Endogenous Growth Models

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Endogenous Technological Change

- First generation endogenous growth models: human capital, AK, learning-by-doing, externalities.
- Second generation (Romer, 1987 and 1990): new varieties of products (or processes) that increase the division of labor.
- Alternative by Grossman and Helpman (1991) : product innovation.
- Schumpeterian growth models by Aghion and Howitt (1992) and Grossman and Helpman (1991).

Ideas as Engine of Growth

- Technology: the way inputs to the production process are transformed into output.

- Technological progress due to new ideas:

    1. Products.

    2. Managerial practices.

    3. Business models.

- Why (and under what circumstances) are resources are spent on the development of new ideas?

Historians of Science versus Economists

- Many historians of science focus on the autonomous role of science in developing inventions and progress (the "Newton paradigm").
- However, economists emphasize the role of profit.
- Classical study of Schmookler (*Invention and Economic Growth*, 1963): innovation is determined by the size of the market.
- Examples:
  1. Horseshoe, many innovations in the late 19^{th} century and early 20^{th} century, stop afterwards.
  2. Air conditioners sold at Sears, between 1960 and 1980 and between 1980 and 1990.
  3. Drugs for Malaria versus drugs for male impotence.

Ideas

- What is an idea?
- What are the basic characteristics of an idea?
    1. Ideas are *nonrivalrous* goods.
    2. Ideas are, at least partially, *excludable*.
- Nonrivalrousness: implies that cost of providing the good to one more consumer, the *marginal cost* of this good, is constant at zero. Production process for ideas is usually characterized by substantial fixed costs and low marginal costs. Think about software.
- Excludability: required so that firm can recover fixed costs of development. Existence of intellectual property rights like patent or copyright laws are crucial for the private development of new ideas.

Intellectual Property Rights and the Industrial Revolution

- Ideas engine of growth.
- Intellectual property rights needed for development of ideas.
- Sustained growth recent phenomenon.
- Coincides with establishment of intellectual property rights.

Data on Ideas

- Measure technological progress directly through ideas.

- Measure ideas via measuring patents.

- Measure ideas indirectly by measuring resources devoted to development of ideas.

Household

- Representative household with a utility function:

$$U(0) = \int_0^\infty e^{-\rho t} \frac{c(t)^{1-\sigma}}{1-\sigma} dt$$

- $L = 1$, no population growth $(c(t) = C(t))$.
- Asset evolution:

$$\dot{a}(t) = r(t)\, a(t) + w(t) - c(t)$$

- Euler equation:

$$\frac{\dot{C}(t)}{C(t)} = \frac{1}{\sigma}(r(t) - \rho)$$

- I will skip being explicit about initial and transversality conditions.

Final Good Producer I

- Competitive producer with technology:

$$Y(t) = \frac{1}{\alpha} \left(\int_0^{N(t)} x(v, t)^{\alpha} \, dv \right) L^{1-\alpha}$$

- The elasticity of substitution τ among different intermediate goods is just

$$\alpha = \frac{\tau - 1}{\tau} \Rightarrow \tau = \frac{1}{1 - \alpha}$$

- Price of final good normalized to 1.
- $x(v, t)$: input, fully depreciated in production.
- $p(v, t)$: price of inputs.

Final Good Producer II

- Problem of final good producer:

$$\max_{[x(v,t)]_0^{N(t)},L} \left\{ \begin{array}{c} \frac{1}{\alpha}\left(\int_0^{N(t)} x(v,t)^{\alpha} dv\right) L^{1-\alpha} \\ -\int_0^{N(t)} p(v,t)\, x(v,t)\, dv - w(t) L \end{array} \right\}$$

- Hence, necessary conditions are:

$$x(v,t)^{\alpha-1} L^{1-\alpha} - p(v,t) = 0 \Rightarrow$$

$$x(v,t) = p(v,t)^{\frac{1}{\alpha-1}} L$$

and

$$(1-\alpha)\frac{Y(t)}{L} = w(t)$$

Varieties Producers I

- Each varieties producer is a monopolist in its own type.
- Production of inputs at (constant) marginal cost α (we can always define the units of the final good and the units of each variety to get this result).
- Then, per unit profit is $p(v, t) - \alpha$.

Varieties Producers II

- Given demand function:

$$x(v,t) = p(v,t)^{\frac{1}{\alpha-1}} L$$

we have:

$$\max_{p(v,t)} \left(p(v,t) - \alpha\right) x(v,t) =$$

$$\max_{p(v,t)} \left(p(v,t) - \alpha\right) p(v,t)^{\frac{1}{\alpha-1}} L$$

$$\propto \max_{p(v,t)} p(v,t)^{\frac{\alpha}{\alpha-1}} - \alpha p(v,t)^{\frac{1}{\alpha-1}}$$

Note that this problem is static!

Varieties Producers III

- Optimality condition:

$$\frac{\alpha}{\alpha-1}p(v,t)^{\frac{1}{\alpha-1}}-\frac{\alpha}{\alpha-1}p(v,t)^{\frac{1}{\alpha-1}-1}=0\Rightarrow$$
$$p(v,t)=p=1$$

Classical condition (mark-up over marginal cost). Same for all producers.

- Demand is then:

$$x(v,t)=x=L$$

Same for all producers.

- Hence, total profit

$$\pi(v,t)=\pi=(p(v,t)-\alpha)\,x(v,t)=(1-\alpha)\,L$$

Innovation

- Innovation:

$$\dot{N}(t) = \eta Z(t)$$

with some initial $N(0)$.

- Free entry into inputs market.
- Innovation gives you a perpetual patent.
- Value of a patent:

$$V(\upsilon, t) = \int_t^\infty e^{-\int_t^s r(s')ds'} \pi(\upsilon, s)\, ds$$

- Optimality condition:

$$r(t) V(\upsilon, t) - \dot{V}(\upsilon, t) = \pi(\upsilon, t) \Rightarrow$$
$$r(t) V(\upsilon, t) - \dot{V}(\upsilon, t) = (1-\alpha) L$$

Aggregate Constraints

- Free entry determines $Z(t)$:

$$\eta V(\upsilon, t) = 1$$

- Assets:

$$a = \int_0^{N(t)} V(\upsilon, t)\, d\upsilon$$

- Aggregate resource constraint:

$$Y(t) = C(t) + X(t) + Z(t)$$

Equilibrium

A equilibrium is a sequence of allocations $\{Y(t), C(t), X(t), Z(t)\}_{t=0}^{\infty}$, available varieties $\{N(t)\}_{t=0}^{\infty}$, quantities and prices for varieties $\left\{[p(\upsilon, t), x(\upsilon, t)]_0^{N(t)}\right\}_{t=0}^{\infty}$, and input prices $\{r(t), w(t)\}_{t=0}^{\infty}$ such that:

- Given input prices, $\{r(t), w(t)\}_{t=0}^{\infty}$, the representative household maximizes its utility.
- Given prices for varieties $\left\{[p(\upsilon, t)]_0^{N(t)}\right\}_{t=0}^{\infty}$, and wages $\{w(t)\}_{t=0}^{\infty}$, the final good producer maximizes.
- Given demand function, the varieties producers set up prices of varieties to maximize profits.
- Free entry determines $Z(t)$.
- Markets clear:

$$a = \int_0^{N(t)} V(\upsilon, t)\, d\upsilon$$

$$Y(t) = C(t) + X(t) + Z(t)$$

Solving for Equilibrium I

- First, note that

$$x(v,t) = x = L$$

we have:

$$Y(t) = \frac{1}{\alpha}\left(\int_0^{N(t)} x(v,t)^{\alpha} dv\right) L^{1-\alpha} = \frac{1}{\alpha} N(t) L$$

Increases in varieties increase productivity of labor. Hence:

$$g_Y = g_N$$

Also,

$$X(t) = \alpha \int_0^{N(t)} x(v,t)\, dv = \alpha \int_0^{N(t)} L dv = \alpha N(t) L = \alpha^2 Y(t)$$

Solving for Equilibrium II

- Second, we have

$$w(t) = (1-\alpha)\frac{Y(t)}{L} = \frac{1-\alpha}{\alpha}N(t)$$

- Third, since

$$\eta V(v,t) = 1$$

we have:

$$V(v,t) = \frac{1}{\eta}$$
$$\dot{V}(v,t) = 0$$

and then:

$$r(t)V(v,t) - \dot{V}(v,t) = (1-\alpha)L \Rightarrow$$
$$r(t) = r = \eta(1-\alpha)L$$

Solving for Equilibrium III

- Fourth

$$\frac{\dot{C}(t)}{C(t)} = \frac{1}{\sigma}\left(r(t) - \rho\right) \Rightarrow$$
$$g_C = \frac{1}{\sigma}\left(\eta\left(1-\alpha\right)L - \rho\right)$$

Now, this is a model without transitional dynamics, and hence:

$$g_Y = g_N = g_C = \frac{1}{\sigma}\left(\eta\left(1-\alpha\right)L - \rho\right)$$

Solving for Equilibrium IV

- Finally,

$$g_N = \frac{\dot{N}(t)}{N(t)} = \eta \frac{Z(t)}{N(t)} = \frac{1}{\sigma}\left(\eta(1-\alpha)L - \rho\right)$$

and we get:

$$\begin{aligned} Z(t) &= \frac{1}{\sigma}\left((1-\alpha)L - \frac{\rho}{\eta}\right)N(t) \\ &= \frac{1}{\sigma}\left((1-\alpha) - \frac{\rho}{\eta L}\right)\alpha Y(t) \end{aligned}$$

Since

$$X(t) = \alpha^2 Y(t)$$

we find, to close the model

$$C(t) = \left[1 - \alpha^2 - \frac{1}{\sigma}\left((1-\alpha) - \frac{\rho}{\eta L}\right)\right]Y(t)$$

The Social Planner's Problem I

- The Social planner's problem can be written as:

$$\max_{\{c(t),N(t)\}_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t} \frac{C(t)^{1-\sigma}}{1-\sigma} dt$$

$$\text{s.t. } Y(t) = C(t) + X(t) + Z(t)$$

- We can rewrite the resource constraint in terms of net output:

$$\begin{aligned} \widetilde{Y}(t) &= \frac{1}{\alpha}\left(\int_0^{N(t)} x(v,t)^{\alpha} dv\right) L^{1-\alpha} - \int_0^{N(t)} \alpha x(v,t)\, dv \\ &= C(t) + Z(t) \end{aligned}$$

The Social Planner's Problem II

- Maximizing net output is a static problem with optimality conditions:

$$x(v,t)^{\alpha-1} L^{1-\alpha} - \alpha = 0 \Rightarrow x(v,t) = x = \alpha^{\frac{1}{\alpha-1}} L$$

and then:

$$\widetilde{Y}^*(t) = (1-\alpha)\,\alpha^{\frac{1}{\alpha-1}} N^*(t) L$$

and

$$Y^*(t) = \frac{1}{\alpha^{\frac{1}{1-\alpha}}} N^*(t) L$$

- Compare with equilibrium output:

$$Y^*(t) = \frac{1}{\alpha^{\frac{1}{1-\alpha}}} N^*(t) L > Y(t) = \frac{1}{\alpha} N(t) L$$

The Social Planner's Problem III

- Now, we can rewrite the whole problem as:

$$\max_{\{c(t)\}_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t} \frac{C(t)^{1-\sigma}}{1-\sigma} dt$$

$$\text{s.t. } \dot{N}(t) = \eta \left[(1-\alpha) \alpha^{\frac{1}{\alpha-1}} N(t) L - C(t) \right]$$

- The Hamiltonian:

$$\mathcal{H}(N, C, \mu) = \frac{C(t)^{1-\sigma}}{1-\sigma} + \mu(t) \eta \left[(1-\alpha) \alpha^{\frac{1}{\alpha-1}} N(t) L - C(t) \right]$$

with optimality conditions:

$$\begin{aligned} \mathcal{H}_C(N, C, \mu) &= C^*(t)^{-\sigma} - \mu(t) \eta = 0 \\ \mathcal{H}_N(N, C, \mu) &= \mu(t) \eta (1-\alpha) \alpha^{\frac{1}{\alpha-1}} L = \rho \mu(t) - \dot{\mu}(t) \end{aligned}$$

The Social Planner's Problem IV

- Then:

$$\mu(t) = \frac{C^*(t)^{-\sigma}}{\eta}$$

$$\dot{\mu}(t) = -\sigma \frac{C^*(t)^{-\sigma-1}}{\eta} \dot{C}^*(t)$$

$$-\frac{\dot{\mu}(t)}{\mu(t)} = \sigma \frac{\dot{C}^*(t)}{C^*(t)}$$

and we get:

$$\frac{\dot{C}^*(t)}{C^*(t)} = \frac{1}{\sigma}\left[\eta(1-\alpha)\alpha^{\frac{1}{\alpha-1}} L - \rho\right]$$

The Social Planner's Problem V

- In comparison, in equilibrium we get:

$$\frac{\dot{C}(t)}{C(t)} = \frac{1}{\sigma}\left[\eta(1-\alpha)L - \rho\right]$$

Since $\alpha^{\frac{1}{\alpha-1}} > 1$, the growth rate under a social planner is higher than in the equilibrium.

- Intuition.

- Policy remedies.

The Social Planner's Problem VI

- Also

$$g_N^* = \frac{\dot{N}^*(t)}{N^*(t)} = \eta \frac{Z^*(t)}{N^*(t)} = \frac{1}{\sigma}\left(\eta(1-\alpha)\alpha^{\frac{1}{\alpha-1}}L - \rho\right)$$

and we get:

$$Z^*(t) = \frac{1}{\sigma}\left((1-\alpha)\alpha^{\frac{1}{\alpha-1}} - \frac{\rho}{\eta L}\right)\alpha Y^*(t)$$

- Two sources of higher growth:

1. Higher output, $Y^*(t)$.
2. Higher fraction of output on R&D.

Growth with Knowledge Spillovers

- We substitute:

$$\dot{N}(t) = \eta Z(t)$$

by:

$$\dot{N}(t) = \eta N(t) L_R(t)$$

- Labor used in production:

$$L_Y(t) = L - L_R(t)$$

- Rest of the analysis is pretty much the same as before.

Scale Effects

- Note that:

$$\frac{Y(t)}{L} = \frac{1}{\alpha} N(t)$$

  and

$$\frac{\dot{N}(t)}{N(t)} = \frac{1}{\sigma} \left(\eta (1 - \alpha) L - \rho \right)$$

- Scale effect on L.
- Problems:
  1. Size of markets do not seem to matter that much.
  2. With population growth, the economy will explode.
  3. Increases in observed R&D without increases in long-run growth rates.

Growth without Scale Effects

- Modification by Jones (1995).
- Instead of

$$\dot{N}(t) = \eta N(t) L_R(t)$$

we have :

$$\dot{N}(t) = \eta N^{\phi}(t) L_R(t)$$

where $\phi < 1$.

- Growth rate:

$$g = \frac{n}{1-\phi}$$

where n is population growth.