structures in Theorem 1 yield the same Bayesian equilibrium predictions. Thus, the predictions based on a redundant structure reflect the equilibria on a full parameter space. Interestingly, the measure-theoretic property of separativity also clarifies the difference between the mathematical definition of non-redundancy and its interpretation in terms of belief hierarchies.

We also obtain a partial result on the bound of the additional uncertainties through the strategy space. Any given equilibrium on a redundant type structure can be obtained from some non-redundant type structure with payoff and action parameters as basic uncertainties. The latter non-redundant structure does not introduce new equilibrium predictions, and moreover, it preserves the hierarchies of beliefs over payoffs. However, the result is only partial because it does not tell us whether or not all the equilibria on the redundant structure can be obtained from a single such non-redundant structure with the belief preserving property. We shall discuss in detail how the straightforward idea of pasting these structures together will not fully deliver the result.[3]

The rest of the paper is organized as follows. In Section 2, we set up the framework, prove several basic properties, and present the main theorems. In Section 3, we study the Bayesian equilibrium. In Section 4, we review the related literature and situate the current work in a discussion. All the omitted proofs are in Appendix A.

2. Type structures and incomplete information

2.1. Preliminaries

We carry out the analysis in a purely measure-theoretic setup. Heifetz and Samet [20] first studied this topology-free formalism. Let (X, Σ) be a measurable space with a σ-field Σ. We also write $\Sigma(X)$ as the σ-field of X for any given measurable space X when the context is clear. Let $\Delta(X)$ denote the measurable space of all countably additive probability measures on X. We endow $\Delta(X)$ with the σ-field Σ_Δ generated by all sets of the form $b^p(E) = \{\mu \in \Delta(X): \mu(E) \geqslant p\}$, where $E \in \Sigma$ and $0 \leqslant p \leqslant 1$. We consider any product of measurable spaces with the product σ-field and any subspace of a measurable space with the relative σ-field. These assumptions are met if X is a separable metric space with the Borel σ-field, and $\Delta(X)$ is endowed with the Borel σ-field generated by the weak topology.[4]

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[2] See [6] for rationalizable bidding behavior in auctions.

[3] Note that our result does not imply that the analyst should always include actions in the state space in order to conduct Bayesian equilibrium analysis. In that case he would have to calculate the players' action choices before making his predictions. If all beliefs over opponents' action choices are allowed, then the resulting model will introduce more predictions than the given redundant structure (see Battigalli and Siniscalchi [5]).

[4] This Borel σ-field coincides with Σ_Δ (Bertsekas and Shreve [7, Proposition 7.25]). See also Battigalli and Siniscalchi [4, Lemma 1 and footnote 5] for a similar conclusion with regard to the Polish topology.

The measure-theoretic formulation sheds light on an important feature of type structures, *separativity*.

Definition 1. We call a measurable space (X, Σ) separative if for every pair of distinct points in X there is a measurable set containing one point and not the other. In this case, we call Σ a separative σ-field.[5]

We adopt the following conventions:

- For a given measure $\mu \in \Delta(X)$ and a measurable map $f : X \to Y$, let μf^{-1} be the image measure of μ under f. That is, $\mu f^{-1} \in \Delta(Y)$ with $\mu f^{-1}(A) = \mu(f^{-1}(A))$ for any measurable subset A of Y.
- If $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$, then (f_1, f_2) is the map $f : X_1 \times X_2 \to Y_1 \times Y_2$ with $f(x_1, x_2) = (f(x_1), f(x_2))$, $x_1 \in X_2$ and $x_2 \in X_2$, unless otherwise stated.
- We write Id_X as the identity map on X, that is, $\mathrm{Id}_X(x) = x$ for any $x \in X$.
- If $\mu \in \Delta(X_1 \times X_2)$, then $\mathrm{Marg}_{X_1} \mu$ is the marginal probability measure of μ on X_1, that is, $(\mathrm{Marg}_{X_1} \mu)(E) = \mu(E \times X_2)$ for each measurable subset E of X_1.
- Given a product space $\prod_{\theta \in \Theta} X_{\theta \in \Theta}$ where Θ is an index set, we write Proj_{X_θ} or Proj_θ as the canonical projection, that is, $\mathrm{Proj}_\theta(x) = x_\theta$ for each $x = (x_\theta)_{\theta \in \Theta}$ in the product space. For each $\theta^* \in \Theta$, we denote $X_{-\theta^*}$ as $\prod_{\theta \in \Theta \setminus \{\theta^*\}} X_\theta$.
- For $x \in X$, we denote by $\delta_x \in \Delta(X)$ the evaluation at x: for any measurable subset E of X, $\delta_x(E)$ is 1 if $x \in E$ and 0 otherwise.

The concept of separativity is crucial in this paper. Here are some useful facts.

Lemma 1 *(Properties of separativity).*

(1) *The measurable space* $(\Delta(X), \Sigma_\Delta)$ *is separative* (*even though* X *may not be*).
(2) *A measurable space* (X, Σ) *is separative if and only if* $\delta_x \neq \delta_y$ *for any distinct points* x *and* y.
(3) *The product of any separative measurable spaces is separative.*
(4) *If* $f : (X, \Sigma(X)) \to (Y, \Sigma(Y))$ *is measurable and injective, and* $(Y, \Sigma(Y))$ *is separative, then* $(X, \Sigma(X))$ *is separative.*
(5) *If all singletons in* X *are measurable sets, then* X *is separative, but not vice versa.*

Proof. See Appendix A.1. □

2.2. *Type structures*

Let $I = \{1, 2, \ldots, n\}$ be a finite set of n individuals.

[5] Separativity is a weak separating property of a collection of sets. The following observation related to topological spaces is helpful: any Borel space induced by a topology satisfying the T_1 separation axiom (Royden [25, p. 178]), including the Hausdorff topology, is separative. Separativity is weaker than the assumption that singletons are measurable; see Part (5) of Lemma 1 below.

Definition 2. An X-based type structure is a collection $\langle X, (T_i)_{i\in I}, (m_i)_{i\in I}\rangle$, or $\langle X, T, m\rangle$ for short, such that

(1) X and T_i, $i \in I$, are measurable spaces with σ-fields $\Sigma(X)$ and $\Sigma(T_i)$, respectively.
(2) For each $i \in I$, m_i is a measurable function $m_i : T_i \to \Delta(X \times T)$.
(3) For each $i \in I$ and $t_i \in T_i$, the marginal measure of $m_i(t_i)$ on T_i is δ_{t_i}.

A type structure $\langle X, T, m\rangle$ is separative if T_i, with the σ-field $\Sigma(T_i)$, is separative for each $i \in I$.

Separative type structures impose the separativity condition only on the type spaces T_i's because separativity of the parameter space X is not a property of types. This distinction is important as we shall see in Proposition 2 and Theorem 1. From part (2) of Lemma 1, T_i is separative if and only if $\delta_{t_i} \neq \delta_{t_i'}$ for any distinct types t_i and t_i' of player i. That is, for each other type t_i', type t_i can distinguish itself from t_i' by naming an event E on which t_i and t_i' have different beliefs. Note that the choice of E can depend on t_i'. If we impose the stronger assumption that all singletons are events (see part (5) of Lemma 1), then type t_i can find one single event to distinguish itself from all other types at the same time. This event can simply be $\{t_i\}$. If a type structure is not separative, then some type would not be able to distinguish itself from the other types.

Heifetz and Samet [20] interpret condition (3) in Definition 2 as a "self-conscious" condition, and they do not impose separativity. In the existing topological formulation, singletons are assumed to be measurable, which implies separativity by part (5) of Lemma 1.[6] It is therefore natural for us to treat separative type structures as a strict subclass of all type structures and to examine their implications.

Following Mertens and Zamir [24] and Heifetz and Samet [20], we introduce the notion of type morphism that links two type structures.

Definition 3. Consider two X-based type structures $\langle X, (T_i)_{i\in I}, (m_i)_{i\in I}\rangle$ and $\langle X, (T_i')_{i\in I}, (m_i')_{i\in I}\rangle$. Let $\tau_i : T_i \to T_i'$ be a measurable function for each i. Then the map $\tau = (\tau_1, \ldots, \tau_n)$ from T to T' is an X-based type morphism if for each $i \in I$ and $t_i \in T_i$,

$$m_i'\big(\tau_i(t_i)\big) = m_i(t_i)(\mathrm{Id}_X, \tau)^{-1},$$

where (Id_X, τ) is a map from $X \times T$ to $X \times T'$ according to our convention.[7] We call τ an X-based type isomorphism if τ is an isomorphism.

Definition 4. An X-based type structure $\langle X, T^*, m^*\rangle$ is *universal* if for every X-based type structure $\langle X, T, m\rangle$, there is a unique *X-based* type morphism from T to T^*.

[6] The coarsest topology appearing in the type space literature that I am aware of is Hausdorff, in which singletons are Borel sets; see Mertens, Sorin, and Zamir [23].

[7] Note that (Id_X, τ) is a well-defined jointly measurable function by the following result, which is an easy adaptation of Aliprantis and Border [1, Lemma 4.48]: let $(X_1, \Sigma(X_1))$, $(X_2, \Sigma(X_2))$, $(Y_1, \Sigma(Y_1))$, and $(Y_2, \Sigma(Y_2))$ be measurable spaces, and let $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ be measurable. Then the induced map $f = (f_1, f_2)$ from $(X_1 \times X_2, \Sigma(X_1) \otimes \Sigma(X_2))$ to $(Y_1 \times Y_2, \Sigma(Y_1) \otimes \Sigma(Y_2))$ is jointly measurable.

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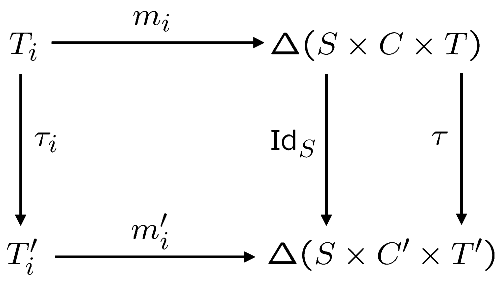

Fig. 2. The S-based marginalized type morphism preserves the marginalized belief structure.

Heifetz and Samet [20] prove the existence of the X-based universal type structure in a measure-theoretic framework where T^* is identified with a subspace of the space of belief hierarchies.

One special form of type structures $\langle S \times C, T, m\rangle$ is of particular interest, where S is interpreted as a partial description of the full parameter space $S \times C$ and we will refer to it as a *partial parameter space*. Note that when C is a singleton, the type structure $\langle S \times C, (T_i)_{i\in I}, (m_i)_{i\in I}\rangle$ is identified with the type structure $\langle S, (T_i)_{i\in I}, (m_i)_{i\in I}\rangle$. We define a marginalized type morphism from one type structure to the other that preserves the structure of beliefs over the common partial parameter space S.

Definition 5. Consider two type structures $\langle S \times C, (T_i)_{i\in I}, (m_i)_{i\in I}\rangle$ and $\langle S \times C', (T'_i)_{i\in I}, (m'_i)_{i\in I}\rangle$. Let $\tau_i : T_i \to T'_i$ be a measurable function for each i. Then the map $\tau = (\tau_1, \dots, \tau_n)$ from T to T' is an S-based marginalized type morphism if for each $i \in I$ and $t_i \in T_i$,

$$\mathrm{Marg}_{S\times T'}\, m'_i\big(\tau_i(t_i)\big) = \big(\mathrm{Marg}_{S\times T}\, m_i(t_i)\big)(\mathrm{Id}_S, \tau)^{-1},$$

where (Id_S, τ) is a map from $S \times T$ to $S \times T'$ according to our convention. We call τ an S-based marginalized type isomorphism if τ is an isomorphism.

The diagram, see Fig. 2, illustrates the definition. If both C and C' are singletons, i.e., S is identified with the full parameter space, then an S-based marginalized type morphism coincides with the standard notion of type morphisms.

2.3. *Belief hierarchies*

Given a measurable space X, the space of X-based belief hierarchies is defined inductively. The space of the first order beliefs for player i is $H_i^1(X) = \Delta(X)$. For each $k \geqslant 1$, denote $H^k(X) = \prod_{i\in I} H_i^k(X)$. The space of $(k+1)$th order beliefs for player i is $H_i^{k+1}(X) = \Delta(X \times \prod_{l=1}^{k} H^l(X))$. Hence, the space of X-based belief hierarchies is $H_i(X) = \prod_{k=1}^{\infty} H_i^k(X)$. $H_i(X)$ with the product σ-field is separative (see Lemma 1, (1) and (3)).

For an $(S \times C)$-based type structure $M = \langle S \times C, T, m\rangle$, we are interested in two different kinds of belief hierarchies, the $(S \times C)$-based hierarchy—the full hierarchy based on the full parameter space $S \times C$—and the S-based hierarchy—the partial hierarchy based on the partial parameter space S. The belief hierarchies are derived inductively; see, e.g., Battigalli and Siniscalchi [4] and Heifetz and Samet [20].

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2.3.1. Full hierarchy

In the "full hierarchy" map $\bar{h} = (\bar{h}_i)_{i\in I}$, where $\bar{h}_i : T_i \to H_i(S \times C)$, $\bar{h}_i$ associates an $(S \times C)$-based belief hierarchy with each type of player i. The map $\bar{h}_i$ is defined inductively. The first order belief map for player i, $\bar{h}_i^1 : T_i \to H_i^1(S \times C)$, is defined by

$$\bar{h}_i^1(t_i) = m_i(t_i)(\mathrm{Proj}_{S\times C})^{-1},$$

where $\mathrm{Proj}_{S\times C} : S \times C \times T \to S \times C$ is the projection mapping. Thus, for any $t_i \in T_i$ and any measurable set $E \subset S \times C$,

$$\bar{h}_i^1(t_i)(E) = m_i(t_i)(E \times T).$$

Write $\bar{h}^k = (\bar{h}_i^k)_{i\in I}$ for $k \geqslant 1$. We shall define the $(k+1)$th order belief map. Let us first define $\bar{p}^k : S \times C \times T \to S \times C \times \prod_{l=1}^k H^l(S \times C)$ with $\bar{p}^k(s, c, t) = (s, c, \bar{h}^1(t), \ldots, \bar{h}^k(t))$. That is, for each "state of the world" $(s, c, t) \in S \times C \times T$, the map $\bar{p}^k$ specifies the "state of nature" (s, c) and the beliefs up to order k for type profile t. Inductively, define the $(k+1)$th order belief map for player i, $\bar{h}_i^{k+1} : T_i \to H_i^{k+1}(S \times C)$, as

$$\bar{h}_i^{k+1}(t_i) = m_i(t_i)(\bar{p}^k)^{-1}. \tag{1}$$

Finally, define $\bar{h}_i(t_i) = (\bar{h}_i^1(t_i), \bar{h}_i^2(t_i), \ldots)$. In Appendix A.2, we show that all the functions involved in this inductive definition are measurable.

2.3.2. Partial hierarchy

The "partial hierarchy" map $h_i : T_i \to H_i(S)$, which generates the belief hierarchies over the partial parameter space S, is defined similarly by replacing $H_i^k(S \times C)$ with $H_i^k(S)$ and $\mathrm{Proj}_{S\times C}$ with Proj_S in the definition above.

The first order belief map for player i, $h_i^1 : T_i \to H_i^1(S)$, is given by

$$h_i^1(t_i) = m_i(t_i)(\mathrm{Proj}_S)^{-1}.$$

Write $h^k = (h_i^k)_{i\in I}$ for $k \geqslant 1$. Define function $p^k : S \times C \times T \to S \times \prod_{l=1}^k H^l(S)$ with $p^k(s, c, t) = (s, h^1(t), \ldots, h^k(t))$. Inductively,

$$h_i^{k+1}(t_i) = m_i(t_i)\bigl(p^k\bigr)^{-1}. \tag{2}$$

Finally, write $h_i(t_i) = (h_i^1(t_i), h_i^2(t_i), \ldots)$. Intuitively, the S-based hierarchies are obtained by "marginalizing" the $(S \times C)$-based hierarchies onto the partial space S.

The following result says that an S-based marginalized type morphism preserves S-based hierarchies.

Proposition 1. *If $M = \langle S \times C, T, m\rangle$ and $M' = \langle S \times C', T', m'\rangle$ are two type structures with S-based partial belief hierarchy maps h and h' respectively, and $\tau : T \to T'$ is an S-based marginalized type morphism from M to M', then $h_i' \circ \tau_i = h_i$ for each $i \in I$.*

Proof. See Appendix A.3. □

2.4. Redundant and non-redundant structures

For a type structure $\langle X, T, m\rangle$, let $\sigma(\bar{h}_i)$ be the smallest σ-field of the subsets of T_i for which the X-based full hierarchy map $\bar{h}_i$ is measurable. Note that by definition, $\bar{h}_i$ is measurable, and hence $\Sigma(T_i)$, the σ-field on T_i, is finer than $\sigma(\bar{h}_i)$. The next definition formalizes the notion of non-redundancy.

Definition 6. A type structure $\langle X, T, m\rangle$ is non-redundant if for each $i \in I$, $\sigma(\bar{h}_i)$ is a separative σ-field on T_i. A type structure is redundant if it is not non-redundant.

The following result links the formal definition with our familiar intuition: non-redundancy means no two types of a player have the same belief hierarchies over the full parameter space.

Proposition 2 *(Non-redundancy and separativity).*

(1) *$\langle X, T, m\rangle$ is non-redundant if and only if the X-based full hierarchy map $\bar{h}_i : T_i \to H_i(X)$ is injective.*
(2) *A non-redundant type structure is separative.*
(3) *An X-based universal type structure $\langle X, T^*, m^*\rangle$ is separative.*

Proof. See Appendix A.4. □

Remark. Definition 6 and part (1) of Proposition 2 are similar to, but different from, their counterparts in Mertens and Zamir [24, Definition 2.4 and Proposition 2.5, respectively]. This relation is noteworthy as it reveals the different roles played by the measurable structures of the parameter space X and of the type spaces T_i. The two definitions of non-redundancy are similar in that they have the same intuitive motivation, i.e., the injectivity of the hierarchy maps. The definitions differ in that Mertens and Zamir's treatment imposes stronger requirements. Mertens and Zamir assume that the parameter space X is Hausdorff. Therefore X is separative as well. We distinguish the measurable structures of the parameter space X and of the type spaces T_i. As a result, the "if and only if" result in part (1) of Proposition 2 links the formal definition of non-redundancy and its motivation, while the converse of Mertens and Zamir's Proposition 2.5 does not hold.

We are now ready to introduce the central concept of this paper.

Definition 7. Fix an S-based type structure $M = \langle S, T, m\rangle$. An $(S \times C)$-based structure $\overline{M} = \langle S \times C, \overline{T}, \overline{m}\rangle$ is called an *expansion* of M via C if $\overline{M}$ is non-redundant and there is an S-based marginalized type isomorphism $\tau : T \to \overline{T}$.

In conjunction with Proposition 1, this definition implies that if M' is an expansion of M via C, then the S-based partial hierarchies induced by M' coincide with the S-based full hierarchies induced by M through τ. It turns out that separativity is necessary and sufficient for a type structure to have an expansion.

Theorem 1. *An S-based redundant type structure $M = \langle S, T, m\rangle$ has an expansion $\overline{M} = \langle S \times C, \overline{T}, \overline{m}\rangle$ if and only if M is separative.*

By this theorem, any S-based type structure M that is redundant in terms of S-based hierarchies can be interpreted as a non-redundant type structure based on an enlarged set of parameters

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$S \times C$. This solves the conceptual difficulties of redundant structures: they capture certain "hidden variables" that the players know but the analyst does not know.[8]

In subsequent proofs, we shall construct type structures by deriving new probability measures from the probability measures on the semirings of measurable rectangles via the Caratheodory Extension Procedure. The following crucial lemma is used to verify the measurability of functions whose ranges are probability measures.[9]

Lemma 2. *Let (X, Σ) be a measurable space and $\mathcal{S}$ be a semiring that generates Σ, $\sigma(\mathcal{S}) = \Sigma$. Let $\mathcal{S}_\Delta$ be the σ-field on $\Delta(X)$ generated by sets of the form $b^p(E) = \{\mu \in \Delta(X)\colon\ \mu(E) \geqslant p\}$ for $E \in \mathcal{S}$ and $0 \leqslant p \leqslant 1$. Then $\Sigma_\Delta = \mathcal{S}_\Delta$.*

Proof. See Appendix A.5. □

Now we are ready to prove Theorem 1.

Proof of Theorem 1. The necessity of separativity is straightforward. If $\overline{M}$ is an expansion of M, then there is an S-based marginalized type isomorphism $\tau_i : T_i \to \overline{T}_i$. Since $\overline{M}$ is non-redundant, $\overline{T}_i$, with the σ-field $\Sigma(\overline{T}_i)$, is separative by part (2) of Proposition 2. By part (4) of Lemma 1, T_i, with the σ-field $\Sigma(T_i)$, has to be separative.

We divide the proof for sufficiency into several steps. In Step 1, we define the candidate type structure $\overline{M}$. For any given M, we set $\overline{T} = C = T$. The measure $\overline{m}_i(t_i)$ is defined along the diagonal of $C \times T$ (see Eq. (3) below). In Step 2, we verify that $\overline{M}$ constructed in Step 1 is indeed a well-defined structure. Lemma 2 is used to show the measurability of $\overline{m}_i$. Separativity is used in Step 3 to establish the non-redundancy of the structure $\overline{M}$.

Step 1. We define the candidate type structure.

Set $C = T$ and $\overline{T}_i = T_i$. To construct a candidate type structure $\overline{M} = \langle S \times C, (T_i)_{i \in I}, (\overline{m}_i)_{i \in I}\rangle$, we shall define the map $\overline{m}_i : T_i \to \Delta(S \times C \times T)$. The strategy is to define a set function (with some abuse of notation, we write it as $\overline{m}_i$) on the semiring $\mathcal{S} = \Sigma(S) \times \Sigma(C) \times \Sigma(T)$, and then extend it uniquely to the product σ-field.

For each $t_i \in T_i$, and each measurable rectangle $D \times E \times F \in \mathcal{S}$, define

$$\overline{m}_i(t_i)(D \times E \times F) = m_i(t_i)\big(D \times (E \cap F)\big). \tag{3}$$

This $\overline{m}_i(t_i)$ defines a probability measure on the semiring $\mathcal{S}$. We shall verify its countable additivity below. Consider a sequence of a pairwise disjoint rectangles $D_k \times E_k \times F_k$ in $\mathcal{S}$ such that

$$\bigcup_{k=1}^{\infty}(D_k \times E_k \times F_k) = D \times E \times F \in \mathcal{S}.$$

Then $\{D_k \times (E_k \cap F_k)\}$ is a pairwise disjoint sequence in $\Sigma(S \times C)$ and

$$D \times (E \cap F) = \bigcup_{k=1}^{\infty}\big(D_k \times (E_k \cap F_k)\big).$$

Therefore, by definition and countable additivity of $m_i(t_i)$,

[8] See [10,11,21] for a related discussion.

[9] This generalizes Heifetz and Samet's result [20, Lemma 4.5] for generating fields.

$$\begin{aligned}\overline{m}_i(t_i)(D\times E\times F) &= m_i(t_i)\big(D\times(E\cap F)\big)\\ &= \sum_{k=1}^{\infty} m_i(t_i)\big(D_k\times(E_k\cap F_k)\big)\\ &= \sum_{k=1}^{\infty} \overline{m}_i(t_i)(D_k\times E_k\times F_k).\end{aligned}$$

Therefore, $\overline{m}_i(t_i)$ is a countably additive probability measure on the semiring $\mathcal{S}$. By the Caratheodory Extension Theorem (see [1, Theorem 9.22]), $\overline{m}_i(t_i)$ extends uniquely to a probability measure on $\Sigma(S\times C\times T)$, which we shall still write as $\overline{m}_i(t_i)$.

Step 2. We verify that $\overline{M}=\langle S\times C,(T_i)_{i\in I},(\overline{m}_i)_{i\in I}\rangle$ constructed in Step 1 is indeed a well-defined type structure.

(i) $\overline{m}_i: T_i\to\Delta(S\times C\times T)$ is measurable with respect to the natural σ-fields.

Since $\Sigma(S\times C\times T)$ is generated by the semiring $\mathcal{S}=\Sigma(S)\times\Sigma(C)\times\Sigma(T)$, by Lemma 2 we only need to check the measurability on the measurable rectangles; that is, for any measurable rectangle $D\times E\times F\in\mathcal{S}$ and $0\leqslant p\leqslant 1$,

$$(\overline{m}_i)^{-1}\big(\{\mu\colon\ \mu(D\times E\times F)\geqslant p\}\big) \tag{4}$$

is a measurable subset of T_i.

Rewrite (4) as $\{t_i\colon\ \overline{m}_i(t_i)(D\times E\times F)\geqslant p\}$. By the definition of $\overline{m}_i$, expression (4) is precisely the set $\{t_i\colon\ m_i(t_i)(D\times(E\cap F))\geqslant p\}$, which is a measurable subset of T_i by the measurability of m_i.

(ii) The marginal measure of $\overline{m}_i(t_i)$ on T_i is δ_{t_i}.

In the type structure M, we have $\mathrm{Marg}_{T_i}\, m_i(t_i)=\delta_{t_i}$ for each $t_i\in T_i$. That is, for each measurable subset $E_i\subset T_i$ such that $t_i\in E_i$, $m_i(t_i)(S\times T_{-i}\times E_i)=1$. Thus, in the type structure $\overline{M}$, by the definition of $\overline{m}_i$,

$$\overline{m}_i(t_i)(S\times C\times T_{-i}\times E_i)=m_i(t_i)(S\times T_{-i}\times E_i)=1.$$

That is, for any $t_i\in T_i$ in the type structure $\overline{M}$, $\mathrm{Marg}_{T_i}\,\overline{m}_i(t_i)=\delta_{t_i}$, as required.

This proves that $\overline{M}=\langle S\times C,(T_i)_{i\in I},(\overline{m}_i)_{i\in I}\rangle$ is a type structure.

Step 3. We verify that $\overline{M}$ is an expansion of M. To avoid confusing "T" in the role of the parameter space with "T" in the role of the type spaces $\overline{M}$, we write $c_i: T_i\to C_i$ as the identity map.

(a) We shall show that $\overline{M}$ is non-redundant. It suffices to show that $\bar{h}_i: T_i\to H_i(S\times C)$ is injective by Proposition 2. Consider two distinct types t_i and t_i'. By the separativity of T_i (and hence C_i), we can find a measurable set $K_i\subset C_i$ such that $c_i(t_i)\in K_i$ and $c_i(t_i')\notin K_i$ (recall that $c_i: T_i\to C_i$ is the identity). Therefore,

$$\bar{h}_i^1(t_i)(S\times C_{-i}\times K_i)=\overline{m}_i(t_i)(S\times C_{-i}\times K_i\times T)=m_i(t_i)(S\times T_{-i}\times K_i)=1. \tag{5}$$

The second equality of (5) follows from the fact that the marginal measure of $m_i(t_i)$ on T_i is δ_{t_i} and $t_i=c_i(t_i)\in K_i$. On the other hand,

$$\bar{h}_i^1\big(t_i'\big)(S\times C_{-i}\times K_i)=\overline{m}_i\big(t_i'\big)(S\times C_{-i}\times K_i\times T)=m_i\big(t_i'\big)(S\times T_{-i}\times K_i)=0.$$

Therefore, $\bar{h}_i$ is injective, as required.

(b) We shall show that the identity map $\mathrm{Id}_T : T \to T$ is the required type isomorphism from $M = \langle S, T, m\rangle$ and $\overline{M} = \langle S \times C, T, \overline{m}\rangle$. Namely,

$$\mathrm{Marg}_{S\times T}\, \overline{m}_i\big(\mathrm{Id}_{T_i}(t_i)\big) = m_i(t_i)(\mathrm{Id}_S, \mathrm{Id}_T)^{-1} \tag{6}$$

and

$$\big(\mathrm{Marg}_{S\times T}\, \overline{m}_i(t_i)\big)(\mathrm{Id}_S, \mathrm{Id}_T)^{-1} = m_i\big(\mathrm{Id}_{T_i}(t_i)\big). \tag{7}$$

By definition, $\overline{m}_i(\mathrm{Id}_{T_i}(t_i))(E \times C \times F) = \overline{m}_i(t_i)(E \times C \times F) = m_i(t_i)(E \times F)$ for any measurable rectangle $E \times C \times F$ with $E \in \Sigma(S)$ and $F \in \Sigma(T)$. We just proved that $\overline{m}_i(t_i)(C \times \cdot)$ agrees with $m_i(t_i)(\cdot)$ on the semiring $\Sigma(S) \times \Sigma(T)$. Therefore, $\overline{m}_i(t_i)(B \times C) = m_i(t_i)(B)$ or $\mathrm{Marg}_{S\times T}\, \overline{m}_i(t_i) = m_i(t_i)$ for any measurable subset $B \in \Sigma(S \times T)$ by [8, Theorem 10.3]. This proves Eqs. (6) and (7).

Thus, $\overline{M}$ constructed above is an expansion of M. □

3. Games of incomplete information

Consider a game $G = (S; (A_i)_{i\in I}; (g_i)_{i\in I})$ with a measurable parameter space S, strategy space $A = \prod_{i\in I} A_i$, and bounded *jointly* measurable payoff functions $g_i : S \times A \to \mathbb{R}$. Append to the game a type structure $M = \langle S, T, m\rangle$. Write $G[M]$ for the Bayesian game associated with G and M.

Let $(\beta_i)_{i\in I}$ be a tuple of (behavioral) *strategies* $\beta_i : T_i \times \Sigma(A_i) \to [0, 1]$ such that (1) for every $t_i \in T_i$, $\beta_i(t_i, \cdot) : \Sigma(A_i) \to [0, 1]$ is a probability measure and (2) for every $E \in \Sigma(A_i)$, $t_i \mapsto \beta_i(t_i, E)$ is measurable. The profile $(\beta_i)_{i\in I}$ is a *Bayesian equilibrium* of $G[M]$ if for any $i \in I$ and any $a_i' \in A_i$,

$$\int_{S\times T} \int_A g_i(s, a) \prod_{j\in I} \beta_j\big(t_j', da_j\big)\, dm_i(t_i) \geqslant \int_{S\times T} \int_A g_i\big(s, a_i', a_{-i}\big) \prod_{j\in I\setminus\{i\}} \beta_j\big(t_j', da_j\big)\, dm_i(t_i).$$

In the expression above, $\int_{S\times T} \int_A g_i(s, a) \prod_{j\in I} \beta_j(t_j', da_j)\, dm_i(t_i)$ is player i's expected payoff from $G[M]$ when the players play according to $(\beta_i)_{i\in I}$, and $\int_A g_i(s, a) \prod_{j\in I} \beta_j(t_j, da_j)$ is the shorthand for

$$\int_{A_1} \cdots \left[\int_{A_n} g_i(s, a)\beta_1(t_1, da_1)\right] \cdots \beta_n(t_n, da_n).$$

Let $\overline{M}$ be an expansion of M via C. Let $\overline{G} = (S \times C; (A_i)_{i\in I}; (\overline{g}_i)_{i\in I})$ be such that $\overline{g}_i(s, c, a) = g_i(s, a)$ for all $(s, a) \in S \times A$. We call $\overline{G}[\overline{M}]$ an expanded game of $G[M]$.

We adopt the following convention: when the context is clear, we shall call β an equilibrium on type structure M as well as an equilibrium of the game $G[M]$. The following theorem shows that any equilibrium associated with a redundant type structure can be obtained from its non-redundant expansions, and vice versa. Together with Theorem 1, this result says that the set of predictions obtained from a redundant structure reflects the set of equilibria based on a full parameter space.

Theorem 2. *Let $\overline{M} = \langle S \times C, \overline{T}, \overline{m}\rangle$ be an expansion of $M = \langle S, T, m\rangle$ via C, and $\tau : T \to \overline{T}$ is the associated S-based marginalized type isomorphism. For any Bayesian equilibrium $(\beta_i)_{i\in I}$ of M, the strategy profile $(\overline{\beta}_i)_{i\in I}$, defined via $\overline{\beta}_i(\bar{t}_i, \cdot) = \beta_i(\tau_i^{-1}(\bar{t}_i), \cdot)$ for each $i \in I$ and $\bar{t}_i \in \overline{T}_i$, is a Bayesian equilibrium of $\overline{M}$. Conversely, for any Bayesian equilibrium $(\overline{\beta}_i)_{i\in I}$ of $\overline{M}$, the strategy profile $(\beta_i)_{i\in I}$, defined via $\beta_i(t_i, \cdot) = \overline{\beta}_i(\tau_i(t_i), \cdot)$ for each $i \in I$ and $t_i \in T_i$, is a Bayesian equilibrium of M.*

Proof. See Appendix A.6. □

In Theorem 1, the set of "hidden variables" C can be very large depending on the redundant type structure being considered, because we require that there be an S-based type isomorphism between a redundant structure and its non-redundant expansions. A question one may ask is whether it is possible to construct a space C depending only on the size of the game through which the whole set of Bayesian equilibria of *any* given S-based redundant type structure can be obtained. Intuitively, the action space A could be a candidate for such C. After all, a rational player's equilibrium play in *any given equilibrium* must be compatible with his belief about the payoff functions and about his opponents' action choices in that equilibrium. We go one step further and conjecture that a single $(S \times A)$-based non-redundant type structure will preserve *all* the equilibria of a given redundant type structure without introducing new equilibria.

Conjecture. *For any S-based redundant type structure $M = \langle S, T, m\rangle$, there exist an $(S \times A)$-based non-redundant type structure $\overline{M} = \langle S \times A, \overline{T}, \overline{m}\rangle$ and an S-based surjective marginalized type morphism $\tau : T \to \overline{T}$ such that*:

(i) *For any equilibrium β on M, there exists an equilibrium $\overline{\beta}$ on $\overline{M}$ such that $\overline{\beta}_i(\tau_i(t_i), \cdot) = \beta_i(t_i, \cdot)$ for any $t_i \in T_i$.*
(ii) *For any equilibrium $\overline{\beta}$ on $\overline{M}$, there exists an equilibrium β on M such that $\beta_i(t_i, \cdot) = \overline{\beta}_i(\tau_i(t_i), \cdot)$ for any $t_i \in T_i$.*

As a consequence of Proposition 1, the existence of the surjective marginalized type morphism τ would guarantee that M and $\overline{M}$ induce the same set of S-based belief hierarchies. Condition (i) of the conjecture says that any equilibrium on M induces an equilibrium on $\overline{M}$, while condition (ii) says that any equilibrium on $\overline{M}$ induces an equilibrium on M. In summary, the conjecture states that $\overline{M}$ preserves S-based belief hierarchies and Bayesian equilibrium predictions of M.[10]

We report a partial result in this paper, but we are unable to prove or disprove the conjecture. We first develop the following simple tool that shall be useful to deliver measurability.

Lemma 3. *Fix a type structure $M = \langle X, T, m\rangle$. Then $M' = \langle X, T', m'\rangle$ with the following two properties is a type structure*:

(1) *$T_i' \subset T_i$ and T_i' is endowed with the relative σ-field $\Sigma(T_i') = \{T_i' \cap E_i\colon\ E_i \in \Sigma(T_i)\}$.*
(2) *For each $i \in I$ and $t_i \in T_i'$, $m_i(t_i)$ has a support in $X \times T'$, and for each measurable subset G of $X \times T$, $m_i'(t_i)((X \times T') \cap G) = m_i(t_i)(G)$.*

Furthermore, M' is separative whenever M is separative.

[10] Note that since C is fixed to be A for any type structure, τ cannot be taken as an isomorphism in general. For a trivial example, consider S and A as singletons. Then the non-redundancy of $\overline{M}$ requires that $\overline{T}$ be a singleton, but the redundant type space T can be any measurable space; there is no isomorphism from T to $\overline{T}$.

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Proof. See Appendix A.7. □

In words, Lemma 3 says that if $m_i(t_i)$ has a support in $X \times T'$ for each player i and $t_i \in T_i' \subset T_i$, then we can define a natural σ-field on T_i' and restrict $m_i(t_i)$ to this σ-field so that $\langle X, T', m' \rangle$ is a well-defined type structure.

Definition 8. We call M' a sub-structure of M if the conditions in Lemma 3 are met.

Definition 9. A type structure $M = \langle X, T, m \rangle$ is a union of a family of its sub-structures $M^\theta = \langle X, T^\theta, m^\theta \rangle$, $\theta \in \Theta$, if $T_i = \bigcup_{\theta \in \Theta} T_i^\theta$ for each $i \in I$.

Note that T^θ's are pairwise distinct but they are not necessarily pairwise disjoint. For instance, a type space T^θ could be a strict subset of another type space $T^{\theta'}$. Constructing a new type structure by taking the union of a family of small structures was first discussed in Battigalli and Siniscalchi [5, pp. 27–28 and footnote 33].

Theorem 3. *For any S-based redundant type structure $M = \langle S, T, m \rangle$, there exist an $(S \times A)$-based non-redundant type structure $\overline{M} = \langle S \times A, \overline{T}, \overline{m} \rangle$ that is a union of a family of its sub-structures $\overline{M}^\theta = \langle S \times A, \overline{T}^\theta, \overline{m}^\theta \rangle$, $\theta \in \Theta$, and a family of S-based surjective marginalized type morphisms $\tau^\theta : T \to \overline{T}^\theta$ such that the following hold*:

(1) *For any equilibrium β on M, there exist a $\theta \in \Theta$ and an equilibrium $\overline{\beta}^\theta$ on $\overline{M}^\theta$ such that $\overline{\beta}_i^\theta(\tau_i^\theta(t_i), \cdot) = \beta_i(t_i, \cdot)$ for any $t_i \in T_i$.*
(2) *For any $\theta \in \Theta$ and any equilibrium $\overline{\beta}^\theta$ on $\overline{M}^\theta$, there exists an equilibrium β on M such that $\beta_i(t_i, \cdot) = \overline{\beta}_i^\theta(\tau_i^\theta(t_i), \cdot)$ for any $t_i \in T_i$.*

The existence of an S-based surjective marginalized type morphism implies that M, $\overline{M}$ and every component of $\overline{M}$ contain the same set of S-based belief hierarchies (by Proposition 1). Since $\overline{M}$ is non-redundant, two different sub-structures of $\overline{M}$ necessarily contain different sets of $(S \times A)$-based hierarchies. Part (1) of the result says that any equilibrium of the original redundant structure M can be obtained from some sub-structure of the non-redundant structure $\overline{M}$. Part (2) further implies that this sub-structure does not introduce new equilibrium predictions. Note that $\overline{M}$, the union of those sub-structures, preserves all the equilibria of M only when we conduct equilibrium analysis on each of the sub-structures.[11] The result is only partial in view of the aforementioned conjecture. Theorem 3 implies property (ii) of the conjecture: $\overline{M}^\theta$ preserves the S-based belief hierarchies and it does not introduce new equilibria; however, we still do not know whether a *single* such structure preserves *all* the equilibria

[11] In some contexts, it might seem reasonable to conduct equilibrium analysis within sub-structures to make predictions because of the non-existence of equilibria on the universal (or a large) type structure (see Simon [26] for an example of non-existence of measurable equilibria on compact Polish type spaces with finite games). But this is not necessarily a negative answer for the conjecture—one might be able to get existence on the $(S \times A)$-based type structure through the existence on the original type structure M. Our equilibrium dependent construction in the proof gives us existence on each sub-structure $\overline{M}^\theta$, but it also limits the bite of the theorem.

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of M.[12] That is, property (i) of the conjecture is not guaranteed. Therefore, the conjecture is still open.[13]

The idea of constructing a new structure that preserves Bayesian equilibrium predictions through its sub-structures is similar to the idea proposed by Battigalli and Siniscalchi [5, pp. 27–28 and footnote 33]. We impose more restrictions here: the new structure must be non-redundant in terms of $(S \times A)$-based beliefs and it has to preserve the equilibria of an exogenously given type structure.

Proof of Theorem 3 (*sketch*). Here we first preview the idea. The technical details are in Appendix A.9. Unlike the proof of Theorem (1), the parameter set C is exogenously fixed to be A. The construction of $\overline{M} = \langle S \times A, \overline{T}, \overline{m} \rangle$ and its sub-structures $\overline{M}^\theta = \langle S \times A, \overline{T}^\theta, \overline{m}^\theta \rangle$ must meet five conditions: (I) $\overline{M}$ and $\overline{M}^\theta$ are non-redundant; (II) all type sets, $\overline{T}_i^\theta$ and $\overline{T}_i$, are measurable; (III) $\overline{M}$ expands M by introducing new $(S \times A)$-based beliefs but $\overline{M}$ does not have new S-based belief hierarchies; (IV) as a prediction, $\overline{M}$ contains each equilibrium of M; (V) $\overline{M}$ does not introduce new equilibrium predictions.

The tools we use in the proof are the sub-structure technique derived in Lemma 3 and an extension of Fubini's Theorem (Appendix A.8). We identify $\overline{M}^\theta$ and $\overline{M}$ as sub-structures of the (non-redundant) universal structure $\langle S \times A, \overline{M}^*, \overline{T}^* \rangle$ from which $\overline{M}^\theta$ and $\overline{M}$ inherit the measurability. Conditions (i) and (ii) are met. To avoid introducing new S-based beliefs into $\overline{M}$, we construct $\overline{m}$ by integrating m with the equilibrium β according to the extension of Fubini's Theorem (note that the equilibrium profile β is technically a Markov kernel). This operation preserves both S-based beliefs and the original equilibrium β. Since Fubini's Theorem generalizes the properties of the integrals with independent measures, the induced $(S \times A)$-based beliefs will not bring about new correlations that are relevant to Bayesian equilibrium. Conditions (III), (IV), and (V) are thus met.

We sketch the nine steps of the proof. All the measurability arguments and the proofs for the claims made below are in Appendix A.9.

Step 1. For any equilibrium β on $M = \langle S, T, m \rangle$, we construct an "intermediate" $(S \times A)$-based type structure $\Psi^\beta = \langle S \times A, T, \psi^\beta \rangle$, where for each $i \in I$ and $t_i \in T_i$, ψ_i^β is the unique

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[12] In particular, an equilibrium on $\overline{M}^\theta$ might not be an equilibrium on $\overline{M}$ (or even on $\overline{M}^{\theta'}$ when $\overline{T}^\theta$ is distinct from $\overline{T}^{\theta'}$).

[13] A recount of Theorem 3 might illustrate the heart of the conjecture. Theorem 3 and its proof are equilibrium dependent. More specifically, for each given equilibrium on the original redundant structure, we identify an $(S \times A)$-based sub-structure of the universal structure. The equilibrium dependent construction preserves the original equilibria and hence guarantees the existence on the new type structure. Working with sub-structures of the universal structure guarantees measurability as well. But starting with one particular equilibrium, we cannot guarantee that the whole equilibrium set is preserved. Condition (ii) of the conjecture will be implied by Theorem 3 if there exists a maximal sub-structure $\overline{M}^{\theta^*}$ that subsumes the equilibria on other sub-structures. However, proving this might require further knowledge on the equilibrium set. It seems that, to prove the conjecture fully, we need an equilibrium independent approach, but then the existence of a measurable equilibrium (through the surjective type *morphism*) becomes rather difficult and additional topological assumptions might be needed.

These difficulties do not appear in Theorem 2 because we work with a type isomorphism. With additional topological assumptions, we believe the conjecture is true if A is "large" enough. For example, if A is the Hilbert cube or larger (see Aliprantis and Border [1, p. 85]), then any given separable metrizable (redundant) type spaces can be embedded in A by Urysohn Metrization Theorem (see, [1, p. 86]). But this is not interesting because we have taken A as "the largest separable metric space" and hence we can work with the type isomorphism as compared to some to-be-defined morphism. The conjecture is still open for general A.

extension of the following measure on the semiring of measurable rectangles $\{E \times F \times G$: $E \in \Sigma(S), F \in \prod_{j \in I} \Sigma(A_j), G \in \Sigma(T)\}$:

$$\psi_i^\beta(t_i)(E \times F \times G) = \int_{E \times G} \prod_{j \in I} \beta_j\big(t'_j, F_j\big)\, dm_i(t_i).$$

By virtue of Fubini's Theorem, this operation does not introduce new correlations relevant to Bayesian equilibrium and it enables the recovery of β_i from ψ_i^β (see Steps 3 and 5).

Step 2. The structure $\Psi^\beta = \langle S \times A, T, \psi^\beta \rangle$ can be redundant. We pick a non-redundant sub-structure $\overline{M}^\beta = \langle S \times A, \overline{T}^\beta, \overline{m}^\beta \rangle$ from the universal structure $\langle S \times A, \overline{T}^*, \overline{m}^* \rangle$ so that there is an $(S \times A)$-based surjective type morphism τ^β from Ψ^β to $\overline{M}^\beta$. We show that τ^β is an S-based surjective type morphism from M to $\overline{M}^\beta$.

Step 3. Define a strategy profile $\bar{\beta}^\beta$ on $\overline{M}^\beta$ as follows. For each $i \in I$ and $\bar{t}_i \in \overline{T}_i^\beta$, let $\bar{\beta}_i^\beta(\bar{t}_i, \cdot) := \beta_i(f_i(\bar{t}_i), \cdot)$, where f_i is a selector of $(\tau_i^\beta)^{-1}$. We shall show that $\bar{\beta}_i^\beta$ is independent of the choice of f_i, and hence $\bar{\beta}_i^\beta(\tau_i^\beta(t_i), \cdot) = \beta_i(t_i, \cdot)$ for any $t_i \in T_i$.

Step 4. We show that $\bar{\beta}^\beta$ defined in Step 3 is an equilibrium on $\overline{M}^\beta$.

Step 5. We shall show that $\overline{M}^\beta$ does not introduce more equilibrium predictions than M. For any equilibrium $\bar{\beta}$ of $\overline{M}^\beta$, define a strategy profile $\beta^{\bar{\beta}}$ on M as follows. For each $i \in I$ and $t_i \in T_i$, $\beta_i^{\bar{\beta}}(t_i, \cdot) := \bar{\beta}_i(\tau_i^\beta(t_i), \cdot)$. We show that $\beta^{\bar{\beta}}$ is an equilibrium on M.

Next, we construct Θ, $\overline{M}^\theta$, τ^θ, and $\overline{M}$.

Step 6. Let B be the set of equilibria on M. Set $\Theta := \{\overline{M}^\beta\colon \beta \in B\}$ and for each $\theta \in \Theta$, set $\overline{M}^\theta := \theta$. By the definition of Θ, for each θ there is a unique point $\beta^\theta \in B$ such that $\theta = \overline{M}^{\beta^\theta}$. Set $\tau^\theta := \tau^{\beta^\theta}$. This simple operation removes all the duplication of $\overline{M}^\beta$.

Step 7. We define $\overline{M} = \langle S \times A, \overline{T}, \overline{m} \rangle$ so that $\overline{T} = \bigcup_{\theta \in \Theta} \overline{T}^\theta$, $\overline{T}$ inherits the relative σ-field from $\overline{T}^*$, and $\overline{m}$ is the restriction of $\overline{m}^*$, where $\langle S \times A, \overline{T}^*, \overline{m}^* \rangle$ is the $(S \times A)$-based universal structure.

Step 8. We still need to show that each $\overline{M}^\theta$ is a sub-structure of $\overline{M}$. By the construction in Step 2, $\overline{T}^\theta$ inherits the σ-field from $\overline{T}^*$. We shall show this σ-field is the same as the σ-field that $\overline{T}^\theta$ inherits from $\overline{T}$.

Step 9. We conclude that Θ, $\overline{T}^\theta$, τ^θ, and $\overline{T}$ constructed above satisfy all the requirements of the theorem. □

4. Concluding remarks

4.1. Interpretation of the results

The key implication of our results is that the analyst should use non-redundant type structures if he is sure of the basic uncertainties that the players entertain. The players reason within a particular language that captures their belief hierarchies. A type structure is the analyst's model to represent such language. In a redundant type structure, however, the players cannot even describe and distinguish their own types; that is, the players know less than what the redundant type structure captures.

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If the analyst is not sure of the basic uncertainties that the players entertain, he might make use of the redundant type structures. A redundant type structure can be identified with a non-redundant structure with an enlarged space of uncertainties. Only in a non-redundant structure with a rich set of basic uncertainties can a player distinguish his own types using his language. Since a type structure is a formulation of the game situation, it should be understood independently of solution concepts.

If, in addition, we fix the payoff structures and impose the solution concept of Bayesian equilibrium, we obtain a partial result toward bounding the cardinality of the enlarged space of uncertainties by the space of actions—but the question is still open whether there exists a single non-redundant type structure that preserves all the equilibria of a given redundant structure. However, the result should not be overstated. In particular, it does not imply that in making Bayesian equilibrium predictions the analyst can avoid the issue of redundant types by incorporating the players' actions into the type structure. To build the type structures, the analyst needs to specify the players' beliefs about their opponents' action choices, which are precisely what the analyst tries to predict.

In developing the theory in this paper, we also identify a crucial measure-theoretic property of type structures: separativity. As shown in Theorem 1, separativity is both necessary and sufficient for a redundant type structure to have a non-redundant expansion. Somewhat surprisingly, it clarifies the gap between the mathematical definition of non-redundancy and its interpretation in terms of belief hierarchies (see Remark 2).

4.2. Related literature

Researchers have explored the issue of redundant types using different approaches. Battigalli and Siniscalchi [5] provide a unified framework for the analysis of both dynamic and static games with incomplete information, in which actions are treated as basic uncertainties. They show that the Bayesian equilibrium outcomes in all type structures with compatible restrictions on beliefs are precisely those predicted in their framework. Ely and Peski [16] define a different notion of hierarchy that incorporates conditional beliefs. They show that this new notion of hierarchy captures all the information that is relevant to the solution concept of interim rationalizability, but not to Bayesian equilibrium. Dekel, Fudenberg, and Morris [12,13] propose a different solution concept of "interim correlated rationalizability" which is invariant over the class of type structures modeling the same set of belief hierarchies. In their definition, a player could believe that his opponents' action choices depend on both their types and the unknown payoff parameters.

The interpretation of redundant types in this paper is solution-concept independent. We emphasize the question of whether a player can describe and distinguish his own types using the language that the types try to capture. In a parallel paper, [22] shows how to construct a redundant type structure from a non-redundant type structure through a state-dependent correlating mechanism, which is an incomplete information analog of the correlating device on complete information games (see [2,3,14,17]).

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Appendix A

A.1. Proof of Lemma 1

Proof. (1) Consider two distinct probability measures $\mu_1, \mu_2 \in \Delta(X)$. Since $\mu_1 \neq \mu_2$, there exists $E \in \Sigma$ and $0 \leqslant p \leqslant 1$ such that $\mu_1(E) \geqslant p > \mu_2(E)$. This implies that $\mu_1 \in b^p(E)$ and $\mu_2 \notin b^p(E)$. Thus, μ_1 and μ_2 can be separated by the measurable set $b^p(E) \in \Sigma_\Delta$. $(\Delta(X), \Sigma_\Delta)$ is separative.

(2) If (X, Σ) is separative, then for any distinct points x and y there exists $E \in \Sigma$ such that $x \in E$ and $y \notin E$. Thus $\delta_x(E) = 1$ and $\delta_y(E) = 0$. $\delta_x \neq \delta_y$. Conversely, if $\delta_x \neq \delta_y$, then there exists $E \in \Sigma$ such that $\delta_x(E) = 1$ and $\delta_y(E) = 0$, and hence $x \in E$ and $y \notin E$. That is, (X, Σ) is separative.

(3) Let Θ be an arbitrary index set and X_θ be separative for each $\theta \in \Theta$. Consider the product space $X = \prod_{\theta \in \Theta} X_\theta$ with the usual product σ-field, i.e., the smallest σ-field such that each canonical projection Proj_θ is measurable. Consider two different points $x = (x_\theta)_{\theta \in \Theta}$ and $x' = (x'_\theta)_{\theta \in \Theta}$ in the product space. Since the two points are distinct, $x_{\theta^*} \neq x'_{\theta^*}$ for some $\theta^* \in \Theta$. Since X_{θ^*} is separative, it contains a measurable subset E so that $x_{\theta^*} \in E$ and $x'_{\theta^*} \notin E$. Therefore, $x \in (\mathrm{Proj}_{\theta^*})^{-1}(E)$ and $x' \notin (\mathrm{Proj}_{\theta^*})^{-1}(E)$. It follows immediately that the product space X is separative.

(4) Consider two distinct points x and x' in X. Then $f(x) \neq f(x')$ since f is injective. By the separativity of $(Y, \Sigma(Y))$, there exists measurable subset $E \subset Y$ such that $f(x) \in E$ and $f(x') \notin E$. Therefore $f^{-1}(E)$ is a measurable subset of X containing x and not x'. Thus, $(X, \Sigma(X))$ is separative.

(5) If all singletons are measurable, then X is separative by definition. The converse is not true. Here is a counterexample. Let $\{0, 1\}$ be the two-point set with the discrete σ-field. It is separative. By (3), the product space $\{0, 1\}^{[0,1]}$ with the product σ-field is separative as well. Consider the collection of its measurable subsets Σ that contains all the subsets $E \subset \{0, 1\}^{[0,1]}$ such that either $\mathrm{Proj}_\theta E \in \{\{0\}, \{1\}\}$ or $\mathrm{Proj}_\theta E^c \in \{\{0\}, \{1\}\}$ for at most countably many $\theta \in [0, 1]$.

Note that $\{0, 1\}^{[0,1]} \in \Sigma$, $\emptyset \in \Sigma$, and Σ is closed under complements and countable unions, and hence it is a sub-σ-field of the product σ-field. Furthermore, by the definition of Σ, Proj_θ is measurable on Σ for each θ. Since the product σ-field is the smallest σ-field for which each Proj_θ is measurable, Σ is just the product σ-field on $\{0, 1\}^{[0,1]}$. But Σ does not contain singletons (see, e.g., Dudley [15, p. 223]). □

A.2. Measurability of hierarchy maps in Section 2.3

Let $f : X \to Y$ be a measurable map and $g : \Delta(X) \to \Delta(Y)$ be the corresponding image map; that is, $g(\mu) = \mu f^{-1}$ for each $\mu \in \Delta(X)$ according to our convention. Let $e : X \to \Delta(X)$ be another measurable map.

Lemma 4. *g is measurable.*

Proof. We only need to show that for $F = \{\upsilon \in \Delta(Y)\colon \upsilon(E) \geqslant p\}$, where $0 \leqslant p \leqslant 1$ and E is a measurable subset of Y, $g^{-1}(F)$ is measurable in $\Delta(X)$, since F generates the σ-field of $\Delta(Y)$. By definition, we have

$$\begin{aligned} g^{-1}(F) &= \{\mu \in \Delta(X)\colon\ \mu f^{-1} \in F\} \\ &= \{\mu \in \Delta(X)\colon\ \mu(f^{-1}(E)) \geqslant p\}. \end{aligned}$$

Since f is measurable, $f^{-1}(E)$ is a measurable subset of X. Hence $g^{-1}(F)$ is a measurable subset of $\Delta(X)$ by the definition of σ-field on $\Delta(X)$. Therefore, g is measurable. □

The following is immediate.

Corollary 1. *$x \mapsto e(x) f^{-1}$ is measurable.*

Lemma 5. *$\bar{h}_i^k$, h_i^k, $\bar{h}^k$, h^k, $\bar{h}_i$ h_i, $i \in I$, $k = 1, 2, \ldots, \bar{h}$, and h are all measurable.*

Proof. By definition, $\bar{h}_i^1(t_i) = m_i(t_i)(\mathrm{Proj}_{S\times C})^{-1}$. Its measurability is immediate by applying Corollary 1 (replace e by m_i and f by $\mathrm{Proj}_{S\times C}$). Therefore, $\bar{h}^1(t) = (\bar{h}_i^1(t_i))_{i\in I}$ is measurable (see [1, Lemma 4.48]). Since $\bar{p}^1(s, c, t) = (s, c, \bar{h}^1(t))$, $\bar{p}^1$ is measurable as well. Suppose $\bar{h}_i^l$, $\bar{h}^l$, and $\bar{p}^l$ are measurable, $i \in I$ and $l = 1, \ldots, k$. We complete the proof by induction. Consider $l = k + 1$. By definition and Corollary 1, we have $\bar{h}_i^{k+1}(t_i) = m_i(t_i)(\bar{p}^k)^{-1}$, and hence $\bar{h}_i^{k+1}$ and $\bar{h}^{k+1}$ are measurable. Now $\bar{p}^{k+1}(s, c, t) = (s, c, \bar{h}^1(t), \ldots, \bar{h}^{k+1}(t))$ is measurable. The induction is complete.

Since $\bar{h}_i(t_i) = (\bar{h}_i^1(t_i), \bar{h}_i^2(t_i), \ldots)$ and $\bar{h}(t) = (\bar{h}^1(t), \bar{h}^2(t), \ldots)$, both $\bar{h}_i$ and $\bar{h}$ are measurable.

The measurability of the partial hierarchy maps can be proved in the same way. □

A.3. Proof of Proposition 1

Proof. By Definition 5,

$$\mathrm{Marg}_{S\times T'}\, m_i'(\tau_i(t_i)) = (\mathrm{Marg}_{S\times T}\, m_i(t_i))(\mathrm{Id}_S, \tau)^{-1}. \tag{8}$$

Therefore, for any $t_i \in T_i$, and any measurable subset E_0 of S,

$$m_i'(\tau_i(t_i))(E_0 \times C' \times T') = m_i(t_i)(E_0 \times C \times T). \tag{9}$$

From (9) and the definition of the first order partial belief hierarchy map, $h_i'^1 \circ \tau_i = h_i^1$. Suppose $h_i'^l \circ \tau_i = h_i^l$ for each $i \in I$ and $1 \leqslant l \leqslant k$. Consider measurable sets $E_0 \subset S$ and $E_l \subset H^l(S)$ for $1 \leqslant l \leqslant k$. Then

$$(h^l)^{-1}(E_l) = (h'^l \circ \tau)^{-1}(E_l) = \tau^{-1}((h'^l)^{-1}(E_l)). \tag{10}$$

Recall that $p^k(s, c, t) = (s, h^1(t), \ldots, h^k(t))$ for each $(s, c, t) \in S \times C \times T$, then

$$(p^k)^{-1}\left(\prod_{l=0}^{k} E_l\right) = E_0 \times C \times \bigcap_{l=1}^{k} (h^l)^{-1}(E_l).$$

Similarly, $p'^k(s, c', t') = (s, h'^1(t'), \ldots, h'^k(t'))$, and

$$(p'^k)^{-1}\left(\prod_{l=0}^{k} E_l\right) = E_0 \times C \times \bigcap_{l=1}^{k} (h'^l)^{-1}(E_l),$$

where p'^k and h'^k are the corresponding maps on M'. Therefore,

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$$m_i'\big(\tau_i(t_i)\big)\big(p'^k\big)^{-1}\left(\prod_{l=0}^{k} E_l\right) = m_i'\big(\tau_i(t_i)\big)\left(E_0 \times C \times \bigcap_{l=1}^{k}(h'^l)^{-1}(E_l)\right)$$

$$= \big(\mathrm{Marg}_{S\times T'}\, m_i'\big(\tau_i(t_i)\big)\big)\left(E_0 \times \bigcap_{l=1}^{k}\big(h'^l\big)^{-1}(E_l)\right). \tag{11}$$

Similarly,

$$m_i(t_i)\big(p^k\big)^{-1}\left(\prod_{l=0}^{k} E_l\right) = \big(\mathrm{Marg}_{S\times T}\, m_i(t_i)\big)\left(E_0 \times \bigcap_{l=1}^{k}\big(h^l\big)^{-1}(E_l)\right). \tag{12}$$

By (8), 10, (11), and (12),

$$m_i'\big(\tau_i(t_i)\big)\big(p'^k\big)^{-1}\left(\prod_{l=0}^{k} E_l\right) = m_i(t_i)\big(p^k\big)^{-1}\left(\prod_{l=0}^{k} E_l\right).$$

Since all the sets of the form $\prod_{l=0}^{k} E_l$ generate the σ-field on $S \times \prod_{l=1}^{k} H^l(S)$, we have

$$m_i'\big(\tau_i(t_i)\big)\big(p'^k\big)^{-1} = m_i(t_i)\big(p^k\big)^{-1}.$$

By the definition of $h_i'^{k+1}$ and h_i^{k+1}, $h_i'^{k+1} \circ \tau_i = h_i^{k+1}$. The induction is complete. □

A.4. Proof of Proposition 2

Proof. By definition, the hierarchy map $\bar{h}_i$ is injective if and only if $\bar{h}_i(t_i) \neq \bar{h}_i(t_i')$ whenever $t_i \neq t_i'$. Since $H_i(X)$ with the product σ-field is separative even though $(X, \Sigma(X))$ may not be (Lemma 1), $\bar{h}_i(t_i) \neq \bar{h}_i(t_i')$ if and only if there exists a measurable subset E of $H_i(X)$ such that $\bar{h}_i(t_i) \in E$ and $\bar{h}_i(t_i') \notin E$. Now let us consider the σ-field $\sigma(\bar{h}_i)$. It consists exactly of sets $(\bar{h}_i)^{-1}(E)$ for any measurable subset E of $H_i(X)$ (Billingsley [8, Theorem 20.1]). Thus, $\sigma(\bar{h}_i)$ is separative if and only if for any distinct points t_i and t_i' in T_i there exists a measurable subset E of $H_i(X)$ such that $t_i \in (\bar{h}_i)^{-1}(E)$ and $t_i' \notin (\bar{h}_i)^{-1}(E)$. In summary, we have shown that $\sigma(\bar{h}_i)$ is separative if and only if $\bar{h}_i$ is injective. Part (1) then follows from Definition 6.

Since $\bar{h}_i$ is measurable on T_i, the σ-field $\Sigma(T_i)$ is finer than $\sigma(\bar{h}_i)$, the smallest σ-field such that $\bar{h}_i$ is measurable. Therefore, $(T_i, \Sigma(T_i))$ is separative whenever $\sigma(\bar{h}_i)$ is. By Definition 6, if the underlying type structure $\langle X, T, m\rangle$ is non-redundant, $\sigma(\bar{h}_i)$ is separative, and hence $(T_i, \Sigma(T_i))$ is separative, as required by (2). Note that the result is immediate from part (5) of Lemma 1 if in addition all singletons are measurable.

Heifetz and Samet [20, Section 5, Lemma 5.4, and Theorem 5.5] construct a non-redundant universal structure $\langle X, T^*, m^*\rangle$.[14] It is separative by (2). Since the universal structure is unique up to a type isomorphism (this follows from the definition of the universal type structure), the result follows immediately from Lemma 1(4). □

A.5. Proof of Lemma 2

Proof. Note that $\mathcal{S}_\Delta \subset \Sigma_\Delta$ by definition. To prove the other direction, consider $\mathcal{L} = \{E \in \Sigma\colon b^p(E) \in \mathcal{S}_\Delta, 0 \leqslant p \leqslant 1\}$. $\Sigma_\Delta \subset \mathcal{S}_\Delta$ will follow if $\Sigma \subset \mathcal{L}$. Notice that $\mathcal{S} \subset \mathcal{L}$ and $\mathcal{S}$ is a

14 In their construction, T_i^* is identified as a subspace of $H_i(X)$, which is separative.

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π-system. It suffices to show $\mathcal{L}$ is a λ-system, as $\Sigma = \sigma(\mathcal{S}) \subset \mathcal{L}$ by Dynkin's π–λ Theorem (see [8, Theorem 3.2]).

(1) $X \in \mathcal{L}$ is obvious.

(2) Suppose $E \in \mathcal{L}$. Then $b^p(E) \in \mathcal{S}_\Delta$ for each $0 \leqslant p \leqslant 1$. Note that

$$\{\mu\colon \mu(E) > 1-p\} = \bigcup_{k=1}^{\infty}\left\{\mu\colon \mu(E) \geqslant 1-p+\frac{1}{k}\right\} \in \mathcal{S}_\Delta.$$

Therefore $b^p(E^c) = \{\mu\colon \mu(E^c) \geqslant p\} = \{\mu\colon \mu(E) \leqslant 1-p\} = \Delta(X) - \{\mu\colon \mu(E) > 1-p\} \in \mathcal{S}_\Delta$. Therefore $E^c \in \mathcal{L}$.

Next we show that $\mathcal{L}$ is closed under a countable union of pairwise disjoint sets. As an intermediate step, we shall first show the following:

(3) Consider a finite sequence $\{E_j\}_{j=1}^m$ of pairwise disjoint sets in $\mathcal{L}$. For any integer $k > 0$, consider the set

$$\left\{\mu\colon \sum_{j=1}^{m} \mu(E_j) > p - \frac{1}{k}\right\}. \tag{13}$$

Let $\mathbb{Q}$ be the set of rationals in the real line. The set in (13) can be written as

$$\bigcup_{q_1,\dots,q_{m-1}\in\mathbb{Q}} \{\mu\colon \mu(E_1) > q_1\} \cap \dots \cap \{\mu\colon \mu(E_{m-1}) > q_{m-1}\}$$
$$\cap \left\{\mu\colon \mu(E_m) > p - \frac{1}{k} - \sum_{j=1}^{m-1} q_j\right\}. \tag{14}$$

To see this, note that for any μ in (14), we have $\sum_{j=1}^m \mu(E_j) > p - 1/k$ by definition; for any μ in (13) we can find a set of rational numbers $\{q_1, \dots, q_{m-1}\}$ such that $\mu(E_j) > q_j$ for each $j = 1, \dots, m-1$ and $\mu(E_m) > p - 1/k - \sum_{j=1}^{m-1} q_j$, and hence μ is in (14).

Note that (14) is a countable union of finite intersections of measurable sets, and hence is measurable. Therefore the set (13) is measurable.

(4) Consider a pairwise disjoint sequence $\{E_j\}$ in $\mathcal{L}$. By countable additivity and (3),

$$b^p\left(\bigcup_{j=1}^{\infty} E_j\right) = \left\{\mu\colon \sum_{j=1}^{\infty}\mu(E_j) \geqslant p\right\} = \bigcap_{k=1}^{\infty}\bigcup_{m=1}^{\infty}\left\{\mu\colon \sum_{j=1}^{m}\mu(E_j) > p - \frac{1}{k}\right\} \in \mathcal{S}_\Delta.$$

The second equality follows from the fact that $\sum_{j=1}^\infty \mu(E_j) \geqslant p$ if and only if for any small positive number $1/k$, there exists an m so that $\sum_{j=1}^m \mu(E_j) > p - \frac{1}{k}$. Therefore, $\bigcup_{j=1}^\infty E_j \in \mathcal{L}$.

It follows immediately from (1), (2), and (4) that $\mathcal{L}$ is a λ-system. The claim then follows from Dynkin's π–λ Theorem. □

A.6. *Proof of Theorem 2*

Proof. Note that both τ_i^{-1} and τ_i are measurable since τ is a type isomorphism. For a Bayesian equilibrium $(\beta_i)_{i\in I}$ of M, let us write

$$f_i(s,t) := \int_A g_i(s,a) \prod_{j\in I} \beta_j(t_j, da_j)$$

and

$$f_i^{a_i'}(s,t) := \int_A g_i\big(s, a_i', a_{-i}\big) \prod_{j \in I \setminus \{i\}} \beta_j(t_j, da_j).$$

Here, $f_i(s,t)$ is player i's payoff when the payoff-relevant state is s, the type profile is t, and the players follow the equilibrium strategy profile $(\beta_i)_{i\in I}$; $f_i^{a_i'}(s,t)$ is player i's payoff evaluated at (s,t) when player i plays a_i' and the other players follow their equilibrium strategies.

Consider the strategy profile $(\overline{\beta}_i)_{i\in I}$ on $\overline{M}$ defined by $\overline{\beta}_i(\cdot,\cdot) = \beta_i(\tau_i^{-1}(\cdot),\cdot)$. For each $i \in I$ and $(s,\bar{t}) \in S \times \overline{T}$, we have

$$\begin{aligned}\int_A \bar{g}_i\big((s,c),a\big) \prod_{j\in I} \overline{\beta}_j(\bar{t}_j, da_j) &= \int_A g_i(s,a) \prod_{j\in I} \beta_j\big(\tau_j^{-1}(\bar{t}_j), da_j\big) \\ &= \big(f_i \circ (\mathrm{Id}_S, \tau)^{-1}\big)(s,\bar{t}). \end{aligned} \tag{15}$$

That is, $(f_i \circ (\mathrm{Id}_S,\tau)^{-1})(s,\bar{t})$ is i's payoff evaluated at $(s,\bar{t})$ when the players follow $(\overline{\beta}_i)_{i\in I}$.

Similarly,

$$\int_A \bar{g}_i\big((s,c), a_i', a_{-i}\big) \prod_{j\in I\setminus\{i\}} \overline{\beta}_j(\bar{t}_j, da_j) = \big(f_i^{a_i'} \circ (\mathrm{Id}_S,\tau)^{-1}\big)(s,\bar{t}),$$

where the expression $(f_i^{a_i'} \circ (\mathrm{Id}_S,\tau)^{-1})(s,\bar{t})$ on the right-hand side is player i's payoff when i takes a_i' and the other players follow the proposed strategies.

With some abuse of notation, let us treat $f_i^{a_i'} \circ (\mathrm{Id}_S,\tau)^{-1}$ and $f_i \circ (\mathrm{Id}_S,\tau)^{-1}$ as functions on $S \times C \times \overline{T}$ because by Eq. (15) C consists of payoff-irrelevant parameters. Note that $(\overline{\beta}_i)_{i\in I}$ is a Bayesian equilibrium of $\overline{M}$ if and only if for any $i \in I$, $\bar{t}_i \in \overline{T}_i$, and $a_i' \in A_i$,

$$\int_{S\times C\times\overline{T}} f_i \circ (\mathrm{Id}_S,\tau)^{-1} d\overline{m}_i(\bar{t}_i) \geqslant \int_{S\times C\times\overline{T}} f_i^{a_i'} \circ (\mathrm{Id}_S,\tau)^{-1} d\overline{m}_i(\bar{t}_i). \tag{16}$$

By Definition 7, for any $\bar{t}_i \in \overline{T}_i$, $\mathrm{Marg}_{S\times T'}\,\overline{m}_i(\bar{t}_i) = m_i(\tau_i^{-1}(\bar{t}_i))(\mathrm{Id}_S,\tau)^{-1}$. That is, for any $\bar{t}_i \in \overline{T}_i$, and any $E \in \Sigma(S\times\overline{T})$,

$$\overline{m}_i(\bar{t}_i)(E \times C) = m_i\big(\tau_i^{-1}(\bar{t}_i)\big)(\mathrm{Id}_S,\tau)^{-1}(E).$$

Therefore, for $F_i = f_i$ or $f_i^{a_i'}$,

$$\int_{S\times C\times\overline{T}} F_i \circ (\mathrm{Id}_S,\tau)^{-1}\, d\overline{m}_i(\bar{t}_i) = \int_{S\times C\times\overline{T}} F_i \circ (\mathrm{Id}_S,\tau)^{-1}\, dm_i\big(\tau_i^{-1}(\bar{t}_i)\big)(\mathrm{Id}_S,\tau)^{-1}. \tag{17}$$

By the change of variables theorem [8, Theorem 16.13], the right-hand side of (17) can be rewritten as

$$\int_{S\times T} F_i\, dm_i\big(\tau_i^{-1}(\bar{t}_i)\big). \tag{18}$$

Therefore, (16) is equivalent to

$$\int_{S\times T} f_i\, dm_i\big(\tau_i^{-1}(\bar{t}_i)\big) \geqslant \int_{S\times T} f_i^{a_i'}\, dm_i\big(\tau_i^{-1}(\bar{t}_i)\big). \tag{19}$$

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On the other hand, $(\beta_i)_{i\in I}$ is a Bayesian equilibrium of M if and only if for any $i \in I$, $t_i \in T_i$, and $a_i' \in A_i$,

$$\int\limits_{S\times T} f_i \, dm_i(t_i) \geqslant \int\limits_{S\times T} f_i^{a_i'} \, dm_i(t_i). \tag{20}$$

The theorem follows immediately by comparing (20) and (19). □

A.7. Proof Lemma 3

Proof. We break the proof into several steps.

Step 1. We shall show that the product σ-field on $X \times T'$ is $\{(X \times T') \cap G \colon G \in \Sigma(X \times T)\}$, the relative σ-field that $X \times T'$ inherits from $X \times T$.

The σ-field on T' is generated by $T' \cap E$ where $E = E_1 \times \cdots \times E_n$ and $E_i \in \Sigma(T_i)$. On the one hand, the product σ-field on $X \times T'$ is generated by all sets of the form $D \times (T' \cap E)$, $D \in \Sigma(X)$ and $E \in \Sigma(T)$. On the other hand, the σ-field $\{(X \times T') \cap G \colon G \in \Sigma(X \times T)\}$ is generated by all sets of the form $(X \times T') \cap (D \times E)$, $D \in \Sigma(X)$ and $E \in \Sigma(T)$. The claim then follows from the equality $(X \times T') \cap (D \times E) = D \times (T' \cap E)$.

Step 2. We shall show that for each $i \in I$ and $t_i \in T_i'$, $m_i'(t_i)$ is a well-defined probability measure on the σ-field of $X \times T'$. We check only the countable additivity. Let F_k, $k = 1, 2, \ldots,$ be a sequence of pairwise disjoint measurable sets of $X \times T'$. By Step 1, $F_k = (X \times T') \cap G_k$ for some measurable subset G_k of $X \times T$. Furthermore, G_k, $k = 1, 2, \ldots,$ is a pairwise disjoint sequence. By definition,

$$\begin{aligned}
m_i'(t_i)\left(\bigcup_{k=1}^{\infty} F_k\right) &= m_i'(t_i)\left(\bigcup_{k=1}^{\infty}\big((X \times T') \cap G_k\big)\right) \\
&= m_i(t_i)\left(\bigcup_{k=1}^{\infty} G_k\right) \\
&= \sum_{k=1}^{\infty} m_i(t_i)(G_k) \\
&= \sum_{k=1}^{\infty} m_i'(t_i)\big((X \times T') \cap G_k\big) \\
&= \sum_{k=1}^{\infty} m_i'(t_i)(F_k).
\end{aligned}$$

Step 3. We show below that m_i' is measurable. By Step 1, we only need to show that for each $G \in \Sigma(X \times T)$, the set $E_i' := \{t_i \in T_i' \colon m_i'(t_i)((X \times T') \cap G) \geqslant p\}$, where $0 \leqslant p \leqslant 1$, is measurable in T_i'. By condition (2),

$$E_i' = T_i' \cap \{t_i \colon m_i(t_i)(G) \geqslant p\}.$$

Note that $E_i := \{t_i \colon m_i(t_i)(G) \geqslant p\} \in \Sigma(T_i)$ by the measurability of $m_i(t_i)$. Hence $E_i' = T_i' \cap E_i$ is a measurable set in T_i' by condition (1).

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Step 4. We verify that M' is a type structure. Given the previous two steps, we only need to check the third condition of Definition 2: fix $t_i \in T_i'$ and $E_i' \in \Sigma(T_i')$. From condition (1) of the lemma, $E_i' = T_i' \cap E_i$ for some $E_i \in \Sigma(T_i)$. Then by condition (2),

$$\begin{aligned}\delta_{t_i}\big(E_i'\big) &= \big(\mathrm{Marg}_{T_i'} m_i'(t_i)\big)\big(E_i'\big)\\ &= m_i'(t_i)\big(X \times T_{-i}' \times E_i'\big)\\ &= m_i'(t_i)\big(X \times T_{-i}' \times \big(T_i' \cap E_i\big)\big)\\ &= m_i'(t_i)\big((X \times T') \cap (X \times T_{-i} \times E_i)\big)\\ &= m_i(t_i)(X \times T_{-i} \times E_i)\\ &= \big(\mathrm{Marg}_{T_i} m_i(t_i)\big)(E_i)\\ &= \delta_{t_i}(E_i).\end{aligned}$$

Since $E_i' = T_i' \cap E_i$ and $t_i \in T_i'$, $t_i \in E_i'$ if and only if $t_i \in E_i$. Therefore $\delta_{t_i}(E_i') = \delta_{t_i}(E_i)$.

Step 5. By the definition of relative σ-fields, M' is separative if M is separative. □

A.8. A version of Fubini's Theorem

We need the following extension of Fubini's Theorem in the construction of Theorem 3.

Proposition 3. *Let $f : S \times A_1 \times A_2 \times T_1 \times T_2 \to \mathbb{R}^+$ be a jointly measurable function. For $i = 1, 2$, let $\kappa_i(t_i, da) : T_i \times \Sigma(A_i) \to [0, 1]$ be a stochastic kernel, and let $\pi_i : T_i \to \Delta(S \times T)$ be measurable.*

(1) *$F(s, t) = \int_{A_1} \int_{A_2} f(s, a_1, a_2, t_1, t_2)\kappa_2(t_2, da_2)\kappa_1(t_1, da_1)$ is jointly measurable on $S \times T$.*
(2) *There exists a unique map $P_i : T_i \to \Delta(S \times T \times A)$ such that for any $E \in \Sigma(S \times T)$ and $F_i \in \Sigma(A_i)$, $i \in \{1, 2\}$,*

$$P_i(t_i)(E \times F_1 \times F_2) = \int_E \kappa_1\big(t_1', F_1\big)\kappa_2\big(t_2', F_2\big)\, d\pi_i(t_i).$$

(3) *$P_i : T_i \to \Delta(S \times T \times A)$ is measurable.*
(4) *$\int_{S\times T}[\int_{A_1} \int_{A_2} f(s, a, t)\kappa_2(t_2', da_2)\kappa_1(t_1', da_1)]\, d\pi_i(t_i) = \int_{S\times A\times T} f(s, a, t)\, dP_i(t_i)$.*

Let us first prove the following lemma.

Lemma 6. *Suppose $\pi : \Theta \to \Delta(X)$ is measurable with respect to the corresponding σ-fields. Then $\int f(x)\, d\pi(\theta) : \Theta \to R$ is measurable for any non-negative real-valued measurable function f with domain X.*

Proof. For a fixed measurable set $E \subset X$, $\theta \mapsto \pi(\theta)(E)$ is a real-valued measurable function since

$$\big\{\theta\colon \pi(\theta)(E) \geqslant r\big\} = \pi^{-1}\big\{\mu\colon \mu(E) \geqslant r\big\}$$

is a measurable subset of Θ by the measurability of π. For a simple function $g = \sum_{k=1}^m b_k 1_{E_k}$ on X, $\int g(x)\, d\pi(\theta) = \sum_{k=1}^m b_k \pi(\theta)(E_k)$ is a measurable function on Θ. Since f is a pointwise limit of an increasing sequence $\{g_l\}$ of simple functions, $\int f(x)\, d\pi(\theta) = \lim_l \int g_l(x)\, d\pi(\theta)$ for

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each $\theta \in \Theta$ by the monotone convergence theorem, and therefore $\theta \mapsto \int f(x)\,d\pi(\theta)$ is measurable. $\square$

Proof of Proposition 3. (1) It suffices to show $(s,t) \mapsto \int_{A_1}\int_{A_2} 1_E(s,a,t)\kappa_2(t_2,da_2)\,\kappa_1(t_1,da_1)$ is jointly measurable for every measurable subset $E \subset S \times A \times T$. The stated result will then follow from passing to the limit of an increasing sequence of non-negative simple functions. First consider $E = E^1 \times E_1^2 \times E_2^2 \times E^3$ where $E^1 \in \Sigma(S)$, $E_i^2 \in \Sigma(A_i)$ and $E^3 \in \Sigma(T)$. We then have

$$\int_A 1_{E^1 \times E^2 \times E^3}(s,a,t)\kappa_2(t_2,da_2)\kappa_1(t_1,da_1) = 1_{E^1}(s)\kappa_2\big(t_2,E_2^2\big)\kappa_1\big(t_1,E_1^2\big)1_{E^3}(t).$$

Consider $F^r = \{(s,t)\colon 1_{E^1}(s)\kappa_2(t_2,E_2^2)\kappa_1(t_1,E_1^2)1_{E^3}(t) > r\}$. By definition, $F^r = \emptyset$ for $r \geqslant 1$; $F^r = S \times T$ for $r < 0$; for $0 \leqslant r < 1$,

$$F^r = E^1 \times \big(\big\{t\colon \kappa_2\big(t_2,E_2^2\big)\kappa_1\big(t_1,E_1^2\big) > r\big\} \cap E^3\big).$$

Notice that

$$\big\{t\colon \kappa_2\big(t_2,E_2^2\big)\kappa_1\big(t_1,E_1^2\big) > r\big\} = \bigcup_{q \in \mathbb{Q}, q>0,} \big\{t_1\colon \kappa_1\big(t_1,E_1^2\big) > q\big\} \times \big\{t_2\colon \kappa_2\big(t_2,E_2^2\big) > r/q\big\}$$

is measurable in T. Therefore, F^r is measurable in $S \times T$. $(s,t) \mapsto \int_{A_1}\int_{A_2} 1_E(s,a,t)\kappa_2(t_2,da_2)$ $\kappa_1(t_1,da_1)$ is jointly measurable for measurable rectangles $E = E^1 \times E^2 \times E^3$. To show that the result holds for any measurable E in $S \times A \times T$, consider $\mathcal{L} = \{E \in \Sigma(S \times A \times T) : (s,t) \mapsto \int_A 1_E(s,a,t)\kappa_2(t_2,da_2)\ \kappa_1(t_1,da_1)$ is jointly measurable$\}$. We have shown that the π-system $\Sigma(S) \times \Sigma(A_1) \times \Sigma(A_2) \times \Sigma(T)$ of measurable rectangles belongs to $\mathcal{L}$. It will follow from Dynkin's π–λ Theorem that $\Sigma(S \times A \times T) = \mathcal{L}$ if $\mathcal{L}$ is a λ-system. Let us verify that $\mathcal{L}$ is a λ-system. First, it is straightforward to check that $S \times A \times T \in \mathcal{L}$ and that if $E \in \mathcal{L}$ then $E^c \in \mathcal{L}$. Secondly, consider a pairwise disjoint sequence $\{F^k\}$ in $\mathcal{L}$. We have, by the monotone convergence theorem,

$$\int_A 1_{\bigcup_{k=1}^\infty F^k}(s,a,t)\kappa_2(t_2,da_2)\kappa_1(t_1,da_1) = \sum_{k=1}^\infty \int_A 1_{F^k}(s,a,t)\kappa_2(t_2,da_2)\kappa_1(t_1,da_1).$$

From the right-hand side, we see that the left-hand side is jointly measurable in (s,a,t). Thus $\bigcup_{k=1}^\infty F^k \in \mathcal{L}$. Therefore $\mathcal{L}$ is a λ-system.

We only sketch the proof for (2) and (4) since they are not very different from the familiar form of Fubini's Theorem; see, for example, [8, Theorem 18.3 and Exercise 18.20].

(2) For each $t_i \in T_i$, we show $\underline{P}_i(t_i)(E \times F_1 \times F_2) = \int_E \kappa_2(t_2',F_2)\kappa_1(t_1',F_1)\,d\pi_i(t_i)$ is a countably additive probability measure on the semiring $\mathcal{S} = \Sigma(S \times T) \times \Sigma(A_1) \times \Sigma(A_2)$ of measurable rectangles. Suppose $\{E^k \times F_1^k \times F_2^k\}$ is a pairwise disjoint sequence in $\mathcal{S}$ and $\bigcup_{k=1}^\infty E^k \times F_1^k \times F_2^k = E \times F_1 \times F_2 \in \mathcal{S}$. It is immediate that $\bigcup_{k=1}^\infty E^k = E$ and $\bigcup_{k=1}^\infty F_i^k = F_i$, $i = 1,2$. Therefore, for any $\omega \in S \times T$ and $a \in A$,

$$1_{E \times F_1 \times F_2}(\omega,a_1,a_2) = \sum_{k=1}^\infty 1_{E^k \times F_1^k \times F_2^k}(\omega,a_1,a_2) = \sum_{k=1}^\infty 1_{E^k}(\omega)1_{F_1^k}(a_1)1_{F_2^k}(a_2).$$

Integrating with respect to $\kappa_2(t'_2, \cdot)$, then with respect to $\kappa_1(t'_1, \cdot)$, and finally with respect to $\pi_i(t_i)$, we have

$$\int_E \kappa_2(t'_2, F_2)\kappa_1(t'_1, F_1)\, d\pi_i(t_i) = \sum_{k=1}^{\infty} \int_{E^k} \kappa_2(t'_2, F_2^k)\kappa_1(t'_1, F_1^k)\, d\pi_i(t_i).$$

Therefore, $\underline{P}_i(t_i)$ is countably additive. By the Caratheodory Extension Theorem, $\underline{P}_i(t_i)$, for each $t_i \in T_i$, extends uniquely to a measure $P_i(t_i)$ on $\Sigma(S \times T \times A)$.

(3) By Lemma 2, we only need to show that $t_i \mapsto P_i(t_i)(E \times F_1 \times F_2)$ is measurable. This measurability follows from Lemma 6.

(4) The proof is standard: check the result first for indicator functions, then for simple functions, and finally for non-negative real functions. □

A.9. Proof of Theorem 3

Proof. Step 1. For the given type structure $M = \langle S, (T_i)_{i\in I}, (m_i)_{i\in I}\rangle$, denote by $\mathcal{S}$ the semiring of measurable rectangles $\Sigma(S) \times \prod_{j\in I} \Sigma(A_j) \times \Sigma(T)$. Define a type structure $\Psi^\beta = \langle S \times A, T, \psi^\beta\rangle$ in the following way. For each $i \in I$, let ψ_i^β be the unique extension of the following probability measure on the semiring of measurable rectangles $\{E \times F \times G\colon E \in \Sigma(S), F = \prod_{j\in I} F_j \in \prod_{j\in I} \Sigma(A_j), G \in \Sigma(T)\}$:

$$\psi_i^\beta(t_i)(E \times F \times G) = \int_{E\times G} \prod_{j\in I} \beta_j(t'_j, F_j)\, dm_i(t_i). \tag{21}$$

The existence and measurability of ψ_i^β is guaranteed by Proposition 3. Furthermore, by taking $E = S$, $F = A$, $G_j = T_j$ for $j \neq i$ in (21), for any $t_i \in T_i$,

$$\psi_i^\beta(t_i)(S \times A \times T_{-i} \times G_i) = m_i(t_i)(S \times T_{-i} \times G_i). \tag{22}$$

Therefore, $\mathrm{Marg}_{T_i} \psi_i^\beta(t_i) = \mathrm{Marg}_{T_i} m_i(t_i) = \delta_{t_i}$. Thus, $\Psi^\beta = \langle S \times A, T, \psi^\beta\rangle$ is a well-defined type structure. We shall show that Id_T is an S-based marginalized type morphism from M to Ψ^β, and hence Ψ^β preserves the S-based belief hierarchies of M. To see this, take $F = A$ in Eq. (21); then

$$\psi_i^\beta(t_i)(E \times A \times G) = m_i(t_i)(E \times G).$$

That is,

$$\big(\mathrm{Marg}_{S\times T} \psi_i^\beta(t_i)\big)(E \times G) = \big(\mathrm{Marg}_{S\times T} m_i(t_i)\big)(E \times G).$$

Since the measurable rectangles generate the σ-field of $S \times T$, we conclude that

$$\mathrm{Marg}_{S\times T} \psi_i^\beta(t_i) = \mathrm{Marg}_{S\times T} m_i(t_i). \tag{23}$$

By Definition 5, Eq. (23) shows that Id_T is an S-based type morphism from M to Ψ^β.

Step 2. Given $\Psi^\beta = \langle S \times A, T, \psi^\beta\rangle$, we define an $(S \times A)$-based non-redundant type structure $\overline{M}^\beta = \langle S \times A, \overline{T}^\beta, \overline{m}^\beta\rangle$ as follows. According to the construction of Heifetz and Samet [20], Ψ^β is mapped into an $(S \times A)$-based universal type structure $\langle S \times A, \overline{T}^*, \overline{m}^*\rangle$ under a unique $(S \times A)$-based type morphism that we write as τ^β. Write $\overline{T}_i^\beta = \tau_i^\beta(T_i) \subset \overline{T}_i^*$. $\overline{T}_i^\beta$ is endowed with the relative σ-field. For any $i \in I$ and $\bar{t}_i \in \overline{T}_i^\beta$, we shall show that $\overline{m}_i^*(\bar{t}_i)$ has a support in

$S \times A \times \overline{T}^\beta$. To see this, suppose to the contrary that $\overline{m}_i^*(\bar{t}_i)(E) > 0$ for some measurable set E, and $E \subset S \times A \times (\overline{T}^* \backslash \overline{T}^\beta)$. Therefore,

$$(S \times A \times T) \cap \left(\mathrm{Id}_{S\times A}, \tau^\beta\right)^{-1}(E) = \emptyset.$$

Consider $t_i \in T_i$ such that $\tau_i^\beta(t_i) = \bar{t}_i$. Then $\psi_i^\beta(t_i)((\mathrm{Id}_{S\times A}, \tau^\beta)^{-1}(E)) = 0$, because $\psi_i^\beta(t_i)(S \times A \times T) = 1$. This contradicts the fact that $\tau^\beta : T \to \overline{T}^\beta$ is a type morphism.

For any $G \in \Sigma(S \times A \times U^*)$, let

$$\overline{m}_i^\beta(\bar{t}_i)\left(\left(S \times A \times \overline{T}^\beta\right) \cap G\right) = \overline{m}_i^*(\bar{t}_i)(G).$$

Then $\overline{M}^\beta = \langle S \times A, \overline{T}^\beta, \overline{m}^\beta\rangle$ is a well-defined non-redundant type structure by Lemma 3. We shall show that $\tau^\beta : T \to \overline{T}^\beta$ is also an S-based marginalized type morphism. Since τ^β is an $(S \times A)$-based type morphism, it follows from Definition 5 that

$$\mathrm{Marg}_{S\times A\times \overline{T}^\beta}\, \overline{m}_i^\beta\left(\tau_i^\beta(t_i)\right) = \left(\mathrm{Marg}_{S\times A\times T}\, \psi_i^\beta(t_i)\right)\left(\mathrm{Id}_{S\times A}, \tau^\beta\right)^{-1}.$$

That is, for all measurable subsets $E \subset S$, $F = \prod_{j\in I} F_j \in \prod_{j\in I} \Sigma(A_j)$, and $G \in \Sigma(\overline{T}^\beta)$,

$$\overline{m}_i^\beta\left(\tau_i^\beta(t_i)\right)(E \times F \times G) = \psi_i^\beta(t_i)\left(\mathrm{Id}_{S\times A}, \tau^\beta\right)^{-1}(E \times F \times G) \tag{24}$$

$$= \psi_i^\beta(t_i)\left(E \times F \times \left(\tau^\beta\right)^{-1}(G)\right). \tag{25}$$

Set $F = A$ in Eq. (25). We have

$$\left(\mathrm{Marg}_{S\times \overline{T}^\beta}\, \overline{m}_i^\beta\left(\tau_i^\beta(t_i)\right)\right)(E \times G) = \left(\mathrm{Marg}_{S\times T}\, \psi_i^\beta(t_i)\right)\left(E \times \left(\tau^\beta\right)^{-1}(G)\right)$$
$$= \left(\mathrm{Marg}_{S\times T}\, \psi_i^\beta(t_i)\right)\left(\mathrm{Id}_S, \tau^\beta\right)^{-1}(E \times G).$$

Since the measurable rectangles $E \times G$ generate the σ-field on $S \times \overline{T}^\beta$, we conclude that

$$\mathrm{Marg}_{S\times \overline{T}}\, \overline{m}_i^\beta\left(\tau_i^\beta(t_i)\right) = \left(\mathrm{Marg}_{S\times T}\, \psi_i^\beta(t_i)\right)\left(\mathrm{Id}_S, \tau^\beta\right)^{-1}. \tag{26}$$

That is, $\tau^\beta : T \to \overline{T}^\beta$ is an S-based type morphism from Ψ^β to $\overline{M}^\beta$.

By comparing Eqs. (26) and (23), we further have

$$\mathrm{Marg}_{S\times \overline{T}}\, \overline{m}_i^\beta\left(\tau_i^\beta(t_i)\right) = \left(\mathrm{Marg}_{S\times T}\, m_i(t_i)\right)\left(\mathrm{Id}_S, \tau^\beta\right)^{-1}.$$

That is, $\tau^\beta : T \to \overline{T}^\beta$ is also an S-based marginalized type morphism from M to $\overline{M}^\beta$.

Step 3. For the fixed $i \in I$ and $t_i \in T_i$, set $E = S$, $G = T$, and $F_j = A_j$ for each $j \neq i$ in Eq. (21). We have

$$\psi_i^\beta(t_i)(S \times F_i \times A_{-i} \times T) = \beta_i(t_i, F_i). \tag{27}$$

For the same i, set $E = S$, $G = \overline{T}^\beta$, and $F_j = A_j$ for each $j \neq i$ in Eq. (25). We obtain

$$\overline{m}_i^\beta\left(\tau_i^\beta(t_i)\right)\left(S \times F_i \times A_{-i} \times \overline{T}^\beta\right) = \psi_i^\beta(t_i)(S \times F_i \times A_{-i} \times T). \tag{28}$$

Comparing (28) and (27), we conclude that

$$\beta_i(t_i, F_i) = \left(\mathrm{Marg}_{A_i}\, \psi_i^\beta(t_i)\right)(F_i) = \left(\mathrm{Marg}_{A_i}\, \overline{m}_i^\beta\left(\tau_i^\beta(t_i)\right)\right)(F_i). \tag{29}$$

Let $f_i : \overline{T}_i^\beta \to T_i$ be an arbitrary selector of the correspondence $(\tau_i^\beta)^{-1} : \overline{T}_i^\beta \rightrightarrows T_i$. Define a strategy $\bar{\beta}_i^\beta : \overline{T}_i^\beta \times \Sigma(A_i) \to [0, 1]$ for player i in type structure $\overline{M}^\beta$ by

$$\overline{\beta}_i^{\beta}(\bar{t}_i,\cdot)=\beta_i\big(f_i(\bar{t}_i),\cdot\big),\quad \forall \bar{t}_i\in \overline{T}_i^{\beta}. \tag{30}$$

We need to verify that $\overline{\beta}_i^{\beta}(\bar{t}_i,\cdot)$ is measurable and independent of f_i.

From Eqs. (29) and (30), for any $\bar{t}_i\in\overline{T}_i^{\beta}$ and $t_i,t_i'\in(\tau_i^{\beta})^{-1}(\bar{t}_i)$,

$$\overline{\beta}_i^{\beta}(\bar{t}_i,\cdot)=\big(\mathrm{Marg}_{A_i}\,\psi_i^{\beta}(t_i)\big)(\cdot)=\big(\mathrm{Marg}_{A_i}\,\psi_i^{\beta}(t_i')\big)(\cdot)=\big(\mathrm{Marg}_{A_i}\,\overline{m}_i^{\beta}(\bar{t}_i)\big)(\cdot).$$

Therefore, the measurability of $\overline{\beta}_i^{\beta}$ follows from the measurability of $\mathrm{Marg}_{A_i}\,\overline{m}_i^{\beta}(\bar{t}_i)$, and $\overline{\beta}_i^{\beta}$ is independent of the selector. By Eq. (29), $\overline{\beta}_i^{\beta}(\tau_i^{\beta}(t_i),\cdot)=\beta_i(t_i,\cdot)$.

Step 4. We shall show that $\overline{\beta}^{\beta}$ defined in Step 3 is an equilibrium on $\overline{M}^{\beta}$ whenever β is an equilibrium of M. From Step 3, $\overline{\beta}_j^{\beta}(\tau_j^{\beta}(t_j),\cdot)=\beta_j(t_j,\cdot)$. Since τ^{β} is surjective, we need to show for each $i\in I$, $t_i\in T_i$, and $a_i'\in A_i$,

$$\begin{aligned}&\int\limits_{(S\times A)\times\overline{T}^{\beta}}\Big[\int\limits_{A}\bar{g}_i\big((s,a''),a\big)\prod_{j\in I}\overline{\beta}_j^{\beta}\big(\bar{t}_j',da_j\big)\Big]\,d\overline{m}_i^{\beta}\big(\tau_i^{\beta}(t_i)\big)\\&\quad\geqslant\int\limits_{S\times A\times\overline{T}^{\beta}}\int\limits_{A}\bar{g}_i\big((s,a''),a_i',a_{-i}\big)\prod_{j\in I\setminus\{i\}}\overline{\beta}_j^{\beta}\big(\bar{t}_j',da_j\big)\,d\overline{m}_i^{\beta}\big(\tau_i^{\beta}(t_i)\big).\end{aligned}\tag{31}$$

From the change of variables theorem, for any $E\in\Sigma(S\times A\times\overline{T}^{\beta})$ and $F\in\prod_{j\in I}\Sigma(A_j)$,

$$\int\limits_{(\mathrm{Id}_{S\times A},\tau^{\beta})^{-1}(E)}\prod_{j\in I}\beta_j\big(t_j',F_j\big)\,d\psi_i^{\beta}(t_i)=\int\limits_{E}\prod_{j\in I}\overline{\beta}_j^{\beta}\big(\bar{t}_j',F_j\big)\,d\overline{m}_i^{\beta}\big(\tau_i^{\beta}(t_i)\big).\tag{32}$$

Therefore,

$$\int\limits_{(S\times A)\times\overline{T}^{\beta}}\Big[\int\limits_{A}\bar{g}_i\big((s,a''),a\big)\prod_{j\in I}\overline{\beta}_j^{\beta}\big(\bar{t}_j',da_j\big)\Big]\,d\overline{m}_i^{\beta}\big(\tau_i^{\beta}(t_i)\big)$$

$$=\int\limits_{(S\times A)\times T}\Big[\int\limits_{A}\bar{g}_i\big((s,a''),a\big)\prod_{j\in I}\beta_j\big(t_j',da_j\big)\Big]\,d\psi_i^{\beta}(t_i)\tag{33}$$

$$=\int\limits_{(S\times A)\times T}\Big[\int\limits_{A}g_i(s,a)\prod_{j\in I}\beta_j\big(t_j',da_j\big)\Big]\,d\psi_i^{\beta}(t_i)\tag{34}$$

$$=\int\limits_{S\times T}\Big[\int\limits_{A}\Big[\int\limits_{A}g_i(s,a)\prod_{j\in I}\beta_j\big(t_j',da_j\big)\Big]\prod_{j\in I}\beta_j\big(t_j',da_j\big)\Big]\,dm_i(t_i)\tag{35}$$

$$=\int\limits_{S\times T}\Big[\int\limits_{A}g_i(s,a)\prod_{j\in I}\beta_j\big(t_j',da_j\big)\Big]\,dm_i(t_i),\tag{36}$$

where equality (33) follows from Eq. (32) and Fubini's Theorem; equality (35) follows from Eq. (21) and Fubini's Theorem; and equality (36) is obtained by a simple calculation.

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Similarly,

$$\int_{S\times A\times\overline{T}^{\beta}}\int_A \bar{g}_i\big((s,a''),a'_i,a_{-i}\big)\prod_{j\in I\setminus\{i\}}\overline{\beta}^{\beta}_j\big(\bar{t}'_j,da_j\big)\,dm^{\beta}_i\big(\tau^{\beta}_i(t_i)\big)$$
$$=\int_{S\times T}\left[\int_A g_i\big(s,a'_i,a_{-i}\big)\prod_{j\in I\setminus\{i\}}\beta_j\big(t'_j,da_j\big)\right]dm_i(t_i). \tag{37}$$

Since $(\beta_j)_{j\in I}$ is a Bayesian equilibrium on M,

$$\int_{S\times T}\left[\int_A g_i(s,a)\prod_{j\in I}\beta_j\big(t'_j,da_j\big)\right]dm_i(t_i)$$
$$\geqslant\int_{S\times T}\left[\int_A g_i\big(s,a'_i,a_{-i}\big)\prod_{j\in I\setminus\{i\}}\beta_j\big(t'_j,da_j\big)\right]dm_i(t_i).$$

Together with (36) and (37), this implies inequality (31). Therefore, $(\overline{\beta}^{\beta}_j)_{j\in I}$ is a Bayesian equilibrium on $\overline{M}^{\beta}$.

Step 5. We shall show $\overline{M}^{\beta}$ does not introduce new equilibrium predictions. For any equilibrium $\overline{\beta}$ of $\overline{M}^{\beta}$, define a strategy profile $\beta^{\overline{\beta}}$ on M as follows. For each $i\in I$ and $t_i\in T_i$,

$$\beta^{\overline{\beta}}_i(t_i,\cdot):=\overline{\beta}_i\big(\tau^{\beta}_i(t_i),\cdot\big).$$

The measurability of $\beta^{\overline{\beta}}_i$ follows from the measurability of $\overline{\beta}$ and τ^{β}. Thus $\beta^{\overline{\beta}}$ is a well-defined strategy profile. We shall show that $\beta^{\overline{\beta}}$ is an equilibrium on M. That is, we need to show that for each $i\in I$, $t_i\in T_i$, and $a'_i\in A_i$,

$$\int_{S\times T}\left[\int_A g_i(s,a)\prod_{j\in I}\beta^{\overline{\beta}}_j\big(t'_j,da_j\big)\right]dm_i(t_i)$$
$$\geqslant\int_{S\times T}\left[\int_A g_i\big(s,a'_i,a_{-i}\big)\prod_{j\in I\setminus\{i\}}\beta^{\overline{\beta}}_j\big(t'_j,da_j\big)\right]dm_i(t_i). \tag{38}$$

Following the same arguments as in Step 4, the left-hand side of inequality (38) is

$$\int_{S\times A\times\overline{T}^{\beta}}\left[\int_A \bar{g}_i\big((s,a'),a\big)\prod_{j\in I}\overline{\beta}_j\big(\bar{t}'_j,da_j\big)\right]d\overline{m}^{\beta}_i\big(\tau^{\beta}_i(t_i)\big) \tag{39}$$

and the right-hand side of inequality (38) is

$$\int_{S\times A\times\overline{T}^{\beta}}\int_A \bar{g}_i\big((s,a''),a'_i,a_{-i}\big)\prod_{j\in I\setminus\{i\}}\overline{\beta}_j\big(\bar{t}'_j,da_j\big)\,d\overline{m}^{\beta}_i\big(\tau^{\beta}_i(t_i)\big). \tag{40}$$

Since $\overline{\beta}$ is an equilibrium on $\overline{M}^{\beta}$ by assumption, the expression in (39) is greater than or equal to the expression in (40). It follows immediately that $\beta^{\overline{\beta}}$ is an equilibrium on M.

Next we construct Θ, $\overline{M}^{\theta}$, and $\overline{M}$, as required.

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Step 6. Let B be the set of equilibria on M. Following the previous steps, we can construct an $(S \times A)$-based non-redundant type structure $\overline{M}^{\beta}$ for each $\beta \in B$. It could be that $\overline{M}^{\beta^1} = \overline{M}^{\beta^2}$ for some β^1 and β^2. We use the following simple set operations to remove the duplications. Set $\Theta := \{\overline{M}^{\beta}\colon\ \beta \in B\}$. For any $\theta \in \Theta$, let $\overline{M}^{\theta} := \theta$. By the definition of Θ, for each θ there is a unique $\beta^{\theta} \in B$ such that $\theta = \overline{M}^{\beta^{\theta}}$. Set $\tau^{\theta} := \tau^{\beta^{\theta}}$.

Step 7. We now construct $\overline{M}$. Recall that $\langle S \times A, \overline{T}^*, \overline{m}^* \rangle$ is the $(S \times A)$-based universal type structure. From Step 2, $\overline{T}_i^{\theta} \subset \overline{T}^*$. Set $\overline{T}_i := \bigcup_{\theta \in \Theta} \overline{T}_i^{\theta}$, and hence $\overline{T}_i \subset \overline{T}_i^*$. Let $\overline{T}_i$ inherit the relative σ-field from $\overline{T}_i^*$. For any $i \in I$ and $\bar{t}_i \in \overline{T}_i$, $\overline{m}_i^*(\bar{t}_i)$ has a support in $S \times A \times \overline{T}$ (see Step 2). For any $G \in \Sigma(S \times A \times \overline{T}^*)$, set

$$\overline{m}_i(\bar{t}_i)\big((S \times A \times \overline{T}) \cap G\big) := \overline{m}_i^*(\bar{t}_i)(G).$$

By Lemma 3, $\overline{M} = \langle S \times A, \overline{T}, \overline{m} \rangle$ is a well-defined non-redundant type structure.

Step 8. We shall show that $\overline{M}^{\theta} = \langle S \times A, \overline{T}^{\theta}, \overline{m}^{\theta} \rangle$ constructed in Step 6 is a sub-structure of $\overline{M}$. We need to verify the two conditions in Lemma 3. From Step 2, we know that $\overline{M}^{\theta}$ is a sub-structure of $\langle S \times A, \overline{T}^*, \overline{m}^* \rangle$, and $\overline{T}_i^{\theta}$ inherits the relative σ-field from $\overline{T}_i^*$. $\Sigma(\overline{T}_i^{\theta}) = \{\overline{T}_i^{\theta} \cap E_i\colon\ E_i \in \Sigma(\overline{T}_i^*)\}$. We need to verify that this σ-field is the same σ-field that $\overline{T}_i^{\theta}$ inherits from $\overline{T}_i$. Since $\overline{T}_i$ inherits the σ-field from $\overline{T}_i^*$ (Step 7), $\Sigma(\overline{T}_i) = \{\overline{T}_i \cap E_i\colon\ E_i \in \Sigma(\overline{T}_i^*)\}$. The σ-field that $\overline{T}_i^{\theta}$ inherits from $\overline{T}_i$ is $\{\overline{T}_i^{\theta} \cap F_i\colon\ F_i \in \Sigma(\overline{T}_i)\}$. Note that

$$\begin{aligned}
\big\{\overline{T}_i^{\theta} \cap F_i\colon\ F_i \in \Sigma(\overline{T}_i)\big\} &= \big\{\overline{T}_i^{\theta} \cap F_i\colon\ F_i \in \Sigma(\overline{T}_i)\big\} \\
&= \big\{\overline{T}_i^{\theta} \cap F_i\colon\ F_i = \overline{T}_i \cap E_i, E_i \in \Sigma\big(\overline{T}_i^*\big)\big\} \\
&= \big\{\overline{T}_i^{\theta} \cap (\overline{T}_i \cap E_i)\colon\ E_i \in \Sigma\big(\overline{T}_i^*\big)\big\} \\
&= \big\{\overline{T}_i^{\theta} \cap E_i\colon\ E_i \in \Sigma\big(\overline{T}_i^*\big)\big\} \\
&= \Sigma\big(\overline{T}_i^{\theta}\big).
\end{aligned}$$

This confirms the first condition of Lemma 3. We verify the second condition below.

From Step 2, for each $i \in I$ and $\bar{t}_i \in \overline{T}_i^{\theta}$, $\overline{m}_i^*(\bar{t}_i)$ has a support in $S \times A \times \overline{T}^{\theta}$. By the definition of $\overline{m}_i(\bar{t}_i)$ in Step 7, $\overline{m}_i(\bar{t}_i)$ has a support in $(S \times A \times \overline{T}) \cap (S \times A \times \overline{T}^{\theta}) = S \times A \times \overline{T}^{\theta}$. From the proof of Lemma 3 and the definition of relative σ-field, it follows that for each measurable subset G of $S \times A \times \overline{T}$, there is a measurable subset G^* of $S \times A \times \overline{T}^*$ such that $G = (S \times A \times \overline{T}) \cap G^*$. Therefore,

$$\begin{aligned}
\overline{m}_i^{\theta}(\bar{t}_i)\big(\big(S \times A \times \overline{T}^{\theta}\big) \cap G\big) &= \overline{m}_i^{\theta}(\bar{t}_i)\big(\big(S \times A \times \overline{T}^{\theta}\big) \cap \big((S \times A \times \overline{T}) \cap G^*\big)\big) \\
&= \overline{m}_i^{\theta}(\bar{t}_i)\big(\big(S \times A \times \overline{T}^{\theta}\big) \cap G^*\big) \\
&= \overline{m}_i^*(\bar{t}_i)(G^*) \\
&= \overline{m}_i(\bar{t}_i)\big((S \times A \times \overline{T}) \cap G^*\big) \\
&= \overline{m}_i(\bar{t}_i)\big((S \times A \times \overline{T}) \cap G\big) \\
&= \overline{m}_i(\bar{t}_i)(G).
\end{aligned}$$

Therefore, $\overline{m}_i^{\theta}$ is the restriction of $\overline{m}_i$.

Step 9. We claim that Θ, $\overline{M}^{\theta}$, τ^{θ}, and $\overline{M}$ constructed above satisfy the requirements of the theorem. Let us verify condition (1) first. For any equilibrium β on M, pick θ such that $\theta = \overline{M}^{\beta}$ in Step 6, and hence $\overline{M}^{\theta} = \overline{M}^{\beta}$ (again by Step 6). By Step 4, $\bar{\beta}^{\beta}$ is an equilibrium on

$\overline{M}^{\beta} = \overline{M}^{\theta}$. Denote $\bar{\beta}^{\beta}$ by $\bar{\beta}^{\theta}$. By Step 3, $\bar{\beta}_i^{\theta}(\tau_i^{\theta}(t_i), \cdot) = \beta_i(t_i, \cdot)$ for any $t_i \in T_i$, as required. Now let us verify condition (2). For any θ, it follows from Step 5 that $\overline{M}^{\theta}$ does not introduce new equilibrium predictions. $\beta_i(t_i, \cdot) = \bar{\beta}_i^{\theta}(\tau_i^{\theta}(t_i), \cdot)$ defines an equilibrium on M. □

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