Consider a correctly-specified vector autoregression for monthly series $x_{1t}$ and $x_{2t}$:

\[
\begin{pmatrix}
  x_{1t} \\
  x_{2t}
\end{pmatrix}
= \begin{pmatrix}
  c_1 \\
  c_2
\end{pmatrix}
+ \begin{pmatrix}
  0 & \varphi_{12} \\
  0 & \varphi_{22}
\end{pmatrix}
\begin{pmatrix}
  x_{1t-1} \\
  x_{2t-1}
\end{pmatrix}
+ \begin{pmatrix}
  e_{1t} \\
  e_{2t}
\end{pmatrix},
\]

where the innovations are weak WN with diagonal covariance matrix $\Sigma$.

a. Write the system as a dynamic (single-) factor model, with factor $F_t$. If $F_t$ is integrated, are $x_{1t}$ and $x_{2t}$ necessarily integrated as well? Cointegrated?

b. Under what conditions is the process covariance stationary? Invertible? Discuss.

From this point onward, assume that the process is covariance stationary.

c. Derive and discuss the autocorrelation function, $\Gamma(\tau)$. Is it symmetric in $\tau$?

d. Derive and discuss the impulse-response function. Discuss and defend the normalization that you adopt.

e. Characterize the Granger-Sims causal structure of the process.

f. Display a state-space representation for the process.

g. Derive and discuss the spectral density matrix $\Phi(\omega)$ in terms of gain and phase.

h. For this part only, assume Gaussian innovations. How would you evaluate the Gaussian likelihood in the time domain using the Kalman filter? How would you evaluate the Gaussian likelihood in the frequency domain using the sample spectral density? What are the comparative properties of the corresponding MLEs?

i. How would you determine whether conditional variance dynamics are operative in the innovations? Would such conditional heteroskedasticity violate the earlier-made white noise assumption? Suppose that you determine that conditional heteroskedasticity is indeed present. How would you extract series of monthly volatility via (1) a GARCH approach, or (2) a realized volatility approach.

For the remainder of the exam, drop the assumption of correct specification. That is, allow for the possibility that the vector autoregression may be mis-specified in some way.

j. How would you assess the adequacy of the model as fitted to data under a normality assumption? You should consider (1) adequacy of the first-order dynamics, (2) adequacy of the normality assumption, and (3) possible structural change.