Introduction to Uncertainty

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Uncertainty in Macroeconomics

- Modern macro studies stochastic processes of observed variables.

- Two elements:
  1. Dynamics.
  2. Uncertainty.

- We will introduce some basic concepts by building a pure exchange economy with stochastic endowments.

- In this lecture, we will present the expected discounted utility and use it to assess the welfare cost of the business cycle.
Time

- Discrete time $t \in \{0, 1, 2, \ldots\}$.

- Why discrete time?
  1. Economic data is discrete.
  2. Easier math.

- Comparison with continuous time:
  1. Discretize observables.
  2. More involved math (stochastic calculus), but often we have extremely powerful results.

- Calendar versus planning time.
Events

- One event $s_t$ happens in each period.

- $s_t \in S = \{1, 2, ..., N\}$.

Note:

1. $S$ is a finite set. We will later talk about measure theory.

2. $S$ does not depend on time.

- Event history $s^t = (s_0, s_1, ..., s_t) \in S \times ... \times S = S^{t+1}$. 
Probabilities

- Probability of \( s^t \) is \( \pi(s^t) \).

- Conditional probability of \( s_{t+1} \) is \( \pi(s_{t+1} | s^t) \).

- At this moment, we are not imposing any transition probability among states across time.

- Our notation allows the \textit{particular} cases:

\[
\begin{align*}
\pi \left( s_{t+1} | s^t \right) & = \pi \left( s_{t+1} \right) \\
\pi \left( s_{t+1} | s^t \right) & = \pi \left( s_{t+1} | s_t \right)
\end{align*}
\]
Commodity Space

- One good in the economy.

- However, good indexed by event history over infinite time. Hence our commodity space is slightly more complicated (see chapter 15 in SLP).

- Commodity space: $\left( C, \| \cdot \| \right)$.

- We pick $l_\infty$, i.e., the space of sequences $c = (c_0, c_1, ...), c_n \in \mathbb{R}$ that are bounded in the norm:

$$\| c \|_\infty = \sup_i |c_i|$$
Household Preferences

- Preferences admit a representation:

\[ U(c) = \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi(s^t) u(c_t(s^t)) \]

- This is known as the von Neumann-Morgenstern expected utility function.

- Remember:

  1. Key assumptions: continuity and independence axioms.
  2. Linear in probabilities.
  3. Cardinal utility: unique only up to an affine transformation.
Facts about Utility Function I: Time Separability

• Total utility $c$ equals the expected discounted sum of period (or instantaneous) utility $u(c_t(s^t))$.

• The period utility at time $t$ only depends on consumption in period $t$ and not on consumption in other periods.

• This formulation rules out, among other things, habit persistence.

• However, it is easy to relax: recursive utility functions.
Facts about Utility Function II: Time Discounting

- $\beta < 1$ indicates that agents are impatient.

- $\beta$ is called the (subjective) time discount factor.

- The subjective time discount rate $\rho$ is defined by $\beta = \frac{1}{1+\rho}$.

- Assumption: constant over time $\rightarrow$ exponential discounting.

- Alternatives: hyperbolic discounting, endogenous discounting, ...
Facts about Utility Function III: Risk Aversion

- Arrow-Pratt Absolute Risk Aversion:

\[ ARA = -\frac{u''(c)}{u'(c)} \]

Why do we divide by \( u'(c) \)?

- Arrow-Pratt Relative Risk Aversion:

\[ RRA = -\frac{u''(c)}{u'(c)}c \]

Interpretation.
Common Utility Functions

- **Constant Absolute Risk Aversion (CARA):**
  
  \[-e^{-ac}\]

- **Constant Relative Risk Aversion (CRRA):**
  
  \[
  \frac{c^{1-\gamma} - 1}{1 - \gamma} \text{ for } \gamma \neq 1 \\
  \log c \text{ for } \gamma = 1
  \]
  
  (you need to take limits and apply L’Hôpital’s rule).

- **Why CRRA Utility Functions?**

  1. Market price of risk has been roughly constant over the last two centuries.

  2. This observation suggests that risk aversion should be relatively constant over wealth levels.
CRRA Utility Functions

- $\gamma$ plays a dual role controlling risk-aversion and intertemporal substitution.

- Coefficient of Relative Risk-aversion:
  \[
  - \frac{u''(c)}{u'(c)} c = \gamma
  \]

- Elasticity of Intertemporal Substitution:
  \[
  - \frac{u(c_2)/u(c_1)}{c_2/c_1} \frac{d(c_2/c_1)}{d(u(c_2)/u(c_1))} = \frac{1}{\gamma}
  \]

- Advantages and disadvantages.
Cost of Business Cycles

- Simple CRRA utility function already answers many questions.


- Importance of question:
  1. Limits of stabilization policy.
  2. Macroeconomic priorities.
A Process for Consumption

Assume that consumption evolves over time as:

\[ c_t = \mu^t (1 + \lambda) e^{-\frac{1}{2}\sigma_z^2} z_t c \]

where \( \log z_t \sim \mathcal{N} (0, \sigma_z^2) \).

The moment generating function of a lognormal distribution implies:

\[ \mathbb{E} (z_t^m) = e^{\frac{m^2 \sigma_z^2}{2}} \]

Then:

\[ \mathbb{E} \left( e^{-\frac{1}{2}\sigma_z^2} z_t \right) = 1 \]

\[ \mathbb{E} \left( z_t^{1-\gamma} \right) = e^{\frac{1}{2}(1-\gamma)^2 \sigma_z^2} \]
A Compensating Differential

- We want to find the value of $\lambda$ such that:

\[
E \frac{c_t^{1-\gamma} - 1}{1-\gamma} = \frac{(\mu^t c)^{1-\gamma} - 1}{1-\gamma}
\]

- If this condition is true period by period and event by event, it should also be true when we sum up.

- Moreover, the converse is also true: $\lambda$ is the smallest number that makes total utilities over time to be equal. Why? Because of the CRRA and the i.i.d. structure of $z_t$.

- Interpretation: $\lambda$ is the welfare cost of uncertainty, i.e., by how much we need to raise consumption in every period and state.
Finding Compensating Differential

- Dropping irrelevant constants, $\lambda$ solves:

$$\mathbb{E} \left( (1 + \lambda) \left( e^{-\frac{1}{2} \sigma_z^2 z_t} \right) \right)^{1-\gamma} = 1 \Rightarrow$$

$$(1 + \lambda) e^{-\frac{1}{2} \sigma_z^2} \left( \mathbb{E} z_t^{1-\gamma} \right)^{\frac{1}{1-\gamma}} = 1 \Rightarrow$$

$$(1 + \lambda) e^{-\frac{1}{2} \sigma_z^2 + \frac{1}{2} (1-\gamma) \sigma_z^2} = 1 \Rightarrow$$

$$(1 + \lambda) e^{-\frac{1}{2} \gamma \sigma_z^2} = 1$$

- Taking logs: $\lambda \approx \frac{1}{2} \gamma \sigma_z^2$.

- Let us put some numbers here. Using quarterly U.S. data 1947-2006, $\sigma_z^2 = (0.033)^2$. What is $\gamma$?
Size of Risk Aversion

- Most evidence suggests that $\gamma$ is low, between 1 and 3. At most 10.

- Types of evidence:
  1. Questionnaires.
  2. Experiments.
  3. Econometric estimates from observed behavior.

- Two powerful arguments from growth theory international comparisons. We will revisit these points when we talk about asset pricing.

An Estimate of the Cost of the Business Cycle

- Let us take $\gamma = 1$ as a benchmark number. Then, we have:

$$\lambda \approx \frac{1}{2} \gamma \sigma_z^2 = \frac{1}{2} \times 1 \times (0.033)^2 = 0.0005$$

- Even if we take $\gamma = 10$ as an upper bound:

$$\lambda \approx \frac{1}{2} \gamma \sigma_z^2 = \frac{1}{2} \times 10 \times (0.033)^2 = 0.005$$

- These are extremely small numbers.

- Later we will see how this finding is intimately linked with the Equity premium puzzle.

- How could we turn around this result?
We assumed:

1. Representative agent.
2. Exogenous lognormal consumption.
3. Expected utility.

How important are each of these three assumptions?
Representative Agent

- Representative agent: fluctuations are at the margin.

- Lucas is very explicit about the possible costs of inequality.

- We will see in the next lecture that, with complete markets, we will have perfect risk sharing.

- But the interesting question is the effects of business cycles with incomplete markets and heterogeneity.

- Krusell and Smith (2002), loss of 0.001 of average consumption, 65% of households lose when business cycles are removed.
Exogenous Lognormal Consumption

- A combined hypothesis: exogenous consumption + lognormal consumption.

- Exogenous consumption ⇒ Cho and Cooley (2001), business cycles may increase welfare: mean versus spread effect. Same answer if we have New Keynesian models Galí, Gertler, and López-Salido (2007).

- Lognormal consumption ⇒ great depressions? Chatterjee and Corbae (2005): welfare cost of 0.0187. They calibrate a great depression every 87 years.

- A nonparametric approach by Álvarez and Jermann (2004) suggests costs between 0.0008 and 0.0049.
Problems of Expected Utility

- We have representation:

\[
U(c) = \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi(s^t) u(c_t(s^t))
\]

- Three strong assumptions:

  1. Intertemporal elasticity of substitution and risk aversion are determined by just one parameter.
  2. Temporal separability.
  3. Expected utility.

- All are problematic and they may affect our calculations.
Recursive Utility

- Epstein-Zin preferences (1989):

\[ U_t = \left( (1 - \beta) c_t^\rho + \beta (E_t U_{t+1}^\alpha)^{\frac{\rho}{\alpha}} \right)^{\frac{1}{\rho}} \]

separates elasticity of substitution:

\[ \gamma = \frac{1}{1 - \rho} \]

from risk-aversion \( \alpha \).


- Risk in the long run:
  1. Bansal and Yaron (2004): difficult to distinguish a long run component from a random walk.
     Implications for the equity premium.
Temporal Anomalies


- Explanations:

     \[ \sum_{t=0}^{\infty} \delta \beta^t u(c_t) \]

Uncertainty Anomalies

1. Framing effects (Kahneman and Tversky).

2. Allais paradox. Three prizes in a lottery: \( \{0, 1, 10\} \)

   Problem 1: \( L_1 = (0, 1, 0) \) versus \( L_2 = (0.01, 0.89, 0.1) \).

   Problem 2: \( L_3 = (0.89, 0.11, 0) \) versus \( L_4 = (0.9, 0, 0.1) \).

3. Ellsberg paradox.
Ambiguity Aversion

- Knight (1921) risk versus uncertainty.

- Gilboa and Schmidler (1989):
  \[
  \min_{Q \in \mathcal{P}} E_Q u(c)
  \]

- Two possible extensions:
  1. Choice over time.
  2. General class of ambiguity aversion.
Choice over Time

• Epstein and Schneider (2003):

\[ \min_{Q \in \mathcal{P}} \mathbb{E}_{Q} \sum_{t=0}^{\infty} \beta^{t} u(c_t) \]

• Difficult technical assumption ⇒ rectangularity.
Ambiguity and the Variational Representation of Preferences

- Maccheroni, Marinacci, and Rustichini (2006):
  \[
  \min_{Q \in \mathcal{P}} \{ \mathbb{E}_Q u(c) + \phi(Q) \}
  \]
  The function \( u \) represents risk attitudes while the index \( c \) captures ambiguities attitudes.
  They extend it to the intertemporal case.
  One particular example:
  \[
  \min_{Q \in \mathcal{P}} \{ \mathbb{E}_Q u(c) + \theta R(Q\|P) \}
  \]
  Hansen and Sargent’s (2006) research program on robust control.