# Herding and crowding to efficiency

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**Job market paper for Efraim Berkovich**

**Abstract**

In many markets, for instance experience goods markets, buyers cannot always know what surplus they will get at a particular seller. If buyers see past trades and payoffs decrease due to crowding, public information about service quality at every seller grows. We study this value discovery process in an overlapping generations, discrete-time dynamic market with imperfect information and crowding. Short-lived buyers encounter infinite-lived sellers who provide heterogeneous quality service. The number of trades at each seller is public information. At first, buyers do not have information about what is offered at any seller, but they can pay to acquire private information about any single seller. In each period, this market endogenously segments into areas of known quality and unknown quality. In regions of unknown quality, informed buyers drive out uninformed traders, reducing trade and forcing uninformed buyers into regions of known quality—a pattern that superficially resembles risk-aversion, though buyers are risk-neutral. As time goes on, public information grows, in contrast to many herding models where information is lost. This market becomes more efficient over time as fewer buyers incur search costs each period.

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# 1 Introduction

A new entrepreneur seeking accounting, or other professional services for his business, looks for accountants with a fair number of clients already. This instinctive behavior seems optimal in the face of meager private information and serves as the intuitive basis for the way many markets work. In this paper, we model a simple market where buyers have limited private information but can see the past actions of other buyers, and we add a negative payoff externality. We find that it is possible for markets to become more efficient through herding effects.

In discrete time, short-lived buyers face a multitude of long-lived sellers offering services of heterogeneous quality. Buyer payoffs increase in the quality of the service at the seller and decrease in the number of other buyers at the seller. Each buyer has inelastic demand for the service, so actions are discrete. Sellers provide the same quality forever, and, therefore, buyers want to know the quality of all sellers in order to distribute themselves optimally. All buyers see the purchasing decisions of the previous period's buyers. Initially, each buyer has no private information about the quality of the sellers, but he can choose to acquire costly private information. With or without private information, buyers make their purchase decision, consume, and leave.

## 1.1 Market for accountants

As an illustrative example, consider the case of a new entrepreneur seeking accounting services. Suppose that the quality of an accountant is fixed through time. Clients cannot see service quality until after they use the accountant's services. They do see the number of other clients at each accountant. Business owners need accounting services and cannot provide it for themselves. Once an entrepreneur selects an accountant, he would rather purchase the service than leave, even if he does not like the quality of service. Potential buyers of accounting services do have the option to put down a deposit at another accountant, so if the first accountant proves unsatisfactory, they can quickly switch service providers for the same fiscal period. If they do not put down the deposit, they cannot switch. Payoff to entrepreneurs increases in accountant quality, but decreases in the number of other clients at the same provider. At the end of each reporting period, entrepreneurs see the number of clients each accountant had the previous period, choose an accountant, go elsewhere if they do not like the quality of service and have left a deposit with another provider, then pay and consume services. As time goes on, the number of clients at a provider becomes a good signal as to the quality of service.

The number of previous period buyers is a public signal of the quality of a seller. We find that for a large market, the market endogenously segments into regions of publicly known seller quality and regions where the quality is not known exactly but known to be within a particular range. As time and search progresses, the regions of known quality become larger.

When a buyer's information set is partially determined by the prior actions of other buyers, it is rational to act on this public information. However, in many models of herding, informational cascade type effects reduce the amount of social learning and generate an inefficient outcome. Information cascades can occur even when acquiring private information is free. By adding a negative crowding externality to the payoffs of buyers, we find that herding actually leads to increased social learning and efficiency. In some sense, this result is not surprising, as one expects that information cascades become attenuated under negative payoff externalities. We find that, in this market, the public signal eventually separates different quality sellers; it converges to perfect separation in the limit of time if search is costless.

So long as buyers choose to acquire private information, sellers of publicly known quality receive a larger number of buyers than the number they receive in periods when buyers do not search. Therefore, those buyers who buy from known quality sellers get a lower payoff compared to the payoff when no one pays to be informed. One can intuit that if some sellers are over-bought others must be under-bought, specifically the sellers in market segments of low public information. When this inefficiency exists in real world markets, it may be partially a result of limited information rather than being the result of irrational traders, noise traders, or even risk averse traders. The information asymmetry between informed and uninformed traders pushes uninformed traders out of market segments where information matters and lowers payoffs in the segments of full information.

## 1.2 Related literature

The paper adds to the growing literature on herding with payoff externalities. In a recent work, Drehmann *et al.* (2007) perform experimental work on various herding models where they look at a variety of payoff externalities. Their theoretical model differs in that agents choose between two options sequentially, instead of the large market and simultaneous choices in our model.

Many herding models tend to study relatively small markets whereas we take a search theoretic approach to try to describe a larger one. There exists a large literature on search with capacity constraints. For example, Peters (1984) presents a directed search model with rationing of buyers by sellers. Burdett, Shi, and Wright (2001) discuss equilibrium pricing and matching in a similar model. Shimer (2005) presents a directed labor search model with heterogeneous workers and firms and finds assortative matching. Lester (2007) endogenizes the buyer's decision of whether to search or not, as do we in this paper.

Learning from other agents' actions in markets has extensive precedent in the literature. Banerjee (1992), (Bikhchandani *et al.* 1992) and (Welch 1992) describe models of herding where agents obtain a private signal about a choice of options and also observe the prior actions of a number of other agents who faced the same decision. They find that agents begin to ignore their private signal, forming an informational cascade. In an elaboration, Debo, Parlour, and Rajan (2008) describe a model where agents obtain a private signal about a firm's quality and also observe the length of a queue of prior randomly arriving agents. The agents infer the firm's quality from the length of the queue. Vives (1996) notes that informational cascades depend on discrete action spaces, though Huck and Oechssler (1998) find a counter-example.

Herding and learning has been well-studied in relation to asset markets. For instance, Lang and Nakamura (1990) offer a dynamic model of learning in credit markets where a decrease in the number of traders decreases public information and increases risk thereby further driving out traders. They find that market prices have greater volatility than the underlying shocks because of this feedback. The effect of traders being driven out of market segments with low public information parallels effects seen in our model. Similarly, Caplin and Leahy (1994) describe a market model with private information where the actions of agents publicly reveal information resulting in periodic market crashes.

The importance of the precision of the public versus private information was examined by Morris and Shin (2002) who model a game with strategic complementarities where traders have private and public information. They find that public information may be welfare decreasing if it is not sufficiently precise. Conversely, high precision public information relative to private information can be welfare increasing. In our analysis, the increasing refinement of the public signal improves market efficiency in time.

When rational traders lack information, herding may be part of a best response strategy and may improve efficiency. Nofsinger and Sias (1999) study the effects of institutional and individual stock ownership in relation to herding. Quoting Nofsinger and Sias (1999): "Herding and feedback trading have the potential to explain a number of financial phenomena, such as excess volatility, momentum, and reversals in stock prices."[1] They find a strong positive relation between annual changes in institutional ownership and returns. In their view, this relation is consistent with either intra-year feedback trading by institutions or a stronger impact of institutional herding (vs. individual) on returns. The latter hypothesis resembles work by Postlewaite and Kircher (2008) while the former may be more closely associated with the model in our paper, though neither hypothesis excludes the other.

# 2 Model

We consider an overlapping generations model of a market where buyers are alive for two periods and sellers are long-lived. Time proceeds in discrete periods. In the first period of life, buyers do not buy but see the actions of the previous generation of buyers. In their second and last period, buyers can acquire private information about any one seller and then purchase from any one seller[2].

Each generation of homogeneous, anonymous, risk-neutral buyers is the same size and is denoted by $\mathcal{B} = \{1..B\}$. The set of initially homogeneous, long-lived sellers is denoted by $\mathcal{S} = \{1..M\}$. We assume more buyers than sellers and that there are $N$ buyers per seller. Sellers produce and sell services of quality $q \in [0, 1]$. At the start of the game, each seller draws a quality type $q$ from a uniform distribution and produces service of that specific quality forever[3]. Each seller produces $N$ units of the good each period. The quality of a seller $i$ is denoted by $q(i)$ and the number of buyers at a seller $i$ in period $t$ is denoted by $n_t(i)$.

For the buyer, a higher quality service yields higher utility all else equal. We assume buyers have inelastic demand for $N$ units of the service from a seller regardless of quality. A single buyer can purchase from only one seller[4].

The young generation of buyers has no available actions, but they observe the trades of the old generation. For members of the old generation, a buyer's available actions are (a) to sample the quality at a single seller $j$ of their choosing at a cost $c$, and (b) to choose a seller $i$ from whom to buy. For completeness, we note that the buyer can choose not to buy from any seller and can

[1] Quoting further to explain: "Herding is a group of investors trading in the same direction over a period of time; feedback trading involves correlation between herding and lag returns... Most herding models suggest that investors follow some common signal. Feedback trading, a special case of herding, results when lag returns, or variables correlated with lag returns (e.g., earnings momentum, decisions of previous traders, changes in firm characteristics, etc.) act as the common signal."

[2] The model described is close to the one we present in Berkovich and Tayon (2008). That paper studied a class of similar models but for non-rival goods. The lack of capacity constraint led to a sharper herding effect than we see with the crowding constraint in this paper.

[3] One can assume that it is difficult to change the production process, and so sellers cannot change the quality of the produced service once the game begins. Varying quality instead of price is not too far-fetched since in many real-world situations prices are easily visible, whereas quality may be more difficult to determine by the individual buyer. We think of quality as a stand-in for a consumer surplus which depends on quality and price. Assuming, instead, that sellers have homogenous quality and vary the price, we can substitute for price as $p = 1 - q$. If the good is indivisible, then the mathematics is identical. The notion of uncertain quality and known price is similar to Williamson and Wright (1994).

[4] In our model, the binary action of either buying or not buying leads to an over-pricing effect in that searchers cannot "short-sell" a low quality seller. As Nagel (2005) finds, a short-selling restriction leads to an optimism-bias in prices on stocks. In our model, the short-selling restriction prevents full value discovery.

choose not to consume, but these actions are never optimal. Our model differs slightly from many search models in that we expand the buyer's action space by relaxing the requirement that buyers purchase only from a seller they have sampled.

The timing of the game is as follows:

1. Sellers are assigned a quality type by Nature from a uniform distribution. For each seller $i$, $q(i) \sim U[0,1]$.

2. At the start of period $t$, a generation of buyers is born and sees $n_t(i)$ for every seller $i$—the number of old generation buyers at a seller. For $t = 0$, buyers see all sellers as identical.

3. Old generation buyers optionally sample the quality at one seller $j$ at a cost $c$ and then know $q(j)$ exactly.

4. Old buyers pick a seller $i$ from whom to buy. All buyers at seller $i$ and the seller engage in a bargaining subgame.

5. After bargaining splits the surplus, old buyers receive their payoff, and die at the end of the period.

6. A new period begins. Repeat from step 2.

Buyers cannot distinguish sellers within a partition set of their information set, that is all sellers who had $n$ buyers initially are indistinguishable[5]. We assume a matching technology between buyers and sellers in a partition set is such that buyers are matched with sellers based on a uniform distribution. So, if a buyer wants to buy from a seller who had $n$ buyers and there are $K$ such sellers, the probability of that buyer matching with any particular seller in the set is $\frac{1}{K}$. If a buyer samples a seller, he creates a new partition because that seller becomes distinguishable from all other sellers with the same number of buyers.

In order to have a price arise endogenously in our model, we have buyers and seller bargain. The bargaining subgame is contrived in order to lead to a particular result, but other mechanisms and bargaining games that lead to a predictable split of surplus between the seller and buyers generally lead to the same qualitative outcome we obtain here. An alternative simpler formulation is, for instance, to have a fixed market price across all sellers.

The bargaining game is as follows. Within a period, time is divided into sub-periods, and sellers and buyers discount with rate $\delta \in (0,1)$ per sub-period. The total surplus at a seller is defined to be $\frac{qN}{\delta}$. Once all buyers have chosen sellers, the quality at each seller is revealed to the buyers there. Of all the buyers at a seller, one is chosen at random to bargain to set the price for all buyers. That buyer makes an offer to the seller to split the surplus. The seller can accept or reject. If he rejects, he makes a counteroffer. The buyer can accept or reject. If he rejects, both buyer and seller get zero surplus.

[5] The refinement of the information set available to a buyer is critical in determining his actions. A given refinement operator $V$, operates on the state of the world $\theta$ (including all past history) and yields an object that is the buyer's information set; that is, the operator $V$ determines how the states of the world are partitioned into information sets. One type of $V$ is a single partition, the refinement often assumed in standard search literature; buyers see no distinguishing traits for any seller $i$. Another refinement $V$ is where all seller qualities are publicly known (and history is ignored)—the refinement used in directed search literature (in conjunction with a capacity constraint). Our paper analyzes a "herding" refinement where buyers know the number of buyers who purchased goods at each seller in the previous period, formally, $V = \{n_{t-1}(i) : \forall i\}$. Classifying different information set refinements allows for a more generalized approach to information in search where particular classes of refinements may lead to qualitatively similar outcomes. Our interest lies in analyzing the "herding" refinement from an initial state of no information (identical to random search). A different initial information state where parts of the seller space have known quality does not change the basic analysis, nor does it qualitatively affect the end state. If parts of the seller space have known quality at $t = 0$, those sellers receive a known number of buyers and our analysis covers the unknown quality sellers.

The subgame-perfect outcome is that the buyers get $qN$ surplus to share. Since there are $n_t(i)$ buyers at the seller, each buyer's share of the surplus is $\frac{1}{n_t(i)}$, and a buyer's payoff to buying from seller $i$ at time $t$ is given by $u_t(i) = u(q(i), n_t(i)) = \frac{q}{n_t(i)}N$. We created the setup with the multiplier $N$ in order to normalize payoffs as we change the number of buyers.

# 3 Buyers' dynamics

In each period, buyers decide whether to sample or not. We consider only symmetric equilibria and denote by $\sigma$ the probability that a buyer samples. If sampling, the buyer selects a seller $i$ to sample. If not buying from him, the buyer selects another seller $j(q(i))$ from whom to buy. After sampling, buyers use a period decision function $f_t(q(i))$ to (possibly) mix over buying from the realized $i$ versus buying from a random $j$. If not sampling, the buyer just selects a seller $j$ from whom to buy. The expected payoff to sampling in period $t$ is

$$W_S(t) = E\left[f_t(q(i))\, u(q(i), n_t(i)) + (1 - f_t(q(i)))\, u(q(j), n_t(j))\right]$$

and the expected payoff to not sampling is

$$W_N(t) = E\left[u(q(j), n_t(j))\right]$$

In period $t = 0$, the first generation of buyers have no information about the quality of any seller. In $t = 1$, they sample uniformly across all sellers. The period decision function, $f_1(q)$ of whether to buy from the sampled seller depends only on the quality of the seller. To simplify notation, we drop the time-subscript on $f$ when referring to period $t = 1$. Define by $L(q)$, the expected payoff to leaving the sampled seller and buying from a random other seller. Define by $S(q)$, the expected payoff to buying from the sampled seller.

We use the following definition to describe the shape of the decision function $f(q)$.

**Definition 1** *"Flats" are positive measure regions $f^{-1}(0)$ and $f^{-1}(1)$.*

In the first period of the game, if buyers sample, the decision function is increasing in $q$ and has the following shape: flat-gradient-flat.

**Proposition 2** *In $t = 1$, if $\sigma > 0$,*
*(a) $f(q)$ is increasing and strictly increasing when $f(q) \in (0,1)$,*
*(b) a flat exists $\Leftrightarrow W_S > W_N$, and*
*(c) two flats exist, $f^{-1}(0) = [0, q_{lo}]$ and $f^{-1}(1) = [q_{hi}, 1]$, $0 < q_{lo} < q_{hi} < 1$.*

**Proof.** Part (a): Non-samplers (fraction $1 - \sigma$ of the buyers) and "unhappy" samplers (that is, samplers who do not buy from their sampled seller) buy with a uniform distribution across all sellers. We look at three cases:

Case 1: If $f(q) = 1$, then $f(q') = 1\ \forall q' > q$.

Case 2: If $f(q) = 0$, then $f(q') = 0\ \forall q' < q$.

Case 3: Otherwise, $f(q)$ is strictly increasing for $q$ such that $f(q) \in (0,1)$.

Proof for cases 1 and 2 is straightforward and omitted. For case 3, the payoffs to staying and leaving must be the same. Suppose, toward contradiction, that $f(q) \geq f(q')$ for some $q < q'$.All else being equal, the expected payoff to buyers who leave depends on $f$. It must be that $L(q) \geq L(q')$ as more buyers leaving the sampled seller lowers payoffs at the other sellers. We have that $L(q) = S(q)$ and $L(q') = S(q')$ since buyers are mixing between leaving and staying. So $S(q) \geq S(q')$, but this is impossible since $u(q,n)$ strictly increases in $q$ and strictly decreases in $n$ and the expected number of buyers is lower at sellers of quality $q'$.

Part (b): Suppose no flat exists, then all samplers get the same expected payoff because $L(q) = S(q)$ everywhere, so sampling and then purchasing from a random seller gives the same expected payoff as purchasing from a random seller—the expected payoff of a non-sampler. Conversely, payoffs on a flat are increasing linearly in $q$; so samplers get a surplus. Note, since $W_S - c \geq W_N$ in equilibrium, then $c > 0$ implies flats.

Part (c): In the finite-sized game, the payoff to staying at a very low quality seller is decreasing to zero in the quality (regardless of the number of buyers). Therefore, there must be some positive measure region of low quality where samplers leave for sure. Define by $D(q)$ the expected payoff to buying uniformly from a random seller (including the sampled seller) after sampling $q(i) = q$. We have that $D(q) = \frac{1}{M}S(q) + \frac{M-1}{M}L(q)$. Because buyers sample all sellers uniformly, $\int D(q)\,dq = \int L(q)\,dq$. We have that $D(q) < L(q)$ for $q$ in the low flat. Furthermore, $D(q) = L(q)$ for $q$ in the gradient Therefore, it must be that $D(q) > L(q)$ in the high flat and since the low flat is positive measure, the high flat must be positive measure.

We turn to the question of whether both flats exist in the limit of sellers going to infinity and buyer to seller ratio $N$ going to infinity. It is sufficient to consider only when all buyers sample.

Two flats do not exist when taking the limit in $N$ going to infinity but keep the number of sellers fixed at some $M < \infty$. Suppose that there exists a positive measure high flat and that there exists a low flat. A buyer on the low flat knows there is a positive probability that all other sellers are high flat and thus a positive probability that he would be the only buyer at his low flat seller if he stays. Since the payoff to being the sole buyer goes to infinity, he stays. This effect is due to the possibility of unbounded payoffs.

Two flats exist when we fix the ratio of buyers to sellers at $N < \infty$ and take the number of sellers to infinity. At any seller, the arrival rate of "unhappy" samplers approaches a Poisson distribution with some rate $\lambda(N) > 0$. Therefore, the expected payoff at a low flat seller is

$$E\left[u(q,n)\,|q \in \text{ low flat}\right] = qNe^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\frac{1}{k+1}$$

Since the summation converges to a finite number (less than $e^{\lambda}$), then expected payoff is decreasing in $q$ to zero and there exists a low flat.

With the number of sellers at infinity, two flats exist in the limit of $N \to \infty$. It must be that $0 <; \lim_{N\to\infty}\frac{N}{\lambda} < K$, for some $0 < K < \infty$. Clearly, the ratio cannot go to zero. If the ratio goes to infinity, that implies nearly all buyers stay with their sampled seller, which is not individually rational. So,

$$E\left[u(q,n)\,|q \in \text{ low flat}\right] \leq \lim_{\lambda\to\infty} qK\lambda e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\frac{1}{k+1}$$

But $\lim_{\lambda\to\infty} e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^{k+1}}{(k+1)!} = 1$, so there exists a low flat. A high flat then necessarily exists because expected payoffs are bounded. ■

Having a finite number of buyers implies that to solve for equilibrium we must calculate the payoffs to a variety of possible distributions of buyers over sellers. But when taking the limit, the expectation of the payoff simplifies. Therefore, for tractability we proceed with the analysis as if we have a continuum of buyers of measure $B \equiv M$, that is, there are measure one buyers for every seller, and then take $M$ to infinity. The proposition above suggests that since the buyers' decision function is well-behaved in the limit, our analysis converges to the dynamics of the model taking the limit in $M$ and $N$. In this limit formulation, we stipulate that the expected payoff to a buyer at a seller is given by $u(q,n) = \frac{q}{n}$ where $q$ is the quality of the seller and $n$ is the measure of buyers at the seller.

Assuming a finite $M$ number of sellers and a continuum of measure $M$ buyers, in $t = 1$,the payoff to leaving a seller of quality $\tilde{q}$ after sampling is

$$L(\tilde{q}) = \int_0^1 \frac{q}{1-\sigma+\sigma f(q)+\gamma(\tilde{q})} dq$$

where $\gamma(\tilde{q})$ is the expected measure of "unhappy" samplers who arrive to buy from every other seller other than the $\tilde{q}$ seller. It is

$$\gamma(\tilde{q}) = \frac{\sigma}{M-1}\left[(1-f(\tilde{q}))+(M-2)\int_0^1 (1-f(q))\,dq\right]$$

However, as $M \to \infty$, $\gamma(\tilde{q}) \to \gamma$ where

$$\gamma = \sigma \int_0^1 (1-f(q))\,dq$$

The payoff to buying from the sampled seller is

$$S(\tilde{q}) = \frac{\tilde{q}}{1-\sigma+\gamma+\sigma f(\tilde{q})}$$

where $\gamma$ is the measure of "unhappy" samplers who arrive to buy from this seller as before.

For buyers to mix between leaving and staying, it must be that $S(q) = L(q)$ for some $q$. In the limit of $M$, the expected payoff to leaving, $L(\tilde{q})$, goes to the expected payoff to buying randomly from a seller, that is, $L(\tilde{q}) \to W_N$, the payoff to non-samplers. Looking at the ends of the gradient, that is $q_{lo}$ and $q_{hi}$, a simplified form for $W_N$ arises. Since in the limit $S(q_{lo}) = S(q_{hi}) = W_N$, then $W_N = \frac{q_{lo}}{1-\sigma+\gamma} = \frac{q_{hi}}{1+\gamma}$.

The value of the sample or "search again" option can be described in two ways. The option can be described as the right to reject the sampled seller, a "put" option, or it can be described as the right to buy from the sampled seller, a "call" option. Implicit in both definitions is that the outside option of buying from a random seller exists.

**Proposition 3** *In the limit of $M$ and $N$, the value of a put option is $c$ for $\sigma \in (0,1)$ and is greater than $c$ for $\sigma = 1$. Furthermore, the values of a put and call option are equal.*

**Proof.** For $\sigma = 1$, define $W_N$ as the expected payoff to buying from a random seller. We have that $W_S - c \geq W_N$ with equality for $\sigma \in (0,1)$. Since $f(q)$ has flats,

$$\begin{aligned} W_N &= \int_0^{q_{lo}} \tfrac{q}{1-\sigma+\gamma} dq + \int_{q_{lo}}^{q_{hi}} W_N dq + \int_{q_{hi}}^1 \tfrac{q}{1+\gamma} dq \\ W_S &= \int_0^{q_{hi}} W_N dq + \int_{q_{hi}}^1 \tfrac{q}{1+\gamma} dq \end{aligned}$$

Subtracting, we get that $c \leq \int_0^{q_{lo}} \frac{q_{lo}-q}{1-\sigma+\gamma} dq$ where the right-hand side is the value of the put option. Similarly, $c \leq \int_{q_{hi}}^1 \frac{q-q_{hi}}{1+\gamma} dq$ where the right-hand side is the value of the call option. ■

The $\frac{q}{n}$ capacity constraint is zero-sum, meaning total consumer surplus is constant regardless of the distribution of the buyers, provided all sellers have buyers. The total consumer surplus is $N\sum_i q(i)$, so the average buyer's surplus is $\frac{1}{MN} \cdot N\sum_i q(i) = E[q(i)]$. Thus, we have the equation

$$\sigma W_S + (1-\sigma) W_N = \frac{1}{2} \tag{1}$$

For interior $\sigma$, the indifference $W_S - c = W_N$ implies that the value of the search option (for example the put option) can be written as

$$c = \frac{1}{2} \cdot \frac{q_{lo}}{1-\sigma+\gamma} \tag{2}$$

To derive the decision function $f(q)$, we note that the buyers are indifferent between staying and leaving on the gradient, so $W_N = \frac{q(i)}{1-\sigma+\gamma+\sigma f(q(i))}$. Differentiating with respect to $q(i)$ gives $f'(q) = \frac{1-\sigma+\gamma+\sigma f(q)}{\sigma q}$ and with the condition $f(q_{lo}) = 0$, we get[6]

$$f(q) = \frac{\sigma - 1 - \gamma}{\sigma} + q\frac{1-\sigma+\gamma}{\sigma q_{lo}}$$

So, the decision function $f$ is linear on the gradient and $\gamma = \sigma \int_0^1 (1 - f(q))\, dq = \sigma \left(q_{lo} + \frac{q_{hi}-q_{lo}}{2}\right)$. Since $W_N = \frac{q_{lo}}{1-\sigma+\gamma} = \frac{q_{hi}}{1+\gamma}$, one can write $q_{hi}$ in terms of the only $\sigma$ and $q_{lo}$. Substituting for $W_N$ in (1) yields $\sigma c + \frac{q_{lo}}{1-\sigma+\gamma} = \frac{1}{2}$. Solving with (2) gives a unique equilibrium for interior $\sigma$, graphed in Figure 1 . When the value of the search option is strictly greater than $c$, then $\sigma = 1$, and the equation 1 defines $q_{lo}$ uniquely.

The fraction of samplers, $\sigma$, is determined by $c$. When $\sigma = 0$, no one samples and the marginal buyer's reservation quality is $\frac{1}{2}$. The maximum search option value is $\frac{1}{8}$. For $c \geq \frac{1}{8}$, $\sigma = 0$. As $c$ decreases from $\frac{1}{8}$ and $\sigma$ increases, $q_{lo}$ and $q_{hi}$ diverge from $\frac{1}{2}$. For low enough $c$, all buyers sample. There exists a $\bar{c} \approx 0.043689$ such that $\sigma = 1\ \forall c < \bar{c}$. This result differs from the endogenous directed search model in Lester (2007). In that paper, the endogenously determined measure of searchers approached one only if search was costless whereas, in our model, all buyers choose to search even for low but positive costs to searching.

At $t = 1$, the new generation of buyers see three regions: low flat, gradient, and high flat. Buyers know the quality of any seller on the gradient because $f(q)$ is strictly increasing. Therefore, in $t = 2$ searchers only sample sellers in the flats. Searchers use the period decision function $f_2(q)$ which consists of regional decision functions $f_{low}$ and $f_{high}$, that have properties as in $t = 1$.

Consider a parameterization of $t = 1$, the flat $i$ is $[a_i, b_i]$ and has $\mu_i$ fraction of the buyers. It suffices to consider equilibrium strategies where the problem is "regionalized." A multiplicity of equilibrium strategies exist where "unhappy" and non-sampling buyers buy from different regions in varying measures. Since we are interested in equilibrium outcomes, we do not distinguish between these strategies. We use the notation $\sigma_i$ is the fraction of the buyers in the region $i$ who sample

[6] Alternatively, $f(q)$ can be derived from $W_N = \frac{q_{lo}}{1-\sigma+\gamma} = S(q)$.

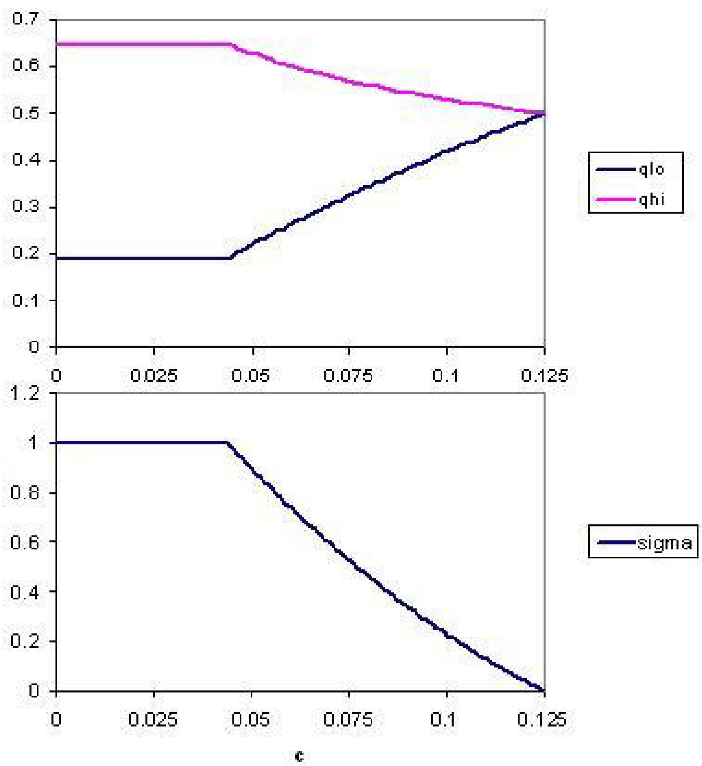


Figure 1: At $t = 1$, equilibrium values for $\sigma, q_{lo}$, and $q_{hi}$ given $c$.

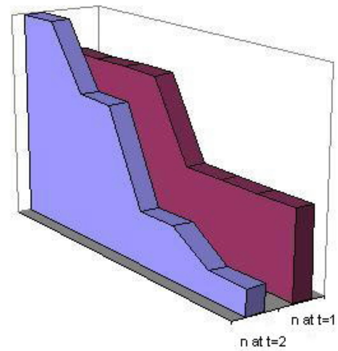


Figure 2: Buyer distribution in $t = 1$ and $t = 2$..

and $\gamma_i$ is the fraction who are "unhappy" samplers, so, for instance, the measure of buyers who search in region $i$ is $\sigma_i \mu_i$ Rewriting equation (1),

$$\sigma_i c + \frac{q_{lo} - a_i}{\frac{\mu_i}{b_i - a_i}(1 - \sigma_i + \gamma_i)} = \frac{\frac{a_i + b_i}{2}}{\frac{\mu_i}{b_i - a_i}}$$

and substituting $k_i = \frac{\mu_i}{b_i - a_i} c$, $\hat{q}_i = q_{lo} - a_i$, and $d_i = a_i + b_i$ yields,

$$\sigma_i k_i + \frac{\hat{q}_i}{1 - \sigma_i + \gamma_i} = \frac{d_i}{2} \tag{3}$$

For interior $\sigma_i$, the value of the search option equation (2), similarly yields

$$k_i = \frac{1}{2} \frac{\hat{q}_i^2}{1 - \sigma_i + \gamma_i} \tag{4}$$

These equations are nearly identical to (1) and (2). Solving them yields unique solutions as before. Additionally, market segments are linked, so $W_N = W_{N,i}$ $\forall i$ and

$$W_{N,i} = \frac{q_{lo,i} - a_i}{\frac{\mu_i}{b_i - a_i}(1 - \sigma_i + \gamma_i)} = \frac{b_j^2 - a_j^2}{2\mu_j} = W_{N,j} \tag{5}$$

for flat region $i$ and gradient region $j$. Since the total measure of buyers is fixed, $\sum \mu_k = 1$.

Sampling continues in subsequent periods until such time as the value of the search option is less than $c$. The market continues to segment into gradient and flat regions as each former flat region becomes transformed into a flat-gradient-flat. Figure 2 provides an illustration of buyer distribution in periods $t = 1$ and $t = 2$.

By our analysis of $t = 1$, we have that the regionalized solution is unique given the endpoints $a$ and $b$, the search cost $c$, and the total measure of buyers in the segment $\mu$. Since $W_{N,k}$ is the same for all regions $k$ in equilibrium, the solution requires finding $\mu_k$ for all regions. Note that even if all buyers sample, so that nominally there are no uninformed buyers, a mass of buyers must still purchase from the gradient in equilibrium.

**Proposition 4** *In the limit of $M$ and $N$, there is a unique equilibrium outcome for $c = 0$ and a finite number of equilibrium outcomes for $c > 0$.*

**Proof.** Consider period $t-1$ where there are multiple regions indexed by $k$. We know that each region has a unique equilibrium outcome given the measure of buyers there, $\mu_k$. The expected payoff to a buyer in region $k$ is defined by $W_k = \max\{W_{N,k}(\mu_k; c, a_k, b_k), W_{S,k}(\mu_k; c, a_k, b_k) - c\}$.

We show that, for two flats $i$ and $j$, there are a finite number of crossings for $W_i(\mu)$ and $W_j(d-\mu)$ given $\mu$, where $d$ is the total mass to be split between them. Since $W_k$ strictly decreases in $\mu$ when $\sigma = 0$ or 1, we only need to check that $W_i(\mu) = W_j(d-\mu)$ for a finite number of points when $\sigma$ is interior. Since $W_i(\mu)$ is smooth when $\sigma$ is interior and has a limit as $\sigma = 0$ and 1, then if there are an infinite number of intersections it must be that $W_i(\mu) = W_j(d-\mu)$ over the entire region $\sigma \in (0,1)$ Define $f(\mu; c, a_i, b_i) = \frac{q_{lo} - a_i}{1 - \sigma_i + \gamma_i}$, so $W_i(\mu) = f(\mu)\frac{b_i - a_i}{\mu}$, so that means

$$\frac{f(\mu; c, a_i, b_i)}{f(d-\mu; c, a_j, b_j)} = \frac{\mu}{d-\mu}\frac{b_j - a_j}{b_i - a_i}$$

We have that $f(0)$ is zero only if $q_{lo} = a$, but we calculate that there is no real solution for $\sigma$ when $q_{l0} \to a$. Therefore, these two smooth functions are coincident over a finite number of intersecting points. There is at least one intersection since $W_i \to \infty$ as $\mu \to 0$.

Where $g$ is a gradient region, the payoff $W_g = \frac{b_g^2 - a_g^2}{2\mu_g}$ strictly decreases in $\mu_g$. Therefore, in equilibrium, the payoff in all gradients strictly decreases in $\mu_g$ (any measure added to any gradient). Since $W_g(\mu)$ and $W_i(1-\mu)$, where $i$ is a flat region, are both (piece-wise) smooth functions, they can only be coincident if they have the same derivative. But they do not, so $W_g(\mu) = W_i(d-\mu)$ in only a finite number of points (and again there is at least one intersection). Therefore, since any two regions have a finite number of intersections, the whole market has a finite number of equilibrium outcomes in any period.

Since for $c = 0$, it must be that $\sigma = 1$ and $W_k$ is strictly decreasing always, then there is exactly one intersection between any two regional payoffs $W_i(\mu)$ and $W_j(d-\mu)$. Thus, there is a unique equilibrium outcome in every period and hence a unique equilibrium outcome for the whole game. For $c > 0$, (we show later) there is a period $T_c < \infty$ such that there is no sampling for $t \geq T_c$ so there are a finite number of equilibrium outcomes for the whole game. ■

In the formulation with a continuum of buyers, buyers know the entire history of the market from the public signal of the current period in the sense that they know how many buyers each seller had in every prior period. Since each gradient seller's quality is known and the equilibrium strategy of past periods is known, buyers can determine which flat (or gradient) that seller was in during every prior period. Current flats sellers are known to be within a quality range which must be within a flat in every prior period. The public signal serves as a perfect substitute for history much as the state variable of money distribution does in Corbae, Temzelides, and Wright (2003) for the directed matching economies (but not the random matching economies) described in that paper. The public signal does not have this property in our model when $N$ is finite. Then, history cannot be perfectly known, rather buyers have a probability distribution over possible histories.

# 4 Welfare

Average consumer surplus is $\frac{1}{2}$, by the zero-sum condition. Average social welfare is average surplus minus average search costs. In $t = 1$, average social welfare is then $\frac{1}{2} - \sigma c$ For interior

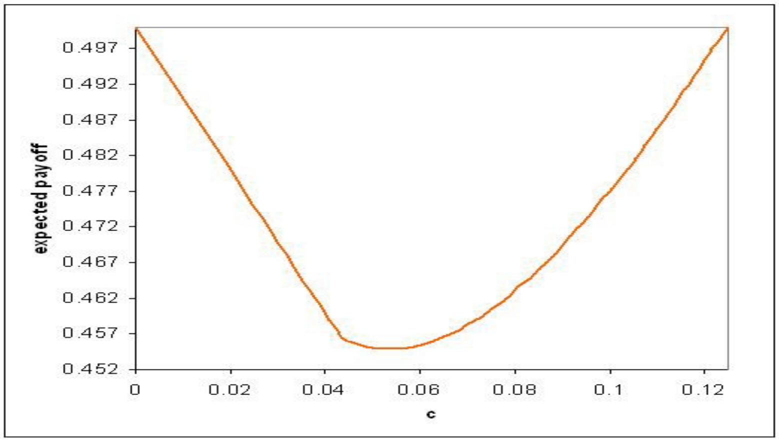


Figure 3: Average welfare (or expected payoff) in $t = 1$ as a function of $c$.

$\sigma$, $W_S = W_N + c$ and so average social welfare is $W_N$. $W_N$ is non-monotonic in $c$. Adding on the solutions where $\sigma = 1$, this equation for social welfare is, in fact, the expected payoff to buyers as a function of $c$.

For interior equilibria, all buyers receive payoff of $W_N$. Given the risk-neutrality of buyers, the centralized assignment problem, or the social planner's problem, is solved by maximizing total welfare, assuming transferable utility. From the standpoint of total welfare, the first-best outcome is when no one pays to sample (or sampling is free). The second-best outcome, the equilibrium we describe, is therefore welfare reducing.

When considering the outcome for a single period, rather than looking at allocational efficiency, we believe it makes more sense to consider informational efficiency. We mean by informational efficiency that the public signal about a seller shows that seller's quality (and hence the payoff to buyers from buying there in equilibrium). An analogous formulation is that informational efficiency is achieved when the variance in surplus over number of buyers, $\frac{q}{n}$, is zero. In this decentralized market, if the market becomes informationally efficient, then it is not individually rational to pay to sample, and so the equilibrium and social planner's outcomes coincide. So long as there are flats, the market is not informationally efficient. Each period when buyers search brings the market closer to informational efficiency.

# 5 Excess trade at the gradient and end of search

Consider $t = 1$, there are three sub-markets: the flats and the informationally efficient gradient. Search in the flats reduces payoff to non-searchers in the flats causing an increased number of non-searchers on the gradient. That is, relative to the efficient amount of buyers, the gradient has too many buyers and the flats have too few buyers. The presence of search reduces payoffs to uninformed buyers and reduces trade in the flats.

**Proposition 5** *If $\sigma > 0$, then, after search, all flats shrink by a fraction bounded away from zero.*

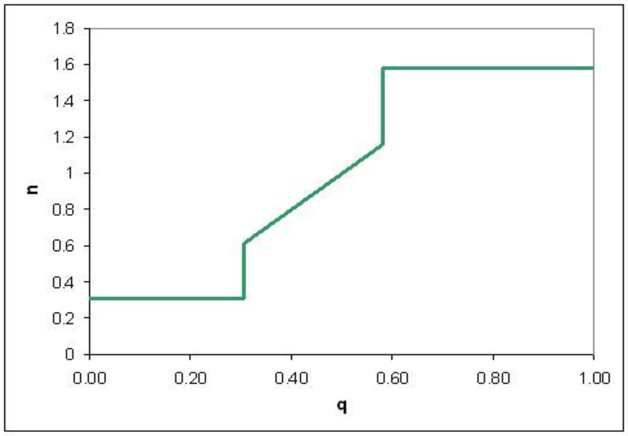


Figure 4: Buyer distribution for $t \geq 2$, for $c = 0.07$ (when search ends after $t = 1$).

*All flats go to length zero if search continues forever.*

**Proof.** Consider a distribution at some period $t$ that has flats. It suffices to consider the case of a given positive measure flat $[a, b]$ that is searched by a positive measure of buyers who all sample. The following period distribution is flat-gradient-flat. The gradient is informationally efficient, in the sense that seller qualities are publicly known. The quality of sellers in the flats is known to be within a certain range. It suffices to show that the gradient in $[a, b]$ is always of measure greater than $\eta\,(b - a)$ for some $\eta > 0$ . To simplify the presentation, we normalize the average measure of sampling buyers at a seller to one. First, note that the shape of the decision function cannot approach all flat. If the decision function goes to all high flat, then a buyer who samples a seller with quality $\frac{a+b}{4}$, and hence an expected payoff to staying of $\frac{a+b}{4}$, can, by leaving, get a higher expected payoff that approaches $\frac{a+b}{2}$. (Similarly, if the decision function goes to all low flat, a buyer who samples a seller with quality $\frac{3(a+b)}{4}$ is better off staying). Therefore, there is a positive measure $\gamma$ bounded away from zero of unhappy samplers. Consider the expected payoffs at the low and high ends of the gradient; they must be equal so $\frac{q_{lo}}{\gamma} = \frac{q_{hi}}{1+\gamma}$. Since $\gamma$ is bounded away from zero, it cannot be that $q_{lo} \to q_{hi}$ and the gradient must be a positive fraction bounded away from zero of the flat $[a, b]$ . ■

When $c$ is greater than the search option value, search ends—that is, buyers stop sampling. As a consequence to the above proposition, the search option goes to zero, if search continues forever. So for any $c > 0$, $\exists T_c$ such that there is no search for all $t \geq T_c$. When search ends, buyers know the quality of each seller in the gradient and they know the average quality in the flats. Since the gradient is informationally efficient, buyers buy from gradient sellers in such measure so that a buyer's payoff at each gradient seller is $\frac{1}{2}$. Each seller in a flat region receives the same number of buyers, with the expected payoff being $\frac{1}{2}$ for a buyer in a flat. If $c = 0$, then search never ends, and the whole market approaches informational efficiency in the limit of time.

**Proposition 6** *When sampling is costless, buyers sample forever.*

*For each seller $i$, $\lim_{t\to\infty} n_t(i) = 2q(i)$ and the payoff to every buyer approaches $\frac{1}{2}$ as $t \to \infty$.*

**Proof.** By the proposition above, we have that all flats converge to zero length. Given any $\varepsilon > 0$, all sellers' qualities are publicly known within $\varepsilon$ at time $t > T_\varepsilon$ for some finite $T_e$. Since all buyers sample, all buyers have an expected payoff of $\frac{1}{2}$ (by the zero-sum condition on consumer surplus). So, for a flat $[\hat{q} - \delta, \hat{q} + \delta]$ at time $t - 1$, the average measure of buyers that buy from a seller in that flat at time $t$ must be $2\hat{q}$. As $\delta \to 0$, $\lim_{t\to\infty} n_t(i) = 2q(i)$ for buyers on the flats. Since all buyers must get the same expected payoff, it must be that $\lim_{t\to\infty} n_t(i) = 2q(i)$ for buyers on the gradient as well. The payoff to every buyer approaches $\frac{1}{2}$. ■

When search goes forever, the limiting distribution of measure $2q(i)$ buyers at seller $i$ is informationally efficient.

# 6 Endogenous entry

To highlight the effect of increased information efficiency on allocational efficiency, we look at a variation of the market where buyers can choose to enter or stay out. When search is costly, the growth in information eventually causes the market to become allocationally efficient, and, as the public signal becomes more accurate, increasingly more buyers choose to enter the market.

Suppose that buyers have an option to receive payoff $r$ by staying out of the market altogether. In the new game, a buyer first decides whether to enter the market and, if he enters, decides whether to pay to sample. We set the total measure of buyers to be $\frac{1}{r}$ so that not all buyers want to enter. In equilibrium, the marginal buyer is indifferent about entry, so the per seller measure $\mu$ of buyers who enter the market must be such that the expected payoff in the market is $r$. Thus, $r = W_N = W_S - c$. Using an argument similar to the proposition above on the finite number of equilibria, we can show that there is a finite number of values for $\mu$ such that $r = W_N = W_S - c$. Since $W_N$ is non-monotonic in $\mu$, there are potentially a finite number of multiple equilibria with different $\sigma$.

By a result above, the search option value strictly decreases in time to zero as long as buyers sample, so that means that, for all $c > 0$, there exists some period $T_c$ such that the search option value is less than $c$. The value of $W_N$ must be $r$ in equilibrium and $W_N = W_S - c$. Thus, for $t \geq T_c$ no buyers pay to sample. Consider $c > 0$, assume buyers sample in $t = 1$, and compare $t = 1$ and $t = T_c$. For the same measure $\mu$ and any $\sigma > 0$, $W_N(\sigma > 0, \mu) < W_N(\sigma = 0, \mu)$. However, since $W_N = r$ and $r$ is fixed, in equilibrium it must be that a larger measure $\mu_{T_c} > \mu_0$ enters the market, where $\mu_t$ is the per seller measure of buyers in period $t$. For $c = 0$, it is the case that $\sigma = 1$ always (that is, buyers always sample) and so $W_S = r$. The expected payoff to entering the market depends solely on the measure of entering buyers. Therefore, the same number of buyers enter every period.

As market information grows, we can say the market becomes "commoditized" since the variance of surplus at a seller decreases. It becomes less advantageous for a buyer to become informed and the expected payoff of uninformed buyers increases. With endogenous entry (and $c > 0$), more buyers come into the market causing payoff losses from crowding that offset the individual buyer gains in information.

# 7 Conclusion

In order to examine markets with difficult to see surplus, such as the market for accounting services, we describe the dynamic evolution of a simple market of experience goods with public and private information. In our model, buyers see the purchasing actions of the previous period's buyers leading to social learning through time. In Berkovich and Tayon (2008), which describes a similar market model but with non-rival goods, most sellers are repeatedly abandoned and public information about them is lost. However, due to the crowding constraint in the model here, buyers spread out across all sellers in every period. The dynamics generate more refined information about the quality of every seller.

Public information about each seller does not grow uniformly for all types of seller. Buyer activity segments the market into regions of known value and regions of unknown, though range bound, value. Buyers pay a cost in order to get private information about the unknown regions. The existence of these informed buyers inhibits uninformed buyers from buying in the market segments of unknown value and drives them to buy from segments with known quality sellers. This excess trade with known quality sellers continues until the cost of searching exceeds the expected return, at which point search ends. Although all traders are risk-neutral, the resulting market dynamics create the impression that uninformed traders are risk-averse because they engage in excess trade with known value sellers.

Thinking more broadly of this model as an asset market with prices being related to the number of buyers, the initial period of no asset specific information may correspond to a shock to an existing market so that prices of individual assets vary from the informationally efficient value. As time proceeds and the public signal contains more information, prices come closer to the value of the underlying asset. Initially, known value assets are over-bought because uninformed traders have nowhere else to go, having been forced out of unknown value market segments by lower returns. Meanwhile, informed traders get higher returns by sifting through the market segments of unknown value assets, thus lowering returns to uninformed traders. This trading pattern looks like a "flight to safety" which eventually dissipates as the public information about the market becomes more refined. In time, prices converge to the underlying value as traders learn about asset values by observing past trades. With endogenous entry to the market, the market becomes commoditized and more traders are willing to come in as time goes on.

In addition to describing value discovery (or price formation), our model offers caution about comparing returns in new asset markets. The return achieved by the average investor in a particular market may be higher, in general, than the return to the average asset in the market because of the higher returns achieved by informed investors. This error of comparison may explain effects such as described by Dimson and Marsh (1999) who find that stock market returns for small UK firms had a premium return of 6% until the effect was publicized after which point they had a discount return of 6%. Securities based on a broad index of a market segment may underperform relative to the return of prior investors in that market segment. One may speculate that the error was also committed recently by investors in securitized mortgage bonds who had the expectation of comparable returns to prior mortgage bond investors. The investors in these securities presumably aimed for the average investor return instead of the average asset return they in fact earned.

# 8 References

Banerjee, Abhijit. “A simple model of herd behavior,” *Quarterly Journal of Economics,* (1992) v.107(3), pp.797-817.

Berkovich, Efraim, and Robert Tayon, Jr. "Search, herding, and limits to efficiency" *manuscript* (2008) .

Bikhchandani, Sushil, David Hirshleifer, and Ivo Welch. "A theory of fads, fashion, custom, and cultural change as informational cascades," *The Journal of Political Economy* (1992), v.100, pp.992–1026.

Burdett, Kenneth, Shouyong Shi, and Randall Wright. "Pricing and Matching with Frictions," *Journal of Political Economy* (2001), v.109, pp. 1060–1085 .

Caplin, Andrew and John Leahy. "Business as Usual, Market Crashes, and Wisdom After the Fact," *American Economic Review* (1994) v.84(3), pp. 548-565

Corbae, Dean, Ted Temzelides, and Randall Wright. "Directed Matching and Monetary Exchange," *Econometrica* (2003) v. 71(3), pp. 731-756

Debo, Laurens. G., Christine A. Parlour, and Uday Rajan. "Inferring Quality from a Queue," *manuscript* (2008)

Dimson, Elroy and Paul Marsh. "Murphy's Law and Market Anomalies," *Journal of Portfolio Management* (1999) v.23 pp.53-69

Drehmann, Mathias, Jorg Oechssler, and Andreas Roider. "Herding with and without payoff externalities — an internet experiment," *International Journal of Industrial Organization* (2007) v. 25 pp. 391–415

Huck, Steffen and Jorg Oechssler. "Informational cascades with continuous action spaces," *Economics Letters* (1998) v. 60(2), pp.163-166

Kircher, Philipp and Andrew Postlewaite, "Strategic firms and endogenous consumer emulation," *Quarterly Journal of Economics*, (2008) v. 123(2), pp. 621-661

Lang, William W. and Leonard I. Nakamura. "The dynamics of credit markets in a model with learning," *Journal of Monetary Economics,* (1990) v. 26(2), pp. 305-318

Lester, Benjamin, "Differential information, limited availability, and prices" *manuscript* (2007).

Morris, Stephen and Hyun Song Shin. "The Social Value of Public Information," *American Economic Review* (2002), v.92, pp.1521-1534.

Nagel, Stefan. "Short sales, institutional investors and the cross-section of stock returns," *Journal of Financial Economics* (2005) v. 78(2), pp. 277-309

Nofsinger, John R. and Richard W. Sias. "Herding and Feedback Trading by Institutional and Individual Investors," The Journal of Finance (1999) v.54(6) pp. 2263-2295

Peters, Michael. "Bertrand Equilibrium with Capacity Constraints and Restricted Mobility," *Econometrica* (1984), v.52,(5) pp. 1117-1127

Shimer, Robert. "The Assignment of Workers to Jobs in an Economy with Coordination Frictions," *Journal of Political Economy* (2005) v. 113(5), pp. 996-1025

Vives, Xavier. "Social learning and rational expectations," *European Economic Review* (1996) v.40, pp.589-601

Welch, Ivo. "Sequential sales, learning and cascades," *The Journal of Finance* (1992), v.47, pp.695–732.

Williamson, Steve and Randall Wright. "Barter and Monetary Exchange Under Private Information," *American Economic Review* (1994) v. 84(1), pp. 104-123

# 9 Appendix: Finite variations

Consider the model with finite buyers and finite sellers. If buyers search in $t = 1$, they search in an infinite number of periods. This observation results from the fact that a finite number of buyers cannot faithfully reproduce any distribution with probability one. Even if buyers achieve informational efficiency, in the following periods buyers cannot replicate the configuration. Without further search, the public signal becomes completely uninformative as $t \to \infty$.

When $N$, the ratio of buyers to sellers, goes to infinity, the public signal of the number of buyers at a seller becomes increasingly less noisy for next period buyers who try to invert from the signal to the quality of the seller. At the limit of $N$, even with finite sellers, the quality of gradient seller can be known from the public signal in $t = 1$. We note that, there exists only one flat in this setup (as explained in a proposition above), but that the decision function is still increasing otherwise.

**Proposition 7** *For finite $M$ sellers and $N$ going to infinity, in $t = 2$, the signal $n_t(i)$ is invertible to $q(k)$ for any seller $k$ in the gradient, publicly if a flat exists and privately by all samplers otherwise.*

**Proof.** Only one flat exists in the limit of $N$ for finite $M$, but the decision function is still increasing so there is a gradient. It is sufficient to consider the case of $\sigma = 1$, since buyers can subtract off the measure of non-samplers from every seller.

In $t = 2$, if a flat is publicly visible (that is, two or more sellers have the same number of buyers), then buyers know that $f(q(i)) = 1$ (say for the high flat) for all $i$ in the flat. If no flat is publicly visible, then at most one seller is in the flat. In that case, buyers sample from an extreme seller (analogous to searching the flats). In that case, the quality of this extreme seller $i$ (highest or lowest depending on the shape of $f$) is known to the samplers and so is $f(q(i))$.

Suppose seller $i$ is this seller so that $f(q(i))$ is known. The total number of buyers is $n_0(i) = f(q(i)) + \frac{1}{M-1}\sum_{j\neq i}(1 - f(q(j)))$. Pick any seller $k$ on the gradient. Then,

$$\begin{aligned} n_0(i) - n_0(k) &= f(q(i)) + \frac{1}{M-1}(1 - f(q(k))) \\ &\quad - f(q(k)) - \frac{1}{M-1}(1 - f(q(k))) \end{aligned}$$

Since $n_0(i)$ and $n_0(k)$ are known publicly and $f(q(i))$ is known, this equation is solved for $f(q(k))$. And since $f(q)$ is strictly increasing on the gradient, it is solved for $q(k)$. ■

With finite sellers, it may be the case that all sellers are in the gradient in $t = 1$. If that event occurs, in $t = 2$ samplers know the quality of every seller by the result above and distribute themselves so that every sampler gets the same payoff. In future periods, the public signal becomes less informative because multiple histories can yield the same buyer distribution. For example, a state of the world where sellers are in a flat in $t = 1$ and are in the gradient in $t = 2$, yielding a market distribution of buyers which can look identical to the distribution resulting from the state of all sellers being in the gradient in $t = 1$. Therefore, after $t = 2$, an informationally efficient distribution is not possible for finite sellers.