

Gambling Reputation:
Repeated Bargaining with Outside Options
*Supplementary Material**

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1 Multiple equilibria in the non-generic case

We consider the non-generic case not covered by Theorem 1 in the main text. Fix any $\delta > \bar{\delta}$ as in the proof of Theorem 1. Also, fix any $C \in (\underline{C}, \bar{C}]$. Let S_0, \dots, S_N, \dots be the solutions to the recursive equation

$$S_n = (1 - \delta)(qH + c_1) + \delta(1 - q)S_{n+1} + \delta qS_{n-1} \quad (1)$$

with the initial conditions $S_0 = (1 - \delta)(qH + c_1) + \delta(1 - q)S_1 + \delta qS_0 = qH + \delta c_1 - (1 - \delta)c_2$, where $N = \sup\{n \in \mathbb{Z} : S_n > C\}$. Define $p^* = \Phi^{-N}(p^{**})$, where $p^{**} = \frac{qH - c_2 - C}{(2q-1)H}$.

The case of $S_{N+1} = C$ is non-generic in the following sense. Note that, in order to have $S_K = C$ for some integer K , C and δ must satisfy one of a family of polynomials that are at most countable (recall that such an integer K goes to ∞ as $\delta \rightarrow 1$). Therefore, for each C , the roots in δ are at most countable.

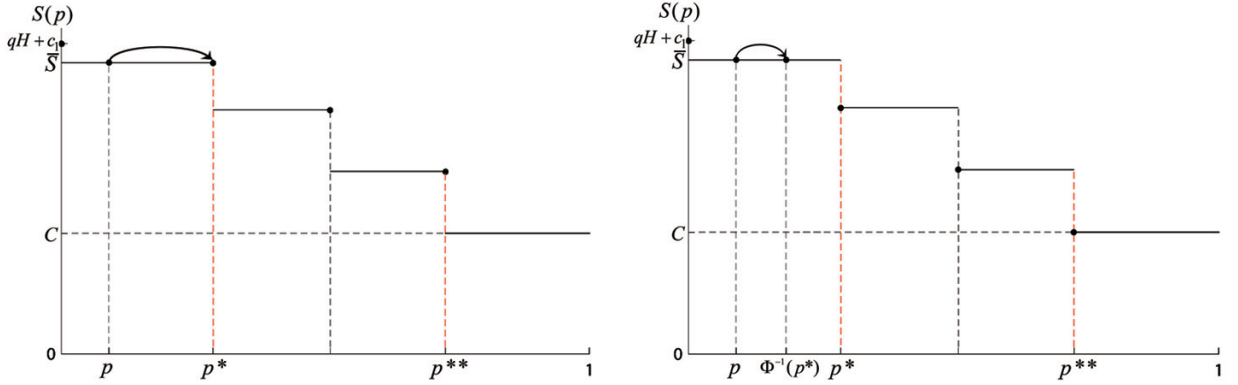
In the non-generic case, two equilibrium outcomes are possible. The first equilibrium is as reported in Theorem 1 such that, at p^{**} , player 2 makes a losing demand for sure and, hence, $S(p^{**}) = S_N$. The other equilibrium is identical to the first equilibrium except for the following:

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- At any $p \in (0, \Phi^{-1}(p^*))$, type B rejects player 1's demand in a way that posterior immediately after rejection moves to $\Phi^{-1}(p^*)$.
- At any $p \in [\Phi^{-1}(p^*), p^*)$, type B rejects player 1's demand for sure.
- At p^* , player 2 demands C for sure and type B accepts it for sure. Thus, in this equilibrium, $S(p^*) = C$.

The next figure illustrates these two equilibria.

Figure 1: Non-generic equilibria



2 When type G 's cutoff is $C \in [0, \underline{C}]$

Proposition 1 *Let $C = \underline{C}$. Fix any $\delta > \frac{c_1+c_2}{(2q-1)H+c_1+c_2}$. There exists a Markov equilibrium. Furthermore, any Markov equilibrium is such that, at any $p \in (0, 1)$, a demand s is serious only if $s \in [qH - c_2, qH + c_1]$ and, moreover, every demand is rejected with a positive probability; thus, type B 's payment is at least $\bar{S} = qH + \delta c_1 - (1 - \delta)c_2$.*

Proof. First, It is straightforward to observe that, with $C = \underline{C}$, there exists a Markov equilibrium such that, at any $p \in (0, 1)$, player 2 demands $qH + c_1$ and type B rejects it with an arbitrary but interior probability.

Next, fix a Markov equilibrium, and consider any $p \in (0, 1)$. Note that

$$p(1 - q)H + (1 - p)qH - c_2 > (1 - q)H - c_2.$$

Thus, player 2 will demand \underline{C} only if type B rejects it for sure. Assume that type B rejects \underline{C} for sure when \underline{C} is demanded. But, it is straightforward to see that accepting such a

demand is profitable since acceptance would then show that he is good, a contradiction. Thus, \underline{C} cannot be demanded in equilibrium. On the other hand, it is clear that any demand below \underline{C} will not be made, while given type G 's behavior any $s \in (\underline{C}, qH - c_2)$ will be demanded only if it is rejected for sure. Thus, a demand s can be serious only if $s \in [qH - c_2, qH + c_1]$.

Fix any $s \in [qH - c_2, qH + c_1]$. This demand cannot be accepted with probability 1. To see this, suppose otherwise. But then, rejection would show that player 1 is good and, hence, the deviation yields continuation payment $(1 - \delta)(qH + c_1) + \delta\underline{C}$, which is less than the equilibrium payment $(1 - \delta)s + \delta(qH + c_1)$ given δ . Thus, s must be rejected and the equilibrium payment must be given by rejection.

Since the good type also rejects every equilibrium demand, rejection (and any subsequent third party signal) can never reveal the good type. Moreover, rejection gives one-period expected payment $qH + c_1$. On the other hand, accepting s leads to continuation payment $(1 - \delta)s + \delta(qH + c_1)$. It therefore follows that $S(p) \geq \bar{S}$. ■

Proposition 2 *Fix $C \in [0, \underline{C})$. Then, the following characterizes all Markov equilibria with sufficiently large δ : player 2 demands a constant demand $s^* \in [qH - c_2, qH + c_1]$ for sure which type B accepts for sure; acceptance leads to a constant demand $qH + c_1$ while rejection (only by type G) leads to a constant demand $s' = (\frac{1}{\delta} - 1)s^* + (2 - \frac{1}{\delta})(qH + c_1)$.*

Proof. Cases 1-3 in the proof of Proposition 1 (Characterization of serious demands) of the main text remain true. But Case 4 are no longer true with $C < \underline{C}$ since, then, even when s^* is rejected and hence player 1 is known to be type G , player 2 will not demand C . Since there are only two possible demands in Case 4, C or s^* , and s^* is accepted by type G with probability 1, we have the following observations in this case.

(1) Player 2 will never make a demand lower than or equal to C . demands lower than C contradict that B plays a cutoff strategy (Lemma 1 in the main text), while a demand C gives player 2 a payoff of at most $pC + (1 - p)(qH - c_2)$. This quantity is strictly less than the payoff from the losing demand $\frac{qH+c_1}{1-\delta}$ because $C < (1 - q)H - c_2$.

(2) The implication of (1) is that type G will always reject player 2's demands.

(3) Therefore, the only demand is $s^* \in (qH - c_2, qH + c_1]$, and it is accepted with probability 1 by type B . Then type B 's expected payment is

$$(1 - \delta)s^* + \delta(qH + c_1). \tag{2}$$

This payment must not be higher than that from rejection (which leads to the revelation of good type), while the payment from accepting $s^*(p) + \varepsilon$ must not be lower than that from rejection (by the definition of $s^*(p)$) to make rejection incentive compatible. Therefore,

$$(1 - \delta)s^* + \delta(qH + c_1) = (1 - \delta)(qH + c_1) + \delta s'$$

where s is the demand made by player 2 when he believes player 1 is type G (note that player 2 can make a very high demand here since the insistent type only accepts a demand below \underline{C} even at $p = 1$). Hence

$$s' = \left(\frac{1}{\delta} - 1\right) s^* + \left(2 - \frac{1}{\delta}\right) (qH + c_1). \quad (3)$$

This implies that $s' \in [s^*(p), qH + c_1]$. Note that type B will accept player 2's demand for sure when $s' < qH + c_1$ in all future periods (but player 2 does not have a deviation because the belief is stuck at 1). It follows from (3) that s^* is independent of belief because s' must be constant (the belief is 1).

(4) Now consider an equilibrium where player 2 demands $s = qH - c_2$. If s is expected to be accepted with probability 1, we need $s' = \left(\frac{1}{\delta} - 1\right) (qH - c_2) + \left(2 - \frac{1}{\delta}\right) (qH + c_1)$ as in (3) above. We now argue that it cannot be accepted by type B with an interior probability. Suppose type B accepts it with an interior probability. Then, because rejection leads to an interior posterior, the lowest possible expected payment for type B after rejection is

$$(1 - \delta)(qH + c_1) + (\delta - \delta^2)(qH - c_2) + \delta^2(qH + c_1).$$

This amount is larger than the payment from immediate acceptance of $qH - c_2$. ■

3 Non-Markov equilibria

First, we establish a folk theorem for the case of complete information with $p = 0$.

Proposition 3 *Suppose that player 1's type is known to be B . Then, we have the following:*

1. *In any subgame perfect equilibrium, player 1's equilibrium expected payment, S , is such that $S \in [qH - c_2, qH + c_1]$.*

2. Fix any $\delta > \frac{1}{2}$. Then, any $S \in [qH - c_2, qH + c_1]$ can be supported as an equilibrium expected payment of player 1.

Proof. 1. Fix any δ and any subgame perfect equilibrium.

First, let us show that $S \geq qH - c_2$. Suppose not, so $S < qH - c_2$. Then, since rejecting any offer gives player 1 (one-period) expected payment of $qH + c_1$, acceptance of an offer strictly below $qH - c_2$ must occur at some history on the equilibrium path. Consider player 2 who makes such an offer. But, clearly, this short-run player can improve his expected payoff by instead making any offer at least $qH - c_2$; player 1's rejection gives him payoff $qH - c_2$. Next, let us show that $S \leq qH + c_1$. Suppose not. But, the bad type can guarantee himself payment of $qH + c_1$ by always rejecting.

2. We know that there exists a Markov equilibrium that supports payment $qH + c_1$. Consider any $S \in [qH - c_2, qH + c_1)$ and the following trigger strategy profile:

- At any history in which no deviation from the equilibrium has been observed, player 2 offers S for sure and player 1 accepts an offer if and only if it is less than or equal to S .
- At any history in which acceptance of an offer higher than S has been observed, player 2 offers $qH + c_1$ for sure and player 1 accepts an offer if and only if it is less than or equal to $qH + c_1$.
- At any other history, player 2 offers S for sure and player 1 accepts an offer if and only if it is less than or equal to S .

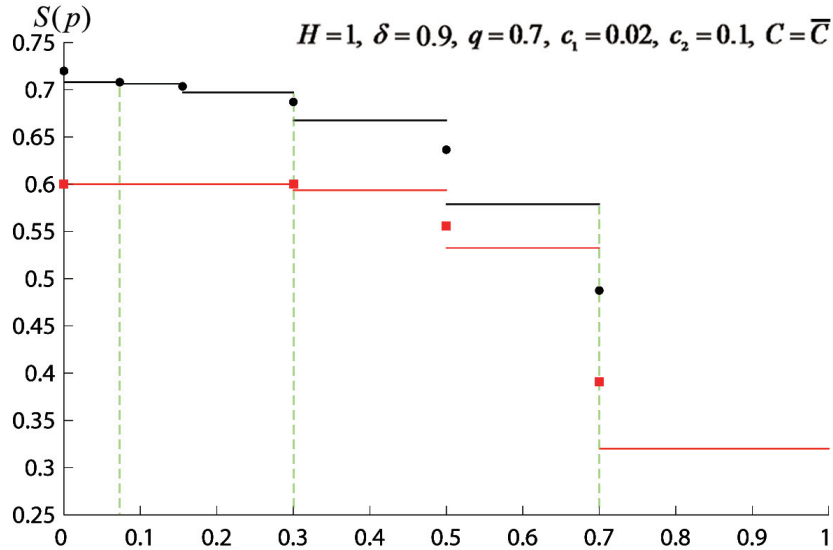
In order to establish that the above profile constitutes a subgame perfect equilibrium, it suffices to consider player 1's incentives when facing a deviating offer $S + \varepsilon$ for small $\varepsilon > 0$. Given the above profile, rejecting the offer yields payment $(1 - \delta)(qH + c_1) + \delta S$, while acceptance leads to $(1 - \delta)(S + \varepsilon) + \delta(qH + c_1)$. Since $\delta > \frac{1}{2}$ and $S < qH + c_1$, it is easily seen that the latter is larger than the former. Thus, player 1 will reject $S + \varepsilon$ for sure. This, in turn, supports optimality of player 2's strategy. ■

Two constructions of non-Markov equilibrium with incomplete information

Fix any $\delta > \bar{\delta}$ as in the proof of Theorem 1 in the main text. Also, fix any $C \in (\underline{C}, \bar{C}]$.

1. At any history/period t with $p_t > 0$, all players play according to the Markov equilibrium of Theorem 1 for belief p_t ; at any history with $p^t = 0$, the continuation strategies are given by the equilibrium in which the bad type obtains payment $S^* \in [qH - c_2, qH + c_1)$ (Proposition 3 above). It is straightforward to see that this non-Markov profile only changes the initial condition for the recursive equation (1) above, from $S(p^0) = \bar{S}$ to $(1 - \delta)(qH - c_2) + \delta S^*$. We draw below the corresponding equilibrium payments for $S^* = qH - c_2$ (lower lines and squares), together with the equilibrium payments of the Markov equilibrium in the top right panel of Figure 8 (Comparative Statics) in the main text (upper lines and dots).

Figure 2: A non-Markov equilibrium



2. At any history/period t with $p_t < p^{**}$ and $p_t = 1$, all players play according to the Markov equilibrium of Theorem 1 for belief p_t . At other histories, the strategies are as follows:

- At any period t such that $p_t \in (p^{**}, 1)$, player 2 offers $C' \in (\underline{C}, \bar{C}]$ and the bad type accept offer s if and only if $s \leq C'$.
- Phase p^{**}

- The phase begins at any period t such that $p_t = p^{**}$ and $p_{t-1} < p^{**}$, and lasts for $K + 1$ periods.
- In the first K periods of the phase, player 2 offers C for sure and both types of player 1 accept it for sure. Belief upon rejection (off-the-equilibrium) is 0.
- In the last period of the phase, player 2 offers $qH - c_2$ for sure and the bad type reject it for sure.

To establish that the above profile constitutes an equilibrium, consider the equilibrium continuation payment of type B at the beginning of Phase p^{**} . It corresponds to

$$(1 - \delta^K)C + \delta^K [(1 - \delta)(qH + c_1) + \delta qS_{N-1} + \delta(1 - q)C'], \quad (4)$$

where S_{N-1} is derived from the recursive equation (1) above. Note that (4) must equal S_N where

$$S_N = (1 - \delta)(qH + c_1) + \delta qS_{N-1} + \delta(1 - q)S_{N+1}. \quad (5)$$

If $S_{N+1} < C'$, we have $(1 - \delta)(qH + c_1) + \delta qS_{N-1} + \delta(1 - q)C' > qS_{N-1} + (1 - q)S_{N+1}$; also, $C < qH + c_1$. Then, there exist a positive integer K and $C' \in (\underline{C}, \overline{C}]$ such that (4) equals (5). The players' behavior during the phase are mutually optimal given the stated off-the-equilibrium beliefs.