Neoclassical Growth Model

Jesús Fernández-Villaverde

University of Pennsylvania

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Neoclassical Growth Model

- Original contribution of Ramsey (1928). That is why sometimes it is known as the Ramsey model.

- Completed by David Cass (1965) and Tjalling Koopmans (1965). That is why sometimes it is known as the Cass-Koopmans model.

- William Brock and Leonard Mirman (1972) introduced uncertainty.

- Finn Kydland and Edward Prescott (1982) used to create the real business cycle research agenda.
Utility Function

- Representative household with a utility function:

\[ u(c(t)) \]

**Observation**

\( u(c) \) is strictly increasing, concave, twice continuously differentiable with derivatives \( u' \) and \( u'' \), and satisfies Inada conditions:

\[
\begin{align*}
\lim_{c \rightarrow 0} u'(c) &= \infty \\
\lim_{c \rightarrow \infty} u'(c) &= 0
\end{align*}
\]
Dynastic Structure

- Population evolves:
  \[ L(t) = \exp(nt) \]
  with \( L_0 = 1 \).

- Intergenerational altruism.

- Intertemporal utility function:
  \[
  U(0) = \int_0^\infty e^{-(\rho-n)t} u(c(t)) \, dt
  \]
  \( \rho \): subjective discount rate, such that \( \rho > n \).
  \( \rho - n \): “effective” discount rate.
Budget Constraint

- Asset evolution:

\[ \dot{a} = (r - \delta - n) a + w - c \]

- Who owns the capital in the economy? Role of complete markets.

- Modigliani-Miller theorems.

- Arrow securities.
No-Ponzi-Game Condition

- No-Ponzi games.
- Historical examples.

Condition:

$$\lim_{t \to \infty} a(t) \exp \left( - \int_0^t (r - \delta - n) \, ds \right) = 0$$
Production Side

- Cobb-Douglas aggregate production function:
  \[ Y = K^\alpha L^{1-\alpha} \]

- Per capita terms:
  \[ y = k^\alpha \]

- From the first order condition of firm with respect to capital \( k \):
  \[ r = \alpha k^{\alpha-1} \]
  \[ w = k^\alpha - k\alpha k^{\alpha-1} = (1 - \alpha) k^\alpha \]

- Interest rate:
  \[ r - \delta \]
Aggregate Consistency Conditions

- Asset market clearing:
  \[ a = k \]

- Implicitly, labor market clearing.

- Resource constraint:
  \[ \dot{k} = k^\alpha - c - (n + \delta) k \]
A competitive equilibrium is a sequence of per capita allocations \( \{c(t), k(t)\}_{t=0}^{\infty} \) and input prices \( \{r(t), w(t)\}_{t=0}^{\infty} \) such that:

- Given input prices, \( \{r(t), w(t)\}_{t=0}^{\infty} \), the representative household maximizes its utility:

\[
\max_{\{c(t), a(t)\}_{t=0}^{\infty}} \int_0^\infty e^{-(\rho-n)t} u(c(t)) \, dt
\]

s.t. \( \dot{a} = (r - \delta - n) a + w - c \)

\[
\lim_{t \to \infty} a(t) \exp \left( -\int_0^t (r - \delta - n) \, ds \right) = 0
\]

\( a_0 = k_0 \)
Competitive Equilibrium II

- Input prices, \( \{ r(t), w(t) \} \) \( \forall t \geq 0 \), are equal to the marginal productivities:

\[
\begin{align*}
  r(t) &= \alpha k(t)^{\alpha - 1} \\
  w(t) &= (1 - \alpha) k(t)^\alpha
\end{align*}
\]

- Markets clear:

\[
\begin{align*}
  a(t) &= k(t) \\
  \dot{k} &= k(t)^\alpha - c(t) - (n + \delta) k(t)
\end{align*}
\]
We can come back now to the problem of the household.

We build the Hamiltonian:

\[ \mathcal{H}(a, c, \mu) = u(c(t)) + \mu(t)((r(t) - n - \delta)a(t) - w(t) - c(t)) \]

where:

1. \( a(t) \) is the state variable.
2. \( c(t) \) is the control variable.
3. \( \mu(t) \) is the current-value co-state variable.
Necessary Conditions

1. Partial derivative of the Hamiltonian with respect to controls is equal to zero:
\[ \mathcal{H}_c (a, c, \mu) = u' (c(t)) - \mu (t) = 0 \]

2. Partial derivative of the Hamiltonian with respect to states is:
\[ \mathcal{H}_a (a, c, \mu) = \mu (t) (r (t) - n - \delta) = (\rho - n) \mu (t) - \dot{\mu} (t) \]

3. Partial derivative of the Hamiltonian with respect to co-states is:
\[ \mathcal{H}_\mu (a, c, \mu) = (r (t) - n - \delta) a (t) - c (t) = \dot{a} (t) \]

4. Transversality condition:
\[ \lim_{t \to \infty} e^{-\rho t} \mu (t) \dot{a} (t) = 0 \]
From the second condition:

\[
\mu (r - n - \delta) = (\rho - n) \mu - \dot{\mu} \Rightarrow \\
(r - n - \delta) = (\rho - n) - \frac{\dot{\mu}}{\mu} \Rightarrow \\
\frac{\dot{\mu}}{\mu} = -(r - \delta - \rho)
\]

From the first condition:

\[u' (c) = \mu\]

and taking derivatives with respect to time:

\[u'' (c) \dot{c} = \dot{\mu} \Rightarrow \]

\[
\frac{u'' (c)}{u' (c)} \dot{c} = \frac{\dot{\mu}}{\mu} = -(r - \delta - \rho)
\]
Now, we can combine both expression:

\[-\sigma \frac{\dot{c}}{c} = -(r - \delta - \rho)\]

where

\[\sigma = - \frac{u''(c)}{u'(c)} c = \frac{d \log \left( \frac{c(s)}{c(t)} \right)}{d \log \left( \frac{u'(c(s))}{u'(c(t))} \right)}\]

is the (inverse of) elasticity of intertemporal substitution (EIS).

Thus:

\[\frac{\dot{c}}{c} = \frac{1}{\sigma} (r - \delta - \rho)\]

This expression is known as the consumer Euler equation.
In the previous equation we have implicitly assumed that $\sigma$ is a constant.

This will be only true of a class of utility functions.

**Constant Relative Risk Aversion (CRRA):**

$$\frac{c^{1-\sigma} - 1}{1 - \sigma} \quad \text{for} \quad \sigma \neq 1$$

$$\log c \quad \text{for} \quad \sigma = 1$$

(you need to take limits and apply L’Hôpital’s rule).

Why is it called CRRA?
Applying Equilibrium Conditions

- First, note that $r = \alpha k^{\alpha-1}$. Then:
  \[
  \frac{\dot{c}}{c} = \frac{1}{\sigma} (\alpha k^{\alpha-1} - \delta - \rho)
  \]

- Second, $k = a$. Then:
  \[
  \dot{a} = (r - \delta - n) a + w - c \Rightarrow \\
  \dot{k} = (\alpha k^{\alpha-1} - \delta - n) k + w - c \Rightarrow \\
  \dot{k} = k^\alpha - c - (n + \delta) k
  \]

  where in the last step we use the fact that $k^\alpha = \alpha k^{\alpha-1} k + w$. 
System of Differential Equations

We have two differential equations:

\[
\frac{\dot{c}}{c} = \frac{1}{\sigma} (\alpha k^{\alpha-1} - \delta - \rho)
\]

\[
\dot{k} = k^\alpha - c - (n + \delta) k
\]

on two variables, \( k \) and \( c \), plus the transversality condition:

\[
\lim_{t \to \infty} e^{-\rho t} \mu \dot{a} = \lim_{t \to \infty} e^{-\rho t} \mu \dot{k} = 0
\]

How do we solve it?
Steady State

- We search for a steady state where $\dot{c} = \dot{k} = 0$.
- Then:

$$\frac{1}{\sigma} \left( \alpha (k^*)^{\alpha-1} - \delta - \rho \right) = 0$$

$$(k^*)^\alpha - c^* - (n + \delta) k^* = 0$$

- System of two equations on two unknowns $k^*$ and $c^*$ with solution:

$$k^* = \left( \frac{\alpha}{\rho + \delta} \right) \frac{1}{1-\alpha}$$

$$c^* = (k^*)^\alpha - (n + \delta) k^*$$

- Note that EIS does not enter into the steady state. In fact, the form of the utility function is irrelevant!
Transitional Dynamics

- The neoclassical growth model does not have a closed-form solution.

- We can do three things:
  
  1. Use a phase diagram.
  2. Solve an approximated version of the model where we linearize the equations.
  3. Use the computer to approximate numerically the solution.
Linearization I

- We can linearize the system

\[
\frac{\dot{c}}{c} = \frac{1}{\sigma} \left( \alpha k^{\alpha - 1} - \delta - \rho \right)
\]

\[
\dot{k} = k^{\alpha} - c - (n + \delta) k
\]

- We get:

\[
\dot{c} \approx \frac{c^* \alpha (\alpha - 1) (k^*)^{\alpha - 2}}{\sigma} (k - k^*) + \frac{\alpha (k^*)^{\alpha - 1}}{\sigma} - \delta - \rho (c - c^*) 
\]

\[
= \frac{c^*}{\sigma} \left( \alpha (\alpha - 1) (k^*)^{\alpha - 2} \right) (k - k^*)
\]

and

\[
\dot{k} \approx \left( \alpha (k^*)^{\alpha - 1} - n - \delta \right) (k - k^*) - (c - c^*)
\]

\[
= (\rho - n) (k - k^*) - (c - c^*)
\]
The behavior of the linearized system is given by the roots (eigenvalues) $\zeta$ of:

$$\det\begin{pmatrix}
\rho - n - \zeta & -1 \\
\frac{c^*}{\sigma} \left( \alpha (\alpha - 1) (k^*)^{\alpha - 2} \right) & -\zeta
\end{pmatrix}$$

Solving

$$-\zeta (\rho - n - \zeta) + \frac{c^*}{\sigma} \left( \alpha (\alpha - 1) (k^*)^{\alpha - 2} \right) = 0 \Rightarrow$$

$$\zeta^2 - \zeta (\rho - n) + \frac{c^*}{\sigma} \left( \alpha (\alpha - 1) (k^*)^{\alpha - 2} \right) = 0$$
Thus:

\[ \zeta = \frac{(\rho - n) \pm \sqrt{1 - 4 \left( \alpha (\alpha - 1) (k^*)^{\alpha - 2} \right)}}{2} \]

and since \( \alpha (\alpha - 1) < 1 \), we have one positive and one negative eigenvalue \( \Rightarrow \) one stable manifold.

We will call \( \zeta_1 \) the positive eigenvalue and \( \zeta_2 \) the negative one.

With some results in differential equations, we can show:

\[ k = k^* + \eta_1 e^{\zeta_1 t} + \eta_2 e^{\zeta_2 t} \]

\[ k - k^* = \eta_1 e^{\zeta_1 t} + \eta_2 e^{\zeta_2 t} \]

where \( \eta_1 \) and \( \eta_2 \) are arbitrary constants of integration.
It must be that $\eta_1 = 0$. If $\eta_1 > 0$, we will violate the transversality condition and $\eta_1 < 0$ will take $k_t$ to 0.

Then, $\eta_2$ is determined by:

$$\eta_2 = k_0 - k^*$$

Hence:

$$k = \left(1 - e^{\xi_2 t}\right) k^* + e^{\xi_2 t} k_0 \Rightarrow$$

$$k - k^* = \eta_2 e^{\xi_2 t} = (k_0 - k^*) e^{\xi_2 t}$$
Linearization V

Also:

\[ \dot{c} = \frac{c^*}{\sigma} \left( \alpha (\alpha - 1) (k^*)^{\alpha-2} \right) (k - k^*) \]

or

\[ c = \frac{c^*}{\sigma} \left( \alpha (\alpha - 1) (k^*)^{\alpha-2} \right) \frac{\eta_2}{\xi_2} e^{\xi_2 t} + c^* \]

where the constant \( c^* \) ensures that we converge to the steady state.

Since \( y = k^\alpha \), we get:

\[ \log y = \alpha \log \left( k^* + (k_0 - k^*) e^{\xi_2 t} \right) \]
Linearization VI

- Taking time derivatives and making $y = y_0$:

\[
\frac{\dot{y}}{y_0} = \frac{\alpha}{k^* + (k_0 - k^*)} e^{\xi_2 t} \left( (k_0 - k^*) \xi_2 e^{\xi_2 t} \right)
\]

\[= \alpha \xi_2 - \alpha \xi_2 \frac{k^*}{k_0}
\]

\[= \alpha \xi_2 - \alpha \xi_2 \left( \frac{y^*}{y_0} \right)^{\frac{1}{\alpha}}
\]

- This suggests to go to the data and run convergence regressions of the form:

\[g_{i,t,t-1} = b^0 + b^1 \log y_{i,t-1} + \epsilon_{i,t}\]

- We need to be careful about interpreting the coefficient $\hat{b}^1$.

- Where does the error come from?
Selecting Parameter Values

- In general computers cannot approximate the solution for arbitrary parameter values.

- How do we determine the parameter values?

- Two main approaches:
  1. Calibration.
  2. Statistical methods: Methods of Moments, ML, Bayesian.

- Advantages and disadvantages.
Calibration as an Empirical Methodology


- Two sources of information:
  1. Well accepted microeconomic estimates.

- Problems of 1 and 2.

- References:
Calibration of the Standard Model

- Parameters: $n$, $\alpha$, $\delta$, $\rho$, and $\sigma$.
- $n$: population growth in the data.
- $\alpha$: capital income. Proprietor’s income?
- $\delta$: in steady state
  \[ \delta k^* = x^* \implies \delta = \frac{x^*}{k^*} \]
- $\rho$: in steady state
  \[ r^* = \alpha \left( \frac{\alpha}{\rho + \delta} \right)^{\frac{\alpha}{1-\alpha}-1} \]

Then, we take $r^*$ from the data and given $\alpha$ and $\delta$, we find $\rho$
- $\sigma$: from microeconomic evidence.
Running Model in the Computer

- We have the system:

\[
\begin{align*}
\dot{c} &= \frac{1}{\sigma} (\alpha k^{\alpha-1} - \delta - \rho) \\
\dot{k} &= k^\alpha - c - (n + \delta) k 
\end{align*}
\]

- Many methods to solve it.

- A simple one is a shooting algorithm.

- A popular alternative: Runge-Kutta methods.
A Shooting Algorithm

Approximate the system by:

\[
\frac{c(t+\Delta t)-c(t)}{\Delta t} = \frac{1}{\sigma} \left( \alpha k(t)^{\alpha-1} - \delta - \rho \right)
\]

\[
\frac{k(t+\Delta t)-k(t)}{\Delta t} = k(t)^{\alpha} - c(t) - (n+\delta)k(t)
\]

for a small \( \Delta t \).

Steps:

1. Given \( k(0) \), guess \( c(0) \).
2. Trace dynamic system for a long \( t \).
3. Is \( k(t) \rightarrow k^* \)? If yes, we got the right \( c(0) \). If \( k(t) \rightarrow \infty \), raise \( c(0) \), if \( k(t) \rightarrow 0 \), lower \( c(0) \).

Intuition: phase diagram.
We can actually work on our system of differential equations a bit more to show a more intimate relation between the Solow and the Neoclassical growth model.

The savings rate is defined as:

\[
s(t) = 1 - \frac{c(t)}{y(t)}
\]

Now

\[
\frac{d}{dt} \left( \frac{c(t)}{y(t)} \right) = \frac{\ddot{c}}{c} - \frac{\dot{y}}{y} = \frac{\ddot{c}}{c} - \alpha \frac{\dot{k}}{k}
\]
If we substitute in the differential equations for $\frac{\dot{c}}{c}$ and $\dot{k}$:

$$\frac{d (c(t) / y(t))}{dt} = \frac{1}{c(t) / y(t)}$$

$$= \frac{1}{\sigma} (\alpha k^{\alpha-1} - \delta - \rho) - \alpha \left( k^{\alpha-1} - \frac{c}{k} - n - \delta \right)$$

$$= \frac{1}{\sigma} (\alpha k^{\alpha-1} - \delta - \rho) - \alpha \left( k^{\alpha-1} - \frac{c}{k} k^{\alpha-1} - n - \delta \right)$$

$$= -\frac{1}{\sigma} (\delta + \rho) + \alpha (n + \delta) + \left( \frac{1}{\sigma} - 1 + \frac{c}{y} \right) \alpha k^{\alpha-1}$$
Then:

$$\frac{d(c(t)/y(t))}{dt} \cdot \frac{1}{c(t)/y(t)} = \frac{1}{\sigma}(\delta + \rho) + \alpha(n + \delta) + \left(\frac{1}{\sigma} - 1 + \frac{c}{y}\right)\alpha k^{\alpha-1}$$

$$\dot{k} = k^\alpha - c - (n + \delta)k$$

is another system of differential equations.

This system implies that the saving rate is monotone (always increasing, always decreasing, or constant).
We find the locus \( \frac{d(c(t)/y(t))}{dt} = 0: \)

\[
\left( \frac{1}{\sigma} - 1 + \frac{c}{y} \right) \alpha k^{\alpha - 1} = \frac{1}{\sigma} \left( \delta + \rho \right) - \alpha \left( n + \delta \right) \Rightarrow
\]

\[
c = 1 - \frac{1}{\sigma} + \left( \frac{1}{\sigma} \left( \delta + \rho \right) - \alpha \left( n + \delta \right) \right) \frac{1}{\alpha} k^{1-\alpha}
\]

Hence, if

\[
\frac{1}{\sigma} \left( \delta + \rho \right) = \alpha \left( n + \delta \right)
\]

the savings rate is constant, and we are back into the basic Solow model!
The Social Planner’s Problem

- The Social planner’s problem can be written as:

\[
\max \{c(t), k(t)\}_{t=0}^{\infty} \int_0^\infty e^{-(\rho-n)t} u(c(t)) \, dt \\
\text{s.t. } \dot{k} = k(t)^\alpha - c(t) - (n+\delta) k(t) \\
\lim_{t \to \infty} k(t) \exp \left( - \int_0^t (r - \delta - n) \, ds \right) = 0 \\
k_0 \text{ given}
\]

- This problem is very similar to the household’s problem.

- We can also apply the optimality principle to the Hamiltonian:

\[
u(c(t)) + \mu(t) \left( k(t)^\alpha - c(t) - (n+\delta) k(t) \right)
\]
Necessary Conditions

1. Partial derivative of the Hamiltonian with respect to controls is equal to zero:

\[ H_c (a, c, \mu) = u' (c (t)) - \mu (t) = 0 \]

2. Partial derivative of the Hamiltonian with respect to states is:

\[ H_a (a, c, \mu) = \mu (t) \left( \alpha k (t)^{\alpha - 1} - n - \delta \right) = (\rho - n) \mu (t) - \dot{\mu} (t) \]

3. Partial derivative of the Hamiltonian with respect to co-states is:

\[ H_{\mu} (a, c, \mu) = k (t)^{\alpha} - c (t) - (n + \delta) k (t) = \dot{k} (t) \]

4. Transversality condition:

\[ \lim_{t \to \infty} e^{-\rho t} \mu (t) \dot{k} (t) = 0 \]
Comparing the Necessary Conditions

- Following very similar steps than in the problem of the consumer we find:

\[
\frac{\dot{c}}{c} = \frac{1}{\sigma} (\alpha k^{\alpha-1} - \delta - \rho)
\]

\[
\dot{k} = k^\alpha - c - (n + \delta) k
\]

\[
\lim_{t \to \infty} e^{-\rho t} \mu(t) \hat{k}(t) = 0
\]

- From the household problem:

\[
\frac{\dot{c}}{c} = \frac{1}{\sigma} (\alpha k^{\alpha-1} - \delta - \rho)
\]

\[
\dot{k} = k^\alpha - c - (n + \delta) k
\]

\[
\lim_{t \to \infty} e^{-\rho t} \mu(t) \hat{k}(t) = 0
\]

- Both problems have the same necessary conditions!