Job Search Models

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Motivation

- Trade in the labor market is a decentralized economic activity:
  1. It takes time and effort.
  2. It is uncoordinated.
- Central points:
  1. Matching arrangements.
  2. Productivity opportunities constantly arise and disappear.

Empirical Observations

- Huge amount of labor turnover.
- Pioneers in this research: Davis and Haltiwanger.
- Micro data:
  1. Current population survey (CPS).
  2. Job opening and labor turnover survey (JOLTS): 16.000 establishments, monthly.
  3. Business employment dynamics (BED): entry and exit of establishments.
  4. Longitudinal employer household dynamics (LEHD): matched data.

Basic Accounting Identity

- For each period t and level of aggregation i:

$$\begin{aligned}\text{Net Employment Change}_{ti} &= \underbrace{\text{Hires}_{ti} - \text{Separations}_{ti}}_{\text{Workers Flows}} \\ &= \underbrace{\text{Creation}_{ti} - \text{Destruction}_{ti}}_{\text{Jobs Flows}}\end{aligned}$$

- Difficult to distinguish between voluntary and involuntary separations.

Four Models of Random Matching

- Pissarides (1985).
- Mortensen and Pissarides (1994).
- Burdett and Mortensen (1998).
- Moen (1997).

Setup

- Pissarides (1985)
- Continuous time.
- Constant and exogenous interest rate r: stationary world.
- No capital (we will change this later).

Workers

- Continuum of measure L of worker. A law of large numbers hold in the economy.
- Workers are identical.
- Linear preferences (risk neutrality).
- Thus, worker maximizes total discounted income:

$$\int_0^\infty e^{-rt} y(t)\, dt$$

where r is the interest rate and $y(t)$ is income per period.

Firms

- Endogenous number of small firms:
  1. One firm=one job.
  2. Competitive producers of the final output at price p.
- Free entry into production:
  1. Perfectly elastic supply of firm operators.
  2. Zero-profit condition.
- Vacancy cost $c > 0$ per unit of time.

Matching Function I

- L workers, u unemployment rate, and v vacancy rate.
- How do we determine how many matches do we have?
- Define matching function:

$$fL = m(uL, vL)$$

where f is the rate of jobs created.

- Increasing in both argument, concave, and constant returns to scale.
- Why CRS?
  1. Argument against decreasing returns to scale: submarkets.
  2. But possibly increasing returns to scale (we will come back to this).
- Then

$$f = m(u, v).$$

Matching Function II

- All matches are random.
- Microfoundation of the matching function? Butters (1977).
- Empirical evidence:

$$f_t = e^{\varepsilon_t} u_t^{0.72} v_t^{0.28}$$

- ε_t is the sum of:
  1. High frequency noise.
  2. Very low frequency movement (for example, demographics).

What if Increasing Returns to Scale?

- Multiple equilibria:
    1. High activity equilibrium.
    2. Low activity equilibrium.
- Diamond (1982), Howitt and McAfee (1987).
- In any case, a matching function implies externalities and opens door to inefficiencies.

Properties of Matching Function I

- Define vacancy unemployment ratio (or market tightness) as:

$$\theta = \frac{v}{u}$$

- Then:

$$q(\theta) = m\left(\frac{u}{v}, 1\right) = m\left(\frac{1}{\theta}, 1\right)$$

- We can show:

1. $q'(\theta) \leq 0$.
2. $\frac{q'(\theta)}{q(\theta)}\theta \in [-1, 0]$.

Properties of Matching Function II

- Since $\frac{f}{v} = \frac{m(u,v)}{v} = q(\theta)$, we have:
  1. $q(\theta)$ is the (Poisson) rate at which vacant jobs become filled.
  2. Mean duration of a vacancy is $\frac{1}{q(\theta)}$.
- Since $\frac{f}{u} = \frac{m(u,v)}{u} = \theta q(\theta)$, we have:
  1. $\theta q(\theta)$ is the (Poisson) rate at which unemployed workers find a job.
  2. Mean duration of unemployment is $\frac{1}{\theta q(\theta)}$.

Externalities

- Note that $q(\theta)$ and $\theta q(\theta)$ depend on market tightness.
- This is called a search or congestion externality.
- Think about a party where you take 5 friends.
- Prices and wages do not play a direct role for the rates.
- Competitive versus search equilibria.

Job Creation and Job Destruction

- Job creation: a firm and a worker match and they agree on a wage.
- Job creation in a period: $fL = u\theta q(\theta) L$.
- Job creation rate: $\frac{u\theta q(\theta)}{1-u}$.
- Job destruction: exogenous at (Poisson) rate λ.
- Job destruction in a period: $\lambda (1-u) L$.
- Job destruction rate: $\frac{\lambda(1-u)}{1-u}$.

Evolution of Unemployment

- Evolution of unemployment:

$$\dot{u} = \lambda \left(1 - u\right) - u\theta q\left(\theta\right)$$

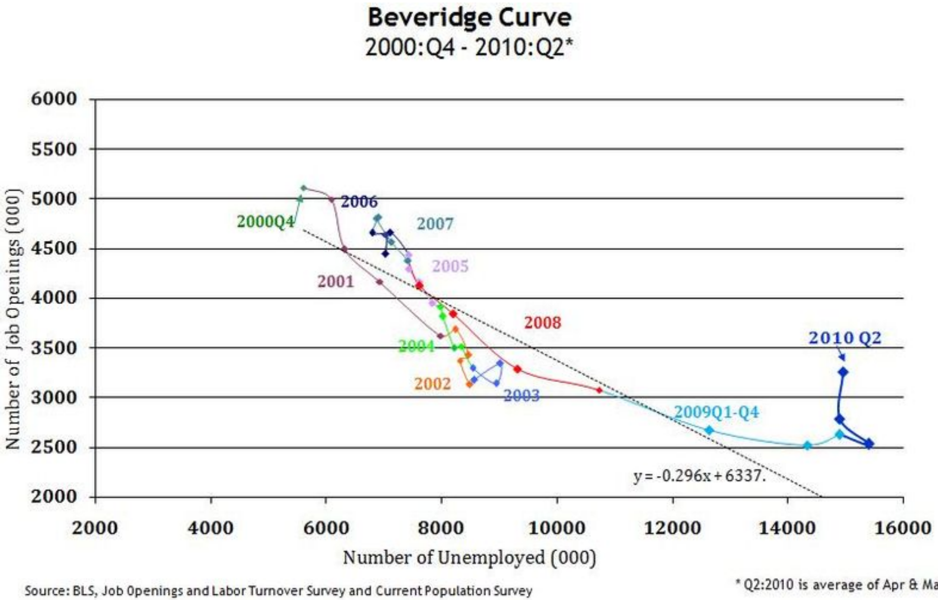
- In steady state:

$$\lambda \left(1 - u\right) = u\theta q\left(\theta\right)$$

  or

$$u = \frac{\lambda}{\lambda + \theta q\left(\theta\right)}$$

- This relation is a downward-slopping and convex to the origin curve: the Beveridge Curve.


Beveridge Curve
2000:Q4 - 2010:Q2*
Number of Job Openings (000)
Number of Unemployed (000)
6000
5500
5000
4500
4000
3500
3000
2500
2000
2000
4000
6000
8000
10000
12000
14000
16000
2000Q4
2006
2007
2005
2001
2008
2004
2002
2003
2010 Q2
2009Q1-Q4
y = -0.296x + 6337.
Source: BLS, Job Openings and Labor Turnover Survey and Current Population Survey
* Q2:2010 is average of Apr & Ma

Labor Contracts and Firm's Value Functions

- Wage w.
- Hours fixed and normalized to 1.
- Either part can break the contract at any time without cost.
- J is the value function of an occupied job.
- V is the value function of a vacant job.
- Then, in a stationary equilibrium:

$$rV = -c + q(\theta)(J - V)$$
$$rJ = p - w - \lambda J$$

- Note $J = \frac{p-w}{r+\lambda}$ and $J' = -\frac{1}{r+\lambda}$.

Job Creation Condition

- Because of free entry

$$V = 0$$
$$J = \frac{c}{q(\theta)}.$$

- Then:

$$p - w - (r + \lambda) J = 0 \Rightarrow$$
$$p - w - (r + \lambda) \frac{c}{q(\theta)} = 0$$

- This equation is know as the job creation condition.
- Interpretation.

Workers I

- Value of not working: z.
- Includes leisure, UI, home production.
- Because of linearity of preferences, we can ignore extra income.
- U is the value function of unemployed worker.
- W is the value function of employed worker.
- Then:

$$rU = z + \theta q(\theta)(W - U)$$
$$rW = w + \lambda (U - W)$$

- Note $W = \frac{w}{r+\lambda} + \frac{\lambda}{r+\lambda} U$ and $W' = \frac{1}{r+\lambda}$.

Workers II

- With some algebra:

$$\left(r+\theta q\left(\theta\right)\right)U-\theta q\left(\theta\right)W=z$$
$$-\lambda U+\left(r+\lambda\right)W=w$$

and

$$U = \frac{\left(r+\lambda\right)z+\theta q\left(\theta\right)w}{\left(r+\theta q\left(\theta\right)\right)\left(r+\lambda\right)-\lambda\theta q\left(\theta\right)} = \frac{\lambda z+\theta q\left(\theta\right)w+rz}{r^2+r\theta q\left(\theta\right)+\lambda r}$$
$$W = \frac{\left(r+\theta q\left(\theta\right)\right)w+\lambda z}{\left(r+\theta q\left(\theta\right)\right)\left(r+\lambda\right)-\lambda\theta q\left(\theta\right)} = \frac{\lambda z+\theta q\left(\theta\right)w+rw}{r^2+r\theta q\left(\theta\right)+\lambda r}$$

- Clearly, for $r>0$, $W>U$ if and only if $w>z$.
- Note that if $r=0$, $W=U$.

Wage Determination I

- We can solve Nash Bargaining Solution:

$$w = \arg\max \left(W - U\right)^{\beta} \left(J - V\right)^{1-\beta}$$

- First order conditions:

$$\beta \frac{W'}{W - U} = -\left(1 - \beta\right) \frac{J'}{J - V}$$

- Since $W' = -J' = \frac{1}{r+\lambda}$ and $V = 0$:

$$W = U + \beta \left(\underbrace{W - U + J}_{\text{surplus of the relation}} \right) = U + \beta S$$

Wage Determination II

- Also

$$W - U = \frac{\beta}{1-\beta} J = \frac{\beta}{1-\beta} \frac{c}{q(\theta)}$$

- Since $J = \frac{p-w}{r+\lambda}$ and $W = \frac{w}{r+\lambda} + \frac{\lambda}{r+\lambda} U$

$$\frac{w}{r+\lambda} - \frac{r}{r+\lambda} U = \beta \left(\frac{w}{r+\lambda} - \frac{r}{r+\lambda} U + \frac{p-w}{r+\lambda} \right) \Rightarrow$$
$$w = rU + \beta (p - rU)$$

- Interpretation.

Wage Determination III

- Now, note:

$$w = rU + \beta (p - rU) \Rightarrow$$
$$w = (1-\beta) rU + \beta p \Rightarrow$$
$$w = (1-\beta)(z + \theta q(\theta)(W - U)) + \beta p \Rightarrow$$
$$w = (1-\beta)\left(z + \theta q(\theta) \frac{\beta}{1-\beta} \frac{c}{q(\theta)}\right) + \beta p \Rightarrow$$
$$w = (1-\beta) z + \beta (p + \theta c)$$

- The last condition is known as the Wage Equation.

Steady State

- Three equations:

$$w = (1-\beta)\, z + \beta\theta c + \beta p$$
$$p - w - (r+\lambda)\,\frac{c}{q(\theta)} = 0$$
$$u = \frac{\lambda}{\lambda + \theta q(\theta)}$$

- Combine the first two conditions:

$$(1-\beta)(p-z) - \frac{r+\lambda+\beta\theta q(\theta)}{q(\theta)} c = 0$$
$$u = \frac{\lambda}{\lambda + \theta q(\theta)}$$

that we can plot in the Beveridge Diagram.

Comparative Statics

- Raise z: higher unemployment because less surplus to firms. Relation with unemployment insurance.
- Changes in matching function.
- Changes in Nash parameter.
- Dynamics?

Efficiency I

- Can the equilibrium achieve social efficiency despite search externalities?
- Social Planner:

$$\max_{u,\theta} \int_0^\infty e^{-rt} \left(p\left(1-u\right) + zu - c\theta u\right) dt$$

$$s.t.\ u = \frac{\lambda}{\lambda + \theta q\left(\theta\right)}$$

- The social planner faces the same matching frictions than the agents.
- First order conditions of the Hamiltonian:

$$-e^{-rt}\left(p - z + c\theta\right) + \mu\left(\lambda + \theta q\left(\theta\right)\right) - \dot{\mu} = 0$$

$$-e^{-rt} cu + \mu u q\left(\theta\right)\left(1 - \eta\left(\theta\right)\right) = 0$$

where μ is the multiplier and $\eta\left(\theta\right)$ is (minus) the elasticity of $q\left(\theta\right)$.

Efficiency II

- From the second equation:

$$\mu = e^{-rt} \frac{cu}{uq(\theta)(1-\eta(\theta))}$$

- Now:

$$\begin{aligned} e^{-rt} cu &= \mu uq(\theta)(1-\eta(\theta)) \\ -rt + \log cu &= \log \mu + \log uq(\theta)(1-\eta(\theta)) \end{aligned}$$

and taking time derivatives:

$$-r = \frac{\dot{\mu}}{\mu} \Rightarrow -\dot{\mu} = r\mu$$

and

$$\begin{aligned} -e^{-rt}(p - z + c\theta) + \mu(\lambda + \theta q(\theta)) - \dot{\mu} &= 0 \Rightarrow \\ -e^{-rt}(p - z + c\theta) + \mu(r + \lambda + \theta q(\theta)) &= 0 \end{aligned}$$

Efficiency III

- Thus we get:

$$
\begin{aligned}
-e^{-rt}\left(p-z+c\theta\right)+e^{-rt}\frac{cu\left(r+\lambda+\theta q\left(\theta\right)\right)}{uq\left(\theta\right)\left(1-\eta\left(\theta\right)\right)} &= 0 \Rightarrow \\
\left(1-\eta\left(\theta\right)\right)\left(p-z\right)-\frac{r+\lambda+\eta\left(\theta\right)\theta q\left(\theta\right)}{q\left(\theta\right)}c &= 0
\end{aligned}
$$

- Remember that the market job creation condition:

$$
\left(1-\beta\right)\left(p-z\right)-\frac{r+\lambda+\beta\theta q\left(\theta\right)}{q\left(\theta\right)}c=0
$$

- Both conditions are equal if, and only if, $\eta\left(\theta\right)=\beta$.

Hosios' Rule

- Imagine that matching function is $m = Au^{\eta}v^{1-\eta}$.
- Then $\eta(\theta) = \eta$.
- We have that efficiency is satisfied if $\eta = \beta$.
- This result is know as the Hosios Rule (Hosios, 1990):
  1. If $\eta > \beta$ equilibrium unemployment is below its social optimum.
  2. If $\eta < \beta$ equilibrium unemployment is above its social optimum.
- Intuition: externalities equal to share of surplus.

Introducing Capital

- Production function $f(k)$ per worker with depreciation rate δ.
- Arbitrage condition in capital market $f'(k) = (r+\delta)$.
- We have four equations:

$$f'(k) = (r+\delta)$$
$$w = (1-\beta)z + \beta\theta c + \beta p(f(k) - (r+\delta)k)$$
$$p(f(k) - (r+\delta)k) - w - (r+\lambda)\frac{c}{q(\theta)} = 0$$
$$u = \frac{\lambda}{\lambda + \theta q(\theta)}$$

Setup

- Mortensen and Pissarides (1994).
- Similar to previous model but we endogeneize job destruction.
- Why? Empirical Evidence from Davis, Haltiwanger, and Schuh (1996).
- Productivity of a job px where x is the idiosyncratic component.
- New x's arrive with Poisson rate λ.
- Distribution is $G(\cdot)$.
- Distribution is memoriless and with bounded support $[0, 1]$.
- Initial draw is $x = 1$. Why?

Policy Function of the Firm

- Value function for a job is $J(x)$.
- Then:
  1. If $J(x) \geq 0$, the job is kept.
  2. If $J(x) < 0$, the job is destroyed.
- There is an R such that $J(R) = 0$.
- This R is the reservation productivity.

Flows into Unemployment

- A law of large numbers hold for the economy.
- Job destruction: $\lambda G(R)(1-u)$.
- Unemployment evolves:

$$\dot{u} = \lambda G(R)(1-u) - u\theta q(\theta)$$

- In steady state:

$$u = \frac{\lambda G(R)}{\lambda G(R) + \theta q(\theta)}$$

Value Functions

- Value functions for the firm:

$$\begin{aligned} rV &= -c + q(\theta)(J(1) - V) \\ rJ(x) &= px - w(x) + \lambda \int_R^1 J(s)\, dG(s) - \lambda J(x) \end{aligned}$$

- Value functions for the worker:

$$\begin{aligned} rU &= z + \theta q(\theta)(W(1) - U) \\ rW(x) &= w(x) + \lambda \int_R^1 W(s)\, dG(s) + \lambda G(R) U - \lambda W(x) \end{aligned}$$

- Because of free entry, $V = 0$ and $J(1) = \frac{c}{q(\theta)}$.
- Also, by Nash bargaining:

$$W(x) - U = \beta(W(x) - U + J(x))$$

Equilibrium Equations

$$u = \frac{\lambda G(R)}{\lambda G(R) + \theta q(\theta)}$$

$$J(R) = 0$$

$$J(1) = \frac{c}{q(\theta)}$$

$$W(x) - U = \beta \left(W(x) - U + J(x) \right)$$

Solving the Model I

- First, repeating the same steps than in the Pissarides model:

$$w(x) = (1-\beta) z + \beta (px + \theta c)$$

- Second:

$$W(R) - U = \beta (W(R) - U + J(R)) = \beta (W(R) - U) \Rightarrow$$
$$W(R) = U$$

- Third:

$$rJ(x) = px - (1-\beta) z - \beta (px + \theta c) + \lambda \int_R^1 J(s)\, dG(s) - \lambda J(x) \Rightarrow$$
$$(r+\lambda) J(x) = (1-\beta) px - (1-\beta) z - \beta\theta c + \lambda \int_R^1 J(s)\, dG(s)$$

Solving the Model II

- At $x = R$

$$(r+\lambda) J(R) = (1-\beta) pR - (1-\beta) z - \beta\theta c + \lambda \int_R^1 J(s)\, dG(s) = 0$$

- Thus:

$$\begin{aligned}
(r+\lambda) J(x) &= (1-\beta) p (x-R) \Rightarrow \\
(r+\lambda) J(1) &= (1-\beta) p (1-R) \Rightarrow \\
(r+\lambda) \frac{c}{q(\theta)} &= (1-\beta) p (1-R) \Rightarrow \\
(1-\beta) p \frac{1-R}{r+\lambda} &= \frac{c}{q(\theta)}
\end{aligned}$$

Solving the Model III

- Note that

$$(r+\lambda) J(x) = (1-\beta) p (x-R) \Rightarrow J(x) = \frac{(1-\beta)}{r+\lambda} p (x-R)$$

- Then

$$(r+\lambda) J(x) = (1-\beta)(px - z) - \beta\theta c + \lambda \int_R^1 J(s) \, dG(s) \Rightarrow$$

$$(r+\lambda) J(x)$$

$$= (1-\beta)(px - z) - \beta\theta c + \frac{\lambda (1-\beta) p}{r+\lambda} \int_R^1 (s-R) \, dG(s)$$

Solving the Model IV

- Evaluate the previous expression at $x = R$ and using the fact that $J(R) = 0$

$$(r+\lambda) J(R) = 0 =$$

$$= (1-\beta)(pR - z) - \beta\theta c + \frac{\lambda(1-\beta)p}{r+\lambda}\int_R^1 (s-R)\, dG(s)$$

$$\Rightarrow$$

$$R - \frac{z}{p} - \frac{\beta}{1-\beta}\theta c + \frac{\lambda}{r+\lambda}\int_R^1 (s-R)\, dG(s) = 0$$

Solving the Model V

- We have two equations on two unknowns, R and θ:

$$(1-\beta)\,p\frac{1-R}{r+\lambda}=\frac{c}{q(\theta)}$$

$$R-\frac{z}{p}-\frac{\beta}{1-\beta}\theta c+\frac{\lambda}{r+\lambda}\int_R^1 (s-R)\,dG(s)=0$$

- The first expression is known as the job creation condition.

- The second expression is known as the job destruction condition.

- Together with $u=\frac{\lambda G(R)}{\lambda G(R)+\theta q(\theta)}$ and $w(x)=(1-\beta)z+\beta(px+\theta c)$, we complete the characterization of the equilibrium.

Efficiency

- Social Welfare:

$$\max_{u,\theta} \int_0^\infty e^{-rt} \left(y + zu - c\theta u \right) dt$$

$$s.t.\ u = \frac{\lambda G(R)}{\lambda G(R) + \theta q(\theta)}$$

where y is the average product per person in the labor market.

- The evolution of y is given by:

$$\dot{y} = p\theta q(\theta) u + \lambda (1-u) \int_R^1 psdG(s) - \lambda y$$

- Again, Hosios' rule.

Motivation

- Burdett and Mortensen (1998).
- Wage dispersion: different wages for the same work.
- Violates the law of one price.
- What is same work? Observable and unobservable heterogeneity.
- Evidence of wage dispersion: Mincerian regression

$$w_i = X_i' \beta + \varepsilon_i$$

- Typical Mincerian regression accounts for 25-30% of variation in the data.

Theoretical Challenge

- Remember Diamond's paradox: elasticity of labor supply was zero for the firm.
- Not all the deviations from a competitive setting deliver wage dispersion.
- Wage dispersion you get from Mortensen-Pissarides is very small (Krusell, Hornstein, Violante, 2007).
- Main mechanism to generate wage dispersion: on-the-job search.

Environment

- Unit measure of identical workers.
- Unit measure of identical firms.
- Each worker is unemployed (state 0) or employed (state 1).
- Poisson arrival rate of new offers λ. Same for workers and unemployed agents.
- Offers come from an equilibrium distribution F.

Previous Assumptions that We Keep

- No recall of offers.
- Job-worker matches are destroyed at rate δ.
- Value of not working: z.
- Discount rate r.
- Vacancy cost c.

Value Functions for Workers

- Utility of unemployed agent:

$$rV_0 = z + \lambda \left[\int \max \left\{ V_0, V_1 \left(w' \right) \right\} dF \left(w' \right) - V_0 \right]$$

- Utility of worker employed at wage w:

$$\begin{aligned} rV_1 \left(w \right) &= w + \lambda \int \left[\max \left\{ V_1 \left(w \right), V_1 \left(w' \right) \right\} - V_1 \left(w \right) \right] dF \left(w' \right) \\ &+ \delta \left[V_0 - V_1 \left(w \right) \right] \end{aligned}$$

- As before, there is a reservation wage w_R such that $V_0 = V_1 \left(w_R \right)$.

- Clearly, $w_R = z$.

Firms Problem

- $G(w)$: distribution of workers.

- Wage posting: Butters (1977), Burdett and Judd (1983), and Mortensen (1990).

- The profit for a firm:

$$\pi(p, w) = \frac{[u + (1-u) G(w)]}{r + \delta + \lambda (1 - F(w))} (p - w)$$

- Firm sets wages w to maximize $\pi(p, w)$. No symmetric pure strategy equilibrium.

- Firms will never post w lower than z.

Unemployment

- Steady state unemployment:

$$\lambda\left(1-F\left(z\right)\right)u=\delta\left(1-u\right)$$

- Then:

$$u=\frac{\delta}{\delta+\lambda\left[1-F\left(z\right)\right]}=\frac{\delta}{\delta+\lambda}$$

where we have used the fact that no firm will post wage lower than z and that F will not have mass points (equilibrium property that we have not shown yet).

Distribution of Workers

- Workers gaining less than w:

$$E(w) = (1-u)\,G(w)$$

- Then:

$$\dot{E}(w) = \lambda F(w)\,u - (\delta + \lambda[1-F(w)])\,E(w)$$

- In steady state:

$$E(w) = \frac{\lambda F(w)}{\delta + \lambda[1-F(w)]}u \Rightarrow$$

$$G(w) = \frac{E(w)}{1-u} = \frac{\delta F(w)}{\delta + \lambda[1-F(w)]}$$

Solving for an Equilibrium I

- Equilibrium objects: u, $F(w)$, λ, $G(w)$.
- Simple yet boring arguments show that $F(w)$ does not have mass points and has connected support.
- First, by free entry:

$$\pi(p,z) = \frac{\delta}{\delta+\lambda}\frac{p-z}{r+\delta+\lambda} = c$$

  which we solve for λ.

- Hence, we also know $u = \frac{\delta}{\delta+\lambda}$.

Solving for an Equilibrium II

- Second, by the equality of profits and with some substitutions:

$$
\begin{aligned}
\pi(p,w) &= \frac{\left[\frac{\delta}{\delta+\lambda}+\left(\frac{\lambda}{\delta+\lambda}\right)\frac{\delta F(w)}{\delta+\lambda[1-F(w)]}\right]}{r+\delta+\lambda(1-F(w))}(p-w) \\
&= \frac{\delta}{\delta+\lambda[1-F(w)]}\frac{p-w}{r+\delta+\lambda(1-F(w))} \\
&= \frac{\delta}{\delta+\lambda}\frac{p-z}{r+\delta+\lambda}
\end{aligned}
$$

- Previous equality is a quadratic equation on $F(w)$.
- To simplify the solution, set $r=0$. Then:

$$
F(w)=\frac{\delta+\lambda}{\delta}\left[1-\left(\frac{p-w}{p-z}\right)^{0.5}\right]
$$

Solving for an Equilibrium III

- Now, we get:

$$G(w) = \frac{\delta}{\lambda}\left[\left(\frac{p-w}{p-z}\right)^{0.5} - 1\right]$$

- Highest wage is $F(w^{\max}) = 1$

$$w^{\max} = \left(1 - \frac{\delta}{\delta+\lambda}\right)^2 p + \left(\frac{\delta}{\delta+\lambda}\right)^2 z$$

- Empirical content.

- Modifications to fit the data.

Competitive Search

- Moen (1997).
- A market maker chooses a number of markets m and determines the wage w_i in each submarket.
- Workers and firms are free to move between markets.
- Two alternative interpretations:
  1. Clubs charging an entry fee. Competition drives fees to zero.
  2. Wage posting by firms.

Workers

- Value functions:

$$
\begin{aligned}
rU_i &= z + \theta_i q(\theta_i)(W_i - U_i) \\
rW_i &= w_i + \lambda(U_i - W_i)
\end{aligned}
$$

- Then:

$$
\begin{aligned}
W_i &= \frac{1}{r+\lambda} w_i + \frac{\lambda}{r+\lambda} U_i \\
rU_i &= z + \theta_i q(\theta_i)\left(\frac{w_i - rU_i}{r+\lambda}\right)
\end{aligned}
$$

- Workers will pick the highest U_i.
- In equilibrium, all submarkets should deliver the same U_i. Hence:

$$
\theta_i q(\theta_i) = \frac{rU - z}{w_i - rU}(r+\lambda)
$$

- Negative relation between wage and labor market tightness.
- If $w_i < rU$, the market will not attract workers and it will close.

Firms

- Value Functions:

$$\begin{aligned} rV_i &= -c + q(\theta_i)(J_i - V_i) \\ rJ_i &= p - w_i - \lambda J_i \end{aligned}$$

- Thus:

$$rV_i = -c + q(\theta_i)\left(\frac{p - w_i}{r + \lambda} - V_i\right)$$

- Each firm solves

$$rV_i = \max_{w_i,\theta_i}\left(-c + q(\theta_i)\left(\frac{p - w_i}{r + \lambda} - V_i\right)\right)$$

$$s.t.\ rU_i = z + \theta_i q(\theta_i)\left(\frac{w_i - rU}{r + \lambda}\right)$$

Equilibrium

- Impose equilibrium condition $V_i = 0$ and solve the dual:

$$rU_i = \max_{w_i,\theta_i} \left(z + \theta_i q(\theta_i) \frac{w_i - rU}{r + \lambda} \right)$$
$$s.t.\ c = q(\theta_i) \frac{p - w_i}{r + \lambda}$$

- Plugging the value of w_i from the constraint into the objective function:

$$rU_i = \max_{\theta_i} \left(z - c\theta_i + \theta_i q(\theta_i) \frac{p - rU}{r + \lambda} \right)$$

- Solution:

$$c = q(\theta_i) \frac{p - rU}{r + \lambda} + \theta_i q'(\theta_i) \frac{p - rU}{r + \lambda}$$

that is unique if $\theta_i q(\theta_i)$ is concave.