A Baseline DSGE Model

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1. Introduction

In these notes, we present a baseline sticky prices-sticky wages model. The basic structure of the economy is as follows. A representative household consumes, saves, holds money, supplies labor, and sets its own wages subject to a demand curve and Calvo's pricing. The final output is manufactured by a final good producer, which uses as inputs a continuum of intermediate goods manufactured by monopolistic competitors. The intermediate good producers rent capital and labor to manufacture their good. Also, these intermediate good producers face the constraint that they can only change prices following a Calvo's rule. Finally, there is a monetary authority that fixes the one-period nominal interest rate through open market operations with public debt.

1.1. Households

There is a continuum of households in the economy indexed by j. The households maximizes the following lifetime utility function, which is separable in consumption, c_{jt}, real money balances, m_{jt}/p_t (where p_t is the price level), and hours worked, l_{jt}:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t d_t \left\{ \log\left(c_{jt} - hc_{jt-1}\right) + \upsilon \log\left(\frac{m_{jt}}{p_t}\right) - \varphi_t \psi \frac{l_{jt}^{1+\gamma}}{1+\gamma} \right\}$$

where β is the discount factor, h is the parameter that controls habit persistence, γ is the inverse of Frisch labor supply elasticity, d_t is an intertemporal preference shock with law of motion:

$$\log d_t = \rho_d \log d_{t-1} + \sigma_d \varepsilon_{d,t} \text{ where } \varepsilon_{d,t} \sim \mathcal{N}(0,1),$$

and φ_t is a labor supply shock with law of motion:

$$\log \varphi_t = \rho_\varphi \log \varphi_{t-1} + \sigma_\varphi \varepsilon_{\varphi,t} \text{ where } \varepsilon_{\varphi,t} \sim \mathcal{N}(0,1).$$

Note that the preference shifters are common for all households. Also, we have selected a utility function (log utility in consumption) whose marginal relation of substitution between consumption and leisure is linear in consumption to ensure the presence of a balanced growth path with constant hours.

Households can trade on the whole set of possible Arrow-Debreu commodities, indexed both by the household j (since the household faces idiosyncractic wage-adjustment risk that we will describe below) and by time (to capture aggregate risk). Our notation a_{jt+1} indicates the amount of those securities that pay one unit of consumption in event $\omega_{j,t+1,t}$ purchased by household j at time t at (real) price $q_{jt+1,t}$. To save on notation, we drop the explicit dependence of $q_{jt+1,t}$ and a_{jt+1} on the event when no ambiguity arises. Summing over different individual assets we can price securities contingent only on aggregate states. Households also hold an amount b_{jt} of government bonds that pay a nominal gross interest rate of R_t.

Then, the $j-th$ household's budget constraint is given by:

$$
\begin{aligned}
& c_{jt} + x_{jt} + \frac{m_{jt}}{p_t} + \frac{b_{jt+1}}{p_t} + \int q_{jt+1,t} a_{jt+1} d\omega_{j,t+1,t} \\
= {} & w_{jt} l_{jt} + \left(r_t u_{jt} - \mu_t^{-1} a\left[u_{jt}\right]\right) k_{jt-1} + \frac{m_{jt-1}}{p_t} + R_{t-1}\frac{b_{jt}}{p_t} + a_{jt} + T_t + \digamma_t
\end{aligned}
$$

where w_{jt} is the real wage, r_t the real rental price of capital, $u_{jt} > 0$ the intensity of use of capital, $\mu_t^{-1} a\left[u_{jt}\right]$ is the physical cost of use of capital in resource terms, μ_t is an investment-specific technological shock to be described momentarily, , T_t is a lump-sum transfer, and $\digamma_t$ are the profits of the firms in the economy. We assume that $a\left[1\right] = 0$, a' and $a'' > 0$.

Investment x_{jt} induces a law of motion for capital

$$
k_{jt} = \left(1-\delta\right) k_{jt-1} + \mu_t \left(1 - S\left[\frac{x_{jt}}{x_{jt-1}}\right]\right) x_{jt}
$$

where δ is the depreciation rate and $S\left[\cdot\right]$ is an adjustment cost function such that $S\left[\Lambda_x\right] = 0$, $S'\left[\Lambda_x\right] = 0$, and $S''\left[\cdot\right] > 0$ where Λ_x is the growth rate of investment along the balance growth path. We will determine that growth rate below. Note our capital timing: we index capital by the time its level is decided. The investment-specific technological shock follows an autoregressive process:

$$
\mu_t = \mu_{t-1} \exp\left(\Lambda_\mu + z_{\mu,t}\right) \text{ where } z_{\mu,t} = \sigma_\mu \varepsilon_{\mu,t} \text{ and } \varepsilon_{\mu,t} \sim \mathcal{N}(0,1)
$$

The value of μ_t is also the inverse of the relative price of new capital in consumption terms.

Given our description of the household's problem, the lagrangian function associated with it is:

$$
\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[
\begin{array}{c}
d_t \left\{ \log\left(c_{jt} - hc_{jt-1}\right) + \upsilon \log\left(\frac{m_{jt}}{p_t}\right) - \varphi_t \psi \frac{l_{jt}^{1+\gamma}}{1+\gamma} \right\} \\
-\lambda_{jt} \left\{ \begin{array}{c} c_{jt} + x_{jt} + \frac{m_{jt}}{p_t} + \frac{b_{jt}}{p_t} + \int q_{jt+1,t} a_{jt+1} d\omega_{j,t+1,t} \\ -w_{jt} l_{jt} - \left(r_t u_{jt} - \mu_t^{-1} a\left[u_{jt}\right]\right) k_{jt-1} - \frac{m_{jt-1}}{p_t} - R_{t-1}\frac{b_{jt-1}}{p_t} - a_{jt} - T_t - \digamma_t \end{array} \right\} \\
-Q_{jt} \left\{ k_{jt} - \left(1-\delta\right) k_{jt-1} - \mu_t \left(1 - S\left[\frac{x_{jt}}{x_{jt-1}}\right]\right) x_{jt} \right\}
\end{array}
\right]
$$

where they maximize over c_{jt}, b_{jt}, u_{jt}, k_{jt}, x_{jt}, w_{jt}, l_{jt} and a_{jt+1} (maximization with respect to money holdings comes from the budget constraint), λ_{jt} is the lagrangian multiplier associated with the budget constraint and Q_{jt} the lagrangian multiplier associated with installed capital.

The first order conditions with respect to c_{jt}, b_{jt}, u_{jt}, k_{jt}, and x_{jt} are:

$$d_t \left(c_{jt} - hc_{jt-1}\right)^{-1} - h\beta \mathbb{E}_t d_{t+1} \left(c_{jt+1} - hc_{jt}\right)^{-1} = \lambda_{jt}$$
$$\lambda_{jt} = \beta \mathbb{E}_t \{\lambda_{jt+1} \frac{R_t}{\Pi_{t+1}}\}$$
$$r_t = \mu_t^{-1} a' \left[u_{jt}\right]$$
$$Q_{jt} = \beta \mathbb{E}_t \left\{(1-\delta) Q_{jt+1} + \lambda_{jt+1} \left(r_{t+1} u_{jt+1} - \mu_{t+1}^{-1} a \left[u_{jt+1}\right]\right)\right\}$$
$$-\lambda_{jt} + Q_{jt}\mu_t \left(1 - S\left[\frac{x_{jt}}{x_{jt-1}}\right] - S'\left[\frac{x_{jt}}{x_{jt-1}}\right]\frac{x_{jt}}{x_{jt-1}}\right) + \beta \mathbb{E}_t Q_{jt+1}\mu_{t+1} S'\left[\frac{x_{jt+1}}{x_{jt}}\right]\left(\frac{x_{jt+1}}{x_{jt}}\right)^2 = 0.$$

We do not take first order conditions with respect to Arrow-Debreau securities since, in our environment with complete markets and separable utility in labor, their equilibrium price will be such that their demand ensures that consumption does not depend on idiosyncractic shocks (see Erceg et. al., 2000).

If we define the (marginal) Tobin's Q as $q_{jt} = \frac{Q_{jt}}{\lambda_{jt}}$, (the ratio of the two lagrangian multipliers, or more loosely the value of installed capital in terms of its replacement cost) we get:

$$d_t \left(c_{jt} - hc_{jt-1}\right)^{-1} - h\beta \mathbb{E}_t d_{t+1} \left(c_{jt+1} - hc_{jt}\right)^{-1} = \lambda_{jt}$$
$$\lambda_{jt} = \beta \mathbb{E}_t \{\lambda_{jt+1} \frac{R_t}{\Pi_{t+1}}\}$$
$$r_t = \mu_t^{-1} a' \left[u_{jt}\right]$$
$$q_{jt} = \beta \mathbb{E}_t \left\{\frac{\lambda_{jt+1}}{\lambda_{jt}} \left((1-\delta) q_{jt+1} + r_{t+1} u_{jt+1} - \mu_{t+1}^{-1} a \left[u_{jt+1}\right]\right)\right\}$$
$$1 = q_{jt}\mu_t \left(1 - S\left[\frac{x_{jt}}{x_{jt-1}}\right] - S'\left[\frac{x_{jt}}{x_{jt-1}}\right]\frac{x_{jt}}{x_{jt-1}}\right) + \beta \mathbb{E}_t q_{jt+1}\mu_{t+1} \frac{\lambda_{jt+1}}{\lambda_{jt}} S'\left[\frac{x_{jt+1}}{x_{jt}}\right]\left(\frac{x_{jt+1}}{x_{jt}}\right)^2.$$

The last equation is important. If $S\left[\cdot\right] = 0$ (i.e., there are no adjustment costs), we get:

$$q_{jt} = \frac{1}{\mu_t}$$

i.e., the marginal Tobin's Q is equal to the replacement cost of capital (the relative price of capital). Furthermore, if $\mu_t = 1$, as in the standard neoclassical growth model, $q_{jt} = 1$.

The first order condition with respect to labor and wages is more involved. The labor used by intermediate good producers to be described below is supplied by a representative, competitive firm that hires the labor supplied by each household j. The labor supplier aggregates the differentiated labor of households with the following production function:

$$l_t^d = \left(\int_0^1 l_{jt}^{\frac{\eta-1}{\eta}} dj \right)^{\frac{\eta}{\eta-1}} \tag{1}$$

where $0 \leq \eta < \infty$ is the elasticity of substitution among different types of labor and l_t^d is the aggregate labor demand.[1]

The labor "packer" maximizes profits subject to the production function (1), taking as given all differentiated labor wages w_{jt} and the wage w_t. Consequently, its maximization problem is:

$$\max_{l_{jt}} w_t l_t^d - \int_0^1 w_{jt} l_{jt} dj$$

whose first order conditions are:

$$w_t \frac{\eta}{\eta - 1} \left(\int_0^1 l_{jt}^{\frac{\eta-1}{\eta}} dj \right)^{\frac{\eta}{\eta-1}-1} \frac{\eta - 1}{\eta} l_{jt}^{\frac{\eta-1}{\eta}-1} - w_{jt} = 0 \qquad \forall j$$

Dividing the first order conditions for two types of labor i and j, we get:

$$\frac{w_{it}}{w_{jt}} = \left(\frac{l_{it}}{l_{jt}} \right)^{-\frac{1}{\eta}}$$

or:

$$w_{jt} = \left(\frac{l_{it}}{l_{jt}} \right)^{\frac{1}{\eta}} w_{it}$$

Hence:

$$w_{jt} l_{jt} = w_{it} l_{it}^{\frac{1}{\eta}} l_{jt}^{\frac{\eta-1}{\eta}}$$

and integrating out:

$$\int_0^1 w_{jt} l_{jt} dj = w_{it} l_{it}^{\frac{1}{\eta}} \int_0^1 l_{jt}^{\frac{\eta-1}{\eta}} dj = w_{it} l_{it}^{\frac{1}{\eta}} \left(l_t^d \right)^{\frac{\eta-1}{\eta}}$$

Now, by the zero profits condition implied by perfect competition $w_t l_t^d = \int_0^1 w_{jt} l_{jt} dj$, we get:

$$w_t l_t^d = w_{it} l_{it}^{\frac{1}{\eta}} \left(l_t^d \right)^{\frac{\eta-1}{\eta}} \Rightarrow w_t = w_{it} l_{it}^{\frac{1}{\eta}} \left(l_t^d \right)^{-\frac{1}{\eta}}$$

[1] Often, papers write $\theta = \frac{\eta-1}{\eta}$ and $\frac{1}{\theta} = \frac{\eta}{\eta-1}$. For that reparametrization, $-\infty < \theta \leq 1$, where as $\theta \to -\infty$ (i.e., as $\eta \to 0$), we go to a Leontieff production function, $\theta = 0$ (i.e., $\eta = 1$) , we have a Cobb-Douglas, and $\theta = 1$ (i.e., $\eta \to \infty$), a linear production function.

and, consequently, the input demand functions associated with this problem are:

$$l_{jt} = \left(\frac{w_{jt}}{w_t}\right)^{-\eta} l_t^d \qquad \forall j \tag{2}$$

This functional form shows the effect of elasticity η, on the demand for $j-th$ type of labor.

To find the aggregate wage, we use again the zero profit condition $w_t l_t^d = \int_0^1 w_{jt} l_{jt} dj$ and plug-in the input demand functions:

$$w_t l_t^d = \int_0^1 w_{jt} \left(\frac{w_{jt}}{w_t}\right)^{-\eta} l_t^d dj \Rightarrow w_t^{1-\eta} = \int_0^1 w_{jt}^{1-\eta} dj$$

to deliver:

$$w_t = \left(\int_0^1 w_{jt}^{1-\eta} dj\right)^{\frac{1}{1-\eta}}.$$

Idiosyncratic risks come about because households set their wages following a Calvo's setting. In each period, a fraction $1-\theta_w$ of households can change their wages. All other households can only partially index their wages by past inflation. Indexation is controlled by the parameter $\chi_w \in [0,1]$. This implies that if the household cannot change her wage for τ periods her normalized wage after τ periods is $\prod_{s=1}^{\tau} \frac{\Pi_{t+s-1}^{\chi_w}}{\Pi_{t+s}} w_{jt}$.

Therefore, the relevant part of the lagrangian for the household is then:

$$\max_{w_{jt}} \mathbb{E}_t \sum_{\tau=0}^{\infty} (\beta\theta_w)^{\tau} \left\{ -d_t \varphi_t \psi \frac{l_{jt+\tau}^{1+\gamma}}{1+\gamma} + \lambda_{jt+\tau} \prod_{s=1}^{\tau} \frac{\Pi_{t+s-1}^{\chi_w}}{\Pi_{t+s}} w_{jt} l_{jt+\tau} \right\}$$

subject to

$$l_{jt+\tau} = \left(\prod_{s=1}^{\tau} \frac{\Pi_{t+s-1}^{\chi_w}}{\Pi_{t+s}} \frac{w_{jt}}{w_{t+\tau}}\right)^{-\eta} l_{t+\tau}^d \qquad \forall j$$

or, substituting the demand function (2), we get:

$$\max_{w_{jt}} \mathbb{E}_t \sum_{\tau=0}^{\infty} (\beta\theta_w)^{\tau} \left\{ \begin{array}{c} \lambda_{jt+\tau} \prod_{s=1}^{\tau} \frac{\Pi_{t+s-1}^{\chi_w}}{\Pi_{t+s}} w_{jt} \left(\prod_{s=1}^{\tau} \frac{\Pi_{t+s-1}^{\chi_w}}{\Pi_{t+s}} \frac{w_{jt}}{w_{t+\tau}}\right)^{-\eta} l_{t+\tau}^d \\ -d_{t+\tau} \varphi_{t+\tau} \psi \frac{\left(\prod_{s=1}^{\tau} \frac{\Pi_{t+s-1}^{\chi_w}}{\Pi_{t+s}} \frac{w_{jt}}{w_{t+\tau}}\right)^{-\eta(1+\gamma)}}{1+\gamma} \left(l_{t+\tau}^d\right)^{1+\gamma} \end{array} \right\}$$

which simplifies to

$$\max_{w_{jt}} \mathbb{E}_t \sum_{\tau=0}^{\infty} (\beta\theta_w)^\tau \left\{ \begin{array}{c} \lambda_{jt+\tau} \left(\prod_{s=1}^{\tau} \frac{\Pi_{t+s-1}^{\chi_w}}{\Pi_{t+s}} \frac{w_{jt}}{w_{t+\tau}} \right)^{1-\eta} w_{t+\tau} l_{t+\tau}^d \\ -d_{t+\tau}\varphi_{t+\tau}\psi \frac{\left(\prod_{s=1}^{\tau} \frac{\Pi_{t+s-1}^{\chi_w}}{\Pi_{t+s}} \frac{w_{jt}}{w_{t+\tau}} \right)^{-\eta(1+\gamma)}}{1+\gamma} \left(l_{t+\tau}^d \right)^{1+\gamma} \end{array} \right\}$$

All households set the same wage because complete markets allow them to hedge the risk of the timing of wage change. Hence, we can drop the jth from the choice of wages and λ_{jt}. The first order condition of this problem is:

$$\mathbb{E}_t \sum_{\tau=0}^{\infty} (\beta\theta_w)^\tau \left\{ \begin{array}{c} (1-\eta)\,\lambda_{t+\tau} \left(\prod_{s=1}^{\tau} \frac{\Pi_{t+s-1}^{\chi_w}}{\Pi_{t+s}} \right)^{1-\eta} \left(\frac{w_t^*}{w_{t+\tau}} \right)^{-\eta} l_{t+\tau}^d \\ +\frac{\eta}{w_t^*} d_{t+\tau}\varphi_{t+\tau}\psi \left(\prod_{s=1}^{\tau} \frac{\Pi_{t+s-1}^{\chi_w}}{\Pi_{t+s}} \frac{w_t^*}{w_{t+\tau}} \right)^{-\eta(1+\gamma)} \left(l_{t+\tau}^d \right)^{1+\gamma} \end{array} \right\} = 0$$

or

$$\frac{\eta-1}{\eta} w_t^* \mathbb{E}_t \sum_{\tau=0}^{\infty} (\beta\theta_w)^\tau \lambda_{t+\tau} \left(\prod_{s=1}^{\tau} \frac{\Pi_{t+s-1}^{\chi_w}}{\Pi_{t+s}} \right)^{1-\eta} \left(\frac{w_t^*}{w_{t+\tau}} \right)^{-\eta} l_{t+\tau}^d =$$
$$\mathbb{E}_t \sum_{\tau=0}^{\infty} (\beta\theta_w)^\tau \left(d_{t+\tau}\varphi_{t+\tau}\psi \left(\prod_{s=1}^{\tau} \frac{\Pi_{t+s-1}^{\chi_w}}{\Pi_{t+s}} \frac{w_t^*}{w_{t+\tau}} \right)^{-\eta(1+\gamma)} \left(l_{t+\tau}^d \right)^{1+\gamma} \right)$$

Now, if we define:

$$f_t^1 = \frac{\eta-1}{\eta} w_t^* \mathbb{E}_t \sum_{\tau=0}^{\infty} (\beta\theta_w)^\tau \lambda_{t+\tau} \left(\prod_{s=1}^{\tau} \frac{\Pi_{t+s-1}^{\chi_w}}{\Pi_{t+s}} \right)^{1-\eta} \left(\frac{w_{t+\tau}}{w_t^*} \right)^{\eta} l_{t+\tau}^d$$

and

$$f_t^2 = \mathbb{E}_t \sum_{\tau=0}^{\infty} (\beta\theta_w)^\tau d_{t+\tau}\varphi_{t+\tau}\psi \left(\prod_{s=1}^{\tau} \frac{\Pi_{t+s}^{\chi_w}}{\Pi_{t+s-1}} \right)^{-\eta(1+\gamma)} \left(\frac{w_{t+\tau}}{w_t^*} \right)^{\eta(1+\gamma)} \left(l_{t+\tau}^d \right)^{1+\gamma}$$

we have that the equality $f_t^1 = f_t^2$ is just the previous first order condition. Note that for those sums to be well defined (and, more generally for the maximization problem to have a solution), we need to assume that $(\beta\theta_w)^\tau \lambda_{t+\tau}$ goes to zero faster than $\left(\prod_{s=1}^{\tau} \Pi_{t+s}^{\chi_w} / \Pi_{t+s-1} \right)^{1-\eta}$ goes to infinity in expectation.

One can express f_t^1 and f_t^2 recursively as:

$$f_t^1 = \frac{\eta - 1}{\eta} \left(w_t^*\right)^{1-\eta} \lambda_t w_t^\eta l_t^d + \beta\theta_w \mathbb{E}_t \left(\frac{\Pi_t^{\chi_w}}{\Pi_{t+1}}\right)^{1-\eta} \left(\frac{w_{t+1}^*}{w_t^*}\right)^{\eta-1} f_{t+1}^1$$

and:

$$f_t^2 = \psi d_t \varphi_t \left(\frac{w_t}{w_t^*}\right)^{\eta(1+\gamma)} \left(l_t^d\right)^{1+\gamma} + \beta\theta_w \mathbb{E}_t \left(\frac{\Pi_t^{\chi_w}}{\Pi_{t+1}}\right)^{-\eta(1+\gamma)} \left(\frac{w_{t+1}^*}{w_t^*}\right)^{\eta(1+\gamma)} f_{t+1}^2.$$

Now, since $f_t^1 = f_t^2$, we can define $f_t = f_t^1 = f_t^2$, such that:

$$f_t = \frac{\eta - 1}{\eta} \left(w_t^*\right)^{1-\eta} \lambda_t w_t^\eta l_t^d + \beta\theta_w \mathbb{E}_t \left(\frac{\Pi_t^{\chi_w}}{\Pi_{t+1}}\right)^{1-\eta} \left(\frac{w_{t+1}^*}{w_t^*}\right)^{\eta-1} f_{t+1}$$

and:

$$f_t = \psi d_t \varphi_t \left(\frac{w_t}{w_t^*}\right)^{\eta(1+\gamma)} \left(l_t^d\right)^{1+\gamma} + \beta\theta_w \mathbb{E}_t \left(\frac{\Pi_t^{\chi_w}}{\Pi_{t+1}}\right)^{-\eta(1+\gamma)} \left(\frac{w_{t+1}^*}{w_t^*}\right)^{\eta(1+\gamma)} f_{t+1}.$$

Since we assume complete markets and separable utility in labor (see Erceg et. al., 2000), we consider a symmetric equilibrium where $c_{jt} = c_t$, $u_{jt} = u_t$, $k_{jt-1} = k_t$, $x_{jt} = x_t$, $\lambda_{jt} = \lambda_t$, $q_{jt} = q_t$, and $w_{jt}^* = w_t^*$. Therefore, the first order conditions associated to the consumer's problems are:

$$d_t \left(c_t - hc_{t-1}\right)^{-1} - h\beta\mathbb{E}_t d_{t+1} \left(c_t - hc_t\right)^{-1} = \lambda_t$$

$$\lambda_t = \beta\mathbb{E}_t\{\lambda_{t+1}\frac{R_t}{\Pi_{t+1}}\}$$

$$r_t = \mu_t^{-1} a' \left[u_t\right]$$

$$q_t = \beta\mathbb{E}_t \left\{\frac{\lambda_{t+1}}{\lambda_t}\left[\left(1-\delta\right) q_{t+1} + \left(r_{t+1}u_{t+1} - \mu_{t+1}^{-1} a\left[u_{t+1}\right]\right)\right]\right\}$$

$$1 = q_t\mu_t \left(1 - S\left[\frac{x_t}{x_{t-1}}\right] - S'\left[\frac{x_t}{x_{t-1}}\right]\frac{x_t}{x_{t-1}}\right) + \beta\mathbb{E}_t q_t \mu_{t+1}\frac{\lambda_{t+1}}{\lambda_t} S'\left[\frac{x_{t+1}}{x_t}\right]\left(\frac{x_{t+1}}{x_t}\right)^2.$$

the budget constraint:

$$c_{jt} + x_{jt} + \frac{m_{jt}}{p_t} + \frac{b_{jt+1}}{p_t} + \int q_{jt+1,t} a_{jt+1} d\omega_{j,t+1,t}$$

$$= w_{jt} l_{jt} + \left(r_t u_{jt} - \mu_t^{-1} a\left[u_{jt}\right]\right) k_{jt-1} + \frac{m_{jt-1}}{p_t} + R_{t-1}\frac{b_{jt}}{p_t} + a_{jt} + T_t + F_t$$

and the laws of motion for f_t:

$$f_t = \frac{\eta - 1}{\eta} \left(w_t^*\right)^{1-\eta} \lambda_t w_t^\eta l_t^d + \beta\theta_w \mathbb{E}_t \left(\frac{\Pi_t^{\chi_w}}{\Pi_{t+1}}\right)^{1-\eta} \left(\frac{w_{t+1}^*}{w_t^*}\right)^{\eta-1} f_{t+1}$$

and:

$$f_t = \psi d_t \varphi_t \left(\frac{w_t}{w_t^*} \right)^{\eta(1+\gamma)} \left(l_t^d \right)^{1+\gamma} + \beta \theta_w \mathbb{E}_t \left(\frac{\Pi_t^{\chi_w}}{\Pi_{t+1}} \right)^{-\eta(1+\gamma)} \left(\frac{w_{t+1}^*}{w_t^*} \right)^{\eta(1+\gamma)} f_{t+1}.$$

We need both laws of motion to be able, later, to solve for all the relevant endogenous variables.

Note that using the zero profits condition for the labor supplier, $w_t l_t^d = \int_0^1 w_{jt} l_{jt} dj$ and the net zero supply of all securities, we have that the aggregate budget constraint can be written as:

$$c_t + x_t + \frac{\int_0^1 m_{jt} dj}{p_t} + \frac{\int_0^1 b_{jt+1} dj}{p_t} =$$

$$w_t l_t^d + \left(r_t u_t - \mu_t^{-1} a \left[u_t \right] \right) k_{t-1} + \frac{\int_0^1 m_{jt-1} dj}{p_t} + R_{t-1} \frac{\int_0^1 b_{jt} dj}{p_t} + T_t + F_t$$

Thus, in a symmetric equilibrium, in every period, a fraction $1 - \theta_w$ of households set w_t^* as their wage, while the remaining fraction θ_w partially index their price by past inflation. Consequently, the real wage index evolves:

$$w_t^{1-\eta} = \theta_w \left(\frac{\Pi_{t-1}^{\chi_w}}{\Pi_t} \right)^{1-\eta} w_{t-1}^{1-\eta} + (1 - \theta_w) w_t^{*1-\eta}.$$

i.e., as a geometric average of past real wage and the new optimal wage. This structure is a direct consequence of the memoryless characteristic of Calvo pricing.

1.2. The Final Good Producer

There is one final good is produced using intermediate goods with the following production function:

$$y_t^d = \left(\int_0^1 y_{it}^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}}. \quad (3)$$

where ε is the elasticity of substitution.

Final good producers are perfectly competitive and maximize profits subject to the production function (3), taking as given all intermediate goods prices p_{ti} and the final good price p_t. As a consequence their maximization problem is:

$$\max_{y_{it}} p_t y_t^d - \int_0^1 p_{it} y_{it} di$$

Following the same steps than for the wages, we find the input demand functions associated with this problem are:

$$y_{it} = \left(\frac{p_{it}}{p_t}\right)^{-\varepsilon} y_t^d \qquad \forall i,$$

where y_t^d is the aggregate demand and the zero profit condition $p_t y_t^d = \int_0^1 p_{it} y_{it} di$ to deliver:

$$p_t = \left(\int_0^1 p_{it}^{1-\varepsilon} di\right)^{\frac{1}{1-\varepsilon}}.$$

1.3. Intermediate Good Producers

There is a continuum of intermediate goods producers. Each intermediate good producer i has access to a technology represented by a production function

$$y_{it} = A_t k_{it-1}^{\alpha} \left(l_{it}^d\right)^{1-\alpha} - \phi z_t$$

where k_{it-1} is the capital rented by the firm, l_{it}^d is the amount of the "packed" labor input rented by the firm, and where A_t follows the following process:

$$A_t = A_{t-1} \exp\left(\Lambda_A + z_{A,t}\right) \text{ where } z_{A,t} = \sigma_A \varepsilon_{A,t} \text{ and } \varepsilon_{A,t} \sim \mathcal{N}(0,1)$$

The parameter ϕ, which corresponds to the fixed cost of production, and $z_t = A_t^{\frac{1}{1-\alpha}} \mu_t^{\frac{\alpha}{1-\alpha}}$ guarantee that economic profits are roughly equal to zero in the steady state. We rule out the entry and exit of intermediate good producers.

Since $z_t = A_t^{\frac{1}{1-\alpha}} \mu_t^{\frac{\alpha}{1-\alpha}}$, we have that

$$z_t = z_{t-1} \exp\left(\Lambda_z + z_{z,t}\right) \text{ where } z_{z,t} = \frac{z_{A,t} + \alpha z_{\mu,t}}{1-\alpha} \text{ and } \Lambda_z = \frac{\Lambda_A + \alpha \Lambda_\mu}{1-\alpha}.$$

Intermediate goods producers solve a two-stages problem. In the first stage, taken the input prices w_t and r_t as given, firms rent l_{it}^d and k_{it-1} in perfectly competitive factor markets in order to minimize real cost:

$$\min_{l_{it}^d, k_{it-1}} w_t l_{it}^d + r_t k_{it-1}$$

subject to their supply curve:

$$y_{it} = \begin{cases} A_t k_{it-1}^{\alpha} \left(l_{it}^d\right)^{1-\alpha} - \phi z_t \text{ if } A_t k_{it-1}^{\alpha} \left(l_{it}^d\right)^{1-\alpha} \geq \phi z_t \\ 0 \text{ otherwise} \end{cases}$$

Assuming an interior solution, the first order conditions for this problem are:

$$\begin{aligned} w_t &= \varrho (1-\alpha) A_t k_{it-1}^{\alpha} \left(l_{it}^d\right)^{-\alpha} \\ r_t &= \varrho \alpha A_t k_{it-1}^{\alpha-1} \left(l_{it}^d\right)^{1-\alpha} \end{aligned}$$

where ϱ is the Lagrangian multiplier or:

$$k_{it-1} = \frac{\alpha}{1-\alpha} \frac{w_t}{r_t} l_{it}^d$$

The real cost is then:

$$w_t l_{it}^d + \frac{\alpha}{1-\alpha} w_t l_{it}^d$$

or:

$$\left(\frac{1}{1-\alpha}\right) w_t l_{it}^d$$

Given that the firm has constant returns to scale, we can find the real marginal cost mc_t by setting the level of labor and capital equal to the requirements of producing one unit of good $A_t k_{it-1}^{\alpha} \left(l_{it}^d\right)^{1-\alpha} = 1$ or:

$$A_t k_{it-1}^{\alpha} \left(l_{it}^d\right)^{1-\alpha} = A_t \left(\frac{\alpha}{1-\alpha} \frac{w_t}{r_t} l_{it}^d\right)^{\alpha} \left(l_{it}^d\right)^{1-\alpha} = A_t \left(\frac{\alpha}{1-\alpha} \frac{w_t}{r_t}\right)^{\alpha} l_{it}^d = 1$$

that implies that:

$$l_{it}^d = \frac{\left(\frac{\alpha}{1-\alpha} \frac{w_t}{r_t}\right)^{-\alpha}}{A_t}$$

Then:

$$mc_t = \left(\frac{1}{1-\alpha}\right) w_t \frac{\left(\frac{\alpha}{1-\alpha} \frac{w_t}{r_t}\right)^{-\alpha}}{A_t}$$

that simplifies to:

$$mc_t = \left(\frac{1}{1-\alpha}\right)^{1-\alpha} \left(\frac{1}{\alpha}\right)^{\alpha} \frac{w_t^{1-\alpha} r_t^{\alpha}}{A_t}$$

Note that the marginal cost does not depend on i: all firms receive the same technology shocks and all firms rent inputs at the same price.

In the second stage, intermediate good producers choose the price that maximizes discounted real profits. To do so, they consider that are under the same pricing scheme than households. In each period, a fraction $1-\theta_p$ of firms can change their prices. All other firms can only index their prices by past inflation. Indexation is controlled by the parameter $\chi \in [0,1]$, where $\chi = 0$ is no indexation and $\chi = 1$ is total indexation.

The problem of the firms is then:

$$\max_{p_{it}} \mathbb{E}_t \sum_{\tau=0}^{\infty} (\beta\theta_p)^\tau \frac{\lambda_{t+\tau}}{\lambda_t} \left\{ \left(\prod_{s=1}^{\tau} \Pi_{t+s-1}^{\chi} \frac{p_{it}}{p_{t+\tau}} - mc_{t+\tau} \right) y_{it+\tau} \right\}$$

subject to

$$y_{it+\tau} = \left(\prod_{s=1}^{\tau} \Pi_{t+s-1}^{\chi} \frac{p_{it}}{p_{t+\tau}} \right)^{-\varepsilon} y_{t+\tau}^{d},$$

where the marginal value of a dollar to the household, is treated as exogenous by the firm. Since we have complete markets in securities and utility separable in consumption, this marginal value is constant across households and, consequently, $\lambda_{t+\tau}/\lambda_t$ is the correct valuation on future profits.

Substituting the demand curve in the objective function and the previous expression, we get:

$$\max_{p_{it}} \mathbb{E}_t \sum_{\tau=0}^{\infty} (\beta\theta_p)^\tau \frac{\lambda_{t+\tau}}{\lambda_t} \left\{ \left(\left(\prod_{s=1}^{\tau} \Pi_{t+s-1}^{\chi} \frac{p_{it}}{p_{t+\tau}} \right)^{1-\varepsilon} - \left(\prod_{s=1}^{\tau} \Pi_{t+s-1}^{\chi} \frac{p_{it}}{p_{t+\tau}} \right)^{-\varepsilon} mc_{t+\tau} \right) y_{t+\tau}^{d} \right\}$$

or

$$\max_{p_{it}} \mathbb{E}_t \sum_{\tau=0}^{\infty} (\beta\theta_p)^\tau \frac{\lambda_{t+\tau}}{\lambda_t} \left\{ \left(\left(\prod_{s=1}^{\tau} \frac{\Pi_{t+s-1}^{\chi}}{\Pi_{t+s}} \frac{p_{it}}{p_t} \right)^{1-\varepsilon} - \left(\prod_{s=1}^{\tau} \frac{\Pi_{t+s-1}^{\chi}}{\Pi_{t+s}} \frac{p_{it}}{p_t} \right)^{-\varepsilon} mc_{t+\tau} \right) y_{t+\tau}^{d} \right\}$$

whose solution p_{it}^* implies the first order condition:

$$\mathbb{E}_t \sum_{\tau=0}^{\infty} (\beta\theta_p)^\tau \frac{\lambda_{t+\tau}}{\lambda_t} \left\{ \left((1-\varepsilon) \left(\prod_{s=1}^{\tau} \frac{\Pi_{t+s-1}^{\chi}}{\Pi_{t+s}} \frac{p_{it}^*}{p_t} \right)^{1-\varepsilon} p_{it}^{*-1} + \varepsilon \left(\prod_{s=1}^{\tau} \frac{\Pi_{t+s-1}^{\chi}}{\Pi_{t+s}} \frac{p_{it}^*}{p_t} \right)^{-\varepsilon} p_{it}^{*-1} mc_{t+\tau} \right) y_{t+\tau}^{d} \right\} = 0$$

or

$$\mathbb{E}_t \sum_{\tau=0}^{\infty} (\beta\theta_p)^\tau \lambda_{t+\tau} \left\{ \left((1-\varepsilon) \left(\prod_{s=1}^{\tau} \frac{\Pi_{t+s-1}^{\chi}}{\Pi_{t+s}} \right)^{1-\varepsilon} \frac{p_{it}^*}{p_t} + \varepsilon \left(\prod_{s=1}^{\tau} \frac{\Pi_{t+s-1}^{\chi}}{\Pi_{t+s}} \right)^{-\varepsilon} mc_{t+\tau} \right) y_{t+\tau}^{d} \right\} = 0$$

where, in the second step, we have dropped irrelevant constants and we have used the fact that we are in a symmetric equilibrium. Note how this expression nests the usual result in the fully flexible prices case $\theta_p = 0$:

$$p_{it}^* = \frac{\varepsilon}{\varepsilon - 1} p_t mc_{t+\tau}$$

i.e., the price is equal to a mark-up over the nominal marginal cost.

Since we only consider a symmetric equilibrium, we can write that $p_{it}^* = p_t^*$ and:

$$\mathbb{E}_t \sum_{\tau=0}^{\infty} (\beta\theta_p)^\tau \lambda_{t+\tau} \left\{ \left((1-\varepsilon) \left(\prod_{s=1}^{\tau} \frac{\Pi_{t+s-1}^\chi}{\Pi_{t+s}} \right)^{1-\varepsilon} \frac{p_t^*}{p_t} + \varepsilon \left(\prod_{s=1}^{\tau} \frac{\Pi_{t+s-1}^\chi}{\Pi_{t+s}} \right)^{-\varepsilon} mc_{t+\tau} \right) y_{t+\tau}^d \right\} = 0$$

To express the previous first order condition recursively, we define:

$$g_t^1 = \mathbb{E}_t \sum_{\tau=0}^{\infty} (\beta\theta_p)^\tau \lambda_{t+\tau} \left(\prod_{s=1}^{\tau} \frac{\Pi_{t+s-1}^\chi}{\Pi_{t+s}} \right)^{-\varepsilon} mc_{t+\tau} y_{t+\tau}^d$$

and

$$g_t^2 = \mathbb{E}_t \sum_{\tau=0}^{\infty} (\beta\theta_p)^\tau \lambda_{t+\tau} \left(\prod_{s=1}^{\tau} \frac{\Pi_{t+s-1}^\chi}{\Pi_{t+s}} \right)^{1-\varepsilon} \frac{p_t^*}{p_t} y_{t+\tau}^d$$

and then the first order condition is $\varepsilon g_t^1 = (\varepsilon - 1) g_t^2$.

As it was the case for f's that we found in the household problem, we need $(\beta\theta_p)^\tau \lambda_{t+\tau}$ to go to zero sufficiently fast in relation with the rate of inflation for g_t^1 and g_t^2 to be well defined and stationary.

Then, we can write the g's recursively:

$$g_t^1 = \lambda_t mc_t y_t^d + \beta\theta_p \mathbb{E}_t \left(\frac{\Pi_t^\chi}{\Pi_{t+1}} \right)^{-\varepsilon} g_{t+1}^1$$

and

$$g_t^2 = \lambda_t \Pi_t^* y_t^d + \beta\theta_p \mathbb{E}_t \left(\frac{\Pi_t^\chi}{\Pi_{t+1}} \right)^{1-\varepsilon} \left(\frac{\Pi_t^*}{\Pi_{t+1}^*} \right) g_{t+1}^2$$

where:

$$\Pi_t^* = \frac{p_t^*}{p_t}.$$

Given Calvo's pricing, the price index evolves:

$$p_t^{1-\varepsilon} = \theta_p \left(\Pi_{t-1}^\chi \right)^{1-\varepsilon} p_{t-1}^{1-\varepsilon} + (1-\theta_p) p_t^{*1-\varepsilon}$$

or, dividing by $p_t^{1-\varepsilon}$,

$$1 = \theta_p \left(\frac{\Pi_{t-1}^\chi}{\Pi_t} \right)^{1-\varepsilon} + (1-\theta_p) \Pi_t^{*1-\varepsilon}$$

1.4. The Government Problem

The government sets the nominal interest rates according to the Taylor rule:

$$\frac{R_t}{R} = \left(\frac{R_{t-1}}{R}\right)^{\gamma_R} \left(\left(\frac{\Pi_t}{\Pi}\right)^{\gamma_\Pi} \left(\frac{\frac{y_t^d}{y_{t-1}^d}}{\Lambda_{y^d}}\right)^{\gamma_y}\right)^{1-\gamma_R} \exp\left(m_t\right)$$

through open market operations that are financed through lump-sum transfers T_t. Those transfers insure that the deficit are equal to zero:

$$T_t = \frac{\int_0^1 m_{jt} dj}{p_t} - \frac{\int_0^1 m_{jt-1} dj}{p_t} + \frac{\int_0^1 b_{jt+1} dj}{p_t} - R_{t-1} \frac{\int_0^1 b_{jt} dj}{p_t}$$

The variables Π represents the target level of inflation (equal to inflation in the steady state), R steady state nominal gross return of capital, and Λ_{y^d} the steady state gross growth rate of y_t^d. The term m_t is a random shock to monetary policy that follows $m_t = \sigma_m \varepsilon_{mt}$ where ε_{mt} is distributed according to $\mathcal{N}(0,1)$. The presence of the previous period interest rate, R_{t-1}, is justified because we want to match the smooth profile of the interest rate over time observed in U.S. data. Note that R is beyond the control of the monetary authority, since it is equal to the steady state real gross returns of capital plus the target level of inflation.

Applying the definition of transfers above, the aggregated budget constraint of households is equal to:

$$c_t + x_t = w_t l_t^d + \left(r_t u_t - \mu_t^{-1} a\left[u_t\right]\right) k_{t-1} + \digamma_t.$$

1.5. Aggregation

First, we derive an expression for aggregate demand:

$$y_t^d = c_t + x_t + \mu_t^{-1} a\left[u_t\right] k_{t-1}$$

With this value, the demand for each intermediate good producer is

$$y_{it} = \left(c_t + x_t + \mu_t^{-1} a\left[u_t\right] k_{t-1}\right) \left(\frac{p_{it}}{p_t}\right)^{-\varepsilon} \qquad \forall i,$$

and using the production function is:

$$A_t k_{it-1}^{\alpha} \left(l_{it}^d\right)^{1-\alpha} - \phi z_t = \left(c_t + x_t + \mu_t^{-1} a\left[u_t\right] k_{t-1}\right) \left(\frac{p_{it}}{p_t}\right)^{-\varepsilon}$$

Since all the firms have the same optimal capital-labor ratio:

$$\frac{k_{it-1}}{l_{it}^d} = \frac{\alpha}{1-\alpha}\frac{w_t}{r_t}$$

and by market clearing

$$\int_0^1 l_{it}^d di = l_t^d$$

and

$$\int_0^1 k_{it-1} di = u_t k_{t-1}.$$

it must be the case that:

$$\frac{k_{it-1}}{l_{it}^d} = \frac{u_t k_{t-1}}{l_t^d}.$$

Then:

$$A_t k_{it-1}^{\alpha} \left(l_{it}^d\right)^{1-\alpha} = A_t \left(\frac{k_{it-1}}{l_{it}^d}\right)^{\alpha} \left(l_{it}^d\right) = A_t \left(\frac{u_t k_{t-1}}{l_t^d}\right)^{\alpha} l_{it}^d$$

Integrating out

$$\int_0^1 A_t \left(\frac{u_t k_{t-1}}{l_t^d}\right)^{\alpha} l_{it}^d di = A_t \left(\frac{u_t k_{t-1}}{l_t^d}\right)^{\alpha} \int_0^1 l_{it}^d di = A_t \left(u_t k_{t-1}\right)^{\alpha} \left(l_t^d\right)^{1-\alpha}$$

and we have

$$A_t \left(u_t k_{t-1}\right)^{\alpha} \left(l_t^d\right)^{1-\alpha} - \phi z_t = \left(c_t + x_t + \mu_t^{-1} a\left[u_t\right] k_{t-1}\right) \int_0^1 \left(\frac{p_{it}}{p_t}\right)^{-\varepsilon} di$$

Define $v_t^p = \int_0^1 \left(\frac{p_{it}}{p_t}\right)^{-\varepsilon} di$. By the properties of the index under Calvo's pricing

$$v_t^p = \theta_p \left(\frac{\Pi_{t-1}^{\chi}}{\Pi_t}\right)^{-\varepsilon} v_{t-1}^p + \left(1-\theta_p\right) \Pi_t^{*-\varepsilon}.$$

we get:

$$c_t + x_t + \mu_t^{-1} a\left[u_t\right] k_{t-1} = \frac{A_t \left(u_t k_{t-1}\right)^{\alpha} \left(l_t^d\right)^{1-\alpha} - \phi z_t}{v_t^p}$$

Now, we derive an expression for aggregate labor demand. We know that

$$l_{jt} = \left(\frac{w_{jt}}{w_t}\right)^{-\eta} l_t^d$$

If we integrate over all households j, we get

$$\int_0^1 l_{jt} dj = l_t = \int_0^1 \left(\frac{w_{jt}}{w_t}\right)^{-\eta} dj l_t^d$$

where l_t is the aggregate labor supply of households.

Define

$$v_t^w = \int_0^1 \left(\frac{w_{jt}}{w_t}\right)^{-\eta} dj.$$

Hence

$$l_t^d = \frac{1}{v_t^w} l_t.$$

Also, as before:

$$v_t^w = \theta_w \left(\frac{w_{t-1}}{w_t} \frac{\Pi_{t-1}^{\chi_w}}{\Pi_t}\right)^{-\eta} v_{t-1}^w + (1-\theta_w) \left(\Pi_t^{w*}\right)^{-\eta}.$$

2. Equilibrium

A definition of equilibrium in this economy is standard and the symmetric equilibrium policy functions are determined by the following equations:

- The first order conditions of the household

$$d_t \left(c_t - h c_{t-1}\right)^{-1} - h\beta \mathbb{E}_t d_{t+1} \left(c_{t+1} - h c_t\right)^{-1} = \lambda_t$$

$$\lambda_t = \beta \mathbb{E}_t \{\lambda_{t+1} \frac{R_t}{\Pi_{t+1}}\}$$

$$r_t = \mu_t^{-1} a' \left[u_t\right]$$

$$q_t = \beta \mathbb{E}_t \left\{\frac{\lambda_{t+1}}{\lambda_t} \left(\left(1-\delta\right) q_{t+1} + r_{t+1} u_{t+1} - \mu_{t+1}^{-1} a \left[u_{t+1}\right]\right)\right\}$$

$$1 = q_t \mu_t \left(1 - S\left[\frac{x_t}{x_{t-1}}\right] - S'\left[\frac{x_t}{x_{t-1}}\right] \frac{x_t}{x_{t-1}}\right) + \beta \mathbb{E}_t q_{t+1} \mu_{t+1} \frac{\lambda_{t+1}}{\lambda_t} S'\left[\frac{x_{t+1}}{x_t}\right] \left(\frac{x_{t+1}}{x_t}\right)^2$$

$$f_t = \frac{\eta - 1}{\eta} \left(w_t^*\right)^{1-\eta} \lambda_t w_t^\eta l_t^d + \beta \theta_w \mathbb{E}_t \left(\frac{\Pi_t^{\chi_w}}{\Pi_{t+1}}\right)^{1-\eta} \left(\frac{w_{t+1}^*}{w_t^*}\right)^{\eta-1} f_{t+1}$$

$$f_t = \psi d_t \varphi_t \left(\Pi_t^{*w}\right)^{-\eta(1+\gamma)} \left(l_t^d\right)^{1+\gamma} + \beta \theta_w \mathbb{E}_t \left(\frac{\Pi_t^{\chi_w}}{\Pi_{t+1}}\right)^{-\eta(1+\gamma)} \left(\frac{w_{t+1}^*}{w_t^*}\right)^{\eta(1+\gamma)} f_{t+1}$$

- The firms that can change prices set them to satisfy:

$$g_t^1 = \lambda_t mc_t y_t^d + \beta\theta_p \mathbb{E}_t \left(\frac{\Pi_t^\chi}{\Pi_{t+1}}\right)^{-\varepsilon} g_{t+1}^1$$
$$g_t^2 = \lambda_t \Pi_t^* y_t^d + \beta\theta_p \mathbb{E}_t \left(\frac{\Pi_t^\chi}{\Pi_{t+1}}\right)^{1-\varepsilon} \left(\frac{\Pi_t^*}{\Pi_{t+1}^*}\right) g_{t+1}^2$$
$$\varepsilon g_t^1 = (\varepsilon - 1) g_t^2$$

where they rent inputs to satisfy their static minimization problem:

$$\frac{u_t k_{t-1}}{l_t^d} = \frac{\alpha}{1-\alpha}\frac{w_t}{r_t}$$
$$mc_t = \left(\frac{1}{1-\alpha}\right)^{1-\alpha} \left(\frac{1}{\alpha}\right)^{\alpha} \frac{w_t^{1-\alpha} r_t^{\alpha}}{A_t}$$

- The wages evolve as:

$$1 = \theta_w \left(\frac{\Pi_{t-1}^{\chi_w}}{\Pi_t}\right)^{1-\eta} \left(\frac{w_{t-1}}{w_t}\right)^{1-\eta} + (1-\theta_w)\left(\Pi_t^{*w}\right)^{1-\eta}$$

and the price level evolves:

$$1 = \theta_p \left(\frac{\Pi_{t-1}^{\chi}}{\Pi_t}\right)^{1-\varepsilon} + (1-\theta_p)\,\Pi_t^{*1-\varepsilon}$$

- Government follow its Taylor rule

$$\frac{R_t}{R} = \left(\frac{R_{t-1}}{R}\right)^{\gamma_R} \left(\left(\frac{\Pi_t}{\Pi}\right)^{\gamma_\Pi} \left(\frac{\frac{y_t^d}{y_{t-1}^d}}{\Lambda_{y^d}}\right)^{\gamma_y}\right)^{1-\gamma_R} \exp(m_t)$$

- Markets clear:

$$y_t^d = \frac{A_t (u_t k_{t-1})^{\alpha} \left(l_t^d\right)^{1-\alpha} - \phi z_t}{v_t^p}$$
$$y_t^d = c_t + x_t + \mu_t^{-1} a\left[u_t\right] k_{t-1}$$

where

$$
\begin{aligned}
l_t &= v_t^w l_t^d \\
v_t^p &= \theta_p \left(\frac{\Pi_{t-1}^{\chi}}{\Pi_t}\right)^{-\varepsilon} v_{t-1}^p + (1-\theta_p)\,\Pi_t^{*-\varepsilon} \\
v_t^w &= \theta_w \left(\frac{w_{t-1}}{w_t}\frac{\Pi_{t-1}^{\chi_w}}{\Pi_t}\right)^{-\eta} v_{t-1}^w + (1-\theta_w)\left(\Pi_t^{*w}\right)^{-\eta}
\end{aligned}
$$

and

$$
k_t - (1-\delta)\,k_{t-1} - \mu_t\left(1 - S\left[\frac{x_t}{x_{t-1}}\right]\right) x_t = 0.
$$

3. Stationary Equilibrium

Since we have growth in this model induced by technological change, most of the variables are growing in average. To solve the model, we need to make variables stationary.

3.1. Manipulating Equilibrium Conditions

First, we work on the first order conditions of the household

$$
d_t\left(\frac{c_t}{z_t} - h\frac{c_{t-1}}{z_{t-1}}\frac{z_{t-1}}{z_t}\right)^{-1} - h\beta\mathbb{E}_t d_{t+1}\left(\frac{c_{t+1}}{z_{t+1}}\frac{z_{t+1}}{z_t} - h\frac{c_t}{z_t}\right)^{-1} = \lambda_t z_t
$$

$$
\lambda_t z_t = \beta\mathbb{E}_t\{\lambda_{t+1} z_{t+1}\frac{z_t}{z_{t+1}}\frac{R_t}{\Pi_{t+1}}\}
$$

$$
\mu_t r_t = a'\left[u_t\right]
$$

$$
q_t\mu_t = \beta\mathbb{E}_t\left\{\frac{\lambda_{t+1}z_{t+1}}{\lambda_t z_t}\frac{z_t}{z_{t+1}}\frac{\mu_t}{\mu_{t+1}}\left[(1-\delta)\,q_{t+1}\mu_{t+1} + \mu_{t+1}r_{t+1}u_{t+1} - a\left(u_{t+1}\right)\right]\right\}
$$

$$
\begin{aligned}
1 = q_t\mu_t &\left(1 - S\left[\frac{\frac{x_t}{z_t}}{\frac{x_{t-1}}{z_{t-1}}}\frac{z_t}{z_{t-1}}\right] - S'\left[\frac{\frac{x_t}{z_t}}{\frac{x_{t-1}}{z_{t-1}}}\frac{z_t}{z_{t-1}}\right]\frac{\frac{x_t}{z_t}}{\frac{x_{t-1}}{z_{t-1}}}\frac{z_t}{z_{t-1}}\right) \\
&+\beta\mathbb{E}_t q_{t+1}\mu_{t+1}\frac{\lambda_{t+1}}{\lambda_t}S'\left[\frac{\frac{x_{t+1}}{z_{t+1}}}{\frac{x_t}{z_t}}\frac{z_{t+1}}{z_t}\right]\left(\frac{\frac{x_{t+1}}{z_{t+1}}}{\frac{x_t}{z_t}}\frac{z_{t+1}}{z_t}\right)^2
\end{aligned}
$$

$$
f_t = \frac{\eta-1}{\eta}\left(\frac{w_t^*}{z_t}\right)^{1-\eta}\lambda_t z_t\left(\frac{w_t}{z_t}\right)^{\eta} l_t^d + \beta\theta_w\mathbb{E}_t\left(\frac{\Pi_t^{\chi_w}}{\Pi_{t+1}}\right)^{1-\eta}\left(\frac{\frac{w_{t+1}^*}{z_{t+1}}z_{t+1}}{\frac{w_t^*}{z_t}z_t}\right)^{\eta-1} f_{t+1}
$$

$$
f_t = \psi d_t\varphi_t\left(\Pi_t^{*w}\right)^{-\eta(1+\gamma)}\left(l_t^d\right)^{1+\gamma} + \beta\theta_w\mathbb{E}_t\left(\frac{\Pi_t^{\chi_w}}{\Pi_{t+1}}\right)^{-\eta(1+\gamma)}\left(\frac{\frac{w_{t+1}^*}{z_{t+1}}z_{t+1}}{\frac{w_t^*}{z_t}z_t}\right)^{\eta(1+\gamma)} f_{t+1}
$$

The firms that can change prices set them to satisfy:

$$g_t^1 = \lambda_t z_t mc_t \frac{y_t^d}{z_t} + \beta\theta_p \mathbb{E}_t \left(\frac{\Pi_t^{\chi}}{\Pi_{t+1}}\right)^{-\varepsilon} g_{t+1}^1$$

$$g_t^2 = \lambda_t z_t \Pi_t^* \frac{y_t^d}{z_t} + \beta\theta_p \mathbb{E}_t \left(\frac{\Pi_t^{\chi}}{\Pi_{t+1}}\right)^{1-\varepsilon} \left(\frac{\Pi_t^*}{\Pi_{t+1}^*}\right) g_{t+1}^2$$

$$\varepsilon g_t^1 = (\varepsilon - 1) g_t^2$$

where they rent inputs to satisfy their static minimization problem:

$$\frac{u_t}{l_t^d}\frac{k_{t-1}}{z_{t-1}\mu_{t-1}} = \frac{\alpha}{1-\alpha}\frac{w_t}{z_t}\frac{1}{r_t\mu_t}\frac{z_t}{z_{t-1}}\frac{\mu_t}{\mu_{t-1}}$$

$$mc_t = \left(\frac{1}{1-\alpha}\right)^{1-\alpha}\left(\frac{1}{\alpha}\right)^{\alpha}\frac{\left(\frac{w_t}{z_t}\right)^{1-\alpha}(r_t\mu_t)^{\alpha}\frac{z_t^{1-\alpha}}{\mu_t^{\alpha}}}{A_t}$$

The wages evolve as:

$$1 = \theta_w \left(\frac{\Pi_{t-1}^{\chi_w}}{\Pi_t}\right)^{1-\eta}\left(\frac{\frac{w_{t-1}}{z_{t-1}}}{\frac{w_t}{z_t}}\frac{z_{t-1}}{z_t}\right)^{1-\eta} + (1-\theta_w)\left(\Pi_t^{*w}\right)^{1-\eta}$$

and the price level evolves:

$$1 = \theta_p \left(\frac{\Pi_{t-1}^{\chi}}{\Pi_t}\right)^{1-\varepsilon} + (1-\theta_p)\,\Pi_t^{*1-\varepsilon}$$

Government follow its Taylor rule

$$\frac{R_t}{R} = \left(\frac{R_{t-1}}{R}\right)^{\gamma_R}\left(\left(\frac{\Pi_t}{\Pi}\right)^{\gamma_\Pi}\left(\frac{\frac{\frac{y_t^d}{z_t}}{\frac{y_{t-1}^d}{z_{t-1}}}\frac{z_t}{z_{t-1}}}{\Lambda_{y^d}}\right)^{\gamma_y}\right)^{1-\gamma_R}\exp\left(m_t\right)$$

Markets clear:

$$\frac{y_t^d}{z_t} = \frac{\mu_{t-1}^{\alpha} z_{t-1}^{\alpha}\frac{A_t}{z_t}\left(u_t\frac{k_{t-1}}{z_{t-1}\mu_{t-1}}\right)^{\alpha}\left(l_t^d\right)^{1-\alpha} - \phi}{v_t^p}$$

but since $\mu_{t-1}^{\alpha} z_{t-1}^{\alpha} = \frac{z_{t-1}}{A_{t-1}}$, we have

$$\frac{y_t^d}{z_t} = \frac{\frac{z_{t-1}}{A_{t-1}}\frac{A_t}{z_t}\left(u_t\frac{k_{t-1}}{z_{t-1}\mu_{t-1}}\right)^{\alpha}\left(l_t^d\right)^{1-\alpha} - \phi}{v_t^p}$$

$$\frac{y_t^d}{z_t} = \frac{c_t}{z_t} + \frac{x_t}{z_t} + \frac{z_{t-1}}{z_t}\frac{\mu_{t-1}}{\mu_t} a\left[u_t\right] \frac{k_{t-1}}{z_{t-1}\mu_{t-1}}$$

where

$$\begin{aligned} l_t &= v_t^w l_t^d \\ v_t^p &= \theta_p \left(\frac{\Pi_{t-1}^{\chi}}{\Pi_t}\right)^{-\varepsilon} v_{t-1}^p + (1-\theta_p)\,\Pi_t^{*-\varepsilon} \\ v_t^w &= \theta_w \left(\frac{\frac{w_{t-1}}{z_{t-1}} z_{t-1}}{\frac{w_t}{z_t} z_t}\frac{\Pi_{t-1}^{\chi_w}}{\Pi_t}\right)^{-\eta} v_{t-1}^w + (1-\theta_w)\left(\Pi_t^{*w}\right)^{-\eta} \end{aligned}$$

and

$$\frac{k_t}{z_t\mu_t}\frac{z_t\mu_t}{z_{t-1}\mu_{t-1}} - (1-\delta)\frac{k_{t-1}}{z_{t-1}\mu_{t-1}} - \frac{\mu_t}{\mu_{t-1}}\frac{z_t}{z_{t-1}}\left(1 - S\left[\frac{\frac{x_t}{z_t} z_t}{\frac{x_{t-1}}{z_{t-1}} z_{t-1}}\right]\right)\frac{x_t}{z_t} = 0.$$

3.2. Change of Variables

We now redefine that variables to obtain a system on stationary variables that we can easily manipulate. Hence, we define $\widetilde{c}_t = \frac{c_t}{z_t}$, $\widetilde{\lambda}_t = \lambda_t z_t$, $\widetilde{r}_t = r_t\mu_t$, $\widetilde{q}_t = q_t\mu_t$, $\widetilde{x}_t = \frac{x_t}{z_t}$, $\widetilde{w}_t = \frac{w_t}{z_t}$, $\widetilde{w}_t^* = \frac{w_t^*}{z_t}$, $\widetilde{k}_t = \frac{k_t}{z_t\mu_t}$, and $\widetilde{y}_t^d = \frac{y_t^d}{z_t}$.

Then, the set of equilibrium conditions are:

- The first order conditions of the household:

$$d_t\left(\widetilde{c}_t - h\widetilde{c}_{t-1}\frac{z_{t-1}}{z_t}\right)^{-1} - h\beta\mathbb{E}_t d_{t+1}\left(\widetilde{c}_{t+1}\frac{z_{t+1}}{z_t} - h\widetilde{c}_t\right)^{-1} = \widetilde{\lambda}_t$$

$$\widetilde{\lambda}_t = \beta\mathbb{E}_t\{\widetilde{\lambda}_{t+1}\frac{z_t}{z_{t+1}}\frac{R_t}{\Pi_{t+1}}\}$$

$$\widetilde{r}_t = a'\left[u_t\right]$$

$$\widetilde{q}_t = \beta\mathbb{E}_t\left\{\frac{\widetilde{\lambda}_{t+1}}{\widetilde{\lambda}_t}\frac{z_t}{z_{t+1}}\frac{\mu_t}{\mu_{t+1}}\left((1-\delta)\,\widetilde{q}_{t+1} + \widetilde{r}_{t+1}u_{t+1} - a\left(u_{t+1}\right)\right)\right\}$$

$$1 = \widetilde{q}_t\left(1 - S\left[\frac{\widetilde{x}_t}{\widetilde{x}_{t-1}}\frac{z_t}{z_{t-1}}\right] - S'\left[\frac{\widetilde{x}_t}{\widetilde{x}_{t-1}}\frac{z_t}{z_{t-1}}\right]\frac{\widetilde{x}_t}{\widetilde{x}_{t-1}}\frac{z_t}{z_{t-1}}\right)$$

$$+\beta\mathbb{E}_t\widetilde{q}_{t+1}\frac{\widetilde{\lambda}_{t+1}}{\widetilde{\lambda}_t}\frac{z_t}{z_{t+1}}S'\left[\frac{\widetilde{x}_{t+1}}{\widetilde{x}_t}\frac{z_{t+1}}{z_t}\right]\left(\frac{\widetilde{x}_{t+1}}{\widetilde{x}_t}\frac{z_{t+1}}{z_t}\right)^2$$

$$f_t = \frac{\eta-1}{\eta}\left(\widetilde{w}_t^*\right)^{1-\eta}\widetilde{\lambda}_t\left(\widetilde{w}_t\right)^{\eta} l_t^d + \beta\theta_w\mathbb{E}_t\left(\frac{\Pi_t^{\chi_w}}{\Pi_{t+1}}\right)^{1-\eta}\left(\frac{\widetilde{w}_{t+1}^*}{\widetilde{w}_t^*}\frac{z_{t+1}}{z_t}\right)^{\eta-1} f_{t+1}$$

$$f_t = \psi d_t\varphi_t\left(\Pi_t^{*w}\right)^{-\eta(1+\gamma)}\left(l_t^d\right)^{1+\gamma} + \beta\theta_w\mathbb{E}_t\left(\frac{\Pi_t^{\chi_w}}{\Pi_{t+1}}\right)^{-\eta(1+\gamma)}\left(\frac{\widetilde{w}_{t+1}^*}{\widetilde{w}_t^*}\frac{z_{t+1}}{z_t}\right)^{\eta(1+\gamma)} f_{t+1}$$

- The firms that can change prices set them to satisfy:

$$g_t^1 = \widetilde{\lambda}_t mc_t \widetilde{y}_t^d + \beta\theta_p \mathbb{E}_t \left(\frac{\Pi_t^\chi}{\Pi_{t+1}}\right)^{-\varepsilon} g_{t+1}^1$$
$$g_t^2 = \widetilde{\lambda}_t \Pi_t^* \widetilde{y}_t^d + \beta\theta_p \mathbb{E}_t \left(\frac{\Pi_t^\chi}{\Pi_{t+1}}\right)^{1-\varepsilon} \left(\frac{\Pi_t^*}{\Pi_{t+1}^*}\right) g_{t+1}^2$$
$$\varepsilon g_t^1 = (\varepsilon - 1) g_t^2$$

where they rent inputs to satisfy their static minimization problem:

$$\frac{u_t \widetilde{k}_{t-1}}{l_t^d} = \frac{\alpha}{1-\alpha} \frac{\widetilde{w}_t}{\widetilde{r}_t} \frac{z_t}{z_{t-1}} \frac{\mu_t}{\mu_{t-1}}$$
$$mc_t = \left(\frac{1}{1-\alpha}\right)^{1-\alpha} \left(\frac{1}{\alpha}\right)^{\alpha} (\widetilde{w}_t)^{1-\alpha} \widetilde{r}_t^{\alpha}$$

- The wages evolve as:

$$1 = \theta_w \left(\frac{\Pi_{t-1}^{\chi_w}}{\Pi_t}\right)^{1-\eta} \left(\frac{\widetilde{w}_{t-1}}{\widetilde{w}_t} \frac{z_{t-1}}{z_t}\right)^{1-\eta} + (1-\theta_w) \left(\Pi_t^{*w}\right)^{1-\eta}$$

and the price level evolves:

$$1 = \theta_p \left(\frac{\Pi_{t-1}^{\chi}}{\Pi_t}\right)^{1-\varepsilon} + (1-\theta_p) \Pi_t^{*1-\varepsilon}$$

- Government follow its Taylor rule:

$$\frac{R_t}{R} = \left(\frac{R_{t-1}}{R}\right)^{\gamma_R} \left(\left(\frac{\Pi_t}{\Pi}\right)^{\gamma_\Pi} \left(\frac{\frac{\widetilde{y}_t^d}{\widetilde{y}_{t-1}^d} \frac{z_t}{z_{t-1}}}{\Lambda_{y^d}}\right)^{\gamma_y}\right)^{1-\gamma_R} \exp(m_t)$$

- Markets clear:

$$\widetilde{y}_t^d = \widetilde{c}_t + \widetilde{x}_t + \frac{z_{t-1}}{z_t} \frac{\mu_{t-1}}{\mu_t} a [u_t] \widetilde{k}_{t-1}$$
$$\widetilde{y}_t^d = \frac{\frac{A_t}{A_{t-1}} \frac{z_{t-1}}{z_t} \left(u_t \widetilde{k}_{t-1}\right)^{\alpha} \left(l_t^d\right)^{1-\alpha} - \phi}{v_t^p}$$

where

$$
\begin{aligned}
l_t &= v_t^w l_t^d \\
v_t^p &= \theta_p \left(\frac{\Pi_{t-1}^{\chi}}{\Pi_t} \right)^{-\varepsilon} v_{t-1}^p + (1-\theta_p) \Pi_t^{*-\varepsilon} \\
v_t^w &= \theta_w \left(\frac{\widetilde{w}_{t-1}}{\widetilde{w}_t} \frac{z_{t-1}}{z_t} \frac{\Pi_{t-1}^{\chi_w}}{\Pi_t} \right)^{-\eta} v_{t-1}^w + (1-\theta_w) \left(\Pi_t^{*w}\right)^{-\eta}
\end{aligned}
$$

and

$$
\widetilde{k}_t \frac{z_t}{z_{t-1}} \frac{\mu_t}{\mu_{t-1}} - (1-\delta) \widetilde{k}_{t-1} - \frac{z_t}{z_{t-1}} \frac{\mu_t}{\mu_{t-1}} \left(1 - S \left[\frac{\widetilde{x}_t}{\widetilde{x}_{t-1}} \frac{z_t}{z_{t-1}} \right] \right) \widetilde{x}_t = 0.
$$

4. Solving the Model

We will solve the model by loglinearizing the equilibrium conditions and applying standard techniques. Before loglinearizing, we need to find the steady-state of the model. We will solve the normalized model defined in the last section. Note that later, when we bring the model to the data, we will need to undo the normalization.

4.1. The Steady-State

Now, we will find the deterministic steady-state of the model. First, let $\widetilde{z} = \exp(\Lambda_z)$, $\widetilde{\mu} = \exp(\Lambda_\mu)$, and $\widetilde{A} = \exp(\Lambda_A)$. Also, given the definition of $\widetilde{c}$, $\widetilde{x}_t$, $\widetilde{w}_t$, $\widetilde{w}_t^*$, and $\widetilde{y}_t^d$, we have that $\Lambda_c = \Lambda_x = \Lambda_w = \Lambda_{w^*} = \Lambda_{y^d} = \Lambda_z$.

Then, in steady-state, the first order conditions of the household can be written as:

$$
d \frac{1}{\widetilde{c} - \frac{h}{\widetilde{z}} \widetilde{c}} - h\beta d \frac{1}{\widetilde{c}\widetilde{z} - h\widetilde{c}} = \widetilde{\lambda}
$$

$$
1 = \beta \frac{1}{\widetilde{z}} \frac{R}{\Pi}
$$

$$
\widetilde{r} = a'[1]
$$

$$
\widetilde{q} = \beta \frac{1}{\widetilde{z}\widetilde{\mu}} \left((1-\delta) \widetilde{q} + \widetilde{r} u - a[1] \right)
$$

$$
1 = \widetilde{q} \left(1 - S[\widetilde{z}] - S'[\widetilde{z}] \widetilde{z}\right) + \beta \frac{\widetilde{q}}{\widetilde{z}} S'[\widetilde{z}] \widetilde{z}^2
$$

$$
f = \frac{\eta - 1}{\eta} \left(\widetilde{w}^*\right)^{1-\eta} \widetilde{\lambda} \widetilde{w}^{\eta} l^d + \beta\theta_w \left(\frac{\Pi^{\chi_w}}{\Pi} \right)^{1-\eta} \widetilde{z}^{\eta-1} f
$$

$$
f = \psi d\varphi \left(\Pi^{w^*}\right)^{-\eta(1+\gamma)} \left(l^d\right)^{1+\gamma} + \beta\theta_w \left(\frac{\Pi^{\chi_w}}{\Pi} \right)^{-\eta(1+\gamma)} \widetilde{z}^{\eta(1+\gamma)} f
$$

the first order conditions of the firm as:

$$g^1 = \widetilde{\lambda} mc \widetilde{y}^d + \beta\theta_p \left(\frac{\Pi^\chi}{\Pi}\right)^{-\varepsilon} g^1$$

$$g^2 = \widetilde{\lambda}\Pi^* \widetilde{y}^d + \beta\theta_p \left(\frac{\Pi^\chi}{\Pi}\right)^{1-\varepsilon} g^2$$

$$\varepsilon g^1 = (\varepsilon - 1) g^2$$

$$\frac{u\widetilde{k}}{l^d} = \frac{\alpha}{1-\alpha}\frac{\widetilde{w}}{\widetilde{r}}\widetilde{z}\widetilde{\mu}$$

$$mc = \left(\frac{1}{1-\alpha}\right)^{1-\alpha}\left(\frac{1}{\alpha}\right)^{\alpha} \widetilde{w}^{1-\alpha}\widetilde{r}^{\alpha}$$

the law of motion for wages and prices as:

$$1 = \theta_w \left(\frac{\Pi^{\chi_w}}{\Pi}\right)^{1-\eta} \widetilde{z}^{-(1-\eta)} + (1-\theta_w)\left(\Pi^{w^*}\right)^{1-\eta}$$

$$1 = \theta_p \left(\frac{\Pi^\chi}{\Pi}\right)^{1-\varepsilon} + (1-\theta_p)\,\Pi^{*1-\varepsilon}$$

and the market clearing conditions as:

$$\widetilde{c} + \widetilde{x} = \widetilde{y}^d$$

$$v^p \widetilde{y}^d = \frac{A}{z}(u\widetilde{k})^{\alpha}(l^d)^{1-\alpha} - \phi$$

$$l = v^w l^d$$

$$v^p = \theta_p \left(\frac{\Pi^\chi}{\Pi}\right)^{-\varepsilon} v^p + (1-\theta_p)\,\Pi^{*-\varepsilon}$$

$$v^w = \theta_w \left(\frac{\Pi^{\chi_w}}{\Pi}\right)^{-\eta} \widetilde{z}^{\eta} v^w + (1-\theta_w)\,(\Pi^{w^*})^{-\eta}$$

$$\widetilde{k}\widetilde{z}\widetilde{\mu} - (1-\delta)\,\widetilde{k} - \widetilde{z}\widetilde{\mu}\,(1 - S\,[z])\,\widetilde{x} = 0.$$

To find the steady-state, we need to choose functional forms for $a\,[\cdot]$ and $S\,[\cdot]$. For $a\,[u]$ we pick: $a\,[u] = \gamma_1\,(u-1) + \frac{\gamma_2}{2}(u-1)^2$. Since in the steady state we have $u = 1$, then $\widetilde{r} = a'\,[1] = \gamma_1$ and $a\,[1] = 0$. The investment adjustment cost function is $S\left[\frac{x_t}{x_{t-1}}\right] = \frac{\kappa}{2}(\frac{x_t}{x_{t-1}} - \Lambda_x)^2$. Then, along the balanced growth path, $S\,[\Lambda_x] = S'\,[\Lambda_x] = 0$. Using this two expressions, we can rearrange the system of equations that determine the steady-state as:

$$(1-h\beta\widetilde{z})\frac{1}{1-\frac{h}{\widetilde{z}}}\frac{1}{\widetilde{c}}=\widetilde{\lambda}$$
$$R=\frac{\Pi\widetilde{z}}{\beta}$$
$$\widetilde{r}=\gamma_1$$
$$\widetilde{r}=\left(1-\frac{\beta}{\widetilde{z}\widetilde{\mu}}(1-\delta)\right)/\frac{\beta}{\widetilde{z}\widetilde{\mu}}$$
$$\left(1-\beta\theta_w\widetilde{z}^{(\eta-1)}\Pi^{-(1-\chi_w)(1-\eta)}\right)f=\frac{\eta-1}{\eta}w^*\widetilde{\lambda}(\Pi^{w*})^{-\eta}l^d$$
$$\left(1-\beta\theta_w\widetilde{z}^{\eta(1+\gamma)}\Pi^{\eta(1-\chi_w)(1+\gamma)}\right)f=\psi\left(\Pi^{w^*}\right)^{-\eta(1+\gamma)}\left(l^d\right)^{1+\gamma}$$
$$\left(1-\beta\theta_p\Pi^{(1-\chi)\varepsilon}\right)g^1=\widetilde{\lambda}mc\widetilde{y}^d$$
$$\left(1-\beta\theta_p\Pi^{-(1-\chi)(1-\varepsilon)}\right)g^2=\widetilde{\lambda}\Pi^*\widetilde{y}^d$$
$$\varepsilon g^1=(\varepsilon-1)g^2$$
$$\frac{\widetilde{k}}{l^d}=\frac{\alpha}{1-\alpha}\frac{\widetilde{w}}{\widetilde{r}}\widetilde{z}\widetilde{\mu}$$
$$mc=\left(\frac{1}{1-\alpha}\right)^{1-\alpha}\left(\frac{1}{\alpha}\right)^{\alpha}\widetilde{w}^{1-\alpha}\widetilde{r}^{\alpha}$$
$$\frac{1-\theta_w\Pi^{-(1-\chi_w)(1-\eta)}\widetilde{z}^{-(1-\eta)}}{1-\theta_w}=\left(\Pi^{w^*}\right)^{1-\eta}$$
$$\frac{1-\theta_p\Pi^{-(1-\chi)(1-\varepsilon)}}{1-\theta_p}=\Pi^{*1-\varepsilon}$$
$$\widetilde{c}+\widetilde{x}=\widetilde{y}^d$$
$$v^p\widetilde{y}^d=\frac{\widetilde{A}}{\widetilde{z}}(\widetilde{k})^{\alpha}(l^d)^{1-\alpha}-\phi$$
$$l=v^wl^d$$
$$\frac{1-\theta_p\Pi^{(1-\chi)\varepsilon}}{1-\theta_p}v^p=\Pi^{*-\varepsilon}$$
$$\frac{\left(1-\theta_w\widetilde{z}^{\eta}\Pi^{(1-\chi_w)\eta}\right)}{1-\theta_w}v^w=(\Pi^{w^*})^{-\eta}$$
$$\widetilde{k}=\frac{\widetilde{z}\widetilde{\mu}}{\widetilde{z}\widetilde{\mu}-(1-\delta)}\widetilde{x}.$$

First, notice that there is some restrictions on γ_1

$$\tilde{r}=\frac{1-\frac{\beta}{\widetilde{z}\widetilde{\mu}}(1-\delta)}{\frac{\beta}{\widetilde{z}\widetilde{\mu}}}=\gamma_1$$

and that the nominal interest rate is:

$$R = \frac{\Pi \widetilde{z}}{\beta}$$

The relationship between inflation and optimal relative prices is:

$$\Pi^* = \left(\frac{1 - \theta_p \Pi^{-(1-\varepsilon)(1-\chi)}}{1 - \theta_p} \right)^{\frac{1}{1-\varepsilon}}$$

expression from which we obtain the following two results:

1. If there is zero price inflation ($\Pi = 1$), then $\Pi^* = 1$.
2. If there is full price indexation ($\chi = 1$), then $\Pi^* = 1$.

From the optimal price setting equations, we get that the marginal cost is:

$$mc = \frac{\varepsilon - 1}{\varepsilon} \frac{1 - \beta \theta_p \Pi^{(1-\chi)\varepsilon}}{1 - \beta \theta_p \Pi^{-(1-\chi)(1-\varepsilon)}} \Pi^*$$

The relationship between inflation and optimal relative wage is:

$$\Pi^{w^*} = \left(\frac{1 - \theta_w \Pi^{-(1-\chi_w)(1-\eta)} \widetilde{z}^{-(1-\eta)}}{1 - \theta_w} \right)^{\frac{1}{1-\eta}}.$$

Then we find that:

$$\widetilde{w} = (1 - \alpha) \left(mc \left(\frac{\alpha}{\widetilde{r}} \right)^{\alpha} \right)^{\frac{1}{1-\alpha}}$$

and the optimal wage evolves:

$$\widetilde{w}^* = \widetilde{w} \Pi^{w^*}$$

Again, we can note the following:

1. If there is zero price inflation ($\Pi = 1$) and zero growth rate ($\widetilde{z} = 1$), then $\Pi^{w^*} = 1$.
2. If there is full wage indexation ($\chi_w = 1$) and zero growth rate ($\widetilde{z} = 1$), then $\Pi^{w^*} = 1$.

We have the two following equations for the wage household decision:

$$\left(1 - \beta \theta_w \widetilde{z}^{\eta - 1} \Pi^{-(1-\chi_w)(1-\eta)}\right) f = \frac{\eta - 1}{\eta} w^* \left(\Pi^{w^*}\right)^{-\eta} \widetilde{\lambda} l^d$$

and

$$\left(1 - \beta \theta_w \widetilde{z}^{\eta(1+\gamma)} \Pi^{\eta(1-\chi_w)(1+\gamma)}\right) f = \psi \left(\Pi^{w^*}\right)^{-\eta(1+\gamma)} \left(l^d\right)^{1+\gamma}.$$

Dividing the second by the first delivers:

$$\frac{1-\beta\theta_w \widetilde{z}^{\eta(1+\gamma)} \Pi^{\eta(1-\chi_w)(1+\gamma)}}{1-\beta\theta_w \widetilde{z}^{(\eta-1)} \Pi^{-(1-\chi_w)(1-\eta)}} = \frac{\psi \left(\Pi^{w^*}\right)^{-\eta\gamma} \left(l^d\right)^{\gamma}}{\frac{\eta-1}{\eta} w^* \widetilde{\lambda}}$$

which defines l^d as a function of $\widetilde{\lambda}$. Now, we will look for another relationship between l^d and $\widetilde{\lambda}$ to have two equations with two unknowns.

Before doing so, we highlight that, again, there are two interesting cases:

1. If there is zero price inflation ($\Pi = 1$) and zero growth rate ($\widetilde{z} = 1$), then we get the static condition that real wages are a markup $\frac{\eta}{\eta-1}$ over the marginal rate of substitution between consumption and leisure.

2. If there is full wage indexation ($\chi_w = 1$) and zero growth rate ($\widetilde{z} = 1$), then we get the static condition that real wages are a markup $\frac{\eta}{\eta-1}$ over the marginal rate of substitution between consumption and leisure.

The expression for the dispersion of prices is given by:

$$v^p = \frac{1-\theta_p}{1-\theta_p \Pi^{(1-\chi)\varepsilon}} \Pi^{*-\varepsilon}$$

where no price inflation or full price indexation delivers no price dispersion in steady-state.

Using the expression for Π^{w*} we find that the wage dispersion in steady-state is:

$$v^w = \frac{1-\theta_w}{1-\theta_w \Pi^{(1-\chi_w)\eta} \widetilde{z}^{\eta}} (\Pi^{w^*})^{-\eta}$$

where no price inflation or full wage indexation combine with zero growth delivers no wage dispersion in in steady-state.

The relationship between labor demand and labor supply is:

$$l = v^w l^d.$$

Note now that

$$\widetilde{y}^d = \frac{\frac{\widetilde{A}}{\widetilde{z}} (\widetilde{k})^{\alpha} (l^d)^{1-\alpha} - \phi}{v^p}.$$

But, since in steady-state $\widetilde{k} = \frac{\widetilde{z}\widetilde{\mu}}{\widetilde{z}\widetilde{\mu}-(1-\delta)} \widetilde{x}$, it is the case that:

$$\widetilde{c} + \frac{\widetilde{z}\widetilde{\mu} - (1-\delta)}{\widetilde{z}\widetilde{\mu}} \widetilde{k} = \widetilde{y}^d = \frac{\frac{\widetilde{A}}{\widetilde{z}} (\widetilde{k})^{\alpha} (l^d)^{1-\alpha} - \phi}{v^p}$$

that allows us to find $\widetilde{k}$ as function of l^d:

$$\frac{\widetilde{k}}{l^d} = \Omega = \frac{\alpha}{1-\alpha}\frac{\widetilde{w}}{\widetilde{r}}\widetilde{z}\widetilde{\mu} \Rightarrow \widetilde{k} = \Omega l^d.$$

Then:

$$\begin{aligned}
\widetilde{c} &= \frac{\frac{\widetilde{A}}{\widetilde{z}}\Omega^{\alpha} l^d - \phi}{v^p} - \frac{\widetilde{z}\widetilde{\mu} - (1-\delta)}{\widetilde{z}\widetilde{\mu}}\Omega l^d \\
&= \left(\frac{\widetilde{A}}{\widetilde{z}}\left(v^p\right)^{-1}\Omega^{\alpha} - \frac{\widetilde{z}\widetilde{\mu} - (1-\delta)}{\widetilde{z}\widetilde{\mu}}\Omega\right) l^d - \left(v^p\right)^{-1}\phi
\end{aligned}$$

Now, we can express the marginal utility of consumption in terms of hours, and get another relationship between l^d as a function of $\widetilde{\lambda}$:

$$(1-h\beta)d\widetilde{z}\left(1-\frac{h}{\widetilde{z}}\right)^{-1}\left(\left(\frac{\widetilde{A}}{\widetilde{z}}\left(v^p\right)^{-1}\Omega^{\alpha} - \frac{\widetilde{z}\widetilde{\mu} - (1-\delta)}{\widetilde{z}\widetilde{\mu}}\Omega\right) l^d - \left(v^p\right)^{-1}\phi\right)^{-1} = \widetilde{\lambda}$$

Using both relationship we can solve for l^d and get:

$$\frac{1-\beta\theta_w \widetilde{z}^{\eta(1+\gamma)}\Pi^{\eta(1-\chi_w)(1+\gamma)}}{1-\beta\theta_w \widetilde{z}^{(\eta-1)}\Pi^{-(1-\chi_w)(1-\eta)}} =$$

$$\frac{\psi\left(\Pi^{w^*}\right)^{-\eta\gamma}\left(l^d\right)^{\gamma}}{\frac{\eta-1}{\eta}w^*(1-h\beta)d\widetilde{z}\left(1-\frac{h}{\widetilde{z}}\right)^{-1}\left(\left(\frac{\widetilde{A}}{\widetilde{z}}\left(v^p\right)^{-1}\Omega^{\alpha} - \frac{\widetilde{z}\widetilde{\mu}-(1-\delta)}{\widetilde{z}\widetilde{\mu}}\Omega\right) l^d - \left(v^p\right)^{-1}\phi\right)^{-1}}.$$

Note that this is nonlinear equation. Therefore we will use a root finder to find l^d.

Once we have l^d, we can solve for capital, investment, output, and consumption as follows:

$$\begin{aligned}
\widetilde{k} &= \Omega l^d \\
\widetilde{x} &= \frac{\widetilde{z}\widetilde{\mu} - (1-\delta)}{\widetilde{z}\widetilde{\mu}}\widetilde{k} \\
\widetilde{y}^d &= \frac{\frac{\widetilde{A}}{\widetilde{z}}(\widetilde{k})^{\alpha}(l^d)^{1-\alpha} - \phi}{v^p} \\
\widetilde{c} &= \left(\frac{\widetilde{A}}{\widetilde{z}}\left(v^p\right)^{-1}\Omega^{\alpha} - \frac{\widetilde{z}\widetilde{\mu} - (1-\delta)}{\widetilde{z}\widetilde{\mu}}\Omega\right) l^d - \left(v^p\right)^{-1}\phi
\end{aligned}$$

4.2. Loglinear approximations

For each variable var_t, we define $\widehat{var}_t = \log var_t - \log var$, where var is the steady-state value for the variable var_t. Then, we can write $var_t = var \exp^{\widehat{var}_t}$.

We start by log linearizing the marginal utility of consumption:

$$d_t \left(\widetilde{c}_t - h\widetilde{c}_{t-1} \frac{z_{t-1}}{z_t} \right)^{-1} - h\beta \mathbb{E}_t d_{t+1} \left(\widetilde{c}_{t+1} \frac{z_{t+1}}{z_t} - h\widetilde{c}_t \right)^{-1} = \widetilde{\lambda}_t. \tag{4}$$

It is helpful to define the auxiliary variable $aux_t = d_t \left(\widetilde{c}_t - h\widetilde{c}_{t-1} \frac{z_{t-1}}{z_t} \right)^{-1}$. Then, we have that:

$$aux_t - h\beta \mathbb{E}_t \widetilde{z}_{t+1} aux_{t+1} = \widetilde{\lambda}_t$$

where $\widetilde{z}_{t+1} = \frac{z_{t+1}}{z_t}$.

Then, (4) can be written as:

$$aux \exp^{\widehat{aux}_t} - h\beta z aux \mathbb{E}_t \exp^{\widehat{aux}_{t+1} + \widehat{\widetilde{z}}_{t+1}} = \widetilde{\lambda} e^{\widehat{\widetilde{\lambda}}_t}$$

which can be loglinearized as:

$$aux \left(\widehat{aux}_t - h\beta z \mathbb{E}_t (\widehat{aux}_{t+1} + \widehat{\widetilde{z}}_{t+1}) \right) = \widetilde{\lambda} \widehat{\widetilde{\lambda}}_t. \tag{5}$$

Using the following two steady state relationship that $aux(1 - h\beta\widetilde{z}) = \widetilde{\lambda}$ and that $\mathbb{E}_t \widehat{\widetilde{z}}_{t+1} = 0$, we can write:

$$\widehat{aux}_t - h\beta z \mathbb{E}_t \widehat{aux}_{t+1} = (1 - h\beta z) \widehat{\widetilde{\lambda}}_t.$$

Next, we loglinearize the auxiliary variable:

$$aux e^{\widehat{aux}_t} = d \exp^{\widehat{d}_t} \left(\widetilde{c} \exp^{\widehat{\widetilde{c}}_t} - \frac{h}{\widetilde{z}} \widetilde{c} \exp^{\widehat{\widetilde{c}}_{t-1} - \widehat{\widetilde{z}}_t} \right)^{-1}.$$

The loglinear approximation is:

$$\begin{aligned} aux \widehat{aux}_t &= d \left(\widetilde{c} - \frac{h}{\widetilde{z}} \widetilde{c} \right)^{-1} \widehat{d}_t - d \left(\widetilde{c} - \frac{h}{\widetilde{z}} \widetilde{c} \right)^{-2} \widetilde{c} \widehat{\widetilde{c}}_t \\ &\quad + d \left(\widetilde{c} - \frac{h}{\widetilde{z}} \widetilde{c} \right)^{-2} \frac{h\widetilde{c}}{\widetilde{z}} \left(\widehat{\widetilde{c}}_{t-1} - \widehat{\widetilde{z}}_t \right). \end{aligned}$$

Making use of the fact that $aux = d \left(\widetilde{c} - \frac{h}{\widetilde{z}} \widetilde{c} \right)^{-1}$, we find:

$$aux \widehat{aux}_t = aux \left(\widehat{d}_t - \left(\widetilde{c} - \frac{h}{\widetilde{z}} \widetilde{c} \right)^{-1} \widetilde{c} \widehat{\widetilde{c}}_t + \left(\widetilde{c} - \frac{h}{\widetilde{z}} \widetilde{c} \right)^{-1} \frac{h\widetilde{c}}{\widetilde{z}} \left(\widehat{\widetilde{c}}_{t-1} - \widehat{\widetilde{z}}_t \right) \right),$$

that simplifies to:

$$aux\widehat{aux}_t = aux\left(\widehat{d}_t - \left(1-\frac{h}{\widetilde{z}}\right)^{-1}\widehat{\widetilde{c}}_t + \left(1-\frac{h}{\widetilde{z}}\right)^{-1}\frac{h}{\widetilde{z}}\left(\widehat{\widetilde{c}}_{t-1} - \widehat{\widetilde{z}}_t\right)\right)$$

$$\widehat{aux}_t = \widehat{d}_t - \left(1-\frac{h}{\widetilde{z}}\right)^{-1}\left(\widehat{\widetilde{c}}_t - \frac{h\widehat{\widetilde{c}}_{t-1}}{\widetilde{z}} + \frac{h\widehat{\widetilde{z}}_t}{\widetilde{z}}\right). \tag{6}$$

Putting the (5) and (6) together:

$$(1-h\beta z)\widehat{\widetilde{\lambda}}_t =$$

$$\widehat{d}_t - \left(1-\frac{h}{\widetilde{z}}\right)^{-1}\left(\widehat{\widetilde{c}}_t - \frac{h\widehat{\widetilde{c}}_{t-1}}{\widetilde{z}} + \frac{h\widehat{\widetilde{z}}_t}{\widetilde{z}}\right) - h\beta z\mathbb{E}_t\left\{\widehat{d}_{t+1} - \left(1-\frac{h}{\widetilde{z}}\right)^{-1}\left(\widehat{\widetilde{c}}_{t+1} - \frac{h\widehat{\widetilde{c}}_t}{\widetilde{z}} + \frac{h\widehat{\widetilde{z}}_{t+1}}{\widetilde{z}}\right)\right\}.$$

After some algebra, we arrive to the final expression:

$$(1-h\beta z)\widehat{\widetilde{\lambda}}_t = \widehat{d}_t - h\beta z\mathbb{E}_t\widehat{d}_{t+1} - \frac{1+h^2\beta}{\left(1-\frac{h}{\widetilde{z}}\right)}\widehat{\widetilde{c}}_t + \frac{h}{\widetilde{z}\left(1-\frac{h}{\widetilde{z}}\right)}\widehat{\widetilde{c}}_{t-1} + \frac{\beta hz}{\left(1-\frac{h}{\widetilde{z}}\right)}\mathbb{E}_t\widehat{\widetilde{c}}_{t+1} - \frac{h}{\widetilde{z}\left(1-\frac{h}{\widetilde{z}}\right)}\widehat{\widetilde{z}}_t$$

This expression helps to understand the role of the habit persistence parameter h. If we set $h=0$ (i.e., no habit), we would get:

$$\widehat{\widetilde{\lambda}}_t = \widehat{d}_t - \widehat{\widetilde{c}}_t$$

where the lags and forward terms drop.

Now, we loglinearize the Euler equation:

$$\widetilde{\lambda}_t = \beta\mathbb{E}_t\{\widetilde{\lambda}_{t+1}\frac{z_t}{z_{t+1}}\frac{R_t}{\Pi_{t+1}}\}.$$

To do so, we write the expression as

$$\widetilde{\lambda}e^{\widehat{\widetilde{\lambda}}} = \beta\mathbb{E}_t\{\widetilde{\lambda}e^{\widehat{\widetilde{\lambda}}_{t+1}}\frac{1}{\widetilde{z}e^{z_{z,t}}}\frac{Re^{\widehat{R}_t}}{\Pi e^{\widehat{\Pi}_{t+1}}}\}$$

By using the fact that

$$R = \frac{\Pi\widetilde{z}}{\beta}$$

we simplify to:

$$e^{\widehat{\widetilde{\lambda}}} = \mathbb{E}_t\{e^{\widehat{\widetilde{\lambda}}_{t+1}}\frac{1}{e^{z_{z,t}}}\frac{e^{\widehat{R}_t}}{e^{\widehat{\Pi}_{t+1}}}\}$$

Now, it is easy to show that:

$$\widehat{\widetilde{\lambda}}_t = \mathbb{E}_t\{\widehat{\widetilde{\lambda}}_{t+1} + \widehat{R}_t - \widehat{\Pi}_{t+1}\}. \tag{9}$$

Let us now consider:

$$\widetilde{r}_t = a' [u_t] .$$

First, we write:

$$\widetilde{r}e^{\widehat{\widetilde{r}}_t} = a' \left[u \exp^{\widehat{u}_t}\right] ,$$

where the loglinear approximation delivers:

$$\widetilde{r}\widehat{\widetilde{r}}_t = a'' [u] \, u\widehat{u}_t.$$

Since $\widetilde{r} = a' [u]$, then:

$$\widehat{\widetilde{r}}_t = \frac{a'' [u] \, u}{a' [u]}\widehat{u}_t,$$

or

$$\widehat{\widetilde{r}}_t = \frac{\gamma_2}{\gamma_1}\hat{u}_t. \tag{10}$$

The next equation to consider relates the shadow price of capital to the return on investment:

$$\widetilde{q}_t = \beta\mathbb{E}_t \left\{ \frac{\widetilde{\lambda}_{t+1}}{\widetilde{\lambda}_t} \frac{1}{\widetilde{z}_{t+1}} \frac{1}{\widetilde{\mu}_{t+1}} \left((1-\delta) \, \widetilde{q}_{t+1} + \widetilde{r}_{t+1}u_{t+1} - a \, [u_{t+1}]\right) \right\}$$

where $\widetilde{\mu}_{t+1} = \frac{\mu_{t+1}}{\mu_t}$. We can we write this expression as

$$\widetilde{q}e^{\widehat{\widetilde{q}}_t} = \frac{\beta}{\widetilde{z}\widetilde{\mu}}\mathbb{E}_t \exp^{\Delta\widehat{\widetilde{\lambda}}_{t+1} - \widehat{\widetilde{z}}_{t+1} - \widehat{\widetilde{\mu}}_{t+1}} \left\{ \begin{array}{c} (1-\delta) \, \widetilde{q}e^{\widehat{\widetilde{q}}_{t+1}} + \\ \widetilde{r}u \exp^{\widehat{\widetilde{r}}_{t+1} + \widehat{u}_{t+1}} - a \left[u \exp^{\widehat{u}_{t+1}}\right] \end{array} \right\}.$$

Loglinearization delivers:

$$\begin{aligned} \widetilde{q}\widehat{\widetilde{q}}_t \;&=\; \frac{\beta}{\widetilde{z}\widetilde{\mu}}\mathbb{E}_t \left(\triangle\widehat{\widetilde{\lambda}}_{t+1} - \widehat{\widetilde{z}}_{t+1} - \widehat{\widetilde{\mu}}_{t+1}\right) \left((1-\delta) \, \widetilde{q} + \widetilde{r}u - a \, [u]\right) \\ &\quad + \frac{\beta}{\widetilde{z}\widetilde{\mu}} \left(\mathbb{E}_t \, (1-\delta) \, \widetilde{q}\widehat{\widetilde{q}}_{t+1} + \widetilde{r}u \left(\widehat{\widetilde{r}}_{t+1} + \widehat{u}_{t+1}\right) - a \, [u] \, u\widehat{u}_{t+1}\right). \end{aligned}$$

Making use of the following steady-state relationships: $u = 1$, $a\,[u] = 0$, $\widetilde{r} = a'\,[u]$, $\widetilde{q} = 1$

and $1=\frac{\beta}{\widetilde{z}\widetilde{\mu}}\widetilde{r}+\frac{\beta}{\widetilde{z}\widetilde{\mu}}(1-\delta)$, and $\mathbb{E}_t\left(-\widehat{\widetilde{z}}_{t+1}-\widehat{\widetilde{\mu}}_{t+1}\right)=0$, the previous expression simplifies to:

$$\widehat{\widetilde{q}}_t=((1-\delta)+\widetilde{r})\frac{\beta}{\widetilde{z}\widetilde{\mu}}\mathbb{E}_t\Delta\widehat{\widetilde{\lambda}}_{t+1}+\frac{\beta(1-\delta)}{\widetilde{z}\widetilde{\mu}}\mathbb{E}_t\widehat{\widetilde{q}}_{t+1}+\frac{\beta}{\widetilde{z}\widetilde{\mu}}\widetilde{r}u\mathbb{E}_t\widehat{\widetilde{r}}_{t+1},$$

that implies:

$$\widehat{\widetilde{q}}_t=\mathbb{E}_t\Delta\widehat{\widetilde{\lambda}}_{t+1}+\frac{\beta(1-\delta)}{\widetilde{z}\widetilde{\mu}}\mathbb{E}_t\widehat{\widetilde{q}}_{t+1}+\left(1-\frac{\beta(1-\delta)}{\widetilde{z}\widetilde{\mu}}\right)\mathbb{E}_t\widehat{\widetilde{r}}_{t+1}. \tag{11}$$

The next equation to loglinearize is:

$$1=\widetilde{q}_t\left(1-S\left[\frac{\widetilde{x}_t}{\widetilde{x}_{t-1}}\frac{z_t}{z_{t-1}}\right]-S'\left[\frac{\widetilde{x}_t}{\widetilde{x}_{t-1}}\frac{z_t}{z_{t-1}}\right]\frac{\widetilde{x}_t}{\widetilde{x}_{t-1}}\frac{z_t}{z_{t-1}}\right)+\beta\mathbb{E}_t\widetilde{q}_{t+1}\frac{\widetilde{\lambda}_{t+1}}{\widetilde{\lambda}_t}\frac{z_t}{z_{t+1}}S'\left[\frac{\widetilde{x}_{t+1}}{\widetilde{x}_t}\frac{z_{t+1}}{z_t}\right]\left(\frac{\widetilde{x}_{t+1}}{\widetilde{x}_t}\frac{z_{t+1}}{z_t}\right)^2$$

which can be rearranged as:

$$\begin{aligned}1 &= \widetilde{q}\exp^{\widehat{\widetilde{q}}_t}\left(1-S\left[\widetilde{z}\exp^{\Delta\widehat{\widetilde{x}}_t+\widehat{\widetilde{z}}_t}\right]-S'\left[\widetilde{z}\exp^{\Delta\widehat{\widetilde{x}}_t+\widehat{\widetilde{z}}_t}\right]\widetilde{z}\exp^{\Delta\widehat{\widetilde{x}}_t+\widehat{\widetilde{z}}_t}\right)+\\ &+\beta\frac{\widetilde{q}}{\widetilde{z}}\mathbb{E}_t\exp^{\widehat{\widetilde{q}}_t-\widehat{\widetilde{z}}_{t+1}+\Delta\widehat{\widetilde{\lambda}}_{t+1}}S''\left[\widetilde{z}\exp^{\Delta\widehat{\widetilde{x}}_{t+1}+\widehat{\widetilde{z}}_{t+1}}\right]\widetilde{z}^2\exp^{2\left(\Delta\widehat{\widetilde{x}}_{t+1}+\widehat{\widetilde{z}}_{t+1}\right)}.\end{aligned}$$

Taking the loglinear approximation (and using the fact that $\widetilde{q}=1$) we get:

$$0=\widehat{\widetilde{q}}_t-S''[\widetilde{z}]\widetilde{z}^2\left(\Delta\widehat{\widetilde{x}}_t+\widehat{\widetilde{z}}_t\right)+\frac{\beta}{z}S''[\widetilde{z}]\widetilde{z}^3\mathbb{E}_t\left(\Delta\widehat{\widetilde{x}}_{t+1}+\widehat{\widetilde{z}}_{t+1}\right).$$

Reorganizing:

$$\kappa\widetilde{z}^2\left(\Delta\widehat{\widetilde{x}}_t+\widehat{\widetilde{z}}_t\right)=\widehat{\widetilde{q}}_t+\beta\kappa\widetilde{z}^2\mathbb{E}_t\Delta\widehat{\widetilde{x}}_{t+1} \tag{12}$$

where κ comes from the adjustment cost function.

We move now into loglinearizing the equations that describe the law of motion of f_t

$$f_t=\frac{\eta-1}{\eta}(\widetilde{w}_t^*)^{1-\eta}\widetilde{\lambda}_t(\widetilde{w}_t)^{\eta}l_t^d+\beta\theta_w\mathbb{E}_t\left(\frac{\Pi_t^{\chi_w}}{\Pi_{t+1}}\right)^{1-\eta}\left(\frac{\widetilde{w}_{t+1}^*}{\widetilde{w}_t^*}\frac{z_{t+1}}{z_t}\right)^{\eta-1}f_{t+1}$$

and

$$f_t=\psi d_t\varphi_t\left(\Pi_t^{*w}\right)^{-\eta(1+\gamma)}\left(l_t^d\right)^{1+\gamma}+\beta\theta_w\mathbb{E}_t\left(\frac{\Pi_t^{\chi_w}}{\Pi_{t+1}}\right)^{-\eta(1+\gamma)}\left(\frac{\widetilde{w}_{t+1}^*}{\widetilde{w}_t^*}\frac{z_{t+1}}{z_t}\right)^{\eta(1+\gamma)}f_{t+1}.$$

The first equation can be written as:

$$
\begin{aligned}
f \exp^{\widehat{f}_t} &= \frac{\eta-1}{\eta}\left(\widetilde{w}^*\right)^{1-\eta}\widetilde{\lambda}\widetilde{w}^{\eta} l^d \exp^{(1-\eta)\widehat{\widetilde{w}}_t^* + \widehat{\widetilde{\lambda}}_t + \eta\widehat{\widetilde{w}}_t + \widehat{l}_t^d} + \\
&\quad \beta\theta_w \Pi^{-(1-\eta)(1-\chi_w)} \widetilde{z}^{\eta-1} f \mathbb{E}_t \exp^{\widehat{f}_t - (1-\eta)\left(\widehat{\Pi}_{t+1} - \chi_w \widehat{\Pi}_t + \Delta\widehat{\widetilde{w}}_{t+1}^* + \widehat{\widetilde{z}}_{t+1}\right)}.
\end{aligned}
$$

Since $\mathbb{E}_t \widehat{\widetilde{z}}_{t+1} = 0$, we can loglinearize the previous expression as:

$$
\begin{aligned}
f \widehat{f}_t &= \frac{\eta-1}{\eta}\left(\widetilde{w}^*\right)^{1-\eta}\widetilde{\lambda}\widetilde{w}^{\eta} l^d \left((1-\eta)\,\widehat{\widetilde{w}}_t^* + \widehat{\widetilde{\lambda}}_t + \eta\widehat{\widetilde{w}}_t + \widehat{l}_t^d\right) + \\
&\quad \beta\theta_w \Pi^{-(1-\eta)(1-\chi_w)} \widetilde{z}^{\eta-1} f \mathbb{E}_t \left(\widehat{f}_{t+1} - (1-\eta)\left((\widehat{\Pi}_{t+1} - \chi_w \widehat{\Pi}_t) + \Delta\widehat{\widetilde{w}}_{t+1}^*\right)\right).
\end{aligned}
$$

Since in the steady-state we have that

$$
1 - \beta\theta_w \widetilde{z}^{\eta-1} \Pi^{-(1-\eta)(1-\chi_w)} = \frac{\frac{\eta-1}{\eta}\left(w^*\right)^{1-\eta}\lambda w^{\eta} l^d}{f},
$$

we can write that:

$$
\begin{aligned}
\widehat{f}_t &= \left(1 - \beta\theta_w \widetilde{z}^{\eta-1} \Pi^{-(1-\eta)(1-\chi_w)}\right)\left((1-\eta)\,\widehat{\widetilde{w}}_t^* + \widehat{\widetilde{\lambda}}_t + \eta\widehat{\widetilde{w}}_t + \widehat{l}_t^d\right) + \\
&\quad \beta\theta_w \Pi^{-(1-\eta)(1-\chi_w)} \widetilde{z}^{\eta-1} \mathbb{E}_t \left(\widehat{f}_{t+1} - (1-\eta)\left(\widehat{\Pi}_{t+1} - \chi_w \widehat{\Pi}_t + \Delta\widehat{\widetilde{w}}_{t+1}^*\right)\right) \qquad (13)
\end{aligned}
$$

Let us know consider the second equation describing the behavior of f. First we can write it as:

$$
\begin{aligned}
f \exp^{\widehat{f}_t} &= \psi d\varphi (l^d)^{1+\gamma} \exp^{\widehat{d}_t + \widehat{\varphi}_t + \eta(1+\gamma)\left(\widehat{\widetilde{w}}_t - \widehat{\widetilde{w}}_t^*\right) + (1+\gamma)\widehat{l}_t^d} + \\
&\quad + \beta\theta_w \widetilde{z}^{\eta(1+\gamma)} \Pi^{\eta(1+\gamma)(1-\chi_w)} f \mathbb{E}_t \exp^{\widehat{f}_{t+1} + \eta(1+\gamma)\left(\widehat{\Pi}_{t+1} - \chi_w \widehat{\Pi}_t + \Delta\widehat{\widetilde{w}}_{t+1}^* + \widehat{\widetilde{z}}_{t+1}\right)}.
\end{aligned}
$$

It can be shown that in steady-state:

$$
1 - \beta\theta_w \widetilde{z}^{\eta(1+\gamma)} \Pi^{\eta(1+\gamma)(1-\chi_w)} = \frac{\psi d\varphi (l^d)^{1+\gamma}}{f}
$$

and since $\mathbb{E}_t \widehat{\widetilde{z}}_{t+1} = 0$, that implies

$$
\begin{aligned}
\widehat{f}_t &= \left(1 - \beta\theta_w \widetilde{z}^{\eta(1+\gamma)} \Pi^{\eta(1+\gamma)(1-\chi_w)}\right)\left(\widehat{d}_t + \widehat{\varphi}_t + \eta\,(1+\gamma)\left(\widehat{\widetilde{w}}_t - \widehat{\widetilde{w}}_t^*\right) + (1+\gamma)\widehat{l}_t^d\right) \\
&\quad + \beta\theta_w \widetilde{z}^{\eta(1+\gamma)} \Pi^{\eta(1+\gamma)(1-\chi_w)} \mathbb{E}_t \left(\widehat{f}_{t+1} + \eta\,(1+\gamma)\left(\widehat{\Pi}_{t+1} - \chi_w \widehat{\Pi}_t + \Delta\widehat{\widetilde{w}}_{t+1}^*\right)\right). \qquad (14)
\end{aligned}
$$

Let us loglinearize the law of motion for g_t^1 and g_t^2. First consider

$$g_t^1 = \widetilde{\lambda}_t mc_t \widetilde{y}_t^d + \beta\theta_p \mathbb{E}_t \left(\frac{\Pi_t^\chi}{\Pi_{t+1}}\right)^{-\varepsilon} g_{t+1}^1$$

that can be rewritten as:

$$g^1 \exp^{\widehat{g}_t^1} = \widetilde{\lambda} mc \widetilde{y}^d \exp^{\widehat{\widetilde{\lambda}}_t + \widehat{mc}_t + \widehat{\widetilde{y}}_t^d} + \beta\theta_p g^1 \Pi^{\varepsilon(1-\chi)} \mathbb{E}_t \exp^{\varepsilon(\widehat{\Pi}_{t+1} - \chi\widehat{\Pi}_t) + \widehat{g}_{t+1}^1}.$$

If we loglinearize that last expression, we get:

$$g^1 \widehat{g}_t^1 = \widetilde{\lambda} mc \widetilde{y}^d \left(\widehat{\widetilde{\lambda}}_t + \widehat{mc}_t + \widehat{\widetilde{y}}_t^d\right) + \beta\theta_p g^1 \Pi^{\varepsilon(1-\chi)} \mathbb{E}_t \left(\varepsilon(\widehat{\Pi}_{t+1} - \chi\widehat{\Pi}_t) + \widehat{g}_{t+1}^1\right).$$

It can shown that in steady state

$$1 - \beta\theta_p \Pi^{\varepsilon(1-\chi)} = \frac{\widetilde{\lambda} mc \widetilde{y}^d}{g^1}$$

therefore,

$$\widehat{g}_t^1 = \left(1 - \beta\theta_p \Pi^{\varepsilon(1-\chi)}\right) \left(\widehat{\widetilde{\lambda}}_t + \widehat{mc}_t + \widehat{\widetilde{y}}_t^d\right) + \beta\theta_p \Pi_t^{\varepsilon(1-\chi)} \mathbb{E}_t \left(\varepsilon(\widehat{\Pi}_{t+1} - \chi\widehat{\Pi}_t) + \widehat{g}_{t+1}^1\right). \qquad (15)$$

Let us now consider:

$$g_t^2 = \widetilde{\lambda}_t \Pi_t^* \widetilde{y}_t^d + \beta\theta_p \mathbb{E}_t \left(\frac{\Pi_t^\chi}{\Pi_{t+1}}\right)^{1-\varepsilon} \left(\frac{\Pi_t^*}{\Pi_{t+1}^*}\right) g_{t+1}^2,$$

that can rewritten as:

$$g^2 \exp^{\widehat{g}_t^2} = \widetilde{\lambda} \Pi^* \widetilde{y}^d \exp^{\widehat{\widetilde{\lambda}}_t + \widehat{\Pi}_t^* + \widehat{\widetilde{y}}_t^d} + \beta\theta_p \Pi^{-(1-\varepsilon)(1-\chi)} g^2 \mathbb{E}_t \exp^{-(1-\varepsilon)\left(\widehat{\Pi}_{t+1} - \chi\widehat{\Pi}_t\right) - \left(\widehat{\Pi}_{t+1}^* - \widehat{\Pi}_t^*\right) + \widehat{g}_{t+1}^2}.$$

If we loglinearize that last expression, we get:

$$\begin{aligned} g^2 \widehat{g}_t^2 &= \widetilde{\lambda} \Pi^* \widetilde{y}^d \left(\widehat{\widetilde{\lambda}}_t + \widehat{\Pi}_t^* + \widehat{\widetilde{y}}_t^d\right) \\ &\quad + \beta\theta_p \Pi^{-(1-\varepsilon)(1-\chi)} g^2 \mathbb{E}_t \left(-(1-\varepsilon)\left(\widehat{\Pi}_{t+1} - \chi\widehat{\Pi}_t\right) - \left(\widehat{\Pi}_{t+1}^* - \widehat{\Pi}_t^*\right) + \widehat{g}_{t+1}^2\right). \end{aligned}$$

It can be shown that:

$$1 - \beta\theta_p \Pi^{-(1-\varepsilon)(1-\chi)} = \frac{\widetilde{\lambda} \Pi^* \widetilde{y}^d}{g^2},$$

therefore:

$$\begin{aligned}\widehat{g}_t^2 &= \left(1-\beta\theta_p\Pi^{-(1-\varepsilon)(1-\chi)}\right)\left(\widehat{\widetilde{\lambda}}_t+\widehat{\Pi}_t^*+\widehat{\widetilde{y}}_t^d\right)\\ &+\beta\theta_p\Pi^{-(1-\varepsilon)(1-\chi)}\mathbb{E}_t\left(-(1-\varepsilon)\left(\widehat{\Pi}_{t+1}-\chi\widehat{\Pi}_t\right)-\left(\widehat{\Pi}_{t+1}^*-\widehat{\Pi}_t^*\right)+\widehat{g}_{t+1}^2\right). \qquad (16)\end{aligned}$$

Note that is easy to show that $\varepsilon g_t^1=(\varepsilon-1)g_t^2$ loglinearizes to:

$$\widehat{g}_t^1=\widehat{g}_t^2. \qquad (17)$$

Now, let us loglinearize the relationship between the capital-labor ratio and the real wage-real interest rate

$$\frac{u_t\widetilde{k}_{t-1}}{l_t^d}=\frac{\alpha}{1-\alpha}\frac{\widetilde{w}_t}{\widetilde{r}_t}\frac{z_t}{z_{t-1}}\frac{\mu_t}{\mu_{t-1}}.$$

It is easy to show that

$$\widehat{u}_t+\widehat{\widetilde{k}}_{t-1}-\widehat{l}_t^d=\widehat{\widetilde{w}}_t-\widehat{\widetilde{r}}_t+\widehat{\widetilde{z}}_t+\widehat{\widetilde{\mu}}_t. \qquad (18)$$

Let us loglinearize the marginal cost

$$mc_t=\left(\frac{1}{1-\alpha}\right)^{1-\alpha}\left(\frac{1}{\alpha}\right)^{\alpha}\left(\widetilde{w}_t\right)^{1-\alpha}\widetilde{r}_t^{\alpha}$$

to get

$$\widehat{mc}_t=(1-\alpha)\widehat{\widetilde{w}}_t+\alpha\widehat{\widetilde{r}}_t. \qquad (19)$$

Let us now concentrate on the aggregate wage law of motion:

$$1=\theta_w\left(\frac{\Pi_{t-1}^{\chi_w}}{\Pi_t}\right)^{1-\eta}\left(\frac{\tilde{w}_{t-1}}{\tilde{w}_t}\frac{z_{t-1}}{z_t}\right)^{1-\eta}+(1-\theta_w)\left(\Pi_t^{w^*}\right)^{1-\eta}$$

the above expression can be rewritten as:

$$1=\theta_w\Pi^{-(1-\chi_w)(1-\eta)}\widetilde{z}^{-(1-\eta)}\exp^{-(1-\eta)(\widehat{\Pi}_t-\chi_w\widehat{\Pi}_{t-1}+\widehat{\Pi}_t^w+\widetilde{z}_t)}+(1-\theta_w)\left(\Pi^{w^*}\right)^{1-\eta}\exp^{(1-\eta)\widehat{\Pi}_t^{w*}},$$

and loglinearized to:

$$\theta_w\Pi^{-(1-\chi_w)(1-\eta)}\widetilde{z}^{-(1-\eta)}(\widehat{\Pi}_t-\chi_w\widehat{\Pi}_{t-1}+\widehat{\Pi}_t^w+\widetilde{z}_t)=(1-\theta_w)\left(\Pi^{w^*}\right)^{1-\eta}\widehat{\Pi}_t^{w*},$$

that we can write as:

$$\frac{\theta_w \Pi^{-(1-\chi_w)(1-\eta)} \widetilde{z}^{-(1-\eta)}}{(1-\theta_w)\left(\Pi^{w*}\right)^{1-\eta}} (\widehat{\Pi}_t - \chi_w \widehat{\Pi}_{t-1} + \widehat{\Pi}_t^w + \widehat{z}_t) = \widehat{\widetilde{w}}_t^* - \widehat{\widetilde{w}}_t. \tag{20}$$

Now we concentrate on the aggregate price law of motion:

$$1 = \theta_p \left(\frac{\Pi_{t-1}^{\chi}}{\Pi_t} \right)^{1-\varepsilon} + (1-\theta_p) \, \Pi_t^{*1-\varepsilon}$$

that can be rewritten as:

$$1 = \theta_p \Pi^{-(1-\varepsilon)(1-\chi)} \exp^{-(1-\varepsilon)(\widehat{\Pi}_t - \chi \widehat{\Pi}_{t-1})} + (1-\theta_p) \left(\Pi^*\right)^{(1-\varepsilon)} \exp^{(1-\varepsilon)\widehat{\Pi}_t^*}$$

that loglinearizes to:

$$\frac{\theta_p \Pi^{-(1-\varepsilon)(1-\chi)}}{(1-\theta_p)\left(\Pi^*\right)^{(1-\varepsilon)}} (\widehat{\Pi}_t - \chi \widehat{\Pi}_{t-1}) = \widehat{\Pi}_t^*. \tag{21}$$

The Taylor rule loglinnearizes to:

$$\widehat{R}_t = \gamma_R \widehat{R}_{t-1} + (1-\gamma_R) \left(\gamma_\Pi \widehat{\Pi}_t + \gamma_y (\triangle \widehat{\widetilde{y}}_t^d + \widehat{\widetilde{z}}_t) \right) + \widehat{m}_t. \tag{22}$$

The market clearing conditions:

$$l_t = v_t^w l_t^d,$$

$$\widetilde{c}_t + \widetilde{x}_t + \frac{a\left[u_t\right] \widetilde{k}_{t-1}}{\widetilde{\mu}_t \widetilde{z}_t} = \widetilde{y}_t^d,$$

and

$$v_t^p \widetilde{y}_t^d = \frac{\widetilde{A}_t}{\widetilde{z}_t} \left(u_t \widetilde{k}_{t-1} \right)^\alpha \left(l_t^d \right)^{1-\alpha} - \phi$$

can be written as:

$$l \exp^{\widehat{l}_t} = v^w l^d \exp^{\widehat{v}_t^w + \widehat{l}_t^d},$$

$$\widetilde{c} \exp^{\widehat{\widetilde{c}}_t} + \widetilde{x} \exp^{\widehat{\widetilde{x}}_t} + \frac{a\left[\widetilde{u} \exp^{\widehat{\widetilde{u}}_t}\right] \widetilde{k} \exp^{\widehat{\widetilde{k}}_{t-1}}}{\widetilde{z}\widetilde{\mu} \exp^{\widehat{\widetilde{\mu}}_t + \widehat{\widetilde{z}}_t}} = \widetilde{y}^d \exp^{\widehat{\widetilde{y}}_t^d},$$

and

$$v^p \widetilde{y}^d \exp^{\widehat{v}_t^p + \widehat{\widetilde{y}}_t^d} = \frac{\widetilde{A}}{\widetilde{z}} \left(u\widetilde{k} \right)^\alpha \left(\widetilde{l}^d \right)^{1-\alpha} \exp^{\widehat{\widetilde{A}}_t - \widehat{\widetilde{z}}_t + \alpha\left(\widehat{u}_t + \widehat{\widetilde{k}}_{t-1} \right) + (1-\alpha) \widehat{l}_t^d} - \phi$$

respectively, where $\widetilde{A}_{t+1} = \frac{A_{t+1}}{A_t}$.

Loglinearizing we get:

$$\widehat{l}_t = \widehat{v}_t^w + \widehat{l}_t^d, \tag{23}$$

$$\tilde{c}\widehat{\tilde{c}}_t + \tilde{x}\widehat{\tilde{x}}_t + \frac{\gamma_1 \widetilde{k}}{\widetilde{z}\widetilde{\mu}}\widehat{u}_t = \tilde{y}^d \widehat{\widetilde{y}}_t^d, \tag{24}$$

and

$$\left(\tilde{y}^d v^p\right)\left(\widehat{v}_t^p + \widehat{\widetilde{y}}_t^d\right) = \frac{\widetilde{A}}{\widetilde{z}}\left(u\widetilde{k}\right)^{\alpha}\left(\widetilde{l}^d\right)^{1-\alpha}\left(\widehat{\widetilde{A}}_t - \widehat{\widetilde{z}}_t + \alpha\left(\widehat{u}_t + \widehat{\widetilde{k}}_{t-1}\right) + (1-\alpha)\,\widehat{l}_t^d\right). \tag{25}$$

Let us consider now

$$v_t^p = \theta_p\left(\frac{\Pi_{t-1}^{\chi}}{\Pi_t}\right)^{-\varepsilon} v_{t-1}^p + (1-\theta_p)\,\Pi_t^{*-\varepsilon}$$

that can be written as

$$v^p \exp^{\widehat{v}_t^p} = \theta_p \Pi^{\varepsilon(1-\chi)} v^p \exp^{\varepsilon(\widehat{\Pi}_t - \chi\widehat{\Pi}_{t-1}) + \widehat{v}_{t-1}^p} + (1-\theta_p)\,\Pi^{*-\varepsilon}\exp^{\varepsilon\widehat{\Pi}_t^*}.$$

Using

$$\left(1-\theta_p\Pi^{\varepsilon(1-\chi)}\right)v^p = (1-\theta_p)\,\Pi^{*-\varepsilon}$$

we get:

$$\widehat{v}_t^p = \theta_p\Pi^{\varepsilon(1-\chi)}\left(\varepsilon(\widehat{\Pi}_t - \chi\widehat{\Pi}_{t-1}) + \widehat{v}_{t-1}^p\right) - \left(1-\theta_p\Pi^{\varepsilon(1-\chi)}\right)\varepsilon\widehat{\Pi}_t^*. \tag{26}$$

Let us know consider the law of motion of the wage dispersion:

$$v_t^w = \theta_w\left(\frac{\Pi_{t-1}^{\chi_w}}{\Pi_t}\frac{1}{\widetilde{z}_t}\frac{\tilde{w}_{t-1}}{\tilde{w}_t}\right)^{-\eta} v_{t-1}^w + (1-\theta_w)\left(\Pi_t^{w*}\right)^{-\eta}.$$

that can be written

$$v^w e^{\widehat{v}_t^w} = \theta_w\Pi^{\eta(1-\chi_w)}\widetilde{z}^{\eta}v^w e^{\eta(\widehat{\Pi}_t - \chi_w\widehat{\Pi}_{t-1} + \widehat{\tilde{w}}_t - \widehat{\tilde{w}}_{t-1} + \widehat{\tilde{z}}_t) + \widehat{v}_{t-1}^w} + (1-\theta_w)\left(\Pi^{w*}\right)^{-\eta} e^{-\eta(\widehat{\tilde{w}}_t^* - \widehat{\tilde{w}}_t)}.$$

Using

$$\left(1-\theta_w\Pi^{\eta(1-\chi_w)}\widetilde{z}^{\eta}\right)v^w = (1-\theta_w)\left(\Pi^{w*}\right)^{-\eta}$$

we get:

$$\widehat{v}_t^w = \theta_w\Pi^{\eta(1-\chi_w)}\widetilde{z}^{\eta}\left(\eta\left(\widehat{\Pi}_t - \chi_w\widehat{\Pi}_{t-1} + \widehat{\widetilde{w}}_t - \widehat{\widetilde{w}}_{t-1} + \widehat{\widetilde{z}}_t\right) + \widehat{v}_{t-1}^w\right) - \left(1-\theta_w\Pi^{\eta(1-\chi_w)}\widetilde{z}^{\eta}\right)\eta(\widehat{\widetilde{w}}_t^* - \widehat{\widetilde{w}}_t). \tag{27}$$

Finally, let us loglinearize the law of motion of capital

$$\tilde{k}_t \widetilde{z}_t \widetilde{\mu}_t = (1-\delta)\,\tilde{k}_{t-1} + \widetilde{\mu}_t \widetilde{z}_t \left(1 - S\left[\frac{\tilde{x}_t}{\tilde{x}_{t-1}}\widetilde{z}_t\right]\right)\tilde{x}_t.$$

If we rearrange terms, we get:

$$\tilde{k}\widetilde{z}\widetilde{\mu}\exp^{\widehat{\tilde{k}}_t+\widehat{\widetilde{z}}_t+\widehat{\widetilde{\mu}}_t} = (1-\delta)\,\tilde{k}\exp^{\widehat{\tilde{k}}_{t-1}} + \widetilde{z}\widetilde{\mu}\exp^{\widehat{\widetilde{z}}_t+\widehat{\widetilde{\mu}}_t}\left(1 - S\left[ze^{\widehat{\widetilde{z}}_t+\triangle\widehat{\tilde{x}}_t}\right]\right)\tilde{x}\exp^{\widehat{\tilde{x}}_t}.$$

Loglinearizing:

$$\tilde{k}\widetilde{z}\widetilde{\mu}\left(\widehat{\tilde{k}}_t + \widehat{\widetilde{z}}_t + \widehat{\widetilde{\mu}}_t\right) = (1-\delta)\,\tilde{k}\widehat{\tilde{k}}_{t-1} + \widetilde{z}\widetilde{\mu}\tilde{x}(\widehat{\widetilde{z}}_t + \widehat{\widetilde{\mu}}_t + \widehat{\tilde{x}}_t).$$

Note that, using $\tilde{k} = \frac{\widetilde{z}\widetilde{\mu}}{\widetilde{z}\widetilde{\mu}-(1-\delta)}\tilde{x}$, we can rearrange the previous expression to get:

$$\widehat{\tilde{k}}_t + \widehat{\widetilde{z}}_t + \widehat{\widetilde{\mu}}_t = \frac{(1-\delta)}{\widetilde{z}\widetilde{\mu}}\widehat{\tilde{k}}_{t-1} + \frac{\widetilde{z}\widetilde{\mu}-(1-\delta)}{\widetilde{z}\widetilde{\mu}}(\widehat{\widetilde{z}}_t + \widehat{\widetilde{\mu}}_t + \widehat{\tilde{x}}_t)$$

or

$$\widehat{\tilde{k}}_t = \frac{(1-\delta)}{\widetilde{z}\widetilde{\mu}}\widehat{\tilde{k}}_{t-1} + \frac{\widetilde{z}\widetilde{\mu}-(1-\delta)}{\widetilde{z}\widetilde{\mu}}\widehat{\tilde{x}}_t - \frac{1-\delta}{\widetilde{z}\widetilde{\mu}}\left(\widehat{\widetilde{z}}_t + \widehat{\widetilde{\mu}}_t\right). \tag{28}$$

4.3. System of Linear Stochastic Difference Equations

We now present the equations in the system as ordered in Uhlig algorithm model2fun.m

Equation 1 The first equation is

$$\frac{\theta_w \Pi^{-(1-\chi_w)(1-\eta)}\widetilde{z}^{-(1-\eta)}}{(1-\theta_w)\left(\Pi^{w^*}\right)^{1-\eta}}(\widehat{\Pi}_t - \chi_w\widehat{\Pi}_{t-1} + \widehat{\Pi}^w_t + \widetilde{z}_t) = \widehat{\widetilde{w}}^*_t - \widehat{\widetilde{w}}_t$$

We make use of the fact that $\widehat{\Pi}^w_t = \widehat{\widetilde{w}}_t - \widehat{\widetilde{w}}_{t-1}$, and define the auxiliary parameter $a_1 = \frac{\theta_w \Pi^{-(1-\chi_w)(1-\eta)}\widetilde{z}^{-(1-\eta)}}{(1-\theta_w)\left(\Pi^{w^*}\right)^{1-\eta}}$. In the Uhlig code this expression appears as $a_1 = \frac{\theta_w \Pi^{-(1-\chi_w)(1-\eta)}\widetilde{z}^{-(1-\eta)}}{(1-\theta_w)\left\{\exp\left[\log(\Pi^{w^*})\right]\right\}^{1-\eta}}$, because when we solve for the steady state of Π^{w^*}, we express it in log-levels.

Then, the equation boils down to:

$$a_1\widehat{\Pi}_t - \chi_w a_1\widehat{\Pi}_{t-1} + a_1\widehat{\widetilde{w}}_t - a_1\widehat{\widetilde{w}}_{t-1} + a_1\widetilde{z}_t = \widehat{\widetilde{w}}^*_t - \widehat{\widetilde{w}}_t$$

and rearranging:

$$a_1\widehat{\Pi}_t - \chi_w a_1\widehat{\Pi}_{t-1} + (1+a_1)\widehat{\widetilde{w}}_t - a_1\widehat{\widetilde{w}}_{t-1} + a_1\widetilde{z}_t - \widehat{\widetilde{w}}^*_t = 0 \tag{29}$$

Equation 2 The second equation is:

$$\frac{\theta_p \Pi^{-(1-\varepsilon)(1-\chi)}}{(1-\theta_p)\left(\Pi^*\right)^{(1-\varepsilon)}}(\widehat{\Pi}_t - \chi \widehat{\Pi}_{t-1}) = \widehat{\Pi}_t^*$$

In order to make notation for compact, define $a_2 = \frac{\theta_p \Pi^{-(1-\varepsilon)(1-\chi)}}{(1-\theta_p)(\Pi^*)^{(1-\varepsilon)}}$. Note that in the file this expression appears as $a_2 = \frac{\theta_p \Pi^{-(1-\varepsilon)(1-\chi)}}{(1-\theta_p)\{\exp[\log(\Pi^*)]\}^{(1-\varepsilon)}}$, because when we solve for the steady state value Π^*, we express it in log-levels. Substituting for a_2:

$$a_2(\widehat{\Pi}_t - \chi \widehat{\Pi}_{t-1}) = \widehat{\Pi}_t^*$$

Rearranging:

$$a_2 \widehat{\Pi}_t - a_2 \chi \widehat{\Pi}_{t-1} - \widehat{\Pi}_t^* = 0 \tag{30}$$

Equation 3 The third equation is

$$\widehat{\widehat{r}}_t = \phi_u \hat{u}_t,$$

where $\phi_u = \gamma_2/\gamma_1$. Then,

$$-\widehat{\widehat{r}}_t + \phi_u \hat{u}_t = 0 \tag{31}$$

Equation 4 The fourth equation is

$$\widehat{g}_t^1 = \widehat{g}_t^2$$

Rearranging:

$$\widehat{g}_t^1 - \widehat{g}_t^2 = 0 \tag{32}$$

Equation 5 The fifth equation is:

$$\widehat{u}_t + \widehat{\widehat{k}}_{t-1} - \widehat{l}_t^d = \widehat{\widehat{w}}_t - \widehat{\widehat{r}}_t + \widehat{\widehat{z}}_t + \widehat{\widehat{\mu}}_t$$

Rearranging:

$$\widehat{u}_t + \widehat{\widehat{r}}_t + \widehat{\widehat{k}}_{t-1} - \widehat{l}_t^d - \widehat{\widehat{w}}_t - \widehat{\widehat{z}}_t - \widehat{\widehat{\mu}}_t \tag{33}$$

Equation 6 The sixth equation is:

$$\widehat{mc}_t = (1-\alpha)\widehat{\widehat{w}}_t + \alpha \widehat{\widehat{r}}_t$$

Rearranging:

$$(1-\alpha)\widehat{\widehat{w}}_t + \alpha \widehat{\widehat{r}}_t - \widehat{mc}_t = 0 \tag{34}$$

Equation 7 The seventh equation is:

$$\widehat{R}_t = \gamma_R \widehat{R}_{t-1} + (1-\gamma_R)\left(\gamma_\Pi \widehat{\Pi}_t + \gamma_y (\triangle \widehat{\widetilde{y}}^d_t + \widehat{\widetilde{z}}_t)\right) + \widehat{m}_t$$

Rearranging:

$$-\widehat{R}_t + \gamma_R \widehat{R}_{t-1} + (1-\gamma_R)\gamma_\Pi \widehat{\Pi}_t + (1-\gamma_R)\gamma_y \widehat{\widetilde{z}}_t + (1-\gamma_R)\gamma_y \widehat{\widetilde{y}}^d_t - (1-\gamma_R)\gamma_y \widehat{\widetilde{y}}^d_{t-1} + \widehat{m}_t = 0 \quad (35)$$

Equation 8 The eighth equation is:

$$\tilde{c}\widehat{\widetilde{c}}_t + \tilde{x}\widehat{\widetilde{x}}_t + \frac{\gamma_1 \widetilde{k}}{\widetilde{z}\widetilde{\mu}}\widehat{u}_t = \tilde{y}^d \widehat{\widetilde{y}}^d_t$$

Note that in the code, because we have solved for the log-steady state, constants enter as follows: $\tilde{c} = \exp\left[\log\left(\tilde{c}\right)\right]$, $\tilde{x} = \exp\left[\log\left(\tilde{x}\right)\right]$, $\tilde{k} = \exp\left[\log\left(\tilde{k}\right)\right]$, $\tilde{y}^d = \exp\left[\log\left(\tilde{y}^d\right)\right]$. Rearranging:

$$\tilde{c}\widehat{\widetilde{c}}_t + \tilde{x}\widehat{\widetilde{x}}_t + \frac{\gamma_1 \widetilde{k}}{\widetilde{z}\widetilde{\mu}}\widehat{u}_t - \tilde{y}^d \widehat{\widetilde{y}}^d_t = 0 \quad (36)$$

Equation 9 The ninth equation is:

$$\left(\tilde{y}^d v^p\right)\left(\widehat{v}^p_t + \widehat{\widetilde{y}}^d_t\right) = \frac{\widetilde{A}}{\widetilde{\widetilde{z}}}\left(u\widetilde{k}\right)^\alpha \left(\widetilde{l}^d\right)^{1-\alpha}\left(\widehat{\widetilde{A}}_t - \widehat{\widetilde{z}}_t + \alpha\left(\widehat{u}_t + \widehat{\widetilde{k}}_{t-1}\right) + (1-\alpha)\,\widetilde{l}^d_t\right)$$

Define the parameter $produc = \frac{\tilde{A}}{\tilde{\tilde{z}}}\left(u\widetilde{k}\right)^\alpha \left(\widetilde{l}^d\right)^{1-\alpha}$. Note that in terms of Uhlig notation, this is defined as $produc = \frac{\tilde{A}}{\tilde{\tilde{z}}}\left\{\exp\left[\log(u)\right]\exp\left[\log(\widetilde{k})\right]\right\}^\alpha \left\{\exp\left[\log(\widetilde{l}^d)\right]\right\}^{1-\alpha}$. Also note that in terms of the code, we have that $\tilde{y}^d = \exp\left[\log\left(\tilde{y}^d\right)\right]$, $v^p = \exp\left[\log\left(v^p\right)\right]$. Substituting:

$$\left(\tilde{y}^d v^p\right)\left(\widehat{v}^p_t + \widehat{\widetilde{y}}^d_t\right) = produc\left(\widehat{\widetilde{A}}_t - \widehat{\widetilde{z}}_t + \alpha\left(\widehat{u}_t + \widehat{\widetilde{k}}_{t-1}\right) + (1-\alpha)\,\widetilde{l}^d_t\right)$$

Rearranging:

$$\begin{aligned}(\tilde{y}^d v^p)\widehat{v}^p_t + (\tilde{y}^d v^p)\widehat{\widetilde{y}}^d_t - (produc)\widehat{\widetilde{A}}_t + (produc)\widehat{\widetilde{z}}_t - (\alpha)(produc)\widehat{u}_t \\ -(\alpha)(produc)\widehat{\widetilde{k}}_{t-1} - (1-\alpha)\,(produc)\widetilde{l}^d_t = 0\end{aligned} \quad (37)$$

Equation 10 The tenth equation is:

$$\widehat{v}_t^p = \theta_p \Pi^{\varepsilon(1-\chi)} \left(\varepsilon(\widehat{\Pi}_t - \chi \widehat{\Pi}_{t-1}) + \widehat{v}_{t-1}^p \right) - \left(1 - \theta_p \Pi^{\varepsilon(1-\chi)}\right) \varepsilon \widehat{\Pi}_t^*$$

Define $a_3 = \beta \theta_p \Pi^{\varepsilon(1-\chi)}$. Then, $\theta_p \Pi^{\varepsilon(1-\chi)} = \frac{a_3}{\beta}$. This type of parameter definition will become clearer when we analyze the price setting equations. Then, substituting:

$$\widehat{v}_t^p = \frac{a_3}{\beta} \varepsilon \widehat{\Pi}_t - \chi \varepsilon \frac{a_3}{\beta} \widehat{\Pi}_{t-1} + \frac{a_3}{\beta} \widehat{v}_{t-1}^p - \left(1 - \frac{a_3}{\beta}\right) \varepsilon \widehat{\Pi}_t^*$$

And rearranging:

$$\frac{a_3 \varepsilon}{\beta} \widehat{\Pi}_t - \frac{a_3 \varepsilon \chi}{\beta} \widehat{\Pi}_{t-1} + \frac{a_3}{\beta} \widehat{v}_{t-1}^p - \left(1 - \frac{a_3}{\beta}\right) \varepsilon \widehat{\Pi}_t^* - \widehat{v}_t^p = 0 \tag{38}$$

Equation 11 The eleventh equation is:

$$\widehat{v}_t^w = \theta_w \Pi^{\eta(1-\chi_w)} \widetilde{z}^{\eta} \left(\eta \left(\widehat{\Pi}_t - \chi_w \widehat{\Pi}_{t-1} + \widehat{\widetilde{w}}_t - \widehat{\widetilde{w}}_{t-1} + \widehat{\widetilde{z}}_t \right) + \widehat{v}_{t-1}^w \right) - \left(1 - \theta_w \Pi^{\eta(1-\chi_w)} \widetilde{z}^{\eta}\right) \eta (\widehat{\widetilde{w}}_t^* - \widehat{\widetilde{w}}_t)$$

First, let's define $a_4 = \theta_w \Pi^{\eta(1-\chi_w)} \widetilde{z}^{\eta}$. Substituting:

$$\widehat{v}_t^w = a_4 \left(\eta \left(\widehat{\Pi}_t - \chi_w \widehat{\Pi}_{t-1} + \widehat{\widetilde{w}}_t - \widehat{\widetilde{w}}_{t-1} + \widehat{\widetilde{z}}_t \right) + \widehat{v}_{t-1}^w \right) - (1 - a_4) \eta (\widehat{\widetilde{w}}_t^* - \widehat{\widetilde{w}}_t)$$

Rearranging:

$$\widehat{v}_t^w = a_4 \eta \widehat{\Pi}_t - \chi_w \eta a_4 \widehat{\Pi}_{t-1} + a_4 \eta \widehat{\widetilde{w}}_t - a_4 \eta \widehat{\widetilde{w}}_{t-1} + a_4 \eta \widehat{\widetilde{z}}_t + a_4 \widehat{v}_{t-1}^w - (1 - a_4) \eta \widehat{\widetilde{w}}_t^* + (1 - a_4) \eta \widehat{\widetilde{w}}_t$$

Then,

$$a_4 \eta \widehat{\Pi}_t - \chi_w \eta a_4 \widehat{\Pi}_{t-1} + \eta \widehat{\widetilde{w}}_t - a_4 \eta \widehat{\widetilde{w}}_{t-1} + a_4 \eta \widehat{\widetilde{z}}_t + a_4 \widehat{v}_{t-1}^w - (1 - a_4) \eta \widehat{\widetilde{w}}_t^* - \widehat{v}_t^w = 0 \tag{39}$$

Equation 12 The twelfth equation is:

$$\widehat{l}_t = \widehat{v}_t^w + \widehat{l}_t^d$$

Which we rearrange in the code to appear as:

$$\widehat{l}_t - \widehat{v}_t^w - \widehat{l}_t^d = 0 \tag{40}$$

Equation 13 The thirteenth equation is:

$$\widehat{\widetilde{k}}_t = \frac{(1-\delta)}{\widetilde{z}\widetilde{\mu}}\widehat{\widetilde{k}}_{t-1} + \frac{\widetilde{z}\widetilde{\mu}-(1-\delta)}{\widetilde{z}\widetilde{\mu}}\widehat{\widetilde{x}}_t - \frac{1-\delta}{\widetilde{z}\widetilde{\mu}}\left(\widehat{\widetilde{z}}_t + \widehat{\widetilde{\mu}}_t\right)$$

which we rearrange to:

$$\frac{(1-\delta)}{\widetilde{z}\widetilde{\mu}}\widehat{\widetilde{k}}_{t-1} + \left[1 - \frac{(1-\delta)}{\widetilde{z}\widetilde{\mu}}\right]\widehat{\widetilde{x}}_t - \frac{1-\delta}{\widetilde{z}\widetilde{\mu}}\left(\widehat{\widetilde{z}}_t + \widehat{\widetilde{\mu}}_t\right) - \widehat{\widetilde{k}}_t = 0 \tag{41}$$

Equation 14 The fourteenth equation is:

$$\widehat{\widetilde{z}}_t = \frac{\widehat{\widetilde{A}}_t + \alpha\widehat{\widetilde{\mu}}_t}{1-\alpha}$$

which we rearrange to:

$$\frac{1}{1-\alpha}\widehat{\widetilde{A}}_t + \frac{\alpha}{1-\alpha}\widehat{\widetilde{\mu}}_t - \widehat{\widetilde{z}}_t = 0 \tag{42}$$

Equation 15 The fifteenth equation is:

$$(1-b\beta\widetilde{z})\widehat{\widetilde{\lambda}}_t = \widehat{d}_t - b\beta\widetilde{z}\mathbb{E}_t\widehat{d}_{t+1} - \frac{1+b^2\beta}{\left(1-\frac{b}{\widetilde{z}}\right)}\widehat{\widetilde{c}}_t + \frac{b}{\widetilde{z}\left(1-\frac{b}{\widetilde{z}}\right)}\widehat{\widetilde{c}}_{t-1} + \frac{\beta b\widetilde{z}}{\left(1-\frac{b}{\widetilde{z}}\right)}\mathbb{E}_t\widehat{\widetilde{c}}_{t+1} - \frac{b}{\widetilde{z}\left(1-\frac{b}{\widetilde{z}}\right)}\widehat{\widetilde{z}}_t$$

Rearranging:

$$\widehat{d}_t - b\beta\widetilde{z}\mathbb{E}_t\widehat{d}_{t+1} - \frac{1+b^2\beta}{\left(1-\frac{b}{\widetilde{z}}\right)}\widehat{\widetilde{c}}_t + \frac{b}{\widetilde{z}\left(1-\frac{b}{\widetilde{z}}\right)}\widehat{\widetilde{c}}_{t-1} + \frac{\beta b\widetilde{z}}{\left(1-\frac{b}{\widetilde{z}}\right)}\mathbb{E}_t\widehat{\widetilde{c}}_{t+1} - \frac{b}{\widetilde{z}\left(1-\frac{b}{\widetilde{z}}\right)}\widehat{\widetilde{z}}_t - (1-b\beta\widetilde{z})\widehat{\widetilde{\lambda}}_t = 0 \tag{43}$$

Equation 16 The sixteenth equation is:

$$\widehat{\widetilde{\lambda}}_t = \mathbb{E}_t\{\widehat{\widetilde{\lambda}}_{t+1} + \widehat{R}_t - \widehat{\Pi}_{t+1}\}$$

which we rearrange to:

$$\mathbb{E}_t\{\widehat{\widetilde{\lambda}}_{t+1} - \widehat{\widetilde{\lambda}}_t + \widehat{R}_t - \widehat{\Pi}_{t+1}\} = 0 \tag{44}$$

Equation 17 The seventeenth equation is:

$$\widehat{\widetilde{q}}_t = \mathbb{E}_t\Delta\widehat{\widetilde{\lambda}}_{t+1} + \frac{\beta(1-\delta)}{\widetilde{z}\widetilde{\mu}}\mathbb{E}_t\widehat{\widetilde{q}}_{t+1} + \left(1 - \frac{\beta(1-\delta)}{\widetilde{z}\widetilde{\mu}}\right)\mathbb{E}_t\widehat{\widetilde{r}}_{t+1}$$

which we rearrange to:

$$\mathbb{E}_t\widehat{\widetilde{\lambda}}_{t+1}-\widehat{\widetilde{\lambda}}_t+\frac{\beta\left(1-\delta\right)}{\widetilde{z}\widetilde{\mu}}\mathbb{E}_t\widehat{\widetilde{q}}_{t+1}+\left(1-\frac{\beta(1-\delta)}{\widetilde{z}\widetilde{\mu}}\right)\mathbb{E}_t\widehat{\widetilde{r}}_{t+1}-\widehat{\widetilde{q}}_t=0 \tag{45}$$

Equation 18 The eighteenth equation is:

$$\kappa\widetilde{z}^2\left(\Delta\widehat{\widetilde{x}}_t+\widehat{\widetilde{z}}_t\right)=\widehat{\widetilde{q}}_t+\beta\kappa\widetilde{z}^2\mathbb{E}_t\Delta\widehat{\widetilde{x}}_{t+1}$$

which we rearrange by undoing the first-difference operator:

$$\widehat{\widetilde{q}}_t+\beta\kappa\widetilde{z}^2\mathbb{E}_t\widehat{\widetilde{x}}_{t+1}-(1+\beta)\kappa\widetilde{z}^2\widehat{\widetilde{x}}_t+\kappa\widetilde{z}^2\widehat{\widetilde{x}}_{t-1}-\kappa\widetilde{z}^2\widehat{\widetilde{z}}_t=0 \tag{46}$$

Equation 19 The nineteenth equation is:

$$\begin{aligned}\widehat{f}_t \;=\;& \left(1-\beta\theta_w\widetilde{z}^{\eta-1}\Pi^{-(1-\eta)(1-\chi_w)}\right)\left(\left(1-\eta\right)\widehat{\widetilde{w}}^*_t+\widehat{\widetilde{\lambda}}_t+\eta\widehat{\widetilde{w}}_t+\widehat{l}^d_t\right)+\\ & \beta\theta_w\Pi^{-(1-\eta)(1-\chi_w)}\widetilde{z}^{\eta-1}\mathbb{E}_t\left(\widehat{f}_{t+1}-\left(1-\eta\right)\left(\widehat{\Pi}_{t+1}-\chi_w\widehat{\Pi}_t+\Delta\widehat{\widetilde{w}}^*_{t+1}\right)\right)\end{aligned}$$

define $a_5=\beta\theta_w\widetilde{z}^{\eta-1}\Pi^{-(1-\eta)(1-\chi_w)}$. Substituting:

$$\widehat{f}_t=\left(1-a_5\right)\left[\left(1-\eta\right)\widehat{\widetilde{w}}^*_t+\widehat{\widetilde{\lambda}}_t+\eta\widehat{\widetilde{w}}_t+\widehat{l}^d_t\right]+a_5\mathbb{E}_t\left[\widehat{f}_{t+1}-\left(1-\eta\right)\left(\widehat{\Pi}_{t+1}-\chi_w\widehat{\Pi}_t+\Delta\widehat{\widetilde{w}}^*_{t+1}\right)\right]$$

Rearranging:

$$\begin{aligned}&\left(1-\eta\right)\widehat{\widetilde{w}}^*_t+\left(1-a_5\right)\widehat{\widetilde{\lambda}}_t+\left(1-a_5\right)\eta\widehat{\widetilde{w}}_t+\left(1-a_5\right)\widehat{l}^d_t+a_5\mathbb{E}_t\widehat{f}_{t+1}\\ &+(\eta-1)a_5\mathbb{E}_t\widehat{\Pi}_{t+1}-(\eta-1)a_5\chi_w\widehat{\Pi}_t-(1-\eta)a_5\mathbb{E}_t\widehat{\widetilde{w}}^*_{t+1}-\widehat{f}_t=0\end{aligned} \tag{47}$$

Equation 20 The twentieth equation is:

$$\begin{aligned}\widehat{f}_t \;=\;& \left(1-\beta\theta_w\widetilde{z}^{\eta(1+\gamma)}\Pi^{\eta(1+\gamma)(1-\chi_w)}\right)\left(\widehat{d}_t+\widehat{\varphi}_t+\eta\left(1+\gamma\right)\left(\widehat{\widetilde{w}}_t-\widehat{\widetilde{w}}^*_t\right)+(1+\gamma)\widehat{l}^d_t\right)\\ & +\beta\theta_w\widetilde{z}^{\eta(1+\gamma)}\Pi^{\eta(1+\gamma)(1-\chi_w)}\mathbb{E}_t\left(\widehat{f}_{t+1}+\eta\left(1+\gamma\right)\left(\widehat{\Pi}_{t+1}-\chi_w\widehat{\Pi}_t+\Delta\widehat{\widetilde{w}}^*_{t+1}\right)\right)\end{aligned}$$

Define $a_6=\beta\theta_w\widetilde{z}^{\eta(1+\gamma)}\Pi^{\eta(1+\gamma)(1-\chi_w)}$. Then,

$$\begin{aligned}\widehat{f}_t \;=\;& \left(1-a_6\right)\left(\widehat{d}_t+\widehat{\varphi}_t+\eta\left(1+\gamma\right)\left(\widehat{\widetilde{w}}_t-\widehat{\widetilde{w}}^*_t\right)+(1+\gamma)\widehat{l}^d_t\right)\\ & +a_6\mathbb{E}_t\left(\widehat{f}_{t+1}+\eta\left(1+\gamma\right)\left(\widehat{\Pi}_{t+1}-\chi_w\widehat{\Pi}_t+\Delta\widehat{\widetilde{w}}^*_{t+1}\right)\right)\end{aligned}$$

Rearranging,

$$\begin{aligned}&(1-a_6)\,\widehat{d}_t+(1-a_6)\,\widehat{\varphi}_t+\eta\,(1+\gamma)\,(1-a_6)\,\widehat{\widetilde{w}}_t-\eta\,(1+\gamma)\,\widehat{\widetilde{w}}_t^*+(1+\gamma)\,(1-a_6)\,\widehat{l}_t^d-\widehat{f}_t\\&\quad+a_6\mathbb{E}_t\widehat{f}_{t+1}+a_6\eta\,(1+\gamma)\,\mathbb{E}_t\widehat{\Pi}_{t+1}-a_6\eta\,(1+\gamma)\,\chi_w\widehat{\Pi}_t+a_6\eta\,(1+\gamma)\,\mathbb{E}_t\widehat{\widetilde{w}}_{t+1}^*=0 \qquad (48)\end{aligned}$$

Equation 21 The twenty first equation is

$$\widehat{g}_t^1=\left(1-\beta\theta_p\Pi^{\varepsilon(1-\chi)}\right)\left(\widehat{\widetilde{\lambda}}_t+\widehat{mc}_t+\widehat{\widetilde{y}}_t^d\right)+\beta\theta_p\Pi_t^{\varepsilon(1-\chi)}\mathbb{E}_t\left(\varepsilon(\widehat{\Pi}_{t+1}-\chi\widehat{\Pi}_t)+\widehat{g}_{t+1}^1\right)$$

As before, define $a_3=\beta\theta_p\Pi^{\varepsilon(1-\chi)}$. Then:

$$(1-a_3)\,\widehat{\widetilde{\lambda}}_t+(1-a_3)\,\widehat{mc}_t+(1-a_3)\,\widehat{\widetilde{y}}_t^d+\varepsilon a_3\mathbb{E}_t\widehat{\Pi}_{t+1}-\chi\varepsilon a_3\widehat{\Pi}_t+a_3\mathbb{E}_t\widehat{g}_{t+1}^1-\widehat{g}_t^1=0 \qquad (49)$$

Equation 22 The twenty second equation is

$$\begin{aligned}\widehat{g}_t^2 \;&=\; \left(1-\beta\theta_p\Pi^{-(1-\varepsilon)(1-\chi)}\right)\left(\widehat{\widetilde{\lambda}}_t+\widehat{\Pi}_t^*+\widehat{\widetilde{y}}_t^d\right)\\&\quad+\beta\theta_p\Pi^{-(1-\varepsilon)(1-\chi)}\mathbb{E}_t\left(-(1-\varepsilon)\left(\widehat{\Pi}_{t+1}-\chi\widehat{\Pi}_t\right)-\left(\widehat{\Pi}_{t+1}^*-\widehat{\Pi}_t^*\right)+\widehat{g}_{t+1}^2\right)\end{aligned}$$

Define: $a_7=\beta\theta_p\Pi^{-(1-\varepsilon)(1-\chi)}$. Substituting:

$$(1-a_7)\,\widehat{\widetilde{\lambda}}_t+\widehat{\Pi}_t^*+(1-a_7)\,\widehat{\widetilde{y}}_t^d+(\varepsilon-1)a_7\mathbb{E}_t\widehat{\Pi}_{t+1}-\chi(\varepsilon-1)a_7\widehat{\Pi}_t-a_7\mathbb{E}_t\widehat{\Pi}_{t+1}^*+a_7\mathbb{E}_t\widehat{g}_{t+1}^2-\widehat{g}_t^2=0$$

Shocks The preference shocks has the following structure:

$$\begin{aligned}\widehat{d}_t &= \rho_d\widehat{d}_{t-1}+\varepsilon_{d,t}\\ \widehat{\varphi}_t &= \rho_\varphi\widehat{\varphi}_{t-1}+\varepsilon_{\varphi,t}\end{aligned}$$

Note that Uhlig does not allow to write something like $\widehat{d}_t=\rho\widehat{d}_{t-1}+\sigma_d\varepsilon_{d,t}$. We declare the variance-covariance matrix later. The following shocks are not defined because of their i.i.d. nature.

$$\begin{aligned}\widehat{\widetilde{\mu}}_t &= z_{\mu,t}\\ \widehat{\widetilde{A}}_t &= z_{A,t}\\ m_t &= \sigma_m\varepsilon_{m,t}\end{aligned}$$

4.4. Solving the Model

Now, let

$$state_t = \left(\widehat{\Pi}_t, \widehat{\widetilde{w}}_t, \widehat{g}_t^1, \widehat{g}_t^2, \widehat{\widetilde{k}}_t, \widehat{R}_t, \widehat{\widetilde{y}}_t^d, \widehat{\widetilde{c}}_t, \widehat{\upsilon}_t^p, \widehat{\upsilon}_t^w, \widehat{\widetilde{q}}_t, \widehat{\widetilde{f}}_t, \widehat{\widetilde{x}}_t, \widehat{\widetilde{\lambda}}_t, \widehat{\widetilde{z}}_t\right)',$$

$$nstate_t = \left(\widehat{\widetilde{r}}_t, \widehat{u}_t, \widehat{\Pi}_t^*, \widehat{l}_t^d, \widehat{mc}_t, \widehat{l}_t, \widehat{\widetilde{w}}_t^*\right)',$$

$$exo_t = \left(z_{\mu,t}, \widehat{d}_t, \widehat{\varphi}_t, z_{A,t}, m_t\right)',$$

and

$$\varepsilon_t = \left(\varepsilon_{\mu,t}, \varepsilon_{d,t}, \varepsilon_{\varphi,t}, \varepsilon_{A,t}, \varepsilon_{m,t}\right)'.$$

Then, we need to write the system defined above in Uhlig's format, i.e.:

$$0 = AA * state_t + BB * state_{t-1} + CC * nstate_t + DD * exo_t,$$

$$0 = \mathbb{E}_t \left(\begin{array}{c} FF * state_{t+1} + GG * state_t + HH * state_{t-1} \\ +JJ * nstate_{t+1} + KK * nstate_t + LL * exo_{t+1} + MM * exo_t \end{array} \right),$$

and

$$exo_{t+1} = NN * exo_t + \Sigma^{1/2} * \varepsilon_{t+1} \text{ with } \mathbb{E}_t \varepsilon_{t+1} = 0.$$

4.4.1. Writing the Model in Uhlig's Form

First, note that from this section on, and in the codes, we define the variables in terms of their loglinear deviation from steady-state. The matrices in Uhlig's notation are as following

$$
AA = \begin{pmatrix}
a_1 & 1+a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_1 \\
a_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 1-\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(1-\gamma_R)\gamma_\Pi & 0 & 0 & 0 & 0 & -1 & (1-\gamma_R)\gamma_y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (1-\gamma_R)\gamma_y \\
0 & 0 & 0 & 0 & 0 & 0 & -\tilde{y}^d & \tilde{c} & 0 & 0 & 0 & 0 & \tilde{x} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \tilde{y}^d v^p & 0 & \tilde{y}^d v^p & 0 & 0 & 0 & 0 & 0 & produc \\
a_3\varepsilon/\beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\eta a_4 & \eta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & \eta a_4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-\frac{(1-\delta)}{\tilde{z}\tilde{\mu}} & 0 & -\frac{(1-\delta)}{\tilde{z}\tilde{\mu}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix}
$$

$$
BB = \begin{pmatrix}
-\chi_w a_1 & -a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\chi a_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \gamma_R & -(1-\gamma_R)\gamma_y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\alpha(produc) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{-a_3\varepsilon\chi}{\beta} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_3}{\beta} & 0 & 0 & 0 & 0 & 0 & 0 \\
-\chi_w \eta a_4 & -\eta a_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1-\delta}{\tilde{z}\tilde{\mu}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

$$CC = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & \phi_u & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & -1 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\gamma_1 \tilde{k}}{\tilde{z}\tilde{\mu}} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\alpha(produc) & 0 & -(1-\alpha)(produc) & 0 & 0 & 0 \\ 0 & 0 & -(1-\frac{a_3}{\beta})\varepsilon & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\eta(1-a_4) \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$DD = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -produc & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -\frac{1-\delta}{\tilde{z}\tilde{\mu}} & 0 & 0 & 0 & 0 \\ \frac{\alpha}{1-\alpha} & 0 & 0 & \frac{1}{1-\alpha} & 0 \end{pmatrix}$$

$$
FF = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\beta b \tilde{z}}{1-\frac{b}{\tilde{z}}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\beta(1-\delta)}{\tilde{z}\tilde{\mu}} & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \beta\kappa\tilde{z}^2 & 0 & 0 \\
a_5(\eta-1) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_5 & 0 & 0 & 0 \\
a_6\eta(1+\gamma) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_6 & 0 & 0 & 0 \\
a_3\varepsilon & 0 & a_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a_7(\varepsilon-1) & 0 & 0 & a_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

$$
GG = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & GG_{1,8} & 0 & 0 & 0 & 0 & 0 & -(1-b\beta\tilde{z}) & GG_{1,15} \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & GG_{4,13} & 0 & -\kappa\tilde{z}^2 \\
-a_5\chi_w(\eta-1) & (1-a_5)\eta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1-a_5 & 0 \\
-a_6\chi_w\eta(1+\gamma) & GG_{6,2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
-a_3\varepsilon\chi & 0 & -1 & 0 & 0 & 0 & 1-a_3 & 0 & 0 & 0 & 0 & 0 & 0 & 1-a_3 & 0 \\
-a_7(\varepsilon-1)\chi & 0 & 0 & -1 & 0 & 0 & 1-a_7 & 0 & 0 & 0 & 0 & 0 & 0 & 1-a_7 & 0
\end{pmatrix}
$$

where

$$
\begin{aligned}
GG_{1,8} &= -\frac{1+\beta b^2}{1-\frac{b}{\tilde{z}}} \\
GG_{1,15} &= -\frac{b}{\tilde{z}(1-\frac{b}{\tilde{z}})} \\
GG_{6,2} &= (1-a_6)\eta(1+\gamma) \\
GG_{4,13} &= -(1+\beta)\kappa\tilde{z}^2
\end{aligned}
$$

$$
HH = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{b}{\tilde{z}(1-\frac{b}{\tilde{z}})} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \kappa\tilde{z}^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

$$
JJ = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 - \frac{\beta(1-\delta)}{\tilde{z}\tilde{\mu}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -a_5(1-\eta) \\
0 & 0 & 0 & 0 & 0 & 0 & a_6\eta(1+\gamma) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -a_7 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

$$
KK = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1-a_5 & 0 & 0 & 1-\eta \\
0 & 0 & 0 & (1-a_6)(1+\gamma) & 0 & 0 & -\eta(1+\gamma) \\
0 & 0 & 0 & 0 & 1-a_3 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

$$
LL = \begin{pmatrix}
0 & -\beta b\tilde{z} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

$$
MM = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 1-a_6 & 1-a_6 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

$$NN = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \rho_d & 0 & 0 & 0 \\ 0 & 0 & \rho_\varphi & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Sigma = \begin{pmatrix} \sigma_\mu^2 & 0 & 0 & 0 & 0 \\ 0 & \sigma_d^2 & 0 & 0 & 0 \\ 0 & 0 & \sigma_\varphi^2 & 0 & 0 \\ 0 & 0 & 0 & \sigma_A^2 & 0 \\ 0 & 0 & 0 & 0 & \sigma_m^2 \end{pmatrix}$$

4.4.2. Writing the Likelihood function

Uhlig's solution method gives us the following solution:

$$state_t = PP * state_{t-1} + QQ * exo_t$$

and

$$nstate_t = RR * state_{t-1} + SS * exo_t.$$

We observe $obs_t = (\log \Pi_t, \log R_t, \triangle \log w_t, \triangle \log y_t)'$. Therefore, to write the likelihood function, we need to write the model in the following state space form:

$$\begin{aligned} S_t &= A * S_{t-1} + B * \varepsilon_t \\ obs_t &= C * S_{t-1} + D * \varepsilon_t \end{aligned}$$

where $S_{t-1} = (1, state_t, state_{t-1}, exo_{t-1})$.

4.4.3. Building the A and B matrices

Hence,

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & PP & 0 & QQ * NN \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & NN \end{bmatrix}$$

and

$$B=\begin{bmatrix} 0 \\ QQ \\ 0 \\ I \end{bmatrix} * \Sigma^{1/2}.$$

4.4.4. Building the C and D matrices

Define the observables vector:

$$obs_t = \left(\log \Pi_t, \log R_t, \triangle \log w_t, \triangle \log y_t\right)'.$$

Note that:

$$\begin{aligned}\widehat{\Pi}_t &= PP\left(1,:\right) * state_{t-1} + QQ\left(1,:\right) * exo_t = \\ & PP\left(1,:\right) * state_{t-1} + QQ\left(1,:\right) * NN * exo_{t-1} + QQ\left(1,:\right) * \Sigma^{1/2} * \varepsilon_t,\end{aligned}$$

$$\begin{aligned}\widehat{R}_t &= PP\left(6,:\right) * state_{t-1} + QQ\left(6,:\right) * exo_t = \\ & PP\left(6,:\right) * state_{t-1} + QQ\left(6,:\right) * NN * exo_{t-1} + QQ\left(6,:\right) * \Sigma^{1/2} * \varepsilon_t,\end{aligned}$$

$$\begin{aligned}\widehat{\widetilde{w}}_t - \widehat{\widetilde{w}}_{t-1} &= PP\left(2,:\right) * state_{t-1} + QQ\left(2,:\right) * exo_t - PP\left(2,:\right) * state_{t-2} - QQ\left(2,:\right) * exo_{t-1} = \\ & PP\left(2,:\right) * state_{t-1} - PP\left(2,:\right) * state_{t-2} \\ & + QQ\left(2,:\right) * NN * exo_{t-1} - QQ\left(2,:\right) * exo_{t-1} + QQ\left(2,:\right) * \Sigma^{1/2} * \varepsilon_t,\end{aligned}$$

and

$$\begin{aligned}\widehat{\widetilde{y}}_t - \widehat{\widetilde{y}}_{t-1} &= PP\left(7,:\right) * state_{t-1} + QQ\left(7,:\right) * exo_t - PP\left(7,:\right) * state_{t-2} - QQ\left(7,:\right) * exo_{t-1} = \\ & PP\left(7,:\right) * state_{t-1} - PP\left(7,:\right) * state_{t-2} \\ & + QQ\left(7,:\right) * NN * exo_{t-1} - QQ\left(7,:\right) * exo_{t-1} + QQ\left(7,:\right) * \Sigma^{1/2} * \varepsilon_t,\end{aligned}$$

Also, remember that $\widehat{\Pi}_t = \log \Pi_t - \log \Pi_{ss}$, $\widehat{R}_t = \log R_t - \log R_{ss}$, $\widehat{\widetilde{w}}_t - \widehat{\widetilde{w}}_{t-1} = \triangle \log \widetilde{w}_t = \triangle \log w_t - \triangle \log z_t$, and $\widehat{\widetilde{y}}_t - \widehat{\widetilde{y}}_{t-1} = \triangle \log \widetilde{y}_t = \triangle \log y_t - \triangle \log z_t$. Since $\triangle \log z_t = \frac{\alpha \Lambda_\mu + \Lambda_A}{1-\alpha} + \frac{\alpha z_{\mu,t} + z_{A,t}}{1-\alpha}$, we have:

$$obs_t = C * S_{t-1} + D * \varepsilon_t.$$

where

$$C=\begin{bmatrix} \log \Pi_{ss} & PP\left(1,:\right) & 0 & QQ\left(1,:\right)*NN \\ \log R_{ss} & PP\left(6,:\right) & 0 & QQ\left(6,:\right)*NN \\ \frac{\alpha\Lambda_{\mu}+\Lambda_{A}}{1-\alpha} & PP\left(2,:\right) & -PP\left(2,:\right) & QQ\left(2,:\right)*\left(NN-I\right) \\ \frac{\alpha\Lambda_{\mu}+\Lambda_{A}}{1-\alpha} & PP\left(7,:\right) & -PP\left(7,:\right) & QQ\left(7,:\right)*\left(NN-I\right) \end{bmatrix}$$

and

$$D=\begin{bmatrix} QQ\left(1,:\right) \\ QQ\left(6,:\right) \\ QQ\left(2,:\right)+\frac{\begin{pmatrix} \alpha & 0 & 0 & 1 & 0 \end{pmatrix}}{1-\alpha} \\ QQ\left(7,:\right)+\frac{\begin{pmatrix} \alpha & 0 & 0 & 1 & 0 \end{pmatrix}}{1-\alpha} \end{bmatrix}*\Sigma^{1/2}$$