Equilibrium with Complete Markets

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Arrow-Debreu versus Sequential Markets

- In previous lecture, we discussed the preferences of agents in a situation with uncertainty.

- Now, we will discuss how markets operate in a simple endowment economy.

- Two approaches: Arrow-Debreu set-up and sequential markets.

- Under some technical conditions both approaches are equivalent.
We have $I$ agents, $i = 1, \ldots, I$.

Endowment:

$$(e^1, \ldots, e^I) = \{e^1_t(s^t), \ldots, e^I_t(s^t)\}_{t=0, s^t \in S^t}$$

Tradition in macro of looking at endowment economies. Why? Consumption, risk-sharing, asset pricing.

Advantages and shortcomings.
Allocation

Definition

An allocation is a sequence of consumption in each period and event for each individual:

\[(c^1, ..., c^I) = \{c^1_t(s^t), ..., c^I_t(s^t)\}_t=0, s^t \in S^t\]

Definition

Feasible allocation: an allocation such that:

\[c^i_t(s^t) \geq 0 \text{ for all } t, s^t \in S^t, \text{ for } i = 1, 2\]

\[\sum_{i=1}^{I} c^i_t(s^t) \leq \sum_{i=1}^{I} e^i_t(s^t) \text{ for all } t, s^t \in S^t\]
Pareto Efficiency

Definition

An allocation \( \{(c^1_t(s^t), ..., c^I_t(s^t))\}_{t=0, s^t \in S^t} \) is Pareto efficient if it is feasible and if there is no other feasible allocation

\[
\{(\tilde{c}^1_t(s^t), ..., \tilde{c}^I_t(s^t))\}_{t=0, s^t \in S^t}
\]

such that

\[
u(\tilde{c}^i) \geq u(c^i) \text{ for all } i
\]

\[
u(\tilde{c}^i) > u(c^i) \text{ for at least one } i
\]

- Ex ante versus ex post efficiency.
Trade takes place at period 0, before any uncertainty has been realized (in particular, before $s_0$ has been realized).

As for allocation and endowment, Arrow-Debreu prices have to be indexed by event histories in addition to time.

Let $p_t(s^t)$ denote the price of one unit of consumption, quoted at period 0, delivered at period $t$ if (and only if) event history $s^t$ has been realized.

We need to normalize one price to 1 and use it as numeraire.
Arrow-Debreu Equilibrium

Definition

An Arrow-Debreu equilibrium are prices \( \{\hat{p}_t(s^t)\}_{t=0, s^t \in S^t}^{\infty} \) and allocations \( \{\hat{c}_t^i(s^t)\}_{t=0, s^t \in S^t}^{\infty} \) such that:

1. Given \( \{\hat{p}_t(s^t)\}_{t=0, s^t \in S^t}^{\infty} \), for \( i = 1, \ldots, I \), \( \{\hat{c}_t^i(s^t)\}_{t=0, s^t \in S^t}^{\infty} \) solves:

\[
\max_{\{c_t^i(s^t)\}_{t=0, s^t \in S^t}^{\infty}} \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi(s^t) u(c_t^i(s^t))
\]

s.t.
\[
\sum_{t=0}^{\infty} \sum_{s^t \in S^t} \hat{p}_t(s^t) c_t^i(s^t) \leq \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \hat{p}_t(s^t) e_t^i(s^t)
\]
\[
c_t^i(s^t) \geq 0 \text{ for all } t
\]

2. Markets clear:

\[
\sum_{i=1}^{I} \hat{c}_t^i(s^t) = \sum_{i=1}^{I} e_t^i(s^t) \text{ for all } t, \text{ all } s^t \in S^t
\]
Welfare Theorems

Theorem

Let \( (\{\hat{c}^i_t(s^t)\}_{t=0}^\infty)_{i=1,...,I} \) be a competitive equilibrium allocation. Then, \( (\{\hat{c}^i_t(s^t)\}_{t=0}^\infty)_{i=1,...,I} \) is Pareto efficient.

Theorem

Let \( (\{\hat{c}^i_t(s^t)\}_{t=0}^\infty)_{i=1,...,I} \) be Pareto efficient. Then, there is an A-D equilibrium with price \( (\hat{p}_t(s^t))_{t=0}^\infty, s^t \in S^t \) that decentralizes the allocation \( (\{\hat{c}^i_t(s^t)\}_{t=0}^\infty)_{i=1,...,I} \).
Sequential Markets Market Structure

- Now we will let trade take place sequentially in spot markets in each period, event-history pair.

- One period contingent IOU’s: financial contracts bought in period $t$ that pay out one unit of the consumption good in $t+1$ only for a particular realization of $s_{t+1} = j$ tomorrow.

- $Q_t(s^t, s_{t+1})$: price at period $t$ of a contract that pays out one unit of consumption in period $t + 1$ if and only if tomorrow’s event is $s_{t+1}$ (zero-coupon bonds).

- $a_{t+1}^i(s^t, s_{t+1})$: quantities of these Arrow securities bought (or sold) at period $t$ by agent $i$.

- These contracts are often called Arrow securities, contingent claims or one-period insurance contracts.
Period-by-period Budget Constraint

- The period $t$, event history $s^t$ budget constraint of agent $i$ is given by

$$c_t^i(s^t) + \sum_{s_{t+1}|s^t} Q_t(s^t, s_{t+1}) a^i_{t+1}(s^t, s_{t+1}) \leq e_t^i(s^t) + a_t^i(s^t)$$

- Note: we only have prices and quantities.

- Many economists use expectations in the budget constraint. We will later see why. However, this is bad practice.
We need to rule out Ponzi schemes.

Tail of endowment distribution:

\[ A_t^i(s^t) = \sum_{\tau=t}^{\infty} \sum_{s^\tau|s^t} \frac{p_\tau(s^\tau)}{p_t(s^t)} e_\tau^i(s^\tau) \]

\(A_t^i(s^t)\) is known as the natural debt limit.

Then:

\[-a_{t+1}^i(s^{t+1}) \leq A_{t+1}^i(s^{t+1})\]
Sequential Markets Equilibrium

Definition

A SME is prices for Arrow securities \( \{ \hat{Q}_t(s^t, s_{t+1}) \}_{t=0}^{\infty}, s^t \in S^t, s_{t+1} \in S \) and allocations \( \left\{ \left( \hat{c}^i(s^t), \{ \hat{a}^i_{t+1}(s^t, s_{t+1}) \}_{s_{t+1} \in S} \right) \right\}_{t=0}^{\infty}, s^t \in S^t \) such that:

1. For \( i = 1, \ldots, l \), given \( \{ \hat{Q}_t(s^t, s_{t+1}) \}_{t=0}^{\infty}, s^t \in S^t, s_{t+1} \in S \), for all \( i \),
\( \{ \hat{c}^i(s^t), \{ \hat{a}^i_{t+1}(s^t, s_{t+1}) \}_{s_{t+1} \in S} \}_{t=0}^{\infty}, s^t \in S^t \) solves:

\[
\max_{\{c^i_t(s^t), \{a^i_{t+1}(s^t, s_{t+1})\}_{s_{t+1} \in S}\}_{t=0}^{\infty}, s^t \in S^t} \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi(s^t) u(c^i_t(s^t))
\]

s.t. \( c^i_t(s^t) + \sum_{s_{t+1} \mid s^t} \hat{Q}_t(s^t, s_{t+1}) a^i_{t+1}(s^t, s_{t+1}) \leq e^i_t(s^t) + a^i_t(s^t) \)

\( c^i_t(s^t) \geq 0 \) for all \( t, s^t \in S^t \)

\( a^i_{t+1}(s^t, s_{t+1}) \geq -A^i_{t+1}(s^{t+1}) \) for all \( t, s^t \in S^t \)
Definition (cont.)

2. For all $t \geq 0$

\[
\sum_{i=1}^{I} \hat{c}_t^i(s^t) = \sum_{i=1}^{I} e_t^i(s^t) \text{ for all } t, s^t \in S^t
\]

\[
\sum_{i=1}^{I} \hat{a}_{t+1}^i(s^t, s_{t+1}) = 0 \text{ for all } t, s^t \in S^t \text{ and all } s_{t+1} \in S
\]
A full set of one-period Arrow securities is sufficient to make markets “sequentially complete.”

Any (nonnegative) consumption allocation is attainable with an appropriate sequence of Arrow security holdings \( \{a_{t+1}(s^t, s_{t+1})\} \) satisfying all sequential markets budget constraints.

Later, when we talk about asset pricing, we will discuss how to use \( Q_t(s^t, s_{t+1} = j) \) to price any other security.
Pareto Problem

- We will extensively exploit the two welfare theorems.

- Negishī’s (1960) method to compute competitive equilibria:
  1. We fix some Pareto weights.
  2. We solve the Pareto problem associated to those weights.
  3. We decentralize the resulting allocation using the second welfare theorem.

- All competitive equilibria correspond to some Pareto weights.
Social Planner’s Problem

We solve the social planners problem:

\[
\max \sum_{i=1}^{l} \alpha_i \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi(s^t) u(c^i_t(s^t))
\]

s.t. \( \sum_{i=1}^{l} c^i_t(s^t) = \sum_{i=1}^{l} e^i_t(s^t) \) for all \( t, s^t \in S^t \)

\( c^i_t(s^t) \geq 0 \) for all \( t, s^t \in S^t \)

where \( \alpha_i \) are the Pareto weights.

Definition

An allocation \( \{c^i_t(s^t)\}_{t=0, s^t \in S^t}^{\infty} \) is Pareto efficient if and only if it solves the social planners problem for some \( (\alpha_i)_{i=1,\ldots,l} \in [0,1] \).
Perfect Insurance

- We write the lagrangian for the problem:

\[
\max \left\{ \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \left( \sum_{i=1}^{l} \alpha_i \beta^t \pi(s^t) u(c^i_t(s^t)) + \lambda_t(s^t) \left( \sum_{i=1}^{l} [e^i_t(s^t) - c^i_t(s^t)] \right) \right) \right\}
\]

where \( \lambda_t(s^t) \) is the state-dependent lagrangian multiplier.

- We forget about the non-negativity constraints and take FOCs:

\[
\alpha_i \beta^t \pi(s^t) u'(c^i_t(s^t)) = \lambda_t(s^t) \text{ for all } i, t, s^t \in S^t
\]

- Then, by dividing the condition for two different agents:

\[
\frac{u'(c^i_t(s^t))}{u'(c^j_t(s^t))} = \frac{\alpha_j}{\alpha_i}
\]

### Definition

An allocation \( \left( \{c^i_t(s^t)\}_{t=0, s^t \in S^t} \right)_{i=1,...,l} \) has perfect consumption insurance if the ratio of marginal utilities between two agents is constant across time and states.
Irrelevance of History

- From previous equation, and making $j = 1$:

$$c_t^i(s^t) = u'^{-1} \left( \frac{\alpha_1}{\alpha_i} u'(c_t^1(s^t)) \right)$$

- Summing over individuals and using aggregate resource constraint:

$$\sum_{i=1}^{l} e_t^i(s^t) = \sum_{i=1}^{l} u'^{-1} \left( \frac{\alpha_1}{\alpha_i} u'(c_t^1(s^t)) \right)$$

which is one equation on one unknown, $c_t^1(s^t)$.

- Then, the pareto-efficient allocation $(\{c_t^i(s^t)\}_{t=0, s^t \in S^t})_{i=1, \ldots, l}$ only depends on aggregate endowment and not on $s^t$. 
Perfect insurance implies proportional changes in marginal utilities as a response to aggregate shocks.

Do we see perfect risk-sharing in the data?

Surprisingly more difficult to answer than you would think.

Let us suppose we have CRRA utility function. Then, perfect insurance implies:

\[
\frac{c_t^i(s^t)}{c_t^j(s^t)} = \left( \frac{\alpha_i}{\alpha_j} \right)^{\frac{1}{\gamma}}
\]
Individual Consumption

Now, $c^i_t(s^t) = \frac{c^j_t(s^t)}{\alpha^j_0} \frac{1}{\alpha^i_0}$ and we sum over $i$

$$\sum_{i=1}^{l} c^i_t(s^t) = \sum_{i=1}^{l} e^i_t(s^t) = \frac{c^j_t(s^t)}{\alpha^j_0} \sum_{i=1}^{l} \frac{1}{\alpha^i_0}$$

Then:

$$c^j_t(s^t) = \frac{\alpha^j_0}{\sum_{i=1}^{l} \frac{1}{\alpha^i_0}} \sum_{i=1}^{l} e^i_t(s^t) = \theta_j y_t(s^t)$$

i.e., each agent consumes a constant fraction of the aggregate endowment.
Individual Level Regressions

- Take logs:
  \[ \log c_j^t(s^t) = \log \theta_j + \log y_t(s^t) \]

- If we take first differences,
  \[ \Delta \log c_j^t(s^t) = \Delta \log y_t(s^t) \]

- Equation we can estimate:
  \[ \Delta \log c_j^t(s^t) = \alpha_1 \Delta \log y_t(s^t) + \alpha_2 \Delta \log e_t^i(s^t) + \varepsilon_t^i \]
Estimating the Equation

- How do we estimate?

\[ \Delta \log c_j^i(s^t) = \alpha_1 \Delta \log y_t(s^t) + \alpha_2 \Delta \log e_t^i(s^t) + \varepsilon_t^i \]

- CEX data.

- We get \( \alpha_2 \) is different from zero (despite measurement error).

- Excess sensitivity of consumption by another name!

- Possible explanation?
Permanent Income Hypothesis

- Build the Lagrangian of the problem of the household $i$:

$$\sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi(s^t) u(c^i_t(s^t)) - \mu_i \sum_{t=0}^{\infty} \sum_{s^t \in S^t} p_t(s^t) (e^i_t(s^t) - c^i_t(s^t))$$

Note: we have only one multiplier $\mu_i$.

- Then, first order conditions are

$$\beta^t \pi(s^t) u'(c^i_t(s^t)) = \mu_i p_t(s^t) \text{ for all } t, s^t \in S^t$$

- Substituting into the budget constraint:

$$\sum_{t=0}^{\infty} \sum_{s^t \in S^t} \frac{1}{\mu_i} \beta^t \pi(s^t) u'(c^i_t(s^t)) (e^i_t(s^t) - c^i_t(s^t)) = 0$$
No Aggregate Shocks

- Assume that $\sum_{i=1}^{l} e_i^t(s^t)$ is constant over time. From perfect insurance, we know then that $c_i^t(s^t)$ is also constant. Let’s call it $\hat{c}_i^t$.

- Then (cancelling constants)

$$\sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi(s^t) \left( e_i^t(s^t) - \hat{c}_i^t \right) = 0 \Rightarrow$$

$$\hat{c}_i^t = (1 - \beta) \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi(s^t) e_i^t(s^t)$$

- Later, we will see that with no aggregate shocks, $\beta^{-1} = 1 + r$.

- Then,

$$\hat{c}_i^t = \frac{r}{1 + r} \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \left( \frac{1}{1 + r} \right)^t \pi(s^t) e_i^t(s^t)$$