The exam is worth 120 points in total.
Do all questions. Start each question on a new page, clearly labeled. **Fully justify** all answers and show all work. Label all diagrams clearly. Write legibly. If you need to make additional assumptions, state them clearly.

1. **(15 points)** Consider the following modification of a two-bidder second-price sealed-bid auction. Bidder 2 receives an advantage as follows: If bidder 2’s bid is at least 80% of bidder 1’s bid, then bidder 2 wins and pays 80% of bidder 1’s bid. If bidder 2’s bid is less than 80% of bidder 1’s bid, then bidder 1 wins and pays 1.25 times bidder 2’s bid. Suppose bidder \( i \) values the object being sold at \( v_i \), \( i = 1, 2 \). Prove that it is a dominant strategy for each bidder to bid his or her valuation. How would your answer change if bidder 1 paid 1.3 times bidder 2’s bid, when bidder 1 wins (but the other rules of the auction are unchanged)?

2. **(15 points)** Players I (the row player) and II (the column player) will play the following stage game:

\[
\begin{array}{ccc}
& L & C & R \\
U & 5, 3 & 3, 5 & 0, -1 \\
D & 3, 5 & 5, 3 & 0, -1 \\
\end{array}
\]

(a) What is the unique Nash equilibrium of this game?

(b) Now suppose that player I has the option of either publicly choosing his action of \( U \) or \( D \) before II chooses, or of making his choice in secret (so that II does not know I’s choice when choosing). The cost to I of publicly choosing before II is 1 util.

i. Describe an extensive form in which player I publicly choosing \( U \) and player II responding with \( C \) is a subgame perfect equilibrium outcome.

ii. Describe an extensive form in which player I publicly choosing \( U \) and player II responding with \( C \) is not a subgame perfect equilibrium outcome.

3. **(30 points)** Bruce and Sheila would like to build a swimming pool. Bruce values the swimming pool at \( v \) and Sheila values the swimming pool at \( v/2 \); the value of \( v \) is commonly known by Bruce and Sheila. The swimming pool costs \( C \) to build, where \( v < C < 3v/2 \). Let \( x \) denote Bruce’s contribution for the cost of the pool and \( y \) denote Sheila’s contribution. Bruce’s payoff function is

\[
U_1(x, y) = \begin{cases} 
  v - x, & \text{if pool is built, i.e., } x + y \geq C, \\
  -x, & \text{if pool is not built, i.e., } x + y < C.
\end{cases}
\]

Sheila’s payoff function is

\[
U_2(x, y) = \begin{cases} 
  v/2 - y, & \text{if pool is built, i.e., } x + y \geq C, \\
  -y, & \text{if pool is not built, i.e., } x + y < C.
\end{cases}
\]
Figure 1: The game for question 4.

(a) Would Bruce be willing to build the swimming pool on his own (i.e., if \( y = 0 \))? Why or why not? \[5 \text{ points}\]

(b) Suppose now that Bruce first contributes \( x \) to the cost of the pool, and then Sheila, observing Bruce’s choices, chooses her contribution \( y \). What is the unique subgame perfect equilibrium? (Be sure to explain why the equilibrium is unique.) \[10 \text{ points}\]

(c) Suppose that Bruce and Sheila simultaneously choose their levels of contributions. Describe all of the pure strategy Nash equilibria. \[15 \text{ points}\]

4. (20 points) In the game shown in Figure 1, the probability that player I is type \( t_1 \) is \( 1/2 \) and the probability that he is type \( t_2 \) is \( 1/2 \). The first payoff is player I’s payoff, and the second is player II’s.

(a) Show that the outcome in which both types of player I play L is sequential. \[10 \text{ points}\]

(b) Describe a sequential equilibrium in which type \( t_1 \) plays R. Prove that it is sequential. \[10 \text{ points}\]

5. (40 points) A financial manager undertakes an infinite sequence of trades on behalf of a client. Each trade takes one period. In each period, the manager can invest in one of a large number of risky assets. By exerting effort \( (a = E) \) in a period (at a cost of \( e > 0 \)), the manager can identify the most profitable risky asset for that period, which generates a high return of \( R = H \) with probability \( p \) and a low return \( R = L \) with probability \( 1 - p \).

In the absence of effort \( (a = S) \), the manager cannot distinguish between the different risky assets. For simplicity, assume the manager then chooses the wrong asset, yielding the low return \( R = L \) with probability 1; the cost of no effort is 0. In each period, the client chooses the level of the fee \( x \in [0, \bar{x}] \) to be paid to the financial manager for that period. Note that there is an exogenous upper bound \( \bar{x} \) on the fee that can be paid in a period. The client and financial manager are risk neutral, and so the client’s payoff in a period is

\[ u_c(x, R) = R - x, \]

while the manager’s payoff in a period is

\[ u_m(x, a) = \begin{cases} x - e, & \text{if } a = E, \\ x, & \text{if } a = S. \end{cases} \]
The client and manager have a common discount factor $\delta$. The client observes the return on the asset prior to paying the fee, but does not observe the manager’s effort choice.

(a) Suppose the client cannot sign a binding contract committing him to pay a fee (contingent or not on the return). Describe the unique sequentially rational equilibrium when the client uses the manager for a single transaction. Are there any other Nash equilibria? [5 points]

(b) Continue to suppose there are no binding contracts, but now consider the case of an infinite sequence of trades. For a range of values for the parameters ($\delta$, $\bar{x}$, $e$, $p$, $H$, and $L$), there is a perfect public equilibrium in which the manager exerts effort on behalf of the client in every period. Describe it and the restrictions on parameters necessary and sufficient for it to be an equilibrium. [15 points]

(c) Compare the fee paid in your answer to part 5(b) to the fee that would be paid by a client for a single transaction,

i. when the client can sign a legally binding commitment to a fee schedule as a function of the return of that period, and [5 points]

ii. when the client can sign a legally binding commitment to a fee schedule as a function of effort. [5 points]

(d) Redo question 5(b) assuming that the client’s choice of fee level and the manager’s choice of effort are simultaneous, so that the fee paid in period $t$ cannot depend on the return in period $t$. Compare your answer with that to question 5(b). [10 points]