Equilibrium of a Sequence of Auctions when Bidders Demand Multiple Items

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Abstract

We construct an ex post perfect equilibrium in a sequence of second-price auctions with two bidders where each bidder's marginal values for additional units are decreasing. This equilibrium implies an increasing path of transaction prices and is ex post efficient.

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1 Introduction

A sequence of English auctions is often used in auction houses to sell several similar items to bidders with demands for multiple items. That is, the items are sold one-at-a-time in separate auctions conducted in sequence. Familiar examples include retail auctions of cases of wine and wholesale auctions of used automobiles. Such sequences of auctions are especially interesting in view of the "declining price anomaly" observed empirically by Ashenfelter (1989).

When multiple items are sold sequentially, and each bidder demands multiple items, it can be difficult to characterize equilibrium bidding strategies. Most research has therefore focused on the special case that each bidder demands a single item or there are only two items for sale; see, e.g., Katzman (1999) and Szentes (2007). Here I focus on the case that there are only two bidders and study a sequence of auctions for several units of a homogeneous commodity, for which each bidder has decreasing marginal values for additional units. Each bidder is assumed to be risk neutral.

We construct explicitly an equilibrium that satisfies the belief-free selection criterion termed as "ex post perfect". This equilibrium implies a nondecreasing path of transaction prices, and it yields an efficient allocation; in particular, it implements the outcome from a Vickrey auction. The structure of the equilibrium bidding strategies is robust to arbitrary probability distributions of bidders' valuations, higher-order uncertainties, different rules of information disclosure, and interdependent values.

The reasoning behind the equilibrium is that each bidder has an incentive to yield earlier items to his opponent and compete only for later items for which his opponent has lower marginal values. Profitable opportunities for such an "intertemporal arbitrage" cannot exist in equilibrium, and the profits for the same bidder from winning in different rounds equalize in equilibrium. This means that bidder i who wins a later item should pay a price equal to the highest bid in an early round where his opponent j wins (since i would have to outbid j if i wants to win in that round). However, j's winning bid is higher than the transaction price in the second-price auction. Therefore, the equilibrium price path is increasing.

The result in this paper suggests that other assumptions are necessary to account for the declining price anomaly observed by Ashenfelter. For example, in a model of two objects and each bidder demands only one unit, McAfee and Vincent (1993) construct a symmetric equilibrium with a declining path of prices. However, for this result to hold, they need increasing absolute risk aversion. See also Krishna (2002) for a survey.

2 A Sequence of Auctions with Two Bidders

Suppose there are 2 bidders and m units of a homogeneous commodity for sale. Each bidder is risk neutral. Bidder i's private valuation is represented by a vector $v_i = (v_i^1, v_i^2, \ldots, v_i^m)$. After having won k items, $k < m$, bidder i's marginal value for a $(k+1)$-st item is v_i^{k+1}. If he wins n items at prices $p^1, \ldots, p^n$ then his payoff is $\sum_{k=1}^n v_i^k - p^k$. We assume that each bidder's marginal values are nonincreasing: $v_i^1 \geq v_i^2 \geq \cdots \geq v_i^m$. The pair (v_1, v_2) of bidders' valuations is drawn from R_+^{2m} according to an arbitrary joint probability distribution. If bidder i only demands (or is restricted to demand) $m_i < m$ units then $v_i^{m_i+1} = \cdots = v_i^m = 0$.

The m units are sold via a sequence of m second-price or ascending-price English auctions. In each round, a unit is sold to the bidder who bids highest in that round at the price equal to his opponent's bid—or in an English auction, the price rises gradually until one bidder drops out at some price; the unit is then sold at that price to the bidder who stays in. Ties are broken arbitrarily, and could depend on the play of the game.

For $1 \leq k \leq m$, let $b_i^k : (v_i, h_i^{k-1}) \mapsto R_+$ be bidder i's (pure) bidding strategy in the k-th round. The history h_i^{k-1} summarizes all information that bidder i has before period k, which includes at least the number of items he has won so far. Let $R_i(k)$ be the number of units that bidder i has won during the first k rounds, thus $R_i(0) = 0$ and $R_{-i}(k) = k - R_i(k)$. $R_i(k)$ depends on the prior history and thus indirectly on the bidding strategies and the tie-breaking rule. We call $\langle R_1(m), R_2(m)\rangle$ an allocation. After k rounds, $k = 1, \ldots, m-1$, there are $m-k$ units left and bidder i's "continuation valuation" for the $m-k$ remaining units is represented by a vector $CV_{i,k} = (v_i^{R_i(k)+1}, \ldots, v_i^{m-k+R_i(k)})$; thus $v_i^{m-k+R_i(k)}$ is bidder i's continuation valuation for the last available item assuming he wins all of the remaining items.

We use the equilibrium selection criterion called "ex post perfect" as in Ausubel (2002).

Definition 1 (ex post perfect equilibrium) $(b_i^k)_{i=1,2,k=1,\ldots,m}$ *is an* ex post perfect *equilibrium if it is a subgame perfect equilibrium of the game with complete and perfect information, i.e., the bidders' valuations* (v_1, v_2) *are common knowledge, and both bidders observe the histories of bids.*

In this definition, the corresponding game of complete and perfect information is obtained simply by augmenting each history with the bidders' valuations and prior bids.

To illustrate, consider an example with two bidders and two units, and hence two rounds. In the second round, for each bidder it is a weakly dominant strategy to bid his continuation value for the remaining unit, depending on whether he won a unit in the first round. Suppose $v_i^k \in [0, 100]$ for $i, k \in \{1, 2\}$ and $v_1 = (100, 0)$. How would bidder 1 bid? Bidder 1 might simply bid 0 in the first round to yield the first item to bidder 2 and then obtain the second item at the price v_2^2 for a profit of $100 - v_2^2$. Such a strategy by bidder 1 is optimal if and only if bidder 2 bids at least v_2^2 in the first round. This shows that a bidder may prefer to yield earlier items to his opponent and compete only for later items for which his opponent has lower marginal values. But in an equilibrium profitable opportunities for such an "intertemporal arbitrage" cannot exist. In this example, bidder 2's optimal response is to bid exactly v_2^2 in the first round and, after winning the first unit, continue to bid v_2^2 for a second unit, thus resulting in an increasing price path $(0, v_2^2)$ and an efficient allocation. Our main result shows that this intuition generalizes to the case that m units are offered for sale.

Proposition 1 *There is an ex post perfect equilibrium in which in each round each bidder bids his continuation value for the last available unit (as if he can win all of the remaining items):*

$$b_i^k(v_i, h_i^{k-1}) = v_i^{m-(k-1)+R_i(k-1)}, \qquad 1 \le k \le m, \quad i = 1, 2\,.$$

Before establishing Proposition 1 we study the useful properties of the candidate bidding strategy.

Proposition 2

(*a*) *The price path is nondecreasing, and it is strictly increasing if there are no ties in the marginal valuations ($v_i^k \neq v_j^l$ unless $i = j$ and $k = l$).*

(*b*) *The bidding strategies induce an ex post efficient allocation.*

(*c*) *The seller's revenue is the same as in a Vickrey auction: the m items are sold at the m prices that are the m smallest values among $(v_1^1, \ldots, v_1^m, v_2^1, \ldots, v_2^m)$.*

(*d*) *Bidder i's ex post payoff is the same in any realized allocation $\langle R_1(m), R_2(m)\rangle$.*[1]

Proof. (*a*) Since $R_i(k-1) + R_{-i}(k-1) = k - 1$,

$$b_i^k\left(v_i, h_i^{k-1}\right) = v_i^{m-(k-1)+R_i(k-1)} = v_i^{m-R_{-i}(k-1)}. \tag{1}$$

$R_{-i}(k)$ is non-decreasing in k, and hence $v_i^{m-R_{-i}(k-1)}$ is non-decreasing in k. The equilibrium price path

$$\min\left\{v_1^{m-R_2(k-1)}, v_2^{m-R_1(k-1)}\right\}$$

[1]Since an allocation is efficient by property (*b*), different allocations can only result from random tie-breaking outcomes. Since there is no restriction on the joint distribution of bidder types, tie could occur with an ex ante positive probability. If, however, bidder 1 always gets the object if tie occurs, then property (*d*) holds trivially.

is non-decreasing. Suppose there are no ties. If bidder 1 is the winner in the k-th round then 1 wins at the price $v_2^{m-R_1(k-1)} < v_1^{m-R_2(k-1)}$; then $R_1(k) = R_1(k-1)+1$ and $R_2(k) = R_2(k-1)$. In the $(k+1)$th round,

$$\begin{aligned} b_1^{k+1}(v_1, h_1^k) &= v_1^{m-R_2(k)} = v_1^{m-R_2(k-1)} > v_2^{m-R_1(k-1)}, \\ b_2^{k+1}(v_2, h_2^k) &= v_2^{m-R_1(k)} = v_2^{m-R_1(k-1)-1} > v_2^{m-R_1(k-1)}. \end{aligned}$$

The price for the $(k+1)$th round is then $\min\{v_1^{m-R_2(k-1)}, v_2^{m-R_1(k-1)-1}\} > v_2^{m-R_1(k-1)}$.

(b) Suppose the equilibrium induces an allocation $\langle R_1(m), R_2(m)\rangle$. Without loss of generality we assume that $v_1^{R_1(m)} \geq v_2^{R_2(m)}$. If this allocation is not ex post efficient then it must be that

$$v_2^{R_2(m)+1} > v_1^{R_1(m)}. \tag{2}$$

Suppose bidder 1 wins his $R_1(m)$-th unit in the k^*-th round. Then $R_1(k^*-1) = R_1(m) - 1$ and hence by equality (1), bidder 2 bids $v_2^{m-R_1(k^*-1)} = v_2^{m-R_1(m)+1} = v_2^{R_2(m)+1}$ and bidder 1 bids $v_1^{m-(k^*-1)+R_1(k^*-1)} = v_1^{m-k^*+R_1(m)} \leq v_1^{R_1(m)}$. Since individual 1 wins in that round, $v_1^{R_1(m)} \geq v_2^{R_2(m)+1}$, which contradicts inequality (2).

(c) This is straightforward from (a) and (b).

(d) Suppose there are two allocations with $R_1(m) = p$ and $R_1'(m) = p - q$ for some $p \geq q \geq 1$. The efficiency of $\langle R_1(m), R_2(m)\rangle$ implies that

$$v_1^p \geq v_2^{m-p+1}. \tag{3}$$

The efficiency of $\langle R_1'(m), R_2'(m)\rangle$ implies that

$$v_2^{m-(p-q)} \geq v_1^{p-q+1}. \tag{4}$$

By decreasing marginal valuations, $v_1^{p-q+1} \geq v_1^p$ and $v_2^{m-p+1} \geq v_2^{m-(p-q)}$. Therefore, inequalities (3) and (4) imply

$$v_1^{p-q+1} = \cdots = v_1^p = v_2^{m-p+1} = \cdots = v_2^{m-(p-q)}. \tag{5}$$

By (c), for allocations $\langle R_1(m), R_2(m)\rangle$ and $\langle R_1'(m), R_2'(m)\rangle$, bidder 1 pays $(v_2^{m-p+1}, v_2^{m-p+2}, \cdots, v_2^m)$ and $(v_2^{m-(p-q)+1}, \cdots, v_2^m)$, respectively. Thus bidder 1's payoff from allocation $\langle R_1(m), R_2(m)\rangle$ is

$$\begin{aligned} \sum_{k=1}^{p} v_1^k - \sum_{l=m-p+1}^{m} v_2^l &= \sum_{k=1}^{p-q} v_1^k - \sum_{l=m-(p-q)+1}^{m} v_2^l + \left(\sum_{k=p-q+1}^{p} v_1^k - \sum_{l=m-p+1}^{m-(p-q)} v_2^l \right) \\ &\overset{(5)}{=} \sum_{k=1}^{p-q} v_1^k - \sum_{l=m-(p-q)+1}^{m} v_2^l. \end{aligned} \tag{6}$$

Expression (6) is exactly bidder 1's payoff from $\langle R_1'(m), R_2'(m)\rangle$. ■

Proof of Proposition 1. We prove the result by induction. If $m = 1$ then $b_i^1(v_i, h_i^0) = v_i^1$ is a dominant strategy equilibrium and hence subgame perfect. Suppose the proposed strategies form an ex post perfect equilibrium for $m = l$, and hence a subgame perfect equilibrium in the associated game with complete and perfect information. Consider the case $m = l + 1$. By following the prescribed strategies in the game with valuations $v_1 = (v_1^1, \cdots, v_1^{l+1})$ and $v_2 = (v_2^1, \cdots, v_2^{l+1})$, bidder 1 gets $R_1(l + 1)$ units at prices

$$v_2^{l+1} \leq v_2^l \leq \cdots \leq v_2^{l+2-R_i(l+1)}. \tag{7}$$

By Proposition 2(d) we can w.l.o.g. assume that $R_1(l + 1)$ is the maximal number of units that bidder 1 can win from the auctions.

Given that individual 2 and the continuation play follow the prescribed strategies, bidder 1's first round bid can change his own first round payoff or the continuation play only if bidder 1 bids other than v_1^{l+1} in such a way that his bid changes the first round winner. We have several cases to study.

Case I: $v_1^{l+1} > v_2^{l+1}$ but bidder 1 bids $\widehat{b}_1^1 < v_2^{l+1}$.

Bidder 1 loses the first round. The continuation game involves $CV_{1,1} = (v_1^1, \cdots, v_1^l)$ and $CV_{2,1} = (v_2^2, \cdots, v_2^{l+1})$. The continuation game consists of l units and hence the prescribed strategies form an ex post perfect equilibrium for the continuation game by the induction hypothesis and allocates the l units efficiently. In the continuation game, the efficiency implies that the best bidder 1 can do is to win $R_1(l+1)$ units with prices $v_2^{l+1}, \cdots, v_2^{l+2-R_1(l+1)}$, which is the same as what he can get by following the equilibrium strategy from the first round. When there are ties among marginal valuations, it is possible that a deviation will affect the match of tying bids in later rounds and hence affect the final allocation. But Proposition 2(d) guarantees that all efficient allocations of the continuation game yield the same continuation payoff.

Case II: $v_1^{l+1} > v_2^{l+1}$ but bidder 1 bids $\widehat{b}_1^1 = v_2^{l+1}$.

If bidder 1 wins the first round then outcomes do not change. If he loses then the argument in Case I applies.

Case III: $v_1^{l+1} < v_2^{l+1}$ but bidder 1 bids $\widehat{b}_1^1 > v_2^{l+1}$.

Bidder 1 wins the first round at the price v_2^{l+1}. The continuation game involves valuations $CV_{1,1} = (v_1^2, \ldots, v_1^{l+1})$ and $CV_{2,1} = (v_2^1, \ldots, v_2^l)$. Since the prescribed strategies yield an efficient allocation in the continuation game, bidder 1 can win $\max\{0, R_1(l + 1) - 1\}$ units in later rounds. If $R_1(l + 1) = 0$, which means $v_1^1 < v_2^{l+1}$, then bidder 1 would lose all later rounds, so his overall payoff is $v_1^1 - v_2^{l+1} < 0$. He is worse off by this deviation. If $R_1(l+1) = 1$, which means $v_1^1 \geq v_2^{l+1}$, then bidder 1 would also lose all later rounds, and his overall payoff is $v_1^1 - v_2^{l+1}$. This is what he can get by following the prescribed strategy from the first round. If $R_1(l+1) > 1$, then bidder 1 will win $R_1(l+1) - 1$ units in later rounds at prices $v_2^l, \ldots, v_2^{l+2-R_1(l+1)}$. So by this deviation individual 1 can win $R_1(l+1)$ units with prices $v_2^{l+1}, v_2^l, \ldots, v_2^{l+2-R_1(l+1)}$. Comparing to expression (7) we see that bidder 1 ends up with the same payoff as following the prescribed strategy from the first round.

Case IV: $v_1^{l+1} < v_2^{l+1}$ but bidder 1 bids $\widehat{b}_1^1 = v_2^{l+1}$.

If bidder 1 loses the first round then the outcomes do not change. If he wins then the argument in Case III applies.

Case V: $v_1^{l+1} = v_2^{l+1}$ but bidder 1 bids $\widehat{b}_1^1 < v_2^{l+1}$.

Bidder 1 loses the first unit. The continuation game involves $v_1' = (v^1, \cdots, v_1^l)$ and $v_2' = (v_2^2, \cdots, v_2^{l+1})$. The efficiency of the allocation in the continuation game implies that bidder 1 can get at most $R_1(l+1)$ units at prices $v_2^{l+1}, \ldots, v_2^{l+2-R_1(l+1)}$. Comparing this to expression (7) we see that bidder 1 cannot be better off by this deviation.

Note that the case " $v_1^{l+1} = v_2^{l+1}$ but bidder 1 bids $\widehat{b}_1^1 > v_2^{l+1}$" does not change the outcome because $R_1(l+1)$ is the maximal number that bidder 1 can win. ■

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