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PIER Working Paper 01-047

"Computing Sunspots in Linear Rational Expectations Models"

by

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Computing Sunspots in Linear Rational Expectations Models*

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October 2001

*We thank Ellen McGrattan, Stephanie Schmitt-Grohé and Martin Uribe for helpful discussions. This paper was completed while the first author was visiting the Research Department of the Federal Reserve Bank of Minneapolis, for whose hospitality he is grateful. The second author gratefully acknowledges financial support from the University Research Foundation of the University of Pennsylvania. A GAUSS program that implements the solution algorithm is available at *http://www.econ.upenn.edu/˜schorf*.

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Abstract

We provide a computationally simple method of including and analyzing the effects of sunspot shocks in linear rational expectations models when the equilibrium is indeterminate. Under non-uniqueness sunspots can affect model dynamics through endogenous forecast errors that do not completely adjust to fundamental shocks alone. We show that sunspot shocks can be modeled as exogenous belief shocks which can be included in the set of fundamentals. By means of a simple example we illustrate that the exact specification of the transmission mechanism of the belief shocks is irrelevant for the solution of the model.

JEL CLASSIFICATION: C62, C63, E00

KEY WORDS: Sunspots, Indeterminacy, Rational Expectations, Computational Methods

1 Introduction

It is well known that linear rational expectations (RE) models can have multiple equilibria. If the equilibrium is not unique it is possible to construct sunspot equilibria in which stochastic disturbances that are unrelated to fundamental shocks influence model dynamics. With indeterminacy the realization of a sunspot variable affects economic agents' beliefs. In response to these shocks, agents adjust their behavior which induces fluctuations that would not be present in a unique RE equilibrium.

Recently, researchers have become interested in indeterminacy in dynamic stochastic general equilibrium (DSGE) models. Most of the current literature, however, is concerned with identifying the structural determinants of indeterminacy[1], which can to a large extent be analyzed in a deterministic environment. Yet, researchers may also be interested in the stochastic properties of sunspot equilibria. For this purpose, it is necessary to characterize how sunspot shocks enter the RE model and how they affect the dynamics of the system.

In this paper, we provide a simple method of computing sunspot equilibria. We show that an arbitrary vector of belief shocks can be added to the vector of fundamental shocks and that the solution method we use automatically selects the correct dimension of indeterminacy. In this framework, belief shocks can be interpreted as non-systematic shocks to endogenous forecast errors. In analytically constructing sunspot equilibria it is often logically convenient to assume that agents change their forecast of a specific variable in response to a sunspot. In this paper we show that the choice of a specific variable is irrelevant.

Since indeterminacy arises when the mapping from endogenous forecast errors to the unstable components of the linear DSGE model is not unique, sunspot shocks help to solve for those forecast errors that cannot be expressed as functions of

[1]Notable exception include Schmitt-Grohé [8], Clarida, Gali, and Gertler [3] and Lubik [6] who study the effects of sunspot shocks on economic dynamics. We formally show in this paper that the heuristic approach taken by these authors turns out to be correct. For a comprehensive survey of the indeterminacy literature see Benhabib and Farmer [1].

fundamental shocks alone. We show that this approach is equivalent to treating belief shocks as part of the underlying stochastic structure.

2 General Form of the Model

We are interested in linear rational expectations models derived from underlying dynamic optimization problems, such as consumers' utility maximization and firms' profit maximization programs. The resulting system of equations typically consists of dynamic and static first-order conditions, budget constraints and laws of motions for exogenous driving processes. Endogenous variables x_t appearing in expectations are handled in the following way: We introduce endogenous forecast errors η by defining $\xi_{t-1} = \mathbb{E}_{t-1}[x_t]$, so that $x_t = \xi_{t-1} + \eta_t$ which is then added to the system. These forecast errors have the property $\mathbb{E}_{t-1}[\eta_t] = 0$, where $\mathbb{E}_{t-1}[.]$ denotes the expectation conditional on time $t-1$ information. This convention will prove to be convenient later on.

We consider the canonical linear rational expectations model of the form:

$$\Gamma_0 y_t = \Gamma_1 y_{t-1} + \Psi z_t + \Pi \eta_t, \tag{1}$$

where y_t is an $n \times 1$ vector of endogenous variables, z_t is a $l \times 1$ vector of exogenous, serially uncorrelated random disturbances[2], and η_t is a $k \times 1$ vector of expectation errors, satisfying $\mathbb{E}_{t-1}[\eta_t] = 0$ for all t. Sims [9] develops a solution algorithm for (1) subject to restrictions on the rate of growth of y_t.[3] Moreover, he derives conditions that ensure the existence and uniqueness of a stable solution. Roughly, a stable solution exists, if one can choose the expectation errors η_t as a function of the exogenous shock z_t to eliminate explosive components of y_t. The solution is unique if the mapping from z_t to η_t is one-to-one. In the case of non-uniqueness, his algorithm provides one particular solution for the RE system. Sims [9, p.13] mentions that "[i]f

[2] This assumption is not very restrictive. The laws of motion for serially correlated disturbances can always be appended to the system and the vector y_t expanded.

[3] Sims' solution method generalizes the method proposed by Blanchard and Kahn [2].

one is interested in generating the full set of non-unique solutions, one has to add back in, as additional "disturbances" the components of $[\eta_t]$ left undetermined".

In the following section we show how to derive the full set of solutions of the linear RE model in the presence of indeterminacy. Section 4 presents a computationally simpler approach of adding sunspot noise in the form of belief shocks to the system and demonstrates that it generates the same set of solutions as in Section 3. Section 5 provides a simple example, while the last Section concludes.

3 Sunspot Solutions

The system can be transformed through a generalized complex Schur decomposition (QZ) of Γ_0 and Γ_1. There exist matrices Q, Z, Λ, and Ω, such that $Q'\Lambda Z' = \Gamma_0$, $Q'\Omega Z' = \Gamma_1$, $Q'Q = Z'Z = I$, and Λ and Ω are uppertriangular. Let $w_t = Z'y_t$ and premultiply (1) by Q to obtain:

$$\begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ 0 & \Lambda_{22} \end{bmatrix} \begin{bmatrix} w_{1,t} \\ w_{2,t} \end{bmatrix} = \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ 0 & \Omega_{22} \end{bmatrix} \begin{bmatrix} w_{1,t-1} \\ w_{2,t-1} \end{bmatrix} + \begin{bmatrix} Q_{1.} \\ Q_{2.} \end{bmatrix} (\Psi z_t + \Pi \eta_t) \quad (2)$$

The second set of equations can be rewritten as:

$$w_{2,t} = \Lambda_{22}^{-1}\Omega_{22}w_{2,t-1} + \Lambda_{22}^{-1}Q_{2.}(\Psi z_t + \Pi\eta_t) \quad (3)$$

Without loss of generality, we assume that the system is ordered and partitioned such that the $m \times 1$ vector $w_{2,t}$ is purely explosive.

A non-explosive solution of the linear RE model (1) for y_t exists if $w_{2,0} = 0$ and for every vector z_t, one can find a vector η_t that offsets the impact of z_t on $w_{2,t}$:

$$\underbrace{Q_{2.}\Psi}_{m\times l}\underbrace{z_t}_{l\times 1} + \underbrace{\Pi Q_{2.}}_{m\times k}\underbrace{\eta_t}_{k\times 1} = \underbrace{0}_{m\times 1}. \quad (4)$$

The vector η_t, however, need not be unique. For instance, if the number of expectation errors l exceeds the number of explosive components m, Eq. (4) does not provide enough restrictions to uniquely determine the elements of η_t. Hence, it is possible to introduce expectation errors that are unrelated to the fundamental uncertainty z_t without destabilizing the system. For the remainder of the paper we will make the following assumption.

Assumption 1 *The linear RE model (1) has at least one non-explosive solution.*

In this section we will characterize the set of expectation errors η_t that are consistent with the stability condition in Equation (4). The following Lemma will subsequently be useful.

Lemma 1 *Statements (i) and (ii) are equivalent.*

(i) For every $z_t \in I\!R^l$ there exists an $\eta_t \in I\!R^k$ such that Equation (4) is satisfied.

(ii) There exists a (real) $k \times l$ matrix λ such that $Q_{2.}\Psi = Q_{2.}\Pi\lambda$.

Lemma 1 simply states that a stable solution exists whenever it is possible to express the columns of $Q_{2.}\Psi$ as a linear combination of the columns of $Q_{2.}\Pi$.[4] The proof is straightforward. Statement (ii) implies (i), because one can choose $\eta_t = -\lambda z_t$. To show the converse, let the j'th column of λ be equal to an η_t that solves Eq. (4) for $z_t = I_{.j}$, where $I_{.j}$ is the j'th column of the $l \times l$ identity matrix, $j = 1, \ldots, l$.

Since the rows of the matrix $Q_{2.}\Pi$ are potentially linearly dependent it is convenient to work with its singular value decomposition:[5]

$$Q_{2.}\Pi = \begin{bmatrix} U_{.1} & U_{.2} \end{bmatrix} \begin{bmatrix} D_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_{.1}' \\ V_{.2}' \end{bmatrix} = \underbrace{U}_{m\times m} \underbrace{D}_{m\times k} \underbrace{V'}_{k\times k} = \underbrace{U_{.1}}_{m\times r} \underbrace{D_{11}}_{r\times r} \underbrace{V_{.1}'}_{r\times k}, \quad (5)$$

where D_{11} is a diagonal matrix and U and V are orthonormal matrices. Thus, the m explosive components of y_t generate only $r \leq m$ restrictions for the expectation errors η_t.

[4] This insight has been used by Sims [9] to check for the existence of a stable equilibrium.

[5] Singular value decomposition procedures implemented in matrix-oriented software packages such as GAUSS and MATLAB usually return the matrices U, D, and V.

We now exploit the orthonormality of V and express the expectation error η_t as the sum of two orthogonal vectors. The first vector lies in the space spanned by the columns of $V_{.1}$ and the second lies in the column space of $V_{.2}$.

$$\eta_t = (V_{.1}V_{.1}' + V_{.2}V_{.2}')\eta_t = V_{.1}\tilde{\eta}_{1,t} + V_{.2}\tilde{\eta}_{2,t}, \tag{6}$$

where $\tilde{\eta}_{1,t}$ is an $r \times 1$ vector and $\tilde{\eta}_{2,t}$ is $(k-r) \times 1$. The impact of η_t on the unstable component of the linear RE model is given by

$$Q_{2.}\Pi\eta_t = U_{.1}D_{11}V_{.1}'(V_{.1}\tilde{\eta}_{1,t} + V_{.2}\tilde{\eta}_{2,t}) = U_{.1}D_{11}\tilde{\eta}_{1,t}. \tag{7}$$

The second part of the expectation error, $V_{.2}\tilde{\eta}_{2,t}$, does not affect the stability of the linear RE model. If the dimension k of η_t exceeds the number of stability restrictions r, the matrix $V_{.2}$ is nonempty and one can introduce expectations errors $V_{.2}\tilde{\eta}_{2,t}$ that are unrelated to the fundamental shocks z_t without causing an instability.

To offset the effect of the fundamental shocks z_t on the unstable component $w_{2,t}$ of the endogenous variables, $\tilde{\eta}_{1,t}$ has to satisfy

$$Q_{2.}\Psi z_t + Q_{2.}\Pi V_{.1}\tilde{\eta}_{1,t} = 0. \tag{8}$$

According to Lemma 1 there exists a $k \times l$ matrix λ such that the previous equation can be rewritten as

$$\underbrace{U_{.1}D_{11}}_{m\times r}\left(\underbrace{V_{.1}'\lambda z_t}_{r\times 1} + \underbrace{\tilde{\eta}_{1,t}}_{r\times 1}\right) = \underbrace{0}_{m\times 1} \quad . \tag{9}$$

Thus, the solution for $\tilde{\eta}_{1,t}$ is of the form

$$\tilde{\eta}_{1,t} = -V_{.1}'\lambda z_t. \tag{10}$$

Since U is orthonormal and $U_{.1}'U_{.1} = I_{r\times r}$, it is straightforward to verify that $V_{.1}'\lambda$ is uniquely determined by Condition (ii) of Lemma 1:

$$V_{.1}'\lambda = D_{11}^{-1}U_{.1}'Q_{2.}\Psi. \tag{11}$$

The final result is summarized in the following proposition.

Proposition 1 *The full set of solutions to Eq. (4) is*

$$\eta_t = -V_{.1} D_{11}^{-1} U_{.1}' Q_{2.} \Psi z_t + V_{.2} \tilde{\eta}_{2,t}, \tag{12}$$

where $\tilde{\eta}_{2,t} \in I\!R^{k-r}$. If $k = r$ the second term drops out and the solution is unique.

The solution for the expectation errors η_t can be substituted into the linear RE model in Eq. (2) to express y_t as a function of y_{t-1}, the fundamental shocks z_t, and $\tilde{\epsilon}_{2,t}$. In the context of the model we will refer to $\tilde{\eta}_{2,t}$ as normalized reduced form sunspot shocks.[6] The contemporaneous impact of the sunspot shocks onto the transformed endogenous variables is given by

$$\frac{\partial(\Lambda_{11} w_{1,t} + \Lambda_{12} w_{2,t})}{\partial \tilde{\eta}_{1,t}} = \underbrace{Q_{1.} \Pi V_{.2}}_{(n-m)\times(k-r)} \tag{13}$$

The columns of $Q_{1.} \Pi V_{.2}$ contain the directions in which which sunspot shocks can perturb the linear combination of endogenous variables $(\Lambda_{11} w_{1,t} + \Lambda_{12} w_{2,t})$.

The solution method automatically picks the right number of reduced form sunspot shocks that complement the fundamental disturbances in a model where the RE solution is indeterminate. One conceptual difficulty with this approach, however, is how to interpret the additional error term in an economically meaningful way. In the next Section, we introduce the notion of a belief shock and argue that the full set of sunspot solutions can be computed with existing algorithms, such as Sims [9], if the vector of fundamental shocks z_t is augmented by the belief shocks.

4 A Computationally Simple Approach to Sunspots

Suppose that the system is perturbed by a random vector ζ_t in addition to the disturbances z_t and the expectation errors η_t.

$$\Gamma_0 y_t = \Gamma_1 y_{t-1} + \Psi z_t + \Pi(\eta_t + \zeta_t) = \Gamma_1 y_{t-1} + \left[\begin{array}{cc} \Psi & \Pi \end{array}\right] \left[\begin{array}{c} z_t \\ \zeta_t \end{array}\right] + \Pi \eta_t. \tag{14}$$

[6] These sunspot shocks are not uniquely defined. For instance, let M be a $(k-r) \times (k-r)$ orthonormal matrix. We could replace $V_{.2}\tilde{\eta}_{2,t}$ in Equation (12) with $V_{.2}^{*}\tilde{\eta}_{2,t}^{*}$, where $V_{.2}^{*} = V_{.2}M'$ and $\tilde{\eta}_{2,t}^{*} = M\tilde{\eta}_{2,t}$.

We assume that $\mathbb{E}_{t-1}[\zeta_t] = 0$. The vector ζ_t is of the same dimension as η_t and will be interpreted as non-systematic shocks to the endogenous forecasting errors that are not related to the fundamental sources of randomness z_t. We will refer to ζ_t as belief shock. Subsequently, these belief shocks will be treated in the same way as the fundamental shocks z_t. The stability condition (4) changes to:

$$\left[\begin{array}{cc} Q_{2.}\Psi & Q_{2.}\Pi \end{array}\right] \left[\begin{array}{c} z_t \\ \zeta_t \end{array}\right] + Q_{2.}\Pi\eta_t = 0. \tag{15}$$

Proposition 1 implies that one solution of Eq. (15) is

$$\begin{aligned} \eta_t &= -V_{.1}D_{11}^{-1}U_{.1}'\left[\begin{array}{cc} Q_{2.}\Psi & Q_{2.}\Pi \end{array}\right] \left[\begin{array}{c} z_t \\ \zeta_t \end{array}\right] \\ &= -V_{.1}D_{11}^{-1}U_{.1}'Q_{2.}\Psi z_t - V_{.1}V_{.1}'\zeta_t. \end{aligned} \tag{16}$$

Now consider the effect of the endogenous expectation errors η_t onto $(\Lambda_{11}w_{1,t} + \Lambda_{12}w_{2,t})$ in the transformed system (2):

$$\begin{aligned} Q_{1.}\Pi\eta_t &= Q_{1.}\Pi V_{.1}V_{.1}'\eta_t + Q_{1.}\Pi(I - V_{.1}V_{.1}')\eta_t \\ &= -Q_{1.}\Pi V_{.1}V_{.1}'V_{.1}D_{11}^{-1}\lambda_* z_t - Q_{1.}\Pi(I - V_{.1}V_{.1}')V_{.1}D_{11}^{-1}\lambda_* z_t \\ &\quad -Q_{1.}\Pi V_{.1}V_{.1}'V_{.1}V_{.1}'\zeta_t - Q_{1.}\Pi(I - V_{.1}V_{.1}')V_{.1}V_{.1}'\zeta_t \\ &= -Q_{1.}\Pi V_{.1}D_{11}^{-1}\lambda_* z_t - Q_{1.}\Pi V_{.1}V_{.1}'\zeta_t. \end{aligned} \tag{17}$$

The overall contemporaneous impact of ζ_t is

$$\frac{\partial(\Lambda_{11}w_{1,t} + \Lambda_{12}w_{2,t})}{\partial\zeta_t} = Q_{1.}\Pi - Q_{1.}\Pi V_{.1}V_{.1}' = \underbrace{Q_{1.}\Pi V_{.2}V_{.2}'}_{(n-m)\times k}, \tag{18}$$

since orthonormality of V implies that $V_{.1}V_{.1}' + V_{.2}V_{.2} = I_{k\times k}$.

The transformed belief shock $\zeta_t^* = V_{.2}'\zeta_t$ is equivalent to the reduced form sunspot shock $\tilde{\eta}_{2,t}$ in Section 3. As previously, $\Lambda_{11}w_{1,t}+\Lambda_{12}w_{2,t}$ is perturbed in the directions of the columns of $Q_{1.}\Pi V_{.2}$. However, since the dimension k of ζ_t is usually greater than the dimension $k-r$ of $\tilde{\eta}_{2,t}$, there is more than one realization ζ_t of the belief shock that corresponds to a particular $\tilde{\eta}_{2,t}$. Such realizations generate the same equilibrium dynamics. If the solution to Eq. (4) is unique, then $V_{.2}$ is empty and the addition of the forecast-error shocks ζ_t does not affect the RE equilibrium dynamics of the model.

Overall we conclude that augmenting the fundamental shocks by belief shocks generates the full set of stable solutions as characterized in Section 3. Algorithms such as the one described in Sims [9] can be used without modification to compute these solutions.

5 A Simple Example

We illustrate our approach by means of a simple example taken from Lubik and Marzo [7] to which we refer the reader for further details. We consider a reduced-form model of an economy with monopolistically competitive firms that face adjustment costs when changing prices. Goods are produced with variable labor input only. Agents can smooth their consumption stream by purchasing government bonds that pay a nominal rate of interest. The central bank affects the economy's equilibrium by means of an interest rate policy.

The linearized version of such an economy is described by the following equations:

$$\mathbb{E}_t \left[\widetilde{Y}_{t+1}\right] + \sigma \mathbb{E}_t \left[\widetilde{\pi}_{t+1}\right] = \widetilde{Y}_t + \sigma \widetilde{R}_t, \tag{19}$$

$$\beta \mathbb{E}_t \left[\widetilde{\pi}_{t+1}\right] = \widetilde{\pi}_t - \kappa \widetilde{Y}_t, \tag{20}$$

$$\widetilde{R}_t = \psi \widetilde{\pi}_t + \varepsilon_t. \tag{21}$$

All variables are in log-deviations from a unique steady state, where $\widetilde{Y}$ is output (which is equal to consumption in equilibrium), $\widetilde{\pi}$ the inflation rate, and $\widetilde{R}$ the nominal interest rate. ε is a fundamental monetary policy shock. $\sigma > 0$, $\kappa > 0$, and $0 < \beta < 1$ are parameters. Eq. (19) is an intertemporal Euler-equation, while (20) governs inflation dynamics which is derived from firms' optimal price-setting problem. The monetary authority uses the rule (21) in adjusting the nominal interest rate in response to changes in its inflation target. $\psi \geq 0$ measures the elasticity of the interest rate response. Substituting (21) in (19) results in a two-equation system in $\widetilde{Y}$ and $\widetilde{\pi}$ only.

We now transform the reduced-form system (19) - (21) into the canonical form (1). Define $\xi_t^Y = I\!\!E_t\left[\widetilde{Y}_{t+1}\right]$, $\xi_t^\pi = I\!\!E_t\left[\widetilde{\pi}_{t+1}\right]$, and add the following two equations to the system:

$$\widetilde{Y}_t = \xi_{t-1}^Y + \eta_t^Y, \tag{22}$$

$$\widetilde{\pi}_t = \xi_{t-1}^\pi + \eta_t^\pi. \tag{23}$$

This transformation allows us to substitute out the variables appearing in expectations and to introduce the expectation errors η. The canonical system is defined for the vector of endogenous variables $y_t = \left[\widetilde{Y}_t, \widetilde{\pi}_t, \xi_t^Y, \xi_t^\pi\right]'$, fundamental shocks $z_t = [\varepsilon_t]$, and $\eta_t = \left[\eta_t^Y, \eta_t^\pi\right]'$. The dimensions or the vectors are $n = 4$, $k = 2$, and $l = 1$.

To examine the effects of sunspots we can proceed as in Section 4 by adding belief shocks ζ_t^Y, ζ_t^π, respectively, to the two definitional equations above, which expands the exogenous shock vector to $z_t = \left[\varepsilon_t, \zeta_t^Y, \zeta_t^\pi\right]$. It is well established that the RE equilibrium is unique if $\psi > 1$. In this case $r = k = 2$, $V_{.2}$ is the null matrix and any belief shock to output and inflation forecasts does not affect the equilibrium dynamics.

For $0 \leq \psi < 1$ there is a one-dimensional indeterminacy, that is, $r = 1$. Since $V_{.2} = [V_{12}, V_{22}]'$ is now a 2×1 vector sunspot shocks do play a role. It is remarkable, however, that the model solution is isomorphic to the exact location of the sunspot. According to the results in Section 4, the impacts of belief shocks to output and inflation are

$$Q_{1.}\Pi V_{.2} V_{12} \zeta_t^Y \quad \text{and} \quad Q_{1.}\Pi V_{.2} V_{22} \zeta_t^\pi,$$

respectively. Up to a normalization constant, the two shocks have the same effects on the equilibrium dynamics. Consequently, we can model sunspots as influencing beliefs about output, inflation, or both. While ex post it is not possible to infer which forecast was affected by a belief shock, the notion of a belief shock can help us to provide economic intuition about the equilibrium dynamics.

We now conduct the following thought experiment. Suppose that the economy is initially in a non-stochastic steady state. In period $t = 0$ agents observe a sunspot which leads them to believe that the current inflation rate is actually above its steady state value, $\widetilde{\pi}_0 > \pi$, and that adjustment dynamics are monotone thereafter, $\widetilde{\pi}_{t+1} < \widetilde{\pi}_t, t = 0, 1, ...$.[7] We proceed by arguing that this inflation path is not consistent with an RE equilibrium under the anti-inflationary policy $\psi > 1$.

The monetary authority responds to the inflationary belief by raising the nominal interest rate (see Eq. (21)), which in turn increases the expected real rate. From the Euler-equation (19) this implies positive output (and consumption) growth since agents desire to increase current savings. Although the inflation rate is falling towards the steady state, the anti-inflation policy keeps the real rate from declining so that output growth remains positive in future periods. Note that the inflation path implies that initial output $\widetilde{Y}_0$ increases (see Eq. (20)), which is inconsistent with current consumption declining to support an increase in savings. However, since $\widetilde{Y}_0 > Y$ and subsequent output growth is positive. this is obviously not an equilibrium path with steady-state adjustment dynamics. We can therefore rule out a sunspot equilibrium under this policy rule.

Now suppose that monetary policy is not aggressive enough, i.e. $\psi < 1$. Under an inflationary sunspot-belief the expected real rate declines and output growth is negative. The fall in the real rate stimulates current consumption and therefore output, $\widetilde{Y}_0 > Y$. From (20), this is consistent with positive current inflation, which validates the initial assumption of sunspot-driven positive inflation expectations. As inflation falls towards its steady state, the passive interest-rate policy keeps the real rate low, and output returns to the steady state.

Alternatively, we could assume that the sunspot variable influences output expectations so that agents are led to believe that $\widetilde{Y}_0 > Y$, and that adjustment dynamics are such that $\widetilde{Y}_{t+1} < \widetilde{Y}_t, t = 0, 1, ...$ Negative output growth implies a fall in the real rate which is consistent with a weak ($\psi < 1$) monetary policy response

[7] It can be shown that other adjustment dynamics are similarly inconsistent with an equilibrium. See Lubik and Marzo [7] for further details.

and smooth inflation adjustment dynamics. When policy has an anti-inflationary stance negative expected output growth can only be supported by an explosive inflation path that violates the initial assumption of $\widetilde{Y}_0 > Y$.[8]

We conclude that the dynamic effects of sunspot shocks can be analyzed both qualitatively and quantitatively as exogenous disturbances to endogenous forecast errors. Remarkably, the specification of the exact transmission channel of these disturbances is irrelevant as long as a sufficiently large set of fundamental belief shocks is included in the vector of exogenous disturbances. This procedure thus offers the researcher a simple way of studying uniqueness of RE equilibria and sunspot dynamics without explicit reference to the existence calculations made in this paper and elsewhere.

6 Conclusion

In this paper, we provide a convenient method for quantitative theorists to study the implications of sunspot shocks in stochastic general equilibrium models when the RE equilibrium is not unique. We show that sunspot disturbances can be modeled as exogenous shocks to endogenous forecast errors, which have a straightforward economic interpretation as belief shocks. Once the stochastic properties of the driving processes are specified, model statistics can be computed and impulse-response analysis conducted. Furthermore, this paper provides a rigorous justification for common thought experiments in the indeterminacy literature which try to highlight the internal logic of models without unique equilibrium.

This paper might also of interest to empirical researchers. When estimating dynamic stochastic general equilibrium models, it is typical practice[9] to restrict the parameter space to regions where indeterminacy does not occur. But since the possibility of indeterminacy is an integral feature of linear RE models, this practice could lead to serious model misspecification. Our approach demonstrates how to

[8] We can similarly analyze the effects of simultaneous belief shocks in output and inflation.

[9] For instance, Kim [5] or Ireland [4].

properly account for sunspot shocks in empirical models based on linearized DSGE models. Furthermore, this paper provides a framework for investigating the role of sunspot shocks in business cycle fluctuations.

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