Consider the stochastic process, 

\[ y_t = \varphi_1 y_{t-1} + \varphi_2 y_{t-2} + \varepsilon_t \quad \varepsilon_t \sim N(0, \sigma^2). \]

a. Under what conditions is the process covariance stationary? Why?

Suppose now that:

\[ y_t = \varphi_1 y_{t-1} + \varphi_2 y_{t-2} + \varepsilon_t \quad \sim \text{iid} \quad \varepsilon_t \sim N(0, \sigma^2). \]

b. Compute the process’ autocorrelation function, partial autocorrelation function, and spectral density function. Discuss.

c. Under what conditions will the autocorrelation function oscillate with a period greater than 2? Under what conditions will the spectral density have an internal peak? How do the two conditions relate?

d. Rewrite the process with \( \Delta y_t \) on the left, and \( y_{t-1} \) and \( \Delta y_{t-1} \) on the right. Characterize and discuss the limiting distribution of the “t-statistic” associated with the estimated coefficient on \( y_{t-1} \) in the covariance stationary case and in the unit root case.

e. Provide a detailed characterization of the process via its Wold representation. What is the relationship between its Wold representation and spectral density function?

f. How is the variance of a sample mean of the process related to its spectrum at frequency zero?

g. Do the innovations associated with the process’ Wold representation necessarily have constant conditional variance? Is the innovation conditional variance necessarily smaller than the unconditional variance? Why or why not? Are the innovations conditionally Gaussian? Unconditionally Gaussian? Unconditionally symmetric? Unconditionally leptokurtic? Covariance stationary? Strictly stationary?

Suppose now that the innovations are instead governed by:

\[ \varepsilon_t | y_{t-1} \sim N(0, h_t) \quad h_t = 20 + .15 \varepsilon^2_{t-1} + .85 h_{t-1}. \]


i. How would you obtain maximum likelihood estimates of the model?

j. How would you assess the adequacy of this model as fitted to real data by MLE under a normality assumption? You may want to consider, among other things:

(a) Neglected conditional mean dynamics
(b) Neglected conditional variance dynamics
(c) Adequacy of the normality assumption
(d) Structural change.