Solution Methods for Models with Rare Disasters

Jesús Fernández-Villaverde

University of Pennsylvania

Oren Levintal*

Interdisciplinary Center (IDC) Herzliya

April 2, 2017

Abstract

This paper compares different solution methods for computing the equilibrium of dynamic stochastic general equilibrium (DSGE) models with rare disasters along the lines of those proposed by Rietz (1988), Barro (2006), Gabaix (2012), and Gourio (2012). DSGE models with rare disasters require solution methods that can handle the large non-linearities triggered by low-probability, high-impact events with accuracy and speed. We solve a standard New Keynesian model with Epstein-Zin preferences and time-varying disaster risk with perturbation, Taylor projection, and Smolyak collocation. Our main finding is that Taylor projection delivers the best accuracy/speed tradeoff among the tested solutions. We also document that even third-order perturbations may generate solutions that suffer from accuracy problems and that Smolyak collocation can be costly in terms of run time and memory requirements.

Keywords: Rare disasters, DSGE models, solution methods, Taylor projection, perturbation, Smolyak.

JEL classification: C63, C68, E32, E37, E44, G12.

*Correspondence: jesusfv@econ.upenn.edu (Fernández-Villaverde) and oren.levintal@idc.ac.il (Oren Levintal). We thank Marlène Isoré, Pablo Winant, Xavier Gabaix, Tony Smith, the editor Karl Smedders, four referees, and participants at several seminars for comments. David Zarruk Valencia provided superb research assistantship. Fernández-Villaverde gratefully acknowledges financial support from the National Science Foundation under Grant SES 1223271.

1 Introduction

Rietz (1988), Barro (2006), and Gabaix (2012) have popularized the idea that low-probability events with a large negative impact on consumption ("rare disasters") can account for many asset pricing puzzles, such as the equity premium puzzle of Mehra and Prescott (1985). Barro (2006), in particular, argues that a rare disaster model calibrated to match data from 35 countries can reproduce the observed high equity premium, the low risk-free rate, and the stock market volatility. Barro assumed disaster probabilities of 1.7 percent a year and declines in output/consumption in a range of 15 to 64 percent. Barro (2009) can also match the responses of the price/dividend ratio to increases in uncertainty.

Many researchers have followed Barro's lead and formulated, calibrated/estimated, and solved models with disaster probabilities and declines in consumption that are roughly in agreement with Barro's original proposal, including, among others, Barro and Ursúa (2012), Barro and Jin (2011), Nakamura, Steinsson, Barro, and Ursúa (2013), Wachter (2013), and Tsai and Wachter (2015). The approach has also been extended to analyze business cycles (Gourio, 2012), credit risk (Gourio, 2013), and foreign exchange markets (Farhi and Gabaix, 2016 and Gourio, Siemer, and Verdelhan, 2013). These calibrations/estimations share a common feature: they induce large non-linearities in the solution of the model. This is not a surprise. The mechanism that makes rare disasters work is the large precautionary behavior responses induced in normal times by the probability of tail events.

Dealing with these non-linearities is not too challenging when we work with endowment economies. A judicious choice of functional forms and parameterization allows a researcher to derive either closed-form solutions or formulae that can be easily evaluated.

The situation changes, however, when we move to production models, such as those of Gourio (2012, 2013), Andreasen (2012), Isoré and Szczerbowicz (2015), and Petrosky-Nadeau, Zhang, and Kuehn (2015). Suddenly, having an accurate solution is of foremost importance. For example, rare disaster models may help to design policies to prevent disasters (with measures such as a financial stability policy) and to mitigate them (with measures such as bailouts and unconventional monetary policy). The considerable welfare losses associated with rare disasters reported by Barro (2009) suggest that any progress along the lines of having *accurate* quantitative models to evaluate counter-disaster policies is a highly rewarding endeavor.

But we also care about speed. Models that are useful for policy analysis often require estimation of parameter values, which involves the repeated solution of the model, and that the models be as detailed as the most recent generation of dynamic stochastic general equilibrium (DSGE) models, which are indexed by many state variables.

Gourio (2012, 2013) and Petrosky-Nadeau, Zhang, and Kuehn (2015) solve their models with standard projection methods (Judd, 1992). Projection methods are highly accurate (Aruoba, Fernández-Villaverde, and Rubio-Ramírez, 2006), but they suffer from an acute curse of dimensionality. Thus, the previous papers concentrate on analyzing small models. Andreasen (2012) and Isoré and Szczerbowicz (2015) solve more fully-fledged models with third-order perturbations. Perturbation solutions are fast to compute and can handle many state variables. However, there are reasons to be cautious about the properties of these perturbation solutions (see also Levintal, 2015). Perturbations are inherently local and rare disasters trigger equilibrium dynamics that travel far away from the approximation point of the perturbation (even, due to precautionary behavior, in normal times without disasters). Moreover, perturbations may fail to accurately solve for asset prices and risk premia due to the strong volatility embedded in these models.[1]

We get around the limitations of existing algorithms by applying a new solution method, Taylor projection, to compute DSGE models with rare disasters. This method, proposed by Levintal (2016), is a hybrid of Taylor-based perturbations and projections (and hence its name). Like standard projection methods, Taylor projection starts from a residual function created by plugging the unknown decision rules of the agents into the equilibrium conditions of the model and searching for coefficients that make that residual function as close to zero as possible. The novelty of the approach is that instead of "projecting" the residual function according to an inner product, we approximate the residual function around the steady state of the model using a Taylor series, and find the solution that zeros the Taylor series.[2] We show that Taylor projection is sufficiently accurate and fast so as to allow the solution and estimation of rich models with rare disasters, including a New Keynesian model à la Christiano, Eichenbaum, and Evans (2005).

To do so, we propose in Section 2 a standard New Keynesian model augmented with Epstein-Zin preferences and time-varying rare disaster risk. We also present seven simpler versions of the model. In what we will call version 1, we start with a benchmark real business cycle model, also with Epstein-Zin preferences and time-varying rare disaster risk. This

[1] Isoré and Szczerbowicz (2015) address this problem by designing the model such that the detrended variables are independent of the disaster shock. This is possible when the disaster shock scales down the size of the economy, but it does not affect its composition.

[2] The Taylor-projection algorithm is close to how Krusell, Kuruscu, and Smith (2002) solve the generalized Euler equation (GEE) implied by their model. These authors, as we do, postulate a polynomial approximation to the decision rule, plug it into the GEE, take derivatives of the GEE, and solve for the coefficients that zero the resulting derivatives. Coeurdacier, Rey, and Winant (2011), den Haan, Kobielarz, and Rendahl (2015), and Bhandari, Evans, Golosov, and Sargent (2017) propose related solution methods. The approach in Levintal (2016) is, however, backed by theoretical results and more general than in these three previous papers. Also, applying the method to large-scale models requires, as we do in this paper, developing new differentiation tools and exploiting the sparsity of the problem.

model has four state variables (capital, a technology shock, and two additional state variables associated with the time-varying rare disaster risk). Then, we progressively add shocks and price rigidities, until we get to version 8, our complete New Keynesian model with 12 state variables. Our layer-by-layer analysis gauges how accuracy and run time change as new mechanisms are incorporated into the model and as the dimensionality of the state space grows.

In Section 3, we calibrate the model with a baseline parameterization, which captures rare disasters, and with a non-disaster parameterization, where we shut down rare disasters. The latter calibration helps us in measuring the effect of disasters on the accuracy and speed of our solution methods.

In Section 4, we describe how we solve each of the eight versions of the model, with the two calibrations, using perturbation, Taylor projection, and Smolyak collocation. We implement different levels of each of the three solution methods: perturbations from order 1 to 5, Taylor projections from order 1 to 3, and Smolyak collocation from level 1 to 3. Thus, we generate eleven solutions per each of the eight versions of the model and each of the two calibrations, for a total of 176 possible solutions (although we did not find a few of the Smolyak solutions because of convergence constraints).

In Section 5, we present our main results. Our first finding is that first-, second-, and third-order perturbations fail to provide a satisfactory accuracy. This is particularly true for the risk-free interest rate and several impulse response functions (IRFs). Our second finding is that fifth-order perturbations are much more accurate, but they become cumbersome to compute and require a non-trivial run time and some skill at memory management. Our third finding is that second- and third-order Taylor projections offer an outstanding compromise between accuracy and speed. Second-order Taylor projections can be as accurate as Smolyak collocations and, yet, be solved in a fraction of the time. Third-order Taylor projections take longer to run, but their accuracy can be quite high, even in a testbed as challenging as the New Keynesian model with rare disasters. The findings are complemented by Section 6, which documents a battery of robustness exercises.

Finally, we provide an Online Appendix with further details on the model and the solution and a `MATLAB` toolbox to implement the Taylor projection method for a general class of DSGE models.

We postulate, therefore, that a new generation of solution methods, such as Taylor projection (but also, potentially, others such as those in Maliar and Maliar, 2014), can be an important tool in fulfilling the promises of production models with rare disasters. We are ready now to start our analysis by moving into the description of the model.

2 A DSGE model with rare disasters

We build a standard New Keynesian model along the lines of Christiano, Eichenbaum, and Evans (2005). In the model, there is a representative household, a final good producer, a continuum of intermediate good producers subject to Calvo pricing, and a monetary authority that sets up the nominal interest rate following a Taylor rule. Given the goals of this paper and to avoid excessive complexity in the model, we avoid wage rigidities.

We augment the standard New Keynesian model along two dimensions. First, we introduce Epstein-Zin preferences. These preferences have been studied in the context of New Keynesian models by Andreasen (2012), Rudebusch and Swanson (2012), and Andreasen, Fernández-Villaverde, and Rubio-Ramírez (2013), among others. Second, we add a time-varying rare disaster risk. Rare disasters impose two permanent shocks on the real economy: a productivity shock and a capital depreciation shock. When a disaster occurs, technology and capital fall immediately. This specification should be viewed as a reduced form that captures severe disruptions in production, such as those caused by a war or a large natural catastrophe, and failures of firms and financial institutions, such as those triggered by massive labor unrest or a financial panic.

We present first the full New Keynesian model and some of its asset pricing implications. Then, in Subsection 2.7, we describe the simpler versions of the model mentioned in the introduction.

2.1 The household

A representative household's preferences are representable by an Epstein-Zin aggregator between the period utility U_t and the continuation utility V_{t+1}:

$$V_t^{1-\psi} = U_t^{1-\psi} + \beta \mathbb{E}_t \left(V_{t+1}^{1-\gamma} \right)^{\frac{1-\psi}{1-\gamma}} \tag{1}$$

where the period utility over consumption c_t and labor l_t is given by $U_t = e^{\xi_t} c_t \left(1 - l_t\right)^{\nu}$ and $\mathbb{E}_t$ is the conditional expectation operator. The parameter γ controls risk aversion (Swanson, 2012) and the intertemporal elasticity of substitution (IES) is given by $1/\widehat{\psi}$, where $\widehat{\psi} = 1 - (1+\nu)(1-\psi)$ (Gourio, 2012). The intertemporal preference shock ξ_t follows:

$$\xi_t = \rho_\xi \xi_{t-1} + \sigma_\xi \epsilon_{\xi,t}, \;\; \epsilon_{\xi,t} \sim \mathcal{N}(0,1).$$

The household's budget constraint is given by:

$$c_t + x_t + \frac{b_{t+1}}{p_t} = w_t l_t + r_t k_t + R_{t-1}\frac{b_t}{p_t} + F_t + T_t, \tag{2}$$

where x_t is investment in capital, w_t is the wage, r_t is the rental price of capital, F_t are the profits of the firms in the economy, and T_t is a lump-sum transfer from the government. The household trades a nominal bond b_t that pays a gross return of R_t. We transform the nominal bond into real quantities by dividing by the price p_t of the final good. There is, as well, a full set of Arrow securities. With complete markets and a zero net supply condition for those securities, we can omit them from the budget constraint.

Investment x_t induces the law of motion for capital:

$$k_t^* = (1-\delta)\,k_t + \mu_t\left(1 - S\left[\frac{x_t}{x_{t-1}}\right]\right)x_t \tag{3}$$

where

$$\log k_t = \log k_{t-1}^* - d_t\theta_t \tag{4}$$

and

$$S\left[\frac{x_t}{x_{t-1}}\right] = \frac{\kappa}{2}\left(\frac{x_t}{x_{t-1}} - \Lambda_x\right)^2.$$

Here, k_{t-1}^* is the capital decision taken by the household in period $t-1$. Actual capital k_t, however, depends on the disaster shock. Define a binary, i.i.d. random variable d_t that takes values 0 (no disaster) with probability $1-p_d$ or 1 (disaster) with probability p_d. If $d_t = 1$, k_t falls by θ_t. Gourio (2012) interprets θ_t as the permanent capital depreciation triggered by a disaster.

We want, in addition, to capture the idea that the disaster risk can be time-varying. To do so, we add an AR structure to the log of θ_t:

$$\log\theta_t = (1-\rho_\theta)\log\bar{\theta} + \rho_\theta\log\theta_{t-1} + \sigma_\theta\epsilon_{\theta,t},\ \ \epsilon_{\theta,t} \sim \mathcal{N}(0,1). \tag{5}$$

We specify the evolution θ_t in logs to ensure $\theta_t > 0$ for all t. The law of motion in (5) resembles those in models with stochastic volatility (Andreasen, 2012, Fernández-Villaverde, Guerrón-Quintana, and Rubio-Ramírez, 2015, and Gabaix, 2012).

The second term on the right-hand side of equation (3):

$$\mu_t\left(1 - S\left[\frac{x_t}{x_{t-1}}\right]\right)x_t$$

includes two parts: an investment-specific technological shock μ_t that follows:

$$\log \mu_t = \log \mu_{t-1} + \Lambda_\mu + \sigma_\mu \epsilon_{\mu,t}, \;\; \epsilon_{\mu,t} \sim \mathcal{N}(0,1).$$

and a quadratic capital adjustment cost function that depends on investment growth (Christiano, Eichenbaum, and Evans, 2005).

The household maximizes its preferences (1) subject to the budget constraint (2) and the law of motion for capital (3). The optimality conditions for this problem are (see the Online Appendix for details):

$$\mathbb{E}_t \left(M_{t+1} \exp\left(-d_{t+1}\theta_{t+1}\right) \left[r_{t+1} + q_{t+1} \left(1-\delta\right) \right] \right) = q_t \tag{6}$$

$$\begin{aligned} 1 = q_t \mu_t & \left[\left(1 - S\left[\frac{x_t}{x_{t-1}} \right] \right) - S'\left[\frac{x_t}{x_{t-1}} \right] \frac{x_t}{x_{t-1}} \right] \\ & + \mathbb{E}_t \left(M_{t+1} \left[q_{t+1}\mu_{t+1} S'\left[\frac{x_{t+1}}{x_t} \right] \left(\frac{x_{t+1}}{x_t} \right)^2 \right] \right), \end{aligned} \tag{7}$$

$$\nu \frac{c_t}{1-l_t} = w_t, \tag{8}$$

where λ_t is the Lagrange multiplier associated with the budget constraint, q_t is the Lagrange multiplier associated with the evolution law of capital (as a ratio of λ_t), and M_{t+1} is the stochastic discount factor:

$$M_{t+1} = \beta \frac{\lambda_{t+1}}{\lambda_t} \frac{V_{t+1}^{\psi-\gamma}}{\mathbb{E}_t \left(V_{t+1}^{1-\gamma} \right)^{\frac{\psi-\gamma}{1-\gamma}}}.$$

A non-arbitrage condition also determines the nominal gross return on bonds:

$$1 = \mathbb{E}_t M_{t+1} \frac{R_t}{\Pi_{t+1}}.$$

2.2 The final good producer

The final good y_t is produced by a perfectly competitive firm that bundles a continuum of intermediate goods y_{it} using the production function:

$$y_t = \left(\int_0^1 y_{it}^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}} \tag{9}$$

where ε is the elasticity of substitution. The final good producer maximizes profits subject to the production function (9) and taking as given the price of the final good, p_t, and all intermediate goods prices p_{it}. Well-known results tell us that $p_t = \left(\int_0^1 p_{it}^{1-\varepsilon} di \right)^{\frac{1}{1-\varepsilon}}$.

2.3 Intermediate good producers

There is a continuum of differentiated intermediate good producers that combine capital and labor with the production function:

$$y_{i.t} = \max \left\{ A_t k_{i,t}^{\alpha} l_{i,t}^{1-\alpha} - \phi z_t, 0 \right\}. \tag{10}$$

The common neutral technological level A_t follows a random walk with a drift in logs:

$$\log A_t = \log A_{t-1} + \Lambda_A + \sigma_A \epsilon_{A,t} - (1-\alpha) d_t \theta_t, \;\; \epsilon_{A,t} \sim \mathcal{N}(0,1),$$

subject to a Gaussian shock $\epsilon_{A,t}$ and a rare disaster shock d_t with a time-varying impact θ_t. Following Gabaix (2011) and Gourio (2012), disasters reduce physical capital and total output by the same factor. This can be easily generalized at the cost of heavier notation and, possibly, additional state variables. The common fixed cost, ϕz_t, is indexed by a measure of technology, $z_t = A_t^{\frac{1}{1-\alpha}} \mu_t^{\frac{\alpha}{1-\alpha}}$, to ensure that it remains relevant over time.

Intermediate good producers rent labor and capital in perfectly competitive markets with flexible wages and rental rates of capital. However, intermediate good producers set prices following a Calvo schedule. In each period, a fraction $1-\theta_p$ of intermediate good producers reoptimize their prices to $p_t^* = p_{it}$ (the reset price is common across all firms that update their prices). All other firms keep their old prices. Given an indexation parameter χ, this pricing structure yields a Calvo block (see the derivation in the Online Appendix):

$$\frac{k_t}{l_t} = \frac{\alpha}{1-\alpha} \frac{w_t}{r_t} \tag{11}$$

$$g_t^1 = mc_t y_t + \theta_p \mathbb{E}_t M_{t+1} \left(\frac{\Pi_t^{\chi}}{\Pi_{t+1}} \right)^{-\varepsilon} g_{t+1}^1 \tag{12}$$

$$g_t^2 = \Pi_t^* y_t + \theta_p \mathbb{E}_t M_{t+1} \left(\frac{\Pi_t^{\chi}}{\Pi_{t+1}} \right)^{1-\varepsilon} \left(\frac{\Pi_t^*}{\Pi_{t+1}^*} \right) g_{t+1}^2 \tag{13}$$

$$\varepsilon g_t^1 = (\varepsilon - 1) g_t^2 \tag{14}$$

$$1 = \theta_p \left(\frac{\Pi_{t-1}^{\chi}}{\Pi_t} \right)^{1-\varepsilon} + (1-\theta_p) (\Pi_t^*)^{1-\varepsilon} \tag{15}$$

$$mc_t = \left(\frac{1}{1-\alpha} \right)^{1-\alpha} \left(\frac{1}{\alpha} \right)^{\alpha} \frac{w_t^{1-\alpha} r_t^{\alpha}}{A_t}. \tag{16}$$

Here, $\Pi_t \equiv \frac{p_t}{p_{t-1}}$ is the inflation rate in terms of the final good, $\Pi_t^* \equiv \frac{p_t^*}{p_t}$ is the ratio between the reset price and the price of the final good, mc_t is the marginal cost of the intermediate good producer, and g_t^1 and g_t^2 are auxiliary variables that allow us to write this block recursively.

2.4 The monetary authority

The monetary authority sets the nominal interest rate according to the Taylor rule:

$$\frac{R_t}{R} = \left(\frac{R_{t-1}}{R}\right)^{\gamma_R} \left(\left(\frac{\Pi_t}{\Pi}\right)^{\gamma_\Pi} \left(\frac{\frac{y_t}{y_{t-1}}}{\exp\left(\Lambda_y\right)}\right)^{\gamma_y}\right)^{1-\gamma_R} e^{\sigma_m \epsilon_{m,t}} \tag{17}$$

where $\epsilon_{m,t} \sim \mathcal{N}(0,1)$ is a monetary shock, the variable Π is the target level of inflation, and R is the implicit target for the nominal gross return of bonds (which depends on Π, β, and the growth rate Λ_y along the balanced growth path of the model). The proceedings from monetary policy are distributed as a lump sum to the representative household.

2.5 Aggregation

The aggregate resource constraint is given by:

$$c_t + x_t = \frac{1}{v_t^p}\left(A_t k_t^\alpha l_t^{1-\alpha} - \phi z_t\right) \tag{18}$$

where

$$v_t^p = \int_0^1 \left(\frac{p_{it}}{p_t}\right)^{-\varepsilon} di$$

is a measure of price dispersion with law of motion:

$$v_t^p = \theta_p \left(\frac{\Pi_{t-1}^\chi}{\Pi_t}\right)^{-\varepsilon} v_{t-1}^p + (1-\theta_p)\left(\Pi_t^*\right)^{-\varepsilon}.$$

2.6 Asset prices

Rare disasters have a large impact on asset prices. Indeed, this is the reason they have become a popular area of research. Thus, it is worthwhile to review three asset pricing implications of the model. First, the price of a one-period risk-free real bond, q_t^f, is:

$$q_t^f = \mathbb{E}_t\left(M_{t+1}\right).$$

Second, the price of a claim to the stream of dividends $div_t = y_t - w_t l_t - x_t$ (all income minus labor income and investment), which we can call equity, is equal to:

$$q_t^e = \mathbb{E}_t\left(M_{t+1}\left(div_{t+1} + q_{t+1}^e\right)\right).$$

We specified that the household owns the physical capital and rents it to the firm. Given our complete markets assumption, this is equivalent to the firm owning the physical capital and the household owning these claims to dividends. Our ownership convention makes deriving optimality conditions slightly easier. Third, we can define the price-earnings ratio:

$$\frac{q_t^e}{div_t} = \mathbb{E}_t \left(M_{t+1} \frac{div_{t+1}}{div_t} \left(1 + \frac{q_{t+1}^e}{div_{t+1}} \right) \right).$$

All these prices can be solved indirectly, once we have obtained the solution of M_{t+1} and other endogenous variables, or simultaneously. To show the flexibility of Taylor projection, we will solve for q_t^f and q_t^e simultaneously with the other endogenous variables. This approach is necessary, for example, in models with financial frictions, where asset prices can determine real variables.

However, in general, it is not a good numerical strategy to solve simultaneously for volatile asset prices. For instance, the price of a consol fluctuates wildly, especially if the expected return is low or negative. This happens when the disaster risk suddenly rises. The perturbation solution for the price of this asset displays large Taylor coefficients that converge very slowly. Series-based methods may even fail to provide a solution if the variables move outside the convergence domain of their series.

2.7 Stripping down the full model

To examine the computational properties of the solution for models of different size and complexity, we solve eight versions of the model. Version 1 of the model is a benchmark real business cycle model with Epstein-Zin preferences and time-varying disaster risk. Prices are fully flexible, the intermediate good producers do not have market power (i.e., ε goes to infinity), and there are no adjustment costs in investment. Hence, instead of the Calvo block (11)-(16), factor prices are determined by their marginal products:

$$r_t = \alpha A_t k_t^{\alpha-1} l_t^{1-\alpha} \tag{19}$$

$$w_t = (1-\alpha) A_t k_t^{\alpha} l_t^{-\alpha}. \tag{20}$$

The benchmark version consists of four state variables: planned capital k_{t-1}^*, disaster shock d_t, disaster risk θ_t, and technology innovations $\sigma_A \epsilon_{A,t}$. Also, since the model satisfies the classical dichotomy, we can ignore the Taylor rule.

Version 2 of the model introduces investment adjustment costs to version 1, but not the investment-specific technological shock. This adds past investment x_{t-1} as another state variable. We still ignore the monetary part of the model.

Version 3 of the model reintroduces price rigidity. Since we start using the Calvo block (11)-(16), we need two additional state variables: past inflation Π_{t-1} and price dispersion v_{t-1}^p. However, in this version 3, we employ a simple Taylor rule that responds only to inflation. Versions 4 and 5 extend the Taylor rule, so it responds to output growth and the past interest rate. These two versions introduce past output and the past interest rate as additional state variables. But, in all three versions, there are no monetary shocks to the Taylor rule.

Finally, versions 6, 7, and 8 of the model introduce the investment-specific technological shock, the monetary shocks, and the preference shocks. These shocks are added to the vector of state variables one by one. The full model (version 8) contains 12 state variables.

3 Calibration

Before we compute the model, we normalize all relevant variables to obtain stationarity. We follow the normalization scheme in Fernández-Villaverde and Rubio-Ramírez (2006) (see the Online Appendix).

The model is calibrated at a quarterly frequency. When needed, Gaussian shocks are discretized by monomial rules with $2n_\epsilon$ nodes (for n_ϵ shocks). Parameter values are listed in Table 1. Most parameters are taken from Fernández-Villaverde, Guerrón-Quintana, and Rubio-Ramírez (2015), who perform a structural estimation of a very similar DSGE model (hereafter FQR). There are three exceptions. The first exception is Epstein-Zin parameters and the standard deviation of TFP shocks, which we take from Gourio (2012).

The second exception is the three parameters in the Taylor rule, which we calibrate somewhat more conservatively than those in FQR. Specifically, we pick the inflation target to be 2 percent annually, the inflation parameter γ_Π to be 1.3, which satisfies the Taylor principle, and the interest smoothing parameter γ_R to be 0.5. The estimated values of γ_R and γ_Π in FQR are less common in the literature and, when combined with rare disasters, they generate too strong, and empirically implausible, nonlinearities.

The third exception is the parameters related to disasters. In the baseline calibration, we calibrate the mean disaster impact $\bar{\theta}$ such that output loss in a disaster is 40 percent. This is broadly in line with Barro (2006), who estimates an average contraction of 35 percent compared to trend. We do not account for partial recoveries, so the impact of disaster risk may be overstated. For our purposes, this bias makes the model harder to solve because the nonlinearity is stronger. The persistence of disaster risk is set at $\rho_\theta = 0.9$, which is close to Gourio (2012) and Gabaix (2012), although those researchers use slightly different specifications. The standard deviation of the disaster risk is calibrated at $\sigma_\theta = .025$. The four disaster parameters - probability, mean impact, persistence, and standard deviation - have a strong effect on the precautionary saving motive and asset prices. Ideally, these parameters should be jointly estimated, but, to keep our focus, we do not pursue this route. Instead, we choose parameter values that generate realistic risk premia and that are broadly consistent with the previous literature.

We also consider an alternative no-disaster calibration, where we set the mean and standard deviation of the disaster impact very close to zero, while keeping all the other parameter values as in the baseline calibration in Table 1. We do so to benchmark our results without disasters and gauge the role of large risks regarding accuracy and computational time.

4 Solution methods

Given that we deal with models with up to 12 state variables, we only investigate solution methods that scale well in terms of the dimensionality of the state space. This eliminates, for example, value function iteration or tensor-based projection methods. The three methods left on the table are perturbation (a particular case of which is linearization), Taylor projection, and Smolyak collocation.[3] The methods are implemented for different polynomial orders. More concretely, we aim to compute 176 solutions, with 11 solutions per each of the eight versions of the model -perturbations from order 1 to 5, Taylor projections from order 1 to 3, and Smolyak collocation from level 1 to 3- and the two calibrations described above, the baseline calibration and the no-disaster calibration. As we will point out below, we could not find a few of the Smolyak collocation solutions.

Perturbation and Smolyak collocation are well-known. They are described in detail in Fernández-Villaverde, Rubio-Ramírez, and Schorfheide (2016). In comparison, Taylor projection is a new method recently proposed by Levintal (2016). We discuss the three methods briefly in the next pages (see also an example in the Online Appendix). But, first, we need to introduce some notation by casting the model in the form:

$$\mathbb{E}_t f\left(y_{t+1}, y_t, x_{t+1}, x_t\right) = 0 \tag{21}$$

$$y_t = g\left(x_t\right) \tag{22}$$

$$x_{t+1} = h\left(x_t\right) + \eta\epsilon_{t+1}, \tag{23}$$

where x_t is a vector of n_x state variables, y_t is a vector of n_y control variables, $f : \mathbb{R}^{2n_x+2n_y} \rightarrow$

[3] Judd, Maliar, and Maliar (2011) offer an alternative, simulation-based solution method. Maliar and Maliar (2014) survey the recent developments in simulation methods. We abstract from simulation methods, because the Smolyak collocation method is already satisfactory in terms of computational costs. For larger models, simulation methods may be more efficient than Smolyak collocation, although we will later comment on why we conjecture that, for our class of models, simulation methods may face challenges.

$\mathbb{R}^{n_x+n_y}$, $g : \mathbb{R}^{n_x} \to \mathbb{R}^{n_y}$, $h : \mathbb{R}^{n_x} \to \mathbb{R}^{n_x}$, η is a known matrix of dimensions $n_x \times n_\epsilon$, and ϵ is a $n_\epsilon \times 1$ vector of zero mean shocks. The first equation gathers all expectational conditions, the second one maps states into controls, and the last one is the law of motion for states. Equations (21)-(23) constitute a system of $n_y + n_x$ functional equations in the unknown policy functions g and h. In practical applications, some of the elements of h are known (e.g., the evolution of the exogenous state variables), so the number of unknown functions and equations is smaller.

4.1 Perturbation

Perturbation introduces a parameter σ that controls the volatility of the model. Specifically, equation (22) is replaced by $y_t = g(x_t, \sigma)$ and equation (23) with $x_{t+1} = h(x_t, \sigma) + \sigma\eta\epsilon_{t+1}$. At $\sigma = 0$, the economy boils down to a deterministic model, whose steady state, $\bar{x}$, (assuming it exists) can often be easily calculated. Then, by applying the implicit function theorem, we recover the derivatives of the policy functions g and h with respect to x and σ. Having these derivatives, the policy functions are approximated by a Taylor series around $\bar{x}$. To capture risk effects, the Taylor series must include at least second-order terms.

High-order perturbation solutions have been developed and explored by Judd (1998), Gaspar and Judd (1997), Jin and Judd (2002), and Aruoba, Fernández-Villaverde, and Rubio-Ramírez (2006), among others. Obtaining perturbation solutions is easy for low orders, but cumbersome at high orders, especially for large models. In this paper, we use the perturbation algorithm presented in Levintal (2015), which allows solving models with non-Gaussian shocks up to the fifth order. We also reduce computational time by adopting the algorithm proposed by Kameník (2005) to solve the Sylvester equation that arises in perturbation methods.

4.2 Smolyak collocation

Collocation is one of the projection methods introduced by Judd (1992). The policy functions $g(x)$ and $h(x)$ are approximated by polynomial functions $\widehat{g}(x, \Theta_g)$ and $\widehat{h}(x, \Theta_h)$, where Θ_g and Θ_h are the polynomial coefficients of $\widehat{g}$ and $\widehat{h}$, respectively. Let $\Theta = (\Theta_g, \Theta_h)$ denote a vector of size n_Θ of all polynomial coefficients. Substituting in equation (21) yields a residual function $R(x_t, \Theta)$:

$$R(x_t, \Theta) = \mathbb{E}_t f\left(\widehat{g}\left(\hat{h}(x_t, \Theta_h) + \eta\epsilon_{t+1}, \Theta_g\right), \widehat{g}(x_t, \Theta_g), \widehat{h}(x_t, \Theta_h) + \eta\epsilon_{t+1}, x_t\right). \tag{24}$$

Collocation methods evaluate the residual function $R(x, \Theta)$ at N points $\{x_1, \ldots, x_N\}$, and find the vector Θ for which the residual function is zero at all points. This requires solving a nonlinear system for Θ:

$$R(x_i, \Theta) = 0, \ \ \forall i = 1, \ldots, N. \tag{25}$$

The number of grid points N is chosen such that the number of conditions is equal to the number of coefficients to be solved (n_Θ).

Since DSGE models are multidimensional, the choice of the basis function is crucial for computational feasibility. We follow Krüger and Kubler (2004) by using Smolyak polynomials of levels 1, 2, and 3 as the basis function. These approximation levels vary in the size of the basis function. The level 1 approximation contains $1 + 2n_x$ terms, the level 2 contains $1 + 4n_x + (4n_x (n_x - 1))/2$ terms, and the level 3 contains $1 + 8n_x + 12n_x (n_x - 1)/2 + 8n_x (n_x - 1)(n_x - 2)/6$ terms. The Smolyak approximation level is different from the polynomial order, as it contains higher order terms. For instance, an approximation of level 1 contains quadratic terms. Hence, the number of terms in a Smolyak basis of level k is larger than the number of terms in a kth-order complete polynomial.[4]

The first step of this approach is to construct the grid $\{x_1, \ldots, x_N\}$. The bounds of the grid affect the accuracy of the solution. For a given basis function, a wider grid reduces accuracy, because the same approximating function has to fit a larger domain of the state space. We would like to have a good fit at points that the model is more likely to visit, at the expense of other less likely points.

Disaster models pose a special challenge for grid-based methods because the disaster periods are points of low likelihood, but with a large impact. Hence, methods that build a grid over a high probability region (Maliar and Maliar, 2014) may not be appropriate for disaster models. For this reason, we choose a more conservative approach and construct the grid by a hypercube. Specifically, we obtain a third-order perturbation solution, which is computationally cheap, and use it to simulate the model. Then, we take the smallest hypercube that contains all the simulation points (including the disaster periods) and build a Smolyak grid over the hypercube. In the level-3 Smolyak approximations, we had to increase the size of the hypercube by up to 60 percent; otherwise, the Jacobian would be severely ill-conditioned (we use the Newton method; see below). Our grid method is extremely fast, so we ignore its computational costs in our run time comparisons.[5]

[4] We use the codes by Judd, Maliar, Maliar, and Valero (2014) to construct the Smolyak polynomials and the corresponding grid. We also employ their codes of monomial rules to discretize Gaussian shocks.

[5] Judd, Maliar, Maliar, and Valero (2014) propose replacing the hypercube with a parallelotope that encloses the ergodic set. This technique may increase accuracy if the state variables are highly correlated. In our case, the correlation between the state variables is low (piecewise correlation is 0.14 on average), so the potential gain from this method is small, while computational costs are higher. More recently, Maliar and Maliar (2014, 2015) have proposed new types of grids. Given the dimensionality of our problem and the feasibility of using a Newton algorithm with analytic derivatives to solve for Θ, these techniques, which carry computational costs of their own, are unlikely to perform better than our implementation.

The final, and most demanding, step is to solve the nonlinear system (25). Previous studies have used time iteration, e.g., Krüger and Kubler (2004), Malin, Krüger, and Kubler (2011), and Fernández-Villaverde, Gordon, Guerrón-Quintana, and Rubio-Ramírez (2015), but this method can be slow. More recently, Maliar and Maliar (2014) have advocated the use of fixed-point iteration. For the size of our models (up to 12 state variables), a Newton method with analytic Jacobian performs surprisingly well. The run time of the Newton method is faster than that of the fixed-point methods reported in the literature for models of similar size: e.g., see Judd, Maliar, Maliar, and Valero (2014). Moreover, the Newton method ensures convergence if the initial guess is sufficiently good, whereas fixed-point iteration does not guarantee convergence even if it starts near the solution. Our initial guess is a third-order perturbation solution, which proves to be sufficiently accurate for our models. Thus, the Newton method converges in just a few iterations.[6]

Our implementation of Smolyak collocation yields a numerically stable system. By comparison, derivative-free solvers (e.g., Maliar and Maliar, 2015) gain more flexibility in the choice of basis functions and grids, but lose the convergence property of Newton-type solvers, which are especially convenient in our case because we have access to a good initial guess.

4.3 Taylor projection

Taylor projection is a new type of projection method proposed by Levintal (2016). As with standard projection methods, the policy functions $g(x)$ and $h(x)$ are approximated by polynomial functions $\widehat{g}(x, \Theta_g)$ and $\widehat{h}(x, \Theta_h)$, where $\Theta = (\Theta_g, \Theta_h)$ is a vector of size n_Θ of all polynomial coefficients. In our application, we use simple monomials as the basis for our approximation, but one could employ a more sophisticated basis. Given these polynomial functions, we build the residual function $R(x, \Theta)$ exactly as in equation (24). As with standard projection methods, the goal is to find Θ for which the residual function $R(x, \Theta)$, defined by equation (24), is approximately zero over a certain domain of the state space that is of interest.

To do so, one can approximate $R(x, \Theta)$ in the neighborhood of x_0 by a kth-order Taylor series about x_0. In our application, we select x_0 to be the deterministic steady state of the model, but nothing forces us to make that choice. This flexibility in the selection of x_0 is an advantage of Taylor projection with respect to standard perturbation, which is constrained to take the Taylor series expansion of the decision rules of the economy around the deterministic steady state of the model.

[6]We work on a Dell computer with an Intel(R) Core(TM) i7-5600U Processor and 16GB RAM, and our codes are written in `MATLAB/MEX`.

More concretely, if all the Taylor coefficients up to the kth-order are zero, then $R(x, \Theta) \approx 0$ in the neighborhood of x_0. This amounts to finding values for Θ that make the residual function and all its derivatives with respect to the state variables up to the kth-order zero at x_0. Formally, Θ solves:

$$\begin{aligned}
&R(x_0, \Theta) = 0 \\
&\left.\frac{\partial R(x, \Theta)}{\partial x_i}\right|_{x_0} = 0, \ \forall i = 1, \ldots, n_x \\
&\left.\frac{\partial^2 R(x, \Theta)}{\partial x_{i_1} \partial x_{i_2}}\right|_{x_0} = 0, \ \forall i_1, i_2 = 1, \ldots, n_x \\
&\vdots \\
&\left.\frac{\partial^k R(x, \Theta)}{\partial x_{i_1} \cdots \partial x_{i_k}}\right|_{x_0} = 0, \ \forall i_1, \ldots, i_k = 1, \ldots, n_x.
\end{aligned} \tag{26}$$

System (26) is solved using the Newton method with the analytic Jacobian. For comparability with Smolyak collocation, we use the same initial guess (the polynomial coefficients implied by a third-order perturbation solution) and the same stopping rule for the Newton method.

Taylor projection offers several computational advantages over standard projection methods. First, a grid is not required. The polynomial coefficients are identified by information that comes from the model derivatives, rather than a grid of points. Second, the basis function is a complete polynomial. This gives additional flexibility over Smolyak polynomials. For instance, interaction terms can be captured by a second-order solution, which has $1 + n_x + n_x(n_x + 1)/2$ terms in the basis function. In Smolyak polynomials, interactions show up only at the level-2 approximation with $1 + 4n_x + (4n_x(n_x - 1))/2$ terms in the basis function (asymptotically four times larger). More terms in the basis function translate into a larger Jacobian, which is the main computational bottleneck of the Newton method. Finally, the Jacobian of Taylor projection is much sparser than the one from collocation. Hence, the computation of the Jacobian and the Newton step is cheaper.

The main cost of Taylor projection is the computation of all the derivatives. The Jacobian requires differentiation of the nonlinear system (26) with respect to Θ. These derivatives can be computed efficiently by the chain rule method developed by Levintal (2016). This method expresses high-order chain rules in compact matrix notation that exploits symmetry, permutations, and repeated partial derivatives. The chain rules can also take advantage of sparse matrix (or tensor) operations. For more details, see Levintal (2016).

5 Results

We are now ready to discuss our results. In three subsections, we will describe our findings regarding accuracy, simulations, and computational costs.

5.1 Accuracy

As proposed by Judd (1992), we assess accuracy by comparing the mean and maximum unit-free Euler errors across the ergodic set of the model. We approximate this ergodic set by simulating the model with the solution that was found to be the most accurate (third-order Taylor projection). The length of the simulation is 10,000 periods starting at the deterministic steady state, from which we exclude the first 100 periods (results were robust to longer burn-in periods). All simulations are buffeted by the same random shocks.

We first report accuracy measures for the no-disasters calibration model to benchmark our results. Tables 2 and 3 report the mean and maximum error for this calibration. As expected, all 11 solutions are reasonably accurate for each of the 8 versions of the model. The mean Euler errors (in log10 units) range from around -2.7 (for a first-order perturbation) to -10.2 (for a level-3 Smolyak). The max Euler errors range from -1.3 (for a first-order perturbation) to -9.2 (for a level-3 Smolyak). These results replicate the well-understood notion that models with weak volatility can be accurately approximated by linearization. See, for a similar result, Aruoba, Fernández-Villaverde, and Rubio-Ramírez (2006).[7]

Tables 4 and 5 report the accuracy measures for the baseline calibration.[8] The accuracy measures change significantly when disasters are introduced into the model. The mean and maximum errors are now, across all solutions, one to three orders of magnitude larger than before. First-order perturbation and Taylor projection solutions are severely inaccurate, with max Euler errors as high as -0.1. Higher-order perturbation solutions are more accurate, but errors are still relatively large. In particular, we find that a third-order perturbation solution is unlikely to be accurate enough, with mean Euler errors between -1.8 and -2.5 and max Euler errors between -1.5 and -1.8. Even a fifth-order perturbation can generate a disappointing mean Euler error of between -1.9 and -3.5. It is interesting to highlight that the higher-order terms introduced in the approximated solution by the fourth- and fifth-order perturbations are larger than in similar models without rare disasters. For example,

[7]We approximate the same set of variables by all methods and use the model equations to solve for the remaining variables. While applying perturbation methods, researchers usually employ the perturbation solution for all variables instead. We avoid that practice because we want to be consistent across all solution methods. See the Online Appendix for details.

[8]The results for the level-1 Smolyak collocation are partial because the Newton solver did not always converge. For the level-3 Smolyak and to avoid ill-conditioned Jacobians, the size of the grid was increased by 30 percent for version 3 of the model and by 60 percent for versions 4-8.

the contribution of the fifth-order correction term associated with the perturbation parameter changes the annualized interest rate by roughly 0.3 percent, which is non-negligible. Levintal (2015) discusses in detail the interpretation of these additional correction terms.

In comparison, second- and third-order Taylor projections deliver a much more solid accuracy, with mean Euler errors between -3.6 and -6.9. The max Euler errors are about two orders of magnitude larger, suggesting that in a few rare cases these solutions are less accurate. We will later explore whether the differences between mean and max Euler errors are economically significant. We can, however, provide some intuition as to why Taylor projection outperforms perturbation. In standard perturbation, we find a solution for the variables of interest by perturbing a volatility of the shocks around zero. In comparison, in Taylor projection (as we would do in a projection), we take account of the true volatility of the shocks. More concretely, we evaluate the residual function and its derivatives at a point such as the deterministic steady state of the state variables (although other points are possible), but all the relevant conditional expectations in the Euler conditions are still exact, not approximated around a zero volatility. In models with strong volatility, such as those with rare disasters, this can make a big difference.

The Smolyak solution is an improvement over the fifth-order perturbation solution, but it is typically less accurate than a Taylor projection of comparable order. How can this happen given the higher-order terms in the polynomials forming the Smolyak solution? Because of the strong nonlinearity generated by rare disasters. The Smolyak method has to extrapolate outside the grid. Since the grid already contains extreme points (rare disasters), extrapolating outside these extreme points introduces even more extreme points (e.g., a disaster period that occurs right after a disaster period). By comparison, Taylor projection evaluates the residual function and its derivatives at one point, which is a normal period. Thus, it has to extrapolate only for next-period likely outcomes, which can be either normal or disaster periods. This reduces the approximation errors that contaminate the solution. Furthermore, Taylor projection takes advantage of the information embedded in the derivatives of the residual function, information that is ignored in projection methods.

To dig deeper, we plot in Figure 1 the model residuals across the ergodic set for fourth- and fifth-order perturbations, for second- and third-order Taylor projection, and level-2 and -3 Smolyak collocation (lower level approximations display similar errors, but of higher magnitude). We show the errors for the last 1,000 periods out of our simulation of 10,000 and for the full model (version 8).

These plots reveal three important differences among the errors of each method. First is the larger magnitude of the errors in perturbation in comparison with the errors in Taylor projection and Smolyak. Second, Taylor projection exhibits very small errors throughout the sample, except for one peak of high errors (and five intermediate ones), which occur around particularly large disaster periods. Since Taylor projection zeros the Taylor series of the residual function, the residuals are small as long as the model stays around the center of the Taylor series (in our case, the deterministic steady state). Namely, Taylor projection yields a locally accurate solution, which deteriorates at points distant from the center. Fortunately, these points are unlikely, even considering the disaster risk. More crucially, the simulated model moments and IRFs (and, thus, the economic implications) of Taylor projection and Smolyak are nearly indistinguishable (see the next subsection). Also, recall that most of the interesting economics of rare disasters is not in what happens after a disaster (the economy sinks!), but on how the possibility of a disaster changes the behavior of the economy in normal times (for example, regarding asset prices). Thus, obtaining good accuracy in normal times, as Taylor projection does, is rather important.

Third, the Smolyak errors are more evenly distributed than the errors from the Taylor projection. This is not surprising: the collocation algorithm minimizes residuals across the collocation points, which represent the ergodic set. This also reflects the uniform convergence of projection methods (Judd, 1998). The disaster periods tilt the solution towards these rare episodes at the expense of the more likely normal states. As a result, the errors in normal states get larger, because the curvature of the basis function is limited. In other words, to get a bit better accuracy in 5 periods than Taylor projection, Smolyak sacrifices some accuracy in 995 periods. Given the evidence that we report below of the moments of the simulations, the shape of the IRFs, and computational time, and the economic logic of the model about the importance of its behavior in normal times outlined above, this sacrifice is not worthwhile. A possible solution to the problem would be to increase the Smolyak order, but again as shown below, the computational costs are too high.

Finally, we can improve the accuracy of Taylor projection by solving the model outside the deterministic steady state (as we will do in Section 6) or at multiple points (as in Levintal, 2016). For instance, we could solve the model also at a disaster period and use this solution when the model visits that point. For these solutions to be accurate, an important condition must hold: the state variables must not change dramatically (in probability) from the current period to the future period. This condition holds when the model is in a normal state, because it is highly likely that it stays at a normal state the next period as well. However, if the model is in a disaster state, it is very likely that it will change to a normal state the next period. Hence, solving the model in a disaster state is prone to higher approximation errors. Nevertheless, a researcher can build the model in such a way that the future state of the economy is likely to be similar to the current state (for instance, by increasing the frequency of the calibration or the persistence of the exogenous shocks).

5.2 Simulations

Our second step is to compare the equilibrium dynamics generated by the different solutions. In particular, we look at two standard outputs from DSGE models: moments from simulations and IRFs.

Rare disasters generate a strong impact on asset prices and risk premia. The solution methods should be able to approximate these effects. Hence, we examine how the different solutions approximate the prices of equity and risk-free bonds. Tables 6 and 7 present the mean risk-free rate and the mean return on equity across simulations generated by the different methods (again, 10,000 periods with a burn-in of 100). We focus on the full model (version 8). By the previous accuracy measures, the most accurate solutions are Taylor projection of orders 2 and 3, and Smolyak collocation of orders 2 and 3. The mean risk-free rate in these four solutions is 1.5-1.6 percent. Despite the differences in mean and maximum Euler errors, from an economic viewpoint, these four solutions yield roughly the same result.

By comparison, perturbation solutions, which have been found to be less accurate, generate a much higher risk-free rate, ranging from 4.6 percent at the first order to 2.1 percent at the fifth order. At the third order (a popular choice when solving models with stochastic volatility), the risk-free rate is 2.7 percent. Thus, perturbation methods fail to approximate accurately the risk-free rate, unless one goes for very high orders. At the fifth order, the approximation errors are relatively small, which is consistent with the results in Levintal (2015). The mean return on equity is more volatile across the different perturbation solutions, but fairly close to the 5.3-5.4 percent obtained by the four accurate solutions.

Differences in real variables can also be significative. Tables 8 and 9 report the simulation averages of (detrended) investment and capital in the model, the two real variables most affected by the precautionary behavior induced by disasters. We can see differences of nearly 5 percent in the average level of investment and capital between, for example, a first-order perturbation and a third-order Taylor projection. A similar exercise appears in Tables 10 and 11, but now in terms of the standard deviation of both variables. While the differences in the standard deviation of investments are small, they are relevant for capital. These differences in asset prices and real quantities may cause, for instance, misleading calibrations or inconsistent estimators, as researchers try to match observed data with model-simulated data.

We next examine IRFs. We focus on the disaster variables, which generate the main nonlinearity in our model. Figure 2 presents the response of the model to a disaster shock. The initial point for each IRF is the stochastic steady state implied by the corresponding solution method (note the slightly different initial levels of each IRF). After the initial shock, all future shocks are zero.[9] The figure plots the response of output, investment, and consumption. In the left panels, we plot three perturbation solutions and a third-order Taylor projection. In the right panels, we plot the three Taylor projections and Smolyak levels 2 and 3 (the mnemonics in the figure should be easy to read). Although the scale of the shock is large and, therefore, it tends to cluster all IRFs, we can see some non-trivial differences in the IRFs from low-order perturbations with respect to all the other IRFs (furthermore, the model is solved for the detrended variables, which are much less volatile).

Figure 3 plots the IRFs of a disaster risk shock (θ_t). We assume that the disaster impact θ_t rises from a contraction of 40 percent to a contraction of 45 percent, which under our calibration is a 3.5 standard deviations event. This small change has a large impact because the model is highly sensitive to the disaster parameters. All solutions generate in response a decline in detrended output, investment, and consumption, but the magnitudes differ considerably. Note that a change in θ_t impacts the expected growth of neutral technology and, therefore, it has an effect even in a first-order perturbation. As before, the left panels of the figure compare the perturbation solutions to a third-order Taylor projection. Low-order perturbation solutions fail to approximate well the model dynamics, although the fifth-order perturbation is relatively accurate. The right panel of Figure 3 shows a similarity of the four most accurate solutions (second- and third-order Taylor projection and Smolyak levels 2 and 3). This figure and the results from Tables 6 and 7 indicate that the solutions generated by a second- and third-order Taylor projection are economically indistinguishable from the solutions from a Smolyak collocation.

Figure 4 shows similar IRFs, but only for the four most accurate solutions. The left panel depicts the same IRFs as in Figure 3 with some zooming in. The right panel shows IRFs for a larger shock, which increases the anticipated disaster impact from 40 percent to 50 percent, a 7 standard deviations event. Barro (2006) points out that, while rare, this is a shock that is sometimes observed in the data. While the differences among the solutions are economically small (the scale is log), there seem to be two clusters of solutions: second-order Taylor projection and Smolyak level-2 and third-order Taylor projection and Smolyak level-3.

We conclude from this analysis that second- and third-order Taylor projections and Smolyak solutions are economically similar. We could not find a significant difference between these solutions. The other solutions are relatively poor approximations, except for the fifth-order perturbation solution, which is reasonably good.

[9]Following conventional usage, the stochastic steady state is defined as the value of the variables to which the model converges after a long sequence of zero realized shocks. The stochastic steady state is different from the deterministic one because in the former the agents consider the possibility of having non-zero shocks (although they are never realized), while, in the latter, the agents understand that they live in a deterministic environment.

5.3 Computational costs

Our previous findings suggest that the second- and third-order Taylor projections and Smolyak solutions are similar. However, when it comes to computational costs, there are more than considerable differences among the solutions. Table 12 reports total run time (in seconds) for each solution. The second-order Taylor projection is the fastest method among the four accurate solutions by a large difference. It takes about 3 seconds to solve the full model with second-order Taylor projection, 148 seconds with third-order Taylor projection, 56 seconds with second-order Smolyak and 7,742 seconds with third-order Smolyak. Given that these solutions are roughly equivalent, this is a remarkable result. Taylor projection allows us to solve large and highly nonlinear models in a few seconds, and potentially to nest the solution within an estimation algorithm, where the model needs to be solved hundreds of times for different parameter values. Also, a second-order Taylor projection takes considerably less time than a fifth-order perturbation (3.4 seconds versus 30.4 seconds for the full model), even if its mean Euler errors are smaller (-3.6 versus -2.2).

The computational advantage of Taylor projection over Smolyak collocation stems from the structure of the Jacobian. Table 13 presents the size and sparsity of the Jacobian of the full model (version 8) for these two methods. The size of the Jacobian of Taylor projection is much smaller than that of Smolyak collocation (for example, for order/level 3, the dimension is 6,825x6,825 vs. 39,735x39,735). As explained in Section 4.3, this is due to the type of basis function used to approximate the endogenous variables. In Taylor projection, the basis function is a complete polynomial, while in Smolyak collocation it is a Smolyak polynomial, which has a larger number of coefficients. Hence, the number of unknown coefficients that need to be solved in collocation is larger than in Taylor projection.

Also, the Jacobian of Taylor projection is sparser than in collocation (for example, for order/level 3, the share of nonzeros is 0.12 vs. 0.24). To exploit this sparsity, the basis function should take the form of monomials centered at x_0, i.e., powers of $x - x_0$. Since the nonlinear system is evaluated only at x_0, all the powers of $x - x_0$ are zero. Consequently, the coefficients associated with those powers have no effect on the nonlinear system, so their corresponding entries in the Jacobian are zero.[10] By comparison, in collocation the nonlinear system is evaluated at many points $x_1, \ldots, x_N$, so the powers of $x - x_0$ are not zero, thereby introducing more nonzero entries to the Jacobian. In large models, the amount of memory required to store these nonzero entries may exceed the available resources.

[10] Levintal (2016) shows that it is possible to increase further the sparsity of the Jacobian of Taylor projection by using an approximate Jacobian that has a smaller number of nonzero elements. We do not use the approximate Jacobian because the computational gains for the size of models we consider are moderate. However, for larger models the computational gains may be substantial; see the examples in Levintal (2016).

The marginal costs of the different methods are extremely heterogeneous. Moving from version 7 to version 8 of the model adds only one exogenous state variable. This change increases the run time of a second-order Taylor projection by 1.1 seconds. By comparison, a third-order Taylor projection takes about 59 more seconds, Smolyak level-2 takes roughly 28 more seconds, and Smolyak level-3 takes 3,206 seconds. Extrapolating these trends forward implies that the differences in computational costs across solutions would increase rapidly with the size of the model.

We conclude that the second-order Taylor projection solution delivers the best accuracy/speed tradeoff among the tested solutions. The run time of this method is sufficiently fast to enable estimation of the model, which would be much more difficult with the other methods tested. For researchers interested in higher accuracy at the expense of higher costs, we recommend the third-order Taylor projection solution, which is faster than a Smolyak solution of comparable order.

Finally, we provide `MATLAB` codes that perform the Taylor projection method for a general class of DSGE models, including the models defined in Section 4. Given these codes, Taylor projection is as straightforward and easy to implement as standard perturbation methods. In comparison, coding a Smolyak collocation requires some degree of skill and care.[11]

6 Robustness analysis

In this section, we briefly report several exercises to document the robustness of our findings. Our central message is how well Taylor projection survives changing different characteristics of the numerical experiments.

Our first robustness exercise replicates our primary results when the ergodic set of the model is approximated by a Smolyak solution of level 3, instead of a third-order Taylor projection. The findings, for example, regarding Euler equation errors (Tables 14 and 15), remain entirely unchanged.

Our second robustness exercise keeps the approximation of the ergodic set by a level-3 Smolyak solution, but it increases the simulation sample size to $T = 100,000$, instead of the default $T = 10,000$. The mean Euler equation errors (Table 16) remain nearly the same, but the maximum Euler errors become, unsurprisingly, larger. With a longer simulation, we have a higher probability of moving to a region of the state space where a solution method will do worse. Interestingly, even with this long simulation, Taylor projection still does a fine job. For model 8, the maximum Euler equation error for third-order Taylor projection is -1.6.

[11]The codes are available at http://economics.sas.upenn.edu/~jesusfv/Matlab_Codes_Rare_Disasters.zip.

Our third robustness exercise increases the parameter controlling risk aversion, γ, to 5. By making the model more non-linear, a higher risk aversion deteriorates the mean Euler equation errors of all solution methods (Table 18) and increases the max Euler equation error (Table 19). A third-order Taylor projection continues to be the most accurate solution method for nearly all cases regarding mean Euler equation errors. This robustness exercise is interesting because, as reported in Table 20, when we solve the model with an accurate method such as Taylor projection or Smolyak, we can generate negative average risk-free interest rates. The risk of a rare disaster is so severe with large risk aversion that the household is willing to accept a negative risk-free interest rate to hedge against it. Furthermore, the return on equity slightly increases with respect to the baseline case (Table 21). The lower risk-free rate and the higher return on equity deliver, for the more accurate methods, an equity premium of over 7 percent.

Our fourth and fifth robustness exercises double the disaster probability (0.0086 compared to 0.0043 in the benchmark model) and the standard deviation of disaster size (σ_θ) (0.05 compared to 0.025) respectively. See Tables 22 and 25 for results. Again, the main findings of the paper are unchanged.

In our sixth and final robustness exercise, the Taylor projection solution is approximated at the stochastic steady state instead of the deterministic steady state.[12] Results are reported in Table 26. The accuracy of the solution increases considerably. For example, for a third-order Taylor projection, we get a mean Euler equation error of -5.4 in version 8 of the model (the most complicated version). This exercise shows that a key advantage of Taylor projections is the ability to approximate outside the deterministic steady state (as we are forced to do with a standard projection). We keep, however, in the benchmark exercise the Taylor projection undertaken at the deterministic steady state as a conservative scenario (computing the stochastic steady state can be, itself, an involved problem).

7 Conclusions

Models with rare disasters have become a popular line of research in macroeconomics and finance. However, rare disasters, by inducing significant nonlinearities, present computational challenges that have been largely ignored in the literature or dealt with only in a nonsystematic fashion. To fill this gap, in this paper, we formulated and solved a New Keynesian model with time-varying disaster risk (including several simpler versions of it). Our findings were as follows. First, low-order perturbation solutions (first, second, and third) do not offer enough accuracy as measured by the Euler errors, computed statistics, or IRFs. A fifth-order per-

[12] We compute the stochastic steady state using a pruned third-order perturbation.

turbation fixes part of the problem, but it is still not entirely satisfactory regarding accuracy and it imposes some serious computational costs. Second, a second-order Taylor projection seems an excellent choice, with a satisfactory balance of accuracy and run time. A third-order Taylor projection can handle a medium size model with even better accuracy, but at a higher cost. Finally, Smolyak collocation methods were accurate, but they were hard to implement (we failed to find a solution on several occasions) and suffered from long run times.

This paper should be read only as a preliminary progress report. There is much more to be learned about the properties of models with rare disasters than we can cover in one paper. However, we hope that our results will stimulate further investigation on the topic.

References

Andreasen, M. (2012): "On the Effects of Rare Disasters and Uncertainty Shocks for Risk Premia in Non-Linear DSGE Models," *Review of Economic Dynamics*, 15(3), 295–316.

Andreasen, M. M., J. Fernández-Villaverde, and J. F. Rubio-Ramírez (2013): "The Pruned State-Space System for Nonlinear DSGE Models: Theory and Empircal Applications," Working Paper 18983, National Bureau of Economic Research.

Aruoba, S. B., J. Fernández-Villaverde, and J. F. Rubio-Ramírez (2006): "Comparing Solution Methods for Dynamic Equilibrium Economies," *Journal of Economic Dynamics and Control*, 30(12), 2477–2508.

Barro, R. J. (2006): "Rare Disasters and Asset Markets in the Twentieth Century," *Quarterly Journal of Economics*, 121(3), 823–866.

——— (2009): "Rare Disasters, Asset Prices, and Welfare Costs," *American Economic Review*, 99(1), 243–64.

Barro, R. J., and T. Jin (2011): "On the Size Distribution of Macroeconomic Disasters," *Econometrica*, 79(5), 1567–1589.

Barro, R. J., and J. F. Ursúa (2012): "Rare Macroeconomic Disasters," *Annual Review of Economics*, 4(1), 83–109.

Bhandari, A., D. Evans, M. Golosov, and T. Sargent (2017): "Fiscal Policy and Debt Management with Incomplete Markets," *Quarterly Journal of Economics*, Forthcoming.

Christiano, L. J., M. Eichenbaum, and C. L. Evans (2005): "Nominal Rigidities and the Dynamic Effects of a Shock to Monetary Policy," *Journal of Political Economy*, 113(1), 1–45.

Coeurdacier, N., H. Rey, and P. Winant (2011): "The Risky Steady State," *American Economic Review*, 101(3), 398–401.

den Haan, W., M. Kobielarz, and P. Rendahl (2015): "Exact Present Solution with Consistent Future Approximation: A Gridless Algorithm to Solve Stochastic Dynamic Models," Discussion Papers 1536, Centre for Macroeconomics (CFM).

Farhi, E., and X. Gabaix (2016): "Rare disasters and exchange rates," *Quarterly Journal of Economics*, 131(1), 1–52.

Fernández-Villaverde, J., G. Gordon, P. A. Guerrón-Quintana, and J. F. Rubio-Ramírez (2015): "Nonlinear Adventures at the Zero Lower Bound," *Journal of Economic Dynamics and Control*, 57, 182–204.

Fernández-Villaverde, J., P. A. Guerrón-Quintana, and J. F. Rubio-Ramírez (2015): "Estimating Dynamic Equilibrium Models with Stochastic Volatility," *Journal of Econometrics*, 185, 216–229.

Fernández-Villaverde, J., and J. F. Rubio-Ramírez (2006): "A Baseline DSGE Model," Discussion paper, University of Pennsylvania.

Fernández-Villaverde, J., J. F. Rubio-Ramírez, and F. Schorfheide (2016): "Solution and Estimation Methods for {DSGE} Models," in *Handbook of Macroeconomics*, ed. by J. B. Taylor, and H. Uhlig, vol. 2, pp. 527–724. Elsevier.

Gabaix, X. (2011): "Disasterization: A Simple Way to Fix the Asset Pricing Properties of Macroeconomic Models," *American Economic Review*, 101(3), 406–09.

——— (2012): "Variable Rare Disasters: An Exactly Solved Framework for Ten Puzzles in Macro-Finance," *Quarterly Journal of Economics*, 127(2), 645–700.

Gaspar, J., and K. L. Judd (1997): "Solving Large-Scale Rational-Expectations Models," *Macroeconomic Dynamics*, 1, 45–75.

Gourio, F. (2012): "Disaster Risk and Business Cycles," *American Economic Review*, 102(6), 2734–66.

Gourio, F. (2013): "Credit Risk and Disaster Risk," *American Economic Journal: Macroeconomics*, 5(3), 1–34.

Gourio, F., M. Siemer, and A. Verdelhan (2013): "International risk cycles," *Journal of International Economics*, 89(2), 471–484.

Isoré, M., and U. Szczerbowicz (2015): "Disaster Risk and Preference Shifts in a New Keynesian Model," Working Papers 2015-16, CEPII research center.

Jin, H.-H., and K. L. Judd (2002): "Perturbation Methods for General Dynamic Stochastic Models," *Manuscript, Hoover Institution.*

Judd, K. L. (1992): "Projection Methods for Solving Aggregate Growth Models," *Journal of Economic Theory*, 58, 410–452.

——— (1998): *Numerical Methods in Economics*. MIT Press, Cambridge.

Judd, K. L., L. Maliar, and S. Maliar (2011): "Numerically Stable and Accurate Stochastic Simulation Methods for Solving Dynamic Models," *Quantitative Economics*, 2, 173–210.

Judd, K. L., L. Maliar, S. Maliar, and R. Valero (2014): "Smolyak Method for Solving Dynamic Economic Models: Lagrange Interpolation, Anisotropic Grid and Adaptive Domain," *Journal of Economic Dynamics and Control*, 44, 92–123.

Kameník, O. (2005): "Solving SDGE Models: A New Algorithm for the Sylvester Equation," *Computational Economics*, 1412.8659v1, 167–187.

Krüger, D., and F. Kubler (2004): "Computing Equilibrium in OLG Models with Stochastic Production," *Journal of Economic Dynamics and Control*, 28, 1411–1436.

Krusell, P., B. Kuruscu, and A. Smith (2002): "Equilibrium Welfare and Government Policy with Quasi-geometric Discounting," *Journal of Economic Theory*, 105(1), 42–72.

Levintal, O. (2015): "Fifth Order Perturbation Solution to DSGE Models," *Manuscript, Interdisciplinary Center Herzliya.*

——— (2016): "Taylor Projection: A New Solution Method for Dynamic General Equilibrium Models," *Manuscript, Interdisciplinary Center Herzliya.*

Maliar, L., and S. Maliar (2014): "Numerical Methods for Large Scale Dynamic Economic Models," in *Handbook of Computational Economics*, ed. by K. Schmedders, and K. L. Judd, vol. 3, pp. 325–477. Elsevier.

——— (2015): "Merging Simulation and Projection Approaches to Solve High-Dimensional Problems with an Application to a New Keynesian Model," *Quantitative Economics*, 6, 1–47.

MALIN, B. A., D. KRÜGER, AND F. KUBLER (2011): "Solving the Multi-country Real Business Cycle Model Using a Smolyak-collocation Method," *Journal of Economic Dynamics and Control*, 35, 229–239.

MEHRA, R., AND E. C. PRESCOTT (1985): "The Equity Premium: A Puzzle," *Journal of Monetary Economics*, 15(2), 145–161.

NAKAMURA, E., J. STEINSSON, R. BARRO, AND J. URSÚA (2013): "Crises and Recoveries in an Empirical Model of Consumption Disasters," *American Economic Journal: Macroeconomics*, 5(3), 35–74.

PETROSKY-NADEAU, N., L. ZHANG, AND L. KUEHN (2015): "Endogenous Disasters," Discussion paper, Carnegie Mellon University.

RIETZ, T. A. (1988): "The Equity Risk Premium: a Solution," *Journal of Monetary Economics*, 22(1), 117–131.

RUDEBUSCH, G., AND E. SWANSON (2012): "The Bond Premium in a DSGE Model with Long-run Real and Nominal Risks," *American Economic Journal: Macroeconomics*, 4, 105–143.

SWANSON, E. T. (2012): "Risk Aversion and the Labor Margin in Dynamic Equilibrium Models," *American Economic Review*, 102(4), 1663–91.

TSAI, J., AND J. A. WACHTER (2015): "Disaster Risk and its Implications for Asset Pricing," Working Paper 20926, National Bureau of Economic Research.

WACHTER, J. A. (2013): "Can Time-Varying Risk of Rare Disasters Explain Aggregate Stock Market Volatility?," *Journal of Finance*, 68(3), 987–1035.

Table 1: Baseline Calibration

Parameter	Value	Source
Leisure preference (ν)	2.33	Gourio (2012)
Risk aversion (γ)	3.8	Gourio (2012)
Inverse IES ($\widehat{\psi}$)	.5	Gourio (2012)
Trend growth of technology (Λ_A)	.0028	FQR (2015)
Std. of technology shocks (σ_A)	.01	Gourio (2012)
Trend growth of investment shock (Λ_μ)	0	
Std. of investment shock (σ_μ)	.0024	FQR (2015)
Discount factor (β)	.99	FQR (2015)
Cobb-Douglas parameter (α)	.21	FQR (2015)
Depreciation (δ)	.025	FQR (2015)
Fixed production costs (ϕ)	0	FQR (2015)
Disaster probability (p_d)	.0043	Gourio (2012)
Mean disaster size ($\bar{\theta}$)	.5108	
Persistence of disaster risk shock (ρ_θ)	.9	
Std. of disaster risk shock (σ_θ)	.025	
Adjustment cost parameter (κ)	9.5	FQR (2015)
Calvo parameter (θ_p)	.8139	FQR (2015)
Automatic price adjustment (χ)	.6186	FQR (2015)
Elasticity of substitution (ϵ)	10	FQR (2015)
Inflation target (Π)	1.005	
Inflation parameter in Taylor rule* (γ_Π)	1.3	
Output growth parameter in Taylor rule (γ_y)	.2458	FQR (2015)
Interest smoothness in Taylor rule* (γ_R)	.5	
Std. of monetary shock ($\sigma_{m,t}$)	.0025	FQR (2015)
Persistence of intertemporal shock (ρ_ξ)	.1182	FQR (2015)
Std. of intertemporal shock (σ_ξ)	.1376	FQR (2015)

Table 2: No disasters: Mean Euler errors (log10) across the ergodic set

Model	State vars.	Perturbation					Taylor projection			Smolyak collocation		
		1st	2nd	3rd	4th	5th	1st	2nd	3rd	1st	2nd	3rd
1. Benchmark with EZ	4	-2.8	-3.3	-5.2	-6.3	-7.1	-3.2	-6.1	-8.6	-3.3	-7.4	-10.2
2. + capital adjustment costs	5	-2.7	-3.9	-5.0	-6.6	-7.1	-2.9	-4.7	-6.5	-2.7	-4.9	-7.0
3. + Calvo	7	-2.7	-3.9	-4.4	-5.9	-6.5	-3.1	-4.9	-6.6	-3.1	-5.2	-6.9
4. + Taylor rule depends on output growth	8	-2.7	-4.0	-4.7	-6.1	-6.9	-3.1	-4.9	-6.7	-3.1	-5.2	-6.4
5. + Taylor rule is smoothed	9	-2.7	-4.0	-4.6	-6.0	-6.7	-3.1	-4.9	-6.7	-3.1	-5.2	-6.4
6. + investment shock	10	-2.7	-4.0	-4.6	-6.1	-6.7	-3.1	-4.9	-6.7	-3.1	-5.2	-6.5
7. + monetary shock	11	-2.7	-4.0	-4.6	-5.9	-6.7	-3.0	-4.8	-6.5	-3.1	-5.1	-6.4
8. + intertemporal preference shock	12	-2.7	-3.9	-4.6	-5.7	-6.6	-2.9	-4.6	-6.3	-3.1	-5.1	-6.4

Table 3: No disasters: Max Euler errors (log10) across the ergodic set

Model	State vars.	Perturbation					Taylor projection			Smolyak collocation		
		1st	2nd	3rd	4th	5th	1st	2nd	3rd	1st	2nd	3rd
1. Benchmark with EZ	4	-1.3	-2.3	-4.2	-5.4	-7.1	-1.3	-3.4	-5.5	-1.8	-6.2	-9.2
2. + capital adjustment costs	5	-1.6	-2.6	-3.8	-5.2	-6.4	-1.6	-2.6	-3.9	-1.2	-2.9	-5.4
3. + Calvo	7	-1.4	-2.5	-3.7	-5.0	-5.7	-1.4	-2.5	-3.7	-1.7	-4.0	-5.0
4. + Taylor rule depends on output growth	8	-1.4	-2.5	-3.7	-5.1	-6.1	-1.4	-2.5	-3.7	-1.7	-3.9	-4.7
5. + Taylor rule is smoothed	9	-1.4	-2.5	-3.8	-5.1	-6.3	-1.4	-2.5	-3.7	-1.7	-3.7	-4.5
6. + investment shock	10	-1.5	-2.6	-3.9	-5.3	-6.3	-1.5	-2.7	-3.9	-1.7	-3.9	-4.6
7. + monetary shock	11	-1.5	-2.6	-3.8	-5.2	-6.1	-1.5	-2.6	-3.8	-1.7	-3.7	-4.6
8. + intertemporal preference shock	12	-1.4	-2.5	-3.7	-5.0	-5.9	-1.4	-2.5	-3.7	-1.7	-3.7	-4.6

Table 4: Disaster models - Mean Euler errors (log10) across the ergodic set

Model	State vars.	Perturbation					Taylor projection			Smolyak collocation		
		1st	2nd	3rd	4th	5th	1st	2nd	3rd	1st	2nd	3rd
1. Benchmark with EZ and disasters	4	-1.7	-2.1	-2.5	-3.0	-3.5	-3.1	-5.2	-6.9	-3.2	-6.0	-8.5
2. + capital adjustment costs	5	-1.6	-2.0	-2.4	-2.9	-3.3	-2.4	-3.9	-5.3	-	-1.6	-3.5
3. + Calvo	7	-1.7	-2.0	-1.8	-1.7	-1.9	-2.4	-3.8	-4.8	-1.0	-2.6	-3.6
4. + Taylor rule depends on output growth	8	-1.8	-2.1	-2.1	-2.0	-2.2	-2.5	-3.9	-5.1	-1.0	-2.6	-3.5
5. + Taylor rule is smoothed	9	-1.8	-2.1	-2.1	-2.1	-2.2	-2.2	-3.6	-4.5	-	-2.6	-3.6
6. + investment shock	10	-1.8	-2.1	-2.1	-2.1	-2.2	-2.2	-3.6	-4.5	-	-2.6	-3.5
7. + monetary shock	11	-1.8	-2.1	-2.1	-2.1	-2.2	-2.2	-3.6	-4.5	-	-2.6	-3.7
8. + intertemporal preference shock	12	-1.8	-2.2	-2.1	-2.1	-2.2	-2.2	-3.6	-4.4	-	-2.5	-3.6

Table 5: Disaster models - Max Euler errors (log10) across the ergodic set

Model	State vars.	Perturbation					Taylor projection			Smolyak collocation		
		1st	2nd	3rd	4th	5th	1st	2nd	3rd	1st	2nd	3rd
1. Benchmark with EZ and disasters	4	-1.5	-1.6	-1.8	-2.1	-2.4	-1.5	-3.0	-3.7	-1.8	-4.7	-6.8
2. + capital adjustment costs	5	-0.1	-0.8	-1.5	-2.0	-2.3	-0.1	-0.8	-1.4	-	-0.4	-1.9
3. + Calvo	7	-0.2	-1.2	-1.7	-1.6	-1.6	-0.3	-1.2	-2.1	-0.1	-1.5	-2.7
4. + Taylor rule depends on output growth	8	-0.1	-1.1	-1.8	-1.8	-1.9	-0.2	-1.1	-1.7	-0.1	-1.4	-2.4
5. + Taylor rule is smoothed	9	-0.1	-1.0	-1.6	-1.7	-1.9	-0.2	-1.1	-1.6	-	-1.4	-2.3
6. + investment shock	10	-0.1	-1.1	-1.7	-1.8	-1.9	-0.2	-1.1	-1.8	-	-1.4	-2.4
7. + monetary shock	11	-0.2	-1.3	-1.7	-1.8	-1.9	-0.3	-1.3	-1.8	-	-1.5	-2.5
8. + intertemporal preference shock	12	-0.3	-1.4	-1.7	-1.8	-1.8	-0.4	-1.4	-2.0	-	-1.4	-2.4

Table 6: Disaster models - Risk-free rate (% annualized) - Simulation average

Model	State vars.	Perturbation					Taylor projection			Smolyak collocation		
		1st	2nd	3rd	4th	5th	1st	2nd	3rd	1st	2nd	3rd
1. Benchmark with EZ and disasters	4	4.6	3.0	1.8	1.1	0.8	0.7	0.7	0.7	0.7	0.7	0.7
2. + capital adjustment costs	5	4.6	2.9	1.6	0.9	0.5	0.5	0.4	0.4	-	0.3	0.4
3. + Calvo	7	4.6	3.0	2.0	1.5	1.1	0.6	0.5	0.5	0.4	0.5	0.5
4. + Taylor rule depends on output growth	8	4.6	3.4	2.6	2.2	1.9	1.5	1.6	1.6	2.0	1.5	1.6
5. + Taylor rule is smoothed	9	4.6	3.3	2.7	2.3	2.0	1.4	1.5	1.5	-	1.5	1.5
6. + investment shock	10	4.6	3.3	2.7	2.3	2.0	1.4	1.5	1.5	-	1.5	1.5
7. + monetary shock	11	4.6	3.3	2.7	2.3	2.1	1.4	1.5	1.6	-	1.5	1.6
8. + intertemporal preference shock	12	4.6	3.3	2.7	2.3	2.1	1.4	1.5	1.6	-	1.5	1.5

Table 7: Disaster models - Return on equity (% annualized) - Simulation average

Model	State vars.	Perturbation					Taylor projection			Smolyak collocation		
		1st	2nd	3rd	4th	5th	1st	2nd	3rd	1st	2nd	3rd
1. Benchmark with EZ and disasters	4	5.1	5.1	5.1	5.1	5.1	5.1	5.1	5.1	5.1	5.1	5.1
2. + capital adjustment costs	5	5.2	5.3	5.5	5.5	5.5	5.5	5.6	5.6	-	5.8	5.6
3. + Calvo	7	5.1	5.3	5.5	5.6	5.6	5.4	5.4	5.4	4.8	5.4	5.4
4. + Taylor rule depends on output growth	8	5.0	5.2	5.4	5.4	5.5	5.3	5.3	5.3	4.4	5.3	5.3
5. + Taylor rule is smoothed	9	5.0	5.1	5.5	5.6	5.7	5.2	5.3	5.3	-	5.3	5.3
6. + investment shock	10	5.0	5.1	5.5	5.7	5.7	5.3	5.3	5.4	-	5.3	5.3
7. + monetary shock	11	5.0	5.2	5.5	5.7	5.7	5.3	5.3	5.4	-	5.3	5.4
8. + intertemporal preference shock	12	5.0	5.2	5.5	5.7	5.7	5.3	5.3	5.4	-	5.3	5.4

Table 8: Detrended investment - Simulation average

Model	State vars.	Perturbation					Taylor projection			Smolyak collocation		
		1st	2nd	3rd	4th	5th	1st	2nd	3rd	1st	2nd	3rd
1. Smallest	4	0.0699	0.0699	0.0696	0.0695	0.0695	0.0696	0.0695	0.0695	0.0695	0.0695	0.0695
2.	5	0.0700	0.0691	0.0682	0.0678	0.0676	0.0661	0.0673	0.0675	-	0.0697	0.0675
3.	7	0.0563	0.0553	0.0545	0.0539	0.0535	0.0533	0.0534	0.0534	0.0507	0.0534	0.0534
4.	8	0.0563	0.0553	0.0545	0.0540	0.0537	0.0537	0.0536	0.0536	0.0499	0.0536	0.0536
5.	9	0.0563	0.0553	0.0543	0.0538	0.0536	0.0528	0.0536	0.0536	-	0.0536	0.0536
6.	10	0.0563	0.0553	0.0544	0.0538	0.0536	0.0528	0.0536	0.0537	-	0.0537	0.0537
7.	11	0.0563	0.0553	0.0543	0.0538	0.0536	0.0528	0.0536	0.0537	-	0.0536	0.0537
8. Largest	12	0.0563	0.0553	0.0543	0.0538	0.0536	0.0529	0.0536	0.0537	-	0.0536	0.0536

Table 9: Detrended capital - Simulation average

Model	State vars.	Perturbation					Taylor projection			Smolyak collocation		
		1st	2nd	3rd	4th	5th	1st	2nd	3rd	1st	2nd	3rd
1. Smallest	4	2.4575	2.4609	2.4509	2.4473	2.4461	2.4458	2.4457	2.4457	2.4458	2.4457	2.4457
2.	5	2.4619	2.4297	2.3992	2.3848	2.3790	2.3285	2.3691	2.3746	-	2.4614	2.3756
3.	7	1.9705	1.9445	1.9166	1.8975	1.8838	1.8695	1.8774	1.8783	1.7718	1.8768	1.8785
4.	8	1.9707	1.9437	1.9162	1.9005	1.8917	1.8828	1.8861	1.8868	1.7495	1.8880	1.8857
5.	9	1.9707	1.9440	1.9110	1.8930	1.8856	1.8535	1.8851	1.8869	-	1.8878	1.8859
6.	10	1.9694	1.9432	1.9101	1.8921	1.8847	1.8525	1.8841	1.8860	-	1.8864	1.8851
7.	11	1.9693	1.9429	1.9099	1.8921	1.8848	1.8528	1.8843	1.8862	-	1.8836	1.8855
8. Largest	12	1.9695	1.9412	1.9085	1.8908	1.8835	1.8531	1.8829	1.8849	-	1.8854	1.8838

Table 10: Detrended investment - Simulation standard deviation

Model	State vars.	Perturbation					Taylor projection			Smolyak collocation		
		1st	2nd	3rd	4th	5th	1st	2nd	3rd	1st	2nd	3rd
1. Smallest	4	0.0056	0.0056	0.0056	0.0056	0.0056	0.0056	0.0056	0.0056	0.0056	0.0056	0.0056
2.	5	0.0054	0.0059	0.0060	0.0059	0.0059	0.0071	0.0058	0.0059	-	0.0062	0.0059
3.	7	0.0052	0.0051	0.0050	0.0049	0.0048	0.0049	0.0048	0.0049	0.0120	0.0049	0.0049
4.	8	0.0054	0.0053	0.0052	0.0051	0.0050	0.0050	0.0050	0.0050	0.0140	0.0051	0.0051
5.	9	0.0054	0.0055	0.0053	0.0051	0.0051	0.0049	0.0051	0.0051	-	0.0051	0.0051
6.	10	0.0054	0.0055	0.0053	0.0051	0.0051	0.0048	0.0050	0.0051	-	0.0051	0.0051
7.	11	0.0053	0.0054	0.0052	0.0050	0.0050	0.0047	0.0050	0.0050	-	0.0051	0.0050
8. Largest	12	0.0054	0.0053	0.0051	0.0050	0.0050	0.0048	0.0050	0.0050	-	0.0050	0.0050

Table 11: Detrended capital - Simulation standard deviation

Model	State vars.	Perturbation					Taylor projection			Smolyak collocation		
		1st	2nd	3rd	4th	5th	1st	2nd	3rd	1st	2nd	3rd
1. Smallest	4	0.0652	0.0653	0.0649	0.0647	0.0647	0.0648	0.0647	0.0647	0.0648	0.0647	0.0647
2.	5	0.1165	0.1168	0.1151	0.1140	0.1135	0.1104	0.1137	0.1132	-	0.1580	0.1125
3.	7	0.0945	0.0955	0.0932	0.0912	0.0903	0.0865	0.0901	0.0900	0.2003	0.0918	0.0899
4.	8	0.0958	0.0973	0.0951	0.0933	0.0924	0.0886	0.0920	0.0920	0.2325	0.0935	0.0920
5.	9	0.0959	0.0981	0.0955	0.0935	0.0925	0.0850	0.0921	0.0922	-	0.0936	0.0923
6.	10	0.0953	0.0974	0.0950	0.0932	0.0923	0.0857	0.0920	0.0921	-	0.0938	0.0921
7.	11	0.0980	0.1003	0.0978	0.0961	0.0953	0.0885	0.0951	0.0951	-	0.0969	0.0951
8. Largest	12	0.0972	0.0987	0.0961	0.0942	0.0934	0.0868	0.0931	0.0932	-	0.0950	0.0931

Table 12: Run time (seconds)

Model	State vars.	Perturbation					Taylor projection			Smolyak collocation		
		1st	2nd	3rd	4th	5th	1st	2nd	3rd	1st	2nd	3rd
1. Benchmark with EZ and disasters	4	0.0	0.0	0.0	0.0	0.2	0.4	0.5	1.1	0.2	0.4	1.2
2. + capital adjustment costs	5	0.0	0.0	0.0	0.1	0.7	0.4	0.7	1.4	-	0.8	5.1
3. + Calvo	7	0.0	0.0	0.0	0.2	2.7	0.4	1.0	6.6	0.3	2.7	82.9
4. + Taylor rule depends on output growth	8	0.0	0.0	0.0	0.3	4.8	0.4	1.1	12.0	0.3	4.7	302.5
5. + Taylor rule is smoothed	9	0.0	0.0	0.0	0.5	8.1	0.4	1.4	25.9	-	10.6	679.9
6. + investment shock	10	0.0	0.0	0.1	0.7	13.7	0.4	1.8	48.4	-	19.9	1634.0
7. + monetary shock	11	0.0	0.0	0.1	1.0	20.5	0.4	2.3	89.0	-	27.4	4535.9
8. + intertemporal preference shock	12	0.0	0.0	0.1	1.3	30.4	0.4	3.4	148.1	-	55.6	7741.6

We work on a Dell computer with an Intel(R) Core(TM) i7-5600U Processor and 16GB RAM.

Table 13: Jacobian of the full model (version 8)

Order/level	Taylor projection	Smolyak collocation
	Dimension of Jacobian	
1	195×195	375×375
2	$1,365 \times 1,365$	$4,695 \times 4,695$
3	$6,825 \times 6,825$	$39,735 \times 39,735$
	Nonzero elements	
1	7,342	49,728
2	269,290	5,878,132
3	5,601,050	374,482,434
	Share of nonzeros	
1	0.19	0.35
2	0.14	0.27
3	0.12	0.24

Table 14: Robustness 1: Mean Euler errors - Benchmark parameterization

Model	State vars.	Perturbation					Taylor projection			Smolyak collocation		
		1st	2nd	3rd	4th	5th	1st	2nd	3rd	1st	2nd	3rd
1. Benchmark with EZ and disasters	4	-1.7	-2.1	-2.5	-3.0	-3.5	-3.1	-5.2	-6.9	-3.2	-6.0	-8.5
2. + capital adjustment costs	5	-1.6	-2.0	-2.4	-2.9	-3.3	-2.4	-3.9	-5.3	-	-1.6	-3.5
3. + Calvo	7	-1.7	-2.0	-1.8	-1.7	-1.9	-2.4	-3.8	-4.8	-1.0	-2.6	-3.6
4. + Taylor rule depends on output growth	8	-1.8	-2.1	-2.1	-2.0	-2.2	-2.5	-3.9	-5.1	-1.0	-2.6	-3.5
5. + Taylor rule is smoothed	9	-1.8	-2.1	-2.1	-2.1	-2.2	-2.2	-3.6	-4.5	-	-2.6	-3.6
6. + investment shock	10	-1.8	-2.1	-2.1	-2.1	-2.2	-2.2	-3.6	-4.5	-	-2.6	-3.5
7. + monetary shock	11	-1.8	-2.1	-2.1	-2.1	-2.2	-2.2	-3.6	-4.5	-	-2.6	-3.7
8. + intertemporal preference shock	12	-1.8	-2.2	-2.1	-2.1	-2.2	-2.2	-3.6	-4.4	-	-2.5	-3.6

The ergodic set is approximated by simulating the Smolyak solution (level 3) for T=10,000 periods. The table reports mean errors across the ergodic set.

Table 15: Robustness 1: Max Euler errors - Benchmark parameterization

Model	State vars.	Perturbation					Taylor projection			Smolyak collocation		
		1st	2nd	3rd	4th	5th	1st	2nd	3rd	1st	2nd	3rd
1. Benchmark with EZ and disasters	4	-1.5	-1.6	-1.8	-2.1	-2.4	-1.5	-3.0	-3.7	-1.8	-4.7	-6.8
2. + capital adjustment costs	5	-0.1	-0.8	-1.5	-2.0	-2.3	-0.1	-0.8	-1.4	-	-0.4	-1.9
3. + Calvo	7	-0.2	-1.2	-1.7	-1.6	-1.6	-0.3	-1.2	-2.1	-0.1	-1.5	-2.7
4. + Taylor rule depends on output growth	8	-0.1	-1.1	-1.8	-1.8	-1.9	-0.2	-1.1	-1.7	-0.1	-1.4	-2.4
5. + Taylor rule is smoothed	9	-0.1	-1.0	-1.6	-1.7	-1.9	-0.2	-1.1	-1.6	-	-1.4	-2.4
6. + investment shock	10	-0.1	-1.1	-1.7	-1.8	-1.9	-0.2	-1.1	-1.8	-	-1.4	-2.4
7. + monetary shock	11	-0.2	-1.3	-1.7	-1.8	-1.9	-0.3	-1.3	-1.8	-	-1.5	-2.5
8. + intertemporal preference shock	12	-0.3	-1.4	-1.7	-1.8	-1.8	-0.4	-1.4	-2.0	-	-1.4	-2.4

The ergodic set is approximated by simulating the Smolyak solution (level 3) for T=10,000 periods. The table reports max errors across the ergodic set.

Table 16: Robustness 2: Mean Euler errors - Benchmark parameterization

Model	State vars.	Perturbation					Taylor projection			Smolyak collocation		
		1st	2nd	3rd	4th	5th	1st	2nd	3rd	1st	2nd	3rd
1. Benchmark with EZ and disasters	4	-1.7	-2.1	-2.5	-3.0	-3.5	-3.0	-5.2	-6.9	-3.2	-6.0	-8.5
2. + capital adjustment costs	5	-1.6	-2.0	-2.4	-2.9	-3.3	-2.4	-3.9	-5.3	-	-1.6	-3.5
3. + Calvo	7	-1.7	-2.0	-1.8	-1.7	-1.9	-2.4	-3.7	-4.8	-1.0	-2.6	-3.6
4. + Taylor rule depends on output growth	8	-1.8	-2.1	-2.1	-2.0	-2.2	-2.5	-3.9	-5.0	-1.0	-2.6	-3.5
5. + Taylor rule is smoothed	9	-1.8	-2.1	-2.1	-2.1	-2.2	-2.2	-3.6	-4.5	-	-2.6	-3.6
6. + investment shock	10	-1.8	-2.1	-2.1	-2.1	-2.2	-2.2	-3.6	-4.5	-	-2.6	-3.5
7. + monetary shock	11	-1.8	-2.1	-2.1	-2.1	-2.2	-2.2	-3.6	-4.5	-	-2.6	-3.7
8. + intertemporal preference shock	12	-1.8	-2.1	-2.1	-2.1	-2.2	-2.2	-3.6	-4.3	-	-2.5	-3.6

The ergodic set is approximated by simulating the Smolyak solution (level 3) for T=100,000 periods. The table reports mean errors across the ergodic set.

Table 17: Robustness 2: Max Euler errors - Benchmark parameterization

Model	State vars.	Perturbation					Taylor projection			Smolyak collocation		
		1st	2nd	3rd	4th	5th	1st	2nd	3rd	1st	2nd	3rd
1. Benchmark with EZ and disasters	4	-1.3	-1.6	-1.8	-2.0	-2.3	-1.3	-2.9	-3.5	-1.7	-4.0	-6.0
2. + capital adjustment costs	5	-0.2	-0.8	-1.4	-1.9	-2.2	-0.2	-0.7	-1.2	-	-0.3	-1.8
3. + Calvo	7	-0.1	-1.2	-1.6	-1.5	-1.6	-0.1	-1.2	-2.0	0.0	-1.4	-2.5
4. + Taylor rule depends on output growth	8	0.0	-1.1	-1.8	-1.8	-1.9	0.0	-1.1	-1.9	0.0	-1.4	-2.3
5. + Taylor rule is smoothed	9	0.0	-1.0	-1.7	-1.8	-1.9	-0.1	-1.1	-1.8	-	-1.4	-2.2
6. + investment shock	10	0.0	-1.0	-1.7	-1.8	-1.9	-0.1	-1.1	-1.8	-	-1.2	-2.2
7. + monetary shock	11	0.0	-1.1	-1.7	-1.7	-1.9	-0.1	-1.1	-1.7	-	-1.5	-2.3
8. + intertemporal preference shock	12	0.0	-1.0	-1.6	-1.7	-1.8	-0.1	-1.1	-1.6	-	-1.3	-2.3

The ergodic set is approximated by simulating the Smolyak solution (level 3) for T=100,000 periods. The table reports max errors across the ergodic set.

Table 18: Robustness 3: Mean Euler errors - Risk aversion parameter $\gamma = 5$

Model	State vars.	Perturbation					Taylor projection			Smolyak collocation		
		1st	2nd	3rd	4th	5th	1st	2nd	3rd	1st	2nd	3rd
1. Benchmark with EZ and disasters	4	-1.4	-1.6	-1.9	-2.3	-2.8	-2.9	-4.8	-6.4	-3.1	-5.5	-7.9
2. + capital adjustment costs	5	-1.3	-1.6	-1.9	-2.3	-2.6	-2.1	-3.4	-4.5	-	-1.7	-3.3
3. + Calvo	7	-1.4	-1.6	-1.4	-1.2	-1.2	-1.9	-3.1	-3.7	-1.1	-2.3	-3.1
4. + Taylor rule depends on output growth	8	-1.4	-1.7	-1.6	-1.5	-1.5	-2.2	-3.3	-4.1	-1.0	-2.4	-3.1
5. + Taylor rule is smoothed	9	-1.4	-1.7	-1.6	-1.5	-1.5	-1.9	-3.0	-3.4	-1.0	-2.3	-2.9
6. + investment shock	10	-1.4	-1.7	-1.6	-1.5	-1.5	-1.9	-3.0	-3.4	-1.0	-2.3	-2.9
7. + monetary shock	11	-1.4	-1.7	-1.6	-1.5	-1.5	-1.9	-3.0	-3.4	-1.0	-2.5	-3.1
8. + intertemporal preference shock	12	-1.4	-1.7	-1.6	-1.5	-1.5	-1.9	-2.9	-3.4	-0.7	-2.3	-2.9

Table 19: Robustness 3: Max Euler errors - $\gamma = 5$

Model	State vars.	Perturbation					Taylor projection			Smolyak collocation		
		1st	2nd	3rd	4th	5th	1st	2nd	3rd	1st	2nd	3rd
1. Benchmark with EZ and disasters	4	-1.1	-1.2	-1.4	-1.5	-1.8	-1.5	-2.5	-3.2	-1.8	-4.1	-6.2
2. + capital adjustment costs	5	-0.2	-1.0	-1.3	-1.5	-1.7	0.0	-0.8	-1.4	-	-0.5	-1.9
3. + Calvo	7	-0.2	-1.1	-1.2	-1.0	-1.0	-0.3	-1.2	-2.1	-0.1	-1.7	-2.3
4. + Taylor rule depends on output growth	8	-0.1	-1.1	-1.4	-1.3	-1.3	-0.2	-1.1	-1.8	-0.1	-1.6	-2.3
5. + Taylor rule is smoothed	9	-0.1	-1.0	-1.3	-1.2	-1.4	-0.4	-1.1	-1.6	-0.1	-1.6	-2.1
6. + investment shock	10	-0.1	-1.0	-1.3	-1.3	-1.4	-0.4	-1.2	-1.6	0.0	-1.6	-2.1
7. + monetary shock	11	-0.1	-1.2	-1.3	-1.3	-1.3	-0.4	-1.4	-1.7	0.1	-1.7	-2.2
8. + intertemporal preference shock	12	-0.2	-1.3	-1.3	-1.3	-1.3	-0.5	-1.4	-1.7	-0.1	-1.6	-2.1

Table 20: Robustness 3: Risk-free rate (% annualized), $\gamma = 5$

Model	State vars.	Perturbation					Taylor projection			Smolyak collocation		
		1st	2nd	3rd	4th	5th	1st	2nd	3rd	1st	2nd	3rd
1. Benchmark with EZ and disasters	4	4.6	2.5	0.3	-1.4	-2.4	-3.0	-3.1	-3.1	-3.0	-3.1	-3.1
2. + capital adjustment costs	5	4.6	2.3	0.0	-1.8	-2.9	-3.4	-3.6	-3.6	-	-3.6	-3.6
3. + Calvo	7	4.6	2.4	0.7	-0.3	-1.0	-3.3	-3.4	-3.4	-3.1	-3.4	-3.4
4. + Taylor rule depends on output growth	8	4.6	2.9	1.5	0.5	-0.1	-1.7	-1.6	-1.5	-1.0	-1.6	-1.5
5. + Taylor rule is smoothed	9	4.6	2.8	1.7	0.9	0.4	-2.6	-1.8	-1.6	-1.6	-1.6	-1.6
6. + investment shock	10	4.6	2.8	1.7	0.9	0.4	-2.6	-1.8	-1.6	-1.8	-1.6	-1.6
7. + monetary shock	11	4.6	2.8	1.7	1.0	0.4	-2.5	-1.8	-1.6	-1.5	-1.6	-1.6
8. + intertemporal preference shock	12	4.6	2.8	1.7	1.0	0.4	-2.5	-1.8	-1.6	-2.0	-1.6	-1.6

Table 21: Robustness 3: Return on equity (% annualized), $\gamma = 5$

Model	State vars.	Perturbation					Taylor projection			Smolyak collocation		
		1st	2nd	3rd	4th	5th	1st	2nd	3rd	1st	2nd	3rd
1. Benchmark with EZ and disasters	4	5.1	5.1	5.2	5.2	5.2	5.2	5.2	5.2	5.2	5.2	5.2
2. + capital adjustment costs	5	5.2	5.4	5.7	5.8	5.9	5.1	6.0	6.0	-	6.2	6.0
3. + Calvo	7	5.1	5.3	5.7	6.1	6.4	5.6	5.6	5.6	5.5	5.6	5.6
4. + Taylor rule depends on output growth	8	5.0	5.2	5.6	5.9	6.0	5.5	5.6	5.6	5.2	5.5	5.6
5. + Taylor rule is smoothed	9	5.0	5.2	5.9	6.4	6.7	5.5	5.5	5.6	5.0	5.6	5.7
6. + investment shock	10	5.0	5.2	5.9	6.5	6.7	5.5	5.5	5.6	4.8	5.6	5.7
7. + monetary shock	11	5.0	5.2	5.9	6.5	6.7	5.5	5.5	5.6	8.0	5.6	5.7
8. + intertemporal preference shock	12	5.0	5.2	5.9	6.5	6.7	5.7	5.5	5.6	2.9	5.7	5.7

Table 22: Robustness 4: Mean Euler errors - Disaster probability=0.0086

Model	State vars.	Perturbation					Taylor projection			Smolyak collocation		
		1st	2nd	3rd	4th	5th	1st	2nd	3rd	1st	2nd	3rd
1. Benchmark with EZ and disasters	4	-1.5	-1.8	-2.2	-2.7	-3.3	-2.9	-5.0	-6.6	-3.2	-5.7	-8.2
2. + capital adjustment costs	5	-1.4	-1.7	-2.2	-2.6	-2.9	-2.0	-3.3	-4.4	-	-1.5	-3.0
3. + Calvo	7	-1.4	-1.7	-1.2	-1.1	-1.4	-1.9	-3.1	-3.7	-1.0	-2.0	-2.8
4. + Taylor rule depends on output growth	8	-1.5	-1.8	-1.5	-1.5	-1.7	-2.1	-3.2	-4.0	-1.0	-2.2	-3.0
5. + Taylor rule is smoothed	9	-1.5	-1.8	-1.5	-1.5	-1.7	-	-3.0	-3.5	-	-2.1	-2.8
6. + investment shock	10	-1.4	-1.8	-1.5	-1.5	-1.7	-	-3.0	-3.5	-	-2.2	-2.9
7. + monetary shock	11	-1.5	-1.8	-1.5	-1.5	-1.7	-	-3.0	-3.4	-	-2.3	-3.1
8. + intertemporal preference shock	12	-1.5	-1.8	-1.5	-1.5	-1.7	-1.9	-2.9	-3.4	-	-2.2	-2.8

Table 23: Robustness 4: Max Euler equation errors - Disaster probability=0.0086

Model	State vars.	Perturbation					Taylor projection			Smolyak collocation		
		1st	2nd	3rd	4th	5th	1st	2nd	3rd	1st	2nd	3rd
1. Benchmark with EZ and disasters	4	-1.2	-1.4	-1.6	-1.8	-2.1	-1.5	-2.8	-3.5	-1.8	-4.5	-6.5
2. + capital adjustment costs	5	-0.2	-0.9	-1.5	-1.8	-2.0	0.0	-0.8	-1.4	-	-0.3	-1.7
3. + Calvo	7	-0.1	-1.2	-1.1	-1.0	-1.1	-0.2	-1.3	-2.1	-0.1	-1.5	-2.0
4. + Taylor rule depends on output growth	8	-0.1	-1.2	-1.3	-1.3	-1.4	-0.2	-1.2	-1.8	-0.2	-1.5	-2.1
5. + Taylor rule is smoothed	9	0.0	-1.0	-1.2	-1.4	-1.5	-	-1.1	-1.6	-	-1.5	-1.9
6. + investment shock	10	-0.1	-0.9	-1.2	-1.4	-1.5	-	-1.2	-1.6	-	-1.5	-2.1
7. + monetary shock	11	-0.1	-1.1	-1.2	-1.4	-1.4	-	-1.3	-1.7	-	-1.6	-2.1
8. + intertemporal preference shock	12	-0.1	-1.1	-1.2	-1.3	-1.3	-0.5	-1.3	-1.7	-	-1.4	-1.9

Table 24: Robustness 5: Mean Euler errors - $\sigma_\theta = 0.05$

Model	State vars.	Perturbation					Taylor projection			Smolyak collocation		
		1st	2nd	3rd	4th	5th	1st	2nd	3rd	1st	2nd	3rd
1. Benchmark with EZ and disasters	4	-1.7	-2.1	-2.5	-3.0	-3.5	-2.9	-4.6	-5.9	-2.9	-4.6	-6.9
2. + capital adjustment costs	5	-1.6	-2.0	-2.4	-2.8	-3.2	-2.3	-3.7	-4.8	-	-	-3.0
3. + Calvo	7	-1.7	-2.0	-1.8	-1.7	-1.8	-2.3	-3.5	-4.5	-	-2.5	-2.6
4. + Taylor rule depends on output growth	8	-1.7	-2.1	-2.1	-2.0	-2.2	-2.4	-3.6	-4.7	-	-2.3	-
5. + Taylor rule is smoothed	9	-1.7	-2.1	-2.1	-2.1	-2.2	-2.1	-3.4	-4.3	-	-2.3	-
6. + investment shock	10	-1.7	-2.1	-2.1	-2.1	-2.2	-2.1	-3.4	-4.3	-	-2.3	-
7. + monetary shock	11	-1.7	-2.2	-2.1	-2.1	-2.2	-2.1	-3.4	-4.2	-	-2.3	-2.7
8. + intertemporal preference shock	12	-1.7	-2.2	-2.1	-2.1	-2.2	-2.1	-3.4	-4.1	-	-2.3	-

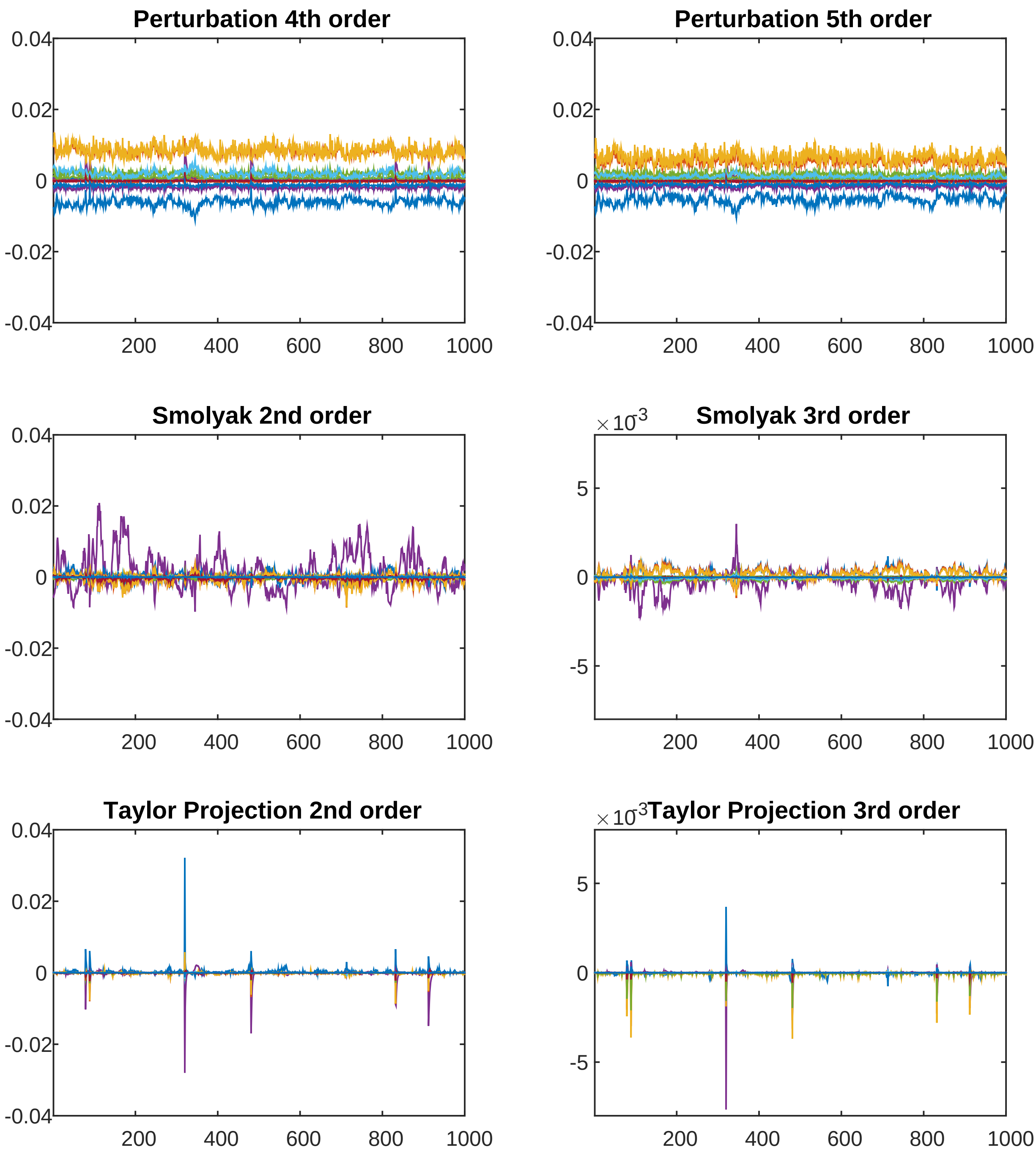
Table 25: Robustness 5: Max Euler errors - $\sigma_\theta = 0.05$

Model	State vars.	Perturbation					Taylor projection			Smolyak collocation		
		1st	2nd	3rd	4th	5th	1st	2nd	3rd	1st	2nd	3rd
1. Benchmark with EZ and disasters	4	-1.2	-1.3	-1.4	-1.5	-1.7	-1.5	-2.0	-2.4	-1.8	-3.2	-5.2
2. + capital adjustment costs	5	0.0	-0.5	-1.0	-1.5	-1.7	0.0	-0.5	-0.9	-	-	-1.6
3. + Calvo	7	0.1	-0.7	-1.4	-1.4	-1.4	0.1	-0.7	-1.8	-	-1.3	-1.8
4. + Taylor rule depends on output growth	8	0.2	-0.6	-1.5	-1.5	-1.6	0.1	-0.6	-1.5	-	-1.2	-
5. + Taylor rule is smoothed	9	0.2	-0.5	-1.5	-1.6	-1.6	0.1	-0.6	-1.5	-	-1.2	-
6. + investment shock	10	0.1	-0.5	-1.5	-1.6	-1.6	0.1	-0.6	-1.5	-	-1.2	-
7. + monetary shock	11	0.1	-0.7	-1.6	-1.7	-1.6	0.1	-0.8	-1.7	-	-1.3	-1.7
8. + intertemporal preference shock	12	0.1	-0.9	-1.6	-1.6	-1.6	0.0	-1.0	-1.8	-	-1.2	-

Table 26: Robustness 6: Taylor projection at the stochastic steady state - Mean and max Euler equation errors (EEE)

Model	State vars.	mean EEE			max EEE		
		1st	2nd	3rd	1st	2nd	3rd
1. Benchmark with EZ and disasters	4	-3.1	-5.2	-6.9	-1.4	-3.0	-3.7
2. + capital adjustment costs	5	-2.7	-4.2	-5.7	-0.1	-0.8	-1.3
3. + Calvo	7	-2.7	-4.2	-5.6	-0.2	-1.3	-2.2
4. + Taylor rule depends on output growth	8	-2.8	-4.3	-5.7	-0.2	-1.1	-1.8
5. + Taylor rule is smoothed	9	-2.7	-4.2	-5.6	-0.2	-1.1	-1.6
6. + investment shock	10	-2.7	-4.2	-5.6	-0.2	-1.1	-1.9
7. + monetary shock	11	-2.7	-4.2	-5.6	-0.2	-1.4	-1.9
8. + intertemporal preference shock	12	-2.6	-4.1	-5.4	-0.3	-1.4	-2.1

Figure 1: Model residuals across the ergodic set



This figure depicts the unit-free residuals of the model equilibrium conditions (version no. 8) for six different solution methods. The residuals are computed across a fixed sample of 1000 points, which represent the ergodic set of the model. Each plot contains 15 lines for the 15 equations of the model. Note that the scale of the 3rd-order Smolyak and the 3rd-order Taylor projection is different from the other plots.

Figure 2: Impulse response functions to a disaster shock

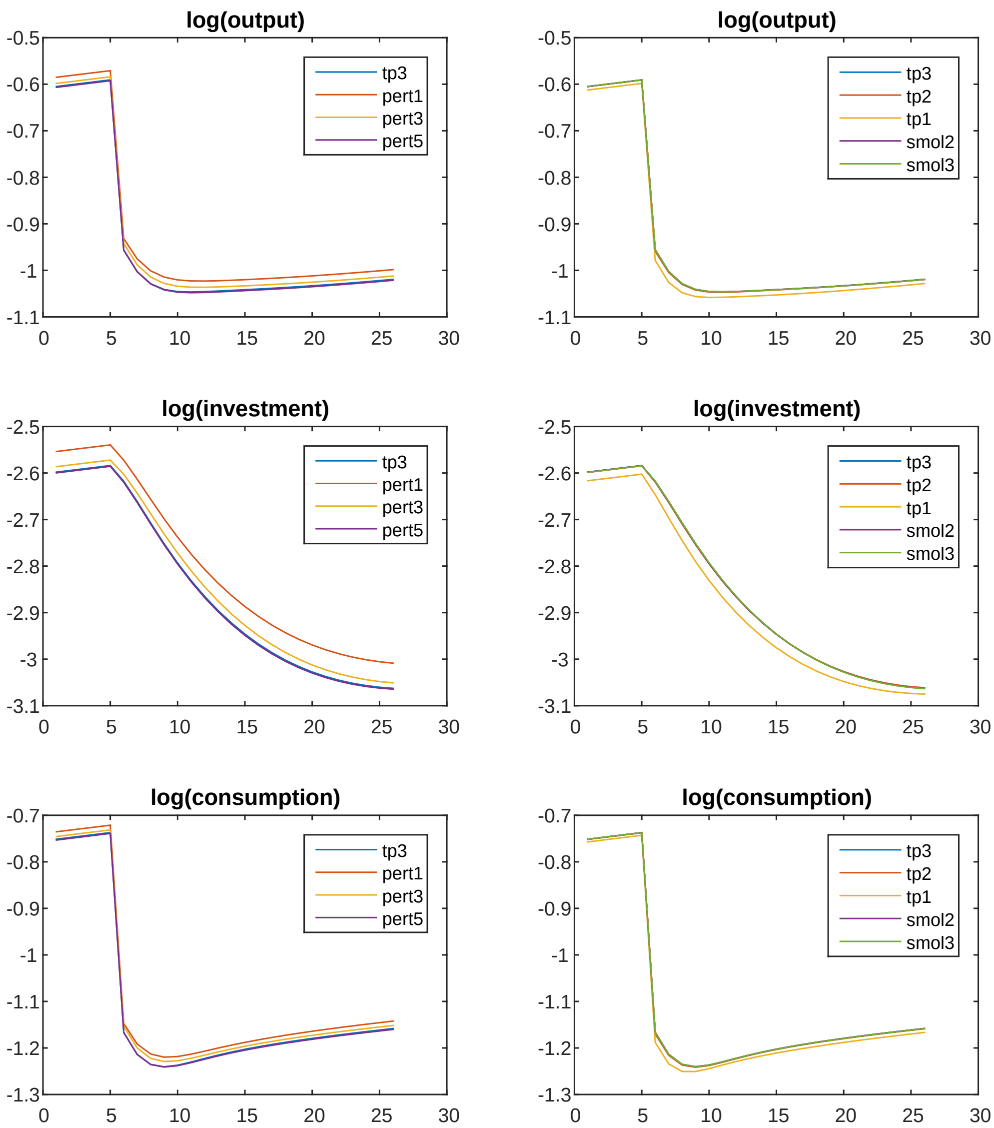


Figure 3: Impulse response functions to a disaster risk shock

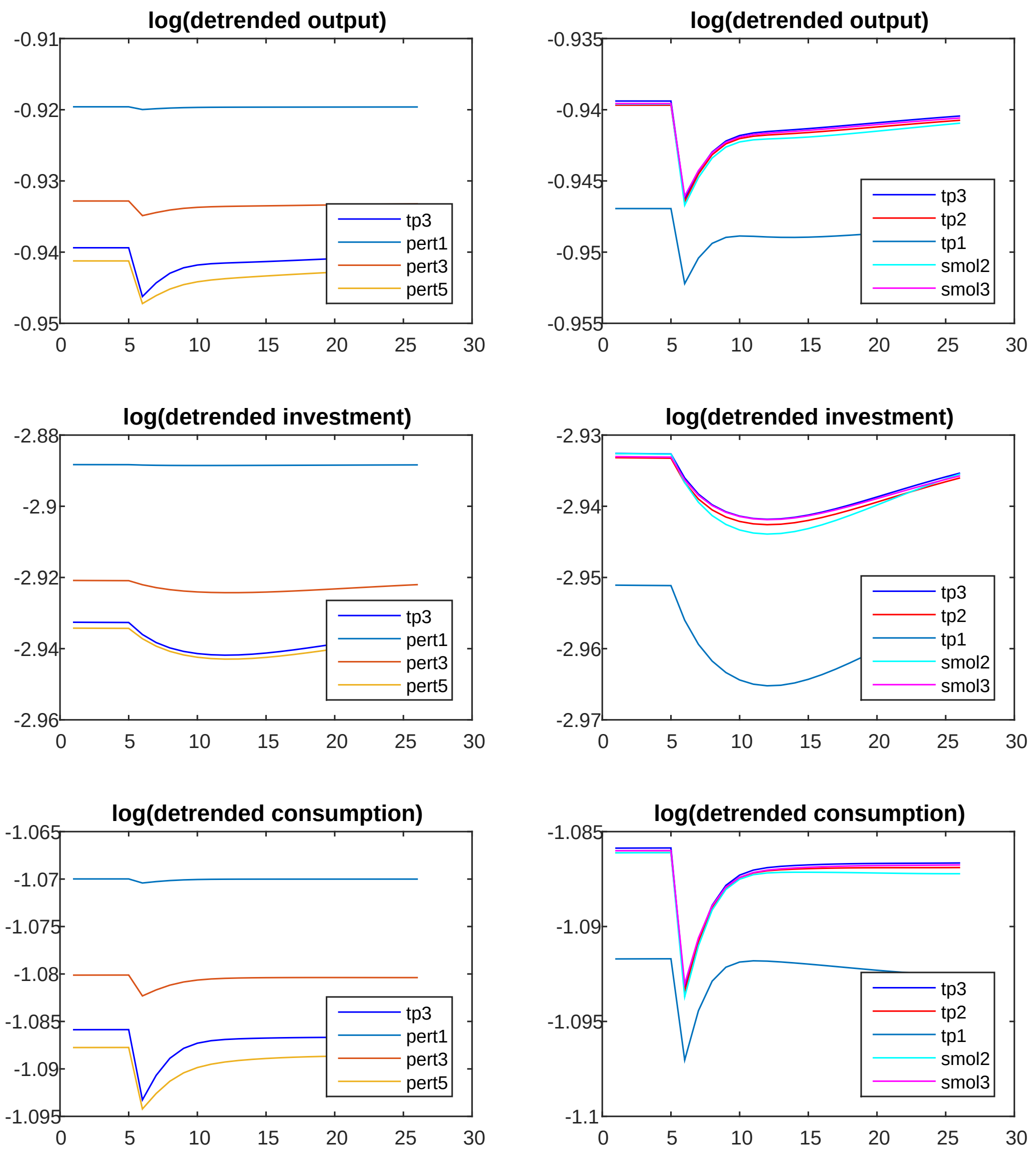
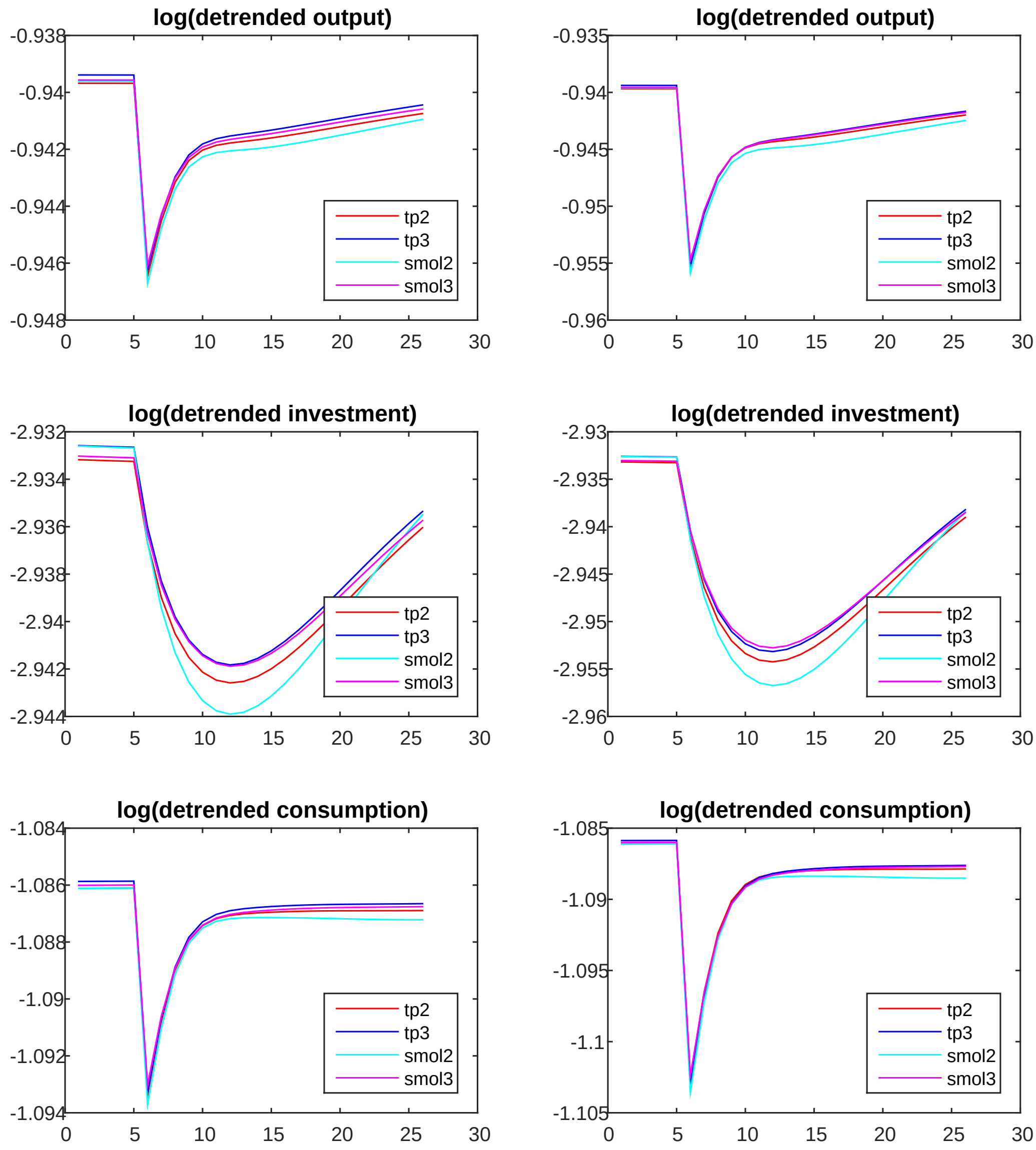


Figure 4: Impulse response functions to small (left) and big (right) disaster risk shocks



8 Online Appendix

In this appendix, we present the Euler conditions of the model, we develop the pricing Calvo block, we introduce the stationary representation of the model, we define the variables that we include in our simulaton, and we develop a simple example of how to implement Taylor projection in comparison with perturbation and projection.

8.1 Euler conditions

Define the household's maximization problem as follows:

$$\max_{c_t, k_t^*, x_t, l_t} \left\{ U_t^{1-\psi} + \beta \mathbb{E}_t \left(V_{t+1}^{1-\gamma} \right)^{\frac{1-\psi}{1-\gamma}} \right\}$$
$$\text{s.t. } c_t + x_t - w_t l_t - r_t k_t - F_t - T_t = 0$$
$$k_t^* - (1-\delta) k_t - \mu_t \left(1 - S \left[\frac{x_t}{x_{t-1}} \right] \right) x_t = 0$$
$$k_{t+1} = k_t^* \exp\left(-d_{t+1} \theta_{t+1} \right).$$

The value function V_t depends on the household's actual stock of capital k_t and on past investment x_{t-1}, as well as on aggregate variables and shocks that the household takes as given. Thus, let us use $V_{k,t}$ and $V_{x,t}$ to denote the derivatives of V_t with respect to k_t and x_{t-1} (assuming differentiability). These derivatives are obtained by the envelope theorem:

$$(1-\psi) V_t^{-\psi} V_{k,t} = \lambda_t r_t + Q_t (1-\delta) \tag{27}$$
$$(1-\psi) V_t^{-\psi} V_{x,t-1} = Q_t \mu_t S' \left[\frac{x_t}{x_{t-1}} \right] \left(\frac{x_t}{x_{t-1}} \right)^2, \tag{28}$$

where λ_t and Q_t are the Lagrange multipliers associated with the budget constraint and the evolution law of capital (they enter the Lagrangian in negative sign). We exclude the third constraint from the Lagrangian and substitute it directly in the value function or the other constraints, whenever necessary.

Differentiating the Lagrangian with respect to c_t, k_t^*, x_t, and l_t yields the first-order conditions:

$$(1-\psi) U_t^{-\psi} U_{c,t} = \lambda_t \tag{29}$$
$$(1-\psi) \beta \mathbb{E}_t \left(V_{t+1}^{1-\gamma} \right)^{\frac{\gamma-\psi}{1-\gamma}} \mathbb{E}_t \left(V_{t+1}^{-\gamma} V_{k,t+1} \exp\left(-d_{t+1} \theta_{t+1} \right) \right) = Q_t \tag{30}$$

$$\lambda_t = Q_t \mu_t \left[\left(1 - S \left[\frac{x_t}{x_{t-1}} \right] \right) - S' \left[\frac{x_t}{x_{t-1}} \right] \frac{x_t}{x_{t-1}} \right] +$$
$$+ (1-\psi) \beta \mathbb{E}_t \left(V_{t+1}^{1-\gamma} \right)^{\frac{\gamma-\psi}{1-\gamma}} \mathbb{E}_t \left(V_{t+1}^{-\gamma} V_{x,t+1} \right) \tag{31}$$
$$(1-\psi) U_t^{-\psi} U_{l,t} = -\lambda_t w_t. \tag{32}$$

Substituting the envelope conditions (27)-(28) and defining:

$$q_t = \frac{Q_t}{\lambda_t}$$

yields equations (6)-(8) in the main text.

8.2 The Calvo block

The intermediate good producer that is allowed to adjust prices maximizes the discounted value of its profits. Fernández-Villaverde and Rubio-Ramírez (2006, pp. 12-13) derive the first-order conditions of this problem for expected utility preferences, which yield the recursion:

$$\bar{g}_t^1 = \lambda_t mc_t y_t + \beta \theta_p \mathbb{E}_t \left(\frac{\Pi_t^{\chi}}{\Pi_{t+1}} \right)^{-\epsilon} \bar{g}_{t+1}^1$$
$$\bar{g}_t^2 = \lambda_t \Pi_t^* y_t + \beta \theta_p \mathbb{E}_t \left(\frac{\Pi_t^{\chi}}{\Pi_{t+1}} \right)^{1-\epsilon} \left(\frac{\Pi_t^*}{\Pi_{t+1}^*} \right) \bar{g}_{t+1}^2.$$

To adjust these conditions to Epstein-Zin preferences, divide by λ_t to have:

$$\frac{\bar{g}_t^1}{\lambda_t} = mc_t y_t + \beta \theta_p \mathbb{E}_t \frac{\lambda_{t+1}}{\lambda_t} \left(\frac{\Pi_t^{\chi}}{\Pi_{t+1}} \right)^{-\epsilon} \frac{\bar{g}_{t+1}^1}{\lambda_{t+1}} \tag{33}$$
$$\frac{\bar{g}_t^2}{\lambda_t} = \Pi_t^* y_t + \beta \theta_p \mathbb{E}_t \frac{\lambda_{t+1}}{\lambda_t} \left(\frac{\Pi_t^{\chi}}{\Pi_{t+1}} \right)^{1-\epsilon} \left(\frac{\Pi_t^*}{\Pi_{t+1}^*} \right) \frac{\bar{g}_{t+1}^2}{\lambda_{t+1}}. \tag{34}$$

Note that $\beta \frac{\lambda_{t+1}}{\lambda_t}$ is the stochastic discount factor in expected utility preferences. In Epstein-Zin preferences the stochastic discount factor is given instead by (2.1). Substituting and defining $g_t^1 = \frac{\bar{g}_t^1}{\lambda_t}$, $g_t^2 = \frac{\bar{g}_t^2}{\lambda_t}$ yields (11)-(16). The other conditions in the Calvo block follow directly from Fernández-Villaverde and Rubio-Ramírez (2006, pp. 12-13).

8.3 The stationary representation of the model

To stationarize the model we define: $\tilde{c}_t = \frac{c_t}{z_t}$, $\tilde{\lambda}_t = \lambda_t z_t^{\psi}$, $\tilde{r}_t = r_t \mu_t$, $\tilde{q}_t = q_t \mu_t$, $\tilde{x}_t = \frac{x_t}{z_t}$, $\tilde{w}_t = \frac{w_t}{z_t}$, $\tilde{k}_t = \frac{k_t}{z_t \mu_t}$, $\tilde{k}_t^* = \frac{k_t^*}{z_t \mu_t}$, $\tilde{y}_t = \frac{y_t}{z_t}$, $\tilde{U}_t = \frac{U_t}{z_t}$, $\tilde{U}_{l,t} = \frac{U_{l,t}}{z_t}$, $\tilde{V}_t = \frac{V_t}{z_t}$, $\hat{A}_t = \frac{A_t}{A_{t-1}}$, $\hat{\mu}_t = \frac{\mu_t}{\mu_{t-1}}$,

$\hat{z}_t = \frac{z_t}{z_{t-1}}$. Other re-scaled endogenous variables will be introduced below when we list the model conditions. Last, the detrended utility variables are normalized by their steady-state value to avoid scaling problems.

We define the following exogenous state variables to make them linear in the shocks

$$d_{t+1} = \mu^d + \left(\epsilon_{d,t+1} - \mu^d\right) \tag{35}$$

$$\log \theta_{t+1} = (1-\rho_\theta)\log \bar{\theta} + \rho_\theta \log \theta_t + \sigma_\theta \epsilon_{\theta,t+1} \tag{36}$$

$$z_{A,t+1} = \sigma_A \epsilon_{A,t+1} \tag{37}$$

$$\log \hat{\mu}_{t+1} = \Lambda_\mu + \sigma_\mu \epsilon_{\mu,t+1} \tag{38}$$

$$m_{t+1} = \sigma_m \epsilon_{m,t+1} \tag{39}$$

$$\xi_{t+1} = \rho_\xi \xi_t + \sigma_\xi \epsilon_{\xi,t+1} \tag{40}$$

The disaster state variable, d_t, is determined by the disaster shock $\epsilon_{d,t+1}$, which takes the values 1 or 0. The mean of this shock is μ^d. Since the mean is nonzero, the shock is demeaned in (35). The state variable $\log \theta_t$ is the log disaster size. The state variable $z_{A,t}$ is introduced to capture Gaussian productivity innovations to $\log \hat{A}_t$. The state variable $\log \hat{\mu}_t$ denotes the growth of investment technology. Finally, m_t and ξ_t are the monetary shock and the time preference shock, respectively.

The following variables depend only on the exogenous variables:

$$\log \hat{A}_t = \Lambda_A + z_{A,t} - (1-\alpha)\, d_t \theta_t$$
$$\log \hat{z}_t = \frac{1}{1-\alpha} \log \hat{A}_t + \frac{\alpha}{1-\alpha} \log \hat{\mu}_t.$$

The model conditions are given by the following equations:

$$\left(\frac{\tilde{V}_t}{\tilde{V}^{ss}}\right)^{1-\psi} = \left(\frac{\tilde{U}_t}{\tilde{U}^{ss}}\right)^{1-\psi} \left(\frac{\tilde{U}^{ss}}{\tilde{V}^{ss}}\right)^{1-\psi} + \beta \mathbb{E}_t \left(\left(\frac{\tilde{V}_{t+1}}{\tilde{V}^{ss}}\right)^{1-\gamma} \hat{z}_{t+1}^{1-\gamma}\right)^{\frac{1-\psi}{1-\gamma}} \tag{41}$$

$$\tilde{U}_t = \tilde{c}_t (1-l_t)^\nu e^{\xi_t} \tag{42}$$

$$U_{c,t} = (1-l_t)^\nu e^{\xi_t} \tag{43}$$

$$\tilde{U}_{l,t} = -\nu \tilde{c}_t (1-l_t)^{\nu-1} e^{\xi_t} \tag{44}$$

$$(1-\psi)\left(\tilde{U}_t\right)^{-\psi} \tilde{U}_{l,t} = -\tilde{\lambda}_t \tilde{w}_t \tag{45}$$

$$(1-\psi)\left(\tilde{U}_t\right)^{-\psi} U_{c,t} = \tilde{\lambda}_t \tag{46}$$

$$M_{t+1}=\beta\frac{\tilde{\lambda}_{t+1}}{\tilde{\lambda}_t}\left(\hat{z}_{t+1}\right)^{-\psi}\frac{\left(\tilde{V}_{t+1}/\tilde{V}^{ss}\right)^{\psi-\gamma}\left(\hat{z}_{t+1}\right)^{\psi-\gamma}}{\mathbb{E}_t\left(\left(\tilde{V}_{t+1}/\tilde{V}^{ss}\right)^{1-\gamma}\left(\hat{z}_{t+1}\right)^{1-\gamma}\right)^{\frac{\psi-\gamma}{1-\gamma}}} \tag{47}$$

$$\mathbb{E}_t\left(M_{t+1}\exp\left(-d_{t+1}\theta_{t+1}\right)\frac{1}{\hat{\mu}_{t+1}}\left[\tilde{r}_{t+1}+\tilde{q}_{t+1}\left(1-\delta\right)\right]\right)=\tilde{q}_t \tag{48}$$

$$1=\tilde{q}_t\left[1-S\left[\frac{\tilde{x}_t}{\tilde{x}_{t-1}}\hat{z}_t\right]-S'\left[\frac{\tilde{x}_t}{\tilde{x}_{t-1}}\hat{z}_t\right]\frac{\tilde{x}_t}{\tilde{x}_{t-1}}\hat{z}_t\right]+\mathbb{E}_t\left(M_{t+1}\tilde{q}_{t+1}S'\left[\frac{\tilde{x}_{t+1}}{\tilde{x}_t}\hat{z}_{t+1}\right]\left(\frac{\tilde{x}_{t+1}}{\tilde{x}_t}\hat{z}_{t+1}\right)^2\right) \tag{49}$$

$$\tilde{y}_t=\tilde{c}_t+\tilde{x}_t \tag{50}$$

$$\tilde{k}_t^*-\left(1-\delta\right)\tilde{k}_t-\left(1-S\left[\frac{\tilde{x}_t}{\tilde{x}_{t-1}}\hat{z}_t\right]\right)\tilde{x}_t=0 \tag{51}$$

$$\tilde{k}_t=\frac{\tilde{k}_{t-1}^*}{\hat{z}_t\hat{\mu}_t}\exp\left(-d_t\theta_t\right) \tag{52}$$

$$\tilde{q}_t^e=\mathbb{E}_t\left(M_{t+1}\hat{z}_{t+1}\left(\tilde{div}_{t+1}+\tilde{q}_{t+1}^e\right)\right) \tag{53}$$

$$\widetilde{div}_t=\tilde{y}_t-\tilde{w}_tl_t-\tilde{x}_t \tag{54}$$

$$q_t^f=\mathbb{E}_tM_{t+1} \tag{55}$$

$$\tilde{g}_t^1=mc_t\tilde{y}_t^d+\theta_p\mathbb{E}_tM_{t+1}\left(\frac{\Pi_t^{\chi}}{\Pi_{t+1}}\right)^{-\epsilon}\tilde{g}_{t+1}^1\hat{z}_{t+1} \tag{56}$$

$$\tilde{g}_t^2=\Pi_t^*\tilde{y}_t^d+\theta_p\mathbb{E}_tM_{t+1}\left(\frac{\Pi_t^{\chi}}{\Pi_{t+1}}\right)^{1-\epsilon}\left(\frac{\Pi_t^*}{\Pi_{t+1}^*}\right)\tilde{g}_{t+1}^2\hat{z}_{t+1} \tag{57}$$

$$\epsilon\tilde{g}_t^1=\left(\epsilon-1\right)\tilde{g}_t^2 \tag{58}$$

$$1=\theta_p\left(\frac{\Pi_{t-1}^{\chi}}{\Pi_t}\right)^{1-\epsilon}+\left(1-\theta_p\right)\left(\Pi_t^*\right)^{1-\epsilon} \tag{59}$$

$$mc_t=\left(\frac{1}{1-\alpha}\right)^{1-\alpha}\left(\frac{1}{\alpha}\right)^{\alpha}\tilde{w}_t^{1-\alpha}\tilde{r}_t^{\alpha} \tag{60}$$

$$\frac{\tilde{k}_t}{l_t}=\frac{\alpha}{1-\alpha}\frac{\tilde{w}_t}{\tilde{r}_t} \tag{61}$$

$$\tilde{y}_t=\frac{\frac{\hat{A}_t}{\hat{z}_t}\left(\tilde{k}_{t-1}^*\exp\left(-d_t\theta_t\right)\right)^{\alpha}l_t^{1-\alpha}-\phi}{v_t^p} \tag{62}$$

$$v_t^p=\theta_p\left(\frac{\Pi_{t-1}^{\chi}}{\Pi_t}\right)^{-\epsilon}v_{t-1}^p+\left(1-\theta_p\right)\left(\Pi_t^*\right)^{-\epsilon} \tag{63}$$

$$\frac{R_t}{R}=\left(\frac{R_{t-1}}{R}\right)^{\gamma_R}\left(\left(\frac{\Pi_t}{\Pi}\right)^{\gamma_\Pi}\left(\frac{\frac{\tilde{y}_t}{\tilde{y}_{t-1}}\hat{z}_t}{\exp\left(\Lambda_y\right)}\right)^{\gamma_y}\right)^{1-\gamma_R}e^{m_t} \tag{64}$$

$$1=\mathbb{E}_tM_{t+1}\frac{R_t}{\Pi_{t+1}}. \tag{65}$$

We define the state of the economy by the endogenous variables $\log\tilde{k}_{t-1}^*$ $\log\tilde{x}_{t-1}$, $\log\Pi_{t-1}$, $\log v_{t-1}^p$, $\log\tilde{y}_{t-1}$ and $\log R_{t-1}$, and the exogenous variables $d_t,\log\theta_t,z_{A,t},\log\hat{\mu}_t$ m_t and ξ_t.

In the flexible price versions of the model, we use the following pricing conditions instead of (56)-(60):

$$\tilde{r}_t = \alpha \hat{A}_t \hat{\mu}_t \left(\tilde{k}^*_{t-1} \exp\left(-d_t \theta_t\right) \right)^{\alpha - 1} l_t^{1-\alpha} \tag{66}$$

$$\tilde{w}_t = (1-\alpha) \frac{\hat{A}_t}{\hat{z}_t} \left(\tilde{k}^*_{t-1} \exp\left(-d_t \theta_t\right) \right)^{\alpha} l_t^{-\alpha} \tag{67}$$

$$\tilde{y}_t = \frac{\hat{A}_t}{\hat{z}_t} \left(\tilde{k}^*_{t-1} \exp\left(-d_t \theta_t\right) \right)^{\alpha} l_t^{1-\alpha} - \phi. \tag{68}$$

8.4 Simulation variables

The benchmark version of the model approximates the endogenous control variables: $\log \mathbb{E}_t \left(\left(\frac{\tilde{V}_{t+1}}{\tilde{V}^{SS}} \hat{z}_{t+1} \right)^{1-\gamma} \right)$, $\log \frac{l_t}{1-l_t}$, $\log \tilde{q}^e_t$, $\log q^f_t$, and $\log \tilde{k}^*_t$. The first variable is an auxiliary variable introduced into the system. The other model variables can be expressed as functions of the approximated variables and the given state variables. We apply a change of variables to ensure that variables are bound within their natural domain. For instance, if $x > 0$, we approximate $\log x$. Similarly, labor l_t must be between 0 and 1 so we approximate $\log \frac{l_t}{1-l_t}$ instead.

The second version with capital adjustment costs approximates, in addition, the variables $\log \tilde{q}_t$ and $\log \tilde{x}^{back}_{t+1}$, which are both determined in period t. The notation ^{back} denotes the past value of the variable, e.g. $\tilde{x}^{back}_t \equiv \tilde{x}_{t-1}$. This is required when the past value of a control variable (e.g. past investment) is an endogenous state variable.

The third version with Calvo pricing approximates, in addition, the variables: $\log \tilde{w}_t$, $\log \tilde{x}_t$, $\log \tilde{g}^1_t$, $\log \Pi_t + \log \Pi^*_t$, $\log \Pi^{back}_{t+1}$, and $\log v^{p,back}_{t+1}$, all determined in period t. We approximate $\log \Pi_t + \log \Pi^*_t$ instead of approximating separately $\log \Pi_t$ and $\log \Pi^*_t$. It can be shown that this transformation ensures that Π_t is always positive, while keeping the number of approximated variables as small as possible.

The fourth version with a Taylor rule that depends on output growth approximates, in addition, $\log \tilde{y}^{back}_{t+1}$, which is determined in period t.

The fifth version with a smoothed Taylor rule approximates, in addition, the variable $\log R^{back}_{t+1}$, which is determined in period t.

The other versions add only exogenous variables, so the number of approximated variables does not change.

8.5 Perturbation vs. Taylor projection: a simple example

As we mention in the main text, in standard perturbation, we find a solution for the variables of interest by perturbing a volatility of the shocks around zero. In comparison, in Taylor projection (as we would do in a projection), we take account of the true volatility of the shocks.

An example should clarify this point. Imagine that we are dealing with the stochastic neoclassical growth model with fixed labor supply, full depreciation, and no persistence of the productivity shock (these two assumptions allows us to derive simple analytic expressions).

The social planner problem of this model can be written as:

$$\max \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \log c_t$$
$$\text{s.t. } c_t + k_{t+1} = e^{z_t} k_t^{\alpha},\ z_t \sim \mathcal{N}(0, \sigma),$$

where $\mathbb{E}_0$ is the conditional expectation operator, β is the discount factor, c_t is consumption, k_t is capital, and z_t is the productivity shock with volatility σ.

To ease the presentation, we will switch now to the recursive notation, where we drop the time subindex and where for an arbitrary variable x, we have that $x' = x_{t+1}$. Thus, consumption can be written in terms of the policy function $c_t = c(k, z)$ and from the resource constraint of the economy $k' = e^z k^{\alpha} - c(k, z)$.

If we substitute $c(k, z)$ and $k' = e^z k^{\alpha} - c(k, z)$ in the Euler equation of the model, we get:

$$-\frac{1}{c(k,z)} + \alpha\beta\mathbb{E}_t \frac{e^{z'}\left(e^z k^{\alpha} - c(k,z)\right)^{\alpha-1}}{c\left(e^z k^{\alpha} - c(k,z), z'\right)} = 0.$$

From this Euler equation, we can find the deterministic steady state of the model:

$$k_{ss} = (\alpha\beta)^{\frac{1}{1-\alpha}}$$
$$c_{ss} = k_{ss}^{\alpha} - k_{ss}.$$

In a first-order perturbation, we postulate an approximation for the policy function of the form:

$$c(k, z) = \theta_0 + \theta_1 (k - k_{ss}) + \theta_2 z,$$

where we are already taking advantage of the certainty equivalence property of first-order approximations to drop the term on σ.

If we plug this policy function into the equilibrium conditions before, we get:

$$-\frac{1}{\theta_0+\theta_1(k-k_{ss})+\theta_2 z}+\alpha\beta\mathbb{E}_t\frac{e^{z'}\left(e^z k^\alpha-\theta_0-\theta_1(k-k_{ss})-\theta_2 z\right)^{\alpha-1}}{\theta_0+\theta_1\left(e^z k^\alpha-\theta_0-\theta_1(k-k_{ss})-\theta_2 z-k_{ss}\right)+\theta_2 z'}=0.$$

To find θ_0, we first evaluate the previous expression at the deterministic steady-state value of the state variables ($k=k_{ss}$ and $z=0$):

$$-\frac{1}{\theta_0}+\alpha\beta\mathbb{E}_t\frac{e^{z'}\left(k_{ss}^\alpha-\theta_0\right)^{\alpha-1}}{\theta_0+\theta_1\left(k_{ss}^\alpha-\theta_0-k_{ss}\right)+\theta_2 z'}=0$$

and then take $\sigma\to 0$ to get:

$$-\frac{1}{\theta_0}+\alpha\beta\frac{\left(k_{ss}^\alpha-\theta_0\right)^{\alpha-1}}{\theta_0+\theta_1\left(k_{ss}^\alpha-\theta_0-k_{ss}\right)}=0.$$

This equation has a zero at $\theta_0=c_{ss}=k_{ss}^\alpha-k_{ss}$. This result is natural: the leading constant term of a perturbation around the deterministic steady state of the policy function of an endogenous variable is just the steady-state value of such a variable (in practice, this result is just assumed without solving for it explicitly).

To find θ_1 and θ_2, we take derivatives of the Euler equation with respect to capital and productivity, evaluate them at the deterministic steady state, and take $\sigma\to 0$, and solve for the unknown coefficients. The algebra is straightforward, but tedious. Note, however, that the procedure is recursive: we solve first for θ_0, and when this coefficient is known, for θ_1 and θ_2.

In a Taylor projection, up to first order, we also postulate:

$$c=\theta_0+\theta_1(k-k_{ss})+\theta_2 z.$$

In this Taylor projection, we will take our approximation around $(k_{ss},0)$ to make the comparison with perturbation easier, but other approximation points are possible.

As before, we substitute in the equilibrium condition:

$$-\frac{1}{\theta_0+\theta_1(k-k_{ss})+\theta_2 z}+\alpha\beta\mathbb{E}_t\frac{e^{z'}\left(e^z k^\alpha-\theta_0-\theta_1(k-k_{ss})-\theta_2 z\right)^{\alpha-1}}{\theta_0+\theta_1\left(e^z k^\alpha-\theta_0-\theta_1(k-k_{ss})-\theta_2 z-k_{ss}\right)+\theta_2 z'}=0, \quad (69)$$

and evaluate it at the deterministic steady-state value of the state variables ($k=k_{ss}$ and $z=0$):

$$-\frac{1}{\theta_0}+\alpha\beta\mathbb{E}_t\frac{e^{z'}\left(k_{ss}^\alpha-\theta_0\right)^{\alpha-1}}{\theta_0+\theta_1\left(k_{ss}^\alpha-\theta_0-k_{ss}\right)+\theta_2 z'}=0.$$

But now we do not let $\sigma\to 0$. Note, in particular, that this means we still have an expectation operator $\mathbb{E}_t$ and a z'. Furthermore, it also means that we must simultaneously solve for θ_0, θ_1, and θ_2, and not recursively as in perturbation. To do so, we take derivatives of equation 69 with respect to k and z and evaluate them at the deterministic steady state. This operation gives us three equations (69 and the two derivatives) on three unknowns (θ_0, θ_1, and θ_2) that can be solved with a standard Newton algorithm.

In general, the presence of the expectation operator will imply that the θ_0 from first-order perturbation and the θ_0 from Taylor projection will be different. To see this, we can plug $\theta_0 = c_{ss}$ after equation 69 and verify that:

$$-\frac{1}{c_{ss}} + \alpha\beta k_{ss}^{\alpha-1}\mathbb{E}_t \frac{e^{z'}}{c_{ss} + \theta_2 z'} \neq 0$$

unless $\sigma = 0$.

To further illustrate this point, we will implement a simple calibration of the model with $\alpha = 0.3$ and $\beta = 0.96$. Productivity, instead of being a normal distribution as before, is now a two-point process:

$$z = \left[log(0.4), -\frac{0.1}{0.9} * log(0.4)\right]$$
$$Prob = [0.1, 0.9].$$

This calibration assumes 10% probability of a 60% fall in TFP and 90% probability of a 10.7% increase (the mean of z is still zero).

The solutions for the θ's are reported in Table 27. Note the difference between the Taylor projection θ's and the perturbation θ's.

Table 27: Solution for θ's

Parameter	Taylor projection	Perturbation
θ_0	0.4872	0.4176
θ_1	0.9326	0.7417
θ_2	0.5252	0.4176

8.6 Taylor projection vs. projection: a simple example

We can continue our previous example with the stochastic neoclassical growth model with full depreciation, but now comparing Taylor projection with a standard projection.

The first steps of a Taylor projection and a standard projection are the same. In both cases we postulate a policy function:

$$c = \theta_0 + \theta_1 (k - k_{ss}) + \theta_2 z.$$

For this example and to make the comparison with perturbation easier, we center the policy function around k_{ss}, even if other approximation points are possible.

As we did in previous cases, we substitute in the equilibrium condition to get a residual function:

$$\begin{aligned} R(k, z, \theta_0, \theta_1, \theta_2) = & -\frac{1}{\theta_0 + \theta_1 (k - k_{ss}) + \theta_2 z} \\ & + \alpha\beta\mathbb{E}_t \frac{e^{z'} \left(e^z k^\alpha - \theta_0 - \theta_1 (k - k_{ss}) - \theta_2 z\right)^{\alpha - 1}}{\theta_0 + \theta_1 \left(e^z k^\alpha - \theta_0 - \theta_1 (k - k_{ss}) - \theta_2 z - k_{ss}\right) + \theta_2 z'}, \end{aligned} \tag{70}$$

but we do not impose that this residual function is zero. Instead, we express it as an explicit function of k, z, θ_0, θ_1, and θ_2.

In Taylor projection, we find the values of θ_0, θ_1, and θ_2 that solve:

$$R(k_{ss}, 0, \theta_0, \theta_1, \theta_2) = 0 \tag{71}$$

$$\left.\frac{\partial R(k, z, \theta_0, \theta_1, \theta_2)}{\partial k}\right|_{k_{ss},0} = 0, \tag{72}$$

$$\left.\frac{\partial R(k, z, \theta_0, \theta_1, \theta_2)}{\partial z}\right|_{k_{ss},0} = 0, \tag{73}$$

In comparison, projection selects three points (k_1, z_1), (k_2, z_2), and (k_3, z_3) (one of these points can be $(k_{ss}, 0)$; there are different choices of how to undertake this selection) and finds the values of θ_0, θ_1, and θ_2 that solve:

$$R(k_1, z_1, \theta_0, \theta_1, \theta_2) = 0 \tag{74}$$

$$R(k_2, z_2, \theta_0, \theta_1, \theta_2) = 0 \tag{75}$$

$$R(k_3, z_3, \theta_0, \theta_1, \theta_2) = 0 \tag{76}$$

In both cases, we have three equations (71-73 for Taylor projection, 74-76 for projection) in three unknowns $(\theta_0, \theta_1, \theta_2)$ that come from the residual function $R(k, z, \theta_0, \theta_1, \theta_2)$, but in the former case we deal with the level and two partial derivatives of the function at one point and in the latter we deal with the level of the function at three different points.