Fall Prelim Exam
Friday 08/22/08, Time limit: 120 minutes

Instructions:
(i) The total number of points is 100, the number of points for each problem is given below.
(ii) The exam is closed book and closed notes.
(iii) To receive full credit for your answers you have to explain your calculations.
    You may state additional assumptions.
Problem 1: Minimum Distance Estimation (50 Points)

Consider the following model of inflation $\pi_t$:

$$\pi_t = \theta \sum_{j=0}^{\infty} \beta^j E_t[mc_{t+j}] + \eta_t. \quad (1)$$

$mc_t$ corresponds to real marginal costs, $\beta = 0.99$ is the known discount factor, $\eta_t$ is an iid $N(0, \sigma_\eta^2)$ disturbance, and $\theta$ is a parameter to be estimated. (1) is a so-called New Keynesian Phillips curve. You can assume that inflation and marginal costs are measured in log-deviations from some steady state.

Let $y_t = [\pi_t, mc_t]'$ and assume that $y_t$ evolves according to a VAR(1) of the form

$$y_t = \Phi y_{t-1} + u_t, \quad u_t \sim N(0, \Sigma). \quad (2)$$

Define the selection matrices $M_1' = [1, 0]$ and $M_2' = [0, 1]$ such that

$$\pi_t = M_1'y_t, \quad mc_t = M_2'y_t.$$  

We are interested in estimating $\theta$, exploiting the fact that (1) imposes restrictions on (2).

(i) (6 Points) Provide conditions on $\Phi$ that guarantee that $y_t$ is stationary.

(ii) (7 Points) Derive the likelihood function, $p(Y_T|\Phi, \Sigma)$ for the VAR model in (2) for a set of observations $Y_T = [y_1, \ldots, y_T]$ conditional on an initial observation $y_0$.

(iii) (7 Points) Derive the maximum likelihood estimator $\hat{\Phi}$ of $\Phi$ and show that it is consistent.

(iv) (8 Points) Show that (1) implies that

$$M_1'y_t = \theta M_2'(I - \beta \Phi)^{-1}y_{t-1} + \eta_t \quad (3)$$

and

$$\left[M_1'\Phi - \theta M_2'(I - \beta \Phi)^{-1}\Phi\right]y_{t-1} = 0. \quad (4)$$

(v) (8 Points) Now consider the following minimum-distance (MD) estimator of $\theta$, which is based on the maximum likelihood estimator of $\Phi$:

$$\hat{\theta} = \arg\min_{\theta \in \Theta} \left[M_1'\hat{\Phi} - \theta M_2'(I - \beta \hat{\Phi})^{-1}\hat{\Phi}\right]W_T\left[M_1'\hat{\Phi} - \theta M_2'(I - \beta \hat{\Phi})^{-1}\hat{\Phi}\right]'\left(M_1'\hat{\Phi} - \theta M_2'(I - \beta \hat{\Phi})^{-1}\hat{\Phi}\right), \quad (5)$$

where $\{W_T\}$ is a sequence of symmetric $2 \times 2$ weight matrices. Show that $\hat{\theta}$ is a consistent estimator of $\theta$. 

(vi) (7 Points) Choose $W_T$ such that $\hat{\Phi} W_T \hat{\Phi}' = I$ to simplify the objective function of the MD estimator. Here $I$ denotes the identity matrix. Show that we can write the estimator as

$$\hat{\theta} = g(\hat{\Phi})$$

and characterize the function $g(\cdot)$.

(vii) (7 Points) Suppose that the sampling distribution of the maximum likelihood estimator is given by

$$\sqrt{T} vec(\hat{\Phi} - \Phi) \Rightarrow \mathcal{N}(0, \Omega).$$

Starting from (6), derive the limit distribution of $\hat{\theta}$. 
Problem 2: Bayesian Analysis (50 Points)

Consider the AR(2) model

\[ y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + u_t, \quad u_t \sim iid N(0, 1), \quad (7) \]

which can be expressed as a linear regression model

\[ y_t = x_t' \phi + u_t, \]

where \( x_t' = [y_{t-1}, y_{t-2}] \) and \( \phi' = [\phi_1, \phi_2] \). The prior distribution is of the form

\[ \phi \sim N(0, \tau^2 I). \quad (8) \]

(i) (10 Points) Derive the posterior distribution of \( \phi \).

(ii) (8 Points) Show that the marginal data density associated with the AR(2) can be expressed as

\[ p(Y_T | AR(2)) = \int p(Y_T | \phi_1, \phi_2) p(\phi_1) d\phi_1, \]

and \( p(Y_T | \phi_1, \phi_2) \) is the likelihood function associated with (7).

(iii) (8 Points) An AR(1) model can be obtained by restricting \( \phi_2 = 0 \) in (7) and letting \( \phi_1 \sim N(0, \tau^2) \). Show that the posterior odds of the AR(1) versus the AR(2) model can be expressed as

\[ \frac{\pi_T(AR(1))}{\pi_T(AR(2))} = \frac{\pi_0(AR(1)) p(Y_T | AR(1))}{\pi_0(AR(2)) p(Y_T | AR(2))} = \frac{\pi_0(AR(1))}{\pi_0(AR(2))} \cdot \frac{p(\phi_2 = 0 | Y_T)}{p(\phi_2 = 0)}, \]

where \( \pi_0(AR(1))/\pi_0(AR(2)) \) are the prior odds, and \( p(\phi_2 = 0 | Y_T) \) and \( p(\phi_2 = 0) \) are the marginal posterior and prior distributions of \( \phi_2 \) evaluated at \( \phi_2 = 0 \) under the AR(2) model.

(iv) (8 Points) Provide an explicit formula for the posterior odds ratio in (iii). What happens to the odds as we let the prior variance increase?

(v) (8 Points) Suppose you would like to forecast \( h \) periods ahead under a quadratic loss function using the AR(2) model. Let \( \hat{\phi}_T \) be the posterior mean of \( \phi \). Is the following iterative predictor

\[ \hat{y}_{T+h} = \hat{\phi}_{1,T} \hat{y}_{T+h-1} + \hat{\phi}_{2,T} \hat{y}_{T+h-2} \]

optimal from the Bayesian perspective? Explain your answer in detail.

(vi) (8 Points) Suppose that you are uncertain whether to use an AR(1) or AR(2) model. Derive a one-step ahead predictive density \( p(y_{T+1} | Y_T) \) that reflects both model and parameter uncertainty.