

Higher-Order Beliefs and Epistemic Conditions for Nash Equilibrium

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Abstract

Aumann and Brandenburger (Econometrica 1995) provide tight sufficient epistemic conditions for Nash equilibrium. In contrast with the conventional wisdom that Nash equilibrium involves common beliefs, they show that first-order beliefs suffice for the play of Nash equilibrium in two-player games. We show that if we disentangle beliefs types and payoff types in their framework, then without a second-order belief about payoffs, even common belief about rationality and conjectures is not sufficient for Nash equilibrium. The key of our construction is the correlation between beliefs about payoffs and beliefs about the opponent's actions.

1 Introduction

In an inspiring paper of epistemic foundations for non-cooperative games, Aumann and Brandenburger (Econometrica 1995) provide tight sufficient epistemic conditions for Nash equilibrium. In contrast with the conventional wisdom that Nash equilibrium involves common

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beliefs, they show that first-order beliefs suffice for the play of Nash equilibria in two-player games.

In this paper, we report the following observation. In the framework of Aumann and Brandenburger, types play two roles: they are both beliefs types and payoff types. We find that if we disentangle the two, then, without a higher-order belief about payoffs, even common belief about everything else is not sufficient for the play of a Nash equilibrium. We notice that the distinction of belief types and payoff types is not completely unusual. It is allowed in Mertens and Zamir's (1985) formulation of belief spaces, and this distinction plays a role in important applications such as mechanism design (See, e.g., Bergemann and Morris 2005).

Let us be more specific. To obtain the sufficient conditions for Nash equilibrium that are as spare as possible, the strategic situation is often modeled as an incomplete information belief system. In particular, players can be uncertain about the payoff structure of the game g being played. In Aumann and Brandenburger's framework, there is a common knowledge that each player i knows his payoff component g_i , or put it differently, a player's payoff function is measurable with respect to his belief types. As we shall demonstrate, this property rules out correlation between beliefs about payoffs and beliefs about actions. It is this correlation that motivate our result.

We follow Aumann and Brandenburger (1995) in defining the finite belief space but relax the aforementioned property. Readers familiar with their formulation shall skip this section. A *game form* consists of a finite set of players $\{1, \dots, n\}$ and an action set A_i for each player i . Let $A := A_1 \times \dots \times A_n$. A *game* prescribes a payoff function $g_i : A \rightarrow \mathbb{R}$ for each i . A (finite) belief system is defined as follows. For each i , let S_i be a finite set of *types* of i . Let $S := S_1 \times \dots \times S_n$ be the set of *states of the world*. At each state $s \in S$,

- a game $\mathbf{g}(s) = (\mathbf{g}_1(s), \dots, \mathbf{g}_n(s))$ is played, where $\mathbf{g}_i(s)$ is player i 's payoff function in the game.
- an action profile $\mathbf{a}(s) = (\mathbf{a}_1(s), \dots, \mathbf{a}_n(s))$ is played, where $\mathbf{a}_i(s)$ is player i 's action.

- player i holds a belief $p(\cdot; s_i)$ over S such that $p(S_{-i} \times \{s_i\}; s_i) = 1$, where $S_{-i} := S_1 \times \dots \times S_{i-1} \times S_{i+1} \times \dots \times S_n$.

In this model, $\mathbf{a}_i(s)$ and $p(\cdot; s_i)$ are measurable with respect to player i 's type s_i —a player “knows” what he does and what he believes. We depart from Aumann and Brandenburger (1995) by allowing the possibility that \mathbf{g}_i is measurable with respect to s_i . That is, payoff and belief types are not necessarily perfectly correlated.

Note that we have used bold symbols to denote functions (such as \mathbf{a}_i , \mathbf{a} , \mathbf{g}_i , and \mathbf{g}). So, for instance, $[s : \mathbf{a}(s) = a]$ is set of states at which action profile a is played. For simplicity, the event is written as $[a]$. We say an *event* E is true at state s if $s \in E$. Player i 's *conjecture* ϕ_i is his belief over opponents' actions A_{-i} . Let $\phi_i(s)$ be player i 's *conjecture at state* s , and by definition, $\phi_i(s)(a_{-i}) = p([a_{-i}]; s_i)$. Player i is *rational* at s if his action at s maximizes his expected payoff at s . Let R_i be the event that player i is rational. By definition,

$$R_i := \left\{ s : \int \mathbf{g}_i(t)(\mathbf{a}_i(t), \mathbf{a}_{-i}(t))p(dt; s_i) \geq \int \mathbf{g}_i(t)(a_i, \mathbf{a}_{-i}(t))p(dt; s_i), \forall a_i \in A_i \right\}.$$

Set $R := R_1 \cap \dots \cap R_n$. Player i is said to *believe* an event E (has occurred) at s if $p(E; s_i) = 1$.¹ Define $B_i E$ as the set of states at which i believes E . Set $B^1 E := B_1 E \cap \dots \cap B_n E$; if $s \in B^1 E$, then E is *mutually believed* at s . Set $CBE := B^1 E \cap B^1 B^1 E \cap \dots$; If $s \in CBE$, then E is commonly believed at s .

2 Main Result

Here is our main observation.

Proposition 1. *Suppose in a two person game there is common belief of rationality and of conjectures, and there is mutual belief, but not higher-order belief, of payoff g . Then the conjectures are not guaranteed to form a Nash equilibrium of g ; furthermore, actions in the support of the conjectures are not guaranteed to be rationalizable actions of g .*

¹We use belief instead of knowledge as the recent epistemic literature makes this distinction.

Proof. To prove this result, it is enough to construct an example with finite games and a finite belief space.

	c	d	
C	1, 1	2, 0	
D	2, 1	0, 0	
	g		

	c	d
C	0, 0	0, 0
D	0, 0	0, 1
	g'	

Figure 1: Payoffs

s^1	s^2
s^3	s^4

$\frac{1}{4}$	$\frac{1}{4}$
$\frac{1}{4}$	$\frac{1}{4}$

Figure 2: The state space consists of four states and there is a common prior that assigns equal probability to each state. Player 1's types are represented by two rows and player 2 two columns. That is, player 1 can distinguish $\{s^1, s^2\}$ from $\{s^3, s^4\}$ but he cannot tell s^1 apart from s^2 or s^3 apart from s^4 ; player 2 can distinguish $\{s^1, s^3\}$ from $\{s^2, s^4\}$ but he cannot tell s^1 apart from s^3 or s^2 apart from s^4 .

	c	d
C	g	g
D	g	g'

Figure 3: The payoffs and players' actions at each state; e.g., at state s^1 , (C, c) is played and the underlying payoff is g .

There are two possible 2×2 games (payoffs) g and g' , as constructed in Figure 1. Player 1 is the row player and player 2 the column player.

The belief system is constructed in Figure 2: the state space is $S = \{s^1, s^2, s^3, s^4\}$ and there is a common uniform prior on the four states: each player has two types, player 1's types are represented by two rows — that is, player 1 can distinguish $\{s^1, s^2\}$ from $\{s^3, s^4\}$ but he cannot tell s^1 apart from s^2 or s^3 apart from s^4 ; player 2's types are represented by two columns.

Furthermore, the actual game played in each state is constructed by Figure 3: $\mathbf{g}(s^1) = \mathbf{g}(s^2) = \mathbf{g}(s^3) = g$ and $\mathbf{g}(s^4) = g'$, that is, the game g is played at states s_1, s_2 , and s_3 ; the game g' is played at state s_4 . The action profiles are $\mathbf{a}(s^1) = (C, c)$, $\mathbf{a}(s^2) = (C, d)$, $\mathbf{a}(s^3) = (D, c)$, and $\mathbf{a}(s^4) = (D, d)$ (see Figure 3). According to this specification, at each state $s \in S$, player 1's conjecture about player 2's action choice is $\phi^1(s) = \frac{1}{2}c + \frac{1}{2}d$ and player 2's conjecture about player 1's action choice is $\phi^2(s) = \frac{1}{2}C + \frac{1}{2}D$.

We shall show the following claims.

1. $s^1 \in [\mathbf{g} = g]$; that is, g is the underlying payoff at s^1 .

This follows directly from the construction (see Figure 3).

2. $s^1 \in B^1[g]$; that is, player 1 believes at s^1 that g is played.

This follows directly from the construction (see Figure 3).

3. $s^1 \in CB[\phi^1 = \frac{1}{2}c + \frac{1}{2}d]$ and $s^1 \in CB[\phi^2 = \frac{1}{2}C + \frac{1}{2}D]$; that is, at s^1 , there is a common belief at s^1 about player 1's and player 2's conjectures.

From Figure 3, player 1's conjecture about player 2's action choice is $\phi^1 = \frac{1}{2}c + \frac{1}{2}d$ at each of the four states. Therefore, there is a common belief about ϕ^1 . The same argument holds for $\phi^2 = \frac{1}{2}C + \frac{1}{2}D$.

4. The conjecture profile $(\frac{1}{2}C + \frac{1}{2}D, \frac{1}{2}c + \frac{1}{2}d)$ is not a Nash equilibrium of g .

From the construction of g (Figure 1), g is dominance solvable with (D, c) as the unique Nash equilibrium.

5. C is not a rationalizable action for player 1 in g .

This is because C is strictly dominated by D in g .

6. At s^1 , CBR holds. That is, there is a common belief of rationality at s^1 !

To see this last claim, first observe that player 1 is rational at every state: at states s^1 and s^2 , player 1 believes g is played, and his action C is a best response against his

conjecture about player 2's action choice, $\phi^1 = \frac{1}{2}c + \frac{1}{2}d$; more importantly, at states s^3 and s^4 , player 1 believes that player 2 plays c when the game is g and d when the game is g' — that is, *player 1's belief about the payoffs and his opponent's actions are correlated*, and player 1's expected payoff is 1 from playing D and 0.5 from C . Hence player 1's action D is a rational choice. Secondly, player 2 is rational at every state: at states s^1 and s^3 , player 2 believes that g is played, and c is his dominant strategy; at states s^2 and s^4 , player 2 believes that player 1 takes C when g is played and D when g' is played — again, *player 2's belief about the payoffs and his opponent's actions are correlated*, and player 2's expected payoff is 1 from playing d and 0.5 from c . Hence his action d is a rational choice. Since both players are rational at each state in the belief system, common belief of rationality holds.

The six steps show that at s^1 , there is common belief of rationality and of conjectures, and there is mutual belief of payoff g . Nevertheless, a player could believe his opponent plays a strictly dominated strategy of g , and the conjectures do not form a Nash equilibrium of g . ■

Remark 1. *The key of our construction is the correlation between beliefs of payoffs and of action choices.*

3 A Positive Result

It turns out that to obtain Nash equilibrium, we need the second-order belief about payoffs.

Lemma 1. *Let g be a game, ϕ a profile of conjectures. Suppose that at some state s , the following events are true: $B_i[\phi]$, and for some $i \neq j$, $B_i R_j$ and $B_i B_j[g_j]$. Let a_j be an action of j such that $\phi_i(a_j) > 0$. Then a_j maximizes g_j against ϕ_j .*

Proof. We follow the argument of Lemma 4.2 in Aumann and Brandenburger (1995). Lemma 4.1 in their paper implies that at s , when $B_i[\phi_i]$ is true then player i 's conjecture is indeed ϕ_i . For any a_j such that $\phi_i(a_j) > 0$, i attributes positive probability at s to $[a_j]$, and by

assumption i attributes probability 1 at s to each of the following events: R_j , $B_j[g_j]$, and $[\phi_j]$. Therefore, there is a state t at which all four events are true: player j plays a_j , j is rational, j believes his payoff function to be g_j , and j holds conjecture ϕ_j . Therefore, a_j maximizes g_j against ϕ_j . ■

The role of the second-order belief $B_i B_j[g_j]$ is clear. It “smooths out” the correlations between the beliefs of g_j and of ϕ_j as in the example above. Correlations could still exist in higher orders, and we need higher-order beliefs to smooth out these higher correlations, but they are irrelevant as far as a “local” characterization of Nash equilibrium is concerned.

The theorems in Aumann and Brandenburger (1995) hold by replacing mutual belief of payoff with second-order knowledge of payoff. We summarize them here. The proof follows their arguments step by step, except that we need replace their Lemma 4.2 with our lemma above. The details are hence omitted.

Proposition 2. *A. Let g be a 2-person game, ϕ a 2-tuple of conjectures. Suppose at some state s , the following events are true: $B^1[\phi]$, $B_i R_j$, and $B_i B_j[g_j]$ for $i \neq j$, then ϕ is a Nash equilibrium of g .*

B. Let g be an n -player game, ϕ a n -tuple of conjectures. The players have a common prior which assigns positive probability to a state s . Suppose the following are true at state s : $CB[\phi]$, and for each j , there is some $i \neq j$ such that $B_i B_j[g_j]$ and $B_i[R_j]$, then ϕ is a Nash equilibrium of g .

The sufficient conditions for conjectures when viewed as a mixed strategy profile to be a Nash equilibrium is noteworthy. $B_i R_j$ says that player j is believed to be rational by his opponents (examples can be easily constructed to show j might not be rational) and $B_i B_j[g_j]$ says that j 's opponents believe that j believes his payoff is g_j (examples can be easily constructed to show j might not believe so).

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