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"Memory Utility"

by

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Memory Utility*

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Abstract. People often consume non-durable goods in a way that seems inconsistent with preferences for smoothing consumption over time. We suggest that such patterns of consumption can be better explained if one takes into account the memories that consumption generates. A memorable good, such as a honeymoon or a vacation, is a good whose mental consumption outlives its physical consumption. We consider a model in which a consumer enjoys physical consumption as well as memories. Memories are generated only by some goods, and only when their consumption exceeds customary levels by a sufficient margin. We offer axiomatic foundations for the structure of the utility function and study optimal consumption in a dynamic model. The model shows how rational consumers, taking into account their future memories, would make optimal choices that rationalize lumpy patterns of consumption.

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1 Introduction

1.1 Memory Consumption

When asked whether they would prefer an increasing intertemporal consumption stream, such as $(10, 12, 14)$, or the analogous decreasing consumption stream $(14, 12, 10)$, people commonly prefer the first. This seemingly intuitive choice reverses the ranking given by the standard discounted sum of stationary utilities, or

$$U(c_0, c_1, \ldots, c_T) = \sum_{t=0}^{T} \delta^t u(c_t), \tag{1}$$

where c_t is consumption in period t, δ is the (stationary) discount factor, and u is the (stationary) utility function, typically assumed to be concave.[1]

When confronted with this contrast, a person might explain that, after having consumed at the relatively high level of 14, the medium consumption (12) is disappointing, and the lower consumption (10) is even more disappointing. By contrast, the increasing consumption stream puts one on a positive-change track. Indeed, Kahneman and Tversky [10] have emphasized that people often react to changes in consumption more than to absolute levels. In line with previous contributions (Helson [9] and Markowitz [12]), they suggest that people form reference points and evaluate current consumption relative to these reference points. This idea is consistent with modifications of the standard model according to which the consumer is viewed as maximizing

$$U(c_0, c_1, \ldots, c_T) = \sum_{t=0}^{T} \delta^t u(c_t, \Lambda_t), \tag{2}$$

where Λ_t designates a habituation level, aspiration level, or reference point that is determined (at least partially) by past consumption levels $(c_0, c_1, \ldots, c_{t-1})$.

Elaborations of the standard model along the lines of (2) encounter difficulties when confronted by an example of a young couple who (not atypically) spend a quarter of their combined annual income on a wedding and honeymoon. Such a large expenditure at the very beginning of their life as a couple seems to violate the preference for consumption smoothing generated

[1] Here and in the sequel, we allow $T = \infty$ when using this notation.

by (1). It also runs contrary to optimal management of one's reference point that arises out of (2): the more spectacular the honeymoon, the bleaker will future consumption appear in comparison.

Why do people spend large amounts of money on non-durable goods, such as vacations, trips, celebrations, and honeymoons, contrary both to consumption smoothing as in (1) and to optimal reference point management as in (2)? Our view is that (1) and (2) both fail to capture the effect of memories. When a couple gets married, they can already envisage themselves leafing through their wedding albums in the near future, telling their children about their honeymoon in the more distant future, and generally deriving pleasure from their consumption long after it has physically ended. Indeed, the unusually large wedding expenditure is an essential ingredient in generating the memories that the couple will enjoy later—it is important that the festivities lie sufficiently outside their ordinary experience—and a substantial part of the cost is typically devoted to items (including photography and keepsakes) designed to reinforce such memories.

Combining these considerations, an individual who takes account of the effect of current consumption on future utility has reasons to consume less than her customary level as well as reasons to consume more. Consuming less will nudge her customary consumption level downwards, with a positive impact on future utility—she may prefer to get the 10 out of the way first. At the same time, consuming more today may engender memories that will be savored tomorrow, putting a premium on higher consumption today—the 14 may enable an experience she will treasure for years. Our goal in this paper is to offer and examine a simple model that captures both effects.

We analyze a model of dynamic consumer choice that includes the effect of past consumption in generating rewarding memories as well as in determining "customary" consumption levels that help set the bar for generating more such memories in the future. Our model is a minimal extension of the standard dynamic choice model, containing the latter as a special case. This makes it straightforward to identify and quantify the differences in consumer behavior that arise because past consumption affects future utility, and to link these differences to the features of the model. Our model also retains the tractability of the standard model, positioning the model for use in applied work, as in Hai, Krueger and Postlewaite [8].

We make specific assumptions about the particular way that past consumption affects future utilities. These assumptions lie behind the tractability of our model, and it is important to understand how restrictive they are. We accordingly provide an axiomatization of the assumed form of preferences over consumption streams that provides a foundation on which we can impose functional form assumptions.

We emphasize that it is not our aim to set out a complete model of memories. There are visits to our grandparents on holidays that are especially memorable, but that our model ignores. Rather our aim is to augment the standard model to accommodate memories that affect economic behavior.

1.2 Relation to the Literature

The suggestion that one can get pleasure in the future from memories of the past dates back (at least) to Adam Smith's [15, p. 152] observation that "We can entertain ourselves with memories of past pleasures...." The idea that a consumer develops a notion of customary consumption that affects her current well-being and that depends on past consumption is widespread, and is perhaps most familiar from models of habit formation (see Attanasio [1] for a survey). Strotz [17] was one of the first to incorporate the utility from past consumption in a model of utility maximization, though as the title of his classic paper on dynamic consistency suggests, consumer choice is problematic when memory is modeled as he does.

This paper is closely related to Hai, Krueger and Postlewaite [8], who introduce the notion of memorable goods and examine the implications of memorable goods for evaluating the (excess) volatility of consumption. Hai, Krueger and Postlewaite set out a model in which past consumption affects the future through the two channels in our model. That paper provides empirical support for the importance of memorable goods. In particular, it shows that the excess sensitivity to foreseen income shocks in Souleles [16] was largely due to expenditures on memorable goods. Our contribution is to analyze a more general set of preferences that can generate memory utility and to provide a theoretical foundation for particularly tractable such utility functions.

2 The Model

The following four subsections develop our model. Section 2.1 introduces a distinction between two types of goods that we will refer to as ordinary goods and memorable goods. Section 2.2 introduces the structure that motivates the characterization of the latter as memorable goods. Section 2.3 shows that the resulting utility maximization problem has a solution. Section 2.4 rearranges the utility representation into a more useful form.

2.1 Ordinary Goods and Memorable Goods

The point of departure for our model is a distinction between two types of goods, which we refer to as an ordinary good (good 1) and a memorable good (good 2). We consider a consumer who consumes these two goods in each of periods $t = 0, 1, 2, \ldots$ and refer to $x_{it} \in \mathbb{R}_+$ $(i = 1, 2)$ as the quantity of good i consumed in period t. Because good 2 generates memory utility, the utility in period t depends on current consumption of good 1 but also on all past consumption of good 2. That is, utility in period t is given by a function

$$\tilde{u}_t(x_{1t}, x_{20}, \ldots, x_{2t}).$$

We will assume (and derive axiomatically in Section 5) a decomposition of the function $\tilde{u}_t$ as

$$\tilde{u}_t(x_{1t}, x_{20}, \ldots, x_{2t}) = u(x_{1t}, x_{2t}) + \tilde{v}_t(x_{20}, \ldots, x_{2t}), \tag{3}$$

where $u : \mathbb{R}^2_+ \to \mathbb{R}$ and $\tilde{v}_t : \mathbb{R}^{t+1}_+ \to \mathbb{R}$. It is straightforward to generalize the analysis to the case in which good 1 is a bundle of ordinary goods and good 2 is a bundle of memorable goods.

The intertemporal objective is the discounted sum of the functions $\tilde{u}(x_{1t}, x_{20}, \ldots, x_{2t})$ given by (3):

$$\sum_{t=0}^{T} \delta^t \tilde{u}_t(x_{1t}, x_{20}, \ldots, x_{2t}) = \sum_{t=0}^{T} \delta^t \left[u(x_{1t}, x_{2t}) + \tilde{v}_t(x_{20}, \ldots, x_{2t}) \right]. \tag{4}$$

We allow T to be finite, but will be particularly interested in the case in which T is infinity.

The most general formulation for allowing nonseparabilities in utility would simply presume that the agent has preferences over infinite consumption streams of the form $\{(x_{10}, x_{20}), (x_{11}, x_{21}), \ldots\}$. One could then apply standard assumptions to ensure that these preferences can be represented by a utility function defined on the space of such consumption streams.

We have built additional structure into (3)–(4). First, we assume that preferences over intertemporal consumption streams are captured by a utility function that is the discounted sum of functions $\tilde{u}_t$, each of which depends upon only current and past consumption. Second, we split each function $\tilde{u}_t$ into two parts, one of which is a function only of current consumption, namely $u(x_{1t}, x_{2t})$, and one of which is a function of the current and all past consumption of good 2, namely $\tilde{v}_t(x_{20}, \ldots, x_{2t})$. We can think of the function u as the counterpart of the typical utility function in a discounted-sum-of-utilities formulation such as (1), and the function $\tilde{v}_t$ as capturing nonseparabilities in preferences. Notice that $\tilde{v}_t$ and hence $\tilde{u}_t$ need the subscript t, because in different periods they will be a function of different arguments.

The decomposition of utility of a consumption stream into a discounted sum of instantaneous utility functions is familiar.[2] The transition from a period-t utility function defined over sequences of the form $\{(x_{10}, x_{20}), (x_{11}, x_{21}), \ldots, (x_{1t}, x_{2t})\}$ to the form given on the right side of (3) gives meaning to the distinction between the two types of goods. The foundations for this distinction are given in Section 5.

2.2 Making Memorable Goods Memorable

Without some additional structure, the intertemporal utility function given by (4) is capable of capturing a variety of nonseparabilities in intertemporal preferences. For example, depending on the functions involved, we might interpret this as a familiar model of habit formation or addiction.

The role of the ordinary goods in (3)–(4) is straightforward. This section introduces additional assumptions that allow us to interpret the utility implications of memorable goods as indeed arising out of considerations having to do with memory.

2.2.1 The Two Faces of Memory

We would like the model to focus attention on the two aspects of utility highlighted in Section 1. First, the previous consumption of memorable goods $\{x_{20}, \ldots, x_{2t-1}\}$ enters the period-t utility function $\tilde{v}_t(x_{20}, \ldots, x_{2t})$ through the accumulation of memories of utility produced by the past consumption of such goods. One may enjoy fond memories of a vacation, wedding, or special night out long after they have occurred. Second, whether the new consumption of memorable goods produces memories that can be consumed in the future depends on how this consumption compares to the consumer's customary consumption, with memories generated by extraordinary consumption levels. A dinner in the type of restaurant one visits weekly is unlikely to generate memory utility, while a rare treat in a five-star restaurant may contribute to utility long after the evening is finished.

[2] We do not offer an axiomatic derivation of this functional form. Koopmans's [11] axioms do not directly apply, because each x_{2t} appears also in future instantaneous utility values. Thus, the axioms need to be adjusted to apply to utility-equivalent bundles, where a change in x_{2t} is compensated by changes in future variables so that only the instantaneous utility at period t is affected, at which point the argument follows familiar lines.

We capture these two forces in a parsimonious form by introducing two state variables. We assume that

$$\tilde{v}_t(x_{20}, \ldots, x_{2t}) = \hat{v}(x_{2t}, \Upsilon_t, \Lambda_t), \tag{5}$$

where we interpret $\Upsilon_t \in \mathbb{R}_+$ as identifying a stock of memory utility at time t and $\Lambda_t \in \mathbb{R}_+$ as identifying the customary level of consumption of the memory good at time t.

2.2.2 Consuming Memories

This section addresses the role of Υ in the function $\hat{v}$, making assumptions that allow us to interpret Υ as capturing the consumption of memories.

Assumption 1. *There exists a function* $\breve{v} : \mathbb{R}^2_+ \to \mathbb{R}$ *and a constant* $\nu \in (0,1)$ *such that*

$$\begin{aligned} \tilde{u}_t(x_{1t}, x_{20}, \ldots, x_{2t}) &= u(x_{1t}, x_{2t}) + \hat{v}(x_{2t}, \Upsilon_t, \Lambda_t) \\ &= u(x_{1t}, x_{2t}) + \Upsilon_t + \breve{v}(x_{2t}, \Lambda_t) \\ &= u(x_{1t}, x_{2t}) + \sum_{\tau=0}^{t-1} \nu^{t-\tau} \breve{v}(x_{2\tau}, \Lambda_\tau) + \breve{v}(x_{2t}, \Lambda_t). \end{aligned} \tag{6}$$

The first equality simply inserts (5) into (3). The second equality assumes that memories of past consumption enter $\hat{v}$ quasilinearly, so that we can write the contribution of memory utility to current utility as the sum of utilities of previously memorable consumption (given by Υ_t) plus the current memory-utility generation (given by $\breve{v}(x_{2t}, \Lambda_t)$). The final equality indicates that Υ_t is a discounted sum of past contributions to memory utility, or

$$\Upsilon_t = \sum_{\tau=0}^{t-1} \nu^{t-\tau} \breve{v}(x_{2\tau}, \Lambda_\tau).$$

2.2.3 Producing Memories

This section addresses the role of Λ in the function $\hat{v}$, in the process describing the generation of memory utility. We first assume that goods 1 and 2 are indeed "goods," in the sense that increased consumption increases current utility. In addition, we assume that the consumption of x_2 generates memory utility, in the form of a value $\breve{v}_t(x_{2t}, \Lambda_t) > 0$, if but only if the consumption x_{2t} is sufficiently large relative to the customary level of consumption of good 2.

Assumption 2.

[2.1] The function u is strictly increasing.

[2.2] The function $\breve{v}$ is increasing in x_2 and decreasing in Λ, and is strictly increasing in x_2 and strictly decreasing in Λ whenever $\breve{v}(x_2, \Lambda) > 0$.

[2.3] There exists $\gamma > 1$ such that $x_2 \leq \gamma\Lambda \Longrightarrow \breve{v}_1(x_2, \Lambda) = 0$.

[2.4] The customary consumption level Λ_t evolves according to, for $\lambda \in [0,1]$,

$$\Lambda_t = \lambda\Lambda_{t-1} + (1-\lambda)x_{2t-1}.$$

Assumption [2.4] indicates that memory utility is generated whenever the consumption of the memorable good is sufficiently larger than the customary level. Assumption [2.4] indicates that the customary level of memory-good consumption drifts in the direction of current consumption.

2.3 Existence of Optimal Consumption Plans

This section introduces the budget constraint and establishes that an optimal consumption plan exists.

We assume that the consumer has income I in each period, and that the rate at which she can borrow and save is equal to her discount factor, δ. Goods 1 and 2 are measured in units such that their prices are each 1, allowing us to write the consumer's intertemporal budget constraint as

$$\sum_{t=0}^{\infty} \delta^t (I - x_{1t} - x_{2t}) = 0.$$

Let Y_t denote the largest expenditure the consumer can make in period t, given that she can borrow any future income and spend any saved income, but has already paid for her previous consumption. Hence, we have

$$Y_0 = \frac{I}{1-\delta}$$

and

$$Y_t = [Y_{t-1} - x_{1t-1} - x_{2t-1}]\,\frac{1}{\delta}.$$

The intertemporal budget constraint implies that $Y_t \geq 0$. Any expenditure larger than Y_{t-1} in period $t-1$ is impossible, being sufficiently large that the consumer's savings and discounted future income would not suffice to pay for it, which in turn ensures $Y_t \geq 0$. There is an upper bound $\overline{Y}_t$

which the consumer achieves by spending nothing on consumption in periods $\{0,\ldots,t-1\}$, given by

$$\overline{Y}_t \;=\; \frac{I}{\delta^t}+\frac{I}{\delta^{t-1}}+\ldots+\frac{I}{\delta^2}+\frac{I}{\delta}+\frac{I}{1-\delta} \;=\; \frac{I}{\delta^t(1-\delta)}.$$

We thus have $Y_t \in [0,\overline{Y}_t]$. Notice that $\overline{Y}_t$ grows arbitrarily large as does t.

The consumer's objective is then to maximize

$$\sum_{t=0}^{\infty}\delta^t\left[u(x_{1t},x_{2t})+\Upsilon_t+\breve{v}_\tau(x_{2t},\Lambda_t)\right] \tag{7}$$

$$s.t. \quad Y_{t+1}=[Y_t-x_{1t}-x_{2t}]\frac{1}{\delta}\geq 0 \tag{8}$$

$$\Upsilon_t=\nu(\Upsilon_{t-1}+\breve{v}(x_{2t-1},\Lambda_{t-1})) \tag{9}$$

$$\Lambda_t=\lambda\Lambda_{t-1}+(1-\lambda)x_{2t-1} \tag{10}$$

$$(x_{1t},x_{2t})\in X^2, \tag{11}$$

given initial values (Y_0,Λ_0), where X^2 is the set of feasible values of (x_{1t},x_{2t}).

The following standard assumptions about constituent utility functions u and $\breve{v}$ suffice to ensure that optimal consumption plans exist. Weaker assumptions would suffice–for example, differentiability is not essential, but it is convenient.

Assumption 3.

[3.1] The function u is continuously differentiable. The function $\breve{v}$ is continuously differentiable when it is positive.

[3.2] For all sequences $\{x_\tau\}_{\tau=0}^{\infty}$ with either $\lim_{\tau\to\infty}x_{1\tau}=\infty$ or $\lim_{\tau\to\infty}x_{2\tau}=\infty$, either it is the case that $\lim_{\tau\to\infty}\frac{du(x_{1\tau},x_{2\tau})}{dx_{1\tau}}=0$ or it is the case that $\lim_{\tau\to\infty}\frac{du(x_{1\tau},x_{2\tau})}{dx_{2\tau}}=0$.

[3.3] For all sequences $\{x_\tau\}_{\tau=0}^{\infty}$ with $\lim_{\tau\to\infty}x_{2\tau}=\infty$, we have $\lim_{\tau\to\infty}\frac{d\breve{v}(x_{2\tau},\Lambda)}{dx_{2\tau}}=0$ uniformly in Λ.

Assumption 3.1 imposes familiar smoothness conditions. Assumptions 3.2–3.3 impose versions of diminishing marginal utility assumptions. Assumption 3.2 indicates that arbitrarily large values of consumption ensure that at least one of the marginal utilities of u is arbitrarily small, and the final assumption imposes a similar requirement for the function $\hat{v}$. The uniform convergence requirement in Assumption 3.3 may appear to be quite stringent. However, we will typically think of $\frac{d^2\hat{v}(x_{2\tau},\Lambda_\tau)}{dx_{2\tau}d\Lambda_t}\leq 0$. It will then suffice for the condition in Assumption 3.3 to hold when $\Lambda=0$.

The consumer's intertemporal maximization problem has a solution:

Proposition 1. *Let Assumptions 1–3 hold. Then there exists an optimal consumption plan* $x^* : \mathbb{R}^3_+ \to \mathbb{R}^2_+$*, identifying values of* (x_{1t}, x_{2t}) *in each period* t *as a function of* $(Y_t, \Upsilon_t, \Lambda_t)$*. Moreover, there exists a continuous value function* $V : \mathbb{R}^3_+ \to \mathbb{R}$ *such that the maximization problem can be written as*

$$V(Y_t, \Upsilon_t, \Lambda_t) \quad = \quad \max_{x_{1t}, x_{2t}} u(x_{1t}, x_{2t}) + \Upsilon_t + \breve{v}(x_{2t}, \Lambda_t) + \delta V(Y_{t+1}, \Upsilon_{t+1}, \Lambda_{t+1})$$

subject to the constraints given in (8)–(11), given initial values $(Y_0, \Upsilon_0, \Lambda_0)$*.*

Given its additive from in (6), the variable Υ affects the value of V, but not the optimal continuation strategy.

This proposition would be immediate from Assumptions 3.1 and 3.2 if the consumption set X were compact (e.g., Sundaram [18, Theorem 12.19]). The proof of Proposition 1 is then completed by the following lemma.

Lemma 2. *Let Assumption 3 hold. Then there exists a finite* $\overline{x}$ *such that any consumption plan featuring a period* t *in which* $x_{it} > \overline{x}$ *for either* $i = 1, 2$ *is dominated by a consumption plan in which* $x_{it} \leq \overline{x}$ *for* $i = 1, 2$ *and for all* t*.*

The proof, contained in Section 7.1, first notes that the consumer's marginal utilities in the first period are bounded below, even if the consumer concentrates all of her consumption in the first period. We then use Assumptions 3.2–3.3 to argue that any unbounded consumption plan must eventually feature a marginal utility smaller than the bound from the first period. The consumer can then increase utility by shifting consumption to the first period, ensuring that the plan in question is not optimal. The intertemporal links created by memorable goods introduce only slight complications in this otherwise quite familiar line of argument.

2.4 Simplifications

This section introduces three simplifications. We are interested in cases in which one would expect consumption to be smoothed. We accordingly assume:

Assumption 4. *The utility function* u *is strictly concave. The utility function* $\breve{v}$ *is strictly concave in* x_2 *on that part of its domain in which it is positive.*

The following assumption is invoked for Proposition 4 below, but is not necessary for the results prior to that:

Assumption 5. *The functions u and $\breve{v}$ are homogeneous of degree $\alpha < 1$.*

The final simplification is purely a matter of notation. It is helpful to rearrange the intertemporal objective as follows:

$$\begin{aligned}
& \sum_{t=0}^{\infty} \delta^t \left[u(x_{1t}, x_{2t}) + \sum_{\tau=0}^{t-1} \nu^{t-\tau} \breve{v}(x_{2\tau}, \Lambda_\tau) + \breve{v}_\tau(x_{2t}, \Lambda_t) \right] \\
= & \sum_{t=0}^{\infty} \delta^t u(x_{1t}, x_{2t}) + \sum_{t=0}^{\infty} \delta^t \sum_{\tau=0}^{t} \nu^{t-\tau} \breve{v}(x_{2\tau}, \Lambda_\tau) \\
= & \sum_{t=0}^{\infty} \delta^t u(x_{1t}, x_{2t}) + \sum_{\tau=0}^{\infty} \sum_{t=\tau}^{\infty} \delta^t \nu^{t-\tau} \breve{v}(x_{2\tau}, \Lambda_\tau) \\
= & \sum_{t=0}^{\infty} \delta^t u(x_{1t}, x_{2t}) + \sum_{\tau=0}^{\infty} \delta^\tau \frac{1}{1-\delta\nu} \breve{v}(x_{2\tau}, \Lambda_\tau) \\
= & \sum_{t=0}^{\infty} \delta^t u(x_{1t}, x_{2t}) + \sum_{\tau=0}^{\infty} \delta^\tau v(x_{2\tau}, \Lambda_\tau) \\
= & \sum_{t=0}^{\infty} \delta^t \left[u(x_{1t}, x_{2t}) + v(x_{2t}, \Lambda_t) \right].
\end{aligned}$$

The first expression is taken from (7). The first equality distributes the initial summation. The next equality interchanges the order of summation in the double sum. The next equality then simplifies the second sum in the double sum. The following equality introduces the function $v = \frac{1}{1-\delta\nu}\breve{v}$. The final equality collects the terms in a single summation.

This formulation has the advantage of focussing attention on the periods in which memory utility is generated, while clearing from view (but not neglecting) the subsequent enjoyment of those memories. The function v is proportional to $\breve{v}$, and hence inherits the properties of $\breve{v}$ given in Assumptions 3–5.

3 A Two-Period Example

Let $T = 1$, so there are two periods, numbered 0 and 1. The agent's objective is

$$\max_{x_{10}, x_{20}, x_{11}, x_{21}} \left\{ u(x_{10}, x_{20}) + \hat{v}(x_{20}, \Lambda_0) + \delta \left[u(x_{11}, x_{21}) + \nu \hat{v}(x_{20}, \Lambda_0) + \hat{v}(x_{21}, \Lambda_1) \right] \right\},$$

subject to the budget constraint

$$x_{10} + x_{20} + \delta(x_{11} + x_{21}) = Y_0,$$

where Y_0 is the discounted present value of the agent's income and we maintain our assumption that goods 1 and 2 are measured in such units that their prices are each 1.

We make the example more concrete by assuming

$$\begin{aligned} u(x_1, x_2) &= \frac{x_1^\alpha}{\alpha} + \frac{x_2^\alpha}{\alpha} \\ v(x_2, \Lambda) &= \xi \max\left\{0, \frac{x_2^\alpha}{\alpha} - \gamma\frac{\Lambda^\alpha}{\alpha}\right\} \end{aligned}$$

for $\xi > 0$ and $\gamma > 1$. The functions u and v are homogeneous of degree α (and hence satisfy Assumption 4). We assume $\alpha \in (0, 1)$, with $\alpha < 1$ ensuring that the functions are concave, and $\alpha > 0$ ensuring that utilities are nonnegative and hence 0 is a relevant comparison for the maximum in the specification of v.

We use two variations of this model to illustrate two aspects of memory utility.

3.1 No Acclimatization

First, we let $\Lambda_1(x_{20}) = \Lambda_0$ for all x_{20}. This corresponds to setting $\lambda = 1$ in our specification of the dynamics governing Λ. In this case, the customary level of memorable-good consumption is fixed for this consumer, and does not respond to her consumption path, allowing us to focus on the consumption of memories.

We consider three types of consumption plans. In the first, the consumer generates no memory utility. The consumer's utility is then given by

$$\frac{x_{10}^\alpha}{\alpha} + \frac{x_{20}^\alpha}{\alpha} + \delta\left[\frac{x_{11}^\alpha}{\alpha} + \frac{x_{21}^\alpha}{\alpha}\right].$$

The first-order conditions for utility maximization give $x_{10}^{\alpha-1} = x_{20}^{\alpha-1} = x_{11}^{\alpha-1} = x_{21}^{\alpha-1}$, and we can solve for

$$x_{10} = x_{20} = x_{11} = x_{21} = \frac{Y_0}{2 + 2\delta}.$$

Let $\underline{V}(Y_0)$ be the indirect utility function, given the constraint that the consumer never generates memory utility and holding Λ_0 fixed. The envelope theorem then gives[3]

$$\frac{d\underline{V}}{dY_0} = \left(\frac{Y_0}{2+2\delta}\right)^{\alpha-1}.$$

Notice that $\underline{V}(Y_0)$ is concave.

In the second consumption plan, the consumer generates memory utility in only a single period. It is immediate that the consumer will do so in period 0. It is here that we see implications of the durability of memory. Given that the customary level of memorable-good consumption is fixed, and given that memories are to be generated in only one period, then that period will be the first, so as to take advantage of lingering memories in the second period. The consumer's utility is then

$$\frac{x_{10}^{\alpha}}{\alpha} + \frac{x_{20}^{\alpha}}{\alpha} + \left(\frac{x_{20}^{\alpha}}{\alpha} - \gamma\frac{\Lambda_0^{\alpha}}{\alpha}\right)\xi(1+\delta\nu) + \delta\left[\frac{x_{11}^{\alpha}}{\alpha} + \frac{x_{21}^{\alpha}}{\alpha}\right].$$

The first-order conditions for utility maximization give $x_{10}^{\alpha-1} = x_{20}^{\alpha-1}[1+\xi+\xi\nu\delta] = x_{11}^{\alpha-1} = x_{21}^{\alpha-1}$, and we can solve for

$$\begin{aligned} x_{10} = x_{11} = x_{21} &= \frac{\theta Y_0}{\theta(1+2\delta)+1} \\ x_{20} &= \frac{Y_0}{\theta(1+2\delta)+1}, \end{aligned}$$

where

$$\theta = [1+\xi+\xi\nu\delta]^{\frac{1}{\alpha-1}} \in (0,1).$$

Compared to the previous case, the generation of memory utility in period 0 prompts an increase in x_{20}, because the generation of memories increases the marginal utility of x_{20}, and a corresponding decrease in all other variables so as to preserve the budget constraint. Let $V(Y_0)$ be the indirect utility function given the constraint that the consumer generate memory utility in period 0 (only). The envelope theorem then gives[4]

$$\frac{dV}{dY_0} = \left(\frac{\theta Y_0}{\theta(1+2\delta)+1}\right)^{\alpha-1}.$$

[3]The domain of this indirect utility function is restricted to income levels Y_0 sufficiently small that no memory utility is generated when consumption is perfectly smooth.

[4]The domain of this indirect utility function is restricted to income levels Y_0 sufficiently large that income utility can be generated in the first period, but not so large that the consumption bundle solving the resulting first-order conditions would also generate memory utility in the second period.

Notice that $V(Y_0)$ is concave and is more steeply sloped than $\underline{V}$.

Now suppose that the consumer generates memory utility in both periods. The consumer's utility is then

$$\frac{x_{10}^{\alpha}}{\alpha}+\frac{x_{20}^{\alpha}}{\alpha}+\left(\frac{x_{20}^{\alpha}}{\alpha}-\gamma\frac{\Lambda_0^{\alpha}}{\alpha}\right)\xi(1+\delta\nu)+\delta\left[\frac{x_{11}^{\alpha}}{\alpha}+\frac{x_{21}^{\alpha}}{\alpha}+\left(\frac{x_{21}^{\alpha}}{\alpha}-\gamma\frac{\Lambda_0^{\alpha}}{\alpha}\right)\xi\right].$$

The first-order conditions for utility maximization give $x_{10}^{\alpha-1}=x_{20}^{\alpha-1}[1+\xi+\xi\nu\delta]=x_{11}^{\alpha-1}=x_{21}^{\alpha-1}(1+\xi)$, and we can solve for

$$\begin{aligned} x_{10}=x_{11} &= \frac{\theta\phi Y_0}{(1+\delta)\theta\phi+\phi+\delta\theta} \\ x_{20} &= \frac{\phi Y_0}{(1+\delta)\theta\phi+\phi+\delta\theta} \\ x_{21} &= \frac{\theta Y_0}{(1+\delta)\theta\phi+\phi+\delta\theta} \end{aligned}$$

where θ is as before and

$$\phi=[1+\xi]^{\frac{1}{\alpha-1}}\in(0,1).$$

Compared to the previous case, the generation of memory utility in period 0 prompts an additional decrease in $x_{10}=x_{11}$, since the generation of memories in both periods increases yet further the marginal utility gains from doing so. Let $\overline{V}(Y_0)$ be the indirect utility function given the constraint that the consumer generate memory utility in both periods. The envelope theorem then gives[5]

$$\frac{d\overline{V}}{dY_0}=\left(\frac{\theta\phi Y_0}{(1+\delta)\theta\phi+\phi+\delta\theta}\right)^{\alpha-1}.$$

Notice that $\overline{V}(Y_0)$ is concave and is more steeply sloped than V.

Figure 1 illustrates the indirect utility functions for this example.

3.2 Rapid Acclimatization

Now we examine the case in which $\lambda=0$, and so $\Lambda_1=x_{20}$. Hence, the first-period consumption of the memorable good sets the second-period customary level. The consumer's acclimatization to past consumption of memorable

[5]The domain of this indirect utility function is restricted to income levels Y_0 sufficiently large that it is possible to generate memory utility in both periods.

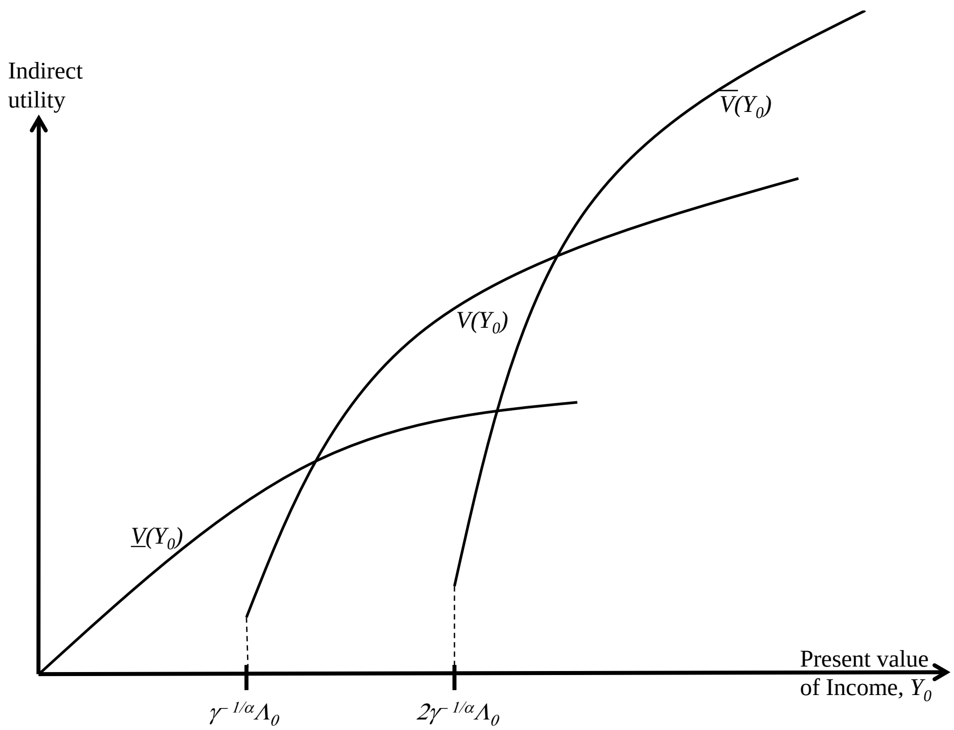

Figure 1: Indirect utility functions for the first specification of the example, featuring no acclimatization. The three indirect utility functions correspond to the case in which no memory utility is generated ($\underline{V}$), memory utility is generated only in the first period (V), and memory utility is generated in both periods ($\overline{V}$).

goods is thus immediate, allowing us to focus on the role of the customary level in the production of memories.

Suppose first that the consumer achieves no memory utility. The consumer's utility is then

$$\frac{x_{10}^{\alpha}}{\alpha} + \frac{x_{20}^{\alpha}}{\alpha} + \delta \left[\frac{x_{11}^{\alpha}}{\alpha} + \frac{x_{21}^{\alpha}}{\alpha} \right].$$

Acclimatization is irrelevant in this case, and as in Section 3.1, the first order conditions imply $x_{10}^{\alpha-1} = x_{20}^{\alpha-1} = x_{11}^{\alpha-1} = x_{21}^{\alpha-1}$.

Suppose next that the consumer generates memory utility only in the second period. Unlike the previous case of no acclimatization, it may be optimal to generate memory utility only in the second period. In particular, if the initial customary level Λ_0 is quite large, generating memory utility in period 0 will require x_{20} to be prohibitively large. The consumer may fare better to choose a small value of x_{20} and then exploit the resulting smaller value of Λ_1 to generate memory utility in period 1. The utility level is then

$$\frac{x_{10}^{\alpha}}{\alpha} + \frac{x_{20}^{\alpha}}{\alpha} + \delta \left[\frac{x_{11}^{\alpha}}{\alpha} + \frac{x_{21}^{\alpha}}{\alpha} + \xi \left(\frac{x_{20}^{\alpha}}{\alpha} - \gamma \frac{x_{21}^{\alpha}}{\alpha} \right) \right].$$

The first-order conditions imply $x_{10}^{\alpha-1} = (1-\delta\xi\gamma)x_{20}^{\alpha-1} = x_{11}^{\alpha-1} = (1+\xi)x_{21}^{\alpha-1}$.

We can now observe that neither of the payoffs involved in these two consumption plans depends on the level Λ_0, and hence which of these two consumption plans gives a higher payoff is independent of the state variables (Y_0, Λ_0).[6] If choosing between these two plans, the results will be:

- No memory utility if γ is large, ξ small and α small.
- Second-period memory utility if γ is small, ξ large, and α large.

Perhaps the only comparative static that is not obvious here is that concerning α. In order to achieve memory utility in the second period, the consumer decreases x_{20} and increases x_{21}. This distortion is relatively less costly as α is large.

Now consider another comparison, that between memory utility in the first period only and memory utility in both periods. When achieving memory utility only in the first period, the consumer's utility is

$$\frac{x_{10}^{\alpha}}{\alpha} + \frac{x_{20}^{\alpha}}{\alpha} + \left(\frac{x_{20}^{\alpha}}{\alpha} - \gamma\frac{\Lambda_0^{\alpha}}{\alpha}\right)\xi(1+\delta\nu) + \delta\left[\frac{x_{11}^{\alpha}}{\alpha} + \frac{x_{21}^{\alpha}}{\alpha}\right],$$

and the first-order conditions give $x_{10}^{\alpha-1} = (1+\xi+\xi\delta\nu)x_{20}^{\alpha-1} = x_{11}^{\alpha-1} = x_{21}^{\alpha-1}$. When achieving memory utility in both periods, the consumer's utility is

$$\frac{x_{10}^{\alpha}}{\alpha} + \frac{x_{20}^{\alpha}}{\alpha} + \left(\frac{x_{20}^{\alpha}}{\alpha} - \gamma\frac{\Lambda_0^{\alpha}}{\alpha}\right)\xi(1+\delta\nu) + \delta\left[\frac{x_{11}^{\alpha}}{\alpha} + \frac{x_{21}^{\alpha}}{\alpha}\right] + \delta\xi\left(\frac{x_{20}^{\alpha}}{\alpha} - \gamma\frac{x_{20}^{\alpha}}{\alpha}\right)$$

and the first-order conditions give $x_{10}^{\alpha-1} = (1+\xi+\xi\delta\nu-\delta\xi\gamma)x_{20}^{\alpha-1} = x_{11}^{\alpha-1} = (1+\xi)x_{21}^{\alpha-1}$. We now notice that the *comparison* between these two payoffs does not depend on Λ_0, which appears additively in each expression, and does not depend on Y_0. We will have:

- Memory utility only in the first period if γ is large and α small.
- Memory utility on both periods if γ is small and α large.

We thus have a number of potential special cases, depending on parameters. Let us consider the case in which γ is small and α large. Then the

[6] It is obvious that Λ_0 does not affect this comparison. The homogeneity of the utility function ensures that if a consumption plan generating no memory utility (for example) is optimal for a given level Y_0, then a scaled version of this plan is optimal at any alternative level Y_0'.

consumer will either generate memory utility only in the second period or will generate memory utility in both periods, depending on (Y_0, Λ_0). For relatively small values of Λ_0 (or, equivalently, large values of Y_0), the consumer will generate memory utility in both periods. For larger values of Λ_0 (smaller values of Y_0), the consumer will generate memory utility only in period 2. Delaying memory utility until period 2 has the obvious disadvantage that there is one fewer periods to enjoy the memory. It has the advantage that the customary level can be decreased in order to make the generation of memories more effective. This advantage is more pronounced the larger is Λ_0 and the smaller is Y_0.

When memory utility is generated only in the second period, we can solve the first-order conditions to obtain

$$\begin{aligned} x_{10} = x_{11} &= \frac{\theta\phi Y_0}{(1+\delta)\theta\phi + \phi + \delta\theta} \\ x_{20} &= \frac{\phi Y_0}{(1+\delta)\theta\phi + \phi + \delta\theta} \\ x_{21} &= \frac{\theta Y_0}{(1+\delta)\theta\phi + \phi + \delta\theta} \end{aligned}$$

where

$$\begin{aligned} \theta &= [1 - \delta\xi\gamma]^{\frac{1}{\alpha-1}} \in (0,1) \\ \phi &= [1 + \xi]^{\frac{1}{\alpha-1}} \in (0,1). \end{aligned}$$

When memory utility is generated in both periods, we have

$$\begin{aligned} x_{10} = x_{11} &= \frac{\theta\phi Y_0}{(1+\delta)\theta\phi + \phi + \delta\theta} \\ x_{20} &= \frac{\phi Y_0}{(1+\delta)\theta\phi + \phi + \delta\theta} \\ x_{21} &= \frac{\theta Y_0}{(1+\delta)\theta\phi + \phi + \delta\theta} \end{aligned}$$

where

$$\begin{aligned} \theta &= [1 + \xi + \delta\xi\nu - \delta\xi\gamma]^{\frac{1}{\alpha-1}} \in (0,1) \\ \phi &= [1 + \xi]^{\frac{1}{\alpha-1}} \in (0,1). \end{aligned}$$

A comparison shows that the x_{10} and x_{11} are smaller when memory utility is generated only in the second period (when Λ_0 is large and Y_0 is

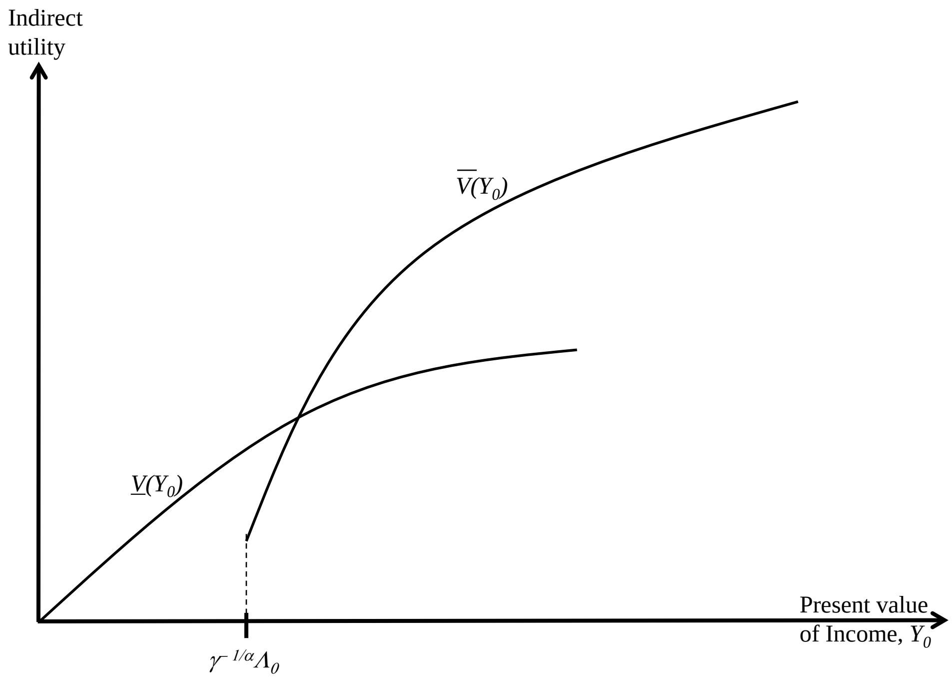

Figure 2: Indirect utility functions for the second specification in the example, featuring immediate acclimatization. The two indirect utility functions correspond to the case of memory utility only in the second period ($\underline{V}$) and memory utility in both periods ($\overline{V}$).

small) than in the case when memory utility is generated in both periods (when Λ_0 is small Y_0 is large). Hence, by familiar arguments, the indirect utility function $\underline{V}(y_0)$ for the case of memory utility only in the second period is steeper than the indirect utility function $\overline{V}(Y_0)$ for the case of memory utility in both periods. We can represent the combined indirect utility function in Figure 2.

Notice that this function has the concave-convex-concave shape suggested by Friedman and Savage [7]. As Friedman and Savage [7] note, a consumer characterized by such utility functions might appear to be both risk averse and risk loving, in the sense that there are both insurance policies and lotteries that the consumer would find attractive. However, Figure 1 shows that each potentially optimal configuration of memory utility generation potentially gives rise to another hump in the utility function. When there are many periods there will be many such plans, and hence the resulting utility function may look quite unlike that presented in Friedman and Savage [7]. We return to this comparison in Section 4.2.

4 Characterization of Optimal Consumption Plans

We now turn to a characterization of optimal consumption plans in the presence of memorable goods. We are particularly interested in the ability of memory utility to account for seemingly excessive lumpiness in consumption. We accordingly consider a case in which consumption would be perfectly smoothed in the absence of memory utility. In particular, a conventional utility-maximization model without memory utility will generate perfect consumption smoothing if either the horizon is finite and deterministic, or it is random with a stationary continuation probability.

The two-period example presented in Section 3 demonstrates why we cannot expect perfect consumption smoothing in the presence of memory utility and a finite, deterministic lifetime. The reasoning is straightforward. Memory utility generated in the first period of a two-period model is enjoyed in both periods, while memory utility generated in the final period can necessarily be enjoyed only in that period. This provides a natural tendency to front-load consumption and hence memory utility. It is then no surprise that young people spend a relatively larger share of their income on weddings than do senior citizens. This section accordingly focuses attention on the infinite-horizon model, interpreted as a model with a random lifetime and stationary continuation probabilities.

4.1 Benchmark: Memoryless Consumption

To provide a comparison, let us recall the familiar special case in which there is no memory utility. We assume that the relevant parts of Assumptions 3–4 hold, so that u is differentiable, increasing and concave.

The consumer's problem is

$$\max_{\{x_{1t},x_{2t}\}_{t=0}^{\infty}} \quad \sum_{t=0}^{\infty} \delta^t u(x_{1t}, x_{2t})$$

$$s.t. \quad \sum_{\tau=0}^{\infty} \delta^t (x_{1t} + x_{2t}) = Y_0.$$

The first-order conditions for this maximization problem call for the marginal utilities to be equalized across goods and across periods. The concavity of u then ensures that we have perfect consumption smoothing. There exist quantities x_1^* and x_2^* with $x_1^* + x_2^* = I$ such that $x_{1t} = x_1^*$ and $x_{2t} = x_2^*$ for all t.

4.2 No Acclimatization

We first consider a special case, namely that in which $\lambda = 1$, so that there is no acclimatization. This case is particularly easy to characterize—either the consumer will perfectly smooth consumption, or the consumer's indirect utility function will be effectively linear. In the latter case, consumption may not be perfectly smoothed, but this lack of smoothing is inconsequential. The path of consumption will be drawn from a set of optimal consumption paths, with the linearity of the indirect utility function ensuring that all such plans have equivalent utilities. These results show that the tendency of the customary consumption level to drift toward actual consumption plays an essential role in the link between memory utility and lumpy consumption.

Suppose that the customary level of consumption is perfectly persistent, so that $\Lambda_t = \Lambda$ for all t, regardless of history. The utility function is constant across periods in this case, and is given by

$$u(x_1, x_2)$$

when $x_2 \leq \gamma\Lambda$ and is given by

$$u(x_2, x_2) + v(x_2, \Lambda)$$

when $x_2 \geq \gamma\Lambda$. Let c_t be the total amount spent on consumption in period t. Then we can define single-period indirect utility functions $\underline{w}(c)$ for the case in which no memory utility is generated in the period in question and $\overline{w}(c)$ for the case in which memory utility is generated. Given the stationarity of Λ, we can write these solely as a function of c. In particular, no intertemporal considerations are involved in deriving these functions. We illustrate the indirect utility functions in Figure 3. Let $w(c) = \max\{\underline{w}(c), \overline{w}(c)\}$. We can use these indirect utility functions to characterize the optimal consumption plan in this case.

Let $\hat{w}$ be the smallest concave function larger than $w(c)$. Then $\hat{w}$ is given by the upper envelope of the utility functions $\underline{w}$, $\overline{w}$ and the dashed tangent in Figure 3, and $\underline{c}$ and $\overline{c}$ are the points of intersection of the tangent and the functions $\underline{w}$ and $\overline{w}$.

Our strategy is now as follows. We derive the optimal consumption plan for the function $\hat{w}$. This is relatively straightforward, since we have a utility function that is fixed across periods and concave. The details of this plan will depend on the level of income. For each level of income, we have an optimal consumption plan and an induced sequence of utilities, given utility function $\hat{w}$. We then show that either the original induced sequence

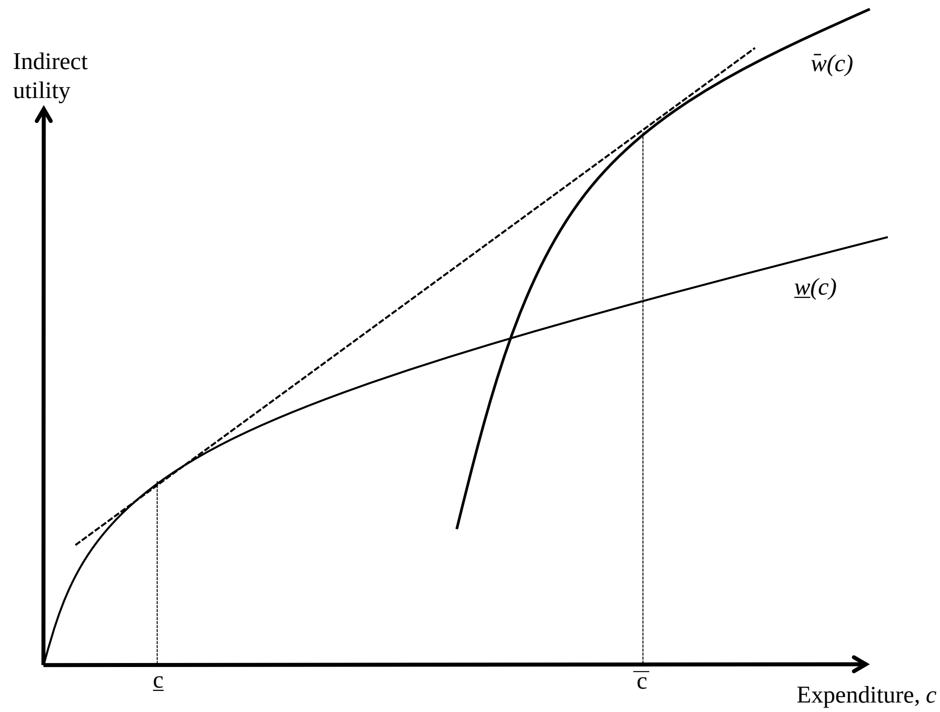

Figure 3: Indirect utility functions for the case of perfectly persistent memory. Expenditure on consumption within a period is denoted by c, and $\underline{w}(c)$ and $\overline{w}c)$ denote the maximal utility in that period given that memory utility is not ($\underline{w}$) or is ($\overline{w}$) generated.

of utilities, or in some cases a different but feasible sequence, gives the same total (discounted) utility under utility function w. This ensures that the resulting plan is optimal for w.

The first observation is that since $\hat{w}$ is concave, it is always optimal to equalize consumption across periods. Let

$$\frac{\hat{c}}{1-\delta} = Y_0.$$

Then $\hat{c}$ is the unique consumption level consistent with the consumer's income and consuming the same amount in each period. This now leads to three cases.

If $\hat{c} \leq \underline{c}$, then consuming $\hat{c}$ in each period under utility function w gives $w(\hat{c}) = \hat{w}(\hat{c})$, and hence we have an optimal consumption plan for utility w. In this case, the consumer's income is too low to make it worth ever securing memory utility. The fixed customary level Λ ensures that, even though the consumer never generates memory utility, the customary level never falls to a point that would make memory utility worthwhile.

If $\hat{c} \geq \underline{c}$, then consuming $\hat{c}$ in each period under utility function w gives $w(\hat{c}) = \hat{w}(\hat{c})$, and hence we have an optimal consumption plan for utility w.

In this case, the consumer's income is sufficiently large that the consumer generates memory utility in every period, with the customary level Λ never increasing as a result.

Suppose $\hat{c} \in (\underline{c}, \overline{c})$. Now we do not have $w(\hat{c}) = \hat{w}(\hat{c})$, since $\hat{w}(\hat{c})$ falls on the line segment that "concavifies" w. However, this line segment is linear. As a result, we can replace the sequence that consumes $\hat{c}$ in every period with a sequence that consumes $\underline{c}$ in some periods and $\overline{c}$ in others. The latter is feasible, and we argue that when this sequence is evaluated with the utility function w, it gives the same discounted utility sum as does the constant sequence $\hat{c}$ evaluated under the utility function $\hat{w}$. As we have argued, this suffices for the result.

In particular, given two periods t and $t' > t$, the consumer is indifferent over pairs $(c_t, c_{t'})$ that satisfies $\underline{c} \leq c_t, c_{t'} \leq \overline{c}$ and $c_t + \delta^{t'-t} c_{t'} = \hat{c}(1 + \delta^{t'-t})$. This in turn means that the consumer is indifferent over variations in c_t and c_t' that satisfy

$$\frac{dc_{t'}}{dc_t} = -\frac{1}{\delta^{t'-t}},$$

which is precisely the rate at which these two can be traded off in order to preserve feasibility. This in turn implies that *any* feasible consumption plan that features only $\underline{c}$ and $\overline{c}$ is optimal—when this sequence is evaluated with the utility function w, it gives the same discounted utility sum as does the constant sequence $\hat{c}$ evaluated under the utility function $\hat{w}$. At one extreme in the collection of such sequences is a plan that first consumes only $\overline{c}$, until switching to the perpetual consumption of $\underline{c}$. This is a consumer who first binges on memory consumption, and then forsakes it entirely. At the other extreme is a plan that first consumes only $\underline{c}$, until switching to the perpetual consumption of $\overline{c}$. This is a consumer who delays gratification. We have thus established the following:

Proposition 3. *Let Assumptions 1–4 hold, and let $\lambda = 1$, so that the level of customary memorable-good consumption shows no acclimatization. Then either*

(i) the consumer never generates memory utility (if Y_0 is sufficiently small);

(ii) the consumer always generates memory utility (if Y_0 is sufficiently large); or

(iii) there will exist expenditure levels $\underline{c} < \overline{c}$ such that any consumption plan that satisfies the budget constraint and exhibits only the consumption levels $\underline{c}$ and $\overline{c}$ is optimal.

The third case includes an infinite number of consumption plans that distribute $\underline{c}$ and $\overline{c}$ seemingly arbitrarily across periods, with the only constraint being that the resulting consumption plan exhausts the consumer's budget.

The utility functions shown in Figure 3 exhibit the concave-convex-concave shape discussed by Friedman and Savage. If we assumed that the consumer uses the same function w for utility maximization under risk, then we would apparently have an explanation for the simultaneous purchases of insurance and lotteries. However, in the presence of an infinite horizon, this consumer would have no interest in buying a lottery. No fair (or worse than fair) lottery can offer the consumer the possibility of memory consumption on terms better than the consumer can achieve by shifting consumption across periods. The infinite horizon is important in this argument, and distinguishes this argument from that of Section 3.

More generally, the link between convex utility functions and a willingness to gamble rests on some "friction" in the consumer's ability to transfer consumption across periods. One such friction is a finite lifetime, appearing in Friedman and Savage [7] in the form of a single-period horizon.

4.3 Acclimatization

We now turn to the case $\lambda \in [0, 1)$, so that acclimatization occurs, including the special case of $\lambda = 0$, or immediate acclimatization. We maintain Assumptions 1–5 throughout.

We first argue that memory utility is not in general a transient phenomenon. This section establishes conditions under which the consumer avails herself of memory utility infinitely often.

Let $x_1^*(Y_0)$ and $x_2^*(Y_0)$ be the consumption quantities that would be optimal, in period 0 and every subsequent period, if we assumed that the function v is identically equal to zero. The stock Λ is irrelevant in this case, and these quantities can be written solely as a function of Y_0. As we have noted in Section 4.1, these quantities will be constant across time periods, and so no time subscripts are needed. We are of course interested in the case in which v is nonzero, and $x_1^*(Y_0)$ and $x_2^*(Y_0)$ will be useful for the analysis of this case.

Definition 1. *We say that memory utility is* felicitous *if, when* $\Lambda_0 = x_2^*(Y_0)$, *the optimal consumption plan calls for the generation of memory utility at least once, and yields a utility strictly higher than never generating memory utility.*

The interpretation of the condition that $\Lambda_0 = x_2^*(Y_0)$ is that the consumer's initial customary level of consumption of the memorable good matches the level of consumption that would be relevant if memories were never generated. Memory utility is felicitous if, in this circumstance, the consumer would find it optimal to at least sometimes generate memory utility.

Assumption 5 requires that the functions u and $\breve{v}$ be homogeneous of degree $\alpha < 1$. Nothing to this point needed this assumption, but we use it in what follows. An implication is that if $\{x^*_{1t}, x^*_{2t}\}_{t=0}^{\infty}$ is an optimal consumption plan given (Y_0, Λ_0), then $\{\alpha x^*_{1t}, \alpha x^*_{2t}\}_{t=0}^{\infty}$ is an optimal consumption plan given $(\alpha Y_0, \alpha \Lambda_0)$ for any $\alpha > 0$. Hence, the question of whether memory utility if felicitous does not depend on Y_0, ensuring that the property of felicity is well defined.

It is less obvious that felicity is a useful concept, since it is defined in terms of endogenous objects. However, we can easily find (less insightful) conditions on primitives ensuring that memory utility if felicitous. For example, let $\{v_n\}$ be a sequence of functions, satisfying the properties placed on v by Assumptions 3–5, and suppose that the sequence is pointwise increasing and pointwise unbounded for any argument (x_2, Λ_0) with $x_2 > \Lambda_0$.[7] Then there exists a value N such that for all $n \geq N$, letting $v = v_n$ ensures that memory utility is felicitous. Hence, memory utility is felicitous if the technology for generating memory utility is sufficiently productive.

We can then show that if memory utility is felicitous, not only does an optimal consumption plan exhibit memory utility, but it does so infinitely often:

Proposition 4. *Let $\lambda \in [0, 1)$, let Assumptions 1–5 hold, and let memory utility be felicitous. Then in an optimal consumption plan, memory utility is generated infinitely often.*

The proof, in Section 7.2, begins by supposing that memory utility is generated at most finitely many times. After the last generation of memory utility, the problem of maximizing the continuation utility is equivalent to the memoryless utility maximization problem considered in Section 4.1. Then, we note that the continuation consumption plan must exhibit the consumption of some bundle (x^*_1, x^*_2) in every period, and hence $\lim_{t\to\infty} \Lambda_t = x^*_2$. But then the homogeneity imposed by Assumption 5 ensures that the optimal continuation consumption plan must be proportional to the original plan, which combines with the assumption that memory utility is felicitous to ensure that memory utility is once again generated.

[7] Notice that along the sequence $\{v_n\}$, the factor γ determining the extent to which the consumption of memory must exceed the customary level in order to generate memory utility is shrinking.

In the setting of Section 4.2, with no acclimatization, the initial customary level Λ_0 plays a key role in determining whether the optimal consumption plan exhibits the generation of memories. The optimal consumption plan calls for the generation of memories if Λ_0 is sufficiently small and does not if Λ_0 is sufficiently large. In the current setting, the initial level Λ_0 plays no role in determining whether memory utility is felicitous, and hence plays no role in determining whether optimal consumption plans exhibit the generation of memories. Notice that if memory utility is *not* felicitous, then the optimal consumption plan will exhibit the generation of memories if Λ_0 is sufficiently small, but need not do so infinitely often.

On the other side, and once again in contrast to the case of no acclimatization, even if memories are sometimes optimally generated, they are not generated in every period:

Proposition 5. *Let $\lambda \in [0, 1)$ and let Assumptions 1–5 hold. Then for every T, there exists a period $t > T$ in which memory utility is not generated.*

The proof is straightforward, and so we offer only a sketch of the argument. If memory utility is generated in period t, we have

$$\begin{aligned} \Lambda_{t+1} &= \lambda\Lambda_t + (1-\lambda)x_{2t} \\ &\geq \lambda\Lambda_t + (1-\lambda)\gamma\Lambda_t \\ &= [\lambda + (1-\lambda)\gamma]\Lambda_t. \end{aligned}$$

Hence, if there exists a time after which memory utility is generated in every period, then the customary level Λ_t must grow without bound (since $\lambda + (1-\lambda)\gamma > 1$), as must the consumption level x_{2t}. We have already seen, as the essential Lemma used in proving Proposition 1, that optimal consumption plans are bounded.

Paired with Proposition 4, this result indicates that an optimal consumption plan must generate memory utility infinitely often, but must intersperse this generation with periods in which no memory utility is generated. A key feature of the latter is that they allow the customary level to decline to the point that memory utility can again be generated. Consumption thus switches back and forth between periods in which memory utility is generated and periods in which the customary level of memorable good consumption is allowed to decline.

We can say something about the intervals in which memory utility is not generated. Section 7.3 proves:

Proposition 6. *Let $\lambda \in [0, 1)$ and let Assumptions 1–5 hold. Suppose that the optimal consumption plan generates memory utility in period t' and*

$t'' > t'$, but not in the intervening periods. Then over the course of the periods $(t'+1, \dots t''-1)$, the marginal utility of good 1 remains constant, while the marginal utility of good 2 increases.

It is a standard result that as long as $\delta > 0$, the marginal utility of good 1 is optimally equalized across periods. If not, the discounted sum of utilities could be increased by shifting the consumption of good 1 from low-marginal-utility to high-marginal-utility periods. Much the same intuition holds for good 2. In this case, however, the relevant marginal utility considerations involve not only the immediate marginal utility in the period of consumption, but also the marginal effect on the customary level of good-2 consumption in each future period in which memory utility is generated. Indeed, the optimality conditions for good 2 trade off the immediate utility-enhancing effects of increased consumption against the utility-decreasing effects of higher future customary levels. The difference between periods t and $t+1$ (with $t' < t < t+1 < t''$) is that in the case of the latter, these future impacts on customary levels are stronger and closer. This makes it all the more important to attenuate these future effects in period $t+1$, leading to a higher marginal utility.

These forces are especially convenient to illustrate when the switching back and forth between periods in which memory utility is generated and intervals with such generation induces a perfect cycle.[8] Such a cycle is characterized by a number n, with memory utility being generated every n periods. Let a sequence of such periods be numbered $1, 2, \dots, n$ with memory utility generated in period 1. Then we can let the consumption levels in these n periods be denoted by $((x_{11}, x_{12}), \dots, (x_{1n}, x_{2n}))$. Then there exists a wealth level Y, intuitively giving the current discounted value of expenditures over the n periods beginning with memory utility generation, and a level of customary consumption Λ, giving the customary consumption at the beginning of each period in which memory utility is generated, such

[8]The indivisibilities created by finite periods lengths can preclude such cycles, though examples of such particulary tractable equilibria are readily constructed.

that the optimal consumption plan must satisfy:

$$\max_{((x_{11},x_{21}),\ldots,(x_{1n},x_{2n}))} \quad u(x_{11},x_{21}) + v(x_{21},\Lambda_1) + \delta u(x_{12},x_{22}) + \ldots + \delta^{n-1}u(x_{1n},x_{2n})$$

$$\begin{aligned} s.t. \quad Y &= x_{11} + x_{21} + \delta(x_{12} + x_{22}) + \delta^2(x_{13} + x_{23}) + \ldots + \delta^{n-1}(x_{1n} + x_{2n}) \\ \Lambda &= \lambda^n\Lambda + \lambda^{n-1}(1-\lambda)x_{21} + \lambda^{n-2}(1-\lambda)x_{22} + \\ &\qquad \ldots + \lambda(1-\lambda)x_{2n-1} + (1-\lambda)x_{2n}. \end{aligned}$$

This maximization problem says nothing about what determines n, Y and Λ, but nonetheless the optimal stationary policy must solve this maximization problem.

Letting ζ be the multiplier on the first constraint and ψ the multiplier on the second, we can formulate the first-order conditions as

$$\begin{aligned} u_1(x_{11},x_{21}) + \zeta &= 0 \\ u_1(x_{12},x_{22}) + \zeta &= 0 \\ &\vdots \\ u_1(x_{1n-1},x_{2n-1}) + \zeta &= 0 \\ u_1(x_{1n},x_{2n}) + \zeta &= 0 \end{aligned}$$

and

$$\begin{aligned} u_2(x_{11},x_{21}) + v_2(x_{21},\Lambda_1) + \zeta + \psi\lambda^{n-1}(1-\lambda) &= 0 \\ u_2(x_{12},x_{22}) + \zeta + \psi\lambda^{n-2}\delta^{-1}(1-\lambda) &= 0 \\ &\vdots \\ u_2(x_{1n-1},x_{2n-1}) + \zeta + \psi\lambda\delta^{-(n-2)}(1-\lambda) &= 0 \\ u_2(x_{1n},x_{2n}) + \zeta + \psi\delta^{-(n-1)}(1-\lambda) &= 0. \end{aligned}$$

The marginal utility of good 1 is equalized across periods. This reflects a standard consumption-smoothing argument. The marginal utility of good 2 increases as the next bout of memory utility draws near. Reducing the consumption of good 2 reduces the customary level against which the next instance of memory utility is measured. The closer is the next instance of memory utility, the more valuable is this reduction, and hence the larger the marginal utility of good 2. This gives us a consumption pattern for good 2 that peaks with the generation of memory utility, then takes a drop, and then declines until the next generation of memory utility.

If the utility function exhibits a positive cross partial derivative, then the consumption pattern for memory goods will spill over into a similarly cyclic behavior for the consumption of good 1. Hence, memory utility can induce cycles in the consumption of goods that are inherently nonmemorable.

5 Foundations

A key feature of our model is that past consumption can affect utility through acclimatization and memory. This calls for a model in which the utility function at time t depend not only on current values of the products, x_{1t} and x_{2t}, but also on their past values. However, such a function allows for a wide variety of history-dependent utility functions. In order to focus on the effects that are of interest to us, we suggested the instantaneous utility function given in (3), which is the sum of two functions. One function depends only on the goods consumed at present, x_{1t} and x_{2t}, capturing the standard, non-memory-related utility, and the other function depends only on past and current values of x_2, capturing the effects of acclimatization and memory.

It is not entirely clear what we assume by this functional form. According to the classical notion of separability the utility function is the sum of two (or more) functions, each of which has a disjoint set of variables. But what is assumed by a summation of two functions whose sets of variables are *not* disjoint? Clearly, not every function can be so written. Yet, such functions do not satisfy the conditions of separability.

The purpose of this section if to axiomatize a functional form as in (3). Axioms on presumably observed preferences (interpreted as the instantaneous preferences at time t) that imply that such a decomposition of the utility function is possible would clarify what is assumed by the model. In this case, we couple standard requirements of weak order, continuity and nontriviality with an axiom called cross-consistency, directing attention to the latter as capturing our departure from standard models. The axiomatization may in turn facilitate further analysis and testing of the model, as it may be easier to design and conduct empirical or experimental exercises that focus on this axiom than taking on the entire memory utility package at once. Finally, we believe that, when interpreting x_2 as the memory-generating good, the axioms we impose are quite plausible, supporting our belief that the functional form is neither too peculiar nor ad hoc.

Our result applies to more general set-ups, and axiomatizes *quasi-separable* utility functions, defined as utility functions that can be written as the sum of two functions, each of which depends on a proper subset of the variables, where these subsets are not disjoint.

5.1 The Setting

Let X, Y, Z be convex subsets of Euclidean spaces. Denote their product by

$$A = X \times Y \times Z$$

and endow it with the product topology. We are interested in binary relations $\succsim \subset A \times A$ that can be represented by maximization of a function

$$f(x, y, z)$$

that can be written as

$$f(x, y, z) = u(x, y) + v(y, z)$$

where

$$u : X \times Y \to \mathbb{R}$$

and

$$v : Y \times Z \to \mathbb{R}$$

are continuous, non-constant functions.

In the memory-good application, X is the bundle of ordinary goods; Y is the bundle of memory goods consumed at present; and Z consists of bundles of memory goods consumed in the past (or the corresponding levels of Υ_t and Λ_t defined by them). Clearly, the same structure can be used for other applications as well.

5.2 The Axioms

For a binary relation $\succsim \subset A \times A$ (with $A = X \times Y \times Z$) we state the following axioms:

A1. Weak order: $\succsim$ is complete and transitive.

A2. Continuity: For every $a \in A$, the sets $\{b \in A \mid b \succ a\}$, $\{b \in A \mid a \succ b\}$ are open.

A3. Cross-Consistency: For every $y_0, y_1 \in Y$, every $x_1, x_2, x_3, x_4 \in X$, and every $z_1, z_2, z_3, z_4 \in Z$, if

$$\begin{array}{rcl} (x_1, y_0, z_1) & \succsim & (x_3, y_1, z_3) \\ (x_2, y_0, z_1) & \precsim & (x_3, y_1, z_4) \\ (x_1, y_0, z_2) & \precsim & (x_4, y_1, z_3) \end{array}$$

then

$$(x_2, y_0, z_2) \precsim (x_4, y_1, z_4)$$

A4. Essentiality: For every $y \in Y$, there exist $x_1, x_2 \in X$ and $z \in Z$ such that $(x_1, y, z) \succ (x_2, y, z)$ and there exist $x \in X$ and $z_1, z_2 \in Z$ such that $(x, y, z_1) \succ (x, y, z_2)$.

These axioms are both obvious and intuitive, with the exception of Cross-Consistency. To understand the meaning of Cross-Consistency, consider first the pair of preferences

$$\begin{array}{rcl} (x_1, y_0, z_1) & \succsim & (x_3, y_1, z_3) \\ (x_2, y_0, z_1) & \precsim & (x_3, y_1, z_4) \,. \end{array}$$

For the sake of the argument, imagine that preferences are monotonic in all coordinates, that x_2 is better than x_1, and that z_4 is better than z_3. On the left side, x_1 was replaced by x_2. On the right side, z_3 was replaced by z_4. As a result, the right side, which used to be not as highly ranked as the left side, became at least as good as the (modified) left side. This means, intuitively, that the difference in utility between z_4 and z_3 (at the level y_1) is at least as high as the difference in utility between x_2 and x_1 (at the level y_0).

Next consider the pair

$$\begin{array}{rcl} (x_1, y_0, z_1) & \succsim & (x_3, y_1, z_3) \\ (x_1, y_0, z_2) & \precsim & (x_4, y_1, z_3) \,. \end{array}$$

Again, to understand the intuition, assume that z_2 is better than z_1 and the same holds for x_4 and x_3. By similar reasoning, the difference in utility between x_4 and x_3 (at the level y_1) is at least as high as the difference in utility between z_2 and z_1 (at the level y_0).

Finally, consider the first and the last comparisons:

$$(x_1, y_0, z_1) \succsim (x_3, y_1, z_3)$$

and

$$(x_2, y_0, z_2) ??? (x_4, y_1, z_4)$$

On the left side we observe two improvements: x_1 was replaced by x_2 *and* z_1 was replaced by z_2. However, there are also two improvements on the hand side: x_3 was replaced by x_4 and z_3 was replaced by z_4. Moreover, the left side improvement occur at the level $y = y_0$ and those on the right side occurred at the level $y = y_1$. But these are precisely the levels of y for which we have some information from the first two comparisons. And since we know that the z_3-z_4 improvement (at y_1) beats the x_1-x_2 improvement (at y_0) *and* that the x_3-x_4 improvement (at y_1) beats the z_1-z_2 improvement (at y_0), we expect that the addition of the (respective) former will beat the addition of the (respective) latter, that is, that $(x_2, y_0, z_2) \precsim (x_4, y_1, z_4)$.

We thus find Cross-Consistency a reasonably compelling property. Section 7.4 exploits this line of argument to achieve a rather straightforward demonstration that Cross-Consistency is necessary for our representation. The main point of Proposition 7 is that with the additional mild assumptions imposed by the other axioms, Cross-Consistency is also sufficient for the representation. These additional assumptions are that $\succsim$ is a continuous weak order—which is necessarily the case for any continuous representation—and that $\succsim$ satisfies a certain sensitivity assumption, so that, given any value of y, neither of the other variables will be immaterial. (We comment on the importance of this assumption in the course of the proof.)

Remark 1. One may consider a weaker version of Cross-Consistency that is restricted to a single y level, that is, a version that requires $y_0 = y_1$. As will be clear from the proof, such a version implies Debreu's [6] "Double Cancellation" axiom and is the basic driving force behind additive separability at each level of y. It is not hard to see, however, that such a weaker version would not suffice for our purposes. For example, assume that x, y, z are positive real variables, and that $\succsim$ is defined by maximization of

$$f(x, y, z) = y \log(x + z).$$

Clearly, at each level of y preferences are defined by maximization of $\log(x + z)$ or, equivalently, of $x + z$ and are therefore separable. Yet, it is not hard to see that such preferences cannot be represented by $u(x, y) + v(y, z)$ over the entire space, as they will not satisfy the necessary condition of Cross-Consistency.

5.3 A Representation Result

Section 7.4 proves:

Proposition 7. *The relation* $\succsim \subset A \times A$ *satisfies A1-A4 if and only if there are continuous functions*

$$\begin{aligned} u &: X \times Y \to \mathbb{R} \\ v &: Y \times Z \to \mathbb{R} \end{aligned}$$

such that $\succsim$ *is represented by*

$$f(x,y,z) = u(x,y) + v(y,z)$$

and such that, for each $y \in Y$*, neither* $u(\cdot, y)$ *nor* $v(y, \cdot)$ *is a constant. Furthermore, in this case u and v are unique in the following sense:* u' *and* v' *also satisfy the representation above iff there are* $\alpha > 0$*, a continuous function* $\beta : Y \to \mathbb{R}$*, and* $\gamma \in \mathbb{R}$ *such that*

$$\begin{aligned} u'(x,y) &= \alpha u(x,y) + \beta(y) \\ v'(y,z) &= \alpha v(y,z) - \beta(y) + \gamma. \end{aligned}$$

When interpreting this representation result in the context of memory utility, one may take the elements of Z to be vectors of past consumption, so that preferences are represented by the function

$$u(x_{1t}, x_{2t}) + \tilde{v}_t(x_{20}, \ldots, x_{2t}).$$

The specific assumptions we impose on $\tilde{v}_t$, namely, the way that it depends on $x_{20}, \ldots, x_{2(t-1)}$ only through Υ_t, Λ_t, are then function form assumptions. We could seek axiomatic foundations for this functional form, but do not expect such an axiomatization to add significantly to our understanding of the model.

6 Discussion

6.1 Related Models

6.1.1 Memorable Goods vs. Durable Goods

Memory utility shares many of the properties of a durable good, but there are also some important differences. Most notably, models of durable good consumption typically contain no counterpart of one of our key characteristics of memory utility, the customary level of consumption. In the standard model, an expenditure on a durable good generates a stream of benefits that are independent of past expenditures. The flow of benefits from the purchase of a sixty inch flat screen television is the same whether this is the first television one ever owned or whether it is a replacement of the previous sixty inch flat screen television that failed last week. In contrast, an expenditure on the memorable good generates valuable memories only if the expenditure is sufficiently above the customary level. A consumer who plans to spend a large amount on the memorable good next period might be better off if her consumption this period decreased, something not possible with durable goods.

6.1.2 Indivisibilities

One might be concerned that the patterns we associate with memory utility simply reflect indivisibilities in consumption goods. There may be a spike in consumption because there is a minimum amount one has to spend on a wedding, or carnival costume, or vacation. We notice, however, that people spend a wide range of sums on (for example) weddings or vacations, and that these sums appear to be correlated with the income levels or consumption patterns of the purchasers. This is what we would expect of an memory-utility explanation, but not a indivisibility explanation.

6.1.3 Addiction

The presence of the stock Λ_t in our memory utility specification prompts a comparison to models of addiction, with Becker and Murphy [2] being the obvious comparison. In their model, the utility function in period t is given by

$$u(y_t, c_t, \Lambda_t),$$

where y_t is the consumption of a nonaddictive good, c_t is the consumption of an addictive good, and Λ_t is the stock of the addictive good. The stock Λ_t is allowed to enter the utility function with either a positive or negative sign, so that an increased stock of consumption may decrease utility (perhaps with something like smoking) or increase utility (perhaps with something like exercise). In addition, the cross derivative u_{cA} may be either positive or negative, so that an increased stock may either enhance or attenuate the urge for current consumption. The primary difference is that the function u in the addiction model as assumed to be concave. This allows a straightforward optimization in each period, and contrasts with the nonconvexities that appear in our case.

6.1.4 Habit Formation

We incorporated the dependence of period t utility by adding to the utility from instantaneous utility of the expenditure in period t a function $\tilde{v}_t$ that depends on past expenditures,

$$\tilde{v}_t(x_{20}, \ldots, x_{2t}) = \hat{v}(x_{20}, \Upsilon_t, \Lambda_t),$$

where we interpret $\Upsilon_t \in \mathbb{R}_+$ as identifying a stock of memories at time t and $\Lambda_t \in \mathbb{R}_+$ as identifying the customary level of consumption of the memory good at time t. If we eliminate Υ from the model, then the effect of past consumption on the immediate utility from expenditure on the memorable good x_{2t} depends only on Λ_t. There then exist simple specifications for the evolution of Λ for which the model becomes a model of habit formation. Thus, our model nests a habit formation model (as well as, obviously, the standard model).

6.2 Applications of a Memory Utility Model

6.2.1 Permanent Income Hypothesis

The standard intertemporal consumption model suggests that optimal consumption should be smooth. In particular, expected but temporary jumps in income have little effect on permanent income, and so should have little effect on consumption. For example, expected tax refund receipts should lead to little immediate increase in consumption. In contrast, Souleles [16] documents that there is excess sensitivity of consumption to such refunds.

Hai, Krueger and Postlewaite [8] demonstrate that if the model in Souleles is extended along the lines of our model, there is essentially no excess sensitivity. Much of what appears to be a puzzling current consumption binge in response to temporary income shocks can be interpreted as the generation of memory utility, which in turn generates a relatively smooth intertemporal pattern of increases in utility.

6.2.2 Retirement Saving

It is a familiar lament in the popular press that Americans save too little for retirement (see, for example, the opening quotation in Scholz, Seshadri and Khitatrakun [14]). The conclusions of more careful analyses are less clear. Scholz, Seshadri and Khitatrakun [14], for example, argue that only about twenty percent of Americans are undersaving, and even then not by vast amounts.

Our analysis of memorable goods brings a new dimension to this discussion, and a new reason to suspect that reports of undersaving may be overstated. The flow of utility in retirement years includes the flow of utility from earlier memorable consumption. Decreasing consumption expenditures as one moves into and through retirement, often taken as a sign of undersaving, may simply reflect the optimal management of memory utility. This suggests a number of empirical projects. For example, with sufficient data, one could look for a correlation between early expenditure on memorable goods and decreases in expenditure later in life that our model would suggest.

6.2.3 Memories as a Substitute for Saving

Taking account of memorable goods in a consumer's utility maximization problem changes how we think about saving and investment in general. Separate from the question of retirement savings, economists have puzzled over why the savings rate in the U.S. has dropped substantially over the past half century (e.g., Parker [13]). The decrease in the savings rate is sometimes interpreted as an indication that consumers care less about the future than they once did. However, over the period in which savings rates decreased, expenditures on vacations increased, providing a hint of a link between the generation of memory utility and savings.

A consumer who shifts expenditure from nonmemorable goods to memorable goods is doing something akin to saving—making current choices that increase her future utility. Any estimate of intertemporal preferences from longitudinal consumption that ignores the memory component of nondurable consumption will result in an upward bias of the consumer's discount rate. There is evidence that recreation is a luxury good (though perhaps becoming less so; see Costa [4]). If memory goods more generally are luxury goods, then estimates of discount factors may become more problematic for higher incomes. It would be interesting to revisit estimates of discount factors with an empirical strategy incorporating memorable goods.

6.2.4 Memorable Goods and Risk Aversion

Ignoring the memory component of consumption will complicate estimates of risk aversion as well as estimates of discount rates. We discussed above the Friedman-Savage anomaly of agents who both gamble and insure. If one accepts our model of memory utility, a nonconvexity arises naturally, and the insurance-and-gambling behavior is less surprising. Section 4.3 showed that a consumer who understands the (possibly large) memory utility that accompanies a big increase in consumption may optimally reduce decrease current consumption for some time so as to be able to afford the memorable event. For our infinitely lived consumer, this intertemporal substitution is a sufficiently powerful tool for managing memory utility as to obviate the need to take on risk. However, a real-world finitely-lived consumer might realize that she will not live long enough to acquire the resources needed to generate a large burst of memory utility. A fair (or even mildly unfair) lottery with a large upside possibility may be part of an optimal plan for such a consumer.

6.3 Extensions

We touch here on one of the many possible extensions of our model. We have laid out the model in which higher-than-customary consumption leads to memory utility that is added to the direct contemporaneous utility from consumption. This captures well the positive memories that stem from high expenditures, but the basic ideas can be extended to cover negative memories as well. Most of us have at some time stayed at hotels that are memorable, but not in a positive way. If we were advising a friend on choosing a hotel for his honeymoon we would suggest paying a premium to be sure that the hotel didn't fall below expectations, since if the experience is negative it won't soon be forgotten.

Accommodating negative memories would be straightforward. The function v aggregates the flow of utility stemming from memorable experiences that is then added on to the direct utility from consumption. One can allow the function to take on negative values for unpleasant experiences, which then have an ongoing drag on my future utility. As with the positive memory formation that we have modeled, the natural way to proceed would be to say that if the expenditure on the memorable good falls sufficiently below the customary level, negative memories are formed.

7 Appendix: Proofs

7.1 Proof of Lemma 2

We first note that in any equilibrium, we have $(x_{10}, x_{20}) \in [0, Y_0]^2$. This implies that there is a lower bound ε on $\frac{du(x_{10},x_{20})}{dx_{10}}$, the marginal utility of good 1 in the first period.

Next, we note that there is an $\hat{x} \geq 0$ such that for any consumption (x_1, x_2) with either $x_1 > \hat{x}$ or $x_2 > \hat{x}$, it must be that either $\frac{du(x_1,x_2)}{dx_1} < \varepsilon/2$ or it is the case that $\frac{du(x_1,x_2)}{dx_2} < \varepsilon/2$. If this is not the case, then we can find a sequence $(x_{1\tau}, x_{2\tau})$ contradicting Assumption 3.3.

The next step is to note that there exists a value $\breve{x}$ such that if $x_2 > \breve{x}$, then $\frac{d\breve{v}(x_2,\Lambda)}{dx_2} < \frac{\varepsilon}{2}(1-\delta)$ for any Λ (where we see later in the proof why the $(1-\delta)$ is needed to cope with the durability of memory utility). This follows from Assumption 3.4.

Let $\overline{x} = \max\{\hat{x}, \breve{x}\}$.

Now suppose we have a candidate equilibrium in which, for some t, we have consumption bundle (x_{1t}, x_{2t}) with $x_{1t} > \overline{x}$ or $x_{2t} > \overline{x}$. We argue that there exists a superior consumption plan in which consumption is unchanged in all periods except 0 and t, and in which consumption of each good in periods 0 and t falls short of $\overline{x}$. Iterating this argument yields the result.

In period t, the derivative of u with respect to either good 1 or good 2 must fall short of $\varepsilon/2$. If it is the derivative with respect to good 1 that has this property, then we have

$$u_1(x_{10}, x_{20}) - u_1(x_{1t}, x_{2t}) > 0. \tag{12}$$

Now consider an alternative strategy that duplicates that of the candidate equilibrium strategy with the possibly exception of x_{10} and x_{1t}. The latter two variables are allowed to vary as long as they satisfy the budget constraint. The budget constraint in turn requires

$$x_{10} + \delta^t x_{1t} = k$$

for some constant k. This allows us to define an implicit function $x_{1t} = f(x_{10})$ whose derivative is given by $-\delta^{-t}$. This allows us to interpret (12) as the derivative of the discounted sum of utility with respect to x_{i0}, under the constraint that $x_{1t} = f(x_{i0})$, ensuring the budget constraint is satisfied. The fact that this derivative is positive ensures that we can increase utility by decreasing x_{1t} and increasing x_{10}.

Now suppose that it is good 2 for which the derivative is small in period t. Then

$$\begin{aligned} u_1(x_{10}, x_{20}) - \delta^t \left[u_2(x_{1t}, x_{2t}) + \sum_{\tau=t}^{\infty} \delta^{\tau-t} \breve{v}_1(x_{2t}, \Lambda_t) \right] &= \\ u_1(x_{10}, x_{20}) - \delta^t \left[u_2(x_{1t}, x_{2t}) + \frac{1}{1+\delta} \breve{v}_1(x_{2t}, \Lambda_t) \right] &> 0. \end{aligned}$$

Our choice of the period t ensures the inequality in the second line. Moreover, it is clear from the first line that this sum is an upper bound on the derivative of the discounted sum of utility with respect to x_{10}, under the constraint that $x_{2t} = f(x_{10})$. (In particular, it captures the effect of varying x_{10} on first-period utility, and then the effect of varying x_{2t} on utility in period t and every subsequent period. It is only an upper bound, because the derivative reduced by the effect of x_{2t} on values of Λ_τ for values $\tau > t$, which we have not incorporated in our calculation.) The fact that this derivative is positive again ensures that we can increase utility by decreasing x_{2t} and increasing x_{10}.

This allows us to construct a sequence of improvements that continues until x_{1t} and x_{2t} each are no larger than $\overline{x}$ for all t, yielding the result.

7.2 Proof of Proposition 4

Let $\{\hat{x}_{1t}, \hat{x}_{2t}\}_{t=0}^{\infty}$ be the optimal consumption plan. By assumption this plan involves the generation of memory utility in some period, though we cannot be sure of which period. An alternative consumption plan is to never secure memory utility, in which case the optimal plan would be $\{(x_1^*, x_2^*), (x_1^*, x_2^*), \ldots\}$. We assume that at least one generation of memory utility is strictly optimal, giving

$$\{\hat{x}_{1t}, \hat{x}_{2t}\}_{t=0}^{\infty} \quad \succ \quad ((x_1^*, x_2^*), (x_1^*, x_2^*), \ldots). \tag{13}$$

The optimal consumption plan induces a sequence of stocks $\{\hat{\Lambda}_t\}_{t=0}^{\infty}$ and wealths $\{\hat{Y}_t\}_{t=0}^{\infty}$.

Suppose the optimal plan involved the generation of memory utility only finitely many times. Then there is a period T such that the consumer enters period $T+1$ with wealth Y_{T+1}, and thereafter consumes

$$\left(x_1^* \frac{Y_T}{Y_0}, x_2^* \frac{Y_T}{Y_0} \right)$$

in each period. As a result, the stock Λ_t becomes arbitrarily close to $x_2^* \frac{Y_T}{Y_0}$. It then suffices to show that if a period t' arrives in which $\Lambda_{t'} = x_2^* \frac{Y_T}{Y_0}$, then the consumer will indulge in memory utility on some subsequent period.[9] To see that this is the case, notice that once such a t' has arrived, the

[9]We can only be assured that $\Lambda_{t'}$ can be made arbitrarily close to $x_2^* \frac{Y_T}{Y_0}$, not precisely equal to it, but the fact that our original preference is strict allows us to look at the case where they are equal.

continuation sequences

$$\left\{\hat{x}_{1t}\frac{Y_T}{Y_0}\right\}_{t=0}^{\infty}, \left\{\hat{x}_{2t}\frac{Y_T}{Y_0}\right\}_{t=0}^{\infty}, \left\{\hat{\Lambda}_t\frac{Y_T}{Y_0}\right\}_{t=0}^{\infty}, \left\{\hat{Y}_t\frac{Y_T}{Y_0}\right\}_{t=0}^{\infty}$$

are feasible and, by definition, involves the generation of some memory utility. We then need only argue that

$$\left(\left\{\hat{x}_{1t}\frac{Y_T}{Y_0}\right\}_{t=0}^{\infty}, \left\{\hat{x}_{2t}\frac{Y_T}{Y_0}\right\}_{t=0}^{\infty}\right) \succ \left(\left(\hat{x}_1^*\frac{Y_T}{Y_0}, \hat{x}_2^*\frac{Y_T}{Y_0}\right), \left(\hat{x}_1^*\frac{Y_T}{Y_0}, \hat{x}_2^*\frac{Y_T}{Y_0}\right), \ldots\right).$$

But this follows from (13) and the homogeneity invoked in Assumption 5.

7.3 Proof of Proposition 6

Fix an optimal consumption plan, and let no memory utility be generated in periods t and $t+1$. Let the periods after $t+1$ in which memory utility is generated be $\{t_\tau\}_{\tau=0}^{\infty}$. Then the derivative of the period-t continuation discounted sum of utility with respect to x_{2_t} is given by

$$u_2(x_{1t}, x_{2t}) + \delta^{\tau_0 - t}\sum_{k=0}^{\infty}\delta^{\tau_k-\tau_0}v_2(x_{2\tau_k}, \Lambda_{\tau_k})\frac{d\Lambda_{\tau_k}}{dx_{2t}},$$

while the corresponding derivative with respect to x_{2t+1} is given by

$$u_2(x_{1t+1}, x_{2t+1}) + \delta^{\tau_0-(t+1)}\sum_{k=0}^{\infty}\delta^{\tau_k-\tau_0}v_2(x_{2\tau_k}, \Lambda_{\tau_k})\frac{d\Lambda_{\tau_k}}{dx_{2t+1}}.$$

In each case the first term captures the immediate effect of consuming good 2 on current consumption, and the summation captures the effect on the future customary levels of consumption, which come into play each time memory utility is generated. We can rewrite these derivatives as

$$u_2(x_{1t}, x_{2t}) + \delta^{\tau_0 - t}\sum_{k=0}^{\infty}\delta^{\tau_k-\tau_0}v_2(x_{2\tau_k}, \Lambda_{\tau_k})(1-\lambda)\lambda^{\tau_k-t-1},$$

and

$$u_2(x_{1t+1}, x_{2t+1}) + \delta^{\tau_0-(t+1)}\sum_{k=0}^{\infty}\delta^{\tau_k-\tau_0}v_2(x_{2\tau_k}, \Lambda_{\tau_k})(1-\lambda)\lambda^{\tau_k-(t+1)-1}.$$

The necessary conditions for utility maximization are that these two derivatives be equal. Noting that $v_2 < 0$, this gives

$$u_2(x_{1t}, x_{2t}) < u_2(x_{1t+1}, x_{2t+1}).$$

7.4 Proof of Proposition 7

7.4.1 Necessity

Assume that $u(x, y) + v(y, z)$ represents $\succsim$. A1 and A2 follow immediately. As for A3, assume that

$$\begin{aligned} (x_1, y_0, z_1) &\succsim (x_3, y_1, z_3) \\ (x_2, y_0, z_1) &\precsim (x_3, y_1, z_4) \\ (x_1, y_0, z_2) &\precsim (x_4, y_1, z_3) . \end{aligned}$$

The first preference statement implies

$$u(x_1, y_0) + v(y_0, z_1) \geq u(x_3, y_1) + v(y_1, z_3),$$

or, equivalently,

$$-u(x_1, y_0) - v(y_0, z_1) \leq -u(x_3, y_1) - v(y_1, z_3),$$

while the other two yield

$$u(x_2, y_0) + v(y_0, z_1) \leq u(x_3, y_1) + v(y_1, z_4)$$

$$u(x_1, y_0) + v(y_0, z_2) \leq u(x_4, y_1) + v(y_1, z_3) .$$

Summing up the last three inequalities we obtain

$$u(x_2, y_0) + v(y_0, z_2) \leq u(x_4, y_1) + v(y_1, z_4)$$

which implies

$$(x_2, y_0, z_2) \precsim (x_4, y_1, z_4) .$$

Finally, observe that A4 holds provided that $u(\cdot, y)$ and $v(y, \cdot)$ are not constant for any y.

7.4.2 Sufficiency – Part I: Construction

In this subsection we construct u, v such that $u(x, y) + v(y, z)$ represents preferences. We will fix $x_0 \in X$ and construct these functions so that

$$u(x_0, y) = 0 \qquad \forall y \in Y.$$

This will prove useful in showing that the functions so constructed are continuous (in Part II), as well as in proving the uniqueness result.

Step 0: Preliminaries: Cross Consistency has a natural counterpart, with the direction of all preference signs reversed:

Reverse Cross Consistency: For every $y_0, y_1 \in Y$, every $x_1, x_2, x_3, x_4 \in X$, and every $z_1, z_2, z_3, z_4 \in Z$, if

$$\begin{array}{rcl} (x_1, y_0, z_1) & \precsim & (x_3, y_1, z_3) \\ (x_2, y_0, z_1) & \succsim & (x_3, y_1, z_4) \\ (x_1, y_0, z_2) & \succsim & (x_4, y_1, z_3) \end{array}$$

then

$$(x_2, y_0, z_2) \succsim (x_4, y_1, z_4) \, .$$

Note that the two conditions are equivalent. (To see this, it suffices to exchange the notation between $y_0 \leftrightarrow y_1$, $x_1 \leftrightarrow x_3$, $x_2 \leftrightarrow x_4$, $z_1 \leftrightarrow z_3$, $z_2 \leftrightarrow z_4$.)

Similarly, one can have the indifference version of the axiom:

Indifference Cross Consistency: For every $y_0, y_1 \in Y$, every $x_1, x_2, x_3, x_4 \in X$, and every $z_1, z_2, z_3, z_4 \in Z$, if

$$\begin{array}{rcl} (x_1, y_0, z_1) & \sim & (x_3, y_1, z_3) \\ (x_2, y_0, z_1) & \sim & (x_3, y_1, z_4) \\ (x_1, y_0, z_2) & \sim & (x_4, y_1, z_3) \end{array}$$

then

$$(x_2, y_0, z_2) \sim (x_4, y_1, z_4) \, .$$

This version follows from the conjunction of Cross Consistency and Reverse Cross Consistency, and hence from each of these alone.

Step 1: Additive representation for any fixed y: For $y \in Y$, define

$$A_y = \{(x, y, z) \in A \,|x \in X, z \in Z\} \, .$$

Restricting attention to A_y, for each $y \in Y$, we note that $\succsim$ is a continuous weak order (basically, on $X \times Z$). For a relation $\succsim$ on $X \times Z$ we will be interested in following condition:[10]

[10] See also Blaschke [3] and Thomsen [19] for the related "hexagon" condition.

Double Cancellation: For every $f, g, h \in X$ and every $p, r, q \in Z$, if

$$(f, p) \succsim (g, q)$$

and

$$(h, p) \precsim (g, r)$$

then

$$(h, q) \precsim (f, r) .$$

In particular, the following lemma states that $\succsim$ on A_y satisfies Double Cancellation.

Lemma 8. *For each $y \in Y$, every $f, g, h \in X$ and every $p, r, q \in Z$, if*

$$(f, y, p) \succsim (g, y, q)$$

and

$$(h, y, p) \precsim (g, y, r)$$

then

$$(h, y, q) \precsim (f, y, r) .$$

Proof: Given $f, g, h \in X$ and $p, r, q \in Z$ that satisfy $(f, y, p) \succsim (g, y, q)$, and $(h, y, p) \precsim (g, y, r)$, define (i)

$$y_0 = y_1 = y$$

(ii)

$$\begin{aligned} x_1 &= x_4 = f \\ x_2 &= h \quad x_3 = g \end{aligned}$$

and (iii)

$$\begin{aligned} z_1 &= p \\ z_2 &= z_3 = q \\ z_4 &= r. \end{aligned}$$

Observe that

$$\begin{aligned} (x_1, y_0, z_1) &= (f, y, p) \succsim (g, y, q) = (x_3, y_1, z_3) \\ (x_2, y_0, z_1) &= (h, y, p) \precsim (g, y, r) = (x_3, y_1, z_4) \end{aligned}$$

and clearly also

$$(x_1, y_0, z_2) = (f, y, q) \sim (f, y, q) = (x_4, y_1, z_3)$$

hence Cross Consistency can be invoked to conclude that

$$(x_2, y_0, z_2) \precsim (x_4, y_1, z_4)$$

which means

$$(x_2, y_0, z_2) = (h, y, q) \precsim (f, y, r) = (x_4, y_1, z_4)\,.$$

■

Thus, Double Cancellation on each A_y follows from Cross Consistency. It follows from Debreu [6] that $\succsim$ has an additively separable continuous representation: there are continuous $u_y : X \to \mathbb{R}$ and $v_y : Z \to \mathbb{R}$ such that, for every $x, x' \in X$ and every $z, z' \in Z$,

$$(x, y, z) \succsim (x', y, z')$$

iff

$$u_y(x) + v_y(z) \geq u_y(x') + v_y(z')\,.$$

Further, thanks to A4, these u_y, v_y are unique up to multiplication by a positive constant and the addition of a constant.[11] In other words, if $u'_y : X \to \mathbb{R}$ and $v'_y : Z \to \mathbb{R}$ also represent $\succsim$ on A_y as above, there must be $\alpha_y > 0$ and $\beta_{yu}, \beta_{yv} \in \mathbb{R}$ such that

$$u'_y(x) = \alpha_y u_y(x) + \beta_{yu}$$

and

$$v'_y(z) = \alpha_y v_y(z) + \beta_{yv.}$$

Finally, by setting $\beta_{yu} = -\alpha_y u_y(x_0)$ we may assume without loss of generality that $u_y(x_0) = 0$.

[11] If one of the two variables $x.z$ does not affect preferences on A_y, its function is a constant—hence unique up to a cardinal transformation, but the representation of the other variable becomes only ordinal. Similarly, If A4 doesn't hold, one of the two functions might be an non-continuous monotone transformation of a continuous function.

Step 2: Additive representation on the entire space: For each y, choose two continuous functions $u_y : X \to \mathbb{R}$ and $v_y : Z \to \mathbb{R}$ that represent $\succsim$ on A_y as above, with $u_y(x_0) = 0$. Recall that these functions are unique up to multiplication (of both) by (the same) $\alpha_y > 0$ and an addition of a constant to v_y. We now wish to show that we may take such an affine transformation of u_y, v_y for each $y \in Y$ so that the resulting functions rank alternatives as does $\succsim$ also across different values of y, (that is, for any two alternatives $a \in A_y, b \in A_{y'}$ even if $y \neq y'$) and that these functions are continuous (over all of A).

For a generic $a \in A$, let $a_X \in X$, $a_Y \in Y$, and $a_Z \in Z$ be its components, so that $(a_X, a_Y, a_Z) = a$.

To visualize the construction, consider, for each $y \in Y$, the image of the functions u_y, v_y. Define

$$I(y) = \{(u_y(x), v_y(z)) \,|\, x \in X, z \in Z\} \subset \mathbb{R}^2.$$

Note that, because u_y, v_y are continuous functions on convex subsets of Euclidean spaces, their images of these functions are convex. That is

$$\begin{aligned} I_u(y) &\equiv \{u_y(x) \,|\, x \in X\} \subset \mathbb{R} \\ I_v(y) &\equiv \{v_y(z) \,|\, z \in Z\} \subset \mathbb{R} \end{aligned}$$

are (potentially infinite) intervals in $\mathbb{R}$, and $I(y) = I_u(y) \times I_v(y)$.

We may imagine indifference curves in $I(y)$, which are downward sloping straight lines with slope -1.

We define $y, y' \in Y$ to be *close* if there exist $x, x' \in X$ and $z, z' \in Z$ such that $(x, y, z) \sim (x', y', z')$ while $(u_y(x), v_y(z))$ is in the interior of $I(y)$ and $\left(u_{y'}(x'), v_{y'}(z')\right)$ is in the interior of $I(y')$.

Because $\succsim$ is known to be a continuous weak order on all of A, it can be represented by a continuous function $W : A \to \mathbb{R}$ (see Debreu [5]). Restricting attention to A_y for any $y \in Y$, W is an increasing monotone transformation of $u_y + v_y$. The function W will allow us to simplify the notation in some of the following arguments, though it doesn't serve any particular role and, clearly, anything stated in the language of W can also be stated in the language of $\succsim$.

Observe that, for any $y \in Y$, $W(A_y)$ is a (potentially infinite) interval in $\mathbb{R}$ with a nonempty interior (due to A4). Moreover, if $y, y' \in Y$ are close, then $W(A_y) \cap W\left(A_{y'}\right)$ is also a (potentially infinite) interval in $\mathbb{R}$ with a nonempty interior.

Lemma 9. *Assume that $y, y' \in Y$ are close. Then there are unique $\alpha_{y'} > 0$ and $\beta_{y'} \in \mathbb{R}$ such that, by defining*

$$u(x,y) = u_y(x); \quad u(x,y') = \alpha_{y'} u_{y'}(x)$$

and

$$v(y,z) = v_y(z); \quad v(y',z) = \alpha_{y'} v_{y'}(z) + \beta_{y'}$$

we obtain u and v such that $u(x,y) + v(y,z)$ represents $\succsim$ on $A_y \cup A_{y'}$.

Thus, the lemma states that we can fix the arbitrarily chosen u_y, v_y for one value, y, and choose an affine positive transformation of the functions for the other, y', and thus obtain a function that represents preferences not only within each subspace $A_y, A_{y'}$ but also across them.

Proof: Let $y, y' \in Y$ be close. Denote by I° the interior of $W(A_y) \cap W(A_{y'})$ which is known to be a nonempty (potentially infinite) interval in $\mathbb{R}$. Clearly, for any $\alpha_{y'} > 0$ and $\beta^u_{y'}, \beta^v_{y'} \in \mathbb{R}$

$$u(x,y) = u_y(x); \quad u(x,y') = \alpha_{y'} u_{y'}(x) + \beta^u_{y'}$$

and

$$v(y,z) = v_y(z); \quad v(y',z) = \alpha_{y'} v_{y'}(z) + \beta^v_{y'}$$

would result in a function $u(x,y) + v(y,z)$ that represents $\succsim$ on A_y as well as on $A_{y'}$. We need to make sure that such a function correctly represents $\succsim$ between $a \in A_y$ and $b \in A_{y'}$. To this end, we first focus on the W-inverse images of I°, that is on

$$\begin{aligned} \hat{A}_y &\equiv \{ a \in A_y \mid W(a) \in I^\circ \} \\ \hat{A}_{y'} &\equiv \{ b \in A_{y'} \mid W(b) \in I^\circ \}. \end{aligned}$$

Consider a value $\xi = u_{y'}(x') + v_{y'}(z') \in \mathbb{R}$ for some $(x', y', z') \in \hat{A}_{y'}$. We claim that there exists a unique $\eta \in \mathbb{R}$ such that, for any $(x,y,z) \in \hat{A}_y$, $(x,y,z) \sim (x',y',z')$ if and only if $u_y(x) + v_y(z) = \eta$. Indeed, this follows from the fact that $u_y(x) + v_y(z)$ represents $\succsim$ on A_y, $u_{y'}(x') + v_{y'}(z')$ – on $A_{y'}$, and from transitivity. Hence there exists a function $g : \mathbb{R} \to \mathbb{R}$ such that $(x,y,z) \sim (x',y',z')$ if and only if

$$u_y(x) + v_y(z) = g\left(u_{y'}(x') + v_{y'}(z')\right).$$

Further, by transitivity, g is increasing (and strictly increasing in the relevant domain) and

$$(x,y,z) \succsim (x',y',z')$$

iff

$$u_y(x) + v_y(z) \geq g\left(u_{y'}\left(x'\right) + v_{y'}\left(z'\right)\right).$$

We wish to show that g is affine on the relevant domain, that is on $u_{y'}(x') + v_{y'}(z')$ where $W((x', y', z')) \in I^{\circ}$. To this end, consider three equally-spaced points in its domain,

$$\xi, \xi + \delta, \xi + 2\delta.$$

We wish to show that the values of g for these points are also equally spaced, that is, that

$$g(\xi + 2\delta) - g(\xi + \delta) = g(\xi + \delta) - g(\delta).$$

Choose $x_1, x_2 \in X$, $z_1, z_2 \in Z$ such that

$$\begin{aligned} u_{y'}(x_1) + v_{y'}(z_1) &= \xi \\ u_{y'}(x_2) + v_{y'}(z_1) &= \xi + \delta \\ u_{y'}(x_1) + v_{y'}(z_2) &= \xi + \delta. \end{aligned}$$

Note that such a selection is possible since there are points in $A_{y'}$ with $u_{y'}(x) + v_{y'}(z) = \xi + 2\delta$. (Note that the selection of x_1, x_2, z_1, z_2 should be done simultaneously: there may be points x_1, z_1 close to the boundary of $A_{y'}$ for which such an x_2 or such a z_2 will not exist.)

Similarly, denote

$$\eta = g(\xi)$$

$$\varepsilon = g(\xi + \delta) - g(\delta) > 0$$

and choose $x_3, x_4 \in X$, $z_3, z_4 \in Z$ such that

$$\begin{aligned} u_y(x_3) + v_y(z_3) &= \eta \\ u_y(x_4) + v_y(z_3) &= \eta + \varepsilon \\ u_y(x_3) + v_y(z_4) &= \eta + \varepsilon. \end{aligned}$$

Clearly, we have

$$\begin{aligned} \left(x_1, y', z_1\right) &\sim (x_3, y, z_3) \\ \left(x_2, y', z_1\right) &\sim (x_3, y, z_4) \\ \left(x_1, y', z_2\right) &\sim (x_4, y, z_3). \end{aligned}$$

By Indifference Cross Consistency, we also have

$$\left(x_2, y', z_2\right) \sim (x_4, y, z_4)$$

which implies

$$g(\xi + 2\delta) = \eta + 2\varepsilon.$$

Thus g is affine on the relevant domain, and there are $\alpha_{y'} > 0$ and $\beta_{y'} \in \mathbb{R}$ such that

$$\begin{aligned} u_y(x) + v_y(z) &= g\left(u_{y'}\left(x'\right) + v_{y'}\left(z'\right)\right) \\ &= \alpha_{y'} u_{y'}\left(x'\right) + \alpha_{y'} v_{y'}\left(z'\right) + \beta_{y'} \end{aligned}$$

whenever $(x, y, z) \sim (x', y', z')$.

Observe that in defining u and v we have some freedom in deciding how to split $\beta_{y'}$ between them. In fact, for any $\beta^u_{y'}, \beta^v_{y'} \in \mathbb{R}$ such that $\beta^u_{y'} + \beta^v_{y'} = \beta_{y'}$ we can define

$$u(x, y) = u_y(x); \quad u\left(x, y'\right) = \alpha_{y'} u_{y'}(x) + \beta^u_{y'}$$

and

$$v(y, z) = v_y(z); \quad v\left(y', z\right) = \alpha_{y'} v_{y'}(z) + \beta^v_{y'}$$

and observe that $u(x, y) + v(y, z)$ represents $\succsim$ for any $a \in \hat{A}_y$ and $b \in \hat{A}_{y'}$. However, to stick to the normalization by which $u(x_0, \cdot) = 0$, we choose $\beta^u_{y'} = 0$ (recall that $u_y(x_0) = u_{y'}(x_0) = 0$) and $\beta^v_{y'} = \beta_{y'}$.

Next consider $a \in A_y \backslash \hat{A}_y$. If there exists $b \in A_{y'}$ such that $a \sim b$ (which might be possible if a and or b are $\succsim$-maximal or $\succsim$-minimal in their sub-spaces, A_y or $A_{y'}$, respectively), the proof continues as above, via transitivity. We are therefore left with the case that $a \succ A_{y'}$ or $a \prec A_{y'}$ (using the obvious notation for a relation between an element and every element of a set). But in this case one can choose $c \in \hat{A}_y$ and complete the proof by transitivity. (For example, for $b \in \hat{A}_{y'}$ one chooses $c \sim b$ and argues that $a \succ c \sim b$ occurs when $u(x, y) + v(y, z)$ obtains a higher value for a than both c and b; otherwise we may have [$a \succ A_{y'}$ and $b \precsim A_y$] or [$a \prec A_{y'}$ and $b \succsim A$] etc.) ■

For $y \in Y$, let

$$C(y) = \left\{ y' \in Y \;\middle|\; y \text{ and } y' \text{ are close} \right\}$$

Lemma 10. *For every $y \in Y$ there exist*

$$\begin{aligned} u &: X \times C(y) \to \mathbb{R} \\ v &: C(y) \times Z \to \mathbb{R} \end{aligned}$$

such that $u(x, y) + v(y, z)$ represents $\succsim$ on $\bigcup_{y' \in C(y)} A_{y'}$.

This lemma states that we can have the desired representation not only for every pair of subspaces $A_y, A_{y'}$ where y' is close to y, but also for all of these simultaneously (holding y fixed).

Proof: Let there be given $y \in Y$. For every $y' \in C(y)$ define $u(x, y') + v(y', z)$ as in the Lemma 9.

Consider $a, b \in \bigcup_{y' \in C(y)} A_{y'}$. If $a, b \in A_{y'}$ for some y' the proof is complete. This is also the case if one of them is in A_y. We are therefore left with the case that

$$a \in A_{y'} \backslash A_y \quad b \in A_{y''} \backslash A_y.$$

In this case we know that both a and b are either "above" all of A_y or "below" it. (That is, $a \succ A_y$ or $a \prec A_y$ and the same is true of b.) In case $a \succ A_y \succ b$ or $b \succ A_y \succ a$, transitivity completes the proof. Hence, we are interested in the case $a, b \succ A_y$ or, symmetrically, $a, b \prec A_y$. Without loss of generality assume that $a, b \succ A_y$. Since y' and y'' are both close to y and they both contain alternatives that are better than A_y, they have to be close to each other. In fact, there has to be a nonempty interior of

$$W(A_y) \cap W\left(A_{y'}\right) \cap W\left(A_{y''}\right).$$

Consider two real number in this interior, $\alpha < \beta$, and six alternatives $c, c', d, d', e.e'$ such that $c, c' \in A_y$, $d, d' \in A_{y'}$, $e, e' \in A_{y''}$ and $W(c) = W(d) = W(e) = \alpha$ and $W(c') = W(d') = W(e') = \beta$. By Lemma 9, $u(\cdot, y')$ and $v(y', \cdot)$ are affine transformations of $u_{y'}(\cdot)$ and $v_{y'}(\cdot)$, respectively, and $u(\cdot, y'')$ and $v(y'', \cdot)$ are affine transformations of $u_{y''}(\cdot)$ and $v_{y''}(\cdot)$. However, the equalities above imply that, if we start with $u(\cdot, y')$ and $v(y', \cdot)$ and use Lemma 9 for y' and y'', we will end up with $u(\cdot, y'')$ and $v(y'', \cdot)$. Hence, $u(x, y) + v(y, z)$ represent preferences also on $A_{y'} \cup A_{y''}$ and correctly rank a and b. ■

Lemma 11. *Let there be given $y_1, y_2, ..., y_n \in Y$ such that y_i is close to y_{i+1} for $i = 1, ..., n-1$. Let*

$$C = \bigcup_{i \leq n} C(y).$$

There exist

$$\begin{aligned} u &: X \times C \to \mathbb{R} \\ v &: C \times Z \to \mathbb{R} \end{aligned}$$

such that $u(x, y) + v(y, z)$ represents $\succsim$ on $\bigcup_{y' \in C} A_{y'}$.

Proof: The proof is by induction on n, with the case $n = 1$ established by Lemma 10. Assume, then, that the claim is true for n and consider $n+1$.

Fix u and v as provided for $y_1, y_2, ..., y_n$. Applying Lemma 10 to y_{n+1}, there are u' and v' defined on $C(y_{n+1})$ that represent $\succsim$ (by their sum) over all of $C(y_{n+1})$. The latter includes a nonempty W-value intersection with A_{y_n}, because y_{n+1} and y_n are close. This means that we can use an affine transformation of u' and v' that would be identical to u and v, respectively, over their intersection.

Clearly, the newly-extended u and v will represent preferences over $C(y_{n+1})$. To see that they do so for all of $\bigcup_{y' \in C} A_{y'}$ we use transitivity as before.$\square$

We are finally ready to complete the proof. We argue that there exists a double-sequence

$$..., y_{-2}, y_{-1}, y_0, y_1, y_2, ...$$

such that (i) y_i and y_{i+1} are close for $i \in \mathbb{Z}$; (ii) $Y = \bigcup_{i \in \mathbb{Z}} C(y)$.

To see that this is the case, use the range of the function W as follows: first, consider a bounded interval $[-M, M]$ in the range of W. The interior of $W(A_y)$ for all $y \in Y$ is an open cover of $[-M, M]$, and thus has a finite subcover. From such a subcover one can choose a finite sequence $y_1, y_2, ..., y_n \in Y$ such that y_i is close to y_{i+1} for $i = 1, ..., n-1$ and that $[-M, M] \subset \bigcup_{i \leq n} W(A_y)$. Then, by induction on M one generated the sequence $..., y_{-2}, y_{-1}, y_0, y_1, y_2, ...$ such that (i) y_i and y_{i+1} are close for $i \in \mathbb{Z}$; (ii) $W(A) = \bigcup_{i \in \mathbb{Z}} W(A_y)$. Finally, one considers Lemma 11 and notes that in its inductive proof the functions u and v are defined as extensions of the same functions in previous steps. Repeating this argument for the doubly-infinite sequence completes the proof of existence of u and v. $\blacksquare$

7.4.3 Sufficiency – Part II: Continuity

We now turn to prove that the functions constructed above are continuous. Observe that for this to be true, one has to rely on the specific construction by which $u(x_0, y) = 0$, which guarantees that $u(x_0, \cdot)$ is continuous in y. Indeed, it is easy to see that by defining

$$\begin{aligned} u'(x,y) &= \alpha u(x,y) + \beta(y) \\ v'(y,z) &= \alpha v(y,z) - \beta(y) \end{aligned}$$

for a discontinuous $\beta(\cdot)$, one can represent $\succsim$ by $u'(x,y) + v'(y,z)$ where neither u' nor v' are continuous, though their sum is.

Step 1: Continuity of $u+v$: It is convenient to rely on the continuous function W that represents $\succsim$. Since $u(x,y)+v(y,z)$ and W both represent $\succsim$, there exists a monotonically increasing $\phi:\mathbb{R}\to\mathbb{R}$ such that

$$u(x,y)+v(y,z)=\phi(W(x,y,z))$$

for all $(x,y,z)\in X\times Y\times Z$. We claim that ϕ is continuous. Assume that it isn't, and that there exists $\xi_n\to\xi$ and $\varepsilon>0$ such that

$$\phi(\xi_n)<\phi(\xi)-\varepsilon \tag{14}$$

or,

$$\phi(\xi_n)>\phi(\xi)+\varepsilon. \tag{15}$$

Consider first case (14). As ξ is in the domain of ϕ, it is in the range of W and thus $\xi\in W(A_y)$ for some y's. For each one of these, it has to be the case that $\xi=\min W(A_y)$. Otherwise, we can find $a_n,a\in A_y$ such that $a_n\to a$ but $\phi(W(a_n))$ fails to converge to $\phi(W(a))$, which is impossible because $u(x,y)+v(y,z)$ is continuous on each A_y separately.

As the range of W is connected, and it is the union of open intervals $\{W(A_y)\}_y$, it follows that

$$\xi=\min\cup_y W(A_y)$$

in which case (14) is impossible.

Similarly, in case (15) we conclude that $\xi=\max\cup_y W(A_y)$, and a contradiction results again. We therefore conclude that $u(x,y)+v(y,z)=\phi(W(x,y,z))$ is a continuous function on $A=X\times Y\times Z$.

Step 2: Continuity of u,v: We now wish to show that u is continuous on $X\times Y$. Clearly, this will mean that u is continuous on $X\times Y\times Z$ and therefore that so is

$$v(y,z)=\phi(W(x,y,z))-u(x,y).$$

To this end we show that $u(x,y)$ is a continuous function of y, and that it is uniformly continuous with respect to x:

Lemma 12. *Let there be given $\tilde{x}\in X$ and $\tilde{y}\in Y$. For every $\varepsilon>0$ there exists $\delta>0$ such that for all $x\in X$ and $y\in Y$, if*

$$|x-\tilde{x}|,|y-\tilde{y}|<\delta$$

then

$$|u(x,y)-u(x,\tilde{y})|<\varepsilon.$$

Proof: Assume not. Then there are $\tilde{x} \in X$, $\tilde{y} \in Y$ and $\varepsilon > 0$ such that $\forall \delta > 0$ there are x, y that are δ-close to $\tilde{x}, \tilde{y}$, respectively, but that the converse inequality holds. We can therefore choose a sequence $\{(x_n, y_n)\}$ such that $(x_n, y_n) \to (\tilde{x}, \tilde{y})$ as $n \to 0$ but, for every n,

$$|u(x_n, y_n) - u(x_n, \tilde{y})| \geq \varepsilon. \tag{16}$$

Choose $\tilde{z} \in Z$. As $f(x, y, z) = u(x, y) + v(y, z)$ is continuous, there exists N such that for all $n \geq N$ we have

$$|f(x_n, y_n, \tilde{z}) - f(\tilde{x}, \tilde{y}, \tilde{z})| < \varepsilon/10. \tag{17}$$

Also, as $u(\cdot, y)$ is continuous for every y, $u(x_n, \tilde{y}) \to u(\tilde{x}, \tilde{y})$ as $n \to 0$ and we can assume that for all $n \geq N$ we also have

$$|u(x_n, \tilde{y}) \to u(\tilde{x}, \tilde{y})| < \varepsilon/10. \tag{18}$$

Consider

$$\begin{aligned} & |f(x_n, y_n, \tilde{z}) - f(\tilde{x}, \tilde{y}, \tilde{z})| \\ = \ & |u(x_n, y_n) + v(y_n, \tilde{z}) - u(\tilde{x}, \tilde{y}) - v(\tilde{y}, \tilde{z})| \\ = \ & |[u(x_n, y_n) - u(x_n, \tilde{y})] + [u(x_n, \tilde{y}) - u(\tilde{x}, \tilde{y})] + [v(y_n, \tilde{z}) - v(\tilde{y}, \tilde{z})]| . \end{aligned}$$

By (18) we know that the middle square brackets denote a small expression, as does the sum of the three square brackets (by (17)). However, the first square brackets is at least ε by (16). This means that the last expression should also be relatively large. Specifically, for every n it follows that

$$|v(y_n, \tilde{z}) - v(\tilde{y}, \tilde{z})| \geq \varepsilon/2. \tag{19}$$

Consider now the sequence $\{(x_0, y_n, \tilde{z})\}_n$ and observe that $(x_0, y_n, \tilde{z}) \to (x_0, \tilde{y}, \tilde{z})$ as $n \to 0$. By continuity of f we should have

$$u(x_0, y_n) + v(y_n, \tilde{z}) \to u(x_0, \tilde{y}) + v(\tilde{y}, \tilde{z}).$$

However, $u(x_0, y_n) = u(x_0, \tilde{y}) = 0$[12] while (19) shows that $v(y_n, \tilde{z})$ fails to converge to $v(\tilde{y}, \tilde{z})$, which is a contradiction.□

We finally show that u is continuous. Assume that $(x_n, y_n) \to (x, y)$ as $n \to 0$ for some point $(x, y) \in X \times Y$. Writing

$$\begin{aligned} & u(x, y) - u(x_n, y_n) \\ = \ & [u(x, y) - u(x_n, y)] + [u(x_n, y) - u(x_n, y_n)], \end{aligned}$$

[12] Observe that the crucial fact is only that $u(x_0, \cdot)$ is continuous, while in our construction it was guaranteed to be constant.

we observe that the first brackets converges to 0 because $u(\cdot, y)$ is continuous (for each y separately) and the second one converges to 0 because of Lemma 12. ■

7.4.4 Uniqueness

Given a representation by u, v, it is straightforward that

$$\begin{aligned} u'(x,y) &= \alpha u(x,y) + \beta(y) \\ v'(y,z) &= \alpha v(y,z) - \beta(y) + \gamma \end{aligned} \tag{20}$$

also represent preferences for every $\alpha > 0$, a continuous function $\beta : Y \to \mathbb{R}$ and $\gamma \in \mathbb{R}$.

Conversely, the construction of the functions showed that, given the normalization $u(x_0, \cdot) = 0$, the only degrees of freedom left are multiplication of both u and v by a positive constant ad shifting of v by an additive one.

However, relaxing the constraint $u(x_0, \cdot) = 0$, one can replace it by any continuous function β so that $u(x_0, \cdot) = \beta(y)$. Conversely, denoting $\beta(y) = u(x_0, \cdot)$ one observes that the transformation (20) holds.

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