

Directed Search and Optimal Production *

Athanasios Geromichalos[†]
Department of Economics
University of Pennsylvania

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Abstract

I consider a model of directed search in which strategic sellers advertise general trading mechanisms. A mechanism determines the number of buyers that will get served and the side payments as a function of ex post realized demand. After observing these advertisements buyers simultaneously visit exactly one seller. Each buyer's expected utility depends on the visiting decisions of other buyers. This dependence becomes especially interesting since the buyers cannot coordinate their visiting strategies. Despite the presence of strategic interaction among the sellers all symmetric equilibria are constrained efficient but not payoff equivalent. Therefore, authorities should intervene in this type of market to redistribute surplus and not to improve efficiency. As markets grow infinitely large all equilibria yield the same profit. For the large market case I provide conditions under which only a very simple class of mechanisms is posted in equilibrium.

Keywords: directed search, efficiency, multiplicity of equilibrium

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[†]Contact: geromich@sas.upenn.edu

1 Introduction

In many markets sellers face a stochastic demand, and buyers visit sellers without knowing their total number of customers. If the sellers have capacity constraints, they might not be able to serve all customers who visit their store (if too many show up). On the other hand, if the nature of the good is such that a minimum number of customers is required for the service to be profitable, sellers might not be willing to serve the customers that visit their store (if too few show up). In both cases, if the markets are characterized by frictions, in the sense that buyers pay a cost to visit more than one seller, some buyers might not get served due to the inability to coordinate their visiting strategies.¹

To illustrate this economic problem through a concrete example, consider the market for boat trips to the Greek islands during the low season. Ferries operated by different companies depart every morning to the same destinations. Suppose that a trip is profitable only if x or more passengers travel. Since demand is low, there is a chance that some ferries will get visited by fewer than x customers. Often ferry owners have a policy (known to the customers) of cancelling the trip in that event. The crucial feature of this market is that once a passenger visits a ferry and is informed that the trip is cancelled, it is too late to take another ferry. If there are more than x buyers that want to travel on a certain day, they would like to coordinate their visiting decisions so that they all travel. Clearly, coordination in this market is difficult.

A question that arises from the preceding example is whether it is optimal to announce in advance the cancellation of a trip. Ex post, if fewer than x customers visit a certain location, it is obvious that the owner should dock the boat. However, these announcements are made ex ante. If all owners follow such a strategy, one of them might have an incentive to deviate and advertise that her company never cancels trips. Such a policy might be ex ante profitable, since it may induce passengers to visit the deviant owner's ferry in order to secure their trip. This could lead to a situation in which all sellers advertise that they never cancel trips, which is ex post inefficient.²

The ferry example is one of many markets with stochastic demand, lack of coordination among buyers, and frictions. In some instances buyers failing to be served is not the only economic consideration. Visiting a seller with many customers often implies a consumption externality that might be negative or positive, depending on the nature of the good. Hence, in this class of markets, buyers care both about the probability of being served at a specific seller and about the utility they will obtain

¹ In a frictionless market the lack of coordination is irrelevant, since buyers who do not get served at the first store they visit can go to other stores at no cost.

² This situation resembles the standard Prisoner's Dilemma, in which each player's fear that the other player will not coordinate, leads to an inefficient equilibrium outcome.

if they get served. Restaurants, bars, and airline companies are just a few examples of businesses that operate in such environments. Studying the welfare properties of these markets is important, and currently there are no models that are well suited for this purpose.

In this paper I provide a general framework that builds upon the directed search literature (see for example Montgomery (1991), Lagos (2000), and Burdett, Shi, and Wright (2001)), and aims to analyze the efficiency properties, the production decisions, and the price determination in markets with the following characteristics. There are a few sellers who behave strategically. Each seller faces a stochastic demand, which she can affect through a public advertisement. Frictions are captured by the fact that once a buyer visits a seller it is technically impossible to visit another seller. The expected utility for a buyer from visiting a specific seller depends on the total number of buyers who visit the seller, and buyers cannot coordinate their visiting strategies.

Sellers compete with each other for customers by advertising general trading mechanisms. A mechanism determines, as a function of ex post realized demand, the number of buyers that will get served and the side payments. Buyers care about net expected utility. Therefore, sellers can increase the probability with which buyers visit them by advertising low prices, promising buyers a high probability of getting served, and/or guaranteeing a high utility of consumption for the buyers who will get served. When sellers choose their advertisements they face the following trade off: ex post, they wish to maximize profit, and ex ante, they promise to potential buyers a high surplus in order to attract a large expected number of visitors.

Since each seller chooses her production and price advertisements strategically in order to direct customers to her store and away from rival sellers' stores, one might expect inefficient outcomes to arise. However, I show that constrained efficiency is always achieved in symmetric equilibrium. Efficiency is constrained by the lack of coordination among buyers.³ Hence, the model predicts that ferry owners will indeed announce that the trip will be cancelled if a small number of customers shows up. Another interesting result is that in small markets, indeterminacy of equilibria arises (see also Coles and Eeckhout 2003). Continua of equilibrium prices exist, which are equally efficient, but not payoff equivalent.

In related literature, Hawkins (2006) also provides an efficiency result. In that paper it is assumed that sellers take as given that they must offer customers a certain level of expected surplus.⁴ Hence, the element of strategic interaction among sellers is absent. This paper shows that efficiency does not result from the competitive behavior of sellers as it is implied in Hawkins (2006). Prescott (1975) considers a model of

³ This means that the only way to improve upon the equilibrium allocation is to direct each buyer to a specific seller.

⁴ This is assumption is often adopted in the literature, and it is known as the market utility assumption. See Montgomery (1991), Acemoglu and Shimer (1999a),(1999b), and Galenianos and Kircher (2008).

the market for hotel rooms, in which sellers also have local monopoly power and face a stochastic demand. He shows that the market will provide the efficient amount of rooms. Prescott assumes that a buyer who does not get served at some seller can visit other sellers without incurring any cost (no frictions). The model I consider provides an analogous result for a market characterized by frictions.

The model is very tractable, and a closed form solution for the matching function is found for any number of buyers and sellers. Also, equilibrium prices are characterized but not uniquely determined. Although sellers are allowed to advertise general mechanisms, common practices, such as announcing a fixed price or an auction, describe equilibrium behavior.⁵ Consistent with findings in Virag (2008), I show that if sellers can charge an arbitrarily large entry fee, they can extract the whole market surplus in some equilibria. This result, along with the efficiency result, have considerable implications for economic policy. In particular, the authorities should intervene in this type of market only to redistribute surplus and not to improve efficiency.

I also consider the case of large markets. I show that the indeterminacy of equilibria that characterizes small markets vanishes when the number of traders in the market approaches infinity. As markets get infinitely large, the complexity of mechanisms available to the sellers increases dramatically.⁶ I present a set of fairly weak restrictions on preferences and technology, under which only a very simple class of mechanisms is posted in equilibrium. The model is well suited to examine the effect of changes in supply on the number of successful matches along the intensive and the extensive margin. Keeping the number of units per buyer fixed across economies, the probability with which buyers get served gets bigger as we move to economies with less sellers (and bigger production per seller). Hence, the number of successful matches is more responsive along the intensive margin. Related results are also presented in Burdett, Shi, and Wright (2001) and Lester (2008).

The rest of this paper is organized as follows. In Section 2 I present the basic model and define equilibrium. Section 3 examines the benchmark case of two buyers and two sellers. Section 4 generalizes the results of Section 3 and provides discussion on the main findings. In Section 5 I consider the case of large markets. Section 6 provides some concluding remarks and Section 7 presents a brief discussion of possible extensions of the model.

⁵ There is large literature on competing auctions when buyers have private independent values. This includes McAfee (1993), Peters (1997), Burguet and Sakovics (1999), and Julien, Kennes, and King (2000). Also, Epstein and Peters (1999), and Martimort and Stole (2002) provide revelation principle related results for competing mechanisms.

⁶ Since sellers advertise mechanisms based on ex post realized demand, when there is an infinite number of buyers in the market, a seller's advertisement becomes an infinite sequence.

2 The Model

There are n buyers and m sellers in the market. Both buyers and sellers are risk neutral. All buyers are identical and anonymous, and each wishes to purchase one unit of an indivisible good. Each seller can produce $x \leq \xi$ units of the good at cost $c(x)$. Unless otherwise specified, $\xi = n$, i.e. a seller can potentially accommodate every buyer in the market. There is no fixed cost. Buyers' utility from consuming the good depends on the number of customers who get served at some specific location. More precisely, if a seller who gets visited by $y \leq n$ customers serves $x \leq y$ of them, the utility obtained by each customer is $u(x)$, with $u(x) \geq 0$ for every $x \leq n$. At this stage it is not necessary to place any restrictions on the functions $c(x)$ and $u(x)$.

The exchange process consists of two stages. At the first stage, each seller posts an advertisement which describes the trading mechanism that will be followed at her store, taking as given the mechanisms of her $m - 1$ competitors. A mechanism describes at every contingency, i.e. as a function of ex post realized demand, how many and which buyers get served and any side payments. Sellers are allowed to post any direct mechanism that treats the buyers symmetrically.⁷ At the second stage, buyers observe all the advertisements and choose a probability of visiting each seller, taking as given the strategies of other buyers. Once buyers show up at their preferred location, trade takes place according to the publicly advertised mechanisms. Sellers commit to their advertisements.

Since buyers are anonymous and identical, a seller's mechanism can be fully characterized by a pair of vectors which describe prices and production at every possible contingency. A trading mechanism for seller j is defined as $M^j \equiv \{\mathbf{p}^j, \mathbf{k}^j\}$, where $\mathbf{p}^j = (p_1^j, p_2^j, \dots, p_n^j, e^j)$ and $\mathbf{k}^j = (k_1^j, k_2^j, \dots, k_n^j)$. For all $x = 1, 2, \dots, n$, $k_x^j \leq x$ determines the number of buyers that will get served conditional on x buyers showing up at seller j 's store. Similarly, for all $x = 1, 2, \dots, n$, p_x^j is the price paid to seller j by all customers who get served, conditional on the fact that this seller gets visited by x buyers. The term e^j denotes the entry fee paid by all customers. I refer to \mathbf{p}^j as the price scheme and to \mathbf{k}^j as the production plan posted by seller j .

Buyers can walk away from the trading process at any time and obtain utility equal to zero (but due to frictions they cannot visit another seller). This assumption has some important implications for the model. First, in order to have buyers participate in the trading process the expected utility generated by the posted mechanisms has to be non-negative. Second, ex-post participation constraints for the buyers have

⁷ In Virag (2008), the author also considers indirect mechanisms, where each seller can condition her advertisement on the advertisements posted by the other sellers. He restricts attention to equilibria that have a very simple form and shows that sellers can extract the full surplus from the buyers. Therefore, in order to achieve collusion, the sellers do not need to consider equilibria in more sophisticated mechanisms.

to be imposed. Here this requires $p_x^j \leq u(k_x^j)$.⁸ There is no assumption that prevents prices from being smaller than the marginal cost or even negative. For the entry fee to have a meaning, I assume that buyers do not know the number of customers that visited the seller before they pay the fee. Also, buyers pay the entry fee before they know whether they will get served or not.⁹

The production plan \mathbf{k}^j chosen by seller j satisfies $\mathbf{k}^j \in K = \prod_{x=1}^n K_x$, with $K_x = \{0, \dots, x\}$ for all $x \geq 1$. For all $x = 1, 2, \dots, n$, $p_x^j \leq u(k_x^j)$ and $e^j \in \mathbb{R}$. I consider mechanisms that lead to non-negative expected profit. If seller j advertises $M^j = \{\mathbf{p}^j, \mathbf{k}^j\}$ and gets visited by x customers, each customer obtains (ex post) utility equal to $u(k_x^j) - p_x^j - e^j$, if she gets served and $-e^j$ otherwise. In this case the seller's profit is given by $k_x^j p_x^j + x e^j - c(k_x^j)$. Next, consider the ex ante payoffs. Suppose that a seller who advertises a mechanism $M = \{\mathbf{p}, \mathbf{k}\}$, gets visited by an arbitrary buyer with probability θ . The expected utility of a buyer who visits that seller is given by

$$U(\theta, M) = \sum_{i=1}^n \binom{n-1}{i-1} (1-\theta)^{n-i} \theta^{i-1} \frac{k_i}{i} [u(k_i) - p_i] - e, \quad (1)$$

and the expected profit for that seller is given by

$$\pi(\theta, M) = \sum_{i=1}^n \binom{n}{i} (1-\theta)^{n-i} \theta^i [k_i p_i + i e - c(k_i)], \quad (2)$$

where $\binom{y}{x}$ denotes the number of ways with which one can choose x out of y objects (the binomial coefficient).

As it is common in the directed search literature, I focus on symmetric equilibria in which all buyers use the same mixed strategy.¹⁰

Definition 1. A subgame perfect equilibrium is a collection of mechanisms M^j , $j = 1, \dots, m$ and a strategy $s : \prod_{j=1}^m M^j \rightarrow \Delta_m$, such that:

i) Given the posted mechanisms, the strategies $s^i = s$, $i = 1, \dots, n$ maximize buyers' expected utility, and

ii) Given buyers' strategy s , M^j is a best response to the mechanisms announced by other sellers, for all $j = 1, \dots, m$.

The term Δ_m denotes the unit simplex that captures the probabilities with which (all) buyers visit each seller.

⁸ Clearly, the ex post utility of a buyer that consumes the good at seller j , when that seller gets visited by x customers, is $u(k_x^j)$ and not $u(x)$. This means that the utility depends on realized production, while the price that a buyer pays depends on realized ex post demand.

⁹ If buyers know the state of the world (the total number of customers at a specific seller) before they pay the entry fee, they would prefer to walk away and not trade in some cases. Then, one would have to consider mechanisms that yield non-negative utility in every state.

¹⁰ In Burdett, Shi, and Wright (2001) the authors provide a coherent explanation of why the mixed strategy equilibrium is the natural type of equilibrium to consider in this type of models. For a detailed discussion on pure strategy equilibria in a similar environment see Coles and Eeckhout (2003).

Definition 2. Let $\sigma(x) \equiv xu(x) - c(x)$ denote the surplus generated if a seller serves x customers, conditional on x or more showing up at that store. Refer to $\sigma : \mathbb{N} \rightarrow \mathbb{R}$ as the ex post surplus function.

3 The 2×2 Case

3.1 Exogenous \mathbf{k}

In this section I analyze the benchmark case $n = m = 2$. Let the sellers be labelled A and B . First, I assume that sellers cannot choose how many units they produce. Thus, I derive the prices that would emerge in equilibrium for a given realization of \mathbf{k} . Once these prices are established, sellers can choose their production endogenously, and it is straightforward to examine the conditions under which different values of \mathbf{k} survive in equilibrium. I focus on symmetric equilibria for the sellers. Assume for simplicity that sellers always serve at least one customer.¹¹ Then, with $n = 2$, we can only have $\mathbf{k} = (1, 1)$ or $\mathbf{k} = (1, 2)$. I refer to $\mathbf{k} = (1, 1)$ as the rationing case (because sellers ration one customer if two show up), and to $\mathbf{k} = (1, 2)$ as the no frictions case, because any buyer who visits a seller with $\mathbf{k} = (1, 2)$ gets served with probability 1.

First suppose $\mathbf{k} = (1, 1)$. Sellers take this as given and choose a price scheme, $\mathbf{p}^j = (p_1^j, p_2^j, e^j)$, $j = A, B$. Let θ be the probability with which an arbitrary buyer visits seller A . Also, let U_j be the expected utility for a buyer that visits seller j . Using (1) one obtains

$$\begin{aligned} U_A &= (1 - \theta)[u(1) - p_1^A] + \frac{\theta}{2}[u(1) - p_2^A] - e^A, \\ U_B &= \theta[u(1) - p_1^B] + \frac{1}{2}(1 - \theta)[u(1) - p_2^B] - e^B. \end{aligned} \quad (3)$$

The objective of seller A is to maximize profits,

$$\pi_A = \theta^2[p_2^A + 2e^A - c(1)] + 2\theta(1 - \theta)[p_1^A + e^A - c(1)],$$

subject to $U_A = U_B$. After some some manipulations the objective of seller A can be written only as a function of the variable θ . In particular, seller A wishes to

$$\max_{\theta} \left\{ 2\theta \left[u(1) - \frac{\theta}{2}u(1) - U_B \right] - \theta(2 - \theta)c(1) \right\}.$$

Taking the first-order condition with respect to θ in the problem above, yields

$$u(1) - \frac{\theta}{2}u(1) - U_B + \theta \left[-\frac{u(1)}{2} - \frac{\partial U_B}{\partial \theta} \right] - (1 - \theta)c(1) = 0. \quad (4)$$

¹¹ It is shown later that this is an equilibrium play if $\sigma(1) > 0$.

Applying total differentiation with respect to θ in (3), yields $\frac{\partial U_B}{\partial \theta} = u(1)/2 - p_1^B + p_1^B/2$. By symmetry, $p_1^A = p_1^B = p_1$, $p_2^A = p_2^B = p_2$, $e^A = e^B = e$, and $\theta = 1/2$. Imposing these conditions in (4) yields

$$p_1 + e = \frac{1}{2}[u(1) + c(1)].$$

The equilibrium prices are not uniquely pinned down. There exists a continuum of symmetric equilibria indexed by the pair of parameters (α, ϵ) . Every triplet $(p_1, p_2, e) = ([u(1) + c(1)]/2 - \epsilon, \alpha, \epsilon)$ that satisfies

$$\alpha \leq u(1), \epsilon \geq \frac{1}{2}[c(1) - u(1)], \text{ and } \alpha + 2\epsilon \in [2c(1) - u(1), 2u(1) - c(1)], \quad (5)$$

constitutes a symmetric equilibrium in the model with $n = m = 2$ and $\mathbf{k} = (1, 1)$.¹² Figure 1 illustrates the set of all equilibrium prices.

These equilibria are not, in general, payoff equivalent. Sellers achieve maximum expected profit when $\alpha + 2\epsilon = 2u(1) - c(1)$. An interesting case arises when $\alpha = u(1)$. I refer to this case as the auction equilibrium, because it replicates the outcome of an auction among identical buyers. In Coles and Eeckhout (2003) this is the unique optimal equilibrium for the sellers. Here, the auction equilibrium is preferred by sellers only if $\epsilon = \sigma(1)/2$. Competition in fixed prices also describes equilibrium behavior. Every (α, ϵ) , with $\alpha = [u(1) + c(1)]/2 - \epsilon$ and $\epsilon \in [-\sigma(1)/2, 3\sigma(1)/2]$ is a symmetric equilibrium in which $p_1 = p_2$. There exists an equilibrium in fixed prices in which sellers extract the whole surplus, indicated by point A in Figure 1. Finally, point B in Figure 1 represents the equilibrium studied in Burdett, Shi, and Wright (2001), i.e. an equilibrium with a fixed price schedule and zero entry fee.

The assumption that buyers do not know the total number of visitors at the store when they pay the entry fee is essential for the existence of some equilibria. Consider the equilibrium price scheme $(p_1, p_2, e) = (c(1), u(1), \sigma(1)/2)$. This scheme leads to expected utility equal to zero, and so buyers accept to participate. If buyers know the state of the world before they pay the entry fee, this equilibrium collapses: if the seller gets visited by two buyers and the buyers know that, they prefer to walk away and obtain zero utility, rather than pay a positive fee in order to enter the store and play a mechanism that yields zero with certainty.

Next, I establish equilibrium prices for $\mathbf{k} = (1, 2)$. Assume for now that $\sigma(2) > 0$.¹³ Given the advertised price schedules, $\mathbf{p}^j = (p_1^j, p_2^j, e^j)$, $j = A, B$, the expected utility

¹² The conditions $\alpha \leq u(1)$, $\epsilon \geq [u(1) + c(1)]/2$ guarantee ex post participation of the buyer in every possible state. The condition $\alpha + 2\epsilon \in [2u(1) - c(1), 2c(1) - u(1)]$ guarantees that expected utility and profit are non-negative.

¹³ When the production choice becomes endogenous, it is shown that one does not need to worry about this restriction anymore. If $\sigma(2) < 0$, posting $\mathbf{k} = (1, 2)$ will never arise in equilibrium.

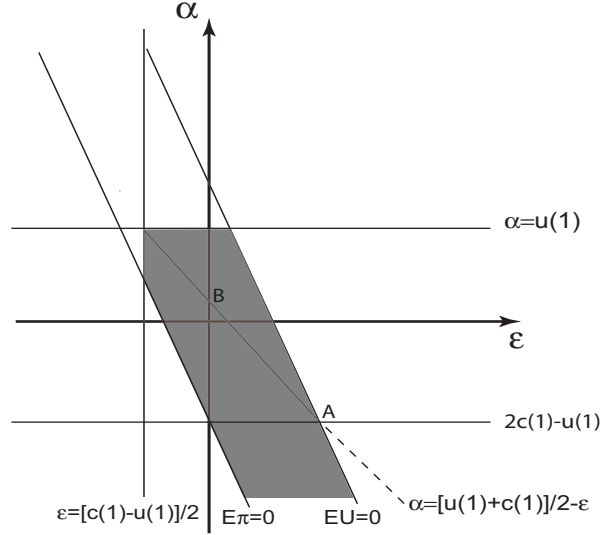


Figure 1: All α, ϵ in the shaded region are consistent with equilibrium.

that an arbitrary buyer obtains by visiting the two sellers is given by

$$\begin{aligned} U_A &= (1 - \theta)[u(1) - p_1^A] + \theta[u(2) - p_2^A] - e^A, \\ U_B &= \theta[u(1) - p_1^B] + (1 - \theta)[u(2) - p_2^B] - e^B. \end{aligned}$$

The objective of seller A is to

$$\max_{\theta} \{2\theta [(1 - \theta)u(1) + \theta u(2) - U_B] - \theta^2 c(2) - 2\theta(1 - \theta)c(1)\}.$$

Taking the first-order condition in this problem and imposing the symmetric equilibrium conditions, yields $p_1 + e = u(1) - u(2) + c(2)/2$. Again, there exists a continuum of symmetric equilibria indexed by the pair (α, ϵ) . Every triplet $(p_1, p_2, e) = (u(1) - u(2) + c(2)/2 - \epsilon, \alpha, \epsilon)$ that satisfies

$$\alpha \leq u(2), \epsilon \geq \frac{1}{2}c(2) - u(2), \text{ and } \alpha + \epsilon \in \left[u(2) - \sigma(1), 2u(2) - \frac{1}{2}c(2) \right], \quad (6)$$

constitutes a symmetric equilibrium for the model with $n = m = 2$ and $\mathbf{k} = (1, 2)$.¹⁴

The equilibrium price p_1 is negatively related to $u(2)$. Intuitively, the willingness

¹⁴ The assumption that $\sigma(2) > 0$ guarantees that the set of equilibria is non empty and that there exists a continuum of equilibrium prices since $2u(2) - c(2)/2 = u(2) + \sigma(2)/2 > u(2)$.

to pay a little more in order to be the only customer at a specific store increases when $u(2)$ falls. As above, the equilibrium price schemes are not payoff equivalent. Sellers achieve maximum expected profit when $\alpha + \epsilon = 2u(2) - c(2)/2$. Although $\alpha = u(2)$ is part of an equilibrium, this is not an auction equilibrium, since here $\mathbf{k} = (1, 2)$, hence, buyers do not have an incentive to bid against each other for the good. Finally, competition in fixed prices does not always describe equilibrium behavior. It can be easily verified that $\sigma(2)/2 \leq 2u(1) - c(1)$ and $3u(2) - c(2) \geq u(1)$ are sufficient conditions for this type of equilibrium to exist.

3.2 Endogenous Determination of \mathbf{k}

So far I have analyzed the determination of prices in symmetric equilibria, assuming that sellers take the value of \mathbf{k} as given. In what follows the choice of \mathbf{k} becomes endogenous. The goal is to determine the conditions under which different values of \mathbf{k} emerge in equilibrium. To that end, suppose that seller B posts $\mathbf{k}^B = (1, 1)$ and a price schedule $(p_1^B, p_2^B, e^B) = ([u(1) + c(1)]/2 - \epsilon, \alpha, \epsilon)$, such that (α, ϵ) satisfy the conditions in (5). Denote this strategy by s^B . Seller A who considers deviating, has two options. The first is a deviation in prices keeping the same \mathbf{k} as seller B , and the second is to announce $\mathbf{k}^A = \mathbf{k}^d = (1, 2)$ and some $\mathbf{p}^d = (p_1^d, p_2^d, e^d)$. Clearly, no deviation in prices can be profitable if $\mathbf{k}^A = (1, 1)$. If such a deviation exists, that would be a contradiction to the fact that the strategies $(p_1^A, p_2^A, e^A) = (p_1^B, p_2^B, e^B) = ([u(1) + c(1)]/2 - \epsilon, \alpha, \epsilon)$, where (α, ϵ) satisfy the conditions in (5), constitute a symmetric equilibrium in the model with $\mathbf{k}^A = \mathbf{k}^B = (1, 1)$.

If there is a profitable deviation, it must involve $\mathbf{k}^A = (1, 2)$. Let t denote the probability with which any given buyer visits seller A if that seller deviates. The expected payoff for a buyer from visiting sellers A and B (respectively) are

$$\begin{aligned} U_d &= (1-t)[u(1) - p_1^d] + t[u(2) - p_2^d] - e^d, \\ U &= t \left[\frac{1}{2}u(1) - \frac{1}{2}c(1) + \epsilon \right] + \frac{1}{2}(1-t)[u(1) - \alpha] - \epsilon. \end{aligned} \quad (7)$$

The deviant seller wants to choose \mathbf{p}^d in order to maximize expected profit, subject to the restriction that buyers are indifferent between visiting the two sellers. As above, seller A 's objective can be written as a function of t . The maximization problem is

$$\max_t \{ 2t [(1-t)u(1) + tu(2) - U] - 2t(1-t)c(1) - t^2c(2) \}, \quad (8)$$

where U is given by (7). After some algebra, one can show that the optimal choice of t is given by

$$t^* = \frac{u(1) + \alpha + 2\epsilon - 2c(1)}{2[2u(1) - 2u(2) + \alpha + 2\epsilon - 3c(1) + c(2)]}, \quad (9)$$

if $\sigma(2) < \sigma(1) + \alpha + 2\epsilon + u(1) - 2c(1)$ and $t^* = 1$, otherwise.¹⁵

In order to find out whether the strategies $\mathbf{k}^A = \mathbf{k}^B = (1, 1)$ and $\mathbf{p}^A = \mathbf{p}^B = ([u(1) + c(1)]/2 - \epsilon, \alpha, \epsilon)$ constitute an equilibrium, one needs to compare the profit that seller A obtains if she follows the prescribed strategy with the profit associated with her best possible deviation. Conditional on seller B playing the strategy s^B , if seller A “follows”, her expected profit is

$$\pi_A^f = \frac{1}{4}[u(1) + \alpha + 2\epsilon - 2c(1)].$$

If seller A deviates to $\mathbf{k}^A = (1, 2)$, she can obtain profit

$$\pi_A^d = 2t^* [(1 - t^*)u(1) + t^*u(2) - U] - 2t^*(1 - t^*)c(1) - t^{*2}c(2),$$

where t^* was described above. If $\sigma(2) < \sigma(1) + \alpha + 2\epsilon + u(1) - 2c(1)$,

$$\pi_A^d = \frac{1}{4} \frac{[u(1) + \alpha + 2\epsilon - 2c(1)]^2}{2u(1) - 2u(2) + \alpha + 2\epsilon - 3c(1) + c(2)}.$$

Otherwise, $t^* = 1$ and $\pi_A^d = \sigma(2) - \sigma(1)$. Comparing π_A^f with the adequate expression for π_A^d , implies that seller A has no incentive to deviate (in the sense that even her best possible deviation is not good enough), if and only if $\sigma(1) \geq \sigma(2)$. If this condition holds, posting $\mathbf{k}^A = \mathbf{k}^B = (1, 1)$ and $\mathbf{p}^A = \mathbf{p}^B = ([u(1) + c(1)]/2 - \epsilon, \alpha, \epsilon)$, where (α, ϵ) satisfy (5), constitutes a symmetric equilibrium.

Given that $\mathbf{k}^B = (1, 1)$, buyers know that if they visit seller B they might get rationed. One might expect seller A to have an incentive to deviate and post $\mathbf{k}^A = (1, 2)$, since such a strategy may be attractive to buyers who do not want to risk getting rationed. However, seller A advertises $\mathbf{k}^A = (1, 2)$ only if $\sigma(1) < \sigma(2)$. If $\sigma(1) \geq \sigma(2)$ we either have $u(1) \approx u(2)$ and $c(2)$ is much bigger than $c(1)$, or $c(1) \approx c(2)$ but $u(2)$ drops dramatically compared to $u(1)$. In the first case, buyers prefer to visit seller A if $\mathbf{k}^A = (1, 2)$, but the cost of producing a second unit is high. In the second case, the additional cost that seller A has to pay to produce a second unit is insignificant. However, buyers do not value the certainty of getting served, because their utility in the event of a double coincidence at seller A is very low.

The question that arises is whether, and under what conditions, an equilibrium with no frictions can emerge in this environment of endogenous determination of production. I repeat the analysis conducted above, but this time I assume that seller

¹⁵ Sufficient conditions for t^* given by (9) to achieve an interior maximum, are $\alpha + 2\epsilon + u(1) - 2c(1) > 0$ and $\sigma(2) < \sigma(1) + \alpha + 2\epsilon + u(1) - 2c(1)$, which means that the ex post surplus of serving two buyers should not be much bigger than the analogous expression for one buyer. From (5), we know that $\alpha + 2\epsilon + u(1) - 2c(1) \geq 0$. If $\alpha + 2\epsilon + u(1) - 2c(1) = 0$ (buyers' optimal equilibrium) and $\sigma(2) < \sigma(1)$, (9) still yields the correct optimal choice, which is $t^* = 0$. If $\sigma(2) \geq \sigma(1) + \alpha + 2\epsilon + u(1) - 2c(1)$, the objective function in (8) becomes an increasing (and convex) function of t for all $t \in [0, 1]$, and the optimal choice is $t^* = 1$.

B follows the (fixed) strategy $\mathbf{k}^B = (1, 2)$ and $\mathbf{p}^B = (u(1) - u(2) + c(2)/2 - \epsilon, \alpha, \epsilon)$, where (α, ϵ) satisfy (6). The only possible profitable deviation for seller A involves $\mathbf{k}^A = \mathbf{k}^d = (1, 1)$. Comparing π_A^f (the profit that seller A obtains if she follows the strategy under consideration) with π_A^d (the profit associated with the best possible deviation), implies that $\pi_A^f \geq \pi_A^d$ if and only if $\sigma(2) \geq \sigma(1)$. If this conditions holds, $\mathbf{k} = (1, 2)$ is posted in equilibrium.

Summarizing, if $\sigma(1) > \sigma(2)$, sellers produce one unit of the good even if both customers show up at their store in the symmetric equilibrium. There exists a continuum of price schemes consistent with this type of equilibrium. These are given by $\mathbf{p}^* = ([u(1) + c(1)]/2 - \epsilon, \alpha, \epsilon)$, where (α, ϵ) satisfy (5). On the other hand, if $\sigma(1) < \sigma(2)$, the symmetric equilibrium has both sellers choose $\mathbf{k} = (1, 2)$ and the equilibrium price scheme is $\mathbf{p}^* = (u(1) - u(2) + c(2)/2 - \epsilon, \alpha, \epsilon)$, where (α, ϵ) satisfy (6). If $\sigma(1) = \sigma(2)$, both types of equilibria coexist. For any parameter values, the emerging equilibria are ex post efficient, in the sense that sellers set $k_2 = 2$, only if the ex post surplus generated by serving two buyers is greater than the analogous expression for one buyer.

Consider now the welfare in the economy. In this model welfare is measured by the expected total surplus, which, in symmetric equilibria, depends only on the emerging \mathbf{k} . If the equilibrium production plan is $\mathbf{k} = \mathbf{k}_{11} = (1, 1)$, the total surplus in the economy is $S(\mathbf{k}_{11}) = \frac{3}{2}\sigma(1)$. If $\mathbf{k} = \mathbf{k}_{12} = (1, 2)$, we have $S(\mathbf{k}_{12}) = \sigma(1) + \frac{1}{2}\sigma(2)$. Clearly, $S(\mathbf{k}_{12}) \geq S(\mathbf{k}_{11})$, if and only if $\sigma(2) \geq \sigma(1)$, which is the condition under which \mathbf{k}_{12} is posted in equilibrium. Therefore, for any parameter values, the emerging equilibrium is not only ex post efficient (in the sense described above), but also ex ante efficient, in the sense that it involves the realization of \mathbf{k} that maximizes the expected total surplus. In the next section I generalize this result for any value of n, m and provide a more detailed discussion about efficiency in this type of market.

4 The $n \times m$ case

4.1 Equilibrium in the General Model

This section establishes a general efficiency result for symmetric equilibria in the model with finite n, m . First, I introduce some necessary definitions.

Definition 3. A production plan $\mathbf{k} = (k_1, k_2, k_3, \dots, k_n)$ is called ex ante constrained efficient, if it maximizes the expected total surplus,

$$S(\mathbf{k}) = n \sum_{i=1}^n \binom{n-1}{i-1} \left(1 - \frac{1}{m}\right)^{n-i} \left(\frac{1}{m}\right)^{i-1} \frac{\sigma(k_i)}{i}. \quad (10)$$

Definition 4. A production plan $\mathbf{k} = (k_1, k_2, k_3, \dots, k_n)$ is called ex post efficient, if for all $i = 1, 2, \dots, n$, $k_i = \arg \max_{\{x \leq i\}} \sigma(x)$. If $\sigma(1) > 0$, an ex post efficient plan always satisfies $k_1 = 1$.

Equation (10) follows from $S(\mathbf{k}) = nU(M) + m\pi(M)$, where $U(M)$ is defined as the expected utility and $\pi(M)$ as the expected profit in the symmetric equilibrium where all sellers post the mechanism M . The expected surplus depends only on the production plan \mathbf{k} . I refer to a production plan that achieves the maximum surplus as constrained (rather than unconstrained) efficient, because the Social Planner is limited by the lack of coordination among buyers. The only way to improve upon this allocation would be to direct each buyer to a specific seller. Regarding Definition 4, a production plan is ex post efficient if it maximizes the total surplus, conditional on the number of buyers that show up at a specific seller.¹⁶

The following Lemma establishes an important result regarding the sellers' behavior. Let S^j denote the strategy set of seller j and s_{-j} a strategy plan for all sellers but j .

Lemma 1. *Fix the strategy of all sellers but j to some arbitrary s_{-j} . For any $M \in S^j$, there exists $\mathbf{p}^* = (p_1^*, p_2^*, \dots, p_n^*, e^*)$, with $p_i^* \leq u(k_i^*)$ for all i , and \mathbf{k}^* an ex post efficient plan, such that*

$$\begin{aligned} \pi_j(M^*, s_{-j}) &\geq \pi_j(M, s_{-j}), \\ \text{s.t. } U(\theta_j, M) &= U(\theta_j, M^*) = U(\theta_h, s_{-j}), \end{aligned}$$

where $\pi_j(M, s_{-j})$ is seller j 's profit if she advertises M , given s_{-j} , and $U(\theta_h, s_{-j})$ is the expected utility that a buyer obtains if she visits seller $h \neq j$. Also, $M^* = \{\mathbf{p}^*, \mathbf{k}^*\}$.

Proof. See the appendix. □

Lemma 1 states that regardless of the strategy followed by other sellers, seller j 's best response involves announcing an ex post efficient production plan. More specifically, for a given s_{-j} and any $M \in S^j$, seller j can always find a price scheme \mathbf{p}^* , which, together with the ex post efficient production \mathbf{k}^* , leaves buyers indifferent

¹⁶ A practical way to identify the ex post efficient production is the following: if one buyer shows up it is always efficient to serve her, since $\sigma(1) > 0$. If two buyers arrive, it is efficient to serve both iff $\sigma(2) \geq \sigma(1)$. Suppose that $\sigma(2) < \sigma(1)$, and so the seller decides to serve only one customer when visited by two. In order to find what is the efficient thing to do in the case that three buyers show up, one only needs to check whether $\sigma(3) \geq \sigma(1)$, and not whether $\sigma(3) \geq \sigma(2)$, since serving two buyers was already proven inefficient. Continuing in this fashion, the process of identifying the ex post efficient \mathbf{k} , can be viewed as a tournament where, in round i , the term $\sigma(i)$ "duels" with the surplus that "won" in the preceding round. Then, the "winner" proceeds to round $i + 1$, in which it "duels" with $\sigma(i + 1)$. This process goes on until $i = n$.

between visiting seller j or any other seller in the second stage and (weakly) improves this seller's profit.¹⁷ An interesting implication of Lemma 1, is that one does not need to worry about sellers committing to their announced production schemes. This is not true regarding prices. As it is outlined in Coles and Eeckhout (2003), commitment to the announced prices is necessary in order to “rule out ex post opportunism, in which sellers, despite the announced price, encourage Bertrand competition once several buyers have turned up”. Unlike prices, sellers do not have an incentive to change the advertised \mathbf{k}^* after observing the number of visiting customers, since this plan is ex post efficient.

Proposition 1. *Strategies $M^* = \{\mathbf{p}^*, \mathbf{k}^*\}$ and $s^* = (1/m, \dots, 1/m)$, constitute a symmetric equilibrium, if and only if \mathbf{k}^* is ex post efficient and \mathbf{p}^* solves*

$$\begin{aligned} & \sum_{i=1}^n H\left(n, \frac{1}{m}, i\right) \left[1 - \frac{f(n, m, i) - 1}{m - 1}\right] k_i^* p_i^* + e^* = \\ & = \sum_{i=1}^n H\left(n, \frac{1}{m}, i\right) \left\{ \frac{m[1 - f(n, m, i)]}{m - 1} k_i^* u(k_i^*) + f(n, m, i) c(k_i^*) \right\}, \end{aligned} \quad (11)$$

with $p_i^* \leq u(k_i^*)$ for all i , and $\pi(M^*), U(M^*) \geq 0$. I have defined $f(n, m, i) \equiv i - (n - i)/(m - 1)$ and $H(n, 1/m, i) \equiv \binom{n-1}{i-1} (1 - 1/m)^{n-i} (1/m)^{i-1} (1/i)$.

If $k_i^* \neq 0$ for all i , a price scheme \mathbf{p}^* that satisfies these conditions always exists.

Proof. See the appendix. □

Proposition 1 establishes indeterminacy of equilibria. There is only one equation that characterizes the equilibrium price schedule, which implies that there are n degrees of freedom in the determination of the equilibrium prices and the entry fee. Given buyers' and rival sellers' strategies, a seller can choose various price schemes that leave the sharing rule of the surplus unchanged. What does change across the various price schemes is the demand elasticity of buyers. Hence, for each price announcement of seller j , the rival sellers have a different best response correspondence. This gives rise to a continuum of equilibria that are not payoff equivalent. Clearly, any sharing rule of the surplus can be supported, i.e. $\pi(M^*) \in [0, S(\mathbf{k}^*)]$.

Another implication of Proposition 1 is that, for all possible parameter values, the equilibria that emerge are ex post efficient. In Section 3, in order to examine whether a strategy profile constitutes an equilibrium, I calculate the prices that emerge under the relevant \mathbf{k} and check for the existence of profitable deviations associated with all alternative production plans. Clearly, this task becomes practically impossible as n

¹⁷ The profit associated with M^* is strictly greater than the one associated with M , if \mathbf{k}^* is the unique ex post efficient production plan and $\mathbf{k} \neq \mathbf{k}^*$.

increases.¹⁸ Proposition 1 provides a simple way of identifying the equilibria in this model. To illustrate this method, suppose that $n = 4$, $m = 2$, $\{u(i)\}_{i=1}^4 = \{1, 1, 1, 1\}$, and $\{c(i)\}_{i=1}^4 = \{0, 0.5, 1.6, 3\}$. The first step is to identify the ex post efficient \mathbf{k}^* . Here $\{\sigma(i)\}_{i=1}^4 = \{1, 1.5, 1.4, 1\}$, hence $\mathbf{k}^* = (1, 2, 2, 2)$. Using these values in (11) one obtains $4p_1^* + 6p_2^* - p_4^* + 8e^* = \frac{15}{2}$. Every $(p_1^*, p_2^*, p_3^*, p_4^*, e^*)$ that satisfies this equation and also $p_1^* + 3p_2^* + 2p_3^* + \frac{1}{2}p_4^* + 8e^* \in [11/8, 13]$, and $p_i^* \leq 1$ for all i , is an equilibrium price schedule.

Two types of equilibria are particularly interesting. The first is the equilibrium in which sellers compete in fixed prices and $e^* = 0$. This type represents the class of mechanisms studied in Burdett, Shi, and Wright (2001). Consider the numeric example introduced above. With $e^* = 0$ and $p_i^* = p^*$ for all i , $p^* = 5/6$. This price satisfies ex post rationality of the buyers and leads to positive expected profit and utility. Therefore, the price scheme $\mathbf{p}^* = (5/6, 5/6, 5/6, 5/6, 0)$ describes equilibrium behavior. The second type of equilibria is the one that replicates the outcome of an auction among identical buyers. If x customers show up at a store and $k_x^* = x$, the buyers are not expected to bid against each other for the good, because they know that there are enough goods everyone. Hence, a necessary condition for the auction equilibrium to exist is ex post shortage of supply. More formally:

Definition 5. Let $\mathbf{k}^* = (k_1^*, k_2^*, \dots, k_n^*)$ be the ex post efficient production schedule. An auction equilibrium exists only if the set $Z \equiv \{x \in \{1, 2, \dots, n\} : k_x^* < x\}$ is non empty. If $Z \neq \emptyset$, an auction equilibrium satisfies $p_i^* = u(k_i^*)$, for all $i \in Z$.

Consider again the numeric example studied above. According to Definition 5, the auction equilibrium has $p_3^* = u(k_3^*)$ and $p_4^* = u(k_4^*)$, implying $p_3^* = p_4^* = 1$. Hence, every triplet (p_1^*, p_2^*, e^*) that satisfies $4p_1^* + 6p_2^* + 8e^* = \frac{17}{2}$, and $p_1^* + 3p_2^* + 8e^* \in [-9/8, 21/2]$ is part of an equilibrium that replicates the outcome of an auction among identical buyers. If $e^* = 0$, sellers can never achieve full extraction of the surplus. This result, which is also outlined in Virag (2008), can be generalized.

Proposition 2. *Suppose sellers cannot charge a positive entry fee. Then, in all symmetric equilibria $U(M^*) > 0$.*

Proof. See the appendix. □

In Coles and Eeckhout (2003) and Virag (2008) it is shown that the optimal equilibrium for the sellers involves price schedules that are increasing in the number of visiting customers. This is not necessarily true in this model. To illustrate

¹⁸ More precisely, for a general number of buyers n and given that $\sigma(1) > 0$, there are $(n/2)(n+1)$ possible values that \mathbf{k} can obtain. Hence, in order to find the equilibrium, one would first have to calculate the (continua of) price schemes associated with each of these values. Then, for a given \mathbf{k} , one would have to check for the existence of profitable deviations associated with each of the remaining $(n/2)(n+1) - 1$ values of \mathbf{k} . This process would have to be repeated $(n/2)(n+1)$ times.

this point, consider the parametric example presented above. The price scheme $(p_1^*, p_2^*, p_3^*, p_4^*, e^*) = (-1/4, -5/12, 1, 1, 3/2)$ is part of an equilibrium that replicates the outcome of an auction among identical buyers. Moreover, the expected utility of a buyer in this equilibrium is equal to zero, and so this is an optimal equilibrium for the sellers. However, this price schedule is not increasing, since $p_1^* > p_2^*$.

Proposition 1 establishes ex post efficiency of the advertised production plans, and provides a simple method to identify the equilibria of the model. As I show in the following Lemma, symmetric equilibria are not only ex post, but also ex ante (constrained) efficient.

Lemma 2. *A production plan \mathbf{k}^* is ex post efficient if and only if it is ex ante constrained efficient.*

Proof. See the Appendix. □

Corollary 1. *Every equilibrium of the model is constrained efficient, in the precise sense that it involves a production plan that maximizes the social surplus.*

To summarize, for any parameter values there exists a (generically) unique production plan that is posted in equilibrium. This plan is ex post efficient, in the sense that it maximizes the total surplus at every possible contingency. Ex post efficiency leads to a simple method of identifying the equilibria of the model. Once the ex post efficient \mathbf{k}^* is found, one only needs to substitute the values $k_1^*, k_2^*, \dots, k_n^*$ in (11). This provides the equilibrium prices. Associated with \mathbf{k}^* , is a continuum of equilibrium price schedules, which are not, in general, payoff equivalent. However, all equilibria are (constrained) efficient, since the ex post efficient production plan also maximizes ex ante expected surplus. Although sellers are allowed to choose very complicated price schemes, common practices, like a fixed price or an auction among buyers, can describe equilibrium behavior.

Ex post efficiency of advertised production plans is a somewhat surprising result. Consider again the case in which $n = m = 2$. The analysis in Section 3 implies that if $\sigma(1) > \sigma(2)$, $\mathbf{k} = (1, 2)$ is never posted in equilibrium. It is clear that if two buyers show up at a store, the seller should maximize (ex post) surplus, which requires serving only one customer. The surprising element of this result is that, ex ante, one might expect the sellers to have an incentive to deviate and post $\mathbf{k}^d = (1, 2)$, in order to attract more customers who value the certainty of being served at the deviant seller. It turns out that sellers never have an incentive to deviate to suboptimal production plans just to attract more customers. Unlike what most oligopolistic models would predict, sellers use their strategic variable \mathbf{k} in such a way that efficiency in the economy is not distorted. In terms of economic policy, this result implies that authorities should intervene in this type of markets only to redistribute surplus, if this is considered necessary, but not to improve efficiency.

4.2 Efficiency and Free Entry

So far I assumed that the number of sellers in the market is exogenous. It is straightforward to show that if the model is extended to allow endogenous determination of m , efficiency is still achieved in symmetric equilibrium. Suppose that a new stage is added in the beginning of the original game, in which sellers decide whether to enter the market or not. If they do not enter their payoff is zero. If they enter they pay a sunk cost, $\psi > 0$, and they participate in the regular two stage game described above. Regardless of the number of sellers that decide to enter, only efficient \mathbf{k} 's will be played in the subgame. Hence, I can determine the optimal number of sellers as $m^* = \arg \max\{S(m, \mathbf{k}^*) - \psi m\}$.¹⁹ For any finite n, m , Proposition 1 suggests that any sharing rule of the surplus can be supported in equilibrium. The efficiency result for the game with free entry is stated below.

Corollary 2. *The following strategies constitute a subgame perfect equilibrium of the game with free entry: in the first stage exactly m^* sellers enter the market. In the second stage active sellers post the mechanism*

$$M = \begin{cases} M_1 = \{\mathbf{p}_1, \mathbf{k}^*\}, & \text{if } m = m^*, \\ M_2 = \{\mathbf{p}_2, \mathbf{k}^*\}, & \text{if } m \neq m^*, \end{cases}$$

where $\mathbf{p}_1, \mathbf{p}_2$ satisfy (11) for $m = m^*$ and $m \neq m^*$, respectively. Also, $\pi(m^*, M_1) \in [\psi, S(m, \mathbf{k}^*)]$ and $\pi(m, M_2) \in [0, \psi)$, for all $m \neq m^*$. Buyers set $s^* = (1/m, \dots, 1/m)$.

The strategies identified in Corollary 2 describe equilibrium behavior both on and off the equilibrium path. Thus, the class of equilibria under consideration is subgame perfect. These equilibria are not symmetric, since some sellers enter the market and others do not. However, buyers and active sellers still play symmetric strategies.

4.3 The Matching Function

The last issue I examine in this section is the number of units sold in the economy. The expected sales per seller are given by $S(n, m) = \sum_{i=1}^n \binom{n}{i} \left(1 - \frac{1}{m}\right)^{n-i} \left(\frac{1}{m}\right)^i k_i^*$. Therefore, in symmetric equilibrium, the number of expected sales in the economy is

$$M(n, m) = m S(n, m) = m \left(1 - \frac{1}{m}\right)^n \sum_{i=1}^n \binom{n}{i} \frac{k_i^*}{(m-1)^i}.$$

¹⁹ The term $S(m, \mathbf{k}^*)$ is the expected total surplus given by (10). When m is exogenous, I denote surplus by $S(\mathbf{k}^*)$ for notational simplicity. Here m is an equilibrium object, hence, I write surplus explicitly as a function of m . The same explanation applies for the term $\pi(m, M)$ in Corollary 2.

The function $M(n, m)$ can be thought of as the matching function. In this setting the matching function depends not only on the number of buyers and sellers in the market, as it is common in the search literature, but also on the production schedule that arises in equilibrium. Clearly, when sellers choose to serve all potential visitors at their store, i.e. $k_i^* = i$ for all i , $M(n, m) = n$, and there are no frictions in the market. At the other extreme, if $\mathbf{k}^* = (1, 1, \dots, 1)$, $M(n, m) = m \left[1 - \left(1 - \frac{1}{m}\right)\right]^n$, which is the number of expected sales derived in Burdett, Shi, and Wright (2001).²⁰

In that paper the authors also examine the arrival rates for buyers and sellers, defined as $A_B \equiv M(n, m)/n$ and $A_S \equiv M(n, m)/m$, and characterize the matching function in terms of its returns to scale. In this setting it is more appropriate to refer to A_B as the probability with which a buyer gets served, and to A_S as the expected sales per seller. I define market tightness as $b \equiv n/m$. First, suppose that the production scheme $\mathbf{k}^* = (1, 1, \dots, 1)$ emerges in equilibrium. For a fixed market tightness, both A_B and A_S are decreasing in m , which implies that $M(n, m)$ exhibits decreasing returns to scale. However, as the market grows large, the matching function has approximately constant returns. If $\mathbf{k}^* = (1, 2, \dots, n)$, $A_S = n/m$ and $A_B = 1$. Therefore, in the no frictions case, the matching function exhibits constant returns to scale even if the market is small.

4.4 Case Study: Greek Ferries in Low Season.

A common practice of Greek ferry owners in the winter, when the demand for trips to the islands is very low, is to advertise that if less than a certain number of passengers show up on a specific day the trip will be cancelled. In this section I present some hypothetical parameter values, that capture the important features of this market, and use the results analyzed so far to show that such advertisements can be supported as equilibrium behavior. Let $m = 3$, $n = 600$, $c(0) = 0$, $c(1) = \dots = c(500) = 200$, and $u(i) = 1$ for all $i \leq n$. Hence, there is no consumption externality and the cost of a trip is constant regardless of the number of passengers.²¹ The capacity of each ferry is given by $\xi = 500$. Buyers observe all firms' advertisements and visit only one location. The ex post surplus function in this market is given by

$$\sigma(i) = \begin{cases} 0, & \text{if } i = 0, \\ i - 200, & \text{if } 1 \leq i \leq 500, \\ 300, & \text{if } i \geq 500. \end{cases}$$

²⁰ In that paper $\mathbf{k} = (1, 1, \dots, 1)$ by default. Hence, it is not surprising that the number of sales in the two models coincide when $\mathbf{k}^* = (1, 1, \dots, 1)$.

²¹ The assumption $c(0) = 0$, implies that if the trip is cancelled the ferry owner does not have to pay the cost of petrol. Also, here I do not examine whether firms should consider exiting this market. The firms know that the low season will not last forever and profits will go up soon.

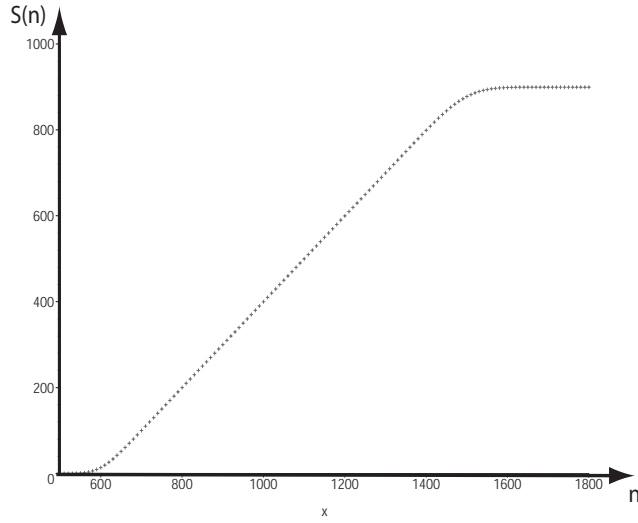


Figure 2: Total surplus as a function of n .

The only production plan that can arise in equilibrium is one that satisfies ex post efficiency. Here it is given by

$$k_i^* = \begin{cases} 0, & \text{if } i \leq 200, \\ i, & \text{if } 200 \leq i \leq 500, \\ 500, & \text{if } i \geq 500. \end{cases}$$

Hence, advertising that “the trip will be cancelled if less than two hundred passengers show up” describes equilibrium behavior, provided that there exist prices that satisfy the conditions described in Proposition 1.²² As pointed out earlier, this result might be somewhat surprising. Given that two sellers play $k_i^* = 0$, for $i \leq 200$, one might expect that the third seller would have an incentive to deviate and advertise that she will run the service regardless of how many passengers show up. Such an advertisement, although ex post inefficient, might be ex ante profitable, since it may attract a lot of passengers who are afraid their trip will be cancelled if they visit one of the non-deviant sellers. Lemma 1 shows that such advertisements cannot be supported in equilibrium.

²² In the proof of Proposition 1 I explain why one should worry about existence of equilibrium in the case where $k_i^* = 0$ for some i . Here for simplicity I ignore equilibrium prices and focus on production schedules.

Figure 2 presents the total surplus in symmetric mixed strategy equilibrium as a function of the number of buyers in the market. For n as big as 550, the coordination problem is so severe that the total surplus is almost zero (because the probability of any boat getting more than two hundred passengers is almost zero). For $n \in [700, 1400]$ the coordination problem is not important and $dS/dn = 1$ (every new customer gets served with certainty). For $n \in [1400, 1550]$, some ferries get more than five hundred customers, hence, rationing starts taking place. Finally, for $n \geq 1550$, $dS/dn = 0$, because all ferries get five hundred passengers with probability arbitrarily close to one.

When $n = 600$ the total surplus is 13.8. This is the maximum possible surplus that can be achieved in symmetric equilibrium with $m = 3$. If only one ferry runs the service, the total surplus increases to 300, because there is no coordination issue. However, the ferry owners know that the low season will not last forever, therefore, they do not find it optimal to exit the market. In this case the authorities could improve welfare by allowing the firms to form an alliance that behaves as a monopoly.²³ If firms can form such an alliance it is reasonable to assume that they can also coordinate on equilibria that leave very small or zero surplus to the buyers. Therefore, economic authorities should let firms behave as a monopoly and intervene only to redistribute some surplus to buyers (if this is considered necessary) and not to improve welfare in the market.

5 Limiting Analysis

5.1 Equilibrium in Large Markets

In Section 4, it was shown that when n, m are finite there exists a continuum of equilibria associated with a specific production scheme. Advertising more buyer surplus in some states and less in some others can leave expected buyer and seller payoff unchanged. This implies that each seller has a continuum of best responses, given buyers' and rival sellers' strategies. A continuum of equilibria exists that are all efficient, but not payoff equivalent. One would expect this indeterminacy to vanish as the market becomes very large. In what follows, I show that this is indeed the case. More specifically, as $n, m \rightarrow \infty$, the expected profit of sellers (or equivalently the expected utility of buyers) collapses to a single limiting value.

Proposition 3. *Suppose that $n, m \rightarrow \infty$ and b is held constant. Also, let $e^* < \infty$ and $p_i^* > -\infty$, for all i . Then, the limit of expected profit for an arbitrary seller in*

²³ In many countries, including Greece, such a behavior is considered illegal and it is forbidden by antitrust laws.

this market is given by

$$\bar{\pi} = b \lim_{n \rightarrow \infty} \left\{ \sum_{i=1}^n H \left(n, \frac{b}{n}, i \right) (1 + b - i) \sigma(k_i^*) \right\}, \quad (12)$$

where $H(n, \frac{b}{n}, i) \equiv \binom{n-1}{i-1} \left(1 - \frac{b}{n}\right)^{n-i} \left(\frac{b}{n}\right)^{i-1} \left(\frac{1}{i}\right)$.

Proof. See the appendix. □

Proposition 3 implies that as the market becomes large, considering only fixed price mechanisms, like in Burdett, Shi, and Wright (2001), or mechanisms that do not include an entry fee, like in Coles and Eeckhout (2003), is without loss of generality. The value of the expected profit at the limit does not depend on which price scheme is played in equilibrium, and it is increasing in b . For given b , $\bar{\pi}$ depends exclusively on the sequence of ex post optimal surplus, $\{\sigma(k_i^*)\}_{i=1}^{\infty}$. With $n \rightarrow \infty$ it is not always possible to assign a sequence representation to the optimal production plan \mathbf{k}^* and effectively describe $\{\sigma(k_i^*)\}_{i=1}^{\infty}$. This issue can be resolved by placing some minor restrictions on the functions u and c , which lead to a special and very tractable class of efficient production schedules. The following proposition states the relevant result.

Proposition 4. *Suppose $\sigma(1) > 0$. If the ex post surplus function, $\sigma(x)$, is strictly quasi concave the unique ex post efficient plan \mathbf{k}^* is fully characterized by the number λ , in the sense that $\mathbf{k}^* = (1, 2, \dots, \lambda, \lambda, \dots)$. A sufficient condition for $\sigma(x)$ to be strictly quasi concave is that $c(x)$ is convex and $xu(x)$ is concave, one of them strictly.*

Proof. See the appendix. □

Proposition 4 indicates that if $\sigma(x)$ is strictly quasi concave, the equilibrium production schedule is fully described by a single number λ , such that if λ or less buyers show up they all get served, but if more than λ show up only λ units are sold and rationing takes place. The case in which $\lambda = \infty$ is not excluded. This is the no frictions case. Henceforth, I restrict attention to preferences and technology that satisfy the assumptions of Proposition 4. Whenever a seller posts the scheme $\mathbf{k}^* = (1, 2, \dots, \lambda, \lambda, \dots)$, I say that the seller chooses maximum production of λ . The sequence $\{\sigma(k_i^*)\}_{i=1}^{\infty}$ has $\sigma(k_i^*) = \sigma(i)$, for $i \leq \lambda$ and $\sigma(k_i^*) = \sigma(\lambda)$, for $i > \lambda$. Next, I examine some specific examples of $\{\sigma(k_i^*)\}_{i=1}^{\infty}$ and obtain closed form solutions for $\bar{\pi}$.

5.2 Closed Form Solutions

First, consider the case in which $\lambda = 1$, and so $\{\sigma(k_i^*)\}_{i=1}^\infty$ is the constant sequence $\{\sigma(1), \sigma(1), \dots\}$.²⁴ One can re-write (12) as

$$\bar{\pi} = b\sigma(1) \lim_{n \rightarrow \infty} \left\{ \sum_{i=1}^n H\left(n, \frac{b}{n}, i\right) (1+b-i) \right\}.$$

After some algebra it can be shown that²⁵

$$\bar{\pi} = \sigma(1) \left(1 - e^{-b} - be^{-b}\right) = \sigma(1) \bar{\pi}_{BSW}, \quad (13)$$

where $\bar{\pi}_{BSW}$ is the limiting value of expected profit found in Burdett, Shi, and Wright (2001). In that paper it is assumed that $u(1) = 1$ and $c(1) = 0$, therefore, the two results coincide.

Next, consider the most realistic case of a strictly convex cost. For simplicity let $u(i) = 1$ for all i and $c(i) = \beta i^2$, $\beta \in (0, 1)$. Here $\sigma(i) = i(1 - \beta i)$. For small values of i , $\sigma(i)$ is increasing and sellers serve all visiting customers. However, as i becomes bigger the convex cost overweighs the linear benefit and sellers do not accommodate more than a certain number of customers. The critical number, after which the marginal cost of serving another customer exceeds the marginal benefit of doing so, is given by $\lambda = \min \left\{ l \in \mathbb{N} : l > \frac{1-\beta}{2\beta} \right\}$. It is straightforward to show that

$$\begin{aligned} \bar{\pi} = & b \lim_{n \rightarrow \infty} \left\{ \sum_{i=1}^{\lambda} \binom{n-1}{i-1} \left(1 - \frac{b}{n}\right)^{n-i} \left(\frac{b}{n}\right)^{i-1} (1+b-i)(1-\beta i) \right\} + \\ & + b\lambda(1-\beta\lambda) \lim_{n \rightarrow \infty} \left\{ \sum_{i=\lambda+1}^n H\left(n, \frac{b}{n}, i\right) (1+b-i) \right\}. \end{aligned}$$

After using some standard results about limits, one can obtain

$$\begin{aligned} \bar{\pi} = & \lambda(1-\beta\lambda) \left(1 - e^{-b} - be^{-b}\right) + e^{-b} \sum_{i=1}^{\lambda} \frac{b^i(1+b-i)}{(i-1)!} \left[1 - \beta i - \frac{\lambda(1-\beta\lambda)}{i}\right] = \\ = & \lambda(1-\beta\lambda) \bar{\pi}_{BSW} + e^{-b} \sum_{i=1}^{\lambda} \frac{b^i(1+b-i)}{(i-1)!} \left[1 - \beta i - \frac{\lambda(1-\beta\lambda)}{i}\right]. \end{aligned} \quad (14)$$

²⁴ This example coincides with the environment described in Burdett, Shi, and Wright (2001). However, in that paper sellers have no choice over their production, while here $\lambda = 1$ because $\sigma(i) < \sigma(1)$, for all $i > 1$.

²⁵ It is easy to prove that $\lim_{n \rightarrow \infty} \left\{ \sum_{i=1}^n H\left(n, \frac{b}{n}, i\right) \right\} = \frac{1}{b}(1 - e^{-b})$ and that $\lim_{n \rightarrow \infty} \left\{ \sum_{i=1}^n \binom{n-1}{i-1} \left(1 - \frac{b}{n}\right)^{n-i} \left(\frac{b}{n}\right)^{i-1} \right\} = 1$. Given these results the expression in (13) follows immediately.

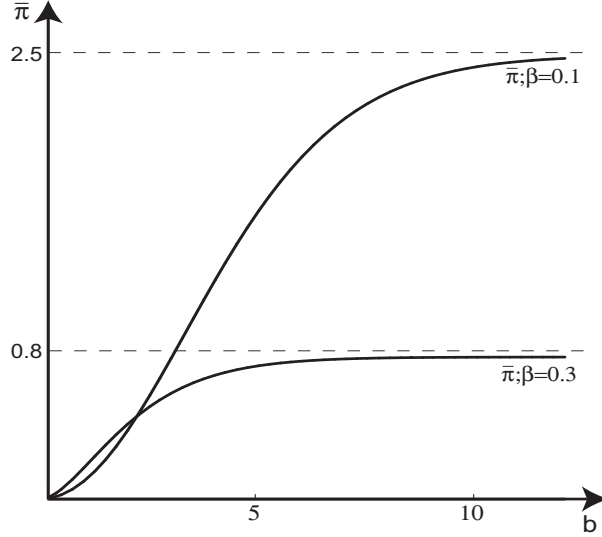


Figure 3: $\bar{\pi}$ as a function of b , for $\beta = 0.3$ and $\beta = 0.1$.

In (14) $\bar{\pi}$ is expressed as a function of the parameters β , λ and the market tightness. The limiting profit in this example can be written as $\bar{\pi} = A\bar{\pi}_{BSW} + B$, where $A \equiv \lambda(1 - \beta\lambda)$ and B is defined as the second term in equation (14). The term A is decreasing in β . Moreover, it can be shown that as b gets very large B goes to zero, implying that²⁶

$$\lim_{b \rightarrow \infty} \bar{\pi} = A \lim_{b \rightarrow \infty} \bar{\pi}_{BSW} + \lim_{b \rightarrow \infty} B = A.$$

Hence, A is the upper bound of the limiting profit for sellers. In Figure 3, $\bar{\pi}$ is plotted as a function of b for two cases. When $\beta = 0.3$ (implying $\lambda = 2$) and when $\beta = 0.1$ (implying $\lambda = 5$).

An interesting case arises when $u(i) = u$ for all i , and total cost is linear, $c(i) = ci$, $c < u$. Then, $\sigma(i) = (u - c)i$.²⁷ Since $u - c > 0$, efficiency requires sellers to choose $\lambda = \infty$. This implies $k_i^* = i$, and so equation (12) yields

$$\bar{\pi} = b(u - c) \lim_{n \rightarrow \infty} \left\{ \sum_{i=1}^n \binom{n-1}{i-1} \left(1 - \frac{b}{n}\right)^{n-i} \left(\frac{b}{n}\right)^{i-1} (1 + b - i) \right\} =$$

²⁶ As it can be seen in Figure 3, $B \rightarrow 0$ and $\bar{\pi}_{BSW} \rightarrow 1$ for very small values of b . For $\beta = 0.3$ this happens for $b \approx 6$.

²⁷ Here $iu(i)$ and $c(i)$ are weakly concave and convex (respectively). However, $\sigma(i) = (u - c)i$ is strictly increasing and, therefore, strictly quasi concave.

$$= b(u - c) \left\{ (1 + b) \lim_{n \rightarrow \infty} \left\{ \sum_{i=1}^n \binom{n-1}{i-1} \left(1 - \frac{b}{n}\right)^{n-i} \left(\frac{b}{n}\right)^{i-1} \right\} - (1 + b) \right\} = 0.$$

This result has an intuitive explanation. When all sellers set $k_i^* = i$, there are no frictions in the market, since all buyers get served with probability 1 at any store they might choose to visit. With no frictions and $n, m \rightarrow \infty$, the model becomes approximately perfectly competitive and equilibrium profit goes to zero at the limit.

5.3 Matching in Large Markets

Finally, consider the number of successful matches in the economy. Since the number of buyers and sellers is infinite, I focus on the variables A_S and A_B introduced in Section 4.

Proposition 5. *Suppose that $\sigma(x)$ is strictly quasi concave and λ denotes the maximum production of sellers in the symmetric equilibrium. Then, as $n, m \rightarrow \infty$ and b is held constant, the expected number of sales per seller is given by*

$$\bar{A}_{S,\lambda}(b) \equiv \lim_{n \rightarrow \infty} A_S = \begin{cases} e^{-b} \sum_{i=1}^{\lambda} \frac{(i-\lambda)b^i}{i!} + \lambda(1 - e^{-b}), & \text{if } \lambda < \infty, \\ b, & \text{if } \lambda = \infty, \end{cases}$$

and the probability with which an arbitrary buyer get served is given by $\bar{A}_{B,\lambda}(b) \equiv \lim_{n \rightarrow \infty} A_B = b^{-1} \bar{A}_{S,\lambda}(b)$.

Proof. See the appendix. □

Proposition 5 implies that for any given b , if $\lambda > \lambda'$, $\bar{A}_{S,\lambda}(b) \geq \bar{A}_{S,\lambda'}(b)$ and $\bar{A}_{B,\lambda}(b) \geq \bar{A}_{B,\lambda'}(b)$.²⁸ Hence, in equilibrium, higher values of λ imply a higher number of sales for sellers and a higher probability of getting served for buyers. As b becomes very large, $\bar{A}_{S,\lambda} \rightarrow \lambda$. Figures 4 and 5 illustrate $\bar{A}_{S,\lambda}$ and $\bar{A}_{B,\lambda}$ for different values of maximum production including $\lambda = \infty$, i.e. the no friction case. To see the effect of different choices of λ on the number of successful matches, suppose that $b = 2$. The expected number of sales (per seller) in a market where sellers choose $\lambda = 1$ is $\bar{A}_{S,1}(2) = 0.8646$. If $\lambda = 2$ the number of sales is given by $\bar{A}_{S,2}(2) = 1.4586$. From the buyer's perspective, in a market with $\lambda = 1$ the probability of getting served is given by $\bar{A}_{B,1}(2) = 0.4323$. But if $\lambda = 2$ this probability increases to $\bar{A}_{B,2}(2) = 0.7293$. If b is small the probability with which an arbitrary buyer gets served in equilibrium is very close to 1, even for values of λ as small as 3. For instance, $\bar{A}_{B,3}(0.6) = 0.9937$.

²⁸ Also, $\lim_{\lambda \rightarrow \infty} \bar{A}_{S,\lambda}(b) = b$ and $\lim_{\lambda \rightarrow \infty} \bar{A}_{B,\lambda}(b) = 1$, which are the arrival rates in a market with no frictions.

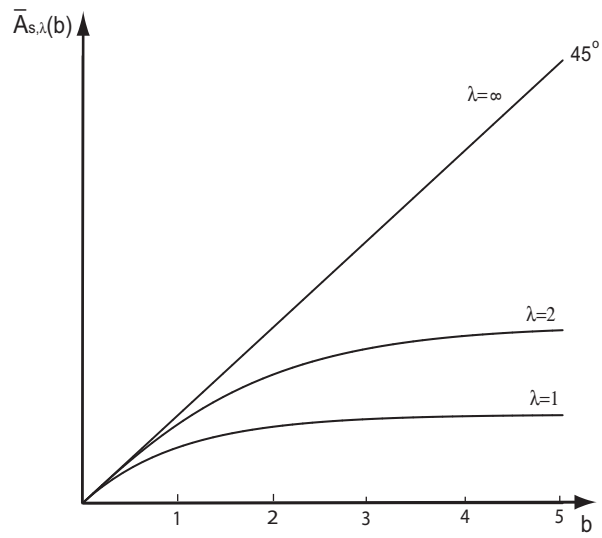


Figure 4: $\bar{A}_{S,\lambda}(b)$ for $\lambda = 1, 2, \infty$.

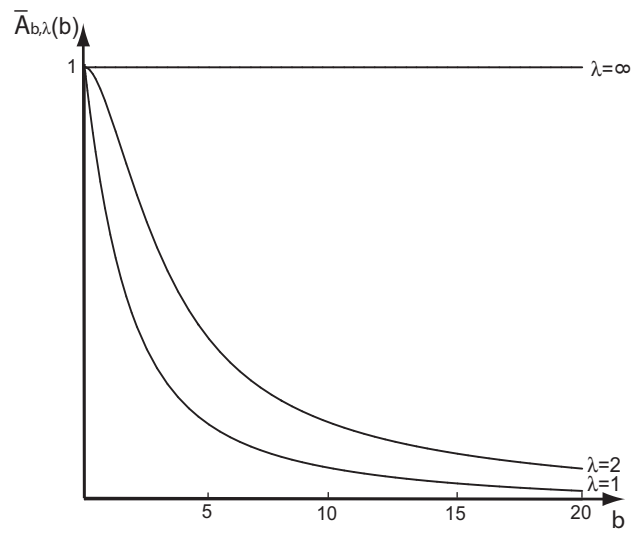


Figure 5: $\bar{A}_{B,\lambda}(b)$ for $\lambda = 1, 2, \infty$.

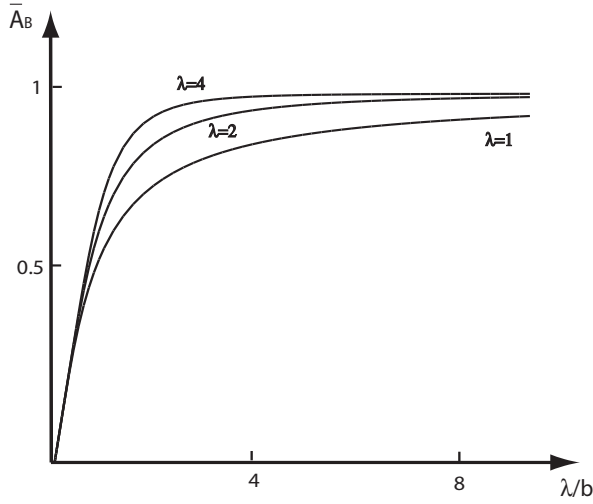


Figure 6: The probability \bar{A}_B for a fixed number of units per buyer.

In Figure 5, the supply of goods in the economy with $\lambda = 2$ is twice as big as the supply in the economy with $\lambda = 1$. Therefore, the distance $\bar{A}_{B,2} - \bar{A}_{B,1}$, for given b , is partly explained by the difference in the number of goods per buyer in the economy. It would be interesting to focus only on the part of this distance that is explained by the intensity of frictions in the economy. Figure 6, depicts the probability with which buyers get served, as a function of the amount of goods per buyer, in economies with $\lambda = 1$, $\lambda = 2$, and $\lambda = 4$. As the figure implies, frictions amount for a lot of unsuccessful matches. In the market with $\lambda = 4$, a buyer gets served with probability close to one for a value of λ/b as low as 3. For the same value of λ/b , the probability of getting served in an economy with $\lambda = 2$ is 0.94.²⁹ This implies that the number of successful matches is more responsive along the intensive margin.

6 Conclusions

In this paper I provide a general framework suitable for the study of markets with frictions, a few strategic sellers who face a stochastic demand, and buyers who cannot

²⁹ If $\lambda = 1$ the probability of getting served when $\lambda/b = 3$ is only 0.85. This probability approaches 1 only for values of λ/b that are bigger than 50 (when $\lambda/b = 50$, $\bar{A}_{B,1} = 0.99$).

coordinate their visiting strategies. In contrast to the predictions of most models of oligopolistic markets, I show that the markets under consideration will provide the socially efficient quantity. Efficiency can be also achieved in equilibrium, in a model with free entry and endogenous determination of the number of sellers. Efficiency is constrained by the lack of coordination among the buyers. The only way to improve upon the equilibrium allocations would be to direct each buyer to a specific seller.

Sellers compete with each other for customers by choosing production and price schedules contingent on the number of visitors. In equilibrium, only ex post efficient production schedules are advertised. Ex post efficiency is a somewhat surprising result. It is clear that once a certain number of customers have shown up at a store, the seller should maximize ex post surplus. However, production plans are announced ex ante. Hence, one might expect sellers to advertise production plans that are not ex post efficient, but they attract a large number of customers, therefore, they are ex ante profitable. It turns out that sellers never have the incentive to advertise ex post inefficient production plans, just to attract more customers.

I show that ex post and ex ante efficiency are equivalent in this model. This implies that the production schedule that is posted in symmetric equilibrium maximizes ex ante expected surplus. Despite the element of strategic interaction among sellers, the production decisions are such that the efficiency of the economy is not distorted. This finding has some important implications for economic policy. More specifically, the authorities should intervene in this type of markets only to redistribute surplus, if this is considered necessary, but not to improve efficiency.

Other results include the existence of a continuum of equilibrium price schemes in small markets. These equilibria are not payoff equivalent. Although sellers are allowed to advertise general pricing mechanisms, common practices, like posting a fixed price or an auction, can describe equilibrium behavior. As the size of a market becomes very large the indeterminacy of equilibria vanishes. This implies that considering less sophisticated mechanisms, like fixed price schemes, is without loss of generality. Finally, I show that the number of successful matches in the economy is more responsive along the intensive margin than it is along the extensive margin.

7 Future Work

An interesting extension of the current model allows the study of overbooking. Suppose that the sellers are airline companies (they could be or bus owners or even restaurants) and face a capacity constraint $\xi \leq n$. There is a new stage in the exchange process, in which after observing each seller's advertisement and before showing up

at a certain location, buyers can costlessly visit the web site of every seller³⁰ On the day of the trip there is a probability $\gamma > 0$ with which buyers get a random shock and decide not to travel. Then, sellers might have an incentive to advertise a production plan with $k_j > \xi$, for some $j > \xi$: although capacity is given by ξ , sellers might find it prudent to sell a number of tickets higher than ξ and refund passengers who end up getting rationed (if too many decide to travel). This incentive becomes stronger as the value of γ increases. This work is in progress.

A Appendix

Proof of Lemma 1. Define the term $H(n, \theta, i) \equiv \binom{n-1}{i-1} (1-\theta)^{n-i} \theta^{i-1} (1/i)$, which is non negative and strictly positive if $\theta > 0$. Combining this definition with equation (1), the expected utility for a buyer from visiting a seller who advertises $M = \{\mathbf{p}, \mathbf{k}\}$ and gets visited by an arbitrary buyer with probability θ can be written as

$$U(\theta, M) = \sum_{i=1}^n H(n, \theta, i) k_i [u(k_i) - p_i] - e. \quad (\text{a.1})$$

The expected profit for a seller who posts M and gets visited by an arbitrary seller with probability θ is given by (2). Using this equation together with the definition of $H(n, \theta, i)$ and the fact that $\binom{n}{i} = \frac{n}{i} \binom{n-1}{i-1}$, implies

$$\pi(\theta, M) = n\theta \left\{ \sum_{i=1}^n H(n, \theta, i) [k_i p_i - c(k_i)] + e \right\}. \quad (\text{a.2})$$

Given s_{-j} , for any $M \in S^j$, indifference in the second stage determines the probabilities with which an arbitrary buyer visits each seller, $(\theta_1, \theta_2, \dots, \theta_{m-1})$. Assume that $\theta_j > 0$ and define $\delta \equiv U(\theta_j, M) = U(\theta_h, s_{-j})$. Clearly, δ is a finite non-negative real number. For any parameter values it is easy to identify the ex post efficient production schedule \mathbf{k}^* . Then, for any $M \in S^j$, one can always find $\mathbf{p}^* = (p_1^*, p_2^*, \dots, p_n^*, e^*)$, with $p_i^* \leq u(k_i^*)$ for all i and $e^* \in \mathbb{R}$, such that the equation

$$G(\mathbf{p}^*) \equiv U(\theta_j, M^*) - \delta = \sum_{i=1}^n H(n, \theta_j, i) k_i^* [u(k_i^*) - p_i^*] - e^* - \delta = 0, \quad (\text{a.3})$$

always has a solution.³¹ To see why this is true, fix the entry fee at some arbitrary $e^* = \tilde{e} > 0$. Consider the price scheme $(p_1^*, p_2^*, \dots, p_n^*, e^*) = (u(k_1^*), u(k_2^*), \dots, u(k_n^*), \tilde{e})$.

³⁰ This new stage is not characterized by frictions. However, frictions are still present in the last stage, i.e. after a buyer physically visits a seller's store.

³¹ In words, for any choice of M one can find a price schedule \mathbf{p}^* such that the mechanisms $M = \{\mathbf{p}, \mathbf{k}\}$ and $M^* = \{\mathbf{p}^*, \mathbf{k}^*\}$, where \mathbf{k}^* is ex post efficient, generate the same expected utility for a buyer who visits seller j , while leaving θ_j unaltered.

Then, $G(u(k_1^*), u(k_2^*), \dots, u(k_n^*), \tilde{e}) = -\tilde{e} - \delta < 0$. Moreover, one can always find prices that make G arbitrary large. Finally, the function G is continuous and decreasing in all its arguments. Combining these observations, one can conclude that the equation $G(\mathbf{p}^*) = 0$ has at least one solution.

Next, let $\hat{\mathbf{p}}^* \in \{\mathbf{p}^* : G(\mathbf{p}^*) = 0\}$. By definition, if seller j advertises the mechanism $\hat{M} = (\hat{\mathbf{p}}^*, \mathbf{k}^*)$, buyers are still indifferent between visiting her or any other seller. Moreover, given s_{-j} , the profit that seller j obtains if she plays this specific strategy is

$$\begin{aligned} \pi_j(\hat{M}, s_{-j}) &= n\theta_j \left\{ \sum_{i=1}^n H(n, \theta_j, i) [k_i^* \hat{p}_i^* - c(k_i^*)] + e^* \right\} = \\ &= n\theta_j \left\{ \sum_{i=1}^n H(n, \theta_j, i) \{k_i^* u(k_i^*) - c(k_i^*) - k_i [u(k_i) - p_i]\} + e \right\}, \end{aligned}$$

where the last equality follows from (a.3) and the definition of the term δ . Next, add and subtract $c(k_i)$ inside every term of the sum in the last expression. This implies

$$\begin{aligned} \pi_j(\hat{M}, s_{-j}) &= n\theta_j \left\{ \sum_{i=1}^n H(n, \theta_j, i) [\sigma(k_i^*) - \sigma(k_i) + p_i k_i - c(k_i)] + e \right\} = \\ &= n\theta_j \sum_{i=1}^n H(n, \theta_j, i) [\sigma(k_i^*) - \sigma(k_i)] + \pi_j(M, s_{-j}), \end{aligned}$$

where the last equality follows from (a.2). Definition 4 and the fact that \mathbf{k}^* is ex post efficient conclude the proof. \square

Proof of Proposition 1. From Lemma 1, the equilibrium can only involve ex post efficient production schedules. Suppose that all sellers but j post the price schedule $\mathbf{p} = (p_1, \dots, p_n, e)$ together with the ex post efficient \mathbf{k}^* . Seller j does not have an incentive to post $\mathbf{k}^d \neq \mathbf{k}^*$. Hence, consider deviations only in the price scheme. If seller j deviates to $\mathbf{p}^d = (p_1^d, \dots, p_n^d, e^d)$, expected profit is given by

$$\pi_j(t, (\mathbf{p}^d, \mathbf{k}^*)) = nt \left\{ \sum_{i=1}^n H(n, t, i) [k_i^* p_i^d - c(k_i^*)] + e^d \right\},$$

where t is the probability with which the deviant seller gets visited by an arbitrary buyer, and the function H was defined above. Seller j wants to maximize this expression, subject to $U(t, (\mathbf{p}^d, \mathbf{k}^*)) = U(\theta, (\mathbf{p}, \mathbf{k}^*))$, where $\theta = (1-t)/(m-1)$ is the probability with which any given buyer visits a non deviant seller. Using (a.1), the condition for indifference of the buyers can be written as

$$\sum_{i=1}^n H(n, t, i) k_i^* p_i^d + e^d = \sum_{i=1}^n H(n, t, i) k_i^* u(k_i^*) - U(\theta, (\mathbf{p}, \mathbf{k}^*)).$$

The term $\sum_{i=1}^n H(n, t, i) k_i^* p_i^d + e^d$ appears both in the objective function and in the constraint. Hence, one can re-write the problem of seller j as

$$\max_t \left\{ nt \left[\sum_{i=1}^n H(n, t, i) \sigma(k_i^*) - U(\theta, (\mathbf{p}, \mathbf{k}^*)) \right] \right\}.$$

The first-order condition yields

$$\sum_{i=1}^n H(n, t, i) \sigma(k_i^*) - U(\theta, (\mathbf{p}, \mathbf{k}^*)) + t \left\{ \sum_{i=1}^n \binom{n-1}{i-1} \frac{\sigma(k_i^*)}{i} (1-t)^{n-i-1} t^{i-2} [-(n-i)t + (i-1)(1-t)] - \frac{\partial U}{\partial t} \right\} = 0 \quad (\text{a.4})$$

To obtain an expression for $\partial U / \partial t$, set $\theta = t$ in (a.1) and apply total differentiation with respect to that variable. One can obtain

$$\frac{\partial U}{\partial t} = \sum_{i=1}^n \binom{n-1}{i-1} \frac{k_i^* [u(k_i^*) - p_i]}{i} (1-t)^{n-i-1} t^{i-2} [-(n-i)t + (i-1)(1-t)] \frac{\partial \theta}{\partial t}, \quad (\text{a.5})$$

and $\frac{\partial \theta}{\partial t} = \frac{-1}{m-1}$. The final step is to impose symmetry conditions. These conditions are $t = \theta = \frac{1}{m}$, $p_i = p_i^d = p_i^*$ for all i , and $e = e^d = e^*$. Combining (a.4), (a.5), and the symmetric equilibrium conditions leads to (11).

It remains to show existence of a \mathbf{p}^* that satisfies the conditions described in Proposition 1. Without loss of generality, I show existence of \mathbf{p}^* that leads to $U(M^*) = 0$.³² In this case $\sum_{i=1}^n H(n, 1/m, i) k_i^* [u(k_i^*) - p_i^*] = e^*$. Using this fact in (11) and after some algebra we have

$$\sum_{i=1}^n H\left(n, \frac{1}{m}, i\right) \left[\frac{1 - f(n, m, i)}{m-1} \right] k_i^* [u(k_i^*) - p_i^*] = \sum_{i=1}^n Q(n, m, i) \sigma(k_i^*) \equiv \eta, \quad (\text{a.6})$$

where $Q(n, m, i) \equiv H(n, 1/m, i) f(n, m, i)$.

I claim that η is strictly positive and finite. The latter follows from the definition of H . The former is not straightforward because the function f is not always positive. If $n < m$, then $f(n, m, i) = (mi - n)/(m-1) > 0$ for all i . But if $n > m$, there exists $\nu \in \mathbb{N}$ with $1 \leq \nu < n$, such that for all $i \leq n$, $f(n, m, i) < 0$ and, therefore, $Q(n, m, i) < 0$. However, one can show that

$$\sum_{i=1}^n Q(n, m, i) = \sum_{i=1}^{\nu} Q(n, m, i) + \sum_{i=\nu+1}^n Q(n, m, i) = \left(\frac{m-1}{m} \right)^{n-1} > 0. \quad (\text{a.7})$$

³² Since e^* enters (11) linearly, one can always decrease its value and achieve an equilibrium with $U(M^*)$ anywhere between zero and $S(\mathbf{k}^*)$.

Also, since \mathbf{k}^* is ex post efficient, $\sigma(k_i^*)$ is non-decreasing in i . Hence,

$$\begin{aligned} -\sum_{i=1}^{\nu} Q(n, m, i) \sigma(k_i^*) &\leq \sigma(k_{\nu}^*) \sum_{i=1}^{\nu} [-Q(n, m, i)] < \\ &< \sigma(k_{\nu}^*) \sum_{i=\nu+1}^n Q(n, m, i) \leq \sum_{i=\nu+1}^n Q(n, m, i) \sigma(k_i^*), \end{aligned}$$

where the strict inequality follows from (a.7). This verifies the claim.

Finally, define the left-hand side of (a.6) as $\Gamma(\mathbf{p}^*)$. This function is continuous in all arguments. If $p_i^* = u(k_i^*)$ for all i , then $\Gamma(\mathbf{p}^*) = 0$. Also, I can choose a very large and negative p_i^* in the states for which $[1 - f(n, m, i)]/(m - 1) > 0$, leading to $\Gamma(\mathbf{p}^*) \rightarrow \infty$. Combining these facts proves that \mathbf{p}^* such that $\Gamma(\mathbf{p}^*) = \eta$ always exists. This concludes the proof. A sufficient condition for existence is that $k_i^* \neq 0$ for all i . However, even if $k_i^* = 0$ for some i , symmetric equilibrium still exists. The problem arises when $k_i^* = 0$ in all states where $[1 - f(n, m, i)]/(m - 1) > 0$. Then $\Gamma(\mathbf{p}^*) \leq 0 < \eta$, and the above proof does not hold. □

Proof of Proposition 2. As I showed in the proof of Proposition 1, $U(M^*) = 0$ implies

$$\sum_{i=1}^n H\left(n, \frac{1}{m}, i\right) k_i^* [u(k_i^*) - p_i^*] = e^* \leq 0. \quad (\text{a.8})$$

Ex post rationality and the definition of H , imply that the left-hand side of (a.8) is non-negative. Therefore, $U(M^*) = 0$ only if $e^* = 0$ and $p_i^* = u(k_i^*)$, all i . The question is whether $\hat{\mathbf{p}}^* = (u(k_1^*), \dots, u(k_n^*), 0)$ can be supported in equilibrium. According to Proposition 1, this will be true if and only if $\hat{\mathbf{p}}^*$, satisfies equation (11) ($\hat{\mathbf{p}}^*$ satisfies every other requirement described in that proposition). Plugging $\hat{\mathbf{p}}^*$ into (11) yields

$$\sum_{i=1}^n Q(n, m, i) \sigma(k_i^*) = 0, \quad (\text{a.9})$$

where $Q(n, m, i)$ is defined in the proof of Proposition 1. In the same proof I show that $\sum_{i=1}^n Q(n, m, i) \sigma(k_i^*) > 0$, hence, I have reached a contradiction. □

Proof of Lemma 2. The total surplus is given by (10). If \mathbf{k}^* is an ex post efficient production schedule, it is straightforward to show that for any $\mathbf{k} \neq \mathbf{k}^*$, $S(\mathbf{k}^*) \geq S(\mathbf{k})$,

and so \mathbf{k}^* is also ex ante efficient. To prove the converse, assume that \mathbf{k}^* maximizes the total surplus, i.e. $S(\mathbf{k}^*) \geq S(\mathbf{k})$ for any $\mathbf{k} \neq \mathbf{k}^*$. One can write

$$\sum_{i \in I_{\mathbf{k}, \mathbf{k}^*}} \binom{n-1}{i-1} \left(1 - \frac{1}{m}\right)^{n-i} \left(\frac{1}{m}\right)^{i-1} \frac{1}{i} [\sigma(k_i^*) - \sigma(k_i)] \geq 0,$$

where $I_{\mathbf{k}, \mathbf{k}^*} \equiv \{i : k_i \neq k_i^*\}$.

There are two possible scenarios. Either $\sigma(k_i^*) - \sigma(k_i) \geq 0$ for all $i \in I_{\mathbf{k}, \mathbf{k}^*}$ or $\sigma(k_i^*) - \sigma(k_i) \geq 0$ for some i and $\sigma(k_i^*) - \sigma(k_i) < 0$ for some others, but the non-negative terms overweight the negative ones. If the former scenario is true, then by definition, \mathbf{k}^* is ex post efficient and the proof is completed. Hence, one needs to show that the latter scenario is excluded. Define $\Psi_{\mathbf{k}, \mathbf{k}^*} \subset I_{\mathbf{k}, \mathbf{k}^*}$, as $\Psi_{\mathbf{k}, \mathbf{k}^*} \equiv \{i \in I_{\mathbf{k}, \mathbf{k}^*} : \sigma(k_i^*) < \sigma(k_i)\}$. Suppose, by a way of contradiction, that there exists \mathbf{k} , such that $\Psi_{\mathbf{k}, \mathbf{k}^*}$ is non empty and \mathbf{k} is ex post efficient. Consider the plan \mathbf{k}' , where

$$k'_i = \begin{cases} k_i, & \text{if } i \in \Psi_{\mathbf{k}, \mathbf{k}^*}, \\ k_i^*, & \text{otherwise.} \end{cases}$$

By construction, for all $i = 1, 2, \dots, n$, $\sigma(k'_i) \geq \sigma(k_i^*)$, with strict inequality for some i 's (the ones in $\Psi_{\mathbf{k}, \mathbf{k}^*}$, which is non empty). This implies that $S(\mathbf{k}') > S(\mathbf{k}^*)$, a contradiction to the fact that \mathbf{k}^* is ex ante efficient. \square

Proof of Proposition 3. Suppose that in the symmetric equilibrium all sellers advertise the mechanism $M^* = \{\mathbf{p}^*, \mathbf{k}^*\}$, where \mathbf{k}^* is ex post efficient. The profit of an arbitrary seller is given by

$$\pi(M^*) = \frac{n}{m} \left\{ \sum_{i=1}^n H\left(n, \frac{1}{m}, i\right) [k_i^* p_i^* - c(k_i^*)] + e^* \right\}, \quad (\text{a.10})$$

Equilibrium prices in a small market satisfy equation (11). Using this fact, one can re-write (a.10) as

$$\begin{aligned} & \sum_{i=1}^n H\left(n, \frac{1}{m}, i\right) \left\{ k_i^* p_i^* \left[1 - b \frac{f(n, m, i) - 1}{n - b} \right] - c(k_i^*) \right\} + e^* = \\ & = \sum_{i=1}^n H\left(n, \frac{1}{m}, i\right) \left\{ \left(1 + \frac{b - \frac{ib}{n}}{1 - \frac{b}{n}} - i \right) \left[\frac{1}{1 - \frac{b}{n}} k_i^* u(k_i^*) - c(k_i^*) \right] \right\}. \end{aligned} \quad (\text{a.11})$$

Define Λ as the left-hand side of equation (a.11). It can be shown that

$$\lim_{n \rightarrow \infty} \Lambda = \lim_{n \rightarrow \infty} \left\{ \sum_{i=1}^n H\left(n, \frac{b}{n}, i\right) [k_i^* p_i^* - c(k_i^*)] \right\} + e^*. \quad (\text{a.12})$$

Using the definition of market tightness and taking the limit as $n \rightarrow \infty$ on equation (a.10) implies that

$$\bar{\pi} = b \lim_{n \rightarrow \infty} \left\{ \sum_{i=1}^n H \left(n, \frac{b}{n}, i \right) [k_i^* p_i^* - c(k_i^*)] + e^* \right\}.$$

The limiting value of profit is equal to the right-hand side of equation (a.12) multiplied by b . Combining this observation with equation (a.11) yields

$$\bar{\pi} = b \lim_{n \rightarrow \infty} \left\{ \sum_{i=1}^n H \left(n, \frac{1}{m}, i \right) \left\{ \left(1 + \frac{b - \frac{ib}{n}}{1 - \frac{b}{n}} - i \right) \left[\frac{1}{1 - \frac{b}{n}} k_i^* u(k_i^*) - c(k_i^*) \right] \right\} \right\},$$

and finally

$$\bar{\pi} = b \lim_{n \rightarrow \infty} \left\{ \sum_{i=1}^n H \left(n, \frac{b}{n}, i \right) (1 + b - i) \sigma(k_i^*) \right\}.$$

□

Proof of Proposition 4. Since the domain of σ is \mathbb{N} , quasi concavity implies that for all $h, i, j \in \mathbb{N}$, with $h < j$ and $h \leq i \leq j$, we have $\sigma(i) > \min\{\sigma(h), \sigma(j)\}$. Suppose, by a way of contradiction, that the ex post efficient schedule \mathbf{k}^* does not have the form described in Proposition 4. The contradictory statement can be re-phrased as follows. Let λ be the smallest number for which $k_\lambda^* = \lambda$ and $k_i^* = \lambda$, where $i = \lambda + 1$.³³ However, there exists $j > i$, such that $k_j^* \neq \lambda$. Since $j > i > \lambda$ and \mathbf{k}^* is ex post efficient, $k_j^* \neq \lambda$ can only imply $k_j^* > \lambda$. Sellers will choose $k_j^* > \lambda$ only if $\sigma(j) > \sigma(\lambda)$. But then, since $j > i > \lambda$ and σ is strictly quasi concave, $\sigma(i) > \min\{\sigma(\lambda), \sigma(j)\} = \sigma(\lambda)$, a contradiction to the fact that $k_i^* = \lambda$. □

Proof of Proposition 5. The expected number of sales per seller is given by

$$\bar{A}_S = \lim_{n \rightarrow \infty} \sum_{i=1}^n \binom{n}{i} \left(1 - \frac{b}{n}\right)^{n-i} \left(\frac{b}{n}\right)^i k_i^*.$$

If $\lambda = \infty$, $k_i^* = i$ for all i and the result is immediate. If $\lambda < \infty$,

$$\bar{A}_S = \lim_{n \rightarrow \infty} \left\{ \sum_{i=1}^{\lambda} \binom{n}{i} \left(1 - \frac{b}{n}\right)^{n-i} \left(\frac{b}{n}\right)^i i + \lambda \sum_{i=\lambda+1}^n \binom{n}{i} \left(1 - \frac{b}{n}\right)^{n-i} \left(\frac{b}{n}\right)^i \right\},$$

³³ If such a number does not exist, then we have the no frictions case, which is part of the class of production schedules under consideration.

Adding and subtracting the term $\lambda \sum_{i=\lambda+1}^n \binom{n}{i} (1 - b/n)^{n-i} (b/n)^i$ in the last expression yields

$$\begin{aligned} \bar{A}_S &= \lim_{n \rightarrow \infty} \left\{ \sum_{i=1}^{\lambda} \binom{n}{i} \left(1 - \frac{b}{n}\right)^{n-i} \left(\frac{b}{n}\right)^i (i - \lambda) + \lambda \sum_{i=1}^n \binom{n}{i} \left(1 - \frac{b}{n}\right)^{n-i} \left(\frac{b}{n}\right)^i \right\} \\ &\equiv \lim_{n \rightarrow \infty} \left\{ \Omega_1(n, b, \lambda) + \lambda \Omega_2(n, b) \right\}. \end{aligned} \quad (\text{a.13})$$

Using some standard results about limits, one can easily show that $\lim_{n \rightarrow \infty} \Omega_2(n, b) = 1 - e^{-b}$. Also,

$$\lim_{n \rightarrow \infty} \Omega_1(n, b, \lambda) = \sum_{i=1}^{\lambda} \frac{(i - \lambda)b^i}{i!} \lim_{n \rightarrow \infty} \left[\frac{n!}{(n - i)! n^i} \left(1 - \frac{b}{n}\right)^{n-i} \right].$$

But since $i \leq \lambda < \infty$, $\lim_{n \rightarrow \infty} \{n! / [(n - i)! n^i]\} = 1$. Also, $\lim_{n \rightarrow \infty} (1 - b/n)^{n-i} = e^{-b}$. Therefore,

$$\lim_{n \rightarrow \infty} \Omega_1(n, b, \lambda) = \sum_{i=1}^{\lambda} \frac{(i - \lambda)b^i}{i!} e^{-b}.$$

Using these observations, one can re-write (a.13) as

$$\bar{A}_S = e^{-b} \sum_{i=1}^{\lambda} \frac{(i - \lambda)b^i}{i!} + \lambda(1 - e^{-b}).$$

□

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