Equilibrium with Complete Markets

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In previous lecture, we discussed the preferences of agents in a situation with uncertainty.

Now, we will discuss how markets operate in a simple endowment economy.

Two approaches: Arrow-Debreu set-up and sequential markets.

Under some technical conditions both approaches are equivalent.
Environment

- We have $I$ agents, $i = 1, \ldots, I$.

- Endowment:

$$ (e^1, \ldots, e^I) = \{ e^i_t(s^i_t), \ldots, e^I_t(s^I_t) \}_{t=0, s^i_t \in S^i_t} $$


- Advantages and shortcomings.
Allocation

Definition

An allocation is a sequence of consumption in each period and event for each individual:

\[(c^1, \ldots, c^I) = \{c^1_t(s^t), \ldots, c^I_t(s^t)\}_{t=0, s^t \in S^t}^\infty\]

Definition

Feasible allocation: an allocation such that:

\[c^i_t(s^t) \geq 0 \text{ for all } t, \text{ all } s^t \in S^t, \text{ for } i = 1, 2\]

\[\sum_{i=1}^I c^i_t(s^t) \leq \sum_{i=1}^I e^i_t(s^t) \text{ for all } t, \text{ all } s^t \in S^t\]
Pareto Efficiency

Definition

An allocation \( \{(c^1_t(s^t), ..., c^I_t(s^t))\}_{t=0, s^t \in S^t} \) is Pareto efficient if it is feasible and if there is no other feasible allocation

\[ \{(\tilde{c}^1_t(s^t), ..., \tilde{c}^I_t(s^t))\}_{t=0, s^t \in S^t} \]

such that

\[ u(\tilde{c}^i) \geq u(c^i) \text{ for all } i \]

\[ u(\tilde{c}^i) > u(c^i) \text{ for at least one } i \]

• Ex ante versus ex post efficiency.
Trade takes place at period 0, *before* any uncertainty has been realized (in particular, before $s_0$ has been realized).

As for allocation and endowment, Arrow-Debreu prices have to be indexed by event histories in addition to time.

Let $p_t(s^t)$ denote the price of one unit of consumption, quoted at period 0, delivered at period $t$ if (and only if) event history $s^t$ has been realized.

We need to normalize one price to 1 and use it as numeraire.
Arrow-Debreu Equilibrium

**Definition**

An Arrow-Debreu equilibrium are prices \( \{ \hat{p}_t(s^t) \}_{t=0}^{\infty}, s^t \in S^t \) and allocations \( \{ \hat{c}_t^i(s^t) \}_{t=0}^{\infty}, s^t \in S^t \) for \( i = 1, \ldots, I \) such that:

1. Given \( \{ \hat{p}_t(s^t) \}_{t=0}^{\infty}, s^t \in S^t \), for \( i = 1, \ldots, I \), \( \{ \hat{c}_t^i(s^t) \}_{t=0}^{\infty}, s^t \in S^t \) solves:

\[
\max_{\{ c_t^i(s^t) \}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi(s^t) u(c_t^i(s^t)) \\
\text{s.t.} \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \hat{p}_t(s^t) c_t^i(s^t) \leq \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \hat{p}_t(s^t) e_t^i(s^t) \\
\quad c_t^i(s^t) \geq 0 \text{ for all } t
\]

2. Markets clear:

\[
\sum_{i=1}^{I} \hat{c}_t^i(s^t) = \sum_{i=1}^{I} e_t^i(s^t) \text{ for all } t, \text{ all } s^t \in S^t
\]
Welfare Theorems

Theorem

Let \( \{ \hat{c}_{t}^{i}(s_{t}) \}_{t=0, s_{t} \in S_{t}}^{\infty} i=1,...,I \) be a competitive equilibrium allocation. Then, \( \{ \hat{c}_{t}^{i}(s_{t}) \}_{t=0, s_{t} \in S_{t}}^{\infty} i=1,...,I \) is Pareto efficient.

Theorem

Let \( \{ \hat{c}_{t}^{i}(s_{t}) \}_{t=0, s_{t} \in S_{t}}^{\infty} i=1,...,I \) be Pareto efficient. Then, there is a an A-D equilibrium with price \( \{ \hat{p}_{t}(s_{t}) \}_{t=0, s_{t} \in S_{t}}^{\infty} \) that decentralizes the allocation \( \{ \hat{c}_{t}^{i}(s_{t}) \}_{t=0, s_{t} \in S_{t}}^{\infty} i=1,...,I \).
Sequential Markets Market Structure

- Now we will let trade take place sequentially in spot markets in each period, event-history pair.

- One period contingent IOU’s: financial contracts bought in period $t$ that pay out one unit of the consumption good in $t + 1$ only for a particular realization of $s_{t+1} = j$ tomorrow.

- $Q_t(s^t, s_{t+1})$: price at period $t$ of a contract that pays out one unit of consumption in period $t + 1$ if and only if tomorrow’s event is $s_{t+1}$ (zero-coupon bonds).

- $a_{t+1}^i(s^t, s_{t+1})$: quantities of these Arrow securities bought (or sold) at period $t$ by agent $i$.

- These contracts are often called Arrow securities, contingent claims or one-period insurance contracts.
The period $t$, event history $s^t$ budget constraint of agent $i$ is given by

$$c^i_t(s^t) + \sum_{s^t+1|s^t} Q_t(s^t, s^t+1)a^i_{t+1}(s^t, s^t+1) \leq e^i_t(s^t) + a^i_t(s^t)$$

Note: we only have prices and quantities.

Many economists use expectations in the budget constraint. We will later see why. However, this is bad practice.
We need to rule out Ponzi schemes.

Tail of endowment distribution:

\[ A^i_t(s^t) = \sum_{\tau = t}^{\infty} \sum_{s^\tau \mid s^t} \frac{p_{\tau}(s^\tau)}{p_t(s^t)} e^{i \tau}(s^\tau) \]

\( A^i_t(s^t) \) is known as the natural debt limit.

Then:

\[ -a^i_{t+1}(s^{t+1}) \leq A^i_{t+1}(s^{t+1}) \]
Sequential Markets Equilibrium

Definition

A SME is prices for Arrow securities \( \{ \hat{Q}_t(s^t, s_{t+1}) \}_{t=0, s^t \in S^t, s_{t+1} \in S} \) and allocations \( \left\{ \left( \hat{c}_t(s^t), \{ \hat{a}_{t+1}(s^t, s_{t+1}) \}_{s_{t+1} \in S} \right) \right\}_{t=0, s^t \in S} \) such that:

1. For \( i = 1, \ldots, l \), given \( \{ \hat{Q}_t(s^t, s_{t+1}) \}_{t=0, s^t \in S^t, s_{t+1} \in S} \), for all \( i \),
   \( \{ \hat{c}_t(s^t), \{ \hat{a}_{t+1}(s^t, s_{t+1}) \}_{s_{t+1} \in S} \}_{t=0, s^t \in S} \) solves:
   \[
   \max_{\{c^i_t(s^t), \{a^i_{t+1}(s^t, s_{t+1})\}_{s_{t+1} \in S}\}_{t=0, s^t \in S}} \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi(s^t) u(c^i_t(s^t))
   \]
   subject to:
   \[
   c^i_t(s^t) + \sum_{s_{t+1} \mid s^t} \hat{Q}_t(s^t, s_{t+1}) a^i_{t+1}(s^t, s_{t+1}) \leq e^i_t(s^t) + a^i_t(s^t)
   \]
   \[
   c^i_t(s^t) \geq 0 \text{ for all } t, s^t \in S^t
   \]
   \[
   a^i_{t+1}(s^t, s_{t+1}) \geq -A^i_{t+1}(s^{t+1}) \text{ for all } t, s^t \in S^t
   \]
Definition (cont.)

2. For all $t \geq 0$

$$
\sum_{i=1}^{l} \hat{c}_t^i(s^t) = \sum_{i=1}^{l} e_t^i(s^t) \text{ for all } t, s^t \in S^t
$$

$$
\sum_{i=1}^{l} \hat{a}_{t+1}^i(s^t, s_{t+1}) = 0 \text{ for all } t, s^t \in S^t \text{ and all } s_{t+1} \in S
$$
A full set of one-period Arrow securities is sufficient to make markets “sequentially complete.”

Any (nonnegative) consumption allocation is attainable with an appropriate sequence of Arrow security holdings \( \{ a_{t+1}(s^t, s_{t+1}) \} \) satisfying all sequential markets budget constraints.

Later, when we talk about asset pricing, we will discuss how to use \( Q_t(s^t, s_{t+1} = j) \) to price any other security.
Pareto Problem

- We will extensively exploit the two welfare theorems.

- Negishi’s (1960) method to compute competitive equilibria:
  1. We fix some Pareto weights.
  2. We solve the Pareto problem associated to those weights.
  3. We decentralize the resulting allocation using the second welfare theorem.

- All competitive equilibria correspond to some Pareto weights.
Social Planner’s Problem

We solve the social planners problem:

\[
\max \left\{ \sum_{t=0}^{\infty} \sum_{i=1}^{I} \alpha_i \sum_{s^t \in S^t} \beta^t \pi(s^t) u(c^i_t(s^t)) \right\}
\]

\[\text{s.t. } \sum_{i=1}^{I} c^i_t(s^t) = \sum_{i=1}^{I} e^i_t(s^t) \text{ for all } t, s^t \in S^t\]

\[c^i_t(s^t) \geq 0 \text{ for all } t, s^t \in S^t\]

where \(\alpha_i\) are the Pareto weights.

Definition

An allocation \(\left\{ c^i_t(s^t) \right\}_{t=0, s^t \in S^t}^{\infty} i=1,\ldots,I\) is Pareto efficient if and only if it solves the social planners problem for some \(\alpha_i\) \(i=1,\ldots,I\) \(\in [0, 1]\).
Perfect Insurance

- We write the lagrangian for the problem:

$$\max_{\{c_t^i(s^t)\}_{t=0, s^t \in S^t}^{\infty}} \sum_{i=1}^I \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \left\{ \sum_{i=1}^I \alpha_i \beta^t \pi(s^t) u(c_t^i(s^t)) - \lambda_t(s^t) \left( \sum_{i=1}^I [e_t^i(s^t) - c_t^i(s^t)] \right) \right\}$$

where $\lambda_t(s^t)$ is the state-dependent lagrangian multiplier.
- We forget about the non-negativity constraints and take FOCs:

$$\alpha_i \beta^t \pi(s^t) u'(c_t^i(s^t)) = \lambda_t(s^t) \text{ for all } i, t, s^t \in S^t$$
- Then, by dividing the condition for two different agents:

$$\frac{u'(c_t^i(s^t))}{u'(c_t^j(s^t))} = \frac{\alpha_j}{\alpha_i}$$

Definition

An allocation $\{c_t^i(s^t)\}_{t=0, s^t \in S^t}^{\infty}$ has perfect consumption insurance if the ratio of marginal utilities between two agents is constant across time and states.
Irrelevance of History

- From previous equation, and making $j = 1$:
  \[ c_t^i(s^t) = u'^{-1}\left(\frac{\alpha_1}{\alpha_i} u'(c_t^1(s^t))\right) \]

- Summing over individuals and using aggregate resource constraint:
  \[ \sum_{i=1}^I e_t^i(s^t) = \sum_{i=1}^I u'^{-1}\left(\frac{\alpha_1}{\alpha_i} u'(c_t^1(s^t))\right) \]

  which is one equation on one unknown, $c_t^1(s^t)$.

- Then, the pareto-efficient allocation $(\{c_t^i(s^t)\}_{t=0, s^t \in S^t}^{\infty})_{i=1,..,I}$ only depends on aggregate endowment and not on $s^t$. 
Perfect insurance implies proportional changes in marginal utilities as a response to aggregate shocks.

Do we see perfect risk-sharing in the data?

Surprisingly more difficult to answer than you would think.

Let us suppose we have CRRA utility function. Then, perfect insurance implies:

$$\frac{c^i_t(s^t)}{c^j_t(s^t)} = \left(\frac{\alpha_i}{\alpha_j}\right)^{\frac{1}{\gamma}}$$
Individual Consumption

Now, \( c^i_t(s^t) = \frac{c^j_t(s^t)}{\alpha_j^\gamma} \alpha_i^\gamma \) and we sum over \( i \)

\[
\sum_{i=1}^{l} c^i_t(s^t) = \sum_{i=1}^{l} e^i_t(s^t) = \frac{c^j_t(s^t)}{\alpha_j^\gamma} \sum_{i=1}^{l} \alpha_i^\gamma
\]

Then:

\[
c^j_t(s^t) = \frac{\alpha_j^\gamma}{\sum_{i=1}^{l} \alpha_i^\gamma} \sum_{i=1}^{l} e^i_t(s^t) = \theta_j y_t(s^t)
\]

i.e., each agent consumes a constant fraction of the aggregate endowment.
Individual Level Regressions

- Take logs:

\[ \log c^j_t(s^t) = \log \theta_j + \log y_t(s^t) \]

- If we take first differences,

\[ \Delta \log c^j_t(s^t) = \Delta \log y_t(s^t) \]

- Equation we can estimate:

\[ \Delta \log c^j_t(s^t) = \alpha_1 \Delta \log y_t(s^t) + \alpha_2 \Delta \log e^i_t(s^t) + \varepsilon_t \]
Estimating the Equation

- How do we estimate?

\[ \Delta \log c^j_t(s^t) = \alpha_1 \Delta \log y_t(s^t) + \alpha_2 \Delta \log e^i_t(s^t) + \varepsilon_t \]

- CEX data.

- We get \( \alpha_2 \) is different from zero (despite measurement error).

- Excess sensitivity of consumption by another name!

- Possible explanation?
Permanent Income Hypothesis

- Build the Lagrangian of the problem of the household $i$:

$$\sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi(s^t) u(c^i_t(s^t)) - \mu_i \sum_{t=0}^{\infty} \sum_{s^t \in S^t} p_t(s^t) (e^i_t(s^t) - c^i_t(s^t))$$

Note: we have only one multiplier $\mu_i$.

- Then, first order conditions are

$$\beta^t \pi(s^t) u'(c^i_t(s^t)) = \mu_i p_t(s^t) \text{ for all } t, s^t \in S^t$$

- Substituting into the budget constraint:

$$\sum_{t=0}^{\infty} \sum_{s^t \in S^t} \frac{1}{\mu_i} \beta^t \pi(s^t) u'(c^i_t(s^t)) (e^i_t(s^t) - c^i_t(s^t)) = 0$$
No Aggregate Shocks

- Assume that $\sum_{i=1}^{I} e_t^i(s^t)$ is constant over time. From perfect insurance, we know then that $c_t^i(s^t)$ is also constant. Let’s call it $\widehat{c}^i$.

- Then (cancelling constants)

$$
\sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi(s^t) (e_t^i(s^t) - \widehat{c}^i) = 0 \Rightarrow
$$

$$
\widehat{c}^i = (1 - \beta) \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi(s^t) e_t^i(s^t)
$$

- Later, we will see that with no aggregate shocks, $\beta^{-1} = 1 + r$.

- Then,

$$
\widehat{c}^i = \frac{r}{1 + r} \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \left(\frac{1}{1 + r}\right)^t \pi(s^t) e_t^i(s^t)
$$