Consider an $N$-variable dynamic factor model for approximating the dynamics of real macroeconomic activity, as pioneered by Geweke, Sargent and Sims, and significantly extended by Stock and Watson, among others:

$$\begin{pmatrix} y_{1t} \\ \vdots \\ y_{Nt} \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_N \end{pmatrix} + \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_N \end{pmatrix} F_t + \begin{pmatrix} \epsilon_{1t} \\ \vdots \\ \epsilon_{Nt} \end{pmatrix}$$

$$F_t = \phi_1 F_{t-1} + \phi_2 F_{t-2} + \eta_t,$$

where $y_t$ is an indicator of real activity (e.g., retail sales growth), $F_t$ is a latent common factor, and all stochastic shocks are white noise with zero mean and constant finite variance, independent at all “own” and “cross” leads and lags.

(1) What is $E(F_t)$? What is $E(y_{it})$?

(2) Under what conditions is $F_t$ covariance stationary? Under what conditions is $y_{it}$ covariance stationary? From this point onward, assume that those conditions are satisfied unless explicitly stated otherwise.

(3) Consider the cross-covariance function of $y_i$ and $y_j$, $\gamma_{y_iy_j}(\tau)$. How is it related to the autocovariance function of $F_t$, $\gamma_F(\tau)$?

(4) Suppose that the polynomial $\Phi(L) = 1 - \phi_1 L - \phi_2 L^2$ has a pair of complex conjugate roots. What does that imply for the behavior of $\gamma_F(\tau)$, as $\tau \to \infty$? What does that imply for the behavior of the spectral density function of $F_t$, $f_F(\omega)$, $\omega \in [0, \pi]$?

(5) Explain concisely how you would estimate the unconditional joint density of the observed variables $(y_{1t}, ..., y_{Nt})'$ using a kernel smoother with a Gaussian kernel function. Under what conditions is your estimator consistent? Is consistency likely to be comforting if $N=12$ and $T=300$? Why or why not?

(6) Cast the system in state space form and display its measurement and transition equations. What can you say about the eigenvalues of the coefficient matrix in the transition equation?

(7) Assume now that all stochastic shocks are Gaussian. Show concisely how to evaluate the Gaussian log likelihood via (a) a time-domain prediction-error decomposition in conjunction with the Kalman filter, and (b) frequency-domain methods. What are comparative merits of the two approaches?

(8) Assume instead that $\eta_t$ follows a conditionally Gaussian GARCH(1,1) process, as for example in a typical financial economic, as opposed to macroeconomic, application. Write down an explicit expression for the process. What conditions are needed to ensure positivity of the unconditional variance? Positivity of the conditional variance? Covariance stationarity? Assuming that those conditions are satisfied, is $\eta_t$ iid? Serially uncorrelated? Serially independent? Unconditionally Gaussian? How does the state space representation of the system change?

(9) Return now to the basic model with strong white noise innovations, and suppose that the polynomial $\Phi(L)$ has two real roots, $r_1 = 1$ and $r_2 = 1.2$. Is $F_t$ I(0) or I(1)? Describe the qualitative shapes of the spectra of $F_t$ and $AF_t$.

(10) Show that all observable variables are I(1). Are they also cointegrated? If so, how many cointegrating relationships are there in the system, and how many common trends, and what are the cointegrating relationships and common trends? Assuming known system parameters, what can be said about the forecasts $\hat{y}_{t+h}$ calculated via the Kalman filter as $h \to \infty$?