Technical Supplement to
“Uniform Inference in Panel Autoregression”

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Abstract

This Supplement comprises six appendices. Appendix SA states results on the asymptotic properties of the Anderson-Hsiao IV estimator and of the pooled ordinary least squares (POLS) estimator and uses these results to provide motivation for the design of the AIP estimator discussed in section 2 of the main paper. Appendix SB gives proofs of the results stated in Appendix SA and also supplies a proof of Lemma A1 in the main paper. Appendix SC analyzes the asymptotic properties of the Anderson-Rubin statistic in the context of the panel autoregression. Appendix SD provides proofs of the key supporting lemmas used in the proofs of the theorems stated in the main paper and in Appendix SA. Additional lemmas and their proofs as well as additional technical details are given in Appendix SE. Finally, Appendix SF provides additional Monte Carlo results comparing the finite sample performance of the AIP estimator with other point estimators of the autoregressive parameter ρ.

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Appendix SA: Asymptotic Properties of the IV and POLS Estimators

In this part of the supplement, we discuss the asymptotic properties of the Anderson-Hsiao IV estimator

\[ \hat{\rho}_{\text{IVD}} = \left[ \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2} \Delta y_{it-1} \right]^{-1} \left[ \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2} \Delta y_{it} \right], \]

and the pooled ordinary least squares (POLS) estimator

\[ \hat{\rho}_{\text{pols}} = \left[ \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \bar{y}_{-1,N,T})^2 \right]^{-1} \left[ \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \bar{y}_{-1,N,T}) y_{it} \right], \]

where \( \bar{y}_{-1,N,T} = N^{-1} (T-1)^{-1} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-1} \). More specifically, we provide an extensive catalogue of the asymptotic behavior of these two estimators under different stable, near unit root, and unit root parameter sequences. The account is fairly comprehensive and subsumes much earlier work, including recent results in Phillips (2014) and Phillips and Han (2015).

Our primary purpose in this discussion is to provide motivation for the AIP (average of IV and POLS) estimator introduced in section 2 of the main paper. As can be seen from the results given below, the Anderson-Hsiao IV estimator performs well when the underlying process is stable, whereas the POLS estimator is the superior estimator when the autoregressive parameter is unity or very nearly unity. As explained in section 2 of our main paper, the design of the AIP estimator seeks to exploit the differential strengths of the IV estimator vis-à-vis the POLS estimator, by creating an estimator that behaves like POLS in the unit root and near unit root regions of the parameter space but which behaves like IV in regions of the parameter space further away from unity.

Formal statements of the results for Anderson-Hsiao IV estimation and POLS estimation are given in Theorems SA-1 and SA-2 below, with Theorem SA-1 providing the asymptotic properties of the former and Theorem SA-2 giving comparable results for the latter. The proofs for these results are provided in the next appendix, Appendix SB.

**Theorem SA-1:**

Let \( Z_1 \) and \( Z_2 \) be independent \( N(0,1) \) random variables. Under Assumptions 1-4, the following statements are true as \( N,T \to \infty \) such that \( N^\kappa / T \to \tau \), for \( \kappa \in (\frac{1}{2}, \infty) \) and \( \tau \in (0, \infty) \).

(a) Suppose that \( \rho_T = 1 \) for all \( T \) sufficiently large. Then,

\[ \sqrt{T} (\hat{\rho}_{\text{IVD}} - \rho_T) \Rightarrow \frac{2Z_1}{Z_2}. \]

(b) Suppose that \( \rho_T = \exp \{-1/q(T)\} \) such that \( T^{1+\frac{1}{\kappa}} \ll q(T) \). Then,

\[ \sqrt{T} (\hat{\rho}_{\text{IVD}} - \rho_T) \Rightarrow \frac{2Z_1}{Z_2}. \]
(c) Suppose that \( \rho_T = \exp\{-1/q(T)\} \) such that \( q(T) \sim T^{1+\frac{1}{2\kappa}} \). Then, without further assumption on the convergence of \( v(T) = T^{1+\frac{1}{2\kappa}}/q(T) \) as \( T \to \infty \),
\[
\sqrt{T} (\hat{\rho}_{IVD} - \rho_T) = O_p(1).
\]
If, in addition, \( T^{1+\frac{1}{2\kappa}}/q(T) \to \lambda \in (0, \infty) \), then
\[
\sqrt{T} (\hat{\rho}_{IVD} - \rho_T) \Rightarrow \frac{2Z_1}{Z_2 - \lambda \sqrt{\pi}/\sqrt{2}}.
\]

(d) Suppose that \( \rho_T = \exp\{-1/q(T)\} \) such that \( T \ll q(T) \ll T^{1+\frac{1}{2\kappa}} \). Then,
\[
\frac{\sqrt{NT^{3/2}}}{q(T)} (\hat{\rho}_{IVD} - \rho_T) \Rightarrow N(0, 8).
\]

(e) Suppose that \( \rho_T = \exp\{-1/q(T)\} \) such that \( q(T) \sim T \). Then,
\[
\sqrt{NT} T^{3/2} \frac{\sqrt{N}}{q(T)} (\hat{\rho}_{IVD} - \rho_T) \Rightarrow N(0, 1),
\]
where
\[
V_{NT} = \frac{1}{NT^2} \sum_{i=1}^{N} E \left[ \left( \frac{\sum_{t=2}^{T} w_{it-1} \epsilon_{it}}{\sigma} \right)^2 \right] = \frac{\sigma^2 q(T)^2}{4T^2} \left[ \exp\left\{-\frac{2T}{q(T)}\right\} + \frac{2T}{q(T)} - 1 \right] \left[ 1 + O\left(\frac{1}{T}\right)\right],
\]
\[
\varpi_T = \sigma^2 \sqrt{1 + \frac{q(T)}{T} \left( \frac{1 - \exp\{-2T/q(T)\}}{2} \right)}.
\]

(f) Suppose that \( \rho_T = \exp\{-1/q(T)\} \) such that \( q(T) \to \infty \) but \( q(T)/T \to 0 \). Then,
\[
\sqrt{NT} (\hat{\rho}_{IVD} - \rho_T) \Rightarrow N(0, 4).
\]

(g) Suppose that \( \rho_T \in \mathcal{G}_{St} = \{ |\rho_T| = \exp\{-1/q(T)\} : q(T) \geq 0 \text{ and } q(T) = O(1) \text{ as } T \to \infty \} \). Then as \( N, T \to \infty \)
\[
\sqrt{\frac{NT}{2(1 + \rho_T)}} (\hat{\rho}_{IVD} - \rho_T) \Rightarrow N(0, 1).
\]

**Theorem SA-2:**
Under Assumptions 1-4, the following statements are true as \( N, T \to \infty \) such that \( N^\kappa/T \to \tau \), for \( \kappa \in \left(\frac{1}{2}, \infty\right) \) and \( \tau \in (0, \infty) \).

(a) Suppose that \( \rho_T = 1 \) for all \( T \) sufficiently large. Then,
\[
\sqrt{NT} (\hat{\rho}_{pols} - \rho_T) \Rightarrow N(0, 2).
\]
(b) Suppose that \( \rho_T = \exp \{-1/q(T)\} \) such that \( T \ll q(T) \). Then,
\[
\sqrt{NT} (\hat{\rho}_{\text{pols}} - \rho_T) \Rightarrow N(0,2).
\]
(c) Suppose that \( \rho_T = \exp \{-1/q(T)\} \) such that \( q(T) \sim T \). Then,
\[
\sqrt{NT} \sqrt{\frac{1}{NT}} (\hat{\rho}_{\text{pols}} - \rho_T) \Rightarrow N(0,2),
\]
where \( \sqrt{V_{NT}} = 2V_{NT}/\sigma^2 \), with \( V_{NT} \) as defined in the statement of part (e) of Theorem SA-1 above.

(d) Suppose that \( \rho_T = \exp \{-1/q(T)\} \) such that \( \frac{q(T)}{T^{1+\kappa}} \ll \rho(T) \ll T \). Then,
\[
\sqrt{NTq(T)} (\hat{\rho}_{\text{pols}} - \rho_T) \Rightarrow N(0,2).
\]

(e) Suppose that \( \rho_T = \exp \{-1/q(T)\} \) such that \( q(T) \sim T^{1+\kappa} = N^{1/3}T^{1/3} \). Then,
\[
\sqrt{NTq(T)} (\hat{\rho}_{\text{pols}} - \rho_T - \frac{2\sigma_a^2}{q(T)^2\sigma^2}) \Rightarrow N(0,2).
\]

(f) Suppose that \( \rho_T = \exp \{-1/q(T)\} \) such that \( q(T) \to \infty \) but \( q(T)/T^{1+\kappa} \to 0 \). Then,
\[
q(T)^2 (\hat{\rho}_{\text{pols}} - \rho_T) = \frac{2\sigma_a^2}{\sigma^2} + o_p(1).
\]

(g) Suppose that \( \rho_T \in \mathcal{G}_S = \{|\rho_T| = \exp \{-\frac{1}{q(T)}\} : q(T) \geq 0 \text{ and } q(T) = O(1) \text{ as } T \to \infty\} \). Then,
\[
\hat{\rho}_{\text{pols}} - \rho_T = \frac{(1 - \rho_T^2)(1 - \rho_T)\sigma_a^2}{(1 - \rho_T^2)\sigma_a^2 + \sigma^2} + o_p(1).
\]

Theorem SA-2 above shows that the POLS estimator is a superior estimator when the underlying process either has a unit root or is a very nearly unit root process, i.e., cases where \( T^{1+\kappa} \ll q(T) \) for \( \kappa \in (\frac{1}{2}, \infty) \). In these cases, the POLS estimator is consistent and asymptotically normal with the fastest possible rate of convergence. POLS is, however, inconsistent when the underlying process is stable. On the other hand, Theorem SA-1 shows that the Anderson-Hsiao IV estimator performs well either when the underlying process is stable or when it can be modelled as a local-to-unity process that represents wider deviation from unity, more specifically, cases where \( q(T) = O(T) \). The Anderson-Hsiao IV estimator is consistent, asymptotically normal, and also free of second order biases in these cases. Unfortunately, in local-to-unity cases where \( T \ll q(T) \), the Anderson-Hsiao estimator suffers from a slower rate of convergence and could also have a nonstandard limiting distribution.

Given these results, it makes sense for us to construct an average of these two estimators to have a weight function that shifts from IV to POLS or vice versa in the (potentially thin) region of the parameter space that is characterized by the collection of localized parameter sequences \( \mathcal{G}_{\gamma} = \{\rho_T = \exp \{-1/q(T)\} : T^{1+\kappa} \ll q(T) \ll T\} \), since this is a region where both estimators perform well. Careful examination of the proof of Theorem 2.1 shows that this desired feature is built into the AIP estimator of the form
\[
\hat{\rho}_{\text{AIP}} = w_{IC}\hat{\rho}_{\text{IVD}} + (1 - w_{IC})\hat{\rho}_{\text{pols}},
\]
with weight function given by
\[ w_{IC} = \frac{1}{1 + \exp \left( \frac{1}{2} \Delta_{IC} \right)} \]
and \( \Delta_{IC} = T_{NT} + \sqrt{N} L(T) \),

where \( L(T) \) denotes some slowly varying function such that \( L(T) \to \infty \) as \( T \to \infty \) and where \( T_{NT} \) is the Studentized statistic for testing the unit root null hypothesis as defined in section 2 of the main paper. In particular, the proof of Theorem 2.1 shows that, asymptotically with probability approaching one, the weight function \( w_{IC} \) for this estimator lies strictly between zero and one only under parameter sequences belonging to the collection \( G_{bdry} \equiv \left\{ \rho_T = \exp \left\{ -1/q(T) \right\} : q(T) \sim T/(L(T))^2 \right\} \) which is of course a subcollection of \( G \cap \kappa \in \left( \frac{1}{2}, \infty \right) \). Moreover, for parameter sequences closer to unity relative to those belonging to \( G_{bdry} \), \( w_{IC} \to 1 \), so that, in these cases, the AIP estimator behaves like POLS in large samples. On the other hand, for parameter sequences further away from unity relative to those belonging to \( G_{bdry} \), \( w_{IC} \to 0 \), resulting in AIP behaving asymptotically like the Anderson-Hsiao IV estimator.

Appendix SB: Proof of Theorems on the AIP, IV and POLS Estimator

Proof of Theorem SA-1:
To show (a), first write
\[ \sqrt{T} (\hat{\rho}_{IVD} - \rho_T) = \left[ \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2} \Delta y_{it-1} \right]^{-1} \left[ \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2} \Delta \varepsilon_{it} \right], \]

(2)

Applying part (a) of Lemmas SD-1 and SD-2 in Appendix SD below, we get
\[ \sqrt{T} (\hat{\rho}_{IVD} - \rho_T) = \left[ \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=3}^{T} w_{it-2} \varepsilon_{it-1} + O_p \left( \frac{1}{\sqrt{T}} \right) \right]^{-1} \left[ \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=4}^{T} \varepsilon_{it-2} \varepsilon_{it-1} + \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} w_{iT-2} \varepsilon_{iT} + O_p \left( \frac{1}{\sqrt{T}} \right) \right] \]
\[ \times \left[ \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=3}^{T} w_{it-2} \varepsilon_{it-1} \right]^{-1} \left[ -\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=4}^{T} \varepsilon_{it-2} \varepsilon_{it-1} + \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} w_{iT-2} \varepsilon_{iT} \right] \]
\[ + O_p \left( \frac{1}{\sqrt{T}} \right). \]

(3)

Moreover, it follows from Lemma SE-24 and part (b) of Lemma SE-20 given in Appendix SE below and from the asymptotic independence of the numerator and denominator terms of (3) that we have the joint weak convergence result
\[ \left( \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=4}^{T} \varepsilon_{it-2} \varepsilon_{it-1} + \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} w_{iT-2} \varepsilon_{iT} \right) \Rightarrow \left( \frac{\sqrt{2} \sigma^2 \mathcal{Z}_1}{(\sigma^2/\sqrt{2}) \mathcal{Z}_2} \right), \]

\[ \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=3}^{T} w_{it-2} \varepsilon_{it-1} \]
as \( N, T \to \infty \). Hence, by the continuous mapping theorem, we deduce that
\[
\sqrt{T}(\hat{\rho}_{IVD} - \rho_T) \Rightarrow 2\frac{Z_1}{Z_2},
\]
as required.

To show part (b), we apply part (b) of Lemmas SD-1 and SD-2 to expression (2) above to obtain
\[
\sqrt{T}(\hat{\rho}_{IVD} - \rho_T) = \left[ \frac{1}{\sqrt{N T}} \sum_{i=1}^{N} \sum_{t=3}^{T} w_{it-2} \varepsilon_{it-1} + O_p \left( \max \left\{ \frac{T^{1+\frac{1}{\pi}}}{q(T), \frac{1}{\sqrt{T}}} \right\} \right) \right]^{-1} \times \left[ -\frac{1}{\sqrt{N T}} \sum_{i=1}^{N} \sum_{t=4}^{T} \varepsilon_{it-2} \varepsilon_{it-1} + \frac{1}{\sqrt{N T}} \sum_{i=1}^{N} w_{iT-2} \varepsilon_{iT} + O_p \left( \frac{1}{\sqrt{T}} \right) \right]
\]
\[
= \left[ \frac{1}{\sqrt{N T}} \sum_{i=1}^{N} \sum_{t=3}^{T} w_{it-2} \varepsilon_{it-1} \right]^{-1} \left[ -\frac{1}{\sqrt{N T}} \sum_{i=1}^{N} \sum_{t=4}^{T} \varepsilon_{it-2} \varepsilon_{it-1} + \frac{1}{\sqrt{N T}} \sum_{i=1}^{N} w_{iT-2} \varepsilon_{iT} \right] + O_p \left( \max \left\{ \frac{T^{1+\frac{1}{\pi}}}{q(T), \frac{1}{\sqrt{T}}} \right\} \right).
\]
The rest of the argument then follows as in part (a) above.

Next, consider show (c). Applying part (c) of Lemmas SD-1 and part (b) of Lemma SD-2 to expression (2) above, we get in this case
\[
\sqrt{T}(\hat{\rho}_{IVD} - \rho_T) = \left[ \frac{1}{\sqrt{N T}} \sum_{i=1}^{N} \sum_{t=3}^{T} w_{it-2} \varepsilon_{it-1} - \frac{T \sqrt{N}}{q(T)} \left[ \frac{1}{N T^2} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-2}^2 \right] + O_p \left( \frac{1}{\sqrt{T}} \right) \right]^{-1} \times \left[ -\frac{1}{\sqrt{N T}} \sum_{i=1}^{N} \sum_{t=4}^{T} \varepsilon_{it-2} \varepsilon_{it-1} + \frac{1}{\sqrt{N T}} \sum_{i=1}^{N} w_{iT-2} \varepsilon_{iT} + O_p \left( \frac{1}{\sqrt{T}} \right) \right]
\]
\[
= \left[ \frac{1}{\sqrt{N T}} \sum_{i=1}^{N} \sum_{t=3}^{T} w_{it-2} \varepsilon_{it-1} - \frac{T^{1+\frac{1}{\pi}}}{q(T)} \frac{\sqrt{N}}{T^\pi} \left[ \frac{1}{N T^2} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-2}^2 \right] \right]^{-1} \times \left[ -\frac{1}{\sqrt{N T}} \sum_{i=1}^{N} \sum_{t=4}^{T} \varepsilon_{it-2} \varepsilon_{it-1} + \frac{1}{\sqrt{N T}} \sum_{i=1}^{N} w_{iT-2} \varepsilon_{iT} \right] + O_p \left( \frac{1}{\sqrt{T}} \right),
\]
as \( N, T \to \infty \). Now, consider the case where \( T^{1+\frac{1}{\pi}}/q(T) \to \lambda \in (0, \infty) \). In this case, making use of Lemma SE-15, part (a) of Lemma SE-20, and the Cramér convergence theorem, we have that
\[
\frac{1}{\sqrt{N T}} \sum_{i=1}^{N} \sum_{t=3}^{T} w_{it-2} \varepsilon_{it-1} - \frac{T^{1+\frac{1}{\pi}}}{q(T)} \frac{\sqrt{N}}{T^\pi} \left[ \frac{1}{N T^2} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-2}^2 \right] \Rightarrow \frac{\sigma^2}{\sqrt{2}} Z_2 - \frac{\lambda^{\frac{1}{\pi}}}{2} \sigma^2,
\]
as \( N, T \to \infty \). Moreover, it follows from part (a) of Lemma SE-22 and the asymptotic independence of the numerator and denominator terms of (4) that we have the joint weak convergence result
\[
\left( \frac{-\frac{1}{\sqrt{N T}} \sum_{i=1}^{N} \sum_{t=4}^{T} \varepsilon_{it-2} \varepsilon_{it-1} + \frac{1}{\sqrt{N T}} \sum_{i=1}^{N} w_{iT-2} \varepsilon_{iT}}{\frac{1}{\sqrt{N T}} \sum_{i=1}^{N} \sum_{t=3}^{T} w_{it-2} \varepsilon_{it-1} - \tau \frac{1}{\sqrt{N T}} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-2}^2} \right) \Rightarrow \left( \sqrt{\frac{\sigma^2}{2}} Z_1 - \frac{\lambda^{\frac{1}{\pi}}}{2} \sigma^2, \right),
\]
as \( N, T \to \infty \). Hence, by continuous mapping we deduce that
\[
\sqrt{T} (\hat{p}_{IVD} - \rho_T) \Rightarrow \frac{\sqrt{2} \sigma^2 Z_1}{(\sigma^2/\sqrt{2}) Z_2 - \lambda \tau \frac{1}{T^2} \sigma^2} \equiv \frac{2Z_1}{Z_2 - \lambda \tau \frac{1}{T^2} \sqrt{2} T^2}.
\]

On the other hand, if \( q(T) \sim T^{1+\frac{1}{\pi}} \) but \( u(T) = T^{1+\frac{1}{\pi}}/q(T) \) does not converge as \( T \to \infty \); then, it is nevertheless the case that the denominator of (4)
\[
\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=3}^{T} w_{it-2} \varepsilon_{it-1} - \frac{T^{1+\frac{1}{\pi}}}{q(T)} \sqrt{\frac{N}{T^2}} \left[ \frac{1}{N} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-2}^2 \right] \neq 0 \text{ wpa } 1,
\]
given that \( Z_2 \) is a continuous random variable. Since the numerator of (4) is \( O_p(1) \), it follows that
\[
\sqrt{T} (\hat{p}_{IVD} - \rho_T) = O_p(1)
\]
in this case as required.

To show (d), first write
\[
\frac{\sqrt{NT^{3/2}}}{q(T)} \left( \hat{p}_{IVD} - \rho_T \right) = \left[ \frac{q(T)}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2} \Delta y_{it-1} \right]^{-1} \left[ \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2} \Delta \varepsilon_{it} \right].
\]

Now, applying part (d) of Lemmas SD-1 and part (b) of Lemma SD-2, we get
\[
\frac{\sqrt{NT^{3/2}}}{q(T)} \left( \hat{p}_{IVD} - \rho_T \right) = \left[ \left( \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-2}^2 \right) + O_p \left( \frac{q(T)}{T^{1+\frac{1}{\pi}}} \right) \right]^{-1} \times \left[ - \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=4}^{T} \varepsilon_{it-2} \varepsilon_{it-1} + \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} w_{iT-2} \varepsilon_{iT} + O_p \left( \frac{1}{\sqrt{T}} \right) \right]
\]
\[
= - \left[ \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-2}^2 \right]^{-1} \left[ - \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=4}^{T} \varepsilon_{it-2} \varepsilon_{it-1} + \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} w_{iT-2} \varepsilon_{iT} \right] + O_p \left( \max \left\{ \frac{q(T)}{T^{1+\frac{1}{\pi}}}, \frac{1}{\sqrt{T}} \right\} \right).
\]

Moreover, it follows from part (a) of Lemma SE-22 and Lemma SE-15 that
\[
- \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=4}^{T} \varepsilon_{it-2} \varepsilon_{it-1} + \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} w_{iT-2} \varepsilon_{iT} \Rightarrow \sqrt{2} \sigma^2 Z_1,
\]
and
\[
\frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-2}^2 \overset{p}{\to} \frac{\sigma^2}{2},
\]
as \( N, T \to \infty \). Hence, by the Cramér convergence theorem, we deduce that
\[
\frac{\sqrt{NT^{3/2}}}{q(T)} \left( \hat{p}_{IVD} - \rho_T \right) \Rightarrow \frac{-2\sqrt{2} \sigma^2 \sigma^2 Z_1}{\sigma^2} \equiv N(0, 8).
\]
as required.

To show part (e), write

$$\frac{V_{NT}}{\omega_T} T^{3/2} \sqrt{\frac{N}{q(T)}} \left( \hat{\rho}_{IVD} - \rho_T \right) = \frac{V_{NT}}{\omega_T} \left[ \frac{q(T)}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2} \Delta y_{it-1} \right]^{-1} \left[ \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2} \Delta \varepsilon_{it} \right],$$

where

$$V_{NT} = \frac{1}{NT^2} \sum_{i=1}^{N} E \left[ \left( \sum_{t=2}^{T} \frac{w_{it-1} \varepsilon_{it}}{\sigma} \right)^2 \right] = \frac{\sigma^2}{4} \frac{q(T)^2}{T^2} \left[ \exp \left\{ -\frac{2T}{q(T)} \right\} + \frac{2T}{q(T)} - 1 \right] \left[ 1 + O \left( \frac{1}{T} \right) \right],$$

$$\omega_T = \sigma^2 \sqrt{1 + \frac{q(T)}{T} \left( 1 - \exp \{-2T/q(T)\} \right)^2}.$$

Now, applying part (e) of Lemma SD-1 and part (c) of Lemma SD-2, we get

$$\frac{V_{NT}}{\omega_T} T^{3/2} \sqrt{\frac{N}{q(T)}} \left( \hat{\rho}_{IVD} - \rho_T \right) = V_{NT} \left[ \left( \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-2}^2 \right) + O_p \left( \frac{1}{\sqrt{N}} \right) \right]^{-1} \times \left[ -\frac{1}{\omega_T \sqrt{NT}} \sum_{i=1}^{N} \sum_{t=4}^{T} \varepsilon_{it-2} \varepsilon_{it-1} + \frac{1}{\omega_T \sqrt{NT}} \sum_{i=1}^{N} w_{iT-2} \varepsilon_{iT} + O_p \left( \frac{1}{\sqrt{T}} \right) \right]$$

$$= \left[ \frac{1}{V_{NT}} \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-2}^2 \right]^{-1} \left[ \frac{1}{\omega_T \sqrt{NT}} \sum_{i=1}^{N} \sum_{t=4}^{T} \varepsilon_{it-2} \varepsilon_{it-1} - \frac{1}{\omega_T \sqrt{NT}} \sum_{i=1}^{N} w_{iT-2} \varepsilon_{iT} \right] + O_p \left( \frac{1}{\sqrt{T}} \right) + O_p \left( \frac{1}{\sqrt{N}} \right).$$

Moreover, it follows from part (b) of Lemma SE-22 and part (a) of Lemma SE-17 that

$$\frac{1}{\omega_T \sqrt{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} \varepsilon_{it-2} \varepsilon_{it-1} - \frac{1}{\omega_T \sqrt{NT}} \sum_{i=1}^{N} w_{iT-2} \varepsilon_{iT} \Rightarrow Z_1 \equiv N(0,1),$$

and

$$\frac{1}{V_{NT}} \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-2}^2 = 1 + \frac{1}{V_{NT}} \left[ \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-2}^2 - V_{NT} \right] = 1 + o_p(1) \xrightarrow{p} 1,$$

as \(N, T \to \infty\). Hence, by the Cramér convergence theorem, we deduce that

$$\frac{V_{NT}}{\omega_T} T^{3/2} \sqrt{\frac{N}{q(T)}} \left( \hat{\rho}_{IVD} - \rho_T \right) = Z_1 \equiv N(0,1),$$

as required.
To show part (f),
\[
\sqrt{NT} (\hat{\rho}_{IVD} - \rho_T) = \left[ \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2} \Delta y_{it-1} \right]^{-1} \left[ \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2} \Delta \varepsilon_{it} \right].
\]

Now, applying part (f) of Lemma SD-1 and part (d) of Lemma SD-2, we get
\[
\sqrt{NT} (\hat{\rho}_{IVD} - \rho_T) = \left[ -\frac{1}{NTq(T)} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-2}^2 + O_p \left( \max \left\{ \sqrt{\frac{q(T)}{NT}}, \frac{1}{q(T)} \right\} \right) \right]^{-1} \times \left[ -\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=4}^{T} \varepsilon_{it-2} \varepsilon_{it-1} + O_p \left( \max \left\{ \sqrt{\frac{q(T)}{T}}, \frac{1}{\sqrt{q(T)}} \right\} \right) \right]
\]
\[
\Rightarrow \left[ -\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=4}^{T} \varepsilon_{it-2} \varepsilon_{it-1} \right]^{-1} \left[ -\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=4}^{T} \varepsilon_{it-2} \varepsilon_{it-1} \right] + O_p \left( \max \left\{ \sqrt{\frac{q(T)}{T}}, \frac{1}{\sqrt{q(T)}} \right\} \right).
\]

Moreover, it follows from part (c) of Lemma SE-22 and part (b) of Lemma SE-17 that
\[
-\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=4}^{T} \varepsilon_{it-2} \varepsilon_{it-1} \Rightarrow \sigma^2 Z_1,
\]
and
\[
-\frac{1}{NTq(T)} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-2}^2 \Rightarrow \sigma^2 \frac{\sigma^2}{2},
\]
as \(N, T \to \infty\). Hence, by the Cramér convergence theorem, we deduce that
\[
\sqrt{NT} (\hat{\rho}_{IVD} - \rho_T) \Rightarrow \left( -\frac{2}{\sigma^2} \right) \sigma^2 Z_1 \equiv N (0, 4).
\]
as required.

Finally, to show part (g), note that by applying part (g) of Lemma SD-1, part (e) of Lemma SD-2, part (c) of Lemma SE-17, and Lemma SE-23; we obtain
\[
\sqrt{\frac{NT}{2(1+\rho_T^2)}} (\hat{\rho}_{IVD} - \rho_T)
\]
\[
= \left[ \frac{1 + \rho_T}{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2} \Delta y_{it-1} \right]^{-1} \left[ \sqrt{\frac{1 + \rho_T}{2}} \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2} \Delta \varepsilon_{it} \right]
\]
\[
= \left[ \frac{1 + \rho_T^2}{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-2}^2 + O_p \left( \max \left\{ \sqrt{\frac{1}{NT}}, \frac{1}{T} \right\} \right) \right]^{-1} \times \sigma^2 \left[ \sqrt{\frac{1 + \rho_T}{2 \sigma^4}} \left( -\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=4}^{T} \varepsilon_{it-2} \varepsilon_{it-1} + (1 - \rho_T) \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=4}^{T} \varepsilon_{it-3} \varepsilon_{it-1} \right) + O_p \left( \frac{1}{\sqrt{T}} \right) \right]
\]
\[
= -\frac{1}{\sigma^2} N (0, \sigma^4) \equiv N (0, 1)
\]
Proof of Theorem SA-2:
To proceed, first write
\[
\hat{p}_{\text{pols}} = \left[ \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \bar{y}_{-1,N,T}) \right]^{-1} \times \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \bar{y}_{-1,N,T}) [a_i (1 - \rho_T) + \rho_T y_{it-1} + \varepsilon_{it}] \\
= \left[ \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \bar{y}_{-1,N,T}) \right]^{-1} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \bar{y}_{-1,N,T}) a_i (1 - \rho_T) \\
+ \rho_T \left[ \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \bar{y}_{-1,N,T}) \right]^{-1} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \bar{y}_{-1,N,T})^2 \\
+ \left[ \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \bar{y}_{-1,N,T}) \right]^{-1} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \bar{y}_{-1,N,T}) \varepsilon_{it},
\]
so that
\[
\hat{p}_{\text{pols}} - \rho_T = (1 - \rho_T) \left[ \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \bar{y}_{-1,N,T}) \right]^{-1} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \bar{y}_{-1,N,T}) a_i \\
+ \left[ \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \bar{y}_{-1,N,T}) \right]^{-1} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \bar{y}_{-1,N,T}) \varepsilon_{it}.
\] (5)

Now, consider part (a), where we take \( \rho_T = 1 \) for all \( T \) sufficiently large. Hence, in this case, we can apply part (a) of Lemmas SD-3, SD-4, and SD-5 given in Appendix SD below, along with Lemma
SE-13 and part (b) of Lemma SE-20 given in Appendix SE to obtain

$$T \sqrt{N} \left( \hat{\rho}_{\text{pols}} - \rho_T \right)$$

$$= (1 - \rho_T) \left[ \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \overline{y}_{-1,N,T})^2 \right]^{-1} \frac{1}{T \sqrt{N}} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \overline{y}_{-1,N,T}) a_i$$

$$+ \left[ \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \overline{y}_{-1,N,T})^2 \right]^{-1} \frac{1}{T \sqrt{N}} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \overline{y}_{-1,N,T}) \varepsilon_{it}$$

$$= \left[ \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1}^2 + O_p \left( \frac{1}{T} \right) + O_p \left( \frac{1}{N} \right) \right]^{-1}$$

$$\times \left[ \frac{1}{T \sqrt{N}} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1} \varepsilon_{it} + O_p \left( \frac{1}{\sqrt{T}} \right) + O_p \left( \frac{1}{\sqrt{N}} \right) \right] + O_p \left( (1 - \rho_T) \max \left\{ \sqrt{N}, \sqrt{T} \right\} \right)$$

$$= \left[ \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1} \right]^{-1} \left[ \frac{1}{T \sqrt{N}} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1} \varepsilon_{it} \right] + O_p \left( \max \left\{ \frac{1}{\sqrt{N}}, \frac{1}{\sqrt{T}} \right\} \right)$$

$$\Rightarrow 2 \sigma^2 N \left( 0, \frac{\sigma^4}{2} \right) \equiv N (0, 2).$$

Next, consider part (b), where we assume that $\rho_T = \exp \{-1/q(T)\}$ such that $T \ll q(T)$. In this case, we apply part (b) of Lemmas SD-3, SD-4, and SD-5 along with Lemma SE-15 and part (a) of Lemma SE-20 to obtain

$$T \sqrt{N} \left( \hat{\rho}_{\text{pols}} - \rho_T \right)$$

$$= (1 - \rho_T) \left[ \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \overline{y}_{-1,N,T})^2 \right]^{-1} \frac{1}{T \sqrt{N}} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \overline{y}_{-1,N,T}) a_i$$

$$+ \left[ \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \overline{y}_{-1,N,T})^2 \right]^{-1} \frac{1}{T \sqrt{N}} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \overline{y}_{-1,N,T}) \varepsilon_{it}$$

$$= \left[ \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1}^2 + O_p \left( \frac{1}{T} \right) + O_p \left( \frac{1}{N} \right) \right]^{-1}$$

$$\times \left[ \frac{1}{T \sqrt{N}} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1} \varepsilon_{it} + O_p \left( \frac{1}{\sqrt{T}} \right) + O_p \left( \frac{1}{\sqrt{N}} \right) \right]$$

$$+ O \left( \frac{1}{q(T)} \right) O_p \left( 1 \right) O_p \left( \max \left\{ \sqrt{N}, \sqrt{T} \right\} \right)$$

$$= \left[ \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1} \right]^{-1} \left[ \frac{1}{T \sqrt{N}} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1} \varepsilon_{it} \right] + O_p \left( 1 \right)$$

$$\Rightarrow 2 \sigma^2 N \left( 0, \frac{\sigma^4}{2} \right) \equiv N (0, 2).$$

Now, we consider part (c), where we take $\rho_T = \exp \{-1/q(T)\}$ such that $q(T) \sim T$. To proceed,
note that
\[ \nabla_{NT} = \frac{2}{\sigma^4} \frac{1}{NT^2} \rho_{Z,N,T}, \]
where \( \rho_{Z,N,T} \) is defined in the statement of Lemma SE-27. It follows from the proof of part (a) of Lemma SE-27 that
\[
\nabla_{NT} = \frac{2}{\sigma^4} \frac{1}{NT^2} N q(T)^2 \frac{\sigma^4}{4} \left[ \exp \left\{ -\frac{2T}{q(T)} \right\} + \frac{2T}{q(T)} - 1 \right] \left[ 1 + O \left( \frac{1}{T} \right) \right] \\
= \frac{1}{2} \frac{q(T)^2}{T^2} \left[ \exp \left\{ -\frac{2T}{q(T)} \right\} + \frac{2T}{q(T)} - 1 \right] \left[ 1 + O \left( \frac{1}{T} \right) \right].
\]
Applying part (c) of Lemmas SD-3, SD-4, and SD-5 along with part (a) of Lemma SE-17 and part (a) of Lemma SE-27; we can get
\[
\sqrt{NT} \nabla_{NT}^{1/2} (\hat{p}_{ols} - \rho_T) \\
= (1 - \rho_T) \nabla_{NT}^{1/2} \left[ \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \bar{y}_{-1,N,T})^2 \right]^{-1/2} \frac{1}{T \sqrt{N}} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \bar{y}_{-1,N,T}) \varepsilon_{it} \\
+ \nabla_{NT}^{1/2} \left[ \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \bar{y}_{-1,N,T})^2 \right]^{-1/2} \frac{1}{T \sqrt{N}} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \bar{y}_{-1,N,T}) \varepsilon_{it} \\
= \nabla_{NT} \left[ \frac{q(T)^2 \sigma^2}{T^2} \frac{\sigma^4}{4} \left[ \exp \left\{ -\frac{2T}{q(T)} \right\} + \frac{2T}{q(T)} - 1 \right] + O_p \left( \max \left\{ \frac{1}{\sqrt{N}}, \frac{1}{T} \right\} \right) \right]^{-1} \\
\times \frac{1}{T \sqrt{N}} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1} \varepsilon_{it} \left[ 1 + O_p \left( \frac{1}{\sqrt{T}} \right) + O_p \left( \frac{1}{\sqrt{N}} \right) \right] \\
+ O \left( \frac{1}{T} \right) O_p \left( \frac{1}{\sqrt{T}} \right) O_p \left( \frac{1}{\sqrt{N}} \right) \max \left\{ \sqrt{N}, \sqrt{T} \right\} \\
= \nabla_{NT} \left[ \frac{q(T)^2 \sigma^2}{T^2} \frac{\sigma^4}{4} \left[ \exp \left\{ -\frac{2T}{q(T)} \right\} + \frac{2T}{q(T)} - 1 \right] + O_p \left( \max \left\{ \frac{1}{\sqrt{N}}, \frac{1}{T} \right\} \right) \right]^{-1} \\
\times \frac{1}{T \sqrt{N}} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1} \varepsilon_{it} \left[ 1 + O_p \left( \frac{1}{\sqrt{T}} \right) + O_p \left( \frac{1}{\sqrt{N}} \right) \right] + O_p \left( \max \left\{ \frac{\sqrt{N}}{T}, \sqrt{T} \right\} \right) \\
= \frac{1}{2} \frac{q(T)^2}{T^2} \left[ \exp \left\{ -\frac{2T}{q(T)} \right\} + \frac{2T}{q(T)} - 1 \right] \left[ 1 + O \left( \frac{1}{T} \right) \right] \\
\times \left[ \frac{q(T)^2 \sigma^2}{T^2} \frac{\sigma^4}{4} \left[ \exp \left\{ -\frac{2T}{q(T)} \right\} + \frac{2T}{q(T)} - 1 \right] + O_p \left( \max \left\{ \frac{1}{\sqrt{N}}, \frac{1}{\sqrt{T}} \right\} \right) \right]^{-1} \\
\times \frac{\sigma^2}{\sqrt{\rho_{Z,N,T}}} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1} \varepsilon_{it} \left[ 1 + O_p \left( \frac{1}{\sqrt{T}} \right) + O_p \left( \frac{1}{\sqrt{N}} \right) \right] + O_p \left( \max \left\{ \frac{\sqrt{N}}{T}, \frac{1}{\sqrt{T}} \right\} \right) \\
= \frac{\sqrt{2}}{\rho_{Z,N,T}} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1} \varepsilon_{it} \left[ 1 + O_p \left( \max \left\{ \frac{\sqrt{N}}{T}, \frac{1}{\sqrt{N}}, \frac{1}{\sqrt{T}} \right\} \right) \right] \\
\Rightarrow \ N \left( 0, 2 \right).
We turn our attention to part (d) where we take $\rho_T = \exp \{-1/q(T)\}$ such that $T^{1+\kappa} \ll q(T) \ll T$. Here, using part (d) of Lemmas SD-3, SD-4, and SD-5 along with part (b) of Lemma SE-17 and part (b) of Lemma SE-27; we obtain

$$
\sqrt{NT}q(T) \left( \hat{\rho}_{\text{pols}} - \rho_T \right)
= (1 - \rho_T) \left[ \frac{1}{NTq(T)} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \overline{y}_{-1,N,T})^2 \right]^{-1} \frac{1}{\sqrt{NT}q(T)} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \overline{y}_{-1,N,T}) a_i
+ \left[ \frac{1}{NTq(T)} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \overline{y}_{-1,N,T})^2 \right]^{-1} \frac{1}{\sqrt{NT}q(T)} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \overline{y}_{-1,N,T}) \varepsilon_{it}
= \frac{2}{\sigma^2} \left[ 1 + O_p \left( \max \left\{ \frac{q(T)}{T}, \frac{1}{q(T)}, \sqrt{\frac{q(T)}{NT}}, \frac{1}{\sqrt{T}} \right\} \right) \right]
\times \left[ \frac{1}{\sqrt{NT}q(T)} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1} \varepsilon_{it} + O_p \left( \max \left\{ \frac{1}{\sqrt{q(T)}}, \sqrt{\frac{q(T)}{NT}}, \frac{1}{\sqrt{T}} \right\} \right) \right]
\Rightarrow \frac{2}{\sigma^2} N \left( 0, \frac{\sigma^4}{2} \right) \equiv N(0,2),
$$

Next, we consider part (e), where we take $\rho_T = \exp \{-1/q(T)\}$ such that $q(T) \sim T^{1+\kappa} = N^{1/3}T^{1/3} \ll T$ for $\kappa \in \left( \frac{1}{4}, \infty \right)$. In this case, we apply part (d) of Lemmas SD-3, SD-4, and SD-5.
along with parts (a) and (b) of Lemma SE-11, and part (b) of Lemmas SE-17 and SE-27 to obtain

\[
\sqrt{NTq(T)} (\hat{\rho}_{\text{pols}} - \rho_T)
\]

\[
= (1 - \rho_T) \left[ \frac{1}{NTq(T)} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \overline{y}_{-1,N,T})^2 \right]^{-1}
\]

\[
\times \frac{1}{\sqrt{NTq(T)}} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \overline{y}_{-1,N,T}) a_i
\]

\[
+ \left[ \frac{1}{\sqrt{NTq(T)}} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \overline{y}_{-1,N,T})^2 \right]^{-1} \frac{1}{\sqrt{NTq(T)}} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \overline{y}_{-1,N,T}) \varepsilon_{it}
\]

\[
= \left( 1 - \left[ 1 - \frac{1}{q(T)} + O \left( \frac{1}{q(T)^2} \right) \right] \right) \left[ \frac{\sigma^2}{2} + O_p \left( \max \left\{ \frac{q(T)}{T}, \frac{1}{q(T)} \sqrt{\frac{NT}{T}}, \frac{1}{\sqrt{T}} \right\} \right) \right]^{-1}
\]

\[
\times \frac{\sqrt{NT}}{\sqrt{q(T)}} \left[ N \sum_{i=1}^{N} a_i^2 - \overline{a}_N^2 \right] + O_p \left( \max \left\{ \sqrt{\frac{q(T)}{T}}, \sqrt{\frac{q(T)N}{T}}, \sqrt{\frac{q(T)}{T}} \right\} \right]
\]

\[
+ \left[ \frac{\sigma^2}{2} + O_p \left( \max \left\{ \frac{q(T)}{T}, \frac{1}{q(T)} \sqrt{\frac{q(T)N}{T}}, \frac{1}{\sqrt{T}} \right\} \right) \right]^{-1}
\]

\[
\times \left[ \frac{1}{\sqrt{NTq(T)}} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1} \varepsilon_{it} + O_p \left( \max \left\{ \frac{1}{\sqrt{q(T)}}, \sqrt{\frac{q(T)}{NT}}, \sqrt{\frac{q(T)}{T}} \right\} \right) \right]
\]

\[
= \frac{\sigma^2}{2} \frac{\sqrt{NT}}{q(T)^{3/2}} \left[ N \sum_{i=1}^{N} a_i^2 - \overline{a}_N^2 \right] \left[ 1 + O_p \left( \max \left\{ \frac{q(T)}{T}, \frac{1}{q(T)} \sqrt{\frac{q(T)N}{T}}, \frac{1}{\sqrt{T}} \right\} \right) \right]
\]

\[
\times \left[ 1 + O_p \left( \max \left\{ \frac{1}{\sqrt{q(T)}}, \sqrt{\frac{N}{q(T)T}} \right\} \right) \right]
\]

\[
+ \frac{2}{\sigma^2} \frac{1}{\sqrt{NTq(T)}} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1} \varepsilon_{it} \left[ 1 + O_p \left( \max \left\{ \frac{q(T)}{T}, \frac{1}{q(T)} \sqrt{\frac{q(T)N}{T}}, \frac{1}{\sqrt{T}} \right\} \right) \right]
\]

\[
\times \left[ 1 + O_p \left( \max \left\{ \frac{1}{\sqrt{q(T)}}, \sqrt{\frac{q(T)N}{T}}, \sqrt{\frac{q(T)}{T}} \right\} \right) \right]
\]

\[
= \left[ \frac{2}{\sigma^2} \frac{\sqrt{NT}}{q(T)^{3/2}} a^2 + \frac{2}{\sigma^2} \frac{2}{\sqrt{2}} Z \right] \left[ 1 + o_p(1) \right]
\]

\[
= \left[ \frac{\sqrt{NTq(T)}}{\sigma^2 q(T)^2} \frac{2\sigma_a^2}{\sigma^2 q(T)^2} + \sqrt{2} Z \right] \left[ 1 + o_p(1) \right]
\]

where $Z \equiv N(0, 1)$. It follows that

\[
\sqrt{NTq(T)} \left( \hat{\rho}_{\text{pols}} - \rho_T - \frac{2\sigma_a^2}{\sigma^2 q(T)^2} \right) \Rightarrow N(0, 2),
\]

as required for part (e).
Now, in part (f), we consider the case where $\rho_T = \exp \{-1/q(T)\}$ such that $q(T) \to \infty$ but $q(T)/T^{1+\frac{1}{q(T)}} \to 0$. In this case, note that applying part (d) of Lemmas SD-3 and SD-5 and part (e) of Lemma SD-4 along with parts (a) and (b) of Lemma SE-11, part (b) of Lemmas SE-17 and SE-27 yield

$$q(T)^2 \left( \bar{\rho}_{pols} - \rho_T \right)$$

$$= (1 - \rho_T) q(T) \left[ \frac{1}{NTq(T)} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \overline{y}_{-1,N,T})^2 \right]^{-1} \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \overline{y}_{-1,N,T}) a_i$$

$$+ \left[ \frac{1}{NTq(T)} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \overline{y}_{-1,N,T})^2 \right]^{-1} \frac{q(T)}{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \overline{y}_{-1,N,T}) \varepsilon_{it}$$

$$= \left( 1 - \left[ 1 - \frac{1}{q(T)} + O \left( \frac{1}{q(T)^2} \right) \right] \right) \left[ \frac{\sigma^2}{2} + O_p \left( \max \left\{ \frac{q(T)}{T}, \frac{q(T)}{T}, \sqrt{\frac{q(T)}{NT}}, \frac{1}{\sqrt{T}} \right\} \right) \right]^{-1}$$

$$\times q(T) \left[ \frac{1}{N} \sum_{i=1}^{N} a_i^2 - \overline{a}_N^2 \right] + O_p \left( \max \left\{ \frac{q(T)}{\sqrt{NT}}, \frac{q(T)}{T} \right\} \right)$$

$$+ \left[ \frac{\sigma^2}{2} + O_p \left( \max \left\{ \frac{q(T)}{T}, \sqrt{q(T)/NT}, \frac{1}{\sqrt{T}} \right\} \right) \right]^{-1}$$

$$\times \frac{q(T)^{3/2}}{\sqrt{NT}} \left[ \frac{1}{\sqrt{NTq(T)}} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1} \varepsilon_{it} + O_p \left( \max \left\{ \frac{1}{\sqrt{q(T)}} \sqrt{\frac{q(T)}{NT}}, \sqrt{\frac{q(T)}{T}} \right\} \right) \right]$$

so that

$$q(T)^2 \left( \bar{\rho}_{pols} - \rho_T \right)$$

$$= \frac{2}{\sigma^2} \left[ \frac{1}{N} \sum_{i=1}^{N} a_i^2 - \overline{a}_N^2 \right] \left[ 1 + O_p \left( \max \left\{ \frac{q(T)}{T}, \sqrt{q(T)/NT}, \frac{1}{\sqrt{T}} \right\} \right) \right] + O_p \left( \frac{q(T)^{3/2}}{\sqrt{NT}} \right)$$

$$= \frac{2\sigma^2}{\sigma^2} + o_p(1),$$

as required for part (f).

Finally, in part (g), we consider the case where $\rho_T \in \mathcal{G}_{St} = \left\{ |\rho_T| = \exp \left\{ -\frac{1}{q(T)} \right\} : q(T) \geq 0 \text{ and } q(T) = O(1) \text{ as } T \to \infty \right\}$. In this case, applying part
(e) of Lemmas SD-3 and SD-5 and part (f) of Lemma SD-4, we obtain

\[
\hat{\rho}_{\text{pols}} - \rho_T = (1 - \rho_T) \left[ 1 - \frac{\rho_T^2}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \left( y_{it-1} - \bar{y}_{1,N,T} \right)^2 \right]^{-1} \frac{1 - \rho_T^2}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \left( y_{it-1} - \bar{y}_{1,N,T} \right) a_i
\]

Thus, we can set \( N = (1 - \rho_T) \sigma_2^2 \left( 1 + o_p(1) \right) \) as required.

\[
\frac{1}{(1 - \rho_T)^2} \left( 1 - \rho_T \right) \sigma_a^2 \left( 1 + o_p(1) \right)
\]

Proof of Lemma A1:

To proceed, note that, in the pathwise asymptotics considered here, \( N \) grows as a monotonically increasing function of \( T \), so that the asymptotics can be taken to be single-indexed with \( T \to \infty \). Hence, as in the statement of the lemma, we can set \( N = N(T) = (\tau T)^{1/\kappa} \) and simplify notation by writing \( C_{\gamma,a,N,T} = C_{\gamma,a,N(T),T} = C_{\gamma,a,T} \).

Consider first part (a), where we take \( \rho_T \in \mathcal{G}_1^p = \{ \rho_T : \rho_T = 1 \text{ for all } T \text{ sufficiently large} \} \). In this case, note that

\[
\Pr(\rho \notin C_{\gamma,a,T} | \rho = \rho_T \in \mathcal{G}_1^p)
\]

\[
= \Pr(\rho \notin C_{\gamma_1,T}^\mathcal{M} | T_1,T \leq -z_{\gamma_1}, T_2,T \leq -z_{\gamma_2} | \rho = \rho_T \in \mathcal{G}_1^p) + \Pr(\rho \notin C_{\gamma_2,T}^\mathcal{M} | T_1,T > -z_{\gamma_1}, T_2,T > -z_{\gamma_2} | \rho = \rho_T \in \mathcal{G}_1^p)
\]

Then, there exists a positive integer \( I_\rho \) such that, for all \( T \geq I_\rho \),

\[
1 - \sqrt{2 \left( \frac{z_{\gamma_1} + z_{\alpha_2}}{\sqrt{N}} \right)} = 1 - \sqrt{2 \left( \frac{z_{\gamma_1} + z_{\alpha_2}}{\tau^{1/\kappa}T^{1/\kappa+1}} \right)} \leq \rho_T = 1,
\]

and

\[
1 - \frac{2 \left( \frac{z_{\gamma_2} + z_{\alpha_2}}{\sqrt{NT}} \right)}{\tau^{1/\kappa}T^{1/\kappa+1}} = 1 - \frac{2 \left( \frac{z_{\gamma_2} + z_{\alpha_2}}{\tau^{1/\kappa}T^{1/\kappa+1}} \right)}{\tau^{1/\kappa}T^{1/\kappa+1}} \leq \rho_T = 1.
\]

Thus, \( \rho_T \in C_{\gamma_1,T}^\mathcal{M} \) and \( \rho_T \in C_{\gamma_2,T}^\mathcal{M} \), for all \( T \) sufficiently large. Hence,
Moreover, by part (b) of Theorem 3.1, \( \limsup_{T \to \infty} \Pr \left( \rho \notin \mathbb{C}^{\text{UR1}}_{\gamma_1, \alpha T} \mid \rho_T = \rho_T \in G_1^\rho \right) = 0 \) and \( \limsup_{T \to \infty} \Pr \left( \rho \notin \mathbb{C}^{\text{UR2}}_{\gamma_2, \alpha T} \mid \rho_T = \rho_T \in G_1^\rho \right) = 0 \).

Moreover, by part (a) of Theorem 3.1, \( \lim_{T \to \infty} \Pr \left( \rho \notin \mathbb{C}^{M}_{\alpha T} \mid \rho = \rho_T \in G_1^\rho \right) = \alpha_1 \). It follows that

\[
\lim \sup_{T \to \infty} \Pr \left( \rho \notin \mathbb{C}_{\gamma, \alpha T} \mid \rho = \rho_T \in G_1^\rho \right) \\
\leq \lim \sup_{T \to \infty} \Pr \left( \rho \notin \mathbb{C}^{M}_{\alpha T} \mid \rho = \rho_T \in G_1^\rho \right) + \lim \sup_{T \to \infty} \Pr \left( \rho \notin \mathbb{C}^{\text{UR1}}_{\gamma_1, \alpha T} \mid \rho = \rho_T \in G_1^\rho \right) \\
+ \lim \sup_{T \to \infty} \Pr \left( \rho \notin \mathbb{C}^{\text{UR2}}_{\gamma_2, \alpha T} \mid \rho = \rho_T \in G_1^\rho \right) \\
= \alpha_1 + 0 + 0 = \alpha_1.
\]

Next, consider part (b), where we take \( \rho_T \in G_2^\rho = \left\{ \rho_T = \exp \left\{ -\frac{1}{q(T)} \right\} : \sqrt{NT} \sim T^{\frac{1}{2\pi^2}} \ll q(T) \right\} \).

Note that, in the present case, \( T\sqrt{N} \sim T^{\frac{1}{2\pi^2}} \ll q(T) \),

\[
1 - \sqrt{2} \left( \frac{z_{\gamma_1} + z_{\alpha_2}}{T\sqrt{N}} \right) = 1 - \sqrt{2} \left( \frac{z_{\gamma_1} + z_{\alpha_2}}{\frac{1}{2\pi^2} T^{\frac{1}{2\pi^2} + 1}} \right) \\
\leq \rho_T = \exp \left\{ -\frac{1}{q(T)} \right\} = 1 - \frac{1}{q(T)} + O \left( \frac{1}{q(T)^2} \right) \leq 1,
\]

and

\[
1 - \frac{2 \left( \frac{z_{\gamma_2} + z_{\alpha_2}}{\sqrt{NT}} \right)}{} = 1 - \frac{2 \left( \frac{z_{\gamma_2} + z_{\alpha_2}}{\frac{1}{2\pi^2} T^{\frac{1}{2\pi^2} + 1}} \right)}{} \\
\leq \rho_T = \exp \left\{ -\frac{1}{q(T)} \right\} = 1 - \frac{1}{q(T)} + O \left( \frac{1}{q(T)^2} \right) \leq 1,
\]

for all \( T \) sufficiently large, so that \( \rho_T \in \mathbb{C}^{\text{UR1}}_{\gamma_1, \alpha T} \) and \( \rho_T \in \mathbb{C}^{\text{UR2}}_{\gamma_2, \alpha T} \) eventually. Hence,

\( \limsup_{T \to \infty} \Pr \left( \rho \notin \mathbb{C}^{\text{UR1}}_{\gamma_1, \alpha T} \mid \rho = \rho_T \in G_2^\rho \right) = 0 \) and \( \limsup_{T \to \infty} \Pr \left( \rho \notin \mathbb{C}^{\text{UR2}}_{\gamma_2, \alpha T} \mid \rho = \rho_T \in G_2^\rho \right) = 0 \).

Moreover, by part (b) of Theorem 3.1, \( \lim_{T \to \infty} \Pr \left( \rho \notin \mathbb{C}^{M}_{\alpha T} \mid \rho = \rho_T \in G_2^\rho \right) = \alpha_1 \). It follows that

\[
\lim \sup_{T \to \infty} \Pr \left( \rho \notin \mathbb{C}_{\gamma, \alpha T} \mid \rho = \rho_T \in G_2^\rho \right) \\
\leq \lim \sup_{T \to \infty} \Pr \left( \rho \notin \mathbb{C}^{M}_{\alpha T} \mid \rho = \rho_T \in G_2^\rho \right) + \lim \sup_{T \to \infty} \Pr \left( \rho \notin \mathbb{C}^{\text{UR1}}_{\gamma_1, \alpha T} \mid \rho = \rho_T \in G_2^\rho \right) \\
+ \lim \sup_{T \to \infty} \Pr \left( \rho \notin \mathbb{C}^{\text{UR2}}_{\gamma_2, \alpha T} \mid \rho = \rho_T \in G_2^\rho \right) \\
= \alpha_1 + 0 + 0 = \alpha_1 < \alpha.
\]

Consider part (c), where we assume

\[
\rho_T \in G_3^\rho = \left\{ \rho_T = \exp \left\{ -\frac{1}{q(T)} \right\} : \sqrt{NT} \sim T^{\frac{1}{2\pi^2}} \ll q(T) \right\} \cap \left( \rho_T \geq 1 - \frac{(z_{\gamma_1} + z_{\alpha_2}) \sqrt{2}}{\sqrt{NT}} \text{ eventually} \right) \right\}.
\]

In this case, \( \rho_T \in \mathbb{C}^{\text{UR1}}_{\gamma_1, \alpha T} \) eventually by assumption. Since \( \sqrt{NT} \ll \sqrt{NT} \sim q(T) \), we also have
\[ \rho_T \in \mathbb{C}_{\gamma_2, \alpha_2, T}^{UR} \text{ eventually. It follows by applying part (b) of Theorem 3.1 that} \]

\[ \lim_{T \to \infty} \sup_{T} \Pr \left( \rho \notin \mathbb{C}_{\gamma, \alpha, T} | \rho = \rho_T \in \mathbb{G}_T^{P} \right) \]

\[ \leq \lim_{T \to \infty} \sup_{T} \Pr \left( \rho \notin \mathbb{C}_{\alpha, T}^{M} | \rho = \rho_T \in \mathbb{G}_T^{P} \right) + \lim_{T \to \infty} \sup_{T} \Pr \left( \rho \notin \mathbb{C}_{\gamma_1, \alpha_2, T}^{UR_1} | \rho = \rho_T \in \mathbb{G}_T^{P} \right) \]

\[ + \lim_{T \to \infty} \sup_{T} \Pr \left( \rho \notin \mathbb{C}_{\gamma_2, \alpha_2, T}^{UR_2} | \rho = \rho_T \in \mathbb{G}_T^{P} \right) \]

\[ = \alpha_1 + 0 + 0 = \alpha_1 < \alpha. \]

We turn our attention now to part (d) where we take

\[ \rho_T \in \mathbb{G}_4^P = \left\{ \rho_T = \exp \left\{ -\frac{1}{q(T)} \right\} : \sqrt{NT} \sim q(T) \cap \rho_T < 1 - \left( \frac{z_{\gamma_1} + z_{\alpha_2}}{\sqrt{NT}} \right) \text{ eventually} \right\}. \]

To proceed, write \( T_{1, T} = \frac{M^{1/2}_y}{{\bar{\rho}}\sqrt{NT}} - \frac{M^{1/2}_y\sqrt{NT}}{\sigma} \). In this case, by part (b) of Lemma SD-3, Lemma SD-12, and part (b) of Theorem SA-2, we have \( T_{1, T} \to N(0, 1) \) and \( \theta_T^* = \bar{\sigma}^{-1}M^{1/2}_y\sqrt{NT} (1 - \rho_T) = 2^{-1/2}\sqrt{NT} (1 - \rho_T) [1 + o_p(1)] \), so that \( T_{1, T} = T_{1, T}^* - \theta_T^* = T_{1, T}^* - \sqrt{NT} (1 - \rho_T)/\sqrt{2} + \xi_T \), where \( \xi_T = o_p(1) \). Now, let \( \epsilon_T \) be a sequence of positive numbers such that, as \( T \to \infty \), \( \epsilon_T \to 0 \) but \( \xi_T/\epsilon_T = o_p(1) \). It follows that

\[ \Pr \left( T_{1, T} > -z_{\gamma_1} \left\{ q(T) \sim T_{\frac{1}{\alpha}}^{\frac{1}{\alpha} - 1} \right\} \cap \left\{ \rho_T < 1 - \left( \frac{z_{\gamma_1} + z_{\alpha_2}}{\sqrt{NT}} \right) \text{ eventually} \right\} \right) \]

\[ = \Pr \left( T_{1, T}^* - \frac{\sqrt{NT} (1 - \rho_T)}{\sqrt{2}} + \xi_T > -z_{\gamma_1} \right) \cap \left\{ \left| \xi_T \right| < \epsilon_T \right\} \cap \left\{ q(T) \sim T_{\frac{1}{\alpha}}^{\frac{1}{\alpha} - 1} \right\} \]

\[ \cap \left\{ \rho_T < 1 - \left( \frac{z_{\gamma_1} + z_{\alpha_2}}{\sqrt{NT}} \right) \text{ eventually} \right\} \]

\[ + \Pr \left( T_{1, T}^* - \frac{\sqrt{NT} (1 - \rho_T)}{\sqrt{2}} + \xi_T > -z_{\gamma_1} \right) \cap \left\{ \left| \xi_T \right| \geq \epsilon_T \right\} \cap \left\{ q(T) \sim T_{\frac{1}{\alpha}}^{\frac{1}{\alpha} - 1} \right\} \]

\[ \cap \left\{ \rho_T < 1 - \left( \frac{z_{\gamma_1} + z_{\alpha_2}}{\sqrt{NT}} \right) \text{ eventually} \right\} \]

\[ \leq \Pr \left( T_{1, T}^* - \frac{\sqrt{NT} (1 - \rho_T)}{\sqrt{2}} > -z_{\gamma_1} + \epsilon_T \right) \cap \left\{ q(T) \sim T_{\frac{1}{\alpha}}^{\frac{1}{\alpha} - 1} \cap \rho_T < 1 - \left( \frac{z_{\gamma_1} + z_{\alpha_2}}{\sqrt{NT}} \right) \text{ eventually} \right\} \]

\[ + \Pr \left( \left| \xi_T \right| \geq \epsilon_T \right) \cap \left\{ q(T) \sim T_{\frac{1}{\alpha}}^{\frac{1}{\alpha} - 1} \right\} \cap \left\{ \rho_T < 1 - \left( \frac{z_{\gamma_1} + z_{\alpha_2}}{\sqrt{NT}} \right) \text{ eventually} \right\} \]

\[ \leq \Pr \left( T_{1, T} > z_{\alpha_2} - \epsilon_T \right) \cap \left\{ q(T) \sim T_{\frac{1}{\alpha}}^{\frac{1}{\alpha} - 1} \right\} \cap \left\{ \rho_T < 1 - \left( \frac{z_{\gamma_1} + z_{\alpha_2}}{\sqrt{NT}} \right) \text{ eventually} \right\} \]

\[ + \Pr \left( \left| \xi_T \right| \geq \epsilon_T \right) \cap \left\{ q(T) \sim T_{\frac{1}{\alpha}}^{\frac{1}{\alpha} - 1} \right\} \cap \left\{ \rho_T < 1 - \left( \frac{z_{\gamma_1} + z_{\alpha_2}}{\sqrt{NT}} \right) \text{ eventually} \right\} \]

\[ \to 1 - \Phi(z_{\alpha_2}) = \alpha_2, \]
where the second inequality above results from the fact that in this case $\rho_T < 1 - \sqrt{2} (z_{\gamma_1} + z_{\alpha_2}) N^{-1/2} T^{-1} \iff \sqrt{NT} (1 - \rho_T) / \sqrt{2 - z_{\gamma_1}} > z_{\alpha_2}$. Moreover, since $\sqrt{NT} \ll \sqrt{NT} \sim q(T)$ in this case, we again have $\rho_T \in \mathcal{C}^{UR2}_{\gamma_2,\alpha_2,T}$ eventually. Hence, applying part (b) of Theorem 3.1, we obtain

$$\lim_{T \to \infty} \sup_{\rho \in \mathcal{C}_{\gamma,\alpha,T}} \Pr(\rho \notin \mathcal{C}_{\gamma,\alpha,T} | \rho = \rho_T) = \Pr(\rho \notin \mathcal{C}^M_{\alpha_1,T}, T_1,T \leq -z_{\gamma_1}, T_2,T \leq -z_{\gamma_2} | \rho = \rho_T) + \Pr(\rho \notin \mathcal{C}^{UR1}_{\gamma_1,\alpha_2,T}, T_1,T > -z_{\gamma_1} | \rho = \rho_T) + \Pr(\rho \notin \mathcal{C}^{UR2}_{\gamma_2,\alpha_2,T}, T_1,T \leq -z_{\gamma_1}, T_2,T > -z_{\gamma_2} | \rho = \rho_T) \leq \lim_{T \to \infty} \sup_{\rho \in \mathcal{C}^M_{\alpha_1,T}} \Pr(\rho = \rho_T) + \lim_{T \to \infty} \sup_{\rho \in \mathcal{C}^{UR1}_{\gamma_1,\alpha_2,T}} \Pr(T_1,T > -z_{\gamma_1} | \rho = \rho_T) + \lim_{T \to \infty} \sup_{\rho \in \mathcal{C}^{UR2}_{\gamma_2,\alpha_2,T}} \Pr(T_1,T \leq -z_{\gamma_1}, T_2,T > -z_{\gamma_2} | \rho = \rho_T) \leq \alpha_1 + \alpha_2 + 0 = \alpha.$$

Consider part (e), where we take $\rho_T \in \mathcal{G}^P_{\delta} = \{ \rho_T = \exp\left\{ -\frac{1}{q(T)} \right\} : \left( \sqrt{NT} \ll q(T) \right) \cap \left( T \ll q(T) \ll T^{1/2} \sim \sqrt{NT} \right) \}$. In this case, note that

$$\theta_T^* = \frac{M_{y/2}}{\sqrt{N}} (1 - \rho_T) = \frac{1}{\sigma \sqrt{2}} \sqrt{NT} \left[ 1 - \exp\left\{ -\frac{1}{q(T)} \right\} \right] [1 + o_p(1)]$$

$$= \frac{1}{\sqrt{2}} \sqrt{NT} \left[ 1 - 1 + \frac{1}{q(T)} + O\left( \frac{1}{q(T)^2} \right) \right] [1 + o_p(1)]$$

$$= \frac{1}{\sqrt{2}} q(T) \left[ 1 + O\left( \frac{1}{q(T)} \right) \right] [1 + o_p(1)] \to \infty, \text{ wpa.1},$$

so that $\frac{q(T)}{\sqrt{NT}} (T_1,T - \theta_T^* + z_{\gamma_1}) \frac{\rho_T}{1/\sqrt{2}} < 0$. Thus,

$$0 \leq \Pr(T_1,T > -z_{\gamma_1} | \rho = \rho_T \in \mathcal{G}^P_{\delta}) = \Pr\left( \frac{q(T)}{\sqrt{NT}} (T_1,T - \theta_T^* + z_{\gamma_1}) + \frac{1}{\sqrt{2}} > \frac{1}{\sqrt{2}} | \rho = \rho_T \in \mathcal{G}^P_{\delta} \right) \leq \Pr\left( \left| \frac{q(T)}{\sqrt{NT}} (T_1,T - \theta_T^* + z_{\gamma_1}) + \frac{1}{\sqrt{2}} \right| > \frac{1}{\sqrt{2}} | \rho = \rho_T \in \mathcal{G}^P_{\delta} \right) \to 0.$$

Moreover, since $\sqrt{NT} \ll q(T)$, we have $\rho_T \in \mathcal{C}^{UR2}_{\gamma_2,\alpha_2,T}$ eventually, as shown previously. Hence, applying part (b) of Theorem 3.1, we deduce that

$$\lim_{T \to \infty} \sup_{\rho \in \mathcal{C}_{\gamma,\alpha,T}} \Pr(\rho \notin \mathcal{C}_{\gamma,\alpha,T} | \rho = \rho_T) \leq \lim_{T \to \infty} \sup_{\rho \in \mathcal{C}^M_{\alpha_1,T}} \Pr(\rho = \rho_T) + \lim_{T \to \infty} \sup_{\rho \in \mathcal{C}^{UR1}_{\gamma_1,\alpha_2,T}} \Pr(T_1,T > -z_{\gamma_1} | \rho = \rho_T) + \lim_{T \to \infty} \sup_{\rho \in \mathcal{C}^{UR2}_{\gamma_2,\alpha_2,T}} \Pr(T_1,T \leq -z_{\gamma_1}, T_2,T > -z_{\gamma_2} | \rho = \rho_T) \leq \alpha_1 + 0 + 0 < \alpha.
Consider part (f), where we take
\[ \rho_T \in \mathcal{G}_6^p = \left\{ \rho_T = \exp \left\{ -\frac{1}{q(T)} \right\} : T \ll q(T) \sim T^{\frac{1}{2}(1+\epsilon)} \sim \sqrt{N_T} \cap \rho_T \geq 1 - \frac{2(z_{\gamma_1} + z_{\alpha_2})}{\sqrt{N_T}} \text{ eventually} \right\}. \]

Following the same argument as part (e) above, we have \( \theta_T^* = \frac{1}{\sqrt{2}q(T)} \left[ 1 + O \left( q(T)^{-1} \right) \right] \left[ 1 + o_T(1) \right] \to \infty \text{ w.p.a.1, so that} \]
\[ \frac{q(T)}{\sqrt{N_T}} \left( T_{1,T}^* - \theta_T^* + z_{\gamma_1} \right) \to -1/\sqrt{2} < 0. \]
It follows that
\[ \Pr \left( T_{1,T} > -z_{\gamma_1} | \rho = \rho_T \in \mathcal{G}_6^p \right) \leq \Pr \left( \left| \frac{q(T)}{\sqrt{N_T}} \left( T_{1,T}^* - \theta_T^* + z_{\gamma_1} \right) + \frac{1}{2} \right| \right) = 1 \Rightarrow \frac{1}{2} \rho = \rho_T \in \mathcal{G}_6^p \to 0. \]
Moreover, in this case, \( \rho_T \in \mathcal{C}_{\gamma_2,\alpha_2,T}^{UR} \) eventually by assumption. Using these results and part (b) of Theorem 3.1, we get
\[ \lim_{T \to \infty} \sup \Pr \left( \rho \notin \mathcal{C}_{\gamma,\alpha} | \rho = \rho_T \in \mathcal{G}_6^p \right) \leq \lim_{T \to \infty} \sup \Pr \left( \rho \notin \mathcal{C}_{\alpha_1} | \rho = \rho_T \in \mathcal{G}_6^p \right) \]
\[ + \lim_{T \to \infty} \sup \Pr \left( \rho \notin \mathcal{C}_{\gamma_2,\alpha_2,T}^{UR} | \rho = \rho_T \in \mathcal{G}_6^p \right) \]
\[ = \alpha_1 + 0 + 0 < \alpha. \]

Consider part (g), where we take
\[ \rho_T \in \mathcal{G}_7^p = \left\{ \rho_T = \exp \left\{ -\frac{1}{q(T)} \right\} : T \ll q(T) \sim T^{\frac{1}{2}(1+\epsilon)} \sim \sqrt{N_T} \cap \rho_T < 1 - \frac{2(z_{\gamma_1} + z_{\alpha_2})}{\sqrt{N_T}} \text{ eventually} \right\}. \]

Write \( T_{2,T} = \tilde{\omega}_{IVL} (\tilde{\rho}_{IVL} - 1) = \tilde{\omega}_{IVL} (\tilde{\rho}_{IVL} - \rho_T) - \tilde{\omega}_{IVL} (1 - \rho_T) = T_{2,T}^* - \vartheta_T^* \). In this case, by part (b) of Lemma SD-9,
\[ T_{2,T}^* = \tilde{\omega}_{IVL} (\tilde{\rho}_{IVL} - \rho_T) \]
\[ = \left( \frac{1}{\sigma^2} \frac{1}{\sqrt{N_T}} \right) \sum_{i=1}^{N} \sum_{t=3}^{T} \Delta y_{it-1} y_{it-1} \left( \sum_{i=1}^{N} \sum_{t=3}^{T} \Delta y_{it-1} y_{it-1} \right)^{-1} \sum_{i=1}^{N} \sum_{t=3}^{T} \Delta y_{it-1} [a_i (1 - \rho_T) + \varepsilon_{it}] \]
\[ = \frac{1}{\sigma^2} \frac{1}{\sqrt{N_T}} \sum_{i=1}^{N} \sum_{t=3}^{T} \Delta y_{it-1} [a_i (1 - \rho_T) + \varepsilon_{it}] \Rightarrow N(0,1). \]

In addition, by part (b) of Lemma SD-8, Lemma SD-12, and the Cramér convergence theorem, we have \( \vartheta_T^* = \left( \tilde{\vartheta}^2 N^{-1} T^{-1} \sum_{i=1}^{N} \sum_{t=3}^{T} \Delta y_{it-1} y_{it-1} \right) \sqrt{N_T} (1 - \rho_T) = \sqrt{N_T} (1 - \rho_T) + o_T(1), \) so that \( T_{2,T} = T_{2,T}^* - \sqrt{N_T} (1 - \rho_T) + \zeta_T, \) where \( \zeta_T = o_T(1). \) Now, let \( \epsilon_T \) be a sequence of positive numbers such that, as \( T \to \infty, \epsilon_T \to 0 \) but \( \zeta_T / \epsilon_T = o_T(1). \) It follows by argument similar to that given in part (d) above
that

\[
\Pr \left( T_{1, T} > -z_{\gamma_1} \mid \mathcal{T}_{1, T} \sim \sqrt{NT} \right) \leq \Pr \left( T_{1, T} > -z_{\gamma_1} \mid \mathcal{T}_{1, T} \sim \sqrt{NT} \right)
\]

where the last inequality is due to the fact that \( \rho_T < 1 - 2N^{-1/2}T^{-1/2} (z_{\gamma_2} + z_{\alpha_2}) \leq \sqrt{NT} (1 - \rho_T) / 2 - z_{\gamma_2} > z_{\alpha_2} \), implying, in turn, that \( \sqrt{NT} (1 - \rho_T) - z_{\gamma_2} > z_{\alpha_2} \). Moreover, similar to the proof of part (e) above, we have in this case \( \lim_{T \to \infty} \Pr \left( T_{1, T} > -z_{\gamma_1} \mid \rho = \rho_T \in \mathcal{G}_T^B \right) = 0 \). Hence, applying these results and applying part (b) of Theorem 3.1, we obtain

\[
\lim_{T \to \infty} \sup_{T} \Pr \left( \rho \notin \mathcal{C}_{\gamma_1, T} \mid \rho = \rho_T \in \mathcal{G}_T^P \right) \leq \lim_{T \to \infty} \sup_{T} \Pr \left( \rho \notin \mathcal{C}_{\alpha_1, T} \mid \rho = \rho_T \in \mathcal{G}_T^P \right) + \lim_{T \to \infty} \sup_{T} \Pr \left( T_{1, T} > -z_{\gamma_1} \mid \rho = \rho_T \in \mathcal{G}_T^P \right)
\]

Consider part (h), where we take \( \rho_T \in \mathcal{G}_S^P = \left\{ \rho_T = \exp \left\{ -\frac{1}{q (T)} \right\} : \sqrt{NT} \ll q (T) \sim T \right\} \). Note that, in this case,

\[
\theta_T^* = \frac{1}{\sigma} \frac{q (T)}{T} \left[ \exp \left\{ -\frac{2T}{q (T)} \right\} + \frac{2T}{q (T)} - 1 \right]^{1/2} \sqrt{NT} \left[ 1 - \exp \left\{ -\frac{1}{q (T)} \right\} \right] \left[ 1 + o_p (1) \right]
\]

so that \( N^{-1/2} \left[ \exp \left\{ -\frac{2T}{q (T)} \right\} + \frac{2T}{q (T)} - 1 \right]^{-1/2} \left( \mathcal{T}_{1, T} \sim \theta_T^* + z_{\gamma_1} \right) \xrightarrow{p} -1/2 < 0 \). Thus,

\[
\Pr \left( T_{1, T} > -z_{\gamma_1} \mid \rho = \rho_T \in \mathcal{G}_S^P \right) \leq \Pr \left( \mathcal{G}_S^P \right) \sim \frac{1}{\sqrt{N}} \left[ \exp \left\{ -\frac{2T}{q (T)} \right\} + \frac{2T}{q (T)} - 1 \right]^{-1/2} \left( \mathcal{T}_{1, T} \sim \theta_T^* + z_{\gamma_1} \right) \xrightarrow{p} \frac{1}{2} \mid \rho = \rho_T \in \mathcal{G}_S^P \right) \to 0.
\]
obtain
\[
\limsup_{T \to \infty} \Pr(\rho \notin \gamma_1, T | \rho = \rho_T \in \mathcal{G}_T^P) \\
\leq \limsup_{T \to \infty} \Pr(\rho \notin \gamma_1, T | \rho = \rho_T \in \mathcal{G}_T^P) + \limsup_{T \to \infty} \Pr(T_{1,T} > -z_{\gamma_1} | \rho = \rho_T \in \mathcal{G}_T^P) \\
+ \limsup_{T \to \infty} \Pr(\rho \notin \gamma_1, T | \rho = \rho_T \in \mathcal{G}_T^P) \\
= \alpha_1 + 0 + 0 < \alpha.
\]

Consider part (i), where we take \( \rho_T \in \mathcal{G}_T^P = \{ \rho_T = \exp\left(-\frac{1}{q(T)}\right) : \sqrt{NT} \sim T^{1+\frac{1}{\alpha}} \ll q(T) \ll T \} \).

Note that, for this case, \( \theta_T^* = \mathcal{M}^{1/2} \sqrt{NT \frac{q(T)}{(1 - \rho_T)}/\sigma} = \sqrt{\frac{NT}{\gamma(T)}} \left[ 1 + O\left(\frac{1}{q(T)}\right) \right] \left[ 1 + o_p(1) \right] \rightarrow \infty \).

Note, where we take \( \rho_T \in \mathcal{G}_T^P = \{ \rho_T = \exp\left(-\frac{1}{q(T)}\right) : \sqrt{NT} \sim T^{1+\frac{1}{\alpha}} \ll q(T) \ll T \} \).

wpa.1, where \( \mathcal{M}^{1/2} \sqrt{NT \frac{q(T)}{(1 - \rho_T)}/\sigma} = \sqrt{\frac{NT}{\gamma(T)}} \left[ 1 + O\left(\frac{1}{q(T)}\right) \right] \left[ 1 + o_p(1) \right] \rightarrow \infty \).

Thus,

\[
\Pr(T_{1,T} > -z_{\gamma_1} | \rho = \rho_T \in \mathcal{G}_T^P) \leq \Pr\left(\sqrt{\frac{q(T)}{NT}} (T_{1,T}^* - \theta_T^* + z_{\gamma_1}) + \frac{1}{\sqrt{2}} > \frac{1}{\sqrt{2}} | \rho = \rho_T \in \mathcal{G}_T^P \right) \rightarrow 0.
\]

Moreover, similar to previous parts, \( \rho_T \in \mathcal{G}_T^P \) eventually since \( \sqrt{NT} \ll q(T) \), so that

\[
\lim_{T \to \infty} \Pr(\rho \notin \gamma_1, T | \rho = \rho_T \in \mathcal{G}_T^P) = 0.
\]

Hence, applying part (d) of Theorem 3.1, we obtain

\[
\limsup_{T \to \infty} \Pr(\rho \notin \gamma_1, T | \rho = \rho_T \in \mathcal{G}_T^P) \\
\leq \limsup_{T \to \infty} \Pr(\rho \notin \gamma_1, T | \rho = \rho_T \in \mathcal{G}_T^P) + \limsup_{T \to \infty} \Pr(T_{1,T} > -z_{\gamma_1} | \rho = \rho_T \in \mathcal{G}_T^P) \\
+ \limsup_{T \to \infty} \Pr(\rho \notin \gamma_1, T | \rho = \rho_T \in \mathcal{G}_T^P) \\
= \alpha_1 + 0 + 0 < \alpha.
\]

Consider part (j), where we take

\[
\rho_T \in \mathcal{G}_T^P = \left\{ \rho_T = \exp\left(-\frac{1}{q(T)}\right) : q(T) \sim T^{1+\frac{1}{\alpha}} \sim \sqrt{NT} \sim T \cap \rho_T \geq 1 - \frac{2(z_{\gamma_2} + z_{\gamma_3})}{\sqrt{NT}} \text{ eventually} \right\}.
\]

Here, following the argument given in part (h) above, we have that \( \Pr(T_{1,T} > -z_{\gamma_1} | \rho = \rho_T \in \mathcal{G}_T^P) = 0. \)

Moreover, note that \( \rho_T \in \mathcal{G}_T^P \) eventually by assumption in this case, so that

\[
\limsup_{T \to \infty} \Pr(\rho \notin \gamma_1, T | \rho = \rho_T \in \mathcal{G}_T^P) = 0. \]

It follows, by applying part (c) of Theorem 3.1, that

\[
\limsup_{T \to \infty} \Pr(\rho \notin \gamma_1, T | \rho = \rho_T \in \mathcal{G}_T^P) \\
\leq \limsup_{T \to \infty} \Pr(\rho \notin \gamma_1, T | \rho = \rho_T \in \mathcal{G}_T^P) + \limsup_{T \to \infty} \Pr(T_{1,T} > -z_{\gamma_1} | \rho = \rho_T \in \mathcal{G}_T^P) \\
+ \limsup_{T \to \infty} \Pr(\rho \notin \gamma_1, T | \rho = \rho_T \in \mathcal{G}_T^P) \\
= \alpha_1 + 0 + 0 < \alpha.
\]

Consider part (k), where we take
\[
\rho_T \in \mathcal{G}_1^P = \left\{ \rho_T = \exp \left\{ -\frac{1}{T} \right\} : \left( q(T) \sim \sqrt{NT} \sim T \right) \cap \left( \rho_T < 1 - \frac{2(z_{\gamma_2} + z_{\alpha_2})}{\sqrt{NT}} \text{ eventually} \right) \right\}.
\]

For this case, write \( T_{2,T} = \widehat{\omega}_{IVL}(\hat{\rho}_{IVL} - 1) = \widehat{\omega}_{IVL}(\hat{\rho}_{IVL} - \rho_T) - \widehat{\omega}_{IVL}(1 - \rho_T) = T_{2,T}^* - \vartheta_T^* \). Applying part (c) of Lemma SD-9, Lemma SD-12, and the Cramér Convergence Theorem; we obtain
\[
T_{2,T}^* = \widehat{\omega}_{IVL}(\hat{\rho}_{IVL} - \rho_T)
= \left( \frac{1}{\sigma^2} \frac{1}{NT} \sum_{i=1}^N \sum_{t=3}^T \Delta y_{it-1} y_{it-1} \right) \sqrt{NT} (\hat{\rho}_{IVL} - \rho_T)
= \frac{1}{\sigma^2} \frac{1}{\sqrt{NT}} \sum_{i=1}^N \sum_{t=3}^T \Delta y_{it-1} [a_i (1 - \rho_T) + \varepsilon_{it}] \Rightarrow N(0, 1),
\]
while, applying part (c) of Lemma SD-8 and Lemma SD-12, we have
\[
\vartheta_T^* = \widehat{\omega}_{IVL}(1 - \rho_T)
= \left( \frac{1}{\sigma^2} \frac{1}{NT} \sum_{i=1}^N \sum_{t=3}^T \Delta y_{it-1} y_{it-1} \right) \sqrt{NT} (1 - \rho_T)
= \left( 1 - \frac{1}{4} \frac{q(T)}{T} \right) \left[ \exp \left\{ -\frac{2T}{q(T)} \right\} + \frac{2T}{q(T)} - 1 \right] \sqrt{NT} (1 - \rho_T) + o_p(1).
\]

These results imply that \( T_{2,T} = T_{2,T}^* - \varphi(T/q(T)) \sqrt{NT} (1 - \rho_T) + \zeta_T \), where \( \zeta_T = o_p(1) \) and where \( \varphi(T/q(T)) = 1 - \frac{1}{4} \frac{q(T)}{T} \left[ \exp \left\{ -\frac{2T}{q(T)} \right\} + \frac{2T}{q(T)} - 1 \right] \). Now, let \( \varepsilon_T \) be a sequence of positive numbers such that, as \( T \to \infty \), \( \varepsilon_T \to 0 \) but \( \zeta_T / \varepsilon_T = o_p(1) \). It follows by arguments similar to part (d) above that
\[
\Pr \left( T_{2,T} > -z_{\gamma_2} \bigg\{ q(T) \sim \sqrt{NT} \sim T \bigg\} \cap \left\{ \rho_T < 1 - \frac{2(z_{\gamma_2} + z_{\alpha_2})}{\sqrt{NT}} \text{ eventually} \right\} \right)
\leq \Pr \left( T_{2,T}^* - \varphi \left( \frac{T}{q(T)} \right) \sqrt{NT} (1 - \rho_T) > -(z_{\gamma_2} + \varepsilon_T) \bigg| \left\{ q(T) \sim \sqrt{NT} \sim T \right\} \cap \left\{ \rho_T < 1 - \frac{2(z_{\gamma_2} + z_{\alpha_2})}{\sqrt{NT}} \text{ eventually} \right\} \right)
\quad + \Pr \left( |\zeta_T| \geq \varepsilon_T \bigg| \left\{ q(T) \sim \sqrt{NT} \sim T \right\} \cap \left\{ \rho_T < 1 - \frac{2(z_{\gamma_2} + z_{\alpha_2})}{\sqrt{NT}} \text{ eventually} \right\} \right)
\quad + \Pr \left( \left| q(T) \sim \sqrt{NT} \sim T \bigg| \left\{ \rho_T < 1 - \frac{2(z_{\gamma_2} + z_{\alpha_2})}{\sqrt{NT}} \text{ eventually} \right\} \right) \right)
\leq \Pr \left( T_{2,T}^* > \frac{\sqrt{NT}(1 - \rho_T)}{2} - (z_{\gamma_2} + \varepsilon_T) \bigg| q(T) \sim \sqrt{NT} \sim T \cap \rho_T < 1 - \frac{2(z_{\gamma_2} + z_{\alpha_2})}{\sqrt{NT}} \text{ eventually} \right)
\quad + \Pr \left( |\zeta_T| \geq \varepsilon_T \bigg| \left\{ q(T) \sim \sqrt{NT} \sim T \right\} \cap \left\{ \rho_T < 1 - \frac{2(z_{\gamma_2} + z_{\alpha_2})}{\sqrt{NT}} \text{ eventually} \right\} \right)
\quad \leq \Pr \left( T_{2,T}^* > z_{\alpha_2} - \varepsilon_T \bigg| q(T) \sim \sqrt{NT} \sim T \right) \cap \left\{ \rho_T < 1 - \frac{2(z_{\gamma_2} + z_{\alpha_2})}{\sqrt{NT}} \text{ eventually} \right\}
\quad + \Pr \left( \left| q(T) \sim \sqrt{NT} \sim T \bigg| \left\{ \rho_T < 1 - \frac{2(z_{\gamma_2} + z_{\alpha_2})}{\sqrt{NT}} \text{ eventually} \right\} \right) \right)
\to 1 - \Phi(z_{\alpha_2}) = \alpha_2,
where the second inequality above follows from the fact that, by Lemma SE-34, $\varphi \left( \frac{T}{\sqrt{\eta(T)}} \right) \sqrt{NT} (1 - \rho_T) \geq \sqrt{NT(1 - \rho_T)}$ for $0 < \frac{T}{\sqrt{\eta(T)}} < \infty$, while the last inequality above is due to the fact that $\rho_T < 1 - \frac{2(z_{\gamma_2} + z_{\alpha_2})}{\sqrt{NT}} \iff \sqrt{NT(1 - \rho_T)} < z_{\gamma_2} > z_{\alpha_2}$. Moreover, similar to the proof of part (h) above, we have $\lim_{T \to \infty} P \left( T_{1,T} > -z_{\gamma_1} | \rho = \rho_T \in G_{11}^P \right) = 0$. Hence, applying part (c) of Theorem 3.1, we have

$$\lim \sup_{T \to \infty} \Pr \left( \rho \notin C_{\gamma,\alpha,T} | \rho = \rho_T \in G_{11}^P \right) \leq \lim \sup_{T \to \infty} \Pr \left( \rho \notin C_{\alpha_1,T}^M | \rho = \rho_T \in G_{11}^P \right) + \lim \sup_{T \to \infty} \Pr \left( T_{1,T} > -z_{\gamma_1} | \rho = \rho_T \in G_{11}^P \right) + \lim \sup_{T \to \infty} \Pr \left( T_{2,T} > -z_{\alpha_2} | \rho = \rho_T \in G_{11}^P \right)$$

$$= \alpha_1 + 0 + \alpha_2 = \alpha.$$

Consider part (i), where we take $\rho_T \in G_{12}^P = \left\{ \rho_T = \exp \left\{-1/q(T) \right\} : \left( q(T) \sim \sqrt{NT} \ll T \right) \cap \left( \rho_T \geq 1 - \frac{2(z_{\gamma_2} + z_{\alpha_2})}{\sqrt{NT}} \text{ eventually} \right) \right\}$. For this case, note that, by arguments similar to those of part (i) above, we have $\Pr \left( T_{1,T} > -z_{\gamma_1} | \rho = \rho_T \in G_{12}^P \right) \to 0$. $\rho_T \in C_{\gamma_2,\alpha_2,T}^{UR}$ eventually by assumption in this case, so that $\lim_{T \to \infty} \Pr \left( \rho \notin C_{\gamma_2,\alpha_2,T}^{UR} | \rho = \rho_T \in G_{12}^P \right) = 0$. Using these results and part (d) of Theorem 3.1, we obtain

$$\lim \sup_{T \to \infty} \Pr \left( \rho \notin C_{\gamma,\alpha,T} | \rho = \rho_T \in G_{12}^P \right) \leq \lim \sup_{T \to \infty} \Pr \left( \rho \notin C_{\alpha_1,T}^M | \rho = \rho_T \in G_{12}^P \right) + \lim \sup_{T \to \infty} \Pr \left( T_{1,T} > -z_{\gamma_1} | \rho = \rho_T \in G_{12}^P \right) + \lim \sup_{T \to \infty} \Pr \left( \rho \notin C_{\gamma_2,\alpha_2,T}^{UR} | \rho = \rho_T \in G_{12}^P \right)$$

$$= \alpha_1 + 0 + 0 < \alpha.$$

Consider part (m), where we take $\rho_T \in G_{13}^P = \left\{ \rho_T = \exp \left\{-1/q(T) \right\} : \left( q(T) \sim \sqrt{NT} \ll T \cap \rho_T < 1 - \frac{2(z_{\gamma_2} + z_{\alpha_2})}{\sqrt{NT}} \text{ eventually} \right) \right\}$. To proceed, again write $T_{2,T} = \tilde{\omega}_{IVL} (\tilde{\rho}_{IVL} - \rho_T) - \tilde{\omega}_{IVL} (1 - \rho_T) = T_{2,T} - \phi_T$). Here, note that $T_{2,T}^{1 / \kappa} (\frac{1}{\kappa^2} - 1) \ll T_{2,T}^{1 / \kappa + 1} \sim \sqrt{NT} \sim q(T)$ for $\kappa \in (\frac{1}{2}, \infty)$, so that by part (d) of Lemma SD-9, Lemma SD-12, and the Cramér convergence theorem, we obtain $T_{2,T} \Rightarrow N(0, 1)$; and, by part (d) of Lemma SD-8, Lemma SD-12, and the Slutsky Theorem,

$$\phi_T^* = \tilde{\omega}_{IVL} (1 - \rho_T) = \left( \frac{1}{\sigma^2} \frac{1}{\sqrt{NT}} \sum_{t=1}^{T} \sum_{i=3}^{T} \Delta y_{it-1} y_{i-1} \right) \sqrt{NT} (1 - \rho_T) = \sqrt{N} (1 - \rho_T) + \rho_p (1),$$

so that $T_{2,T} = T_{2,T} - \sqrt{NT} (1 - \rho_T) / 2 + \zeta_{2,T}$, where $\zeta_{2,T} = o_p (1)$. Now, let $\epsilon_T$ be a sequence of positive numbers such that, as $T \to \infty$, $\epsilon_T \to 0$ but $\zeta_{2,T} / \epsilon_T = o_p (1)$. Similar to the proof of part (k) above, we have

$$\Pr \left( T_{2,T} > -z_{\alpha_2} | \rho = \rho_T \in G_{13}^P \right) \leq \Pr \left( T_{2,T} > z_{\alpha_2} - \epsilon_T | \rho = \rho_T \in G_{13}^P \right) + \Pr \left( |\zeta_T| \geq \epsilon_T | \rho = \rho_T \in G_{13}^P \right)$$

$$\to 1 - \Phi (z_{\alpha_2}) = \alpha_2.$$
Moreover, similar to the proof of part (i) above, we have \( \lim_{T \to \infty} \Pr(T_{1,T} > -z_{\gamma_1} \mid \rho = \rho_T \in G_{13}^p) = 0 \). Hence, applying part (d) of Theorem 3.1, we have

\[
\begin{align*}
\lim_{T \to \infty} \sup \Pr(\rho \notin C_{\gamma,\alpha,T} \mid \rho = \rho_T \in G_{13}^p) & \leq \lim_{T \to \infty} \sup \Pr(\rho \notin C_{\theta_{\alpha,T}}^M \mid \rho = \rho_T \in G_{13}^p) + \lim_{T \to \infty} \Pr(T_{1,T} > -z_{\gamma_1} \mid \rho = \rho_T \in G_{13}^p) \\
& \quad + \lim_{T \to \infty} \Pr(T_{2,T} > -z_{\gamma_2} \mid \rho = \rho_T \in G_{13}^p) \\
& = \alpha_1 + 0 + \alpha_2 = \alpha.
\end{align*}
\]

Consider part (n), where we take \( \rho_T \in G_{14}^p = \left\{ \rho_T = \exp \left\{ -\frac{1}{q(T)} \right\} : T \ll q(T) \ll \sqrt{NT} \right\} \). By arguments similar to part (e) above, we have that \( \Pr(T_{1,T} > -z_{\gamma_1} \mid \rho = \rho_T \in G_{14}^p) \to 0 \). Moreover, write \( T_{2,T} = \hat{\omega}_{IVL}(\hat{\rho}_{IVL} - 1) = \hat{\omega}_{IVL}(\hat{\rho}_{IVL} - \rho_T) - \hat{\omega}_{IVL}(1 - \rho_T) = T_{2,T} - \hat{\vartheta}_T^* \). In this case, by part (b) of Lemma SD-9, Lemma SD-12, and the Cramér convergence theorem, we obtain

\[
\begin{align*}
T_{2,T}^* &= \hat{\omega}_{IVL}(\hat{\rho}_{IVL} - \rho_T) \\
&= \left( \frac{1}{\sigma^2} \frac{1}{NT} \sum_{i=1}^N \sum_{t=3}^T \Delta y_{it-1}y_{it-1} \right) \sqrt{NT}(\hat{\rho}_{IVL} - \rho_T) \\
&= \frac{1}{\sigma^2} \frac{1}{\sqrt{NT}} \sum_{i=1}^N \sum_{t=3}^T \Delta y_{it-1} \left[ a_i (1 - \rho_T) + \varepsilon_{it} \right] \Rightarrow N(0,1),
\end{align*}
\]

In addition, by part (b) of Lemma SD-8 and Lemma SD-12,

\[
\hat{\vartheta}_T^* = \left( \frac{1}{\sigma^2} \frac{1}{NT} \sum_{i=1}^N \sum_{t=3}^T \Delta y_{it-1}y_{it-1} \right) \sqrt{NT}(1 - \rho_T) = \frac{\sqrt{NT}}{q(T)} \left[ 1 + \alpha_p(1) \right] \to \infty \text{ wpa.1.}
\]

It follows that \( \frac{q(T)}{\sqrt{NT}} (T_{2,T}^* - \vartheta_T^* + z_{\gamma_2}) + 1 \overset{p}{\to} 0 \), from which we further deduce that

\[
\begin{align*}
\Pr(T_{2,T} > -z_{\gamma_2} \mid \rho = \rho_T \in G_{14}^p) &= \Pr\left( \frac{q(T)}{\sqrt{NT}} (T_{2,T}^* - \vartheta_T^* + z_{\gamma_2}) + 1 > 1 \mid \rho = \rho_T \in G_{14}^p \right) \\
&\leq \Pr\left( \frac{q(T)}{\sqrt{NT}} (T_{2,T}^* - \vartheta_T^* + z_{\gamma_2}) + 1 > 1 \mid \rho = \rho_T \in G_{14}^p \right) \to 0.
\end{align*}
\]

Using these results and part (b) of Theorem 3.1, we have

\[
\begin{align*}
\lim_{T \to \infty} \sup \Pr(\rho \notin C_{\gamma,\alpha,T} \mid \rho = \rho_T \in G_{14}^p) & \leq \lim_{T \to \infty} \sup \Pr(\rho \notin C_{\theta_{\alpha,T}}^M \mid \rho = \rho_T \in G_{14}^p) + \lim_{T \to \infty} \Pr(T_{1,T} > -z_{\gamma_1} \mid \rho = \rho_T \in G_{14}^p) \\
& \quad + \lim_{T \to \infty} \Pr(T_{2,T} > -z_{\gamma_2} \mid \rho = \rho_T \in G_{14}^p) \\
& \leq \alpha_1 + 0 + 0 = \alpha_1.
\end{align*}
\]

Consider part (o), where we take \( \rho_T \in G_{15}^p = \left\{ \rho_T = \exp \left\{ -\frac{1}{q(T)} \right\} : N^{1/3}T^{1/3} \ll q(T) \sim T \ll \sqrt{NT} \right\} \). By arguments similar to part (h) above, we have \( \Pr(T_{1,T} > -z_{\gamma_1} \mid \rho = \rho_T \in G_{15}^p) \to 0 \). Moreover, write
\[ T_{2,T} = \tilde{w}_{IVL} (\tilde{\rho}_{IVL} - 1) = \tilde{w}_{IVL} (\tilde{\rho}_{IVL} - \rho_T) - \tilde{w}_{IVL} (1 - \rho_T) = T_{2,T}^* - \vartheta_T^*. \] In this case, \( T_{2,T}^* \to N(0, 1) \) by part (c) of Lemma SD-9, Lemma SD-12, and the Cramér convergence theorem. Moreover, by part (c) of Lemma SD-8 and Lemma SD-12,

\[
\vartheta_T^* = \left( \frac{1}{\sigma^2} \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=3}^{T} \Delta y_{it-1} y_{it-1} \right) \sqrt{NT} (1 - \rho_T) = \varphi \left( \frac{T}{q(T)} \right) \sqrt{NT} (1 - \rho_T) [1 + o_p(1)]
\]

where \( \varphi \left( \frac{T}{q(T)} \right) = 1 - \frac{1}{q(T)} \left[ \exp \left( -\frac{2T}{q(T)} \right) + \frac{2T}{q(T)} - 1 \right] \geq \frac{1}{2} \) for all \( T/q(T) > 0 \). It follows that

\[
\sqrt{NT} \left( T_{2,T}^* - \vartheta_T^* + z_{\gamma_2} \right) + \varphi \left( \frac{T}{q(T)} \right) \to 0,
\]

so that by part (d) of Lemma SD-9, Lemma SD-12, and the Cramér convergence theorem, we obtain \( T_{2,T}^* \to N(0, 1) \). In addition, by part (d) of Lemma SD-8 and Lemma SD-12,

\[
\vartheta_T^* = \left( \frac{1}{\sigma^2} \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=3}^{T} \Delta y_{it-1} y_{it-1} \right) \sqrt{NT} (1 - \rho_T) = \frac{\sigma^2}{2} \frac{\sqrt{NT}}{q(T)} [1 + o_p(1)] = \frac{1}{2} \sqrt{NT} \frac{\sqrt{NT}}{q(T)} [1 + o_p(1)] \to \infty \text{ wpa 1},
\]

which gives

\[
\limsup_{T \to \infty} \text{Pr} \left( \rho \notin \mathbb{C}_{\gamma, \alpha, T} | \rho = \rho_T \in G_{15}^p \right) \leq \limsup_{T \to \infty} \text{Pr} \left( \rho \notin \mathbb{C}_{\alpha, T}^M | \rho = \rho_T \in G_{15}^p \right) + \limsup_{T \to \infty} \text{Pr} \left( T_{1,T}^* > -z_{\gamma_1} | \rho = \rho_T \in G_{15}^p \right)
\]

\[
= \limsup_{T \to \infty} \text{Pr} \left( \rho \notin \mathbb{C}_{\alpha, T}^M | \rho = \rho_T \in G_{15}^p \right) = \alpha_1 + 0 + 0 = \alpha_1.
\]

Consider part (p) where we take

\[
\rho_T \in G_{16}^p = \left\{ \rho_T = \exp \left\{ -\frac{1}{q(T)} \right\} : \left\{ N^{1/3} T^{1/3} \sim T^{1+\kappa} \leq q(T) \leq \sqrt{NT} \right\} \cap (q(T) \leq T) \right\}. \]

Here, again, by arguments similar to part (i), we have \( \text{Pr} \left( T_{1,T}^* > -z_{\gamma_1} | \rho = \rho_T \in G_{16}^p \right) \to 0 \). Moreover, note that, in this case, \( T^{1+\kappa} \leq q(T) \) for \( \kappa = \left( \frac{1}{2}, \infty \right) \), so that by part (d) of Lemma SD-9, Lemma SD-12, and the Cramér convergence theorem, we obtain \( T_{2,T}^* \to N(0, 1) \). In addition, by part (d) of Lemma SD-8 and Lemma SD-12,

\[
\vartheta_T^* = \left( \frac{1}{\sigma^2} \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=3}^{T} \Delta y_{it-1} y_{it-1} \right) \sqrt{NT} (1 - \rho_T) = \frac{1}{2} \frac{\sigma^2}{\sigma^2} \frac{\sqrt{NT}}{q(T)} [1 + o_p(1)] = \frac{1}{2} \frac{\sqrt{NT}}{q(T)} [1 + o_p(1)] \to \infty \text{ wpa 1}.
\]

It follows that

\[
\frac{q(T)}{\sqrt{NT}} \left( T_{2,T}^* - \vartheta_T^* + z_{\gamma_2} \right) \to -1/2 < 0,
\]

from which we further deduce that

\[
\text{Pr} \left( T_{2,T}^* > -z_{\gamma_2} | \rho = \rho_T \in G_{16}^p \right) \leq \text{Pr} \left( \left| \frac{q(T)}{\sqrt{NT}} \left( T_{2,T}^* - \vartheta_T^* + z_{\gamma_2} \right) \right| > \frac{1}{2} \right) \to \text{Pr} \left( \rho = \rho_T \in G_{16}^p \right) \to 0.
\]
Hence, using these results and part (d) of Theorem 3.1, we have
\[
\limsup_{T \to \infty} \Pr(\rho \notin \mathbb{C}_{\gamma, \alpha, T} | \rho = \rho_T \in \mathcal{G}_{16}^{p}) \\
\leq \limsup_{T \to \infty} \Pr(\rho \notin \mathbb{C}_{01, T}^{M} | \rho = \rho_T \in \mathcal{G}_{16}^{p}) + \limsup_{T \to \infty} \Pr(T_{1, T} > -z_{\gamma_1} | \rho = \rho_T \in \mathcal{G}_{16}^{p}) \\
+ \limsup_{T \to \infty} \Pr(T_{2, T} > -z_{\gamma_2} | \rho = \rho_T \in \mathcal{G}_{16}^{p}) \\
= \alpha_1 + 0 + 0 = \alpha_1 < \alpha.
\]

Consider part (q) where we take \(\rho_T \in \mathcal{G}_{17}^{p} = \{\rho_T = \exp\left\{-\frac{1}{q(T)}\right\} : q(T) \sim T^{1 + \frac{4\kappa}{3\kappa}} \sim N^{1/3}T^{1/3}\}.\) In this case, write \(T_{1, T} = \hat{T}_{1, T} - \hat{\theta}_T,\) where \(\hat{T}_{1, T} = \hat{\sigma}^{-1}\hat{M}_{yy}^{1/2} \sqrt{NT q(T)} \left(\hat{\rho}_{\text{pols}} - \rho_T - \frac{2\sigma^2}{q(T)^2} \right),\) \(\hat{\theta}_T = \hat{\sigma}^{-1}\hat{M}_{yy}^{1/2} \sqrt{NT q(T)} \left(1 - \rho_T - \frac{2\sigma^2}{q(T)^2} \right),\) and \(\hat{M}_{yy} = N^{-1}T^{-1}q(T)^{-1} \sum_{i=1}^{N} \sum_{i=2}^{T} (y_{it} - \bar{y}_{-1, NT})^2.\) Applying part (e) of Theorem SA-2 along with part (d) of Lemma SD-3 and Lemma SD-12, we have \(\hat{T}_{1, T} \Rightarrow N(0, 1)\) and \(\hat{\theta}_T = \sqrt{\frac{NT}{2q(T)}} [1 + o_p(1)],\) so that \(\sqrt{\frac{q(T)}{NT}} \left(\hat{T}_{1, T} - \hat{\theta}_T + z_{\gamma_1}\right) \overset{p}{\rightarrow} -1/\sqrt{2} < 0.\) Thus,
\[
\Pr(T_{1, T} > -z_{\gamma_1} | \rho = \rho_T \in \mathcal{G}_{17}^{p}) \leq \Pr\left(\left|\sqrt{\frac{q(T)}{NT}} \left(\hat{T}_{1, T} - \hat{\theta}_T + z_{\gamma_1}\right) + \frac{1}{\sqrt{2}}\right| > \frac{1}{\sqrt{2}} | \rho = \rho_T \in \mathcal{G}_{17}^{p}\right) \to 0.
\]
Moreover, note that, in this case, \(T_{1, T}^{1/\left(\frac{4\kappa}{3\kappa}\right)} \ll T^{1 + \frac{4\kappa}{3\kappa}} \sim q(T)\) for \(\kappa \in \left(\frac{1}{2}, \infty\right),\) so by arguments similar to part (o), we have \(\Pr(T_{2, T} > -z_{\gamma_2} | \rho = \rho_T \in \mathcal{G}_{17}^{p}) \to 0.\) Hence, using these results and part (d) of Theorem 3.1, we have
\[
\limsup_{T \to \infty} \Pr(\rho \notin \mathbb{C}_{\gamma, \alpha, T} | \rho = \rho_T \in \mathcal{G}_{17}^{p}) \\
\leq \limsup_{T \to \infty} \Pr(\rho \notin \mathbb{C}_{01, T}^{M} | \rho = \rho_T \in \mathcal{G}_{17}^{p}) + \limsup_{T \to \infty} \Pr(T_{1, T} > -z_{\gamma_1} | \rho = \rho_T \in \mathcal{G}_{17}^{p}) \\
+ \limsup_{T \to \infty} \Pr(T_{2, T} > -z_{\gamma_2} | \rho = \rho_T \in \mathcal{G}_{17}^{p}) \\
= \alpha_1 + 0 + 0 = \alpha_1 < \alpha.
\]
Consider part (r) where we take \(\rho_T \in \mathcal{G}_{18}^{p} = \{\rho_T = \exp\left\{-\frac{1}{q(T)}\right\} : q(T) \to \infty \text{ such that } q(T) / T^{1 + \frac{4\kappa}{3\kappa}} \to 0\}.\) In this case, again represent the unit root statistic \(T_{1, T} = \hat{T}_{1, T} - \hat{\theta}_T,\) where we take \(\hat{T}_{1, T} = \hat{M}_{yy}^{1/2} \sqrt{NT q(T)} \left(\hat{\rho}_{\text{pols}} - \rho_T - \frac{2\sigma^2}{q(T)^2} \right) / \hat{\sigma} = \hat{\sigma}^{-1}\hat{M}_{yy}^{1/2} \sqrt{NT q(T)} \left(\hat{\rho}_{\text{pols}} - \rho_T - \frac{2\sigma^2}{q(T)^2} \right)\) and where \(\hat{\sigma}\) and \(\hat{M}_{yy}\) are as defined previously. Now, applying part (f) of Theorem SA-2 along with part (d) of Lemma SD-3 and Lemma SD-12, we have \(\hat{T}_{1, T} = o_p\left(\frac{\sqrt{NT}}{q(T)^{1/2}}\right) = o_p(1)\) and \(\hat{\theta}_T = \sqrt{\frac{NT}{2q(T)}} [1 + o_p(1)],\) so that \(\sqrt{\frac{q(T)}{NT}} \left(\hat{T}_{1, T} - \hat{\theta}_T + z_{\gamma_1}\right) \overset{p}{\rightarrow} -1/\sqrt{2} < 0.\) Thus,
\[
\Pr(T_{1, T} > -z_{\gamma_1} | \rho = \rho_T \in \mathcal{G}_{18}^{p}) \leq \Pr\left(\left|\sqrt{\frac{q(T)}{NT}} \left(\hat{T}_{1, T} - \hat{\theta}_T + z_{\gamma_1}\right) + \frac{1}{\sqrt{2}}\right| > \frac{1}{\sqrt{2}} | \rho = \rho_T \in \mathcal{G}_{18}^{p}\right) \to 0.
\]
Now, write \(T_{2, T} = \tilde{\omega}_{IVL} (\hat{\rho}_{IVL} - 1) = \tilde{\omega}_{IVL} (\hat{\rho}_{IVL} - \rho_T) - \tilde{\omega}_{IVL} (1 - \rho_T) = T_{2, T}^{\ast} - \vartheta_{T}^{\ast}.\) Consider first the case where \(T_{1, T}^{1/\left(\frac{4\kappa}{3\kappa}\right)} \ll q(T) \ll T^{1 + \frac{4\kappa}{3\kappa}}.\) In this case, part (d) of Lemma SD-9, Lemma SD-12, and

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the Cramér convergence theorem imply that

\[
\frac{q(T)}{\sqrt{NT}} T_{2,T}^* = \frac{q(T)}{\sqrt{NT}} \frac{1}{\sigma^2} \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=3}^{T} \Delta y_{it-1} [a_i (1 - \rho_T) + \varepsilon_{it}] = O_p \left( \frac{q(T)}{\sqrt{NT}} \right).
\]

Next, consider the case where \( q(T) \to \infty \) such that \( q(T)/ T^{1/2}(\frac{1}{\kappa-1}) = O(1) \). In this case, we apply part (e) of Lemma SD-9 and Lemma SD-12 to obtain

\[
\frac{q(T)}{\sqrt{NT}} T_{2,T}^* = \frac{q(T)}{\sqrt{NT}} \frac{1}{\sigma^2} \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=3}^{T} \Delta y_{it-1} [a_i (1 - \rho_T) + \varepsilon_{it}]
= O \left( \frac{q(T)}{\sqrt{NT}} \right) O_p \left( \max \left\{ 1, \frac{T^{1/2}(\frac{1}{\kappa-1})}{q(T)} \right\} \right) = O_p \left( \max \left\{ \frac{q(T)}{\sqrt{NT}}, \frac{1}{T} \right\} \right).
\]

It follows that in either of the two cases above, we have that \( \left( \frac{q(T)}{\sqrt{NT}} \right) T_{2,T}^* = o_p(1) \). Moreover, note that applying part (d) of Lemma SD-8 and Lemma SD-12, we obtain

\[
\frac{q(T)}{\sqrt{NT}} \theta_T^* = \frac{q(T)}{\sqrt{NT}} \left( \frac{1}{\sigma^2} \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=3}^{T} \Delta y_{it-1} y_{it-1} \right) \frac{\sqrt{NT}}{1 - \rho_T} = \frac{1}{\sigma^2} \left[ 1 + o_p(1) \right] = \frac{1}{2} \left[ 1 + o_p(1) \right].
\]

Together, these results imply that \( \left( \frac{q(T)}{\sqrt{NT}} \right) \left( T_{2,T}^* - \theta_T^* + z_{\gamma_2} \right) \overset{p}{\to} -1/2 \), from which we further deduce that

\[
\Pr \left( T_{2,T}^* > -z_{\gamma_2} | \rho = \rho_T \in \mathcal{G}_{18}^P \right) \leq \Pr \left( \left| \frac{q(T)}{\sqrt{NT}} \left( T_{2,T}^* - \theta_T^* + z_{\gamma_2} \right) + \frac{1}{2} \right| > \frac{1}{2} | \rho = \rho_T \in \mathcal{G}_{18}^P \right) \to 0.
\]

Hence, using these results and part (d) of Theorem 3.1, we have

\[
\limsup_{T \to \infty} \Pr \left( \rho \notin \mathcal{C}_{\gamma, \alpha, T} | \rho = \rho_T \in \mathcal{G}_{18}^P \right) \leq \limsup_{T \to \infty} \Pr \left( \rho \notin \mathcal{C}_{\alpha, T}^{M} | \rho = \rho_T \in \mathcal{G}_{18}^P \right) + \limsup_{T \to \infty} \Pr \left( T_{1,T} > -z_{\gamma_1} | \rho = \rho_T \in \mathcal{G}_{18}^P \right) + \limsup_{T \to \infty} \Pr \left( T_{2,T}^* > -z_{\gamma_2} | \rho = \rho_T \in \mathcal{G}_{18}^P \right) = \alpha_1 + 0 + 0 = \alpha_1 < \alpha.
\]

Finally, we consider part (s), where

\[
\rho_T \in \mathcal{G}_{19}^P = \left\{ |\rho_T| = \exp \left\{ -\frac{1}{q(T)} \right\} : q(T) \geq 0 \text{ and } q(T) = O(1) \text{ as } T \to \infty \right\}.
\]

In this case, we decompose \( \overline{T} \) as \( \overline{T} = \overline{T} - \overline{\theta} \), where \( \overline{M}_{yy} = \frac{1}{1-\rho_T^2} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \overline{y}_{-1,NT})^2 \), \( \overline{T}_{1,T} = \tilde{\sigma}^{-1} \overline{M}_{yy}^{1/2} \sqrt{\frac{NT}{1-\rho_T^2}} (\overline{\rho}_{pols} - \rho_T) \), and \( \overline{\theta} = \tilde{\sigma}^{-1} \overline{M}_{yy}^{1/2} \sqrt{\frac{NT}{1-\rho_T^2}} (1 - \rho_T) \). Using part (g) of Theorem SA-2
Thus, along with part (e) of Lemma SD-3 and Lemma SD-12, we have in this case

\[
\tilde{M}_{yy} = (1 - \rho_T^2) \sigma_a^2 + \sigma^2 + o_p(1),
\]

\[
\frac{\tilde{T}_{1,T}}{\sqrt{NT}} = \sigma^{-1} \tilde{M}_{yy}^{1/2} \frac{(\rho_{pob} - \rho_T)}{\sqrt{1 - \rho_T^2}} = \sigma^{-1} \sqrt{\frac{(1 - \rho_T^2) \sigma_a^2 + \sigma^2}{1 - \rho_T^2}} \frac{(1 - \rho_T) \sigma_a^2}{\sigma_a^2 + \sigma^2 / (1 - \rho_T^2)} [1 + o_p(1)]
\]

\[
= \frac{(1 - \rho_T) \sigma_a^2}{\sigma \sqrt{\sigma_a^2 + \sigma^2 / (1 - \rho_T^2)}} [1 + o_p(1)]
\]

\[
\frac{\tilde{\theta}_T}{\sqrt{NT}} = \sigma^{-1} \tilde{M}_{yy}^{1/2} \frac{(1 - \rho_T)}{\sqrt{1 - \rho_T^2}} = \sigma^{-1} \sqrt{\frac{(1 - \rho_T) \sigma_a^2}{1 - \rho_T^2}} (1 + o_p(1)).
\]

Hence,

\[
\frac{T_{1,T}}{\sqrt{NT}} = \frac{\tilde{T}_{1,T}}{\sqrt{NT}} - \frac{\tilde{\theta}_T}{\sqrt{NT}}
\]

\[
= \frac{1}{\sigma} \left[ \sigma_a^2 + \sigma^2 \right]^{-1/2} \left[ (1 - \rho_T) \sigma_a^2 - \left( \sigma_a^2 + \frac{\sigma^2}{1 - \rho_T^2} \right) (1 - \rho_T) \right] [1 + o_p(1)]
\]

\[
= -\frac{1}{\sigma} \left[ \sigma_a^2 + \sigma^2 \right]^{-1/2} \frac{(1 - \rho_T) \sigma_a^2}{1 - \rho_T^2} [1 + o_p(1)]
\]

\[
= -\frac{\sigma \sqrt{1 - \rho_T^2}}{(1 + \rho_T) \sqrt{(1 - \rho_T^2) \sigma_a^2 + \sigma^2}} [1 + o_p(1)]
\]

\[
= -\psi (\sigma^2, \sigma_a^2, \rho_T) [1 + o_p(1)] < 0 \text{ wpa 1.}
\]

where \( \psi (\sigma^2, \sigma_a^2, \rho_T) = \sigma \sqrt{1 - \rho_T^2} \left[ (1 + \rho_T) \sqrt{(1 - \rho_T^2) \sigma_a^2 + \sigma^2} \right]^{-1} \), so that \( \sqrt{NT} \left( \tilde{T}_{1,T} - \tilde{\theta}_T + z_{\gamma_1} \right) \) + \( \psi (\sigma^2, \sigma_a^2, \rho_T) \overset{p}{\rightarrow} 0 \). Moreover, note that, by assumption, \( q(T) = O(1) \), so that for some positive constant \( C_q \), there exists some positive integer \( T^* \) such that for all \( T \geq T^* \), \( q(T) \leq C_q \), from which it follows that \( 0 \leq \rho_T^2 = \exp \left\{ -\frac{2}{\sqrt{T}} \right\} \leq \exp \left\{ -\frac{2}{C_q} \right\} < 1 \). It follows that

\[
\psi (\sigma^2, \sigma_a^2, \rho_T) = \frac{\sigma \sqrt{1 - \rho_T^2}}{(1 + \rho_T) \sqrt{(1 - \rho_T^2) \sigma_a^2 + \sigma^2}} \geq \frac{\sigma \sqrt{1 - \exp \left\{ -\frac{2}{C_q} \right\}}}{2 \sqrt{\sigma_a^2 + \sigma^2}} = \psi (\sigma^2, \sigma_a^2, C_q) > 0.
\]

Thus,

\[
\Pr \left( T_{1,T} > -z_{\gamma_1} \bigg| \rho = \rho_T \in G_{19}^p \right)
\]

\[
= \Pr \left( \frac{1}{\sqrt{NT}} \left( \tilde{T}_{1,T} - \tilde{\theta}_T + z_{\gamma_1} \right) > 0 \bigg| \rho = \rho_T \in G_{19}^p \right)
\]

\[
\leq \Pr \left( \frac{1}{\sqrt{NT}} \left( \tilde{T}_{1,T} - \tilde{\theta}_T + z_{\gamma_1} \right) + \psi (\sigma^2, \sigma_a^2, \rho_T) \bigg| \rho = \rho_T \in G_{19}^p \right) \rightarrow 0.
\]
Next, write $T_{2,T} = \hat{\omega}_{1V}(\hat{\rho}_{1V} - 1) = \hat{\omega}_{1V}(\hat{\rho}_{1V} - \rho_T) - \hat{\omega}_{1V}(1 - \rho_T) = T_{2,T}^* - \vartheta_T^*$. Applying part (f) of Lemma SD-9 and Lemma SD-12, we have

$$
\frac{T_{2,T}^*}{\sqrt{NT}} = \frac{1}{\sqrt{NT}} \left( 1 - \frac{1}{\sigma_N^2} \right) \sum_{i=1}^{N} \sum_{t=3}^{T} \Delta y_{it-1} y_{it-1} \sqrt{NT} (\rho_{1V} - \rho_T)
$$

$$
= \frac{1}{\sqrt{NT} \sigma^2} \left( 1 - \frac{1}{\sigma_N^2} \right) \sum_{i=1}^{N} \sum_{t=3}^{T} \Delta y_{it-1} \left[ a_i (1 - \rho_T) + \varepsilon_{it} \right] [1 + o_p (1)]
$$

$$
= O_p \left( \max \left\{ \frac{1}{\sqrt{NT}}, \frac{1}{T} \right\} \right) = o_p (1),
$$

and, by part (e) of Lemma SD-8 and Lemma SD-12,

$$
\vartheta_T^* = \hat{\omega}_{1V} (1 - \rho_T) = \left( 1 - \frac{1}{\sigma_N^2} \right) \sum_{i=1}^{N} \sum_{t=3}^{T} \Delta y_{it-1} y_{it-1} \sqrt{NT} (1 - \rho_T)
$$

$$
= \frac{1}{\sigma_N^2 (1 + \rho_T)} \sqrt{NT} (1 - \rho_T) [1 + o_p (1)] = \sqrt{NT} \frac{1 - \rho_T}{1 + \rho_T} [1 + o_p (1)] \to \infty \text{ wpa 1.}
$$

It follows that $N^{-1/2} T^{-1/2} \left( T_{2,T}^* - \vartheta_T^* + \varepsilon_{\gamma_2} \right) \overset{P}{\rightarrow} - (1 - \rho_T) / (1 + \rho_T) < 0$, from which we further deduce that

$$
\Pr \left( T_{2,T} > - \varepsilon_{\gamma_2} \right| \rho = \rho_T \in \mathcal{G}_{19}^p \right) \leq \Pr \left( \frac{1}{\sqrt{NT}} \left( T_{2,T}^* - \vartheta_T^* + \varepsilon_{\gamma_2} \right) + \frac{1 - \rho_T}{1 + \rho_T} \right) > \frac{1 - \rho_T}{1 + \rho_T} \Pr \left( \rho = \rho_T \in \mathcal{G}_{19}^p \right) \rightarrow 0.
$$

Using this result and part (e) of Theorem 3.1, we obtain

$$
\limsup_{T \to \infty} \Pr \left( \rho \notin [\gamma_{\alpha,T}] \right| \rho = \rho_T \in \mathcal{G}_{19}^p \right) \leq \limsup_{T \to \infty} \Pr \left( \rho \notin [\gamma_{\alpha,T}] \right| \rho = \rho_T \in \mathcal{G}_{19}^p \right) + \limsup_{T \to \infty} \Pr \left( T_{1,T} > - \varepsilon_{\gamma} \right| \rho = \rho_T \in \mathcal{G}_{19}^p \right)
$$

$$
+ \limsup_{T \to \infty} \Pr \left( T_{2,T} > - \varepsilon_{\gamma_2} \right| \rho = \rho_T \in \mathcal{G}_{19}^p \right)
$$

$$
= \alpha_1 + 0 + 0 = \alpha_1 < \alpha. \quad \Box
$$

**Appendix SC: Asymptotic Properties of the Anderson-Rubin Statistic**

An alternative way to formulate the estimation of $\rho$ in a panel autoregression with individual effects is as an IV regression problem in the sense of Anderson and Hsiao (1981, 1982). The Anderson-Hsiao approach begins by first-differencing the panel $AR(1)$ model, given by equation (3) in the main paper, to obtain

$$
\Delta y_{it} = \rho \Delta y_{it-1} + \Delta \varepsilon_{it}. \quad (6)
$$

The autoregressive parameter $\rho$ is then estimated from (6) using $y_{it-2}$ as the instrument based on the implicit first-stage equation

$$
\Delta y_{it-1} = a_i (1 - \rho_T) + (\rho_T - 1) y_{it-2} + \varepsilon_{it-1}. \quad (7)
$$
It is well known that a weak instrument problem occurs for the Anderson-Hsiao IV estimator when \( \rho_T \) is near unity, as can be seen from equation (7) above. Hence, one may think that the robust confidence procedures which have been developed and used successfully in the weak instrument literature can also be applied in a straightforward manner here to yield asymptotically valid interval estimates in the panel data setting. That turns out not to be the case. In particular, a well-known confidence procedure which is shown to be robust to the effects of weak instruments in the IV regression setup is obtained by inverting the Anderson-Rubin test statistic as shown in Staiger and Stock (1997). This test is also known to have good properties in the just identified case. However, direct application of the Anderson-Rubin procedure based on the IV regression setup given by equations (6) and (7) above does not lead to a confidence interval for \( \rho \) that is asymptotically valid. More specifically, the Anderson-Rubin statistic in this case has the form

\[
A_{NT} (\rho) = \frac{\left( \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2} \Delta \varepsilon_{it} \right)^2 \left( \sum_{i=1}^{N} \sum_{t=2}^{T} \frac{y_{it-2}^2}{\rho} \right)^{-1}}{\sqrt{\frac{1}{N^T-1} \left( \sum_{i=1}^{N} \sum_{t=2}^{T} (\Delta \varepsilon_{it})^2 - \left( \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2} \Delta \varepsilon_{it} \right)^2 \left( \sum_{i=1}^{N} \sum_{t=2}^{T} \frac{y_{it-2}^2}{\rho} \right)^{-1} \right)}}
\]

which depends on \( \rho \) via the term \( \Delta \varepsilon_{it} = \Delta y_{it} - \rho \Delta y_{it-1} \).

The following result gives the asymptotic behavior of \( A_{NT} (\rho) \) under the null hypothesis \( H_0 : \rho = \rho_T \) for alternative sequences \( \{\rho_T\} \).

**Theorem SC-1:**

Suppose that Assumptions 1-4 hold. Then, the following statements are true as \( N, T \to \infty \) such that \( N^\kappa / T = \tau \), for \( \kappa \in \left( \frac{1}{2}, \infty \right) \) and \( \tau \in (0, \infty) \).

(a) Suppose that \( \rho_T = 1 \) for all \( T \) sufficiently large. Then,

\[
TA_{NT} (\rho_T) \Rightarrow 2\chi_1^2
\]

(b) Suppose that \( \rho_T = \exp \{-1/q(T)\} \) such that \( T/q(T) \to 0 \). Then,

\[
TA_{NT} (\rho_T) \Rightarrow 2\chi_1^2
\]

(c) Suppose that \( \rho_T = \exp \{-1/q(T)\} \) such that \( q(T) \sim T \). Then,

\[
TA_{NT} (\rho_T) = \Phi(T) \chi_1^2 + o_p(1)
\]

or

\[
\frac{T}{\Phi(T)} A_{NT} (\rho_T) \Rightarrow \chi_1^2,
\]

where

\[
\Phi(T) = \frac{T}{q(T)} \left[ \frac{2T}{q(T)} + 1 - \exp \left\{ - \frac{2T}{q(T)} \right\} \right] \left[ \exp \left\{ - \frac{2T}{q(T)} \right\} + \frac{2T}{q(T)} - 1 \right]^{-1}.
\]

(d) Suppose that \( \rho_T = \exp \{-1/q(T)\} \) such that \( q(T) \to \infty \) but \( q(T)/T \to 0 \),

\[
q(T) A_{NT} (\rho_T) \Rightarrow \chi_1^2
\]
(e) Suppose that \( \rho_T \in \mathcal{G}_N = \{ \rho_T \mid \rho_T = \exp \left\{ -\frac{1}{q(T)} \right\} : q(T) \geq 0 \text{ and } q(T) = O(1) \text{ as } T \to \infty \} \). Then,

\[
\frac{1 + (1 - \rho_T^2) (\sigma_N^2 + \mu_0^2) / \sigma^2}{1 - \rho_T} A_{NT}(\rho_T) \Rightarrow \chi^2_1.
\]

This result shows that \( A_{NT}(\rho_T) \) is not uniformly convergent. In particular, note that \( A_{NT}(\rho_T) = o_p(1) \) for all unit root and local-to-unity parameter sequences. On the other hand, for parameter sequences associated with a stable panel autoregressive process, \( A_{NT}(\rho_T) \) does not converge in probability to zero and, in fact when appropriately rescaled, converges in distribution to a chi-squared distribution. Hence, the confidence interval obtained by inverting this statistic will not provide the correct asymptotic coverage uniformly over the parameter space \( \rho \in (-1, 1] \).

**Proof of Theorem SC-1:**

To proceed, note first that

\[
\frac{1}{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} E \left[ (\Delta \varepsilon_{it})^2 \right] = \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} E \left[ \varepsilon_{it}^2 - 2 \varepsilon_{it} \varepsilon_{it-1} + \varepsilon_{it-1}^2 \right] = 2\sigma^2 \left( \frac{T-1}{T} \right) = 2\sigma^2 \left[ 1 + O \left( \frac{1}{T} \right) \right].
\]

Moreover,

\[
E \left( \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} (\Delta \varepsilon_{it})^2 - 2\sigma^2 \left( \frac{T-1}{T} \right) \right)^2 = \frac{1}{N^2T^2} \sum_{i=1}^{N} \sum_{t=1}^{T} \left[ E \left( (\Delta \varepsilon_{it})^4 - 4\sigma^4 \left( \frac{T-1}{T} \right)^2 \right) \right] = \frac{1}{N^2T^2} \sum_{i=1}^{N} \sum_{t=2}^{T} \left[ E \left[ \varepsilon_{it}^4 + 6\varepsilon_{it}^2 \varepsilon_{it-1}^2 - 4\varepsilon_{it}^4 \left( \frac{T-1}{T} \right)^2 \right] \right] = \frac{1}{N^2T^2} \sum_{i=1}^{N} \sum_{t=1}^{T} \left[ 2E \left[ \varepsilon_{it}^4 \right] + 2\sigma^4 + \frac{8\sigma^4}{T} - \frac{4\sigma^4}{T^2} \right] = 2 \left( \frac{T-1}{NT} \right) \left[ (E \left[ \varepsilon_{it}^4 \right] + \sigma^4) + \frac{4\sigma^4}{T} - \frac{2\sigma^4}{T^2} \right] = O \left( \frac{1}{NT} \right).
\]

It follows by Markov’s inequality that

\[
\frac{1}{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} (\Delta \varepsilon_{it})^2 = 2\sigma^2 + O \left( \frac{1}{T} \right) + O_p \left( \frac{1}{\sqrt{NT}} \right).
\]
Consider first part (a), where we take $\rho_T = 1$ for all $T$ sufficiently large. In this case, we have by part (a) of Lemma SD-2, Lemma SE-24, and the Cramér Covergence Theorem, that

$$
\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2}\Delta\varepsilon_{it}
= - \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} \varepsilon_{it-2}\varepsilon_{it-1} + \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} w_{iT-2}\varepsilon_{iT} + O_p \left( \frac{1}{\sqrt{T}} \right)
\Rightarrow N(0, 2\sigma^4)
$$

and, by part (a) of Lemma SE-11, and Lemma SE-14, and part (a) of Lemma SE-18 that

$$
\frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2}^2 = \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} (a_i + w_{it-2})^2
= \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-2}^2 + \frac{2}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} a_i w_{it-2} + \frac{T-1}{NT^2} \sum_{i=1}^{N} a_i^2
= \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-2}^2 + O_p \left( \frac{1}{\sqrt{NT}} \right) + O_p \left( \frac{1}{T} \right) \Rightarrow \frac{\sigma^2}{2}
$$

so that

$$
TA_{NT} (\rho_T)
= \frac{1}{T} \left( \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2}\Delta\varepsilon_{it} \right)^2 \left( \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2}^2 \right)^{-1}
= \left[ \left( \sum_{i=1}^{N} \sum_{t=2}^{T} (\Delta\varepsilon_{it})^2 \right) - \left( \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2}\Delta\varepsilon_{it} \right) \left( \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2}^2 \right)^{-1} \left( \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2}\Delta\varepsilon_{it} \right) \left( \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2}^2 \right)^{-1} \right] / (NT - 1)
= \left[ \left( \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2}\Delta\varepsilon_{it} \right) \left( \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2}^2 \right)^{-1} \left( \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2}\Delta\varepsilon_{it} \right) \left( \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2}^2 \right)^{-1} \right]
\Rightarrow 2\sigma^4\chi^2 \left( \frac{\sigma^2}{2} \right)^{-1} \frac{1}{2\sigma^4} \equiv 2\chi^2.
$$

Next, consider part (b), where we take $\rho_T = \exp \{-1/q(T)\}$ such that $T/q(T) \to 0$. In this case, we have, by part (b) of Lemma SD-2, part (a) of Lemma SE-22, and the Cramér convergence theorem, that

$$
\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2}\Delta\varepsilon_{it} = - \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} \varepsilon_{it-2}\varepsilon_{it-1} + \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} w_{iT-2}\varepsilon_{iT} + O_p \left( \frac{1}{\sqrt{T}} \right)
\Rightarrow N(0, 2\sigma^4),
$$

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and, by part (a) of Lemma SE-11, Lemma SE-15, and part (b) of Lemma SE-18 that

\[
\frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2}^2 = \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} (a_i + w_{it-2})^2
\]

\[
= \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-2}^2 + \frac{2}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} a_i w_{it-2} + \frac{T-1}{NT^2} \sum_{i=1}^{N} a_i^2
\]

\[
= \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-2}^2 + O_p \left( \frac{1}{\sqrt{NT}} \right) + O_p \left( \frac{1}{T} \right) \xrightarrow{p} \frac{\sigma^2}{2},
\]

so that

\[
TA_{NT} (\rho_T)
\]

\[
= \left( \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2} \Delta \varepsilon_{it} \right)^2 \left( \frac{1}{\sqrt{NT^2}} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2}^2 \right)^{-1}
\]

\[
= \left[ \left( \sum_{i=1}^{N} \sum_{t=2}^{T} (\Delta \varepsilon_{it})^2 \right) - \left( \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2} \Delta \varepsilon_{it} \right)^2 \left( \frac{1}{\sqrt{NT^2}} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2}^2 \right)^{-1} \right] \quad \text{(NT - 1)}
\]

\[
= \left[ \left( \frac{1}{NT-T} \sum_{i=1}^{N} \sum_{t=2}^{T} (\Delta \varepsilon_{it})^2 \right) - \left( \frac{1}{\sqrt{NT-1}} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2} \Delta \varepsilon_{it} \right)^2 \left( \frac{1}{\sqrt{NT^2}} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2}^2 \right)^{-1} \right] \quad \text{(NT - 1)}
\]

\[
\Rightarrow 2\sigma^4 \chi^2 \left( \frac{\sigma^2}{2} \right) \xrightarrow{\text{d}} 2\chi^2.
\]

Now, consider part (c), where we take \( \rho_T = \exp \{-1/\{q(T)\} \} \) such that \( q(T) \sim T \). In this case, we have, by part (c) of Lemma SD-2 and part (b) of Lemma SE-22, that

\[
\frac{1}{\omega_T \sqrt{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2} \Delta \varepsilon_{it}
\]

\[
= -\frac{1}{\omega_T \sqrt{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} \varepsilon_{it-2} \varepsilon_{it-1} + \frac{1}{\omega_T \sqrt{NT}} \sum_{i=1}^{N} w_{it-2} \varepsilon_{it-1} + O_p \left( \frac{1}{\sqrt{T}} \right)
\]

\[
\Rightarrow N(0,1)
\]

where \( \omega_T = \sigma^2 \sqrt{1+(q(T)/2T)[1-\exp\{-2T/q(T)\}]} \). Moreover, by part (a) of Lemma SE-11, part
(a) of Lemma SE-17, and part (c) of Lemma SE-18 that

$$\frac{1}{Nq(T)^2} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2}$$

$$= \frac{T^2}{q(T)^2} \left[ \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-2}^2 + \frac{2}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} a_i w_{it-2} + \frac{T-1}{NT^2} \sum_{i=1}^{N} a_i^2 \right]$$

$$= \frac{T^2}{q(T)^2} \left[ \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-2}^2 + O_p \left( \frac{1}{\sqrt{NT}} \right) + O_p \left( \frac{1}{T} \right) \right]$$

$$= \frac{\sigma^2}{4} \left[ \exp \left\{ -\frac{2T}{q(T)} \right\} + \frac{2T}{q(T)} - 1 \right] \left[ 1 + O_p \left( \max \left\{ \frac{1}{\sqrt{NT}}, \frac{1}{T} \right\} \right) \right].$$

It follows that

$$T_{AN} (\rho_T)$$

$$= \frac{\omega^2 T^2}{q(T)^2} \left( \frac{1}{\omega_T^2 \sqrt{N} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2}^2} \right)^{-1} \exp \left\{ \frac{1}{Nq(T)^2} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2}^2 \right\} - 1$$

$$= \frac{\omega^2 T^2}{q(T)^2} \left( \frac{1}{\omega_T^2 \sqrt{NT-1} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2}^2} \right)^{-1} \exp \left\{ \frac{1}{Nq(T)^2} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2}^2 \right\} - 1$$

$$= \frac{\omega^2 T^2}{q(T)^2} \left[ \frac{\sigma^2}{4} \left[ \exp \left\{ -\frac{2T}{q(T)} \right\} + \frac{2T}{q(T)} - 1 \right] \right]^{-1} \frac{1}{2\sigma^2} + o_p (1)$$

$$= \frac{T^2}{q(T)^2} \left\{ 4\sigma^4 \left[ 1 + \frac{q(T)}{2T} \left[ 1 + \exp \left\{ -\frac{2T}{q(T)} \right\} \right] \right] \left[ \exp \left\{ -\frac{2T}{q(T)} \right\} + \frac{2T}{q(T)} - 1 \right] \right]^{-1} + o_p (1)$$

$$= \frac{2T^2}{q(T)^2} \left\{ \frac{T}{q(T)} \left[ 1 + \exp \left\{ -\frac{2T}{q(T)} \right\} \right] \right\} \left[ \exp \left\{ -\frac{2T}{q(T)} \right\} \right]^{-1} + o_p (1)$$

$$= \frac{T}{q(T)} \left\{ 1 + \exp \left\{ -\frac{2T}{q(T)} \right\} \right\} \left[ \exp \left\{ -\frac{2T}{q(T)} \right\} \right]^{-1} + o_p (1)$$

$$= \Phi (T) \chi^2 + o_p (1)$$

where

$$\Phi (T) = \frac{T}{q(T)} \left[ 1 - \exp \left\{ -\frac{2T}{q(T)} \right\} \right] \left[ \exp \left\{ -\frac{2T}{q(T)} \right\} \right]^{-1}.$$
Cramér convergence theorem, that
\[ \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2} \Delta \varepsilon_{it} \]
\[ = - \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=4}^{T} \varepsilon_{it-2} \varepsilon_{it-1} + O_{p} \left( \max \left\{ \sqrt{\frac{q(T)}{T}}, \frac{1}{\sqrt{q(T)}} \right\} \right) \]
\[ \Rightarrow N(0, \sigma^4), \]
and, by part (a) of Lemma SE-11, part (b) of Lemma SE-17, and part (d) of Lemma SE-18, that
\[ \frac{1}{NTq(T)} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2}^2 = \frac{1}{NTq(T)} \sum_{i=1}^{N} \sum_{t=2}^{T} (a_i + w_{it-2})^2 \]
\[ = \frac{1}{NTq(T)} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-2}^2 + \frac{2}{NTq(T)} \sum_{i=1}^{N} \sum_{t=2}^{T} a_i w_{it-2} + \frac{T-1}{NTq(T)} \sum_{i=1}^{N} a_i^2 \]
\[ = \frac{1}{NTq(T)} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-2}^2 + O_{p} \left( \frac{1}{\sqrt{NT}} \right) + O_{p} \left( \frac{1}{q(T)} \right) \xrightarrow{p} \frac{\sigma^2}{2}. \]

It follows that
\[ q(T) A_{NT}(\rho_T) \]
\[ = q(T) \sigma^4 \left( \frac{1}{\sigma^2 \sqrt{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2} \Delta \varepsilon_{it} \right)^2 \left( \frac{1}{NTq(T)} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2}^2 \right)^{-1} \]
\[ \leq \left( \sum_{i=1}^{N} \sum_{t=2}^{T} (\Delta \varepsilon_{it})^2 \right) - \left( \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2} \Delta \varepsilon_{it} \right)^2 \left( \frac{1}{NTq(T)} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2}^2 \right)^{-1} \]
\[ \sigma^4 \left( \frac{1}{\sigma^2 \sqrt{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2} \Delta \varepsilon_{it} \right)^2 \left( \frac{1}{NTq(T)} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2}^2 \right)^{-1} \]
\[ \Rightarrow \sigma^4 \chi_1 \left( \frac{\sigma^2}{2} \right)^{-1} = \frac{\chi_1^2}{2}. \]

Finally, consider part (e), where we take \( \rho_T \in \mathcal{G}_{St} \). In this case, we have, by part (e) of Lemma SD-2
and Lemma SE-23, that

\[
\sqrt{\frac{1 + \rho_T}{2\sigma^4}} \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2} \Delta \varepsilon_{it}
= - \sqrt{\frac{1 + \rho_T}{2\sigma^4}} \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=4}^{T} \varepsilon_{it-2} \varepsilon_{it-1}
+ (1 - \rho_T) \sqrt{\frac{1 + \rho_T}{2\sigma^4}} \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=4}^{T} w_{it-3} \varepsilon_{it-1} + o_p(1)
\Rightarrow \, N(0,1),
\]

and, by part (a) of Lemma SE-11, part (c) of Lemma SE-17, and part (e) of Lemma SE-18, that

\[
\frac{1 - \rho_T^2}{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2}^2
= \frac{1 - \rho_T^2}{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} (a_i + w_{it-2})^2
= \frac{1 - \rho_T^2}{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-2}^2 + (1 - \rho_T^2) \frac{T - 1}{NT} \sum_{i=1}^{N} a_i^2 + 2 \frac{1 - \rho_T^2}{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} a_i w_{it-2}
= \frac{1 - \rho_T^2}{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-2}^2 + (1 - \rho_T^2) \frac{T - 1}{NT} \sum_{i=1}^{N} a_i^2 + O_p \left( \frac{1}{\sqrt{NT}} \right)
= \sigma^2 + (1 - \rho_T^2) \left( \sigma_a^2 + \mu_a^2 \right) + o_p(1).
\]

It follows that

\[
1 + \frac{(1 - \rho_T^2) \left( \sigma_a^2 + \mu_a^2 \right)}{\sigma^2} \frac{1}{1 - \rho_T} A_{NT}(\rho_T)
= \frac{2\sigma^4 \left[ 1 + (1 - \rho_T^2)(\sigma_a^2 + \mu_a^2)/\sigma^2 \right] \left( 1 - \rho_T^2 \right)}{(1 - \rho_T)(1 + \rho_T)} \left( \sqrt{\frac{1 + \rho_T}{2\sigma^4}} \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2} \Delta \varepsilon_{it} \right)^2 \left( \frac{1 - \rho_T^2}{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2}^2 \right)^{-1}
= \left[ \left( \sum_{i=1}^{N} \sum_{t=2}^{T} (\Delta \varepsilon_{it})^2 \right) - \left( \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2} \Delta \varepsilon_{it} \right)^2 \left( \frac{1 - \rho_T^2}{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2}^2 \right) \right] / (NT - 1)
= \frac{2\sigma^2 \left[ \sigma^2 + (1 - \rho_T^2) \left( \sigma_a^2 + \mu_a^2 \right) \right] \left( \sqrt{\frac{1 + \rho_T}{2\sigma^4}} \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2} \Delta \varepsilon_{it} \right)^2 \left( \frac{1 - \rho_T^2}{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2}^2 \right)^{-1}}{\left( \sqrt{\frac{1 + \rho_T}{2\sigma^4}} \frac{1}{\sqrt{NT-1}} \sum_{i=1}^{N} \sum_{t=2}^{T} (\Delta \varepsilon_{it})^2 \right)^2 - \left( \frac{1}{\sqrt{NT-1}} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2} \Delta \varepsilon_{it} \right)^2 \left( \frac{1 - \rho_T^2}{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2}^2 \right) \left( \frac{1 - \rho_T^2}{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2}^2 \right)^{-1}}
= \frac{2\sigma^2 \left[ \sigma^2 + (1 - \rho_T^2) \left( \sigma_a^2 + \mu_a^2 \right) \right]}{\sigma^2 + (1 - \rho_T^2) \left( \sigma_a^2 + \mu_a^2 \right)} \frac{1}{2\sigma^2 \lambda^2} + o_p(1)
\Rightarrow \lambda^2 \frac{1}{2}. \square
Appendix SD: Proof of Key Supporting Lemmas

Lemma SD-1:

Under Assumptions 1-4, the following statements are true as as $N, T \to \infty$ such that $N^\kappa/T \to \tau$, for constants $\kappa \in \left(\frac{1}{2}, \infty\right)$ and $\tau \in (0, \infty)$.

(a) If $\rho_T = 1$ for all $T$ sufficiently large, then

$$
\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2} \Delta y_{it-1} = \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-2} \epsilon_{it-1} + O_p \left( \frac{1}{\sqrt{T}} \right).
$$

(b) If $\rho_T = \exp \{ -1/q(T) \}$ such that $T^{1+\frac{1}{q}} \ll q(T)$, then

$$
\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2} \Delta y_{it-1} = \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-2} \epsilon_{it-1} + O_p \left( \max \left\{ \frac{T^{1+\frac{1}{q}}}{q(T)}, \frac{1}{\sqrt{T}} \right\} \right).
$$

(c) If $\rho_T = \exp \{ -1/q(T) \}$ such that $q(T) \sim T^{1+\frac{1}{q}}$, then

$$
\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2} \Delta y_{it-1} = \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-2} \epsilon_{it-1} + O_p \left( \frac{1}{\sqrt{T}} \right).
$$

(d) If $\rho_T = \exp \{ -1/q(T) \}$ such that $T \ll q(T) \ll T^{1+\frac{1}{q}}$, then

$$
\frac{q(T)}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2} \Delta y_{it-1} = - \left[ \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-2}^2 \right] + O_p \left( \frac{q(T)}{\sqrt{NT}} \right).
$$

(e) If $\rho_T = \exp \{ -1/q(T) \}$ such that $q(T) \sim T$, then

$$
\frac{1}{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2} \Delta y_{it-1} = - \frac{T}{q(T)} \left[ \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-2}^2 \right] + O_p \left( \frac{1}{\sqrt{N}} \right).
$$

(f) If $\rho_T = \exp \{ -1/q(T) \}$ such that $q(T) \to \infty$ but $q(T)/T \to 0$, then

$$
\frac{1}{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2} \Delta y_{it-1} = - \left[ \frac{1}{NTq(T)} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-2}^2 \right] + O_p \left( \max \left\{ \sqrt{\frac{q(T)}{NT}}, \frac{1}{\sqrt{T}} \right\} \right).
$$

(g) If $\rho_T \in G_{\text{St}} = \left\{ \rho_T \right\}$ such that $q(T) \geq 0$ and $q(T) = O(1)$ as $T \to \infty$, then

$$
\frac{1 + \rho_T}{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2} \Delta y_{it-1} = - \left( \frac{1 - \rho_T^2}{NT} \right) \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-2}^2 + O_p \left( \max \left\{ \frac{1}{\sqrt{NT}}, \frac{1}{T} \right\} \right).
$$
Proof:
To proceed, we first make some preliminary calculations. Note that
\[
\sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2} \Delta y_{it-1} = \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2} \varepsilon_{it-1} + (1 - \rho_T) \sum_{i=1}^{N} \sum_{t=2}^{T} a_i y_{it-2} - (1 - \rho_T) \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2}^2
\]
\[
= \sum_{i=1}^{N} \sum_{t=2}^{T} (a_i + w_{it-2}) \varepsilon_{it-1} + (1 - \rho_T) \sum_{i=1}^{N} \sum_{t=2}^{T} a_i (a_i + w_{it-2})
\]
\[- (1 - \rho_T) \sum_{i=1}^{N} \sum_{t=2}^{T} (a_i^2 + 2a_i w_{it-2} + w_{it-2}^2)
\]
\[= \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-2} \varepsilon_{it-1} + \sum_{i=1}^{N} \sum_{t=2}^{T} a_i \varepsilon_{it-1} - (1 - \rho_T) \sum_{i=1}^{N} \sum_{t=2}^{T} a_i w_{it-2} - (1 - \rho_T) \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-2}^2.
\]

Now, consider part (a), where we take \(\rho_T = 1\) for all \(T\) sufficiently large. In this case, we can apply part (c) of Lemma SE-11, Lemma SE-14, and part (a) of Lemma SE-18 to obtain
\[
\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2} \Delta y_{it-1} = \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-2} \varepsilon_{it-1} + \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} a_i \varepsilon_{it-1} - (1 - \rho_T) \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} a_i w_{it-2}
\]
\[- (1 - \rho_T) \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-2}^2
\]
\[= \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-2} \varepsilon_{it-1} + O_p \left( \frac{1}{\sqrt{T}} \right) + O_p \left( (1 - \rho_T) \max \left\{ \sqrt{T}, \sqrt{N} \right\} \right)
\]
\[+ O_p \left( (1 - \rho_T) \sqrt{NT} \right)
\]
\[= \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-2} \varepsilon_{it-1} + O_p \left( \frac{1}{\sqrt{T}} \right).
\]
To show parts (b)-(f), we further note that

\[
\sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2} \Delta y_{it-1} \\
= \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-2} \varepsilon_{it-1} + \sum_{i=1}^{N} \sum_{t=2}^{T} a_{i} \varepsilon_{it-1} - (1 - \rho_{T}) \sum_{i=1}^{N} \sum_{t=2}^{T} a_{i} w_{it-2} - (1 - \rho_{T}) \sum_{i=1}^{N} \sum_{t=2}^{T} \varepsilon_{it-1} \]

\[
= \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-2} \varepsilon_{it-1} + \sum_{i=1}^{N} \sum_{t=2}^{T} a_{i} \varepsilon_{it-1} - \left[1 - \exp\left(-\frac{1}{q(T)}\right)\right] \sum_{i=1}^{N} \sum_{t=2}^{T} a_{i} w_{it-2} \\
- \left[1 - \exp\left(-\frac{1}{q(T)}\right)\right] \sum_{i=1}^{N} \sum_{t=2}^{T} \varepsilon_{it-1} \]

\[
= \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-2} \varepsilon_{it-1} + \sum_{i=1}^{N} \sum_{t=2}^{T} a_{i} \varepsilon_{it-1} - \left[1 - \exp\left(-\frac{1}{q(T)}\right)\right] \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-2} \\
- \left[1 - \exp\left(-\frac{1}{q(T)}\right)\right] \sum_{i=1}^{N} \sum_{t=2}^{T} a_{i} \varepsilon_{it-1} \]

\[
= \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-2} \varepsilon_{it-1} + \sum_{i=1}^{N} \sum_{t=2}^{T} a_{i} \varepsilon_{it-1} - \left[1 - \exp\left(-\frac{1}{q(T)}\right)\right] \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-2} \\
- \left[1 - \exp\left(-\frac{1}{q(T)}\right)\right] \sum_{i=1}^{N} \sum_{t=2}^{T} a_{i} \varepsilon_{it-1} \]

\[
= \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2} \Delta y_{it-1} \\
= \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-2} \varepsilon_{it-1} - \frac{T \sqrt{N}}{q(T)} \left[1 + O\left(\frac{1}{q(T)}\right)\right] \\
+ \frac{1}{\sqrt{T}} \left[1 + O\left(\frac{1}{q(T)}\right)\right] \sum_{i=1}^{N} \sum_{t=2}^{T} \frac{\varepsilon_{it-1}}{\sqrt{NT}} \left[1 + O\left(\frac{1}{q(T)}\right)\right] \sum_{i=1}^{N} \sum_{t=2}^{T} \frac{w_{it-2}}{\sqrt{NT}} \left[1 + O\left(\frac{1}{q(T)}\right)\right] \\
= \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-2} \varepsilon_{it-1} + O_{p}\left(\frac{T \sqrt{N}}{q(T)}\right) + O_{p}\left(\frac{1}{\sqrt{T}}\right) + O_{p}\left(\max\left\{\sqrt{T}, \frac{\sqrt{N}}{q(T)}\right\}\right) \\
= \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-2} \varepsilon_{it-1} + O_{p}\left(\max\left\{\frac{T^{1+\frac{1}{2}}}{q(T)}, \frac{1}{\sqrt{T}}\right\}\right),
\]

as required for part (b).

Consider part (c), where we take \(\rho_{T} = \exp\{-1/q(T)\}\) such that \(q(T) \sim T^{1+\frac{1}{2}} \sim T\sqrt{N}\). In this case, applying parts (a) and (c) of Lemma SE-11, Lemma SE-15, and part (b) of Lemma SE-18 to expression (8) to obtain

Next, consider part (b), where we take \(\rho_{T} = \exp\{-1/q(T)\}\) such that \(T^{1+\frac{1}{2}} \ll q(T)\). In this case, we apply parts (a) and (c) of Lemma SE-11, Lemma SE-15, and part (b) of Lemma SE-18 to expression (8) to obtain
expression (8), we have

$$\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2} \Delta y_{it-1}$$

$$= \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-2} \varepsilon_{it-1} - \frac{T}{q(T)} \left[ \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-2}^2 \right] \left[ 1 + O\left( \frac{1}{q(T)} \right) \right]$$

$$+ \frac{1}{\sqrt{T}} \left( \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} a_i \varepsilon_{it-1} \right) - \frac{1}{q(T)} \left[ \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} a_i w_{it-2} \right] \left[ 1 + O\left( \frac{1}{q(T)} \right) \right]$$

$$= \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-2} \varepsilon_{it-1} - \frac{T}{q(T)} \left[ \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-2}^2 \right] + O_p\left( \frac{1}{\sqrt{T}} \right)$$

$$+ O_p \left( \max \left\{ \frac{\sqrt{T}}{q(T)}, \frac{\sqrt{N}}{q(T)} \right\} \right)$$

We turn our attention to part (d), where we take $\rho_T = \exp\{-1/q(T)\}$ such that $T \ll q(T) \ll T^{1+\frac{1}{\kappa}}$. Note that, in this case, we can apply the results given in parts (a) and (c) of Lemma SE-11, Lemma SE-15, and part (b) of Lemma SE-18 to expression (8) to obtain

$$\frac{q(T)}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2} \Delta y_{it-1}$$

$$= - \frac{q(T)}{q(T)} \left[ \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-2}^2 \right] \left[ 1 + O\left( \frac{1}{q(T)} \right) \right] + \frac{q(T)}{T \sqrt{N}} \left[ \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-2} \varepsilon_{it-1} \right]$$

$$+ \frac{q(T)}{\sqrt{NT^{3/2}}} \left( \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} a_i \varepsilon_{it-1} \right) - \frac{q(T)}{q(T) \sqrt{N}} \left[ \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} a_i w_{it-2} \right] \left[ 1 + O\left( \frac{1}{q(T)} \right) \right]$$

$$= \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-2}^2 \left[ 1 + O\left( \frac{1}{q(T)} \right) \right]$$

$$+ O_p \left( \frac{q(T)}{T \sqrt{N}} \right) + O_p \left( \frac{q(T)}{\sqrt{NT^{3/2}}} \right) + O_p \left( \max \left\{ \frac{1}{\sqrt{NT}}, \frac{1}{T} \right\} \right)$$

$$= \left[ \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-2}^2 \right] + O_p \left( \frac{q(T)}{\sqrt{NT}} \right).$$

To show part (e), where we take $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \sim T$. In this case, we make use of the results given in parts (a) and (c) of Lemma SE-11, part (a) of Lemma SE-17, and part (c) of
Lemma SE-18 to deduce

\[
\frac{1}{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2} \Delta y_{it-1} = - \frac{T}{q(T)} \left[ \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-2}^2 \right] \left[ 1 + O \left( \frac{1}{T} \right) \right] + \frac{1}{\sqrt{N}} \left[ \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} \varepsilon_{it-1} \right] \\
+ \frac{1}{\sqrt{NT}} \left( \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} \varepsilon_{it-1} \right) - \frac{1}{NTq(T)} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-2} \left[ 1 + O \left( \frac{1}{T} \right) \right] \\
\]

Consider part (f), where we take \( \rho_T = \exp \left\{ - \frac{1}{q(T)} \right\} \) such that \( q(T) \to \infty \) but \( q(T) / T \to 0 \). In this case, we make use of the results given in parts (a) and (c) of Lemma SE-11, part (b) of Lemma SE-17, and part (d) of Lemma SE-18 to deduce

\[
\frac{1}{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2} \Delta y_{it-1} = - \frac{T}{q(T)} \left[ \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-2}^2 \right] \left[ 1 + O \left( \frac{1}{T} \right) \right] + \frac{1}{\sqrt{N}} \left[ \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-2} \varepsilon_{it-1} \right] \\
+ \frac{1}{\sqrt{NT}} \left( \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} \varepsilon_{it-1} \right) - \frac{1}{NTq(T)} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-2} \left[ 1 + O \left( \frac{1}{T} \right) \right] \\
\]

Finally, to show part (g), note that, in this case, we make use of the results given in parts (a) and
(c) of Lemma SE-11, part (c) of Lemma SE-17, and part (e) of Lemma SE-18 to deduce

\[
\frac{1 + \rho_T}{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2} \Delta y_{it-1} = -\frac{(1 - \rho_T)(1 + \rho_T)}{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-2}^2 - \frac{(1 - \rho_T)(1 + \rho_T)}{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} a_i w_{it-2} + \frac{1 + \rho_T}{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} \alpha_i \varepsilon_{it-1} + \frac{1 + \rho_T}{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-2} \varepsilon_{it-1}
\]

\[
= -\frac{(1 - \rho_T^2)}{NT} \sum_{i=1}^{N} \sum_{t=3}^{T} w_{it-2}^2 + O_p \left( \max \left\{ \frac{1}{\sqrt{NT}}, \frac{1}{T} \right\} \right) + O_p \left( \frac{1}{\sqrt{NT}} \right) + O_p \left( \frac{1}{\sqrt{NT}} \right)
\]

\[
= -\frac{(1 - \rho_T^2)}{NT} \sum_{i=1}^{N} \sum_{t=3}^{T} w_{it-2}^2 + O_p \left( \max \left\{ \frac{1}{\sqrt{NT}}, \frac{1}{T} \right\} \right) \overset{p}{\rightarrow} -\sigma^2,
\]

as required for part (f). □

**Lemma SD-2:**

Under Assumptions 1-4, the following statements are true as \( N, T \to \infty \) such that \( N^\kappa/T \to \tau \), for constants \( \kappa \in \left( \frac{1}{2}, \infty \right) \) and \( \tau \in (0, \infty) \).

(a) If \( \rho_T = 1 \) for all \( T \) sufficiently large, then

\[
\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2} \Delta \varepsilon_{it} = -\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=1}^{T} \varepsilon_{it-2} \varepsilon_{it-1} + \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} w_{it-2} \varepsilon_{it} + O_p \left( \frac{1}{\sqrt{T}} \right).
\]

(b) If \( \rho_T = \exp \{-1/q(T)\} \) such that \( T/q(T) \to 0 \), then

\[
\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2} \Delta \varepsilon_{it} = -\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=1}^{T} \varepsilon_{it-2} \varepsilon_{it-1} + \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} w_{it-2} \varepsilon_{it} + O_p \left( \frac{1}{\sqrt{T}} \right).
\]

(c) If \( \rho_T = \exp \{-1/q(T)\} \) such that \( q(T) \sim T \), then

\[
\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2} \Delta \varepsilon_{it} = -\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=1}^{T} \varepsilon_{it-2} \varepsilon_{it-1} + \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} w_{it-2} \varepsilon_{it} + O_p \left( \frac{1}{\sqrt{T}} \right).
\]

(d) If \( \rho_T = \exp \{-1/q(T)\} \) such that \( q(T) \to \infty \) but \( q(T)/T \to 0 \), then

\[
\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2} \Delta \varepsilon_{it} = -\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=1}^{T} \varepsilon_{it-2} \varepsilon_{it-1} + O_p \left( \max \left\{ \sqrt{\frac{q(T)}{T}}, \frac{1}{\sqrt{q(T)}} \right\} \right).
\]

(e) If \( \rho_T \in \mathcal{G}_{St} = \left\{ \rho_T = \exp \left\{ -\frac{1}{q(T)} \right\} : q(T) \geq 0 \) and \( q(T) = O(1) \) as \( T \to \infty \right\} \), then

\[
\sqrt{\frac{1 + \rho_T}{2\sigma^4}} \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2} \Delta \varepsilon_{it} = \sqrt{\frac{1 + \rho_T}{2\sigma^4}} \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-2} \Delta \varepsilon_{it} + O_p \left( \frac{1}{\sqrt{T}} \right) \Rightarrow N(0, 1).
\]

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Proof:
To proceed, write
\[
\sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2} \Delta \varepsilon_{it} = \sum_{i=1}^{N} \sum_{t=2}^{T} [a_i + w_{it-2}] \Delta \varepsilon_{it}
\]
\[
= \sum_{i=1}^{N} \sum_{t=2}^{T} a_i \Delta \varepsilon_{it} + \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-2} \Delta \varepsilon_{it}.
\] (9)

Applying partial summation we have that
\[
\sum_{i=1}^{N} \sum_{t=1}^{T} w_{it-2} \Delta \varepsilon_{it}
\]
\[
= \sum_{i=1}^{N} \left\{ \sum_{t=4}^{T} [w_{it-3} - w_{it-2}] \varepsilon_{it-1} \right\} + w_{iT-2} \varepsilon_{iT} - w_{i1} \varepsilon_{i2}
\]
\[
= \sum_{i=1}^{N} \left\{ \sum_{t=4}^{T} [\rho_T w_{it-3} + \varepsilon_{it-2}] \varepsilon_{it-1} \right\} + w_{iT-2} \varepsilon_{iT} - w_{i1} \varepsilon_{i2}
\]
\[
= (1 - \rho_T) \sum_{i=1}^{N} \sum_{t=4}^{T} w_{it-3} \varepsilon_{it-1} - \sum_{i=1}^{N} \sum_{t=4}^{T} \varepsilon_{it-2} \varepsilon_{it-1} + \sum_{i=1}^{N} w_{iT-2} \varepsilon_{iT} - \sum_{i=1}^{N} \varepsilon_{i1} \varepsilon_{i2}.
\] (10)

Substituting (10) into (9), we have
\[
\sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2} \Delta \varepsilon_{it}
\]
\[
= - \sum_{i=1}^{N} \sum_{t=4}^{T} \varepsilon_{it-2} \varepsilon_{it-1} + \sum_{i=1}^{N} w_{iT-2} \varepsilon_{iT} + \sum_{i=1}^{N} \sum_{t=2}^{T} a_i \Delta \varepsilon_{it} - \sum_{i=1}^{N} \varepsilon_{i1} \varepsilon_{i2} + (1 - \rho_T) \sum_{i=1}^{N} \sum_{t=4}^{T} w_{it-3} \varepsilon_{it-1}.
\]

Consider first part (a). Here, by assumption, \(\rho_T = 1\) for all \(T\) sufficiently large. Hence, applying parts (g)-(i) of Lemma SE-11 and part (a) of Lemma SE-25 in Appendix SE below, we obtain
\[
\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2} \Delta \varepsilon_{it}
\]
\[
= - \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=4}^{T} \varepsilon_{it-2} \varepsilon_{it-1} + \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} w_{iT-2} \varepsilon_{iT} + \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} a_i \Delta \varepsilon_{it} - \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \varepsilon_{i1} \varepsilon_{i2}
\]
\[
+ (1 - \rho_T) \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=4}^{T} w_{it-3} \varepsilon_{it-1}
\]
\[
= - \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=4}^{T} \varepsilon_{it-2} \varepsilon_{it-1} + \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} w_{iT-2} \varepsilon_{iT} + O_p \left( \frac{1}{\sqrt{T}} \right) + O_p \left( \frac{1}{\sqrt{T}} \right)
\]
\[
+ O_p \left( (1 - \rho_T) \sqrt{T} \right)
\]
\[
= - \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=4}^{T} \varepsilon_{it-2} \varepsilon_{it-1} + \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} w_{iT-2} \varepsilon_{iT} + O_p \left( \frac{1}{\sqrt{T}} \right),
\]
as required.

Next, consider part (b). Here, we consider the case where $\rho_T = \exp \{-1/q(T)\}$ such that $T/q(T) \to 0$. In this case, using the results parts (g)-(i) of Lemma SE-11 and part (b) of Lemma SE-25, we deduce that

$$
\frac{1}{\sqrt{NT}} \sum_{t=1}^T \sum_{i=2}^N y_{it-2} \Delta \varepsilon_{it} = -\frac{1}{\sqrt{NT}} \sum_{i=1}^N \sum_{t=4}^T \varepsilon_{it-2} \varepsilon_{it-1} + \frac{1}{\sqrt{NT}} \sum_{i=1}^N w_{iT-2} \varepsilon_{iT} + O_p \left( \frac{1}{\sqrt{T}} \right) + O_p \left( \frac{\sqrt{T}}{q(T)} \right)
$$

which shows part (b).

Consider part (c), where we take $\rho_T = \exp \{-1/q(T)\}$ such that $q(T) \sim T$. In this case, using the results parts (g)-(i) of Lemma SE-11 and part (c) of Lemma SE-25, we deduce that

$$
\frac{1}{\sqrt{NT}} \sum_{t=1}^T \sum_{i=2}^N y_{it-2} \Delta \varepsilon_{it} = -\frac{1}{\sqrt{NT}} \sum_{i=1}^N \sum_{t=4}^T \varepsilon_{it-2} \varepsilon_{it-1} + \frac{1}{\sqrt{NT}} \sum_{i=1}^N w_{iT-2} \varepsilon_{iT} + O_p \left( \frac{1}{\sqrt{T}} \right) + O_p \left( \frac{1}{\sqrt{q(T)}} \right)
$$

which shows part (c).

Consider now part (d), where we take $\rho_T = \exp \{-1/q(T)\}$ such that $q(T) \to \infty$ but $q(T)/T \to 0$. In this case, we apply parts (g)-(i) of Lemma SE-11, part (d) of Lemma SE-21, and part (d) of Lemma SE-25 to obtain

$$
\frac{1}{\sqrt{NT}} \sum_{t=1}^T \sum_{i=2}^N y_{it-2} \Delta \varepsilon_{it} = -\frac{1}{\sqrt{NT}} \sum_{i=1}^N \sum_{t=4}^T \varepsilon_{it-2} \varepsilon_{it-1} + \frac{1}{\sqrt{NT}} \sum_{i=1}^N w_{iT-2} \varepsilon_{iT} + O_p \left( \frac{1}{\sqrt{T}} \right) + O_p \left( \frac{1}{q(T)} \right)

+ O_p \left( \frac{1}{\sqrt{q(T)}} \right)
$$

as required.
Finally, to show part (e), note that, in this case,

\[
\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2} \Delta \varepsilon_{it} = \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} a_i \Delta \varepsilon_{it} + \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-2} \Delta \varepsilon_{it}
\]

\[
= \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-2} \Delta \varepsilon_{it} + O_p \left( \frac{1}{\sqrt{T}} \right). \tag{11}
\]

Applying partial summation the lead term on the right-hand side of the expression above, we get

\[
\sqrt{\frac{1 + \rho_T}{2\sigma^4}} \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=3}^{T} w_{it-2} \Delta \varepsilon_{it}
\]

\[
= \sqrt{\frac{1 + \rho_T}{2\sigma^4}} \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=4}^{T} \left\{ \sum_{t=4}^{T} (w_{it-3} - w_{it-2}) \varepsilon_{it-1} \right\} + w_{iT-2} \varepsilon_{iT} - w_{i1} \varepsilon_{i2}
\]

\[
= \sqrt{\frac{1 + \rho_T}{2\sigma^4}} \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=4}^{T} \left[ \sum_{j=1}^{t-3} \rho_T^{(t-3-j)} \varepsilon_{ij} - \sum_{j=1}^{t-2} \rho_T^{(t-2-j)} \varepsilon_{ij} \right] \varepsilon_{it-1}
\]

\[
+ \sqrt{\frac{1 + \rho_T}{2\sigma^4}} \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{j=1}^{T-2} \rho_T^{(T-2-j)} \varepsilon_{ij} \varepsilon_{iT} - \sqrt{\frac{1 + \rho_T}{2\sigma^4}} \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \varepsilon_{i1} \varepsilon_{i2}
\]

\[
= \sqrt{\frac{1 + \rho_T}{2\sigma^4}} \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=4}^{T} \left[ \sum_{j=1}^{t-3} \rho_T^{(t-3-j)} \varepsilon_{ij} - \sum_{j=1}^{t-2} \rho_T^{(t-2-j)} \varepsilon_{ij} \right] \varepsilon_{it-1}
\]

\[
- \sqrt{\frac{1 + \rho_T}{2\sigma^4}} \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=4}^{T} \varepsilon_{it-2} \varepsilon_{it-1} + \sqrt{\frac{1 + \rho_T}{2\sigma^4}} \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{j=1}^{T-2} \rho_T^{(T-2-j)} \varepsilon_{ij} \varepsilon_{iT}
\]

\[
- \sqrt{\frac{1 + \rho_T}{2\sigma^4}} \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \varepsilon_{i1} \varepsilon_{i2}
\]

\[
= \sqrt{\frac{1 + \rho_T}{2\sigma^4}} \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=4}^{T} \left[ \sum_{j=1}^{t-3} \rho_T^{(t-3-j)} \varepsilon_{ij} \varepsilon_{it-1} \right] - \sqrt{\frac{1 + \rho_T}{2\sigma^4}} \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=4}^{T} \varepsilon_{it-2} \varepsilon_{it-1}
\]

\[
+ \sqrt{\frac{1 + \rho_T}{2\sigma^4}} \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{j=1}^{T-2} \rho_T^{(T-2-j)} \varepsilon_{ij} \varepsilon_{iT} - \sqrt{\frac{1 + \rho_T}{2\sigma^4}} \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \varepsilon_{i1} \varepsilon_{i2}.
\]
Applying parts (g)-(i) of Lemma SE-11 and part (e) of Lemma SE-21, we then have
\[
\sqrt{\frac{1 + \rho_T}{2\sigma^4}} \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=3}^{T} w_{it-2} \Delta \varepsilon_{it} = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} t_{i-2} \varepsilon_{it-1} + \sqrt{\frac{1 + \rho_T}{2\sigma^4}} \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=4}^{T} \sum_{j=1}^{T-3} \rho_{T}^{(t-3-j)} \varepsilon_{ij} \varepsilon_{it-1}
\]
\[
+ \sqrt{\frac{1 + \rho_T}{2\sigma^4}} \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} w_{iT-2} \varepsilon_{iT} - \sqrt{\frac{1 + \rho_T}{2\sigma^4}} \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \varepsilon_{i1} \varepsilon_{i2}
\]
\[
= \frac{1 + \rho_T}{2\sigma^4} \frac{1}{\sqrt{N}} \sum_{i=1}^{N} (X_{i,T} + Y_{i,T}) + O_p \left( \frac{1}{\sqrt{T}} \right)
\]
Combining this with expression (11) above, we have, by Lemma SE-23,
\[
\frac{1 + \rho_T}{2\sigma^4} \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2} \Delta \varepsilon_{it} = \frac{1 + \rho_T}{2\sigma^4} \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-2} \Delta \varepsilon_{it} + O_p \left( \frac{1}{\sqrt{T}} \right)
\]
\[
= \frac{1 + \rho_T}{2\sigma^4} \frac{1}{\sqrt{N}} \sum_{i=1}^{N} (X_{i,T} + Y_{i,T}) + O_p \left( \frac{1}{\sqrt{T}} \right)
\]
\[
\Rightarrow N(0, 1),
\]
as required. □

**Lemma SD-3:**

Under Assumptions 1-4, the following statements are true as $N, T \to \infty$ such that $N^\kappa/T \to \tau$, for $\kappa \in \left(\frac{1}{2}, \infty\right)$ and $\tau \in (0, \infty)$.

(a) If $\rho_T = 1$ for all $T$ sufficiently large, then
\[
\frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \overline{y}_{1,N,T})^2
\]
\[
= \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1}^2 + O_p \left( \frac{1}{T} \right) + O_p \left( \frac{1}{N} \right) \frac{p}{2} \sigma^2.
\]

(b) If $\rho_T = \exp\{-1/q(T)\}$ such that $T \ll q(T)$, then
\[
\frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \overline{y}_{1,N,T})^2
\]
\[
= \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1}^2 + O_p \left( \frac{1}{T} \right) + O_p \left( \frac{1}{N} \right) \frac{p}{2} \sigma^2.
\]
(c) If $\rho_T = \exp \{ -1/q(T) \}$ such that $q(T) \sim T$, then
\[
\frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \overline{y}_{-1,N,T})^2
= \frac{q(T)^2 \sigma^2}{T^2} \left[ \exp \left\{ - \frac{2T}{q(T)} \right\} + \frac{2T}{q(T)} - 1 \right] + O_{\rho} \left( \max \left\{ \frac{1}{\sqrt{N}}, \frac{1}{\sqrt{T}} \right\} \right).
\]

(d) If $\rho_T = \exp \{ -1/q(T) \}$ such that $q(T) \to \infty$ but $q(T)/T \to 0$, then
\[
\frac{1}{NTq(T)} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \overline{y}_{-1,N,T})^2 = \frac{\sigma^2}{2} + O_{\rho} \left( \max \left\{ \frac{1}{q(T)}, \frac{1}{\sqrt{q(T)NT}}, \frac{1}{\sqrt{T}} \right\} \right).
\]

(e) If $\rho_T \in \mathcal{G}_{St} = \left\{ |\rho_T| = \exp \left\{ -\frac{1}{q(T)} \right\} : q(T) \geq 0 \text{ and } q(T) = O(1) \text{ as } T \to \infty \right\}$, then
\[
\frac{1 - \rho_T^2}{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \overline{y}_{-1,N,T})^2 = (1 - \rho_T^2) \sigma_a^2 + \sigma^2 + o_{\rho}(1),
\]
as $N, T \to \infty$.

**Proof:**
To proceed, first write
\[
y_{it-1} - \overline{y}_{-1,N,T} = a_i + w_{it-1} - \frac{1}{N(T-1)} \sum_{j=1}^{N} \sum_{s=2}^{T} (a_j + w_{is-1})
= a_i - \frac{1}{N} \sum_{j=1}^{N} a_j + w_{it-1} - \frac{1}{N(T-1)} \sum_{j=1}^{N} \sum_{s=2}^{T} w_{is-1}
= a_i - \overline{a}_N + w_{it-1} - \overline{w}_{-1,N,T},
\]
where
\[
\overline{a}_N = \frac{1}{N} \sum_{j=1}^{N} a_j, \quad \overline{w}_{-1,N,T} = \frac{1}{N(T-1)} \sum_{j=1}^{N} \sum_{s=2}^{T} w_{is-1},
\]
so that
\[
\sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \overline{y}_{-1,N,T})^2 = \sum_{i=1}^{N} \sum_{t=2}^{T} (a_i - \overline{a}_N)^2 + \sum_{i=1}^{N} \sum_{t=2}^{T} (w_{it-1} - \overline{w}_{-1,N,T})^2
+ 2 \sum_{i=1}^{N} \sum_{t=2}^{T} (a_i - \overline{a}_N) (w_{it-1} - \overline{w}_{-1,N,T})
= (T-1) \sum_{i=1}^{N} a_i^2 - N(T-1) \overline{a}_N^2 + \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1}^2 - N(T-1) \overline{w}_{-1,N,T}^2
+ 2 \sum_{i=1}^{N} \sum_{t=2}^{T} a_i w_{it-1} - 2N(T-1) \overline{a}_N \overline{w}_{-1,N,T}.
\]

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Consider first part (a), where we take $\rho_T = 1$ for all $T$ sufficiently large. In this case, we apply parts (a) and (b) of Lemma SE-11, part (a) of Lemma SE-18, and part (a) of Lemma SE-26 to obtain

$$
\frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \bar{y}_{-1,N,T})^2
= \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1}^2 + \frac{T-1}{NT^2} \sum_{i=1}^{N} a_i^2 - \frac{N(T-1)}{NT^2} a_N^2
- \frac{N(T-1)}{NT^2} w_{-1,N,T}^2 - 2 \frac{N(T-1)}{NT^2} a_N w_{-1,N,T} + 2 \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} a_i w_{it-1}
= \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1}^2 + O_p \left( \frac{1}{T} \right) + O_p \left( \frac{1}{N} \right) \xrightarrow{p} \frac{\sigma^2}{2},
$$

as required.

Next, consider part (b), where we take $\rho_T = \exp \{-1/q(T)\}$ such that $T \ll q(T)$. Here, we make use of parts (a) and (b) of Lemma SE-11, part (b) of Lemma SE-18, and part (b) of Lemma SE-26 to deduce

$$
\frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \bar{y}_{-1,N,T})^2 = \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1}^2 + O_p \left( \frac{1}{T} \right) + O_p \left( \frac{1}{N} \right) \xrightarrow{p} \frac{\sigma^2}{2},
$$

which is the required result for part (b).

Consider now part (c), where we take $\rho_T = \exp \{-1/q(T)\}$ such that $q(T) \sim T$. To show the stated result for this case, note first that by part (a) of Lemma SE-17, we have

$$
\frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1}^2 = \frac{q(T)^2 \sigma^2}{T^2} \frac{\sigma^2}{4} \left[ \exp \left\{ - \frac{2T}{q(T)} \right\} + \frac{2T}{q(T)} - 1 \right] + O_p \left( \frac{1}{\sqrt{N}} \right) + O_p \left( \frac{1}{\sqrt{T}} \right),
$$

from which we deduce that $\sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1}^2 = O_p (NT^2)$. Using this result and applying parts (a) and (b) of Lemma SE-11, part (c) of Lemma SE-18, and part (c) of Lemma SE-26, we get

$$
\frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \bar{y}_{-1,N,T})^2
= \frac{T-1}{NT^2} \sum_{i=1}^{N} a_i^2 - \frac{N(T-1)}{NT^2} a_N^2 + \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1}^2 - \frac{N(T-1)}{NT^2} w_{-1,N,T}^2
+ 2 \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} a_i w_{it-1} - 2 \frac{N(T-1)}{NT^2} a_N w_{-1,N,T}
= \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1}^2 + O_p \left( \frac{1}{T} \right) + O_p \left( \frac{1}{N} \right)
= \frac{q(T)^2 \sigma^2}{T^2} \frac{\sigma^2}{4} \left[ \exp \left\{ - \frac{2T}{q(T)} \right\} + \frac{2T}{q(T)} - 1 \right] + O_p \left( \max \left\{ \frac{1}{\sqrt{N}}, \frac{1}{\sqrt{T}} \right\} \right).$$
Consider part (d), where we assume that \( \rho_T = \exp \{-1/q(T)\} \) such that \( q(T) \to \infty \) but \( q(T)/T \to 0 \). From part (b) of Lemma SE-17, we obtain

\[
\frac{1}{NTq(T)} \sum_{i=1}^{N} \sum_{t=2}^{T} u_{it-1}^2 = \frac{\sigma^2}{2} + O_p \left( \max \left\{ \frac{1}{q(T)}, \frac{q(T)}{T}, \sqrt{\frac{q(T)}{NT}}, \frac{1}{\sqrt{T}} \right\} \right).
\]

Using this result along with parts (a) and (b) of Lemma SE-11, part (d) of Lemma SE-18, and part (d) of Lemma SE-26, we observe that

\[
\frac{1}{NTq(T)} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \bar{y}_{-1,N,T})^2
\]

\[
= \frac{T-1}{NTq(T)} \sum_{i=1}^{N} a_i^2 - \frac{N(T-1)}{NTq(T)} a_N^2 + \frac{1}{NTq(T)} \sum_{i=1}^{N} \sum_{t=2}^{T} u_{it-1}^2 - \frac{N(T-1)}{NTq(T)} w_{-1,N,T}^2
\]

\[
+2 \frac{1}{NTq(T)} \sum_{i=1}^{N} \sum_{t=2}^{T} a_i u_{it-1} - 2 \frac{N(T-1)}{NTq(T)} \mu_{-1,N,T}
\]

\[
= \frac{1}{NTq(T)} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1}^2 + O_p \left( \frac{1}{q(T)} \right) + O_p \left( \frac{1}{q(T)} \right) + O_p \left( \max \left\{ \frac{q(T)}{NT}, \frac{q(T)}{T^2} \right\} \right)
\]

\[
+ O_p \left( \max \left\{ \frac{1}{\sqrt{NT}}, \frac{1}{T} \right\} \right) + O_p \left( \max \left\{ \frac{1}{\sqrt{NT}}, \frac{1}{T} \right\} \right)
\]

\[
= \frac{\sigma^2}{2} + O_p \left( \max \left\{ \frac{1}{q(T)}, \frac{q(T)}{T}, \sqrt{\frac{q(T)}{NT}}, \frac{1}{\sqrt{T}} \right\} \right)
\]

\[
+ O_p \left( \max \left\{ \frac{1}{q(T)}, \frac{q(T)}{NT}, \frac{1}{\sqrt{NT}} \right\} \right)
\]

\[
= \frac{\sigma^2}{2} + O_p \left( \max \left\{ \frac{1}{q(T)}, \frac{q(T)}{NT}, \sqrt{\frac{q(T)}{NT}}, \frac{1}{\sqrt{T}} \right\} \right),
\]

which shows part (d).

Finally, consider part (e), where we take \( \rho_T \in \mathcal{G}_{st} = \{ |\rho_T| = \exp \{-1/q(T)\} : q(T) \geq 0 \text{ and } q(T) = O(1) \text{ as } T \to \infty \} \). In this case, we make use of parts (a) and (b) of Lemma SE-11, part (c) of Lemma SE-17, and part (e) of Lemmas SE-18 and SE-26 to deduce that
\[
\frac{1 - \rho_T^2}{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \overline{y}_{-1,N,T})^2 \\
= (1 - \rho_T^2) \frac{T - 1}{NT} \sum_{i=1}^{N} a_i^2 - \frac{N(T-1)}{NT} (1 - \rho_T^2) \sigma_N^2 + \frac{(1 - \rho_T^2)}{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1}^2 \\
- (1 - \rho_T^2) \frac{N(T-1)}{NT} \overline{w}_{-1,N,T} + 2 \frac{(1 - \rho_T^2)}{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} a_i w_{it-1} \\
- 2 \frac{(1 - \rho_T^2)}{NT} \frac{N(T-1)}{NT} \overline{w}_{-1,N,T} \\
= (1 - \rho_T^2) \left( \frac{1}{N} \sum_{i=1}^{N} a_i^2 - \overline{a}_N^2 \right) \left[ 1 + O_p \left( \frac{1}{T} \right) \right] + \frac{(1 - \rho_T^2)}{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1}^2 \\
+ O_p \left( \max \left\{ \frac{1}{NT}, \frac{1}{T^2} \right\} \right) + O_p \left( \max \left\{ \frac{1}{\sqrt{NT}}, \frac{1}{T} \right\} \right) \\
= (1 - \rho_T^2) \sigma_a^2 + \sigma^2 + o_p(1)
\]

which completes the proof for part (e). \( \square \)

**Lemma SD-4:**

Under Assumptions 1-4, the following statements are true as \( N,T \to \infty \) such that \( N^\kappa/T \to \tau \), for constants \( \kappa \in \left( \frac{1}{2}, \infty \right) \) and \( \tau \in (0, \infty) \).

(a) If \( \rho_T = 1 \) for all \( T \) sufficiently large, then
\[
\frac{1}{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \overline{y}_{-1,N,T}) a_i = \sigma_a^2 - \mu_a \overline{w}_{-1,N,T} + \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} a_i w_{it-1} + o_p(1) \\
= O_p \left( \max \left\{ 1, \sqrt{\frac{T}{N}} \right\} \right).
\]

(b) If \( \rho_T = \exp \{-1/q(T)\} \) such that \( T \ll q(T) \), then
\[
\frac{1}{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \overline{y}_{-1,N,T}) a_i = \sigma_a^2 - \mu_a \overline{w}_{-1,N,T} + \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} a_i w_{it-1} + o_p(1) \\
= O_p \left( \max \left\{ 1, \sqrt{\frac{T}{N}} \right\} \right).
\]

(c) If \( \rho_T = \exp \{-1/q(T)\} \) such that \( q(T) \sim T \), then
\[
\frac{1}{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \overline{y}_{-1,N,T}) a_i = \sigma_a^2 - \mu_a \overline{w}_{-1,N,T} + \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} a_i w_{it-1} + o_p(1) \\
= O_p \left( \max \left\{ 1, \sqrt{\frac{T}{N}} \right\} \right).
\]
(d) If $\rho_T = \exp \{-1/q(T)\}$ such that $T^{\frac{1+\kappa}{3\kappa}}/q(T) = O(1)$ but $q(T)/T \rightarrow 0$, then
\[
\frac{1}{q(T)^{3/2}} \sqrt{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \bar{y}_{i-1,N,T}) a_i
\]
\[
= \frac{\sqrt{NT}}{q(T)^{3/2}} \left[ \frac{1}{N} \sum_{i=1}^{N} a_i^2 - \bar{a}_N^2 \right] + O_p \left( \max \left\{ \frac{1}{\sqrt{q(T)}}, \sqrt{\frac{N}{q(T)T}} \right\} \right)
\]
\[
= O_p \left( \max \left\{ \frac{\sqrt{NT}}{q(T)^{3/2}}, \frac{1}{\sqrt{q(T)}}, \sqrt{\frac{N}{q(T)T}} \right\} \right).
\]

(e) If $\rho_T = \exp \{-1/q(T)\}$ such that $q(T) \rightarrow \infty$ but $q(T)/T^{\frac{1+\kappa}{3\kappa}} \rightarrow 0$, then
\[
\frac{1}{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \bar{y}_{i-1,N,T}) a_i
\]
\[
= \left[ \frac{1}{N} \sum_{i=1}^{N} a_i^2 - \bar{a}_N^2 \right] + O_p \left( \max \left\{ \frac{q(T)}{\sqrt{NT}}, \frac{q(T)}{T} \right\} \right).
\]

(f) If $\rho_T \in \mathcal{G}_{St} = \{ |\rho_T| = \exp \left\{ -\frac{1}{q(T)} \right\} : q(T) \geq 0 \text{ and } q(T) = O(1) \text{ as } T \rightarrow \infty \}$, then
\[
\frac{1 - \rho_T^2}{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \bar{y}_{i-1,N,T}) a_i = (1 - \rho_T^2) \sigma_a^2 + o_p(1),
\]
as $N, T \rightarrow \infty$.

Proof:
To proceed, first write
\[
\sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \bar{y}_{i-1,N,T}) a_i = \sum_{i=1}^{N} \sum_{t=2}^{T} (a_i - \bar{a}_N + w_{it-1} - \bar{w}_{i-1,N,T}) a_i
\]
\[
= (T - 1) \sum_{i=1}^{N} a_i^2 - N(T-1)\bar{a}_N^2 + \sum_{i=1}^{N} \sum_{t=2}^{T} a_i w_{it-1} - N(T-1)\bar{a}_N \bar{w}_{i-1,N,T}.
\]
Consider first part (a), where we take $\rho_T = 1$ for all $T$ sufficiently large. Here, we apply parts (a)
and (b) of Lemma SE-11, part (a) of Lemma SE-18, and part (a) of Lemma SE-26 to obtain

\[
\frac{1}{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \overline{y}_{-1,N,T}) a_i
\]

\[
= \frac{N (T - 1)}{NT} \frac{1}{N} \sum_{i=1}^{N} a_i^2 - \frac{N (T - 1)}{NT} \overline{a}_N + \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} a_i w_{it-1} - \frac{N (T - 1)}{NT} \overline{a}_{N \overline{w}_{-1,N,T}}
\]

\[
= \left[ \frac{1}{N} \sum_{i=1}^{N} a_i^2 - \overline{a}_N^2 - \overline{a}_{N \overline{w}_{-1,N,T}} \right] \left[ 1 + O \left( \frac{1}{T} \right) \right] + \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} a_i w_{it-1}
\]

\[
= \sigma_a^2 - \mu_a \overline{w}_{-1,N,T} + \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} a_i w_{it-1} + o_p (1)
\]

\[
= O_p (1) + O_p \left( \max \left\{ 1, \sqrt{\frac{T}{N}} \right\} \right) + o_p (1)
\]

\[
= O_p \left( \max \left\{ 1, \sqrt{\frac{T}{N}} \right\} \right).
\]

Next, consider part (b), where we take \( \rho_T = \exp \left\{ -1/q (T) \right\} \) such that \( T \ll q (T) \). In this case, applying parts (a) and (b) of Lemma SE-11, part (b) of Lemma SE-18, and part (b) of Lemma SE-26; we obtain

\[
\frac{1}{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \overline{y}_{-1,N,T}) a_i
\]

\[
= \sigma_a^2 - \mu_a \overline{w}_{-1,N,T} + \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} a_i w_{it-1} + o_p (1)
\]

\[
= O_p (1) + O_p \left( \max \left\{ 1, \sqrt{\frac{T}{N}} \right\} \right) + o_p (1)
\]

\[
= O_p \left( \max \left\{ 1, \sqrt{\frac{T}{N}} \right\} \right),
\]

as required.

Now, we turn our attention to part (c), where we take \( \rho_T = \exp \left\{ -1/q (T) \right\} \) such that \( q (T) \sim T \). In this case, applying parts (a) and (b) of Lemma SE-11 and part (c) of Lemmas SE-18 and SE-26; we
have
\[
\frac{1}{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \overline{y}_{1,N,T}) a_i \\
= \frac{N(T - 1)}{NT} \left[ \frac{1}{N} \sum_{i=1}^{N} a_i^2 - \frac{N(T - 1)}{NT} \bar{a}_N \right] + \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} a_i w_{it-1} - \frac{N(T - 1)}{NT} \bar{a}_N \overline{w}_{1,N,T} \\
= \left[ \frac{1}{N} \sum_{i=1}^{N} a_i^2 - \bar{a}_N^2 - \bar{a}_N \overline{w}_{1,N,T} \right] \left[ 1 + O \left( \frac{1}{T} \right) \right] + \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} a_i w_{it-1} \\
= \frac{\sigma_a^2 - \mu_a \overline{w}_{1,N,T} + 1}{q(T)^3/2} \sum_{i=1}^{N} \sum_{t=2}^{T} a_i w_{it-1} + o_p(1) \\
= O_p(1) + O_p \left( \max \left\{ 1, \frac{\sqrt{T}}{N} \right\} \right) + O_p \left( \max \left\{ 1, \frac{\sqrt{T}}{N} \right\} \right) + o_p(1) \\
= O_p \left( \max \left\{ 1, \frac{\sqrt{T}}{N} \right\} \right)
\]

Consider part (d), where we assume that \( \rho_T = \exp \{-1/q(T)\} \) such that \( T^{1/q(T)} / q(T) = O(1) \) but \( q(T) / T \to 0 \). In this case, applying parts (a) and (b) of Lemma SE-11 and part (d) of Lemmas SE-18 and SE-26; we obtain
\[
\frac{1}{q(T)^3/2} \sqrt{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \overline{y}_{1,N,T}) a_i \\
= \frac{N(T - 1)}{q(T)^3/2} \frac{1}{N} \sum_{i=1}^{N} a_i^2 - \frac{N(T - 1)}{q(T)^3/2} \sqrt{NT} \bar{a}_N \\
+ \frac{1}{q(T)^3/2} \sqrt{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} a_i w_{it-1} - \frac{N(T - 1)}{q(T)^3/2} \sqrt{NT} \bar{a}_N \overline{w}_{1,N,T} \\
= \frac{\sqrt{NT}}{q(T)^3/2} \left[ \frac{1}{N} \sum_{i=1}^{N} a_i^2 - \bar{a}_N^2 \right] \left[ 1 + O \left( \frac{1}{q(T)} \right) \right] + O_p \left( \max \left\{ 1, \frac{\sqrt{N}}{q(T) T} \right\} \right) \\
+ O_p \left( \max \left\{ 1, \frac{\sqrt{N}}{q(T) T} \right\} \right) \\
= \frac{\sqrt{NT}}{q(T)^3/2} \left[ \frac{1}{N} \sum_{i=1}^{N} a_i^2 - \bar{a}_N^2 \right] + O_p \left( \max \left\{ 1, \frac{\sqrt{N}}{q(T) T} \right\} \right) \\
= O_p \left( \max \left\{ 1, \frac{\sqrt{N}}{q(T) T} \right\} \right),
\]

as required for part (d).

Consider part (e), where we assume that \( \rho_T = \exp \{-1/q(T)\} \) such that \( q(T) \to \infty \) but \( q(T) / T^{1/q(T)} \to 0 \). In this case, again applying parts (a) and (b) of Lemma SE-11, and part (d) of Lemmas SE-18 and
SE-26; we obtain

\[
\frac{1}{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \overline{y}_{-1,N,T}) a_i
\]

\[
= \frac{N(T-1)}{NT} \frac{1}{N} \sum_{i=1}^{N} a_i^2 - \frac{N(T-1)}{NT} \overline{a}_N^2
\]

\[
+ \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} a_i w_{it-1} - \frac{N(T-1)}{NT} \overline{\sigma}_{-1,N,T}
\]

\[
= \left[ \frac{1}{N} \sum_{i=1}^{N} a_i^2 - \overline{a}_N^2 \right] \left[ 1 + O_p \left( \frac{1}{T} \right) \right] + O_p \left( \max \left\{ \frac{q(T)}{\sqrt{NT}}, \frac{q(T)}{T} \right\} \right)
\]

\[
+ O_p \left( \max \left\{ \frac{q(T)}{\sqrt{NT}}, \frac{q(T)}{T} \right\} \right)
\]

\[
= \left[ \frac{1}{N} \sum_{i=1}^{N} a_i^2 - \overline{a}_N^2 \right] + O_p \left( \max \left\{ \frac{q(T)}{\sqrt{NT}}, \frac{q(T)}{T} \right\} \right)
\]

which shows part (e).

Finally, consider part (f), where we take

\[
\rho_T \in G_{St} = \left\{ |\rho_T| = \exp \left\{ -\frac{1}{q(T)} \right\} : q(T) \geq 0 \text{ and } q(T) = O(1) \text{ as } T \to \infty \right\}.
\]

In this case, we make use of parts (a) and (b) of Lemma SE-11 and part (e) of Lemmas SE-18 and SE-26 to deduce

\[
\frac{1 - \rho_T^2}{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \overline{y}_{-1,N,T}) a_i = (1 - \rho_T^2) \frac{N(T-1)}{NT} \frac{1}{N} \sum_{i=1}^{N} a_i^2 - (1 - \rho_T^2) \frac{N(T-1)}{NT} \overline{a}_N^2
\]

\[
+ \frac{1 - \rho_T^2}{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} a_i w_{it-1} - (1 - \rho_T^2) \frac{N(T-1)}{NT} \overline{\sigma}_{-1,N,T}
\]

\[
= (1 - \rho_T^2) \left[ \frac{1}{N} \sum_{i=1}^{N} a_i^2 - \overline{a}_N^2 \right] \left[ 1 + O_p \left( \frac{1}{T} \right) \right] + O_p \left( \max \left\{ \frac{1}{\sqrt{NT}}, \frac{1}{T} \right\} \right)
\]

\[
+ O_p \left( \max \left\{ \frac{1}{\sqrt{NT}}, \frac{1}{T} \right\} \right)
\]

\[
= (1 - \rho_T^2) \sigma_a^2 + o_p(1),
\]

which completes the proof of part (f). □

**Lemma SD-5:**

Under Assumptions 1-4, the following statements are true as \(N, T \to \infty\) such that \(N^\kappa/T \to \tau\), for constants \(\kappa \in \left(\frac{1}{2}, \infty\right)\) and \(\tau \in (0, \infty)\).

(a) If \(\rho_T = 1\) for all \(T\) sufficiently large, then

\[
\frac{1}{T \sqrt{N}} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \overline{y}_{-1,N,T}) \varepsilon_{it} = \frac{1}{T \sqrt{N}} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1} \varepsilon_{it} + O_p \left( \frac{1}{T^{1/2}} \right) + O_p \left( \frac{1}{\sqrt{N}} \right).
\]
(b) If $\rho_T = \exp \{-1/q(T)\}$ such that $T \ll q(T)$, then
\[
\frac{1}{T\sqrt{N}} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \overline{y}_{-1,N,T}) \varepsilon_{it} = \frac{1}{T\sqrt{N}} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1} \varepsilon_{it} + O_p \left( \frac{1}{\sqrt{T}} \right) + O_p \left( \frac{1}{\sqrt{N}} \right).
\]

(c) If $\rho_T = \exp \{-1/q(T)\}$ such that $q(T) \sim T$, then
\[
\frac{1}{T\sqrt{N}} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \overline{y}_{-1,N,T}) \varepsilon_{it} = \frac{1}{T\sqrt{N}} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1} \varepsilon_{it} + O_p \left( \frac{1}{\sqrt{T}} \right) + O_p \left( \frac{1}{\sqrt{N}} \right).
\]

(d) If $\rho_T = \exp \{-1/q(T)\}$ such that $q(T) \rightarrow \infty$ but $q(T)/T \rightarrow 0$, then
\[
\frac{1}{\sqrt{NTq(T)}} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \overline{y}_{-1,N,T}) \varepsilon_{it} = \frac{1}{\sqrt{NTq(T)}} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1} \varepsilon_{it} + O_p \left( \max \left\{ \frac{1}{\sqrt{q(T)}}, \frac{\sqrt{q(T)}}{\sqrt{NT}}, \frac{\sqrt{q(T)}}{T} \right\} \right).
\]

(e) If $\rho_T \in \mathcal{G}_T = \{ |\rho_T| = \exp \{-1/q(T)\} : q(T) \geq 0 \text{ and } q(T) = O(1) \text{ as } T \rightarrow \infty \}$, then
\[
\frac{1-N_T^2}{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \overline{y}_{-1,N,T}) \varepsilon_{it} = O_p \left( \frac{1}{\sqrt{NT}} \right),
\]
as $N, T \rightarrow \infty$.

**Proof:**

To proceed, first write
\[
\sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \overline{y}_{-1,N,T}) \varepsilon_{it} = \sum_{i=1}^{N} \sum_{t=2}^{T} (a_{it} - \overline{a}_N + w_{it-1} - \overline{w}_{-1,N,T}) \varepsilon_{it} = \sum_{i=1}^{N} \sum_{t=2}^{T} a_{it} \varepsilon_{it} - N(T-1) \overline{a}_N \overline{\varepsilon}_{N,T} + \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1} \varepsilon_{it} - N(T-1) \overline{w}_{-1,N,T} \overline{\varepsilon}_{N,T}.
\]

Consider first part (a), where we take $\rho_T = 1$ for all $T$ sufficiently large. Here, we make use of parts (b), (c), and (d) of Lemma SE-11, part (b) of Lemma SE-20, and part (a) of Lemma SE-26 to get
\[
\frac{1}{T\sqrt{N}} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \overline{y}_{-1,N,T}) \varepsilon_{it} = \frac{1}{T\sqrt{N}} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1} \varepsilon_{it} + O_p \left( \frac{1}{\sqrt{T}} \right) + O_p \left( \frac{1}{\sqrt{N}} \right) + O_p \left( \max \left\{ \frac{1}{\sqrt{N}}, \frac{1}{\sqrt{T}} \right\} \right)
\]
\[
= \frac{1}{T\sqrt{N}} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1} \varepsilon_{it} + O_p \left( \max \left\{ \frac{1}{\sqrt{N}}, \frac{1}{\sqrt{T}} \right\} \right) = N \left( 0, \frac{\sigma^4}{2} \right),
\]

which shows part (a).

Next, consider part (b), where we take \( \rho_T = \exp \{-1/q(T)\} \) such that \( T \ll q(T) \). In this case, applying parts (b), (c), and (d) of Lemma SE-11, part (a) of Lemma SE-20, and part (b) of Lemma SE-26; and, we obtain

\[
\frac{1}{T \sqrt{N}} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \bar{y}_{i-1,N,T}) \varepsilon_{it} = \frac{1}{T \sqrt{N}} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1} \varepsilon_{it} + \frac{1}{T \sqrt{N}} \sum_{i=1}^{N} \sum_{t=2}^{T} a_i \varepsilon_{it} - \frac{N(T-1)}{T \sqrt{N}} \pi_{N \pi_{N,T}} - \frac{N(T-1)}{T \sqrt{N}} \pi_{-1,N,T \pi_{N,T}}
\]

\[
= \frac{1}{T \sqrt{N}} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1} \varepsilon_{it} + O_p \left( \max \left\{ \frac{1}{\sqrt{N}}, \frac{1}{\sqrt{T}} \right\} \right) \Rightarrow N \left( 0, \frac{\sigma^4}{2} \right),
\]

as required.

We turn our attention to part (c), where we assume that \( \rho_T = \exp \{-1/q(T)\} \) such that \( q(T) \sim T \). In this case, applying parts (b), (c), and (d) of Lemma SE-11, part (c) of Lemma SE-25, and part (c) of Lemma SE-26; we have

\[
\frac{1}{T \sqrt{N}} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \bar{y}_{i-1,N,T}) \varepsilon_{it} = \frac{1}{T \sqrt{N}} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1} \varepsilon_{it} + \frac{1}{T \sqrt{N}} \sum_{i=1}^{N} \sum_{t=2}^{T} a_i \varepsilon_{it} - \frac{N(T-1)}{T \sqrt{N}} \pi_{N \pi_{N,T}} - \frac{N(T-1)}{T \sqrt{N}} \pi_{-1,N,T \pi_{N,T}}
\]

\[
= \frac{1}{T \sqrt{N}} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1} \varepsilon_{it} + O_p \left( \max \left\{ \frac{1}{\sqrt{N}}, \frac{1}{\sqrt{T}} \right\} \right)
\]

which shows part (c).

Now, consider part (d), where we assume that \( \rho_T = \exp \{-1/q(T)\} \) such that \( q(T) \to \infty \) but \( q(T)/T \to 0 \). In this case, applying parts (b), (c), and (d) of Lemma SE-11 and part (d) of Lemmas SE-25 and SE-26; we obtain

\[
\frac{1}{\sqrt{NTq(T)}} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \bar{y}_{i-1,N,T}) \varepsilon_{it} = \frac{1}{\sqrt{NTq(T)}} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1} \varepsilon_{it} + \frac{1}{\sqrt{NTq(T)}} \sum_{i=1}^{N} \sum_{t=2}^{T} a_i \varepsilon_{it} - \frac{N(T-1)}{\sqrt{NTq(T)}} \pi_{N \pi_{N,T}} - \frac{N(T-1)}{\sqrt{NTq(T)}} \pi_{-1,N,T \pi_{N,T}}
\]

\[
= \frac{1}{\sqrt{NTq(T)}} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1} \varepsilon_{it} + O_p \left( \max \left\{ \frac{1}{\sqrt{q(T)}}, \frac{1}{\sqrt{q(T)/T}} \right\} \right) + O_p \left( \max \left\{ \sqrt{\frac{q(T)}{NT}}, \frac{q(T)}{T} \right\} \right) + O_p \left( \max \left\{ \sqrt{\frac{q(T)}{NT}}, \frac{q(T)}{T} \right\} \right)
\]

as required for part (d).

Finally, consider part (e), where we take
\[ \rho_T \in G_{St} = \left\{ |\rho_T| = \exp \left\{ -\frac{1}{q(T)} \right\} : q(T) \geq 0 \text{ and } q(T) = O \left( \frac{1}{T} \right) \text{ as } T \to \infty \right\}. \]

In this case, applying parts (b), (c), and (d) of Lemma SE-11 and part (e) of Lemmas SE-25 and SE-26; we have

\[
\begin{align*}
1 - \rho_T^2 \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \bar{y}_{-1,N,T}) \varepsilon_{it} &= \frac{1 - \rho_T^2}{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} a_i \varepsilon_{it} - \frac{N(T-1)}{NT} (1 - \rho_T^2) \bar{\varepsilon}_{N,T} \\
+ \frac{1 - \rho_T^2}{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1} \varepsilon_{it} - \frac{N(T-1)}{NT} (1 - \rho_T^2) \bar{w}_{-1,N,T} \varepsilon_{N,T} &= \operatorname{O}_p \left( \frac{1}{\sqrt{NT}} \right) + \operatorname{O}_p \left( \frac{1}{\sqrt{NT}} \right) + \operatorname{O}_p \left( \frac{1}{\sqrt{NT�}} \right) \\
&= \operatorname{O}_p \left( \frac{1}{\sqrt{NT}} \right),
\end{align*}
\]

which completes the proof of part (e). \( \square \)

**Lemma SD-6:**

Under Assumptions 1-4, the following statements are true as \( N, T \to \infty \) such that \( N^\kappa / T \to \tau \), for \( \kappa \in \left( \frac{1}{2}, \infty \right) \) and \( \tau \in (0, \infty) \).

(a) If \( \rho_T = 1 \) for all \( T \) sufficiently large, then

\[
\frac{1}{NT} \sum_{i=1}^{N} \sum_{t=4}^{T} (y_{it-3} - y_{it-2})^2 = \sigma^2 + O_p \left( \max \left\{ \frac{1}{\sqrt{NT}}, \frac{1}{T} \right\} \right). 
\]

(b) If \( \rho_T = \exp \left\{ -1/q(T) \right\} \) such that \( T/q(T) \to 0 \),

\[
\frac{1}{NT} \sum_{i=1}^{N} \sum_{t=4}^{T} (y_{it-3} - y_{it-2})^2 = \sigma^2 + O_p \left( \max \left\{ \frac{1}{\sqrt{NT}}, \frac{1}{T} \right\} \right). 
\]

(c) If \( \rho_T = \exp \left\{ -1/q(T) \right\} \) such that \( q(T) \sim T \),

\[
\frac{1}{NT} \sum_{i=1}^{N} \sum_{t=4}^{T} (y_{it-3} - y_{it-2})^2 = \sigma^2 + O_p \left( \max \left\{ \frac{1}{\sqrt{NT}}, \frac{1}{T} \right\} \right). 
\]

(d) If \( \rho_T = \exp \left\{ -1/q(T) \right\} \) such that \( q(T) \to \infty \) but \( q(T)/T \to 0 \),

\[
\frac{1}{NT} \sum_{i=1}^{N} \sum_{t=4}^{T} (y_{it-3} - y_{it-2})^2 = \sigma^2 + O_p \left( \max \left\{ \frac{1}{\sqrt{NT}}, \frac{1}{q(T)} \right\} \right). 
\]

(e) If \( \rho_T \in G_{St} = \left\{ |\rho_T| = \exp \left\{ -\frac{1}{q(T)} \right\} : q(T) \geq 0 \text{ and } q(T) = O \left( \frac{1}{T} \right) \text{ as } T \to \infty \right\}, \) then

\[
\frac{1}{NT} \sum_{i=1}^{N} \sum_{t=4}^{T} (y_{it-3} - y_{it-2})^2 = \frac{2\sigma^2}{1 + \rho_T} + o_p(1). 
\]
Proof of Lemma SD-6:
To show part (a), note that, by assumption here, there exists a positive integer $I_p$ such that for all $T \geq I_p$, $\rho_T = 1$, so that

$$y_{it-3} - y_{it-2} = a_i + w_{it-3} - (a_i + w_{it-2}) = w_{it-3} - w_{it-2} = -\varepsilon_{it-2},$$

for all $T \geq I_p$. Hence, we can apply part (f) of Lemma SE-11 with $g = 2$ to obtain

$$\frac{1}{NT} \sum_{i=1}^{N} \sum_{t=4}^{T} (y_{it-3} - y_{it-2})^2 = \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=4}^{T} \varepsilon_{it-2}^2 = \sigma^2 + O_p \left( \max \left\{ \frac{1}{\sqrt{NT}}, \frac{1}{T} \right\} \right)$$

Now, to show parts (b)-(d), we first write

$$y_{it-3} - y_{it-2} = w_{it-3} - w_{it-2} = (1 - \rho_T) w_{it-3} - \varepsilon_{it-2},$$

from which we obtain

$$\frac{1}{NT} \sum_{i=1}^{N} \sum_{t=4}^{T} (y_{it-3} - y_{it-2})^2 = (1 - \rho_T)^2 \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=4}^{T} w_{it-3}^2 - 2 (1 - \rho_T) \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=4}^{T} w_{it-3} \varepsilon_{it-2}$$

$$+ \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=4}^{T} \varepsilon_{it-2}^2$$

Next, consider part (b), where we take $\rho_T = \exp \left\{ -1/q(T) \right\}$ such that $T/q(T) \to 0$. In this case, applying Lemma SE-15, we obtain

$$(1 - \rho_T)^2 \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=4}^{T} w_{it-3}^2 = T \left( 1 - \exp \left\{ -\frac{1}{q(T)} \right\} \right)^2 \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=4}^{T} w_{it-3}^2$$

$$= T \left( 1 - \left[ 1 - \frac{1}{q(T)} + O \left( \frac{1}{q(T)^2} \right) \right] \right)^2 \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=4}^{T} w_{it-3}^2$$

$$= T \left( \frac{1}{q(T)} + O \left( \frac{1}{q(T)^2} \right) \right)^2 \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=4}^{T} w_{it-3}^2 = \frac{T}{q(T)^2} \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=4}^{T} w_{it-3}^2 \left[ 1 + O \left( \frac{1}{q(T)} \right) \right]$$

$$= O_p \left( \frac{T}{q(T)^2} \right).$$

and, by part (b) of Lemma SE-25

$$2 (1 - \rho_T) \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=4}^{T} w_{it-3} \varepsilon_{it-2} = 2 \left[ \frac{1}{q(T)} + O \left( \frac{1}{q(T)^2} \right) \right] \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=4}^{T} w_{it-3} \varepsilon_{it-2}$$

$$= \frac{2}{q(T)} \left[ 1 + O \left( \frac{1}{q(T)} \right) \right] O_p \left( \frac{1}{\sqrt{N}} \right) = O_p \left( \frac{1}{q(T) \sqrt{N}} \right).$$
Using these results and part (f) of Lemma SE-11 with $g = 2$, we have
\[
\frac{1}{NT} \sum_{i=1}^{N} \sum_{t=4}^{T} (y_{it-3} - y_{it-2})^2
\]
\[
= (1 - \rho_T)^2 \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=4}^{T} \sum_{t=4}^{T} w_{it-3}^2 - 2 (1 - \rho_T) \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=4}^{T} w_{it-3} \varepsilon_{it-2} + \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=4}^{T} \varepsilon_{it-2}^2
\]
\[
= O_p \left( T / q(T)^2 \right) + O_p \left( \frac{1}{q(T) \sqrt{N}} \right) + \sigma^2 + O_p \left( \max \left\{ \frac{1}{\sqrt{NT}}, \frac{1}{T} \right\} \right)
\]
\[
= \sigma^2 + O_p \left( \max \left\{ \frac{1}{\sqrt{NT}}, \frac{1}{T} \right\} \right)
\]

Consider part (c), where we take $\rho_T = \exp \{-1/q(T)\}$ such that $q(T) \sim T$. In this case, applying part (a) of Lemma SE-17, we obtain
\[
(1 - \rho_T)^2 \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=4}^{T} w_{it-3}^2 = T \left( 1 - \exp \left\{ -\frac{1}{q(T)} \right\} \right)^2 \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=4}^{T} \sum_{t=4}^{T} w_{it-3}^2
\]
\[
= \frac{T}{q(T)^2} \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=4}^{T} w_{it-3}^2 \left[ 1 + O \left( \frac{1}{q(T)} \right) \right] = O_p \left( \frac{1}{T} \right),
\]
and, by part (c) of Lemma SE-25,
\[
2 (1 - \rho_T) \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=4}^{T} w_{it-3} \varepsilon_{it-2} = 2 \left[ \frac{1}{q(T)} + O \left( \frac{1}{q(T)^2} \right) \right] \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=4}^{T} w_{it-3} \varepsilon_{it-2} = O_p \left( \frac{1}{T \sqrt{N}} \right).
\]

Using these results along with part (f) of Lemma SE-11 with $g = 2$, we have
\[
\frac{1}{NT} \sum_{i=1}^{N} \sum_{t=4}^{T} (y_{it-3} - y_{it-2})^2 = (1 - \rho_T)^2 \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=4}^{T} \sum_{t=4}^{T} w_{it-3}^2 - 2 (1 - \rho_T) \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=4}^{T} w_{it-3} \varepsilon_{it-2}
\]
\[
+ \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=4}^{T} \varepsilon_{it-2}^2
\]
\[
= O_p \left( \frac{1}{T} \right) + O_p \left( \frac{1}{T \sqrt{N}} \right) + \sigma^2 + O_p \left( \max \left\{ \frac{1}{\sqrt{NT}}, \frac{1}{T} \right\} \right)
\]
\[
= \sigma^2 + O_p \left( \max \left\{ \frac{1}{\sqrt{NT}}, \frac{1}{T} \right\} \right).
\]

We turn our attention now to part (d), where we assume that $\rho_T = \exp \{-1/q(T)\}$ such that $q(T) \to \infty$ but $q(T)/T \to 0$. In this case, applying part (b) of Lemma SE-17, we obtain
\[
(1 - \rho_T)^2 \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=4}^{T} w_{it-3} = q(T) \left( 1 - \exp \left\{ -\frac{1}{q(T)} \right\} \right)^2 \frac{1}{NT q(T)} \sum_{i=1}^{N} \sum_{t=4}^{T} w_{it-3}
\]
\[
= \frac{1}{q(T) \sqrt{N}} \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=4}^{T} w_{it-3}^2 \left[ 1 + O \left( \frac{1}{q(T)} \right) \right] = O_p \left( \frac{1}{q(T)} \right).
\]
and, by part (d) of Lemma SE-25,

\[
2(1 - \rho_T) \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=4}^{T} w_{it-3} \varepsilon_{it-2} = 2 \left[ \frac{1}{q(T)} + O \left( \frac{1}{q(T)^2} \right) \right] \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=4}^{T} w_{it-3} \varepsilon_{it-2}
\]

\[
= \frac{2}{q(T)} \left[ 1 + O \left( \frac{1}{q(T)} \right) \right] O_p \left( \sqrt{\frac{q(T)}{NT}} \right) = O_p \left( \frac{1}{\sqrt{NTq(T)}} \right).
\]

Using these results and part (f) of Lemma SE-11 with \( g = 2 \), we have

\[
\frac{1}{NT} \sum_{i=1}^{N} \sum_{t=4}^{T} (y_{it-3} - y_{it-2})^2 = (1 - \rho_T)^2 \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=4}^{T} w_{it-3}^2 - 2(1 - \rho_T) \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=4}^{T} w_{it-3} \varepsilon_{it-2}
\]

\[
+ \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=4}^{T} \varepsilon_{it-2}^2
\]

\[
= O_p \left( \frac{1}{q(T)} \right) + O_p \left( \frac{1}{\sqrt{NTq(T)}} \right) + \sigma^2 + O_p \left( \max \left\{ \frac{1}{\sqrt{NT}}, \frac{1}{T} \right\} \right)
\]

\[
= \sigma^2 + O_p \left( \max \left\{ \frac{1}{\sqrt{NT}}, \frac{1}{q(T)} \right\} \right).
\]

Finally, consider part (e), where we take \( \rho_T \in \mathcal{G}_{St} \). In this case, applying part (c) of Lemma SE-17, we have

\[
(1 - \rho_T)^2 \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=4}^{T} w_{it-3}^2 = \frac{(1 - \rho_T)^2}{1 - \rho_T^2} \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=4}^{T} w_{it-3}^2 = \frac{(1 - \rho_T)^2}{1 - \rho_T^2} \left[ \sigma^2 + o_p(1) \right]
\]

\[
= \frac{1 - \rho_T}{1 + \rho_T} \sigma^2 + o_p(1).
\]

Moreover, by part (e) of Lemma SE-25, we have

\[
2(1 - \rho_T) \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=4}^{T} w_{it-3} \varepsilon_{it-2} = O_p \left( \frac{1}{\sqrt{NT}} \right).
\]

Hence, applying part (f) of Lemma SE-11, we deduce that

\[
\frac{1}{NT} \sum_{i=1}^{N} \sum_{t=4}^{T} (y_{it-3} - y_{it-2})^2
\]

\[
= \left[ (1 - \rho_T)^2 \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=4}^{T} w_{it-3}^2 - 2(1 - \rho_T) \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=4}^{T} w_{it-3} \varepsilon_{it-2} + \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=4}^{T} \varepsilon_{it-2}^2 \right]
\]

\[
= \frac{1 - \rho_T}{1 + \rho_T} \sigma^2 + o_p(1) + O_p \left( \frac{1}{\sqrt{NT}} \right) + \sigma^2 + O_p \left( \max \left\{ \frac{1}{\sqrt{NT}}, \frac{1}{T} \right\} \right)
\]

\[
= \frac{2\sigma^2}{1 + \rho_T} + o_p(1),
\]

as desired. \( \square \)
Lemma SD-7:

Under Assumptions 1-4, the following statements are true as $N, T \to \infty$ such that $N^\kappa/T \to \tau$, for $\kappa \in \left(\frac{1}{2}, \infty\right)$ and $\tau \in (0, \infty)$.

(a) If $\rho_T = 1$ for all $T$ sufficiently large, then

$$\frac{1}{NT} \sum_{i=1}^{N} y_i^2 T^{-2} = \sigma^2 + O_p \left( \max \left\{ \frac{1}{\sqrt{N}}, \frac{1}{\sqrt{T}} \right\} \right).$$

(b) If $\rho_T = \exp \{ -1/q(T) \}$ such that $T/q(T) \to 0$, then

$$\frac{1}{NT} \sum_{i=1}^{N} y_i^2 T^{-2} = \sigma^2 + O_p \left( \max \left\{ \frac{1}{\sqrt{N}}, 1/(T\sqrt{q(T)}) \right\} \right).$$

(c) If $\rho_T = \exp \{ -1/q(T) \}$ such that $q(T) \sim T$, then

$$\frac{1}{NT} \sum_{i=1}^{N} y_i^2 T^{-2} = \sigma^2 \frac{q(T)}{2T} \left[ 1 - \exp \left\{ - \frac{2T}{q(T)} \right\} \right] + O_p \left( \max \left\{ \frac{1}{\sqrt{N}}, 1/(T\sqrt{q(T)}) \right\} \right).$$

(d) If $\rho_T = \exp \{ -1/q(T) \}$ such that $q(T) \to \infty$ but $q(T)/T \to 0$, then

$$\frac{1}{NT} \sum_{i=1}^{N} y_i^2 T^{-2} = O_p \left( \frac{q(T)}{T} \right) + O_p \left( \sqrt{\frac{q(T)}{NT}} \right) = o_p(1).$$

(e) If $\rho_T \in G_{St} = \{ |\rho_T| = \exp \{ -1/q(T) \} : q(T) \geq 0 \text{ and } q(T) = O(1) \text{ as } T \to \infty \}$, then

$$\frac{1}{NT} \sum_{i=1}^{N} y_i^2 T^{-2} = O_p \left( \frac{1}{T} \right).$$

Proof of Lemma SD-7:

To proceed, consider first part (a), where we take $\rho_T = 1$ for all $T$ sufficiently large. In this case, we apply part (a) of Lemmas SE-11, SE-30, and SE-32 to obtain

$$\frac{1}{NT} \sum_{i=1}^{N} y_i^2 T^{-2} = \sigma^2 + O_p \left( \max \left\{ \frac{1}{\sqrt{N}}, \frac{1}{\sqrt{T}} \right\} \right) + O_p \left( \max \left\{ \frac{1}{\sqrt{NT}}, \frac{1}{T} \right\} \right).$$

Next, consider part (b), where we take $\rho_T = \exp \{ -1/q(T) \}$ such that $T/q(T) \to 0$. In this case, we apply part (a) of Lemmas SE-11 and part (b) of Lemmas SE-30 and SE-32 to obtain

$$\frac{1}{NT} \sum_{i=1}^{N} y_i^2 T^{-2} = \sigma^2 + O_p \left( \max \left\{ \frac{1}{\sqrt{N}}, \frac{T}{\sqrt{q(T)}} \right\} \right) + O_p \left( \max \left\{ \frac{1}{\sqrt{NT}}, \frac{1}{T} \right\} \right) + O_p \left( \frac{1}{T} \right).$$
Consider part (c), where we take $\rho_T = \exp \{-1/q(T)\}$ such that $q(T) \sim T$. In this case, application of part (a) of Lemmas SE-11 and part (c) of Lemmas SE-30 and SE-32 yields

$$
\frac{1}{NT} \sum_{i=1}^{N} y_{iT-2}^2 = \frac{1}{NT} \sum_{i=1}^{N} w_{iT-2}^2 + 2 \frac{1}{NT} \sum_{i=1}^{N} a_i w_{iT-2} + \frac{1}{NT} \sum_{i=1}^{N} a_i^2
$$

$$
= \frac{\sigma^2 q(T)}{2T} \left[ 1 - \exp \left\{ -\frac{2T}{q(T)} \right\} \right] + O_p \left( \max \left\{ \frac{1}{\sqrt{N}}, \frac{1}{\sqrt{T}} \right\} \right) + O_p \left( \max \left\{ \frac{1}{\sqrt{NT}}, \frac{1}{T} \right\} \right) + O_p \left( \frac{1}{T} \right).
$$

Now, we turn our attention to part (d), where we take $\rho_T = \exp \{-1/q(T)\}$ such that $q(T) \to \infty$ but $q(T)/T \to 0$. In this case, we apply part (a) of Lemmas SE-11 and part (d) of Lemmas SE-30 and SE-32 to obtain

$$
\frac{1}{NT} \sum_{i=1}^{N} y_{iT-2}^2 = \frac{1}{NT} \sum_{i=1}^{N} w_{iT-2}^2 + 2 \frac{1}{NT} \sum_{i=1}^{N} a_i w_{iT-2} + \frac{1}{NT} \sum_{i=1}^{N} a_i^2
$$

$$
= O_p \left( \frac{q(T)}{T} \right) + O_p \left( \frac{1}{T} \sqrt{\frac{q(T)}{NT}} \right) + O_p \left( \frac{1}{T} \right) = O_p \left( \frac{q(T)}{T} \right).
$$

Finally, consider part (e), where we take $\rho_T \in \mathcal{G}_{St}$. In this case, we obtain part (a) of Lemmas SE-11 and part (e) of Lemmas SE-30 and SE-32 to obtain

$$
\frac{1}{NT} \sum_{i=1}^{N} y_{iT-2}^2 = \frac{1}{NT} \sum_{i=1}^{N} w_{iT-2}^2 + 2 \frac{1}{NT} \sum_{i=1}^{N} a_i w_{iT-2} + \frac{1}{NT} \sum_{i=1}^{N} a_i^2
$$

$$
= O_p \left( \frac{1}{T} \right) + O_p \left( \frac{1}{\sqrt{NT}} \right) + O_p \left( \frac{1}{T} \right) = O_p \left( \frac{1}{T} \right). \quad \square
$$

**Lemma SD-8:**

Suppose that Assumptions 1-4 hold. Then, the following statements are true as $N, T \to \infty$ such that $N^\kappa/T \to \tau$, for $\kappa \in \left( \frac{1}{2}, \infty \right)$ and $\tau \in (0, \infty)$.

(a) If $\rho_T = 1$ for all $T$ sufficiently large, then

$$
\frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \Delta y_{it} - 1 y_{it-1} = \sigma^2 + O_p \left( \frac{1}{\sqrt{N}} \right).
$$

(b) If $\rho_T = \exp \{-1/q(T)\}$ such that $T \ll q(T)$, then

$$
\frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \Delta y_{it} - 1 y_{it-1} = \sigma^2 + O_p \left( \max \left\{ \frac{T}{q(T)}, \frac{1}{\sqrt{N}} \right\} \right).
$$

(c) If $\rho_T = \exp \{-1/q(T)\}$ such that $q(T) \sim T$, then

$$
\frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \Delta y_{it} - 1 y_{it-1} = \sigma^2 \left( 1 - \frac{1}{4} \frac{q(T)}{T} \left[ \exp \left\{ -\frac{2T}{q(T)} \right\} + \frac{2T}{q(T)} - 1 \right] \right)
$$

$$
+ O_p \left( \max \left\{ \frac{1}{\sqrt{N}}, \frac{1}{\sqrt{T}} \right\} \right).
$$
(d) If $\rho_T = \exp \{-1/q(T)\}$ such that $q(T) \to \infty$ but $q(T)/T \to 0$, then
\[
\frac{1}{NT} \sum_{i=1}^{N} \sum_{t=3}^{T} \Delta y_{it-1} y_{it-1} = \frac{\sigma^2}{2} + O_p \left( \max \left\{ \frac{1}{q(T)}, \frac{q(T)}{NT}, \sqrt{\frac{q(T)}{NT}} \right\} \right).
\]

(e) If $\rho_T \in \mathcal{G}_{St} = \left\{ |\rho_T| = \exp \left\{ -\frac{1}{q(T)} \right\} : q(T) \geq 0 \text{ and } q(T) = O(1) \text{ as } T \to \infty \right\}$, then
\[
\frac{1}{NT} \sum_{i=1}^{N} \sum_{t=3}^{T} \Delta y_{it-1} y_{it-1} = \frac{\sigma^2}{1 + \rho_T} + O_p \left( \frac{1}{T^{1/2}} \right).
\]

**Proof of Lemma SD-8:**

To proceed, first write
\[
\sum_{i=1}^{N} \sum_{t=3}^{T} \Delta y_{it-1} y_{it-1} = \sum_{i=1}^{N} \sum_{t=3}^{T} \left[ a_i (1 - \rho_T) + (\rho_T - 1) y_{it-2} + \epsilon_{it-1} \right] \left[ a_i (1 - \rho_T) + \rho_T y_{it-2} + \epsilon_{it-1} \right]
\]
\[
= (1 - \rho_T)^2 \sum_{i=1}^{N} \sum_{t=3}^{T} a_i^2 + \rho_T (\rho_T - 1) \sum_{i=1}^{N} \sum_{t=3}^{T} y_{it-2}^2 + \sum_{i=1}^{N} \sum_{t=3}^{T} \epsilon_{it-1}^2 + (1 - \rho_T) \left[ 2 \rho_T - 1 \right] \sum_{i=1}^{N} \sum_{t=3}^{T} a_i y_{it-2}
\]
\[
+ 2(1 - \rho_T) \sum_{i=1}^{N} \sum_{t=3}^{T} a_i \epsilon_{it-1} + (\rho_T - 1) \sum_{i=1}^{N} \sum_{t=3}^{T} y_{it-2} \epsilon_{it-1} + \rho_T \sum_{i=1}^{N} \sum_{t=3}^{T} y_{it-2} \epsilon_{it-1}
\]
\[
= (1 - \rho_T)^2 \sum_{i=1}^{N} \sum_{t=3}^{T} a_i^2 + \rho_T (\rho_T - 1) \sum_{i=1}^{N} \sum_{t=3}^{T} a_i^2 + 2 \rho_T (\rho_T - 1) \sum_{i=1}^{N} \sum_{t=3}^{T} a_i w_{it-2}
\]
\[
+ \rho_T (\rho_T - 1) \sum_{i=1}^{N} \sum_{t=3}^{T} w_{it-2}^2 + \sum_{i=1}^{N} \sum_{t=3}^{T} \epsilon_{it-1}^2 + (1 - \rho_T) \left[ 2 \rho_T - 1 \right] \sum_{i=1}^{N} \sum_{t=3}^{T} a_i^2
\]
\[
+ (1 - \rho_T) \left[ 2 \rho_T - 1 \right] \sum_{i=1}^{N} \sum_{t=3}^{T} a_i w_{it-2} + 2(1 - \rho_T) \sum_{i=1}^{N} \sum_{t=3}^{T} a_i \epsilon_{it-1} + (\rho_T - 1) \sum_{i=1}^{N} \sum_{t=3}^{T} \epsilon_{it-1}
\]
\[
+ (\rho_T - 1) \sum_{i=1}^{N} \sum_{t=3}^{T} w_{it-2} \epsilon_{it-1} + \rho_T \sum_{i=1}^{N} \sum_{t=3}^{T} a_i \epsilon_{it-1} + \rho_T \sum_{i=1}^{N} \sum_{t=3}^{T} w_{it-2} \epsilon_{it-1}
\]
\[
= (1 - \rho_T) \left[ 1 - \rho_T - \rho_T + 2 \rho_T - 1 \right] \sum_{i=1}^{N} \sum_{t=3}^{T} a_i^2 + (1 - \rho_T) \left[ 2 \rho_T - 1 - 2 \rho_T \right] \sum_{i=1}^{N} \sum_{t=3}^{T} a_i w_{it-2}
\]
\[
+ \rho_T (\rho_T - 1) \sum_{i=1}^{N} \sum_{t=3}^{T} w_{it-2}^2 + \sum_{i=1}^{N} \sum_{t=3}^{T} \epsilon_{it-1}^2 + ((1 - \rho_T) (2 - 1) + \rho_T) \sum_{i=1}^{N} \sum_{t=3}^{T} a_i \epsilon_{it-1}
\]
\[
+ (\rho_T - 1 + \rho_T) \sum_{i=1}^{N} \sum_{t=3}^{T} w_{it-2} \epsilon_{it-1}
\]
\[
= -(1 - \rho_T) \sum_{i=1}^{N} \sum_{t=3}^{T} a_i w_{it-2} - \rho_T (1 - \rho_T) \sum_{i=1}^{N} \sum_{t=3}^{T} w_{it-2}^2 + \sum_{i=1}^{N} \sum_{t=3}^{T} \epsilon_{it-1}^2 + \sum_{i=1}^{N} \sum_{t=3}^{T} a_i \epsilon_{it-1}
\]
\[
+ (2 \rho_T - 1) \sum_{i=1}^{N} \sum_{t=3}^{T} w_{it-2} \epsilon_{it-1}
\]
Consider part (a), where we take \( \rho_T = 1 \) for all \( T \) sufficiently large. In this case, we can apply parts (c) and (f) of Lemma SE-11, Lemma SE-14, part (a) of Lemma SE-18, and part (a) of Lemma SE-25 to obtain

\[
\frac{1}{NT} \sum_{i=1}^{N} \sum_{t=3}^{T} \Delta y_{it-1} y_{it-1} = \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=3}^{T} \varepsilon_{it-1}^2 + \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=3}^{T} a_i \varepsilon_{it-1} - \frac{(1 - \rho_T)}{NT} \sum_{i=1}^{N} \sum_{t=3}^{T} a_i w_{it-2}
\]

\[
- \frac{\rho_T (1 - \rho_T)}{NT} \sum_{i=1}^{N} \sum_{t=3}^{T} w_{it-2}^2 + \frac{(2 \rho_T - 1)}{NT} \sum_{i=1}^{N} \sum_{t=3}^{T} w_{it-2} \varepsilon_{it-1}
\]

\[
= \sigma^2 + O_p \left( \max \left\{ \frac{1}{\sqrt{NT}}, \frac{1}{T} \right\} \right) + O_p \left( \frac{1}{\sqrt{NT}} \right) + O_p \left( (1 - \rho_T) \max \left\{ \sqrt{\frac{T}{N}}, 1 \right\} \right)
\]

\[
+ O_p \left( \frac{1}{\sqrt{N}} \right)
\]

\[
= \sigma^2 + O_p \left( \frac{1}{\sqrt{N}} \right).
\]

Now, consider part (b) where we take \( \rho_T = \exp \{-1/q(T)\} \) such that \( T \ll q(T) \). Here, applying parts (c) and (f) of Lemma SE-11, Lemma SE-15, part (b) of Lemma SE-18, and part (b) of Lemma SE-25; we obtain

\[
\frac{1}{NT} \sum_{i=1}^{N} \sum_{t=3}^{T} \Delta y_{it-1} y_{it-1} = \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=3}^{T} \varepsilon_{it-1}^2 + \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=3}^{T} a_i \varepsilon_{it-1} - \frac{(1 - \rho_T)}{NT} \sum_{i=1}^{N} \sum_{t=3}^{T} a_i w_{it-2}
\]

\[
- \frac{\rho_T (1 - \rho_T)}{NT} \sum_{i=1}^{N} \sum_{t=3}^{T} w_{it-2}^2 + \frac{(2 \rho_T - 1)}{NT} \sum_{i=1}^{N} \sum_{t=3}^{T} w_{it-2} \varepsilon_{it-1}
\]

\[
= \sigma^2 + O_p \left( \max \left\{ \frac{1}{\sqrt{NT}}, \frac{T}{q(T)} \right\} \right) + O_p \left( \frac{1}{\sqrt{NT}} \right) + O_p \left( \frac{1}{q(T)} \max \left\{ \sqrt{\frac{T}{N}}, 1 \right\} \right)
\]

\[
+ O_p \left( \frac{T}{q(T)} \right) + O_p \left( \frac{1}{\sqrt{N}} \right)
\]

\[
= \sigma^2 + O_p \left( \max \left\{ \frac{T}{q(T)}, \frac{1}{\sqrt{N}} \right\} \right).
\]

Next, consider part (c) where we take \( \rho_T = \exp \{-1/q(T)\} \) such that \( q(T) \sim T \). Here, applying parts (c) and (f) of Lemma SE-11, part (a) of Lemma SE-17, part (c) of Lemma SE-18, and part (c) of...
Lemma SE-25; we obtain
\[
\frac{1}{NT} \sum_{i=1}^{N} \sum_{t=3}^{T} \Delta y_{it-1} y_{it-1} = \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=3}^{T} \varepsilon_{it-1}^2 - \frac{\rho_T (1 - \rho_T)}{NT} \sum_{i=1}^{N} \sum_{t=3}^{T} w_{it-2}^2 - \frac{(1 - \rho_T)}{NT} \sum_{i=1}^{N} \sum_{t=3}^{T} a_i w_{it-2} \\
+ \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=3}^{T} a_i \varepsilon_{it-1} + \frac{(2 \rho_T - 1)}{NT} \sum_{i=1}^{N} \sum_{t=3}^{T} w_{it-2} \varepsilon_{it-1}
\]
\[
= \sigma^2 + O_p \left( \max \left\{ \frac{1}{\sqrt{NT}}, \frac{1}{T} \right\} \right) - \frac{1}{NTq(T)} \sum_{i=1}^{N} \sum_{t=3}^{T} w_{it-2}^2 \left[ 1 + O_p \left( \frac{1}{q(T)} \right) \right] \\
+ O_p \left( \max \left\{ \frac{1}{\sqrt{NT}}, \frac{1}{T} \right\} \right) + O_p \left( \frac{1}{\sqrt{N}} \right) \\
= \sigma^2 \left( 1 - \frac{1}{4} \frac{q(T)}{T} \left[ \exp \left\{ - \frac{2T}{q(T)} \right\} + \frac{2T}{q(T)} - 1 \right] \right) + O_p \left( \max \left\{ \frac{1}{\sqrt{N}}, \frac{1}{\sqrt{T}} \right\} \right).
\]

Consider part (d) where we take \( \rho_T = \exp \left\{ -1/q(T) \right\} \) such that \( q(T) \to \infty \) but \( q(T)/T \to 0 \). Here, applying parts (c) and (f) of Lemma SE-11, part (b) of Lemma SE-17, part (d) of Lemma SE-18, and part (d) of Lemma SE-25; we obtain
\[
\frac{1}{NT} \sum_{i=1}^{N} \sum_{t=3}^{T} \Delta y_{it-1} y_{it-1} = \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=3}^{T} \varepsilon_{it-1}^2 - \frac{\rho_T (1 - \rho_T)}{NT} \sum_{i=1}^{N} \sum_{t=3}^{T} w_{it-2}^2 - \frac{(1 - \rho_T)}{NT} \sum_{i=1}^{N} \sum_{t=3}^{T} a_i w_{it-2} \\
+ \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=3}^{T} a_i \varepsilon_{it-1} + \frac{(2 \rho_T - 1)}{NT} \sum_{i=1}^{N} \sum_{t=3}^{T} w_{it-2} \varepsilon_{it-1}
\]
\[
= \sigma^2 + O_p \left( \max \left\{ \frac{1}{\sqrt{NT}}, \frac{1}{T} \right\} \right) - \frac{1}{NTq(T)} \sum_{i=1}^{N} \sum_{t=3}^{T} w_{it-2}^2 \left[ 1 + O_p \left( \frac{1}{q(T)} \right) \right] \\
+ O_p \left( \max \left\{ \frac{1}{\sqrt{NT}}, \frac{1}{T} \right\} \right) + O_p \left( \frac{1}{\sqrt{N}} \right) \\
= \sigma^2 - \frac{\sigma^2}{2} \left[ 1 + O_p \left( \max \left\{ \frac{1}{q(T)}, \frac{q(T)}{T}, \frac{\sqrt{q(T)}}{NT}, \frac{1}{\sqrt{T}} \right\} \right) \right] + O_p \left( \max \left\{ \frac{\sqrt{q(T)}}{NT}, \frac{1}{\sqrt{T}} \right\} \right)
\]
\[
= \frac{\sigma^2}{2} + O_p \left( \max \left\{ \frac{1}{q(T)}, \frac{\sqrt{q(T)}}{NT}, \frac{1}{\sqrt{T}} \right\} \right)
\]

Finally, consider part (e) where we take \( \rho_T \in \mathcal{G}_\text{st} = \{ |\rho_T| = \exp \left\{ -\frac{1}{q(T)} \right\} : q(T) \geq 0 \text{ and } q(T) = O(1) \text{ as } T \to \infty \} \). Here, applying parts (c) and (f) of Lemma SE-11, part (c) of Lemma SE-17, part (e) of Lemma SE-18, and part (e) of Lemma
SE-25; we obtain

\[
\frac{1}{NT} \sum_{i=1}^{N} \sum_{t=3}^{T} \Delta y_{it-1} y_{it-1} = \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=3}^{T} \frac{\rho^2}{\rho_T} - \frac{\rho_T (1 - \rho_T)}{NT} \sum_{i=1}^{N} \sum_{t=3}^{T} w_{it-2}^2 - \frac{(1 - \rho_T)}{NT} \sum_{i=1}^{N} \sum_{t=3}^{T} a_i \varepsilon_{it-2}
\]

\[+ \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=3}^{T} a_i \varepsilon_{it-1} + \frac{(2\rho_T - 1)}{NT} \sum_{i=1}^{N} \sum_{t=3}^{T} w_{it-2}^2 \]

\[= \sigma^2 + O_p \left( \max \left\{ \frac{1}{\sqrt{NT}}, \frac{1}{T} \right\} \right) - \frac{\rho_T}{1 + \rho_T} \frac{1 - \rho_T^2}{NT} \sum_{i=1}^{N} \sum_{t=3}^{T} w_{it-2}^2 + O_p \left( \frac{1}{\sqrt{NT}} \right) + O_p \left( \frac{1}{1 + \rho_T} \right)
\]

\[= \sigma^2 - \frac{\rho_T \sigma^2}{1 + \rho_T} \left[ 1 + O_p \left( \frac{1}{\sqrt{T}} \right) \right] + O_p \left( \max \left\{ \frac{1}{\sqrt{NT}}, \frac{1}{T} \right\} \right)
\]

\[= \frac{\sigma^2}{1 + \rho_T} + O_p \left( \frac{1}{\sqrt{T}} \right).
\]

**Lemma SD-9:**

Suppose that Assumptions 1-4 hold. Then, the following statements are true as \( N, T \to \infty \) such that \( N^\kappa / T \to \tau \) for some \( \kappa \in \left( \frac{1}{2}, \infty \right) \) and \( \tau \in (0, \infty) \).

(a) If \( \rho_T = 1 \) for all \( T \) sufficiently large, then

\[
\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=3}^{T} \Delta y_{it-1} [a_i (1 - \rho_T) + \varepsilon_{it}] = \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=3}^{T} \varepsilon_{it-1} \varepsilon_{it} + O_p (1 - \rho_T)
\]

\[\Rightarrow N (0, \sigma^4).
\]

(b) If \( \rho_T = \exp \{ -1 / q (T) \} \) such that \( T \ll q (T) \), then

\[
\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=3}^{T} \Delta y_{it-1} [a_i (1 - \rho_T) + \varepsilon_{it}] = \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=3}^{T} \varepsilon_{it-1} \varepsilon_{it} + O_p \left( \frac{\sqrt{T}}{q (T)} \right)
\]

\[\Rightarrow N (0, \sigma^4).
\]

(c) If \( \rho_T = \exp \{ -1 / q (T) \} \) such that \( q (T) \sim T \), then

\[
\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=3}^{T} \Delta y_{it-1} [a_i (1 - \rho_T) + \varepsilon_{it}] = \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=3}^{T} \varepsilon_{it-1} \varepsilon_{it} + O_p \left( \frac{1}{\sqrt{T}} \right)
\]

\[\Rightarrow N (0, \sigma^4).
\]

(d) If \( \rho_T = \exp \{ -1 / q (T) \} \) such that \( q (T) \ll T \) but \( T^{\frac{1}{\kappa - 1}} / q (T) \to 0 \), then

\[
\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=3}^{T} \Delta y_{it-1} [a_i (1 - \rho_T) + \varepsilon_{it}]
\]

\[= \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=3}^{T} \varepsilon_{it-1} \varepsilon_{it} + O_p \left( \max \left\{ \frac{1}{\sqrt{q (T)}}, \frac{T^{\frac{1}{\kappa - 1}}}{q (T)} \right\} \right) \Rightarrow N (0, \sigma^4).
\]

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(e) If \( \rho_T = \exp \{ -1/q(T) \} \) such that \( q(T) \to \infty \) but \( q(T)/T^{1/2}(1-\varepsilon) = O(1) \), then

\[
\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=3}^{T} \Delta y_{it-1} [a_i (1 - \rho_T) + \varepsilon_{it}] = O_p \left( \max \left\{ 1, \frac{T^{1/2}(1-\varepsilon)}{q(T)} \right\} \right).
\]

(f) If \( \rho_T \in \mathcal{G}_t = \left\{ |\rho_T| = \exp \left\{ -\frac{1}{q(T)} \right\} : q(T) \geq 0 \text{ and } q(T) = O(1) \text{ as } T \to \infty \right\} \), then

\[
\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=3}^{T} \Delta y_{it-1} [a_i (1 - \rho_T) + \varepsilon_{it}] = O_p \left( \max \left\{ 1, \sqrt{\frac{N}{T}} \right\} \right) = O_p \left( \max \left\{ 1, T^{1/2}(1-\varepsilon) \right\} \right).
\]

Proof of Lemma SD-9:

To proceed, first write

\[
\sum_{i=1}^{N} \sum_{t=3}^{T} \Delta y_{it-1} [a_i (1 - \rho_T) + \varepsilon_{it}] = (1 - \rho_T) \sum_{i=1}^{N} \sum_{t=3}^{T} [a_i (1 - \rho_T) + (\rho_T - 1) y_{it-2} + \varepsilon_{it-1}] a_i
\]

\[+ \sum_{i=1}^{N} \sum_{t=3}^{T} [a_i (1 - \rho_T) + (\rho_T - 1) y_{it-2} + \varepsilon_{it-1}] \varepsilon_{it}\]

\[= (1 - \rho_T)^2 \sum_{i=1}^{N} \sum_{t=3}^{T} a_i^2 - (1 - \rho_T)^2 \sum_{i=1}^{N} \sum_{t=3}^{T} (a_i + w_{it-2}) a_i + (1 - \rho_T) \sum_{i=1}^{N} \sum_{t=3}^{T} a_i \varepsilon_{it-1}
\]

\[+ (1 - \rho_T) \sum_{i=1}^{N} \sum_{t=3}^{T} a_i \varepsilon_{it} - (1 - \rho_T) \sum_{i=1}^{N} \sum_{t=3}^{T} (a_i + w_{it-2}) \varepsilon_{it} + \sum_{i=1}^{N} \sum_{t=3}^{T} \varepsilon_{it-1} \varepsilon_{it}\]

\[= - (1 - \rho_T)^2 \sum_{i=1}^{N} \sum_{t=3}^{T} a_i w_{it-2} + (1 - \rho_T) \sum_{i=1}^{N} \sum_{t=3}^{T} a_i \varepsilon_{it-1} - (1 - \rho_T) \sum_{i=1}^{N} \sum_{t=3}^{T} w_{it-2} \varepsilon_{it} + \sum_{i=1}^{N} \sum_{t=3}^{T} \varepsilon_{it-1} \varepsilon_{it}.
\]

Consider part (a), where we take \( \rho_T = 1 \) for all \( T \) sufficiently large. In this case, we can apply parts (c) and (h) of Lemma SE-11, part (a) of Lemma SE-18, and part (a) of Lemma SE-25 to obtain

\[
\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=3}^{T} \Delta y_{it-1} [a_i (1 - \rho_T) + \varepsilon_{it}] = \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=3}^{T} \varepsilon_{it-1} \varepsilon_{it} - \frac{(1 - \rho_T)^2}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=3}^{T} w_{it-2} \varepsilon_{it}
\]

\[- \frac{(1 - \rho_T)^2}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=3}^{T} a_i w_{it-2} + \frac{(1 - \rho_T)}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=3}^{T} a_i \varepsilon_{it-1}\]

\[= \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=3}^{T} \varepsilon_{it-1} \varepsilon_{it} + O_p \left( (1 - \rho_T) \sqrt{T} \right) + O_p \left( (1 - \rho_T)^2 \max \left\{ T, \sqrt{NT} \right\} \right) + O_p (1 - \rho_T)
\]

\[= \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=3}^{T} \varepsilon_{it-1} \varepsilon_{it} + O_p \left( (1 - \rho_T) \sqrt{T} \right) \Rightarrow N(0, \sigma^4).
\]

Now, consider part (b) where we take \( \rho_T = \exp \{ -1/q(T) \} \) such that \( T \ll q(T) \). Here, applying
parts (c) and (h) Lemma SE-11, part (b) of Lemma SE-18, and part (b) of Lemma SE-25; we obtain

\[ \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=3}^{T} \Delta y_{it-1} [a_i (1 - \rho_T) + \varepsilon_{it}] \]

\[ = \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=3}^{T} \varepsilon_{it-1} \varepsilon_{it} - \frac{(1 - \rho_T)}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=3}^{T} w_{it-2} \varepsilon_{it} - \frac{(1 - \rho_T)^2}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=3}^{T} a_i w_{it-2} \]

\[ + \frac{(1 - \rho_T)}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=3}^{T} a_i \varepsilon_{it-1} \]

\[ = \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=3}^{T} \varepsilon_{it-1} \varepsilon_{it} + O_p \left( \frac{\sqrt{T}}{q(T)} \right) + O_p \left( \max \left\{ \frac{T}{q(T)^2}, \frac{\sqrt{N}}{q(T)^2} \right\} \right) + O_p \left( \frac{1}{q(T)} \right) \]

\[ = \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=3}^{T} \varepsilon_{it-1} \varepsilon_{it} + O_p \left( \frac{\sqrt{T}}{q(T)} \right) \Rightarrow N(0, \sigma^4). \]

Next, consider part (c) where we take \( \rho_T = \exp \{-1/q(T)\} \) such that \( q(T) \sim T \). Here, applying parts (c) and (h) Lemma SE-11, part (c) of Lemma SE-18, and part (c) of Lemma SE-25; we obtain

\[ \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=3}^{T} \Delta y_{it-1} [a_i (1 - \rho_T) + \varepsilon_{it}] = \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=3}^{T} \varepsilon_{it-1} \varepsilon_{it} - \frac{(1 - \rho_T)}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=3}^{T} w_{it-2} \varepsilon_{it} \]

\[ - \frac{(1 - \rho_T)^2}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=3}^{T} a_i w_{it-2} + \frac{(1 - \rho_T)}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=3}^{T} a_i \varepsilon_{it-1} \]

\[ = \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=3}^{T} \varepsilon_{it-1} \varepsilon_{it} + O_p \left( \frac{1}{\sqrt{T}} \right) + O_p \left( \max \left\{ \frac{1}{T}, \frac{\sqrt{N}}{T^{3/2}} \right\} \right) + O_p \left( \frac{1}{T} \right) \]

\[ = \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=3}^{T} \varepsilon_{it-1} \varepsilon_{it} + O_p \left( \frac{1}{\sqrt{T}} \right) \Rightarrow N(0, \sigma^4). \]

Consider part (d) where we take \( \rho_T = \exp \{-1/q(T)\} \) such that \( q(T) \ll T \) but \( T^{1/2} (1-1) / q(T) \to 0 \). Here, applying parts (c) and (h) Lemma SE-11, part (d) of Lemma SE-18, and part (d) of Lemma SE-25.
Consider part (e) where we take \( \rho_T = \exp \{-1/q(T)\} \) such that \( q(T) \to \infty \) but \( q(T)/T^{\frac{1}{2}(\frac{1}{\kappa}-1)} = O(1) \). Here, applying parts (c) and (h) Lemma SE-11, part (d) of Lemma SE-18, and part (d) of Lemma SE-25; we obtain

\[
\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=3}^{T} \Delta y_{it-1} [a_i (1 - \rho_T) + \varepsilon_{it}]
\]

\[
= \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=3}^{T} \varepsilon_{it-1} \varepsilon_{it} - \frac{(1 - \rho_T)^2}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=3}^{T} \epsilon_{it-2} \varepsilon_{it} - \frac{(1 - \rho_T)^2}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=3}^{T} a_i \epsilon_{it-2} + \frac{(1 - \rho_T)}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=3}^{T} \alpha_i \varepsilon_{it-1} \\
\]

\[
= \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=3}^{T} \varepsilon_{it-1} \varepsilon_{it} + O_p \left( \frac{1}{\sqrt{q(T)}} \right) + O_p \left( \max \left\{ \frac{1}{q(T)}, \frac{1}{q(T)} \sqrt{N/T} \right\} \right) + O_p \left( \frac{1}{q(T)} \right) \\
\]

\[
= \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=3}^{T} \varepsilon_{it-1} \varepsilon_{it} + O_p \left( \max \left\{ \frac{1}{\sqrt{q(T)}}, \frac{T^{\frac{1}{2}(\frac{1}{\kappa}-1)}}{q(T)} \right\} \right) \quad \text{(for } \kappa \in \left( \frac{1}{2}, \infty \right) \text{)} \\
\Rightarrow \quad N(0, \sigma^4).
\]

Finally, consider part (f) where we take \( \rho_T \in \mathcal{G}_{St} = \{ |\rho_T| = \exp \{-\frac{1}{q(T)}\} : q(T) \geq 0 \text{ and } q(T) = O(1) \text{ as } T \to \infty \} \). Here, applying parts
(c) and (h) Lemma SE-11, part (e) of Lemma SE-18, and part (e) of Lemma SE-25; we obtain

\[
\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=3}^{T} \Delta y_{it-1} [a_i (1 - \rho_T) + \varepsilon_{it}]
\]

\[
= \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=3}^{T} \varepsilon_{it-1} \varepsilon_{it} - (1 - \rho_T)^2 \frac{N}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=3}^{T} a_i w_{it-2} \varepsilon_{it} - \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=3}^{T} w_{it-2} \varepsilon_{it}
\]

\[
+ \frac{(1 - \rho_T)}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=3}^{T} a_i \varepsilon_{it-1}
\]

\[
= O_p (1) + O_p \left( \max \left\{ 1, \sqrt{\frac{N}{T}} \right\} \right) + O_p (1) + O_p (1)
\]

\[
= O_p \left( \max \left\{ 1, \sqrt{\frac{N}{T}} \right\} \right). \quad \square
\]

Lemma SD-10:

Let \( \mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_J \) be \( J \) mutually exclusive collections of (parameter) sequences. Now, let \( \{ \rho_{j,T} \} \in \mathcal{G}_{s_j} \) (for \( j = 1, \ldots, J \)), i.e., \( \{ \rho_{j,T} \} \) is a sequence belonging to the collection \( \mathcal{G}_{s_j} \), where \( s_j \in \{ 1, \ldots, J \} \). Define \( T_j = f_j (T) \) (\( j = 1, \ldots, d \)), with \( d \leq J \), where \( f_j (\cdot) : \mathbb{N} \to \mathbb{N} \) is an increasing function in its argument, and let \( \{ \rho_{j,T_j} \} \) denote a subsequence of \( \{ \rho_{j,T} \} \). Furthermore, let \( \mathcal{G}^* \) be a collection of parameter sequences, and for each \( \{ \rho_T \} \in \mathcal{G}^* \), suppose that

\[
\{ \rho_T \} = \bigcup_{j=1}^{d} \{ \rho_{j,T_j} \},
\]

where

\[
\{ \rho_{1,T_1} \} \in \mathcal{G}_{s_1}, \ldots, \{ \rho_{d,T_d} \} \in \mathcal{G}_{s_d},
\]

with \( \mathcal{G}_{s_k} \neq \mathcal{G}_{s_\ell} \) for \( k \neq \ell \) and where

\[
\mathbb{N} = \bigcup_{k=1}^{d} \{ T_k = f_k (T) : T \in \mathbb{N} \}.
\]

Finally, let \( S_T (\rho_T) \) be a sequence of statistics, possibly depending on \( \rho_T \), and let \( \zeta (T) \) be a function of \( T \) such that \( \zeta (T) \to 0 \) as \( T \to \infty \). Then, the following statements hold as \( T \to \infty \).

(a) If for each \( j \in \{ 1, \ldots, d \} \) and each \( \{ \rho_{j,T} \} \in \mathcal{G}_{s_j} \)

\[
S_T (\rho_{j,T}) = O_p (\zeta (T)),
\]

then

\[
S_T (\rho_T) = O_p (\zeta (T)),
\]

for all \( \{ \rho_T \} \in \mathcal{G}^* \).
(b) If for each \( j \in \{1, \ldots, d\} \) and each \( \rho_{j,T} \in \mathcal{G}_{s_k} \)

\[
S_T (\rho_{j,T}) \Rightarrow W, \text{ as } T \to \infty
\]

for some random variable \( W \), then

\[
S_T (\rho_T) \Rightarrow W, \text{ as } T \to \infty,
\]

for all \( \{\rho_T\} \in \mathcal{G}^* \).

**Proof of Lemma SD-10:**

To show (a), note that, by assumption, for any \( \varepsilon > 0 \) and for each \( j \in \{1, \ldots, d\} \), there exists a positive constant \( K_{j,\varepsilon} \) and a positive integer \( L_j \) such that for all \( T \geq L_j \)

\[
\Pr \left( \left| \frac{S_T (\rho)}{\zeta (T)} \right| \geq K_{j,\varepsilon} | \rho = \rho_{j,T} \right) < \varepsilon.
\]

Since, for \( T \geq L_j \), \( T_j = f_j (T) \geq T \geq L_j \) by Lemma SE-33 in Appendix SE below, we further deduce that

\[
\Pr \left( \left| \frac{S_{T_j} (\rho)}{\zeta (T_j)} \right| \geq K_{j,\varepsilon} | \rho = \rho_{j,T_j} \right) < \varepsilon.
\]

for any subsequence \( \{\rho_{j,T_j}\} \in \mathcal{G}_{s_j} \) and for all \( j \in \{1, \ldots, d\} \). Next, let \( L^{\max} = \max \{f_1 (L_1), \ldots, f_d (L_d)\} \) and \( K_{\varepsilon}^{\max} = \max \{K_{1,\varepsilon}, \ldots, K_{d,\varepsilon}\} \). Consider any \( T \geq L^{\max} \), and we must have

\[
T = f_j (T^*),
\]

for some \( j = 1, \ldots, d \) and some \( T^* \in \mathbb{N} \). By Lemma SE-33, we have that

\[
T = f_j (T^*) \geq L^{\max} \geq f_j (L_j) \geq L_j,
\]

from which we also deduce that \( T^* \geq L_j \) since \( f_j (\cdot) \) is a monotonically increasing function. Hence, for any \( \{\rho_T\} \in \mathcal{G}^* \) and for all \( T \geq L^{\max} \)

\[
\Pr \left( \left| \frac{S_T (\rho)}{\zeta (T)} \right| \geq K_{\varepsilon}^{\max} | \rho = \rho_T \right) \leq \Pr \left( \left| \frac{S_{f_j (T^*)} (\rho)}{\zeta (f_j (T^*))} \right| \geq K_{j,\varepsilon} | \rho = \rho_{f_j (T^*)} \right) < \varepsilon.
\]

It follows that

\[
S_T (\rho_T) = O_p (\zeta (T))
\]

for all \( \{\rho_T\} \in \mathcal{G}^* \).

For part (b), note that, in this case, for any \( \varepsilon > 0 \) and for each \( j \in \{1, \ldots, d\} \), there exists a positive integer \( L_j \) such that for all \( T \geq L_j \)

\[
|\Pr (S_T (\rho) \leq x | \rho = \rho_{j,T}) - F_W (x) | < \varepsilon,
\]

for each \( x \) which is a point of continuity of \( F_W (\cdot) \), the cdf of \( W \). Since, for \( T \geq L_j \), \( T_j = f_j (T) \geq T \geq L_j \) by Lemma SE-33 in Appendix SE below, we further deduce that

\[
|\Pr (S_{T_j} (\rho) \leq x | \rho = \rho_{j,T_j}) - F_W (x) | < \varepsilon,
\]
for each \( x \) which is a point of continuity of \( F_W(\cdot) \). Next, let \( L_{\text{max}} = \max \{ f_1(L_1), \ldots, f_d(L_d) \} \). Consider any \( T \geq L_{\text{max}} \), and we must have
\[
T = f_j(T^*),
\]
for some \( j = 1, \ldots, d \) and some \( T^* \in \mathbb{N} \). Again, by Lemma SE-33, we have that
\[
T = f_j(T^*) \geq L_{\text{max}} \geq f_j(L_j) \geq L_j,
\]
and, thus, \( T \geq T^* \geq L_j \) from which it follows that, for any \( \{ \rho_T \} \in \mathcal{G}^* \) and for all \( T \geq L_{\text{max}} \),
\[
\left| \Pr(S_T(\rho) \leq x | \rho = \rho_T) - F_W(x) \right| = \left| \Pr(S_{f_j(T^*)}(\rho) \leq x | \rho = \rho_{f_j(T^*)}) - F_W(x) \right| < \varepsilon,
\]
for each \( x \) which is a point of continuity of \( F_W(\cdot) \), as required. □

**Lemma SD-11:**

Let \( \hat{\rho}_{\text{pre}} \) be the preliminary estimator defined in footnote 4 of the main paper. Suppose that Assumptions 1-4 hold. Then, the following statements are true as \( N, T \to \infty \) such that \( N^\kappa/T = \tau \) for \( \kappa \in \left( \frac{1}{2}, \infty \right) \) and \( \tau \in (0, \infty) \).

(a) If \( \rho_T = 1 \) for all \( T \) sufficiently large, then
\[
\hat{\rho}_{\text{pre}} - \rho_T = O_p \left( \frac{1}{T \sqrt{N}} \right).
\]

(b) If \( \rho_T = \exp\{-1/q(T)\} \) such that \( T/q(T) = O(1) \), then
\[
\hat{\rho}_{\text{pre}} - \rho_T = O_p \left( \frac{1}{T \sqrt{N}} \right).
\]

(c) If \( \rho_T = \exp\{-1/q(T)\} \) such that \( T/(L(T))^2 \ll q(T) \ll T \), then
\[
\hat{\rho}_{\text{pre}} - \rho_T = O_p \left( \frac{1}{\sqrt{NTq(T)}} \right).
\]

(d) If \( \rho_T = \exp\{-1/q(T)\} \) such that \( q(T) \to \infty \) but \( q(T)(L(T))^2/T = O(1) \), then
\[
\hat{\rho}_{\text{pre}} - \rho_T = O_p \left( \frac{1}{\sqrt{NT}} \right).
\]

(e) If \( \rho_T \in \mathcal{G}_{\text{St}} = \left\{|\rho_T| = \exp\left\{-\frac{1}{q(T)}\right\} : q(T) \geq 0 \text{ and } q(T) = O(1) \text{ as } T \to \infty\right\} \), then
\[
\hat{\rho}_{\text{pre}} - \rho_T = O_p \left( \frac{1}{\sqrt{NT}} \right).
\]
Proof of Lemma SD-11:

To proceed, we first consider part (a), where we assume that \( \rho_T = 1 \) for all \( T \) sufficiently large. In this case, by part (a) of Theorems SA-1 and SA-2, respectively, we have

\[
\hat{\rho}_\text{IVD} - 1 = O_p \left( \frac{1}{\sqrt{T}} \right) \quad \text{and} \quad \hat{\rho}_\text{pols} - 1 = O_p \left( \frac{1}{T\sqrt{N}} \right).
\]

Moreover, in this case, there exists a positive integer \( I_\rho \) such that for all \( T \geq I_\rho \)

\[
\frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \overline{y}_{1,NT})^2 \sqrt{NT} (\hat{\rho}_\text{pols} - 1)
\]

Applying part (a) of Lemmas SD-3, Theorem SA-2, and the Cramér convergence theorem, we obtain

\[
\frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \overline{y}_{1,NT})^2 \sqrt{NT} (\hat{\rho}_\text{pols} - \rho_T) \Rightarrow \frac{\sigma^2}{2} \sqrt{2Z} = Z \equiv N(0, \sigma^2),
\]

so that \( \overline{T} = O_p(1) \). It follows that in this case

\[
\frac{\Delta_{\text{IC}}}{\sqrt{N L(T)}} = \frac{T + \sqrt{N L(T)}}{\sqrt{N L(T)}} = 1 + \overline{\eta}_1(N,T),
\]

where

\[
\overline{\eta}_1(N,T) = \frac{T}{\sqrt{N L(T)}} = O_p \left( \frac{1}{\sqrt{N L(T)}} \right),
\]

so that

\[
\overline{w}_{\text{IC}} = \frac{1}{1 + \exp \left\{ \frac{1}{2} \sqrt{N L(T)} \frac{\Delta_{\text{IC}}}{\sqrt{N L(T)}} \right\}} = \exp \left\{ -\frac{1}{2} \sqrt{N L(T)} \left[ 1 + \overline{\eta}_1(N,T) \right] \right\} \left[ 1 + o_p(1) \right].
\]

Applying part (a) of Theorems SA-1 and SA-2, we obtain

\[
\hat{\rho}_\text{pre} - \rho_T
\]

\[
= \overline{w}_{\text{IC}} \hat{\rho}_\text{IVD} + (1 - \overline{w}_{\text{IC}}) \hat{\rho}_\text{pols} - \rho_T
\]

\[
= \overline{w}_{\text{IC}} (\hat{\rho}_\text{IVD} - \rho_T) + (1 - \overline{w}_{\text{IC}}) (\hat{\rho}_\text{pols} - \rho_T)
\]

\[
= \exp \left\{ -\frac{1}{2} \sqrt{N L(T)} \left[ 1 + \overline{\eta}_1(N,T) \right] \right\} (\hat{\rho}_\text{IVD} - \rho_T) \left[ 1 + o_p(1) \right] + (\hat{\rho}_\text{pols} - \rho_T)
\]

\[
= \exp \left\{ -\frac{1}{2} \sqrt{N L(T)} \left[ 1 + \overline{\eta}_1(N,T) + \frac{\ln T}{\sqrt{N L(T)}} \right] \right\} + O_p \left( \frac{1}{T\sqrt{N}} \right)
\]

\[
+ O_p \left( \frac{1}{T\sqrt{N}} \right) \frac{\ln N}{\sqrt{N L(T)}} \]

\[
= O_p \left( \frac{1}{T\sqrt{N}} \right).
\]
Next, we consider part (b), where we take $\rho_T = \exp\{-1/q(T)\}$ such that $T/q(T) \to 0$. We break our analysis here into a number of different subcases:

**Case b(i):** $T \sqrt{N} \ll q(T)$.

In this case, by part (b) of Theorems SA-1 and SA-2, respectively, we have

$$\hat{\rho}_{IVD} - \rho_T = O_p \left( \frac{1}{\sqrt{T}} \right) \quad \text{and} \quad \hat{\rho}_{pols} - \rho_T = O_p \left( \frac{1}{T \sqrt{N}} \right).$$

Moreover, write

$$T = \left( \sum_{i=1}^{N} \sum_{t=1}^{T} (y_{it-1} - \mu_{-1,NT})^2 \right)^{1/2} (\hat{\rho}_{pols} - 1)$$

$$= \sqrt{\frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \mu_{-1,NT})^2 \sqrt{NT} (\hat{\rho}_{pols} - \rho_T)} + \sqrt{\frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \mu_{-1,NT})^2 \sqrt{NT} (\rho_T - 1)}$$

Applying part (b) of Lemmas SD-3, Theorem SA-2, and the Cramér convergence theorem, we obtain

$$\sqrt{\frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \mu_{-1,NT})^2 \sqrt{NT} (\hat{\rho}_{pols} - \rho_T)} \Rightarrow \sqrt{\frac{\sigma^2}{2}} \sqrt{2} Z = Z \equiv N(0, \sigma^2)$$

From part (b) of Lemma SD-3, we also deduce that

$$\sqrt{\frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \mu_{-1,NT})^2 \sqrt{NT} (\rho_T - 1)} = -\sqrt{\frac{\sigma^2}{2}} \sqrt{\frac{NT}{q(T)}} \left[ 1 + o_p(1) \right] = O_p \left( \frac{\sqrt{NT}}{q(T)} \right) = o_p(1).$$

Hence,

$$T = O_p(1) + o_p(1) = O_p(1).$$

The rest of the argument then follows in a manner similar to the proof of part (a) above. Hence, applying part (b) of Theorems SA-1 and SA-2 and following the derivation similar to that given for part (a), we also obtain in this case

$$\hat{\rho}_{pre} - \rho_T = O_p \left( \frac{1}{T \sqrt{N}} \right).$$

**Case b(ii):** $q(T) \sim T \sqrt{N}$

For this case, by part (c) of Theorem SA-1, we have

$$\hat{\rho}_{IVD} - \rho_T = O_p \left( \frac{1}{\sqrt{T}} \right).$$
and, by part (b) of Theorem SA-2, we again obtain
\[ \hat{\rho}_{\text{pols}} - \rho_T = O_p \left( \frac{1}{\sqrt{NT}} \right). \]

Applying part (b) of Lemma SD-3, we have
\[
\sqrt{\frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \bar{y}_{t-1,NT})^2 \sqrt{NT} (\rho_T - 1)}
= -\sqrt{\frac{\sigma^2 \sqrt{NT}}{2 q(T)}} [1 + o_p(1)] = O_p(1),
\]
so that
\[
\mathbb{T} = \sqrt{\frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \bar{y}_{t-1,NT})^2 \sqrt{NT} (\hat{\rho}_{\text{pols}} - \rho_T)}
+ \sqrt{\frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \bar{y}_{t-1,NT})^2 \sqrt{NT} (\rho_T - 1)}
= O_p(1) + O_p(1) = O_p(1).
\]

The rest of the argument is again similar to that of part (a), so that, by applying part (c) of Theorem SA-1 and part (b) of Theorem SA-2, we obtain
\[ \hat{\rho}_{\text{pre}} - \rho_T = O_p \left( \frac{1}{T \sqrt{N}} \right). \]

Case b(iii): \( T \ll q(T) \ll T \sqrt{N} \)

In this case, by part (d) of Theorem SA-1, we have
\[ \hat{\rho}_{\text{IVD}} - \rho_T = O_p \left( \frac{q(T)}{T^{3/2} \sqrt{N}} \right), \]
and, by part (b) of Theorem SA-2, we have.
\[ \hat{\rho}_{\text{pols}} - \rho_T = O_p \left( \frac{1}{\sqrt{NT}} \right). \]

Applying part (b) of Lemma SD-3 and part (b) of Theorem SA-2, we have
\[ \frac{q(T)}{\sqrt{NT}} \mathbb{T} = -\sqrt{\frac{\sigma^2}{2}} + o_p(1). \]

It follows that in this case
\[ \frac{\Sigma_{\text{IC}}}{\sqrt{NL}(T)} = \frac{\mathbb{T} + \sqrt{NL}(T)}{\sqrt{NL}(T)} = 1 + \eta_1(N, T), \]
where
\[\eta_1 (N,T) = \frac{T}{\sqrt{NL} (T)} = T q(T) L(T) = O_p \left( \frac{T}{q(T) L(T)} \right) = o_p (1).\]

Now, write
\[\bar{w}_{IC} = \frac{1}{1 + \exp \left\{ \frac{1}{2} \sqrt{NL} (T) \frac{\sum I}{\sqrt{NL(T)}} \right\}} = \exp \left\{ -\frac{1}{2} \sqrt{NL} (T) [1 + \eta_1 (N,T)] \right\} \left[ 1 + o_p (1) \right].\]

so that by applying part (d) of Theorems SA-1 and part (b) of Theorem SA-2, we obtain
\[
\hat{\rho}_{pre} - \rho_T = \bar{w}_{IC} (\hat{p}_{IVD} - \rho_T) + (1 - \bar{w}_{IC}) (\hat{p}_{pols} - \rho_T)
\[
= \exp \left\{ -\frac{1}{2} \sqrt{NL} (T) [1 + \eta_1 (N,T)] \right\} \left( \hat{p}_{IVD} - \rho_T \right) \left[ 1 + o_p (1) \right] + \left( \hat{p}_{pols} - \rho_T \right) \left[ 1 + o_p (1) \right]
\[
= O_p \left( \exp \left\{ -\frac{1}{2} \sqrt{NL} (T) \left[ 1 + \eta_1 (N,T) + \frac{3 \ln T}{\sqrt{NL} (T)} + \frac{\ln N}{\sqrt{NL} (T)} - \frac{2 \ln q(T)}{\sqrt{NL} (T)} \right] \right\} \right)
\[
+ O_p \left( \frac{1}{T \sqrt{N}} \right) + O_p \left( \exp \left\{ -\frac{1}{2} \sqrt{NL} (T) \left[ 1 + \eta_1 (N,T) + \frac{2 \ln T}{\sqrt{NL} (T)} + \frac{\ln N}{\sqrt{NL} (T)} \right] \right\} \right)
\[
= O_p \left( \frac{1}{T \sqrt{N}} \right).
\]

Case b(iv): \( q(T) \sim T \)

Here, by part (e) of Theorem SA-1, we have
\[\hat{p}_{IVD} - \rho_T = O_p \left( \frac{1}{\sqrt{NT}} \right),\]
and, by the proof of part (c) of Theorem SA-2, we have
\[\hat{p}_{pols} - \rho_T = O_p \left( \frac{1}{\sqrt{NT}} \right).
\]

Moreover, applying part (c) Lemma SD-3 and Theorem SA-2, we obtain
\[\frac{q(T)}{\sqrt{NT}} = -\frac{\sigma q(T)}{2} \sqrt{\exp \left\{ -\frac{2T}{q(T)} \right\} + \frac{2T}{q(T)} - 1 [1 + o_p (1)]} = O_p (1),\]
so that, similar to case b(iii) above,
\[\frac{\sum I}{\sqrt{NL} (T)} = \frac{\bar{T} + \sqrt{NL} (T)}{\sqrt{NL} (T)} = 1 + \eta_1 (N,T),\]
where
\[\eta_1 (N,T) = \frac{T}{\sqrt{NL} (T)} = T q(T) L(T) \sqrt{NT} = O_p \left( \frac{1}{L(T)} \right) = o_p (1),\]

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The rest of the argument follows similar to that given earlier, so that by applying part (e) of Theorem SA-1 along with part (c) of Theorem SA-2, we get

\[
\hat{\rho}_{\text{pre}} - \rho_T = w_{IC} \left( \hat{\rho}_{\text{IVD}} - \rho_T \right) + (1 - w_{IC}) \left( \hat{\rho}_{\text{pols}} - \rho_T \right) \\
= \exp \left\{ -\frac{1}{2} \sqrt{NL(T)} \left[ 1 + \eta_1(N,T) \right] \right\} \left( \hat{\rho}_{\text{IVD}} - \rho_T \right) \left[ 1 + o_p(1) \right] + \left( \hat{\rho}_{\text{pols}} - \rho_T \right) \\
- \exp \left\{ -\frac{1}{2} \sqrt{NL(T)} \left[ 1 + \eta_1(N,T) \right] \right\} \left( \hat{\rho}_{\text{pols}} - \rho_T \right) \left[ 1 + o_p(1) \right] \\
= O_p \left( \exp \left\{ -\frac{1}{2} \sqrt{NL(T)} \left[ 1 + \eta_1(N,T) + \frac{\ln T}{\sqrt{NL(T)}} + \frac{\ln N}{\sqrt{NL(T)}} \right] \right\} \right) + O_p \left( \frac{1}{T\sqrt{N}} \right) \\
+ O_p \left( \exp \left\{ -\frac{1}{2} \sqrt{NL(T)} \left[ 1 + \eta_1(N,T) + \frac{2\ln T}{\sqrt{NL(T)}} + \frac{\ln N}{\sqrt{NL(T)}} \right] \right\} \right) \\
= O_p \left( \frac{1}{T\sqrt{N}} \right),
\]
as required.

To complete the proof of part (b), note that for the pathwise asymptotics considered here \( N \) grows as a monotonically increasing function of \( T \), so that the asymptotics employed here can be taken to be single-indexed with \( T \to \infty \). Hence, we set \( S_T(\rho_T) = \hat{\rho}_{\text{pre}} - \rho_T \) in Lemma SD-10 and apply part (a) of that lemma for the following collection of parameter sequences

\[
\mathcal{G}_1^o = \left\{ \rho_T = \exp \left\{ -\frac{1}{q(T)} \right\} : \sqrt{NT} \sim T^{\frac{1}{2\kappa} + 1} \ll q(T) \right\}, \\
\mathcal{G}_2^o = \left\{ \rho_T = \exp \left\{ -\frac{1}{q(T)} \right\} : \sqrt{NT} \sim T^{\frac{1}{2\kappa} + 1} \sim q(T) \right\}, \\
\mathcal{G}_3^o = \left\{ \rho_T = \exp \left\{ -\frac{1}{q(T)} \right\} : T \ll q(T) \ll T^{\frac{1}{2\kappa} + 1} \sim \sqrt{NT} \right\}, \\
\mathcal{G}_4^o = \left\{ \rho_T = \exp \left\{ -\frac{1}{q(T)} \right\} : q(T) \sim T \right\}.
\]

Let

\[
\mathcal{G}^* = \left\{ \rho_T = \exp \left\{ -\frac{1}{q(T)} \right\} : T/q(T) = O(1) \right\},
\]
and note that, by the calculations given in subcases b(i)-b(iv) above and by part (a) of Lemma SD-10, we have that for every \( \{\rho_T\} \in \mathcal{G}^* \),

\[
S_T(\rho_T) = \hat{\rho}_{\text{pre}} - \rho_T = O_p \left( \frac{1}{T\sqrt{N}} \right),
\]
which is the desired result.

Next, consider part (c), where we take \( T/(L(T))^2 \ll q(T) \ll T \). From part (f) of Theorem SA-1, we see that in the case

\[
\hat{\rho}_{\text{IVD}} - \rho_T = O_p \left( \frac{1}{\sqrt{NT}} \right).
\]
Also, from the proof of part (d) of Theorem SA-2, we deduce that

$$\hat{\rho}_{\text{pols}} = \rho_T + O_p \left( \frac{1}{\sqrt{NTq(T)}} \right)$$

$$= 1 - \frac{1}{q(T)} + O \left( \frac{1}{q(T)^2} \right) + O_p \left( \frac{1}{\sqrt{NTq(T)}} \right)$$

or

$$q(T) (\hat{\rho}_{\text{pols}} - 1) = -1 + O \left( \frac{1}{q(T)} \right) + O_p \left( \frac{\sqrt{q(T)}}{NTq(T)} \right).$$

Using part (d) of Lemma SD-3 and part (d) of Theorem SA-2; it follows that in this case

$$\sqrt{\frac{q(T)}{NT}} = -\frac{\sigma}{\sqrt{2}} + o_p(1),$$

so that

$$\frac{\hat{\Sigma}_{IC}}{\sqrt{N\lambda (T)}} = \frac{\hat{\Sigma}_{IC}}{\sqrt{N\lambda (T)}} = 1 + \eta_1 (N, T),$$

where

$$\eta_1 (N, T) = \frac{\hat{\Sigma}_{IC}}{\sqrt{N\lambda (T)}} = \frac{1}{L(T)} \sqrt{\frac{T}{q(T)}} \sqrt{\frac{q(T)}{NT}} = O_p \left( \frac{1}{L(T)} \sqrt{\frac{T}{q(T)}} \right) = o_p(1).$$

Now, write

$$\overline{w}_{IC} = \frac{1}{1 + \exp \left\{ \frac{1}{2} \sqrt{N\lambda (T)} \frac{\hat{\Sigma}_{IC}}{\sqrt{N\lambda (T)}} \right\} \left[ 1 + \eta_1 (N, T) \right]} = \exp \left\{ -\frac{1}{2} \sqrt{N\lambda (T)} \left[ 1 + \eta_1 (N, T) \right] \right\} [1 + o_p(1)].$$

Hence, by part (f) of Theorem SA-1 and part (d) of Theorem SA-2, we have

$$\hat{\rho}_{\text{pre}} - \rho_T$$

$$= \overline{w}_{IC} (\hat{\rho}_{\text{IVD}} - \rho_T) + (1 - \overline{w}_{IC}) (\hat{\rho}_{\text{pols}} - \rho_T)$$

$$= \exp \left\{ -\frac{1}{2} \sqrt{N\lambda (T)} \left[ 1 + \eta_1 (N, T) \right] \right\} (\hat{\rho}_{\text{IVD}} - \rho_T) \left[ 1 + o_p(1) \right] + (\hat{\rho}_{\text{pols}} - \rho_T)$$

$$- \exp \left\{ -\frac{1}{2} \sqrt{N\lambda (T)} \left[ 1 + \eta_1 (N, T) \right] \right\} (\hat{\rho}_{\text{pols}} - \rho_T) \left[ 1 + o_p(1) \right]$$

$$= O_p \left( \exp \left\{ -\frac{1}{2} \sqrt{N\lambda (T)} \left[ 1 + \eta_1 (N, T) + \frac{\ln T}{\sqrt{N\lambda (T)}} + \frac{\ln N}{\sqrt{N\lambda (T)}} \right] \right\} \right) + O_p \left( \frac{1}{\sqrt{NTq(T)}} \right)$$

$$+ O_p \left( \exp \left\{ -\frac{1}{2} \sqrt{N\lambda (T)} \left[ 1 + \eta_1 (N, T) + \frac{\ln T}{\sqrt{N\lambda (T)}} + \frac{\ln N}{\sqrt{N\lambda (T)}} + \frac{\ln q(T)}{\sqrt{N\lambda (T)}} \right] \right\} \right)$$

$$= O_p \left( \frac{1}{\sqrt{NTq(T)}} \right).$$
as required for part (c).

Now, for part (d), we consider the case where \( q(T) \to \infty \) such that \( q(T) (L(T))^2 / T = O(1) \). It is helpful to break the analysis for this case into a number of subcases.

Case d(i): \( q(T) \sim T / (L(T))^2 \)

Note that, in this case, by part (f) of Theorem SA-1, we have

\[
\hat{\rho}_{IVD} - \rho_T = O_p \left( \frac{1}{\sqrt{NT}} \right),
\]

and, by the proof of part (d) of Theorem SA-2,

\[
\hat{\rho}_{pols} = \rho_T + O_p \left( \frac{1}{\sqrt{NTq(T)}} \right)
= 1 - \frac{1}{q(T)} + O \left( \frac{1}{q(T)^2} \right) + O_p \left( \frac{1}{\sqrt{NTq(T)}} \right).
\]

In addition, similar to part (c) above, we have

\[
\sqrt{\frac{q(T)}{NT}} \Xi = - \frac{\sigma}{\sqrt{2}} + o_p(1),
\]

so that

\[
\frac{\bar{\Sigma}_{IC}}{\sqrt{NL(T)}} = \frac{\Xi + \sqrt{NL(T)}}{\sqrt{NL(T)}}
= 1 + \frac{1}{L(T)} \sqrt{\frac{T}{q(T)}} \sqrt{\frac{q(T)}{NT\Xi}}
= 1 - \frac{\sigma}{L(T)} \sqrt{\frac{T}{2q(T))}} + \eta_2(N,T)
= \xi(T) + \eta_2(N,T),
\]

where \( \eta_2(N,T) = o_p(1) \) and

\[
\xi(T) = 1 - \frac{\sigma}{L(T)} \sqrt{\frac{T}{2q(T)}} = O(1).
\]

Depending on the sequences \( L(T) \) and \( q(T) \), the sequence \( \xi(T) \) could have positive or negative sign eventually or could be zero or have oscillating signs. Nevertheless, since \( \overline{w}_{IC} \in [0,1] \) for all \( N,T \); we have

\[
\hat{\rho}_{pre} - \rho_T
= \overline{w}_{IC} (\hat{\rho}_{IVD} - \rho_T) + (1 - \overline{w}_{IC}) (\hat{\rho}_{pols} - \rho_T)
= O_p(1) O_p \left( \frac{1}{\sqrt{NT}} \right) + O_p(1) O_p \left( \frac{1}{\sqrt{NTq(T)}} \right)
= O_p \left( \frac{1}{\sqrt{NT}} \right).
\]
Case d(ii): $N^{1/3}T^{1/3} \approx T^{14/30} \ll q(T)$ but $q(T)(L(T))^2/T \to 0$

To proceed, first write

$$\Delta_{IC} = T + \sqrt{NL(T)} = T^{0} + \overline{\theta} + \sqrt{NL(T)},$$

where

$$T^{0} = M \sqrt{NTq(T)} (\hat{\rho}_{pols} - \rho_{T}),$$
$$\overline{\theta} = M \sqrt{NTq(T)} (\rho_{T} - 1),$$
$$M_{yy} = \frac{1}{NTq(T)} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \overline{\theta}_{-1,NT})^2.$$

In this case, applying part (d) of Theorem SA-2 and part (d) of Lemmas SD-3, we have

$$T^{0} \Rightarrow N(0,1),$$
$$\overline{\theta} = -\frac{\sigma}{\sqrt{2}} \sqrt{\frac{NT}{q(T)}} [1 + o_{p}(1)];$$
$$\sqrt{NL(T)} = \sqrt{\frac{NT}{q(T)}} \sqrt{q(T) L(T)^2} = o \left( \sqrt{\frac{NT}{q(T)}} \right)$$

so that

$$\frac{\Delta_{IC}}{\sqrt{NT/q(T)}} = \frac{T^{0} + \overline{\theta} + \sqrt{NL(T)}}{\sqrt{NT/q(T)}} = -\frac{\sigma}{\sqrt{2}} + o_{p}(1).$$

Hence,

$$\overline{w}_{IC} = \frac{1}{1 + \exp \left\{ \frac{1}{2} \sqrt{\frac{NT}{q(T)}} \frac{\Delta_{IC}}{\sqrt{NT/q(T)}} \right\}}$$
$$= \frac{1}{1 + \exp \left\{ -\frac{\sigma}{2\sqrt{2}} \sqrt{\frac{NT}{q(T)}} [1 + o_{p}(1)] \right\}}$$
$$= 1 + o_{p}(1).$$

It follows from applying the results of part (f) of Theorem SA-1 and part (d) of Theorem SA-2 that

$$\hat{\rho}_{pre} - \rho_{T} = \overline{w}_{IC} (\hat{\rho}_{TVD} - \rho_{T}) + (1 - \overline{w}_{IC}) (\hat{\rho}_{pols} - \rho_{T})$$
$$= [1 + o_{p}(1)] O_{p} \left( \frac{1}{\sqrt{NT}} \right) + [1 - (1 + o_{p}(1))] O_{p} \left( \frac{1}{\sqrt{NT q(T)}} \right)$$
$$= O_{p} \left( \frac{1}{\sqrt{NT}} \right).$$

Case d(iii): $q(T) \sim T^{14/30} = N^{1/3}T^{1/3}$

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In this case, by part (f) of Theorem SA-1, we have
\[
\hat{\rho}_{IVD} - \rho_T = O_p \left( \frac{1}{\sqrt{NT}} \right),
\]
Also, from the proof of part (e) of Theorem SA-2, we obtain
\[
\hat{\rho}_{pols} - \rho_T = O_p \left( \frac{1}{q(T)^2} \right) = O_p \left( \frac{1}{\sqrt{NTq(T)}} \right)
\]
Next, write
\[
\Delta_{IC} = \Gamma + \sqrt{NL(T)} = \Gamma + \theta + \sqrt{NL(T)},
\]
where
\[
\begin{align*}
\Gamma &= \overline{M}_{yy}^{1/2} \sqrt{NTq(T)} \left( \hat{\rho}_{pols} - \rho_T - \frac{2\sigma^2_T}{q(T)^2 \sigma^2} \right), \\
\theta &= \overline{M}_{yy}^{1/2} \sqrt{NTq(T)} \left( \rho_T + \frac{2\sigma^2_T}{q(T)^2 \sigma^2} - 1 \right), \\
\overline{M}_{yy} &= \frac{1}{NTq(T)} \sum_{i=1}^{N} \sum_{t=2}^{T} \left( y_{it} - \overline{y}_{i-1,NT} \right) ^2.
\end{align*}
\]
In this case, applying part (e) of Theorem SA-2 along with part (d) of Lemmas SD-3, we have
\[
\begin{align*}
\Gamma &\Rightarrow N(0,1), \\
\theta &= -\frac{\sigma}{\sqrt{2}} \sqrt{\frac{NT}{q(T)}} \left[ 1 + o_p(1) \right], \\
\sqrt{NL(T)} &= \sqrt{\frac{NT}{q(T)}} \sqrt{\frac{q(T) L(T)^2}{T}} = o \left( \sqrt{\frac{NT}{q(T)}} \right),
\end{align*}
\]
so that, similar to case d(ii) above,
\[
\begin{align*}
\overline{\Sigma}_{IC} &= \frac{\overline{\Gamma}}{\sqrt{NTq(T)}} + \frac{\theta}{\sqrt{NTq(T)}} + \frac{\sqrt{NL(T)}}{\sqrt{NTq(T)}} \\
&= -\frac{\sigma}{\sqrt{2}} + o_p(1).
\end{align*}
\]
Hence,
\[
\begin{align*}
\overline{w}_{IC} &= \frac{1}{1 + \exp \left\{ \frac{1}{2} \sqrt{\frac{NT}{q(T)}} \frac{\overline{\Sigma}_{IC}}{\sqrt{NTq(T)}} \right\}} \\
&= \frac{1}{1 + \exp \left\{ -\frac{\sigma}{2\sqrt{2}} \sqrt{\frac{NT}{q(T)}} \left[ 1 + o_p(1) \right] \right\}} \\
&= 1 + o_p(1)
\end{align*}
\]
It follows from applying the results of part (f) of Theorem SA-1 and part (e) of Theorem SA-2 that

\[
\hat{\rho}_{\text{pre}} - \rho_T = \mathbb{w}_{\text{IC}} \left( \hat{\rho}_{\text{IVD}} - \rho_T \right) + \left( 1 - \mathbb{w}_{\text{IC}} \right) \left( \hat{\rho}_{\text{pols}} - \rho_T \right)
\]

\[
= \left[ 1 + o_p \left( 1 \right) \right] O_p \left( \frac{1}{\sqrt{NT}} \right) + \left[ 1 - \left( 1 + o_p \left( 1 \right) \right) \right] O_p \left( \frac{1}{q(T)^2} \right)
\]

\[
= O_p \left( \frac{1}{\sqrt{NT}} \right) + o_p \left( \frac{1}{N^{2/3} T^{2/3}} \right)
\]

\[
= O_p \left( \frac{1}{\sqrt{NT}} \right).
\]

Case d(iv): \( q(T) \to \infty \) such that \( q(T)/T^{1/3} \to 0 \)

To proceed, note that, by part (f) of Theorems SA-1 and SA-2, we have

\[
\hat{\rho}_{\text{IVD}} - \rho_T = O_p \left( \frac{1}{\sqrt{NT}} \right) \quad \text{and} \quad \hat{\rho}_{\text{pols}} - \rho_T = O_p \left( \frac{1}{q(T)^2} \right).
\]

Write

\[
\Delta_{\text{IC}} = \mathbb{T} + \sqrt{N} L(T) = \mathbb{T} + \theta + \sqrt{N} L(T),
\]

where

\[
\mathbb{T} = M_{yy}^{1/2} \sqrt{NT} q(T) \left( \hat{\rho}_{\text{pols}} - \rho_T - \frac{2\sigma_a^2}{q(T)^2 \sigma^2} \right)
\]

\[
= M_{yy}^{1/2} \sqrt{NT} \left( \frac{1}{q(T)^{3/2}} \right) q(T)^2 \left( \hat{\rho}_{\text{pols}} - \rho_T - \frac{2\sigma_a^2}{q(T)^2 \sigma^2} \right)
\]

\[
\theta = M_{yy}^{1/2} \sqrt{NT} q(T) \left( \rho_T + \frac{2\sigma_a^2}{q(T)^2 \sigma^2} - 1 \right),
\]

\[
M_{yy} = \frac{1}{NT q(T)} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it-1} - \bar{y}_{1-T})^2.
\]

In this case, applying part (f) of Theorem SA-2 along with part (d) of Lemma SD-3, we have

\[
\mathbb{T} = o_p \left( \frac{\sqrt{NT}}{q(T)^{3/2}} \right),
\]

\[
\theta = -\frac{\sigma}{\sqrt{2}} \sqrt{\frac{NT}{q(T)}} \left[ 1 + o_p \left( 1 \right) \right],
\]

\[
\sqrt{NL(T)} = \sqrt{\frac{NT}{q(T)}} \sqrt{\frac{L(T)^2}{T}} = o \left( \sqrt{\frac{NT}{q(T)}} \right),
\]

so that

\[
\frac{\Delta_{\text{IC}}}{\sqrt{NT/q(T)}} = \frac{\mathbb{T}}{\sqrt{NT/q(T)}} + \frac{\theta}{\sqrt{NT/q(T)}} + \frac{\sqrt{NL(T)}}{\sqrt{NT/q(T)}}
\]

\[
= -\frac{\sigma}{\sqrt{2}} \left[ 1 + \eta_3 \left( N, T \right) \right],
\]

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where \( \eta_3(N,T) = o_p(1) \). Hence,

\[
\begin{align*}
\frac{1}{w_{IC}} &= \frac{1}{1 + \exp \left\{ \frac{1}{2} \sqrt{NT/q(T)} \frac{\bar{\eta}_C}{\sqrt{NT/q(T)}} \right\} } \\
&= \frac{1}{1 + \exp \left\{ -\frac{\sigma}{2\sqrt{2}} \sqrt{NT/q(T)} [1 + \eta_3(N,T)] \right\} } \\
&= 1 + O_p \left( \exp \left\{ -\frac{\sigma}{2\sqrt{2}} \sqrt{NT/q(T)} [1 + \eta_3(N,T)] \right\} \right).
\end{align*}
\]

It follows from applying the results of part (f) of Theorems SA-1 and SA-2 that

\[
\hat{\rho}_{pre} - \rho_T = \bar{w}_{IC} (\hat{\rho}_{IVD} - \rho_T) + (1 - \bar{w}_{IC}) (\hat{\rho}_{pols} - \rho_T) \\
= \left[ 1 + O_p \left( \exp \left\{ -\frac{\sigma}{2\sqrt{2}} \sqrt{NT/q(T)} [1 + \eta_3(N,T)] \right\} \right) \right] \frac{1}{O_p \left( \sqrt{NT} \right)} \\
+ \left[ 1 - \left( 1 + O_p \left( \exp \left\{ -\frac{\sigma}{2\sqrt{2}} \sqrt{NT/q(T)} [1 + \eta_3(N,T)] \right\} \right) \right) \right] \frac{1}{O_p \left( \frac{1}{q(T)} \right)^2} \\
= O_p \left( \frac{1}{\sqrt{NT}} \right) + O_p \left( \exp \left\{ -\frac{\sigma}{2\sqrt{2}} \sqrt{NT/q(T)} \left[ 1 + \eta_3(N,T) + \frac{4\sqrt{2}}{\sigma} \sqrt{\frac{q(T)}{NT} \ln q(T)} \right] \right) \right) \\
= O_p \left( \frac{1}{\sqrt{NT}} \right).
\]

To complete the proof of part (d), again note that because \( N \) grows as monotonically increasing function of \( T \), so that we can consider the asymptotics here as being single-indexed with \( T \to \infty \). Hence, we set \( S_T(\rho_T) = \hat{\rho}_{pre} - \rho_T \) in Lemma SD-10 and apply part (a) of that lemma for the following collection of parameter sequences

\[
\begin{align*}
\mathcal{G}_5^p &= \left\{ \rho_T = \exp \left\{ -\frac{1}{q(T)} \right\} : q(T) \sim T / (L(T))^2 \right\}, \\
\mathcal{G}_6^p &= \left\{ \rho_T = \exp \left\{ -\frac{1}{q(T)} \right\} : N^{1/3} T^{1/3} = T^{1+\alpha \varepsilon} \ll q(T) \text{ but } q(T) (L(T))^2 / T \to 0 \right\}, \\
\mathcal{G}_7^p &= \left\{ \rho_T = \exp \left\{ -\frac{1}{q(T)} \right\} : q(T) \sim T^{1+\alpha \varepsilon} = N^{1/3} T^{1/3} \right\}, \\
\mathcal{G}_8^p &= \left\{ \rho_T = \exp \left\{ -\frac{1}{q(T)} \right\} : q(T) \to \infty \text{ such that } q(T) / T^{1+\alpha \varepsilon} \to 0 \right\}
\end{align*}
\]

Let

\[
\mathcal{G}^* = \left\{ \rho_T = \exp \left\{ -\frac{1}{q(T)} \right\} : q(T) \to \infty \text{ such that } q(T) (L(T))^2 / T = O(1) \right\},
\]

and note that, by the calculations given in subcases d(i)-d(iv) above and by part (a) of Lemma SD-10, we have that for every \( \{ \rho_T \} \in \mathcal{G}^* \),

\[
S_T(\rho_T) = \hat{\rho}_{pre} - \rho_T = O_p \left( \frac{1}{\sqrt{NT}} \right),
\]

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which is the desired result.

Finally, we consider part (e), where we take \( \rho_T \in \mathcal{G}_{St} \). To proceed with this case, note first that from part (g) of Theorems SA-1 and SA-2, we have that

\[
\hat{\rho}_{IVD} - \rho_T = O_P \left( \frac{1}{\sqrt{NT}} \right) \quad \text{and} \quad \hat{\rho}_{pols} - \rho_T = \frac{(1 - \rho_T^2) (1 - \rho_T) \sigma_a^2}{(1 - \rho_T^2) \sigma_a^2 + \sigma^2} [1 + o_p(1)].
\]

Next, define

\[
\tilde{M}_{yy} = \frac{1 - \rho_T^2}{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it} - \bar{y}_{i-1,NT})^2,
\]

\[
\tilde{T} = \tilde{M}_{yy}^{1/2} \sqrt{\frac{NT}{1 - \rho_T^2}} (\hat{\rho}_{pols} - \rho_T),
\]

\[
\tilde{\theta} = \tilde{M}_{yy}^{1/2} \sqrt{\frac{NT}{1 - \rho_T^2}} (\rho_T - 1) = -\tilde{M}_{yy}^{1/2} \sqrt{\frac{NT}{1 + \rho_T}}.
\]

Using part (g) of Theorem SA-2 along with part (e) of Lemmas SD-3, we have in this case

\[
\tilde{M}_{yy} = (1 - \rho_T^2) \sigma_a^2 + \sigma^2 + o_p(1),
\]

\[
\tilde{T} = \tilde{M}_{yy}^{1/2} (\hat{\rho}_{pols} - \rho_T) \sqrt{1 - \rho_T^2},
\]

\[
= \sqrt{\frac{(1 - \rho_T^2) \sigma_a^2 + \sigma^2}{1 - \rho_T^2}} \frac{(1 - \rho_T^2) (1 - \rho_T) \sigma_a^2}{(1 - \rho_T^2) \sigma_a^2 + \sigma^2} [1 + o_p(1)]
\]

\[
= (1 - \rho_T) \sigma_a^2 \sqrt{\frac{(1 - \rho_T^2)}{(1 - \rho_T^2) \sigma_a^2 + \sigma^2} [1 + o_p(1)]}
\]

\[
\tilde{\theta} = \tilde{M}_{yy}^{1/2} (\rho_T - 1) \sqrt{1 - \rho_T^2} = -\sqrt{(1 - \rho_T^2) \sigma_a^2 + \sigma^2} \sqrt{\frac{1 - \rho_T}{1 + \rho_T} [1 + o_p(1)]}.
\]
Hence,

\[
\frac{\ddelta_{IC}}{\sqrt{NT}} = \frac{\ddelta}{\sqrt{NT}} + \frac{\sqrt{NL}(T)}{\sqrt{NT}} = \left[ (1 - \rho_T) \sigma_a^2 \sqrt{\frac{(1 - \rho_T^2)}{(1 - \rho_T^2) \sigma_a^2 + \sigma^2}} - \sqrt{(1 - \rho_T^2) \sigma_a^2 + \sigma^2} \sqrt{\frac{1 - \rho_T}{1 + \rho_T}} \right] [1 + \alpha_p(1)]
\]

\[
= \frac{(1 - \rho_T) \sigma_a^2 \sqrt{(1 + \rho_T)(1 - \rho_T^2) - [(1 - \rho_T^2) \sigma_a^2 + \sigma^2] \sqrt{1 - \rho_T}}}{\sqrt{(1 + \rho_T) [(1 - \rho_T^2) \sigma_a^2 + \sigma^2]}} [1 + \alpha_p(1)]
\]

\[
= -\frac{\sigma^2 \sqrt{1 - \rho_T}}{\sqrt{(1 + \rho_T) [(1 - \rho_T^2) \sigma_a^2 + \sigma^2]}} [1 + \alpha_p(1)]
\]

\[
= -\psi(\rho_T, \sigma^2, \sigma_a^2) [1 + \alpha_p(1)]
\]

(12)

where

\[
\psi(\rho_T, \sigma^2, \sigma_a^2) = \frac{\sigma^2 \sqrt{1 - \rho_T}}{\sqrt{(1 + \rho_T) [(1 - \rho_T^2) \sigma_a^2 + \sigma^2]}}.
\]

Next, note that in this case \(|\rho_T| = \exp\left\{-\frac{1}{q(T)}\right\}\) with \(q(T) = O(1)\), so that there exists a positive constant \(C_q\) and a positive integer \(T^*\) such that for all \(T \geq T^*\)

\[
|\rho_T| \leq \exp\left\{-\frac{1}{C_q}\right\} < 1.
\]

In consequence,

\[
\frac{\ddelta_{IC}}{\sqrt{NT}} \geq \frac{\sigma^2 \sqrt{1 - \rho_T}}{\sqrt{2(\sigma_a^2 + \sigma^2)}} > 0.
\]

Moreover, note that we can rewrite (12) as

\[
\ddelta_{IC} = -\psi(\rho_T, \sigma^2, \sigma_a^2) [1 + \eta_4(N, T)],
\]

where \(\eta_4(N, T) = o_p(1)\), so that

\[
\ddelta_{IC} = \frac{1}{1 + \exp\left\{\frac{1}{2} \sqrt{NT} \frac{\ddelta_{IC}}{\sqrt{NT}}\right\}}
\]

\[
= 1 + O_p \left\{ \exp\left\{-\frac{1}{2} \psi(\rho_T,\sigma^2, \sigma_a^2) \sqrt{NT} [1 + \eta_4(N, T)] \right\} \right\}
\]

\[
= 1 + o_p(1).
\]
It follows from applying part (g) of Theorems SA-1 and SA-2 that

$$\hat{\rho}_{\text{pre}} - \rho_T = \bar{w}_{IC} (\hat{\rho}_{\text{IVD}} - \rho_T) + (1 - \bar{w}_{IC}) (\hat{\rho}_{\text{pols}} - \rho_T)$$

$$= \left[ 1 + O_p \left( \exp \left\{ - \frac{1}{2} \psi (\rho_T, \sigma^2, \sigma^2_a) \sqrt{NT} [1 + \eta_4 (N, T)] \right\} \right) \right] O_p \left( \frac{1}{\sqrt{NT}} \right)$$

$$\quad + \left[ 1 - 1 + O_p \left( \exp \left\{ - \frac{1}{2} \psi (\rho_T, \sigma^2, \sigma^2_a) \sqrt{NT} [1 + \eta_4 (N, T)] \right\} \right) \right] O_p (1)$$

$$= O_p \left( \frac{1}{\sqrt{NT}} \right) + O_p \left( \exp \left\{ - \frac{1}{2} \psi (\rho_T, \sigma^2, \sigma^2_a) \sqrt{NT} [1 + \eta_4 (N, T)] \right\} \right)$$

$$= O_p \left( \frac{1}{\sqrt{NT}} \right) \quad \square$$

**Lemma SD-12:** Let

$$\hat{\sigma}^2 = \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it} - \bar{y}_i - \hat{\rho}_{\text{pre}} [y_{i,t-1} - \bar{y}_{i,-1}])^2,$$

$$\bar{\sigma}^2 = \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} (y_{it} - \bar{y}_i - \hat{\rho}_{\text{AIP}} [y_{i,t-1} - \bar{y}_{i,-1}])^2$$

where \( \hat{\rho}_{\text{AIP}} \) are as defined by (1) and where \( \hat{\rho}_{\text{pre}} \) is as defined in footnote 4 of the main paper. Suppose that Assumptions 1-4 hold; then, as \( N, T \to \infty \) such that \( N^K/T = \tau \), for \( \kappa \in \left( \frac{1}{2}, \infty \right) \) and \( \tau \in (0, \infty) \),

$$\hat{\sigma}^2 = \sigma^2 + O_p \left( \max \left\{ \frac{1}{N}, \frac{1}{T} \right\} \right) = \sigma^2 + O_p \left( \max \left\{ \frac{1}{T^{1/\kappa}}, \frac{1}{T} \right\} \right),$$

$$\bar{\sigma}^2 = \sigma^2 + O_p \left( \max \left\{ \frac{1}{N}, \frac{1}{T} \right\} \right) = \sigma^2 + O_p \left( \max \left\{ \frac{1}{T^{1/\kappa}}, \frac{1}{T} \right\} \right)$$

for every parameter sequences \( \{\rho_T\} \in \mathcal{G}^* = \{\rho_T : -1 < \rho_T \leq 1 \text{ for all } T\} \).

**Proof of Lemma SD-12:**

We will only prove the result for \( \hat{\sigma}^2 \) since the proof for \( \bar{\sigma}^2 \) follows in a similar manner. To proceed,
first write

\[
\hat{\sigma}^2 = \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} \left( y_{it} - \overline{y}_i - \hat{\rho}_{\text{pre}} \left[ y_{it-1} - \overline{y}_{i,-1} \right] \right)^2
\]

\[
= \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} \left( y_{it} - \overline{y}_i - \rho_T \left[ y_{it-1} - \overline{y}_{i,-1} \right] - \left[ \hat{\rho}_{\text{pre}} - \rho_T \right] \left[ y_{it-1} - \overline{y}_{i,-1} \right] \right)^2
\]

\[
= \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} \left( y_{it} - \overline{y}_i - \rho_T \left[ y_{it-1} - \overline{y}_{i,-1} \right] \right)^2
\]

\[- \frac{2}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} \left( y_{it} - \overline{y}_i - \rho_T \left[ y_{it-1} - \overline{y}_{i,-1} \right] \right) \left[ y_{it-1} - \overline{y}_{i,-1} \right]
\]

\[+ \frac{2}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} \left( y_{it-1} - \overline{y}_{i,-1} \right)^2
\]

\[= Q_{1,N,T} + 2 \left[ \hat{\rho}_{\text{pre}} - \rho_T \right] Q_{2,N,T} + \left[ \hat{\rho}_{\text{pre}} - \rho_T \right]^2 Q_{3,N,T}, \text{ (say).}
\]

Note that

\[
Q_{1,N,T}
\]

\[= \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} \left( y_{it} - \overline{y}_i - \rho_T \left[ y_{it-1} - \overline{y}_{i,-1} \right] \right)^2
\]

\[= \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} [a_i + w_{it} - \rho_T (a_i + w_{it-1})]
\]

\[- \frac{1}{T-1} \sum_{s=2}^{T} (a_i + w_{is}) + \rho_T \frac{1}{T-1} \sum_{s=2}^{T} (a_i + w_{is-1}) \right)^2
\]

\[= \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} \left( \varepsilon_{it} - \frac{1}{T-1} \sum_{s=2}^{T} \varepsilon_{is} \right)^2
\]

\[= \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} \varepsilon_{it}^2 - \frac{1}{N} \sum_{i=1}^{N} \varepsilon_i^2
\]

Next, note that

\[
Q_{2,N,T}
\]

\[= \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} \left( y_{it} - \overline{y}_i - \rho_T \left[ y_{it-1} - \overline{y}_{i,-1} \right] \right) \left[ y_{it-1} - \overline{y}_{i,-1} \right]
\]

\[= \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} (\varepsilon_{it} - \overline{\varepsilon}_i) (w_{it-1} - \overline{w}_{i,-1})
\]

\[= \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1} \varepsilon_{it} - \frac{1}{N} \sum_{i=1}^{N} \varepsilon_i \overline{w}_{i,-1}.
\]
In addition,

\[
Q_{3,N,T} = \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} \left( y_{it-1} - \overline{y}_{i-1} \right)^2 \\
= \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} \left( y_{it-1} - \frac{1}{T-1} \sum_{s=2}^{T} \left( a_i + w_{is-1} \right) \right)^2 \\
= \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} \left( w_{it-1} - \frac{1}{T-1} \sum_{s=2}^{T} w_{is-1} \right)^2 \\
= \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} \left[ w_{it-1}^2 - 2w_{it-1} \frac{1}{T-1} \sum_{s=2}^{T} w_{is-1} + \left( \frac{1}{T-1} \sum_{s=2}^{T} w_{is-1} \right)^2 \right] \\
= \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1}^2 - \frac{1}{N} \sum_{i=1}^{N} w_{i-1}^2.
\]

Now, applying the results of parts (e) and (f) of Lemma SE-11, we get

\[
Q_{1,N,T} = \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} \left( y_{it} - \overline{y}_i - \rho_T \left[ y_{it-1} - \overline{y}_{i-1} \right] \right)^2 \\
= \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} \varepsilon_{it}^2 - \frac{1}{N} \sum_{i=1}^{N} \varepsilon_i^2 \\
= \sigma^2 + O_p \left( \max \left\{ \frac{1}{\sqrt{NT}}, \frac{1}{T} \right\} \right) + O_p \left( \frac{1}{T} \right) \\
= \sigma^2 + O_p \left( \max \left\{ \frac{1}{\sqrt{NT}}, \frac{1}{T} \right\} \right).
\]

We divide the rest of the proof into a number of cases.

Case (i): \( \rho_T = 1 \) for all \( T \) sufficiently large.

In this case, we apply part (b) of Lemma SE-20 and part (a) of Lemma SE-29 to obtain

\[
Q_{2,N,T} = \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1} \varepsilon_{it} - \frac{1}{N} \sum_{i=1}^{N} \varepsilon_i w_{i-1} \\
= O_p \left( \frac{1}{\sqrt{N}} \right) + O_p \left( 1 \right) = O_p \left( 1 \right).
\]

In addition, by Lemma SE-14 and part (a) of Lemma SE-28, we get

\[
Q_{3,N,T} = \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1}^2 - \frac{1}{N} \sum_{i=1}^{N} w_{i-1}^2 \\
= O_p \left( T \right) + O_p \left( T \right) = O_p \left( T \right).
\]
Furthermore, in this case,

\[ \hat{\rho}_{\text{pre}} - \rho_T = \mathcal{O}_p \left( \frac{1}{\sqrt{NT}} \right), \]

by part (a) of Lemma SD-11. It follows that

\[
\hat{\sigma}^2 = Q_{1,N,T} + 2 \left[ \hat{\rho}_{\text{pre}} - \rho_T \right] Q_{2,N,T} + \left[ \hat{\rho}_{\text{pre}} - \rho_T \right]^2 Q_{3,N,T}
\]
\[
= \sigma^2 + \mathcal{O}_p \left( \max \left\{ \frac{1}{\sqrt{NT}}, \frac{1}{T} \right\} \right) + \mathcal{O}_p \left( \frac{1}{\sqrt{NT}} \right) \mathcal{O}_p (1) + \mathcal{O}_p \left( \frac{1}{NT^2} \right) \mathcal{O}_p (T)
\]
\[
= \sigma^2 + \mathcal{O}_p \left( \max \left\{ \frac{1}{\sqrt{NT}}, \frac{1}{T} \right\} \right) .
\]

Case (ii): \( \rho_T = \exp \{-1/q(T)\} \) such that \( T/q(T) \rightarrow 0 \)

Applying part (a) of Lemmas SE-20 and part (b) of Lemma SE-29 in this case, we obtain

\[ Q_{2,N,T} = \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1} \varepsilon_{it} - \frac{1}{N} \sum_{i=1}^{N} \varepsilon_{i} w_{i,-1} \]
\[ = \mathcal{O}_p \left( \frac{1}{\sqrt{N}} \right) + \mathcal{O}_p (1) = \mathcal{O}_p (1) . \]

Moreover, applying Lemma SE-15 and part (b) of Lemma SE-28, we get

\[ Q_{3,N,T} = \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} w^2_{it-1} - \frac{1}{N} \sum_{i=1}^{N} w^2_{i,-1} \]
\[ = \mathcal{O}_p (T) + \mathcal{O}_p (T) = \mathcal{O}_p (T) . \]

In addition, from part (b) of Lemma SD-11, we have that in this case

\[ \hat{\rho}_{\text{pre}} - \rho_T = \mathcal{O}_p \left( \frac{1}{\sqrt{NT}} \right) . \]

It follows that

\[ \hat{\sigma}^2 = Q_{1,N,T} + 2 \left[ \hat{\rho}_{\text{pre}} - \rho_T \right] Q_{2,N,T} + \left[ \hat{\rho}_{\text{pre}} - \rho_T \right]^2 Q_{3,N,T} \]
\[ = \sigma^2 + \mathcal{O}_p \left( \max \left\{ \frac{1}{\sqrt{NT}}, \frac{1}{T} \right\} \right) + \mathcal{O}_p \left( \frac{1}{\sqrt{NT}} \right) \mathcal{O}_p (1) + \mathcal{O}_p \left( \frac{1}{NT^2} \right) \mathcal{O}_p (T) \]
\[ = \sigma^2 + \mathcal{O}_p \left( \max \left\{ \frac{1}{\sqrt{NT}}, \frac{1}{T} \right\} \right) . \]

Case (iii): \( \rho_T = \exp \{-1/q(T)\} \) such that \( q(T) \sim T \)

Here, the calculations are similar to case (ii) above, except that we apply part (a) of Lemma SE-27, part (c) of Lemma SE-29, part (a) of Lemma SE-17, part (c) of Lemma SE-28, and part (b) of Lemma SD-11 to obtain

\[ Q_{2,N,T} = \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1} \varepsilon_{it} - \frac{1}{N} \sum_{i=1}^{N} \varepsilon_{i} w_{i,-1} = \mathcal{O}_p (1) , \]
\[ Q_{3,N,T} = \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} w^2_{it-1} - \frac{1}{N} \sum_{i=1}^{N} w^2_{i,-1} = \mathcal{O}_p (T) , \]
\[ \hat{\rho}_{\text{pre}} - \rho_T = \mathcal{O}_p \left( \frac{1}{\sqrt{NT}} \right) . \]
Hence,
\[
\hat{\sigma}^2 = Q_{1,N,T} + 2 \left[ \hat{\rho}_{pre} - \rho_T \right] Q_{2,N,T} + \left[ \hat{\rho}_{pre} - \rho_T \right]^2 Q_{3,N,T}
\]
\[
= \sigma^2 + O_p \left( \max \left\{ \frac{1}{\sqrt{NT}}, \frac{1}{T} \right\} \right) + O_p \left( \frac{1}{\sqrt{NT}} \right) O_p \left( \frac{1}{T} \right) + O_p \left( \frac{1}{NT^2} \right) O_p \left( T \right)
\]
\[
= \sigma^2 + O_p \left( \max \left\{ \frac{1}{N}, \frac{1}{T} \right\} \right).
\]

Case (iv): \( \rho_T = \exp \left\{ -1/q(T) \right\} \) such that \( T/L(T)^2 \ll q(T) \ll T \)

In this case, we apply part (b) of Lemma SE-27 and part (d) of Lemma SE-29 to obtain

\[
Q_{2,N,T} = \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{i,t-1} \varepsilon_{i,t} - \frac{1}{N} \sum_{i=1}^{N} \varepsilon_i \overline{w}_{i,-1}
\]
\[
= \sqrt{\frac{Tq(T)}{N(T-1)^2}} \frac{1}{\sqrt{NTq(T)}} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{i,t-1} \varepsilon_{i,t} - \frac{1}{N} \sum_{i=1}^{N} \varepsilon_i \overline{w}_{i,-1}
\]
\[
= O_p \left( \sqrt{\frac{q(T)}{NT}} \right) + O_p \left( \frac{q(T)}{T} \right) = o_p \left( \sqrt{\frac{q(T)}{T}} \right).
\]

Moreover, applying part (b) of Lemma SE-17 and part (d) of Lemma SE-28, we get

\[
Q_{3,N,T} = \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{i,t-1}^2 - \frac{1}{N} \sum_{i=1}^{N} \overline{w}_{i,-1}^2
\]
\[
= \frac{Tq(T)}{(T-1)NTq(T)} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{i,t-1}^2 - \frac{1}{N} \sum_{i=1}^{N} \overline{w}_{i,-1}^2
\]
\[
= O_p \left( q(T) \right) + O_p \left( \frac{q(T)^2}{T} \right) = O_p \left( q(T) \right).
\]

In addition, by part (c) of Lemma SD-11,

\[
\hat{\rho}_{pre} - \rho_T = O_p \left( \frac{1}{\sqrt{NTq(T)}} \right).
\]

It follows that
\[
\hat{\sigma}^2 = Q_{1,N,T} + 2 \left[ \hat{\rho}_{pre} - \rho_T \right] Q_{2,N,T} + \left[ \hat{\rho}_{pre} - \rho_T \right]^2 Q_{3,N,T}
\]
\[
= \sigma^2 + O_p \left( \max \left\{ \frac{1}{\sqrt{NT}}, \frac{1}{T} \right\} \right) + O_p \left( \frac{1}{\sqrt{NTq(T)}} \right) O_p \left( \sqrt{\frac{q(T)}{T}} \right) + O_p \left( \frac{1}{NTq(T)} \right) O_p \left( q(T) \right)
\]
\[
= \sigma^2 + O_p \left( \max \left\{ \frac{1}{N}, \frac{1}{T} \right\} \right).
\]

Case (v): \( \rho_T = \exp \left\{ -1/q(T) \right\} \) such that \( q(T) \rightarrow \infty \) but \( q(T) L(T)^2 / T = O(1) \)

Here, the calculations are similar to case (iv) above, except that, by part (d) of Lemma SD-11, we have

\[
\hat{\rho}_{pre} - \rho_T = O_p \left( \frac{1}{\sqrt{NT}} \right)
\]
from which it follows that

\[
\tilde{\sigma}^2 = Q_{1, N, T} + 2\left[\tilde{\rho}_{\text{pre}} - \rho_T\right] Q_{2, N, T} + \left[\tilde{\rho}_{\text{pre}} - \rho_T\right]^2 Q_{3, N, T}
\]

\[
= \sigma^2 + O_p\left(\max \left\{\frac{1}{\sqrt{NT}}, \frac{1}{T}\right\}\right) + O_p\left(\frac{1}{\sqrt{NT}}\right) + O_p\left(1\right) + O_p\left(\sqrt{NT}\right)
\]

\[
= \sigma^2 + O_p\left(\max \left\{\frac{1}{N}, \frac{1}{T}\right\}\right).
\]

Case (vi): \(\rho_T \in G_{St} = \{|\rho_T| = \exp\left\{-\frac{1}{q(T)}\right\}: q(T) \geq 0 \text{ and } q(T) = O(1) \text{ as } T \to \infty\}\)

In this case,

\[
Q_{2, N, T} = \frac{1}{N(T-1)} \frac{\sqrt{NT}}{\sqrt{1 - \rho_T^2}} \sqrt{\frac{1 - \rho_T^2}{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1} \varepsilon_{it} - \frac{1}{N} \sum_{i=1}^{N} \varepsilon_{it} w_{i,-1}}
\]

Here, we take \(\rho_T^2 = \exp\left\{-\frac{2}{q(T)}\right\}\) with \(q(T) = O(1)\), so that there exists a positive constant \(C_q\) and a positive integer \(T^*\) such that for all \(T \geq T^*\)

\[
\rho_T^2 = \exp\left\{-\frac{2}{q(T)}\right\} \leq \exp\left\{-\frac{2}{C_q}\right\} < 1,
\]

from which we deduce that for all \(T \geq T^*\)

\[
\frac{1}{\sqrt{1 - \rho_T^2}} \leq \frac{1}{\sqrt{1 - \text{exp}\left\{-2/C_q\right\}}} < \infty
\]

Now, applying part (c) of Lemmas SE-27 and part (e) of Lemma SE-29, we obtain

\[
Q_{2, N, T} = \frac{1}{\sqrt{NT}} \frac{1}{\sqrt{1 - \rho_T^2}} \sqrt{\frac{1 - \rho_T^2}{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1} \varepsilon_{it} \left[1 + O\left(\frac{1}{T}\right)\right] - \frac{1}{N} \sum_{i=1}^{N} \varepsilon_{it} w_{i,-1}}
\]

\[
= O_p\left(\frac{1}{\sqrt{NT}}\right) + O_p\left(\frac{1}{T}\right).
\]

Moreover, write

\[
Q_{3, N, T} = \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1}^2 - \frac{1}{N} \sum_{i=1}^{N} w_{i,-1}^2
\]

\[
= \frac{T}{T-1} \frac{1 - \rho_T^2}{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1}^2 - \frac{1}{N} \sum_{i=1}^{N} w_{i,-1}^2,
\]
and, using the upper bound above, we have that $T \geq T^*$
\[
\frac{1}{1 - \rho_T^2} \leq \frac{1}{1 - \exp \{-2/C_q\}} < \infty.
\]
Hence, applying part (c) of Lemma SE-17 and part (e) of Lemma SE-28, we get

\[
Q_{3,N,T} = \frac{T}{T-1} \frac{1 - \rho_T^2}{N} \sum_{i=1}^N \sum_{t=2}^T w_{i,t-1}^2 - \frac{1}{N} \sum_{i=1}^N w_{i,1}^2 < \infty.
\]

In addition, in this case, we have by part (e) of Lemma SD-11,

\[
\rho_{\text{pre}} - \rho_T = O_p \left( \frac{1}{\sqrt{NT}} \right).
\]

It follows that

\[
\hat{\sigma}^2 = Q_{1,N,T} + 2 \left[ \hat{\rho}_{\text{pre}} - \rho_T \right] Q_{2,N,T} + \left[ \hat{\rho}_{\text{pre}} - \rho_T \right]^2 Q_{3,N,T} = \sigma^2 + O_p \left( \max \left\{ \frac{1}{\sqrt{NT} \cdot \frac{1}{T}} \right\} \right) + O_p \left( \frac{1}{NT} \right) O_p \left( 1 \right)
\]

To complete the proof of this lemma, note that, in the pathwise asymptotics considered here, $N$ grows as a monotonically increasing function of $T$, so that the asymptotics here can be taken to be single-indexed with $T \to \infty$. Hence, we set $S_T(\rho_T) = \hat{\sigma}^2 - \sigma^2$ in Lemma SD-10 and apply part (a) of that lemma for the following collection of parameter sequences

\[
G_1^T = \{ \rho_T : \rho_T = 1 \text{ for all } T \text{ sufficiently large} \}, \\
G_2^T = \{ \rho_T = \exp \left\{ -\frac{1}{q(T)} \right\} : T/q(T) \to 0 \}, \\
G_3^T = \{ \rho_T = \exp \left\{ -\frac{1}{q(T)} \right\} : q(T) \sim T \}, \\
G_4^T = \{ \rho_T = \exp \left\{ -\frac{1}{q(T)} \right\} : T/L(T)^2 \ll q(T) \ll T \}, \\
G_5^T = \{ \rho_T = \exp \left\{ -\frac{1}{q(T)} \right\} : q(T) \to \infty \text{ but } q(T) L(T)^2/T = O(1) \}, \\
G_6^T = G_{\text{sl}} = \left\{ |\rho_T| = \exp \left\{ -\frac{1}{q(T)} \right\} : q(T) \geq 0 \text{ and } q(T) = O(1) \text{ as } T \to \infty \right\}.
\]

Note that the calculations given in cases (i)-(vi) above imply that

\[
\hat{\sigma}^2 = \sigma^2 + O_p \left( \max \left\{ \frac{1}{NT \cdot \frac{1}{T}} \right\} \right) = \sigma^2 + O_p \left( \max \left\{ \frac{1}{T^{1/\kappa} \cdot \frac{1}{T}} \right\} \right).
\]
under every sequence $\rho_T \in \mathcal{G}_s^\sigma$ and for every $\ell \in \{1, 2, \ldots, 6\}$. Now, let $\{\rho_{j,T}\} \in \mathcal{G}_{s_j}^\sigma$ (for $j = 1, \ldots, 6$), i.e., $\{\rho_{j,T}\}$ is a sequence belonging to the collection $\mathcal{G}_{s_j}^\sigma$, where $s_j \in \{1, 2, \ldots, 6\}$. In addition, define $T_j = f_j(T)$ ($j = 1, \ldots, d$), with $d \leq 6$, where $f_j(\cdot) : \mathbb{N} \to \mathbb{N}$ is an increasing function in its argument, and let $\{\rho_{j,T}\}$ denote a subsequence of $\{\rho_{j,T}\}$. Furthermore, let

$$G^* = \{\rho_T : -1 < \rho_T \leq 1 \text{ for all } T\},$$

and note that every parameter sequence $\rho_T \in G^*$ can be represented as

$$\{\rho_T\} = \bigcup_{j=1}^d \{\rho_{j,T}\},$$

where

$$\{\rho_{1,T_1}\} \in \mathcal{G}_{s_1}^\sigma, \ldots, \{\rho_{d,T_d}\} \in \mathcal{G}_{s_d}^\sigma,$$

with $\mathcal{G}_{s_k}^\sigma \neq \mathcal{G}_{s_\ell}^\sigma$ for $k \neq \ell$ and where

$$\mathbb{N} = \bigcup_{k=1}^d \{T_k = f_k(T) : T \in \mathbb{N}\}.$$

Hence, we can apply part (a) of Lemma SD-10 with $\zeta(T) = \max \{T^{-1/\kappa}, T^{-1}\}$ to conclude that for every sequence $\{\rho_T\} \in G^*$,

$$\hat{\sigma}^2 = \sigma^2 + O_p \left( \max \left\{ \frac{1}{N}, \frac{1}{T} \right\} \right) = \sigma^2 + O_p \left( \max \left\{ \frac{1}{N^{1/\kappa}}, \frac{1}{T} \right\} \right),$$

which is the desired result. \qed

**Lemma SD-13:**

Under Assumptions 1-4, the following statements are true as $N, T \to \infty$ such that $N^{\kappa}/T = \tau$ for $\kappa \in \left(\frac{1}{2}, \infty\right)$ and $\tau \in (0, \infty)$.

(a) If $\rho_T = 1$ for all $T$ sufficiently large, then

$$\hat{\omega}_{NT}^2 = 2\sigma^4 + O_p \left( \max \left\{ \frac{1}{\sqrt{N}}, \frac{1}{\sqrt{T}} \right\} \right).$$

(b) If $\rho_T = \exp \{-1/q(T)\}$ such that $T/q(T) \to 0$, then

$$\hat{\omega}_{NT}^2 = 2\sigma^4 + O_p \left( \max \left\{ \frac{1}{\sqrt{N}}, \frac{1}{\sqrt{T}}, \frac{T}{q(T)} \right\} \right).$$

(c) If $\rho_T = \exp \{-1/q(T)\}$ such that $q(T) \sim T$, then

$$\hat{\omega}_{NT}^2 = \sigma^4 + \frac{1}{2}\sigma^4 q(T) \left( 1 - \exp \left\{ -\frac{2T}{q(T)} \right\} \right) + O_p \left( \max \left\{ \frac{1}{\sqrt{N}}, \frac{1}{\sqrt{T}} \right\} \right).$$

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(d) If $\rho_T = \exp \{-1/q(T)\}$ such that $q(T) \to \infty$ but $q(T)/T \to 0$, then

$$\tilde{\omega}_{NT}^2 = \sigma^4 + O_p \left( \max \left\{ \frac{1}{N}, \frac{1}{q(T)}, \frac{q(T)}{T} \right\} \right).$$

(e) If $\rho_T \in G_{St} = \{ |\rho_T| = \exp \{-1/q(T)\} : q(T) \geq 0 \text{ and } q(T) = O(1) \text{ as } T \to \infty \}$, then

$$\tilde{\omega}_{NT}^2 = \frac{2\sigma^4}{1 + \rho_T} + o_p(1).$$

Proof of Lemma SD-13:

To proceed, consider first part (a), where we take $\rho_T = 1$ for all $T$ sufficiently large. In this case, we make use of part (a) of Lemmas SD-6 and SD-7 and Lemma SD-12 to obtain

$$\tilde{\omega}_{NT}^2 = \sigma^2 \left[ \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=4}^{T} (y_{it-3} - y_{it-2})^2 + \frac{1}{NT} \sum_{i=1}^{N} y_{iT-2}^2 \right]$$

$$= \left[ \sigma^2 + O_p \left( \max \left\{ \frac{1}{N}, \frac{1}{T} \right\} \right) \right] \left[ \sigma^2 + O_p \left( \max \left\{ \frac{1}{\sqrt{NT}}, \frac{1}{\sqrt{T}} \right\} \right) \right] + \sigma^2 + O_p \left( \max \left\{ \frac{1}{\sqrt{NT}}, \frac{1}{\sqrt{T}} \right\} \right)$$

$$= 2\sigma^4 + O_p \left( \max \left\{ \frac{1}{\sqrt{N}}, \frac{1}{\sqrt{T}} \right\} \right),$$

as required.

Next, consider part (b), where we take $\rho_T = \exp \{-1/q(T)\}$ such that $T/q(T) \to 0$. In this case, we apply part (b) of Lemmas SD-6 and SD-7 and Lemma SD-12 to obtain

$$\tilde{\omega}_{NT}^2 = \sigma^2 \left[ \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=4}^{T} (y_{it-3} - y_{it-2})^2 + \frac{1}{NT} \sum_{i=1}^{N} y_{iT-2}^2 \right]$$

$$= \left[ \sigma^2 + O_p \left( \max \left\{ \frac{1}{N}, \frac{1}{T} \right\} \right) \right] \times \left[ \sigma^2 + O_p \left( \max \left\{ \frac{1}{\sqrt{NT}}, \frac{1}{\sqrt{T}} \right\} \right) \right] + \sigma^2 + O_p \left( \max \left\{ \frac{1}{\sqrt{NT}}, \frac{1}{\sqrt{T}}, \frac{T}{q(T)} \right\} \right)$$

$$= 2\sigma^4 + O_p \left( \max \left\{ \frac{1}{\sqrt{N}}, \frac{1}{\sqrt{T}}, \frac{T}{q(T)} \right\} \right),$$

as required.

Now, consider part (c), where we take $\rho_T = \exp \{-1/q(T)\}$ such that $q(T) \sim T$. Here, applying
Lemmas part (c) of Lemmas SD-6 and SD-7 and Lemma SD-12, we have

\[
\hat{\omega}^2_{NT} = \tilde{\sigma}^2 \left[ \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=4}^{T} (y_{it-3} - y_{it-2})^2 + \frac{1}{NT} \sum_{i=1}^{N} y_{iT-2}^2 \right]
\]

\[
= \left[ \sigma^2 + O_p \left( \max \left\{ \frac{1}{N}, \frac{1}{T} \right\} \right) \right] \left[ \sigma^2 + O_p \left( \max \left\{ \frac{1}{\sqrt{NT}}, \frac{1}{T} \right\} \right) \right]
\]

\[
+ \left[ \sigma^2 + O_p \left( \max \left\{ \frac{1}{N}, \frac{1}{T} \right\} \right) \right] \left[ \frac{\sigma^2 q(T)}{T} \left( 1 - \exp \left\{ -\frac{2T}{q(T)} \right\} \right) + O_p \left( \max \left\{ \frac{1}{\sqrt{N}}, \frac{1}{\sqrt{T}} \right\} \right) \right]
\]

\[
= \sigma^4 + O_p \left( \max \left\{ \frac{1}{N}, \frac{1}{T} \right\} \right) \left[ \frac{\sigma^2 q(T)}{T} \left( 1 - \exp \left\{ -\frac{2T}{q(T)} \right\} \right) + O_p \left( \max \left\{ \frac{1}{\sqrt{N}}, \frac{1}{\sqrt{T}} \right\} \right) \right]
\]

\[
= \sigma^4 + \frac{1}{2} \sigma^4 \frac{q(T)}{T} \left( 1 - \exp \left\{ -\frac{2T}{q(T)} \right\} \right) + O_p \left( \max \left\{ \frac{1}{\sqrt{N}}, \frac{1}{\sqrt{T}} \right\} \right),
\]

as required.

We turn our attention to part (d), where we assume that \( \rho_T = \exp \left\{ -\frac{1}{q(T)} \right\} \) such that \( q(T) \to \infty \) but \( q(T)/T \to 0 \). In this case, we apply part (d) of Lemmas SD-6 and SD-7 and Lemma SD-12 to obtain

\[
\hat{\omega}^2_{NT} = \tilde{\sigma}^2 \left[ \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=4}^{T} (y_{it-3} - y_{it-2})^2 + \frac{1}{NT} \sum_{i=1}^{N} y_{iT-2}^2 \right]
\]

\[
= \left[ \sigma^2 + O_p \left( \max \left\{ \frac{1}{N}, \frac{1}{T} \right\} \right) \right] \left[ \frac{\sigma^2 q(T)}{T} \left( 1 - \exp \left\{ -\frac{2T}{q(T)} \right\} \right) + O_p \left( \max \left\{ \frac{1}{\sqrt{N}}, \frac{1}{\sqrt{T}} \right\} \right) \right]
\]

\[
= \sigma^4 + \frac{1}{2} \frac{\sigma^2 q(T)}{T} \left( 1 - \exp \left\{ -\frac{2T}{q(T)} \right\} \right) + O_p \left( \max \left\{ \frac{1}{\sqrt{N}}, \frac{1}{\sqrt{T}} \right\} \right)
\]

as required.

Finally, we consider part (e), where we assume that \( \rho_T \in \mathcal{G}_{ST} = \left\{ |\rho_T| = \exp \left\{ -\frac{1}{q(T)} \right\} : q(T) \geq 0 \text{ and } q(T) = O(1) \text{ as } T \to \infty \right\} \).

In this case, we apply part (e) of Lemmas SD-6 and SD-7 and Lemma SD-12 to obtain

\[
\hat{\omega}^2_{NT} = \tilde{\sigma}^2 \left[ \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=4}^{T} (y_{it-3} - y_{it-2})^2 + \frac{1}{NT} \sum_{i=1}^{N} y_{iT-2}^2 \right]
\]

\[
= \left[ \sigma^2 + O_p \left( \max \left\{ \frac{1}{N}, \frac{1}{T} \right\} \right) \right] \left[ 2\sigma^2 \frac{q(T)}{T} + o_p(1) + O_p \left( \frac{1}{T} \right) \right]
\]

\[
= \frac{2\sigma^4}{1 + \rho_T} + o_p(1). \quad \Box
\]
Appendix SE: Additional Lemmas and Technical Details

Lemma SE-1:

Let \(d\) and \(b\) both be some positive integer. The following results hold as \(T \to \infty\).

(a) If \(T/q(T) \to 0\), then

\[
\sum_{t=b}^{T} \sum_{j=1}^{t-b+1} \exp \left\{-d \frac{(t-b+1-j)}{q(T)} \right\} = \frac{T^2}{2} \left[ 1 + O \left( \frac{1}{T} \right) + O \left( \frac{T}{q(T)} \right) \right].
\]

(b) If \(q(T) \sim T\), then

\[
\sum_{t=b}^{T} \sum_{j=1}^{t-b+1} \exp \left\{-d \frac{(t-b+1-j)}{q(T)} \right\} = q(T) \left[ 1 + O \left( \frac{1}{q(T)} \right) + O \left( \frac{T}{q(T)} \right) \right].
\]

(c) If \(q(T) \to \infty\) such that \(q(T)/T \to 0\), then

\[
\sum_{t=b}^{T} \sum_{j=1}^{t-b+1} \exp \left\{-d \frac{(t-b+1-j)}{q(T)} \right\} = Tq(T) \left[ 1 + O \left( \frac{1}{q(T)} \right) + O \left( \frac{q(T)}{T} \right) \right].
\]

Proof of Lemma SE-1:

To proceed, note first that for all \(T \geq b\)

\[
\sum_{t=b}^{T} \sum_{j=1}^{t-b+1} \exp \left\{-d \frac{(t-b+1-j)}{q(T)} \right\} = \sum_{t=b}^{T} \frac{1 - \exp \left\{-d \frac{t-b+1}{q(T)} \right\}}{1 - \exp \left\{-d \frac{1}{q(T)} \right\}}
\]

\[
= \left[ 1 - \exp \left\{-d \frac{1}{q(T)} \right\} \right]^{-1} \left\{ (T-b+1) - \sum_{t=b}^{T} \exp \left\{-d \frac{(t-b+1)}{q(T)} \right\} \right\}
\]

\[
= \left[ 1 - \exp \left\{-d \frac{1}{q(T)} \right\} \right]^{-1} \left\{ (T-b+1) - \exp \left\{-d \frac{1}{q(T)} \right\} \sum_{t=b}^{T} \exp \left\{-d \frac{(t-b)}{q(T)} \right\} \right\}
\]

\[
= \left[ 1 - \exp \left\{-d \frac{1}{q(T)} \right\} \right]^{-1} \times \left\{ (T-b+1) - \exp \left\{-d \frac{1}{q(T)} \right\} \left[ 1 - \exp \left\{-d \frac{1}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{-d \frac{T-b+1}{q(T)} \right\} \right] \right\}.
\]

For part (a), we consider the case \(T/q(T) \to 0\). In this case, note that
\[
\sum_{t=b}^{T} \sum_{j=1}^{t-b+1} \exp \left\{ -\frac{d(t-b+1-j)}{q(T)} \right\} \\
= \left[ 1 - \exp \left\{ -\frac{d}{q(T)} \right\} \right]^{-1} \\
\times \left\{ (T-b+1) - \exp \left\{ -\frac{d}{q(T)} \right\} \left[ 1 - \exp \left\{ -\frac{d}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{ -d \left( \frac{T-b+1}{q(T)} \right) \right\} \right] \right\} \\
= \left[ 1 - \exp \left\{ -\frac{d}{q(T)} \right\} \right]^{-2} \left\{ (T-b+1) \left[ 1 - \exp \left\{ -\frac{d}{q(T)} \right\} \right] \right\} \\
- \exp \left\{ -\frac{d}{q(T)} \right\} \left[ 1 - \exp \left\{ -d \frac{T}{q(T)} \right\} \exp \left\{ \frac{d}{q(T)} (b-1) \right\} \right\} \\
= \left[ 1 - 1 + \frac{d}{q(T)} + O \left( \frac{1}{q(T)^2} \right) \right]^{-2} \left\{ (T-b+1) \left[ 1 - \exp \left\{ -\frac{d}{q(T)} \right\} \right] \right\} \\
- \left[ 1 - 1 + \frac{d}{q(T)} + O \left( \frac{1}{q(T)^2} \right) \right]^{-2} \left\{ 1 - \exp \left\{ -\frac{d}{q(T)} \right\} \right\} \right\} \\
\times \left\{ 1 - \left[ 1 - \frac{dT}{q(T)} + \frac{d^2T^2}{2q(T)^2} + O \left( \frac{T^3}{q(T)^3} \right) \right] \left[ 1 + \frac{d(b-1)}{q(T)} + O \left( \frac{1}{q(T)^2} \right) \right] \right\} \\
= \frac{q(T)^2}{d^2} \left[ 1 + O \left( \frac{1}{q(T)^2} \right) \right] \left\{ \frac{dT}{q(T)} - \frac{d(b-1)}{q(T)} - \frac{d^2T}{2q(T)^2} + O \left( \frac{1}{q(T)^2} \right) \right\} \\
- \left[ 1 - \frac{d}{q(T)} + O \left( \frac{1}{q(T)^2} \right) \right] \left\{ 1 \left[ 1 - \frac{dT}{q(T)} + \frac{d^2T^2}{2q(T)^2} + \frac{d(b-1)}{q(T)} - \frac{d^2(b-1)T}{q(T)^2} \right] \right\} \\
+ O \left( \frac{T^3}{q(T)^3} \right) + O \left( \frac{1}{q(T)^2} \right) \right\} \\
= \frac{q(T)^2}{d^2} \left[ 1 + O \left( \frac{1}{q(T)^2} \right) \right] \left\{ \frac{dT}{q(T)} - \frac{d(b-1)}{q(T)} - \frac{d^2T}{2q(T)^2} + O \left( \frac{T^3}{q(T)^3} \right) \right\} \\
- \left[ 1 - \frac{d}{q(T)} + O \left( \frac{1}{q(T)^2} \right) \right] \left\{ \frac{dT}{q(T)} - \frac{d^2T^2}{2q(T)^2} - \frac{d(b-1)}{q(T)} - \frac{d^2(b-1)T}{q(T)^2} \right\} \\
+ O \left( \frac{T^3}{q(T)^3} \right) + O \left( \frac{1}{q(T)^2} \right) \right\} \\
\]
Next, consider part (b), where we take $q(T) \sim T$. In this case, we obtain by direct calculation

\[
\sum_{t=b}^{T} \sum_{j=1}^{t-b+1} \exp \left\{ -d(t - b + 1 - j) \right\}
= \left[ 1 - \exp \left\{ -\frac{d}{q(T)} \right\} \right]^{-1}
\times \left\{ (T - b + 1) - \exp \left\{ -\frac{d}{q(T)} \right\} \left[ 1 - \exp \left\{ -\frac{d}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{ -d \left( \frac{T - b + 1}{q(T)} \right) \right\} \right] \right\}
= \frac{q(T)}{d} \left[ T - \left( \frac{q(T)}{d} \right) \left[ 1 - \exp \left\{ -\frac{d}{q(T)} \right\} \exp \left\{ \frac{d(b - 1)}{q(T)} \right\} \right] \right] \left[ 1 + O \left( \frac{1}{T} \right) \right]
= \frac{q(T)}{d} \left[ T + \frac{q(T)}{d} \left( \exp \left\{ -\frac{d}{q(T)} \right\} - 1 \right) + O(1) \right] \left[ 1 + O \left( \frac{1}{T} \right) \right]
= \frac{q(T)^2}{d^2} \left[ \exp \left\{ -\frac{dT}{q(T)} \right\} + \frac{dT}{q(T)} - 1 \right] \left[ 1 + O \left( \frac{1}{T} \right) \right]
= O(T^2),
\]

which shows the required result for part (b).

Finally, consider part (c). In this case, $q(T) \to \infty$ such that $q(T)/T \to 0$, and we have
\[
\sum_{t=b}^{T} \sum_{j=1}^{t-b+1} \exp \left\{ -d\frac{(t-b+1-j)}{q(T)} \right\} = \left[ 1 - \exp \left\{ -\frac{d}{q(T)} \right\} \right]^{-1} \\
\times \left\{ (T - b + 1) - \exp \left\{ -\frac{d}{q(T)} \right\} \left[ 1 - \exp \left\{ -\frac{d}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{ -d\left(\frac{T-b+1}{q(T)}\right)\right\} \right] \right\} \\
= \frac{q(T)}{d} \left[ 1 + O\left(\frac{1}{q(T)}\right) \right] \\
\times \left\{ (T - b + 1) - \exp \left\{ -\frac{d}{q(T)} \right\} \left[ 1 - \exp \left\{ -\frac{d}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{ -d\left(\frac{T-b+1}{q(T)}\right)\right\} \right] \right\} \\
= \frac{Tq(T) - q(T)(b-1)}{d} \left[ 1 + O\left(\frac{1}{q(T)}\right) \right] \\
\times \left\{ Tq(T) - q(T)(b-1) \right\} \left[ 1 + O\left(\frac{1}{q(T)}\right) \right] \\
- \frac{q(T)}{d} \left[ 1 + O\left(\frac{1}{q(T)}\right) \right] \left[ 1 - \exp \left\{ -\frac{dT}{q(T)} \right\} \right] \exp \left\{ \frac{d(b-1)}{q(T)} \right\} \\
= \frac{Tq(T)}{d} + O\left(q(T)\right) + O\left(T\right) + O\left(q(T)^2\right) \\
= \frac{Tq(T)}{d} \left[ 1 + O\left(\frac{1}{q(T)}\right) \right] + O\left(q(T)\right),
\]

as required. \( \Box \)

**Lemma SE-2:**

Let \( b \) be a positive integer and let \( g \) be a non-negative integer.

(a) If \( T/q(T) \to 0 \), then

\[
\sum_{t=g+1}^{T} \left[ \sum_{j=1}^{t-g} \exp \left( -b\frac{(t-g-j)}{q(T)} \right) \right]^2 = \frac{T^3}{3} \left[ 1 + O\left(\frac{T}{q(T)}\right) + O\left(\frac{1}{T}\right) \right],
\]

as \( T \to \infty \).

(b) If \( q(T) \sim T \), then

\[
\sum_{t=g+1}^{T} \left[ \sum_{j=1}^{t-g} \exp \left( -b\frac{(t-g-j)}{q(T)} \right) \right]^2 = \frac{q(T)^3}{2b^3} \left[ \frac{2bT}{q(T)} + 4\exp\left( -\frac{bT}{q(T)} \right) - \exp\left( -\frac{2bT}{q(T)} \right) - 3 \right] \left[ 1 + O\left(\frac{1}{T}\right) \right],
\]

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as $T \to \infty$.

(c) If $q(T) \to \infty$ such that $q(T)/T \to 0$, then

$$
\sum_{t=g+1}^{T} \left[ \sum_{j=1}^{t-g} \exp \left( -\frac{b(t-g-j)}{q(T)} \right) \right]^2 = \frac{Tq(T)^2}{b^2} \left[ 1 + O \left( \frac{q(T)}{T} \right) \right],
$$

as $T \to \infty$.

Proof of Lemma SE-2:

To proceed, note first that for all $T \geq g+1$, we have

$$
\sum_{t=g+1}^{T} \left[ \sum_{j=1}^{t-g} \exp \left( -\frac{b(t-g-j)}{q(T)} \right) \right]^2 = \left[ 1 - \exp \left\{ -\frac{b}{q(T)} \right\} \right]^{-2} \sum_{t=g+1}^{T} \left[ 1 - \exp \left\{ -\frac{b(t-g)}{q(T)} \right\} \right]^2
$$

$$
= \frac{q(T)^2}{b^2} \sum_{t=g+1}^{T} \left[ 1 - 2 \exp \left\{ -\frac{b(t-g)}{q(T)} \right\} + \exp \left\{ -2\frac{b(t-g)}{q(T)} \right\} \right] \left[ 1 + O \left( \frac{1}{q(T)} \right) \right]
$$

$$
= \frac{q(T)^2}{b^2} \left[ (T-g) - 2 \exp \left\{ -\frac{b}{q(T)} \right\} \sum_{t=g+1}^{T} \exp \left\{ -\frac{b(t-g-1)}{q(T)} \right\} 
\right. \\
\left. + \exp \left\{ -\frac{2b}{q(T)} \right\} \sum_{t=g+1}^{T} \exp \left\{ -2\frac{b(t-g-1)}{q(T)} \right\} \right] \left[ 1 + O \left( \frac{1}{q(T)} \right) \right]
$$

$$
= \frac{q(T)^2}{b^2} \left[ (T-g) - 2 \exp \left\{ -\frac{b}{q(T)} \right\} \left( 1 - \exp \left\{ -\frac{b}{q(T)} \right\} \right)^{-1} \left( 1 - \exp \left\{ -\frac{b(t-g)}{q(T)} \right\} \right) 
\right. \\
\left. + \exp \left\{ -\frac{2b}{q(T)} \right\} \left( 1 - \exp \left\{ -\frac{2b}{q(T)} \right\} \right)^{-1} \left( 1 - \exp \left\{ -\frac{2b(T-g)}{q(T)} \right\} \right) \right] \left[ 1 + O \left( \frac{1}{q(T)} \right) \right].
$$
Now, consider part (a), where we take \( T/q(T) \to 0 \). Here, we have

\[
\sum_{t=g}^{T} \left[ \sum_{j=1}^{t-g} \exp \left(-\frac{b(t-g-j)}{q(T)}\right) \right]^2
= \frac{q(T)^2}{b^2} \left[ (T-g) - 2 \exp \left\{ -\frac{b}{q(T)} \right\} \left( 1 - \exp \left\{ -\frac{b}{q(T)} \right\} \right) \right]^{-1} \left[ 1 - \exp \left\{ -\frac{bT}{q(T)} \right\} \exp \left\{ \frac{gb}{q(T)} \right\} \right] \\
+ \exp \left\{ -\frac{2b}{q(T)} \right\} \left( 1 - \exp \left\{ -\frac{2b}{q(T)} \right\} \right) \left[ 1 + \frac{2gb}{q(T)} + O \left( \frac{1}{q(T)} \right) \right] \\
= \frac{q(T)^2}{b^2} \left[ (T-g) - \frac{2gb}{b} \left( 1 - \left[ 1 - \frac{bT}{q(T)} + \frac{b^2}{2q(T)^2} \frac{T^2}{q(T)} - \frac{b^3T^3}{6q(T)^3} + O \left( \frac{T^4}{q(T)^4} \right) \right] \left[ 1 + \frac{gb}{q(T)} + O \left( \frac{1}{q(T)} \right) \right] \right] \\
\times \left[ 1 + O \left( \frac{1}{q(T)} \right) \right] \\
+ \frac{q(T)}{2b} \left[ 1 - \left[ 1 - \frac{2bT}{q(T)} + \frac{2b^2T^2}{q(T)^2} - \frac{8b^3T^3}{6q(T)^3} + O \left( \frac{T^4}{q(T)^4} \right) \right] \left[ 1 + \frac{2gb}{q(T)} + O \left( \frac{1}{q(T)} \right) \right] \right] \\
\times \left[ 1 + O \left( \frac{1}{q(T)} \right) \right] \\
= \frac{q(T)^2}{b^2} \left[ (T-g) - \frac{2gb}{b} \left[ \frac{bT}{q(T)} - \frac{1}{2q(T)^2} + \frac{1}{6q(T)^3} - \frac{gb}{q(T)} + \frac{gb^2}{q(T)^2} + O \left( \frac{T^4}{q(T)^4} \right) + O \left( \frac{T^2}{q(T)^3} \right) \right] \right] \\
\times \left[ 1 + O \left( \frac{1}{q(T)} \right) \right] \\
+ \frac{q(T)}{2b} \left[ 2bT \left[ \frac{bT}{q(T)} - 2 \frac{b^2T^2}{q(T)^2} + \frac{4}{3q(T)^3} - \frac{2gb}{q(T)} + \frac{4gb^2}{q(T)^2} + O \left( \frac{T^4}{q(T)^4} \right) + O \left( \frac{T^2}{q(T)^3} \right) \right] \right] \\
\times \left[ 1 + O \left( \frac{1}{q(T)} \right) \right] \\
= \frac{q(T)^2}{b^2} \left[ (T-g) \right] \\
- \frac{2gb}{b} \left[ \frac{bT}{q(T)} - \frac{1}{2q(T)^2} + \frac{1}{6q(T)^3} - \frac{gb}{q(T)} + \frac{gb^2}{q(T)^2} + O \left( \frac{T^4}{q(T)^4} \right) + O \left( \frac{T^2}{q(T)^3} \right) \right] \\
\times \left[ 1 + O \left( \frac{1}{q(T)} \right) \right] \\
+ \frac{q(T)}{2b} \left[ 2bT \left[ \frac{bT}{q(T)} - 2 \frac{b^2T^2}{q(T)^2} + \frac{4}{3q(T)^3} - \frac{2gb}{q(T)} + \frac{4gb^2}{q(T)^2} + O \left( \frac{T^4}{q(T)^4} \right) + O \left( \frac{T^2}{q(T)^3} \right) \right] \right] \\
\times \left[ 1 + O \left( \frac{1}{q(T)} \right) \right]
\]
\[
= \frac{q(T)^2}{b^2}\left\{\left(\frac{T-g}{T}\right) - 2T + \frac{bT^2}{q(T)} - \frac{1}{3} \frac{b^2 T^2}{q(T)^2} + 2g - \frac{2gbT}{q(T)} + O\left(\frac{T^4}{q(T)^3}\right) + O\left(\frac{T^2}{q(T)^2}\right)\right]\times\left[1 + O\left(\frac{1}{q(T)}\right)\right]
\]
\[
+ \left[T - \frac{bT^2}{q(T)} + \frac{2}{3} \frac{b^2 T^3}{q(T)^2} - g + 2 \frac{gbT}{q(T)} + O\left(\frac{T^4}{q(T)^3}\right) + O\left(\frac{T^2}{q(T)^2}\right)\right]\left[1 + O\left(\frac{1}{q(T)}\right)\right]
\]
\[
= \frac{q(T)^2}{b^2}\left\{\frac{1}{3} \frac{b^2 T^3}{q(T)^2} + O\left(\frac{T^4}{q(T)^3}\right) + O\left(\frac{T^2}{q(T)^2}\right)\right\}\left[1 + O\left(\frac{1}{q(T)}\right)\right]
\]
\[
= \frac{T^3}{3}\left[1 + O\left(\frac{T}{q(T)}\right) + O\left(\frac{1}{T}\right)\right],
\]
which completes the proof of part (a).

Next, consider part (b). In this case, \(q(T) \sim T\), and we have

\[
\sum_{t=g+1}^{T} \sum_{j=1}^{t-g} \exp\left(-\frac{b(t-g-j)}{q(T)}\right) ^ 2
\]
\[
= \frac{q(T)^2}{b^2}\left\{(T-g) - 2 \exp\left\{-\frac{b}{q(T)}\right\}\left(1 - \exp\left\{-\frac{b}{q(T)}\right\}\right)^{-1}\left(1 - \exp\left\{-\frac{b(T-g)}{q(T)}\right\}\right)\right\}
\]
\[
+ \exp\left\{-\frac{2b}{q(T)}\right\}\left(1 - \exp\left\{-\frac{2b}{q(T)}\right\}\right)^{-1}\left(1 - \exp\left\{-\frac{2b(T-g)}{q(T)}\right\}\right)\left[1 + O\left(\frac{1}{q(T)}\right)\right]
\]
\[
= \frac{q(T)^2}{b^2}\left\{(T-g) - 2 \frac{q(T)}{b} \left(1 - \exp\left\{-\frac{bT}{q(T)}\right\} + O\left(\frac{1}{T}\right)\right)\left(1 + O\left(\frac{1}{T}\right)\right)\right\}
\]
\[
+ \frac{q(T)}{b}\left(1 - \exp\left\{-\frac{2bT}{q(T)}\right\} + O\left(\frac{1}{T}\right)\right)\left[1 + O\left(\frac{1}{T}\right)\right]\left[1 + O\left(\frac{1}{T}\right)\right]
\]
\[
= \frac{q(T)^3}{b^3}\left\{\frac{bT}{q(T)} - \frac{1}{2} \left(1 - \exp\left\{-\frac{bT}{q(T)}\right\}\right) + \frac{1}{2} \left(1 - \exp\left\{-\frac{2bT}{q(T)}\right\}\right) - 3\right\}\left[1 + O\left(\frac{1}{T}\right)\right]
\]
\[
= \frac{q(T)^3}{b^3}\left\{\frac{bT}{q(T)} + 2 \exp\left\{-\frac{bT}{q(T)}\right\} - \frac{1}{2} \exp\left\{-\frac{2bT}{q(T)}\right\} - 3\right\}\left[1 + O\left(\frac{1}{T}\right)\right]
\]
\[
= \frac{q(T)^3}{b^3}\left\{\frac{bT}{q(T)} + 4 \exp\left\{-\frac{bT}{q(T)}\right\} - \exp\left\{-\frac{2bT}{q(T)}\right\} - 3\right\}\left[1 + O\left(\frac{1}{T}\right)\right]
\]
as required.
Finally, consider part (c), where we take \( q(T) \to \infty \) such that \( q(T)/T \to 0 \). In this case, we have

\[
T \sum_{t=g+1}^{T} \left[ \sum_{j=1}^{t-g} \exp \left( -b \frac{t - g - j}{q(T)} \right) \right]^2
= \frac{q(T)^2}{b^2} \left\{ (T - g) - 2 \frac{q(T)}{b} \left[ 1 + O \left( \frac{1}{q(T)} \right) \right] \left[ 1 + O \left( \exp \left\{ -b \frac{T}{q(T)} \right\} \right) \right] + q(T) \left[ 1 + O \left( \frac{1}{q(T)} \right) \right] \left[ 1 + O \left( \exp \left\{ -2b \frac{T}{q(T)} \right\} \right) \right] \right\}
= \frac{Tq(T)^2}{b^2} \left[ 1 + O \left( \frac{1}{T} \right) + O \left( \frac{q(T)}{T} \right) \right]
= \frac{Tq(T)^2}{b^2} \left[ 1 + O \left( \frac{q(T)}{T} \right) \right],
\]

as required. □

**Lemma SE-3:**

Let \( b \) and \( g \) be fixed positive integers; then, the following statements are true as \( T \to \infty \).

(a) If \( T/q(T) \to 0 \), then

\[
\sum_{j=1}^{T-g} \exp \left\{ -b \frac{T - g - j}{q(T)} \right\}
= T \left[ 1 - g \frac{T}{T} - b \frac{T}{2q(T)} + O \left( \max \left\{ \frac{T^2}{q(T)^2}, \frac{1}{q(T)} \right\} \right) \right]
= O(T)
\]

(b) If \( q(T) \sim T \), then

\[
\sum_{j=1}^{T-g} \exp \left\{ -b \frac{T - g - j}{q(T)} \right\} = \frac{q(T)}{b} \left[ 1 - \exp \left\{ -b \frac{T}{q(T)} \right\} \right] \left[ 1 + O \left( \frac{1}{T} \right) \right].
\]

(c) If \( q(T) \to \infty \) such that \( q(T)/T \to 0 \), then

\[
\sum_{j=1}^{T-g} \exp \left\{ -b \frac{T - g - j}{q(T)} \right\} = \frac{q(T)}{b} \left[ 1 + O \left( \frac{1}{q(T)} \right) \right] = O(q(T)).
\]

**Proof:**

Focus first on part (a), where we consider the case in which \( T/q(T) \to 0 \). In this case,
\[
\sum_{j=1}^{T-g} \exp \left\{ -b \left( \frac{T - g - j}{q(T)} \right) \right\}
\]

\[
= \left[ 1 - \exp \left\{ - \frac{b}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{ -b \left( \frac{T - g}{q(T)} \right) \right\} \right]
\]

\[
= \left[ 1 - \exp \left\{ - \frac{b}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{ - \frac{bT}{q(T)} \right\} \exp \left\{ \frac{bg}{q(T)} \right\} \right]
\]

\[
= \frac{q(T)}{b} \left[ 1 + O \left( \frac{1}{q(T)} \right) \right]
\]

\[
\times \left[ 1 - \left( 1 - \frac{bT}{q(T)} + \frac{1}{2} \frac{b^2 T^2}{q(T)^2} + O \left( \frac{T^3}{q(T)^3} \right) \right) \left( 1 + \frac{gb}{q(T)} + \frac{g^2 b^2}{2q(T)^2} + O \left( \frac{1}{q(T)} \right) \right) \right]
\]

\[
= \frac{q(T)}{b} \left[ 1 - \left( 1 - \frac{bT}{q(T)} + \frac{1}{2} \frac{b^2 T^2}{q(T)^2} + \frac{gb}{q(T)} + O \left( \max \left\{ \frac{T^3}{q(T)^3}, \frac{T}{q(T)^2} \right\} \right) \right) \right]
\]

\[
= T - \frac{1}{2} \frac{bT^2}{q(T)} - g + O \left( \max \left\{ \frac{T^3}{q(T)^3}, \frac{T}{q(T)^2} \right\} \right)
\]

\[
= T \left[ 1 - \frac{g}{T} - \frac{b}{2q(T)} \right] + O \left( \max \left\{ \frac{T^2}{q(T)^2}, \frac{1}{q(T)} \right\} \right)
\]

\[
= O(T),
\]

as required for part (a).

We now turn our attention to part (b), where we take \( q(T) \sim T \). In this case,

\[
= \sum_{j=1}^{T-g} \exp \left\{ -b \left( \frac{T - g - j}{q(T)} \right) \right\}
\]

\[
= \left[ 1 - \exp \left\{ - \frac{b}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{ - \frac{bT}{q(T)} \right\} \exp \left\{ \frac{bg}{q(T)} \right\} \right]
\]

\[
= \frac{q(T)}{b} \left[ 1 + O \left( \frac{1}{T} \right) \right] \left[ 1 - \exp \left\{ - \frac{bT}{q(T)} \right\} \left( 1 + \frac{gb}{q(T)} + \frac{1}{2} \frac{g^2 b^2}{q(T)^2} + O \left( \frac{1}{q(T)} \right) \right) \right]
\]

\[
= \frac{q(T)}{b} \left[ 1 - \exp \left\{ - \frac{bT}{q(T)} \right\} - \frac{gb}{q(T)} \exp \left\{ - \frac{bT}{q(T)} \right\} + \left( \frac{1}{T^2} \right) \right] \left[ 1 + O \left( \frac{1}{T} \right) \right]
\]

\[
= \frac{q(T)}{b} \left[ 1 - \exp \left\{ - \frac{bT}{q(T)} \right\} \right] \left[ 1 + O \left( \frac{1}{T} \right) \right]
\]

\[
= O(T)
\]
Finally, we consider the case \( q(T) \to \infty \) such that \( q(T)/T \to 0 \). In this case,
\[
\sum_{j=1}^{T-2} \exp \left\{ -b \left( \frac{T-g-j}{q(T)} \right) \right\} = \left[ 1 - \exp \left\{ -b \left( \frac{T}{q(T)} \right) \right\} \right]^{-1} \left[ 1 - \exp \left\{ -b \left( \frac{T}{q(T)} \right) \right\} \right] = \frac{q(T)}{b} \left[ 1 + O \left( \frac{1}{q(T)} \right) \right] \left[ 1 + O \left( \exp \left\{ - \frac{bT}{q(T)} \right\} \right) \right] \to 0.
\]

which gives the required result for part (c).

Lemma SE-4:

Let \( d \) be a positive integer; then, the following statements are true as \( T \to \infty \)

(a) If \( T/q(T) \to 0 \), then
\[
\sum_{r=2}^{T-2} \sum_{s=1}^{r-1} \exp \left\{ -d \frac{T-2-r}{q(T)} \right\} \exp \left\{ -d \frac{T-2-s}{q(T)} \right\} = \frac{T^2}{2} \left[ 1 + O \left( \frac{T}{q(T)} \right) \right] \to 0.
\]

(b) If \( q(T) \sim T \), then
\[
\sum_{r=2}^{T-2} \sum_{s=1}^{r-1} \exp \left\{ -d \frac{T-2-r}{q(T)} \right\} \exp \left\{ -d \frac{T-2-s}{q(T)} \right\} = \frac{q(T)^2}{2d^2} \left[ 1 - 2 \exp \left\{ -d \frac{T}{q(T)} \right\} + \exp \left\{ -2dT \right\} \right] \to 0.
\]

(c) If \( q(T) \to \infty \) such that \( q(T)/T \to 0 \), then
\[
\sum_{r=2}^{T-2} \sum_{s=1}^{r-1} \exp \left\{ -d \frac{T-2-r}{q(T)} \right\} \exp \left\{ -d \frac{T-2-s}{q(T)} \right\} = \frac{q(T)^2}{2d^2} \left[ 1 + O \left( \frac{1}{q(T)} \right) \right] \to 0.
\]
Proof of Lemma SE-4:

To proceed, note first that

\[ \sum_{r=2}^{T-1} \sum_{s=1}^{T-1} \exp \left\{ -d \frac{T - 2 - r}{q(T)} \right\} \exp \left\{ -d \frac{T - 2 - s}{q(T)} \right\} \]

\[ = \sum_{r=2}^{T-2} \exp \left\{ -d \frac{T - 2 - r}{q(T)} \right\} \exp \left\{ -d \frac{T - 3 - r}{q(T)} \right\} \sum_{s=1}^{r-1} \exp \left\{ d \left( \frac{s - 1}{q(T)} \right) \right\} \]

\[ = \left[ \exp \left\{ \frac{d}{q(T)} \right\} - 1 \right]^{-1} \sum_{r=2}^{T-2} \exp \left\{ -d \frac{T - 2 - r}{q(T)} \right\} \exp \left\{ -d \frac{T - 3}{q(T)} \right\} \]

\[ \times \left[ \exp \left\{ d \left( \frac{r - 1}{q(T)} \right) \right\} - 1 \right] \]

\[ = \left[ \exp \left\{ \frac{d}{q(T)} \right\} - 1 \right]^{-1} \sum_{r=2}^{T-2} \exp \left\{ -2d \frac{T - 2 - r}{q(T)} \right\} \]

\[ - \left[ \exp \left\{ \frac{d}{q(T)} \right\} - 1 \right]^{-2} \exp \left\{ -2d \frac{T}{q(T)} \right\} \exp \left\{ \frac{7d}{q(T)} \right\} \sum_{r=2}^{T-2} \exp \left\{ d \frac{T - 3 - r}{q(T)} \right\} - 1 \]

\[ = \left[ \exp \left\{ \frac{d}{q(T)} \right\} - 1 \right]^{-1} \sum_{r=2}^{T-2} \exp \left\{ -2d \frac{T - 2 - r}{q(T)} \right\} \]

\[ - \left[ \exp \left\{ \frac{d}{q(T)} \right\} - 1 \right]^{-2} \exp \left\{ -2d \frac{T}{q(T)} \right\} \exp \left\{ \frac{4d}{q(T)} \right\} \]

\[ + \left[ \exp \left\{ \frac{d}{q(T)} \right\} - 1 \right]^{-2} \exp \left\{ -2d \frac{T}{q(T)} \right\} \exp \left\{ \frac{7d}{q(T)} \right\} \]

Now, consider part (a) where we take \( T/q(T) \to 0 \). In this case, note that, applying part (a) of Lemma
SE-3 with $b = 2d$ and $g = 2$, we obtain

$$\sum_{r=2}^{T-2-1} \sum_{s=1}^{T-2} \exp \left\{ -\frac{d}{q(T)} \right\} \exp \left\{ -\frac{T - 2 - r}{q(T)} \right\} \exp \left\{ -\frac{T - 2 - s}{q(T)} \right\}$$

$$= \left[ \exp \left\{ -\frac{d}{q(T)} \right\} - 1 \right]^{-1} \sum_{r=1}^{T-2} \exp \left\{ -\frac{2d}{q(T)} \right\} \exp \left\{ -\frac{T - 2 - r}{q(T)} \right\} - \left[ \exp \left\{ -\frac{d}{q(T)} \right\} - 1 \right]^{-1} \exp \left\{ -\frac{2d}{q(T)} \right\}$$

$$= \frac{q(T)}{d} \left[ 1 - \frac{d}{2q(T)} + O \left( \frac{1}{q(T)^2} \right) \right] \left[ 1 - \frac{2}{2q(T)} - \frac{d}{q(T)} + O \left( \frac{T}{q(T)^2}, \frac{1}{q(T)} \right) \right]$$

$$- \frac{q(T)}{d} \left[ 1 - \frac{d}{2q(T)} + O \left( \frac{1}{q(T)^2} \right) \right]$$

$$\times \left[ 1 - \frac{2dT}{q(T)} + \frac{6d}{q(T)} + \frac{2d^2T^2}{q(T)^2} + O \left( \max \left\{ \frac{T^3}{q(T)^3}, \frac{T}{q(T)^2} \right\} \right) \right]$$

$$- \left[ \frac{d}{q(T)} + \frac{d^2}{2q(T)^2} + \frac{1}{6q(T)^3} + O \left( \frac{1}{q(T)^4} \right) \right]^{-2}$$

$$\times \left[ 1 - \frac{d}{q(T)} + \frac{d^2}{2q(T)^2} + O \left( \frac{T^3}{q(T)^3} \right) \right] \left[ 1 + \frac{4d}{q(T)} + O \left( \frac{1}{q(T)^2} \right) \right]$$

$$+ \left[ \frac{d}{q(T)} + \frac{d^2}{2q(T)^2} + \frac{1}{6q(T)^3} + O \left( \frac{1}{q(T)^4} \right) \right]^{-2}$$

$$\times \left[ 1 - \frac{2dT}{q(T)} + \frac{2d^2T^2}{q(T)^2} + O \left( \frac{T^3}{q(T)^3} \right) \right] \left[ 1 + \frac{7d}{q(T)} + O \left( \frac{1}{q(T)^2} \right) \right]$$
\[ \frac{T q(T)}{d} - 2 \frac{q(T)}{d} - T^2 - q(T) + O \left( \max \left\{ \frac{T^3}{q(T)}, T \right\} \right) \]
\[
- \frac{q(T)^2}{d^2} \left[ 1 + \frac{d}{2q(T)} + \frac{d^2}{6q(T)^2} + O \left( \frac{1}{q(T)^3} \right) \right]^{-2} \]
\[
\times \left[ 1 - \frac{T}{q(T)} + \frac{4d}{q(T)} + \frac{d^2}{2q(T)^2} + O \left( \max \left\{ \frac{T^3}{q(T)^3}, \frac{T}{q(T)^2} \right\} \right) \right] \]
\[
+ \frac{q(T)^2}{d^2} \left[ 1 + \frac{d}{2q(T)} + \frac{d^2}{6q(T)^2} + O \left( \frac{1}{q(T)^3} \right) \right]^{-2} \]
\[
\times \left[ 1 - \frac{2dT}{q(T)} + \frac{7d}{q(T)} + \frac{2d^2T^2}{q(T)^2} + O \left( \max \left\{ \frac{T^3}{q(T)^3}, \frac{T}{q(T)^2} \right\} \right) \right] \]
\[
= \frac{T q(T)}{d} - 2 \frac{q(T)}{d} - T^2 - q(T) + \frac{q(T)}{d} + q(T) - \frac{1}{2} T^2 \]
\[
+ \frac{q(T)^2}{d^2} - \frac{q(T)}{d} + 7 \frac{q(T)}{d} - 2 \frac{T q(T)}{d} + 2 T^2 + O \left( \max \left\{ \frac{T^3}{q(T)^3}, T \right\} \right) \]
\[
= \left[ -T^2 - \frac{1}{2} T^2 + 2 T^2 \right] + \left[ \frac{T q(T)}{d} + \frac{T q(T)}{d} - 2 \frac{T q(T)}{d} \right] + \left[ - \frac{q(T)^2}{d^2} + \frac{q(T)^2}{d^2} \right] \]
\[
+ \left[ -2 \frac{q(T)}{d} - q(T) + \frac{q(T)}{d} - 4 \frac{q(T)}{d} - \frac{q(T)}{d} + 7 \frac{q(T)}{d} \right] + O \left( \max \left\{ \frac{T^3}{q(T)}, T \right\} \right) \]
\[
= \frac{T^2}{2} + O \left( \max \left\{ \frac{T^3}{q(T)}, T \right\} \right) \]
\[
= \frac{T^2}{2} \left[ 1 + O \left( \max \left\{ \frac{T}{q(T)}, \frac{1}{T} \right\} \right) \right] \]

Next, consider part (b) where we take \( q(T) \sim T \). In this case, we apply part (b) of Lemma SE-3
with $b = 2d$ and $g = 2$ to get

$$
\sum_{r=2}^{T-2} \sum_{s=1}^{r-1} \exp \left\{ -d\frac{T-2-r}{T} \right\} \exp \left\{ -d\frac{T-2-s}{T} \right\} \\
= \left[ \exp \left\{ \frac{d}{q(T)} \right\} - 1 \right]^{-1} \sum_{r=1}^{T-2} \exp \left\{ -2d\frac{T-2-r}{q(T)} \right\} \\
- \left[ \exp \left\{ \frac{d}{q(T)} \right\} - 1 \right]^{-1} \exp \left\{ -\frac{2dT}{q(T)} \right\} \exp \left\{ \frac{6d}{q(T)} \right\} \\
- \left[ \exp \left\{ \frac{d}{q(T)} \right\} - 1 \right]^{-2} \exp \left\{ -\frac{dT}{q(T)} \right\} \exp \left\{ \frac{4d}{q(T)} \right\} \\
+ \left[ \exp \left\{ \frac{d}{q(T)} \right\} - 1 \right]^{-2} \exp \left\{ -\frac{2dT}{q(T)} \right\} \exp \left\{ \frac{7d}{q(T)} \right\} \\
= \frac{q(T)}{d} \left[ 1 - \frac{d}{2q(T)} + O \left( \frac{1}{T^2} \right) \right] q(T) \frac{1}{2d} \left[ 1 - \exp \left\{ -\frac{2dT}{q(T)} \right\} \right] \left[ 1 + O \left( \frac{1}{T} \right) \right] \\
- \frac{q(T)}{d} \left[ 1 - \frac{d}{2q(T)} + O \left( \frac{1}{T^2} \right) \right] \exp \left\{ -\frac{2dT}{q(T)} \right\} \left[ 1 + \frac{6d}{q(T)} + O \left( \frac{1}{T^2} \right) \right] \\
- \frac{d}{q(T)} \frac{d^2}{2q(T)^2} + O \left( \frac{1}{q(T)^3} \right) \left[ 1 + \frac{4d}{q(T)} + O \left( \frac{1}{T^2} \right) \right] \\
+ \frac{d}{q(T)} \frac{d^2}{2q(T)^2} + O \left( \frac{1}{q(T)^3} \right) \left[ 1 + \frac{7d}{q(T)} + O \left( \frac{1}{T^2} \right) \right]
$$

as required for part (b).
Finally, consider part (c) where we take $q(T) \to \infty$ such that $q(T)/T \to 0$. In this case, we apply part (c) of Lemma SE-3 with $b = 2d$ and $g = 2$ to obtain

\[
\sum_{r=1}^{T-2} \sum_{s=1}^{T-2} \exp \left\{ -d \frac{T-2-r}{q(T)} \right\} \exp \left\{ -d \frac{T-2-s}{q(T)} \right\} = \left[ \exp \left\{ \frac{d}{q(T)} \right\} - 1 \right]^{-1} \sum_{r=1}^{T-2} \exp \left\{ -2d \frac{T-3-r}{q(T)} \right\} - \left[ \exp \left\{ \frac{d}{q(T)} \right\} - 1 \right]^{-2} \exp \left\{ -d \frac{T}{q(T)} \right\} \exp \left\{ \frac{4d}{q(T)} \right\} - \left[ \exp \left\{ \frac{d}{q(T)} \right\} - 1 \right]^{-2} \exp \left\{ -2d \frac{T}{q(T)} \right\} \exp \left\{ \frac{7d}{q(T)} \right\}
\]

\[
= \frac{q(T)}{d} \left[ 1 - \frac{d}{2q(T)} + O \left( \frac{1}{q(T)^2} \right) \right] \frac{q(T)}{2d} \left[ 1 + O \left( \frac{1}{q(T)} \right) \right] + O \left( q(T) \exp \left\{ - \frac{2dT}{q(T)} \right\} \right) + O \left( q(T)^2 \exp \left\{ - \frac{dT}{q(T)} \right\} \right) + O \left( q(T)^2 \exp \left\{ - \frac{2dT}{q(T)} \right\} \right)
\]

\[
= \frac{q(T)^2}{2d^2} \left[ 1 + O \left( \frac{1}{q(T)} \right) \right] + O \left( \exp \left\{ - \frac{dT}{q(T)} \right\} \right)
\]

\[
= O \left( q(T)^2 \right),
\]

as required, $\square$

Lemma SE-5:

(a) For $r = 0, 1, ..., T - 2$

\[
\sum_{t=r+2}^{T} (t - r - 1) \exp \{ x(t - r) \} = \exp \{ 2x \} - (T - r) \exp \{ x(T - r + 1) \} + (T - r - 1) \exp \{ x(T - r + 2) \} \frac{(1 - \exp \{ x \})^2}{(1 - \exp \{ x \})^2},
\]

for $x \in \mathbb{R}$

(b) Let $r = 0, 1, ..., T - 1$ and let $\alpha$ be a positive integer. In addition, suppose that $\rho \in [0, 1)$. Then,

\[
\sum_{t=r}^{T} (t - r) \rho^{\alpha(t-r)} = \rho^{\alpha} - (T - r + 1) \rho^{\alpha(T-r+1)} + (T - r) \rho^{\alpha(T-r+2)} \frac{(1 - \rho^{\alpha})^2}{(1 - \rho^{\alpha})^2}
\]

Proof of Lemma SE-5:
To show part (a), first set

\[
f(x) = \sum_{t=r+2}^{T} \exp \{x(t-r-1)\} = \sum_{t=r+1}^{T} \exp \{x(t-r-1)\} - 1
\]

\[
= \frac{1 - \exp \{x(T-r)\}}{1 - \exp \{x\}} - 1 \cdot \frac{1 - \exp \{x\}}{1 - \exp \{x\}} = \frac{\exp \{x\} - \exp \{x(T-r)\}}{1 - \exp \{x\}}
\]

Taking a derivative with respect to \(x\), we get

\[
\sum_{t=r+1}^{T} (t-r-1) \exp \{x(t-r-1)\} = f'(x)
\]

\[
= \frac{\exp \{x\} - (T-r) \exp \{x(T-r)\}}{1 - \exp \{x\}} + \frac{(\exp \{x\} - \exp \{x(T-r)\}) \exp \{x\}}{(1 - \exp \{x\})^2}
\]

\[
= (1 - \exp \{x\})^2 \left[ \exp \{x\} - \exp \{2x\} - (T-r) \exp \{x(T-r)\} + (T-r) \exp \{x(T-r+1)\}
\right.
\]

\[
+ \exp \{2x\} - \exp \{x(T-r+1)\}\right]
\]

\[
= \frac{\exp \{x\} - (T-r) \exp \{x(T-r)\} + (T-r-1) \exp \{x(T-r+1)\}}{(1 - \exp \{x\})^2}
\]

It follows that

\[
\sum_{t=r+2}^{T} (t-r-1) \exp \{x(t-r)\}
\]

\[
= \exp \{x\} \sum_{t=b+2}^{T} (t-r-1) \exp \{x(t-r-1)\}
\]

\[
= \exp \{x\} \left[ \exp \{x\} - (T-r) \exp \{x(T-r)\} + (T-r-1) \exp \{x(T-r+1)\} \right]
\]

\[
\frac{(1 - \exp \{x\})^2}{(1 - \exp \{x\})^2}
\]

\[
= \frac{\exp \{2x\} - (T-r) \exp \{x(T-r+1)\} + (T-r-1) \exp \{x(T-r+2)\}}{(1 - \exp \{x\})^2}.
\]

Next, to show part (b), set

\[
f(\rho) = \sum_{t=r}^{T} \rho^{\alpha(t-r)} = \frac{1 - \rho^{\alpha(T-r+1)}}{1 - \rho^{\alpha}}.
\]
Taking a derivative with respect to $\rho$, we get

$$\sum_{t=r}^{n} (t - r) \rho^{\alpha(t-r)} = \frac{\rho}{\alpha} f'(\rho)$$

$$= \alpha \left[ \frac{(1 - \rho^{\alpha(n-r+1)}) \alpha \rho^{\alpha-1}}{(1 - \rho^\alpha)^2} - \frac{\alpha (n-r+1) \rho^{\alpha(n-r+1)-1}}{1 - \rho^\alpha} \right]$$

$$= \frac{\rho}{\alpha} \left[ \frac{\alpha \rho^{\alpha-1} - \alpha \rho^{\alpha(n-r+2)-1} - \alpha (n-r+1) \rho^{\alpha(n-r+1)-1} (1 - \rho^\alpha)}{(1 - \rho^\alpha)^2} \right]$$

$$= \frac{\rho}{\alpha} \left[ \frac{\alpha \rho^{\alpha-1} - \alpha \rho^{\alpha(n-r+2)-1} - \alpha (n-r+1) (\rho^{\alpha(n-r+1)-1} - \rho^{\alpha(n-r+2)-1})}{(1 - \rho^\alpha)^2} \right]$$

$$= \frac{\rho}{\alpha} \left[ \frac{\alpha \rho^{\alpha-1} - \alpha \rho^{\alpha(n-r+2)-1} - \alpha (n-r+1) \rho^{\alpha(n-r+1)-1} + \alpha (n-r+1) \rho^{\alpha(n-r+2)-1}}{(1 - \rho^\alpha)^2} \right]$$

$$= \frac{\rho^\alpha - (n-r+1) \rho^{\alpha(n-r+1)} + (n-r) \rho^{\alpha(n-r+2)}}{(1 - \rho^\alpha)^2},$$

as required. \(\Box\)

**Lemma SE-6:**

Let $b$ be a positive integer, and $g \in \{1, 2\}$.

(a) If $q(T) \to \infty$ such that $q(T)/T \to 0$, then

$$\sum_{t=g+2}^{T} \sum_{s=g+1}^{t-1} \sum_{j=1}^{s-g} \exp \left\{ -b \frac{(t-g-j)}{q(T)} \right\} \exp \left\{ -b \frac{(s-g-j)}{q(T)} \right\} \times \sum_{k=1}^{s-g} \exp \left\{ -b \frac{(t-g-k)}{q(T)} \right\} \exp \left\{ -b \frac{(s-g-k)}{q(T)} \right\}$$

$$= \frac{1}{8} \frac{T q(T)^3}{b^3} \left[ 1 + O \left( \frac{1}{q(T)} \right) + O \left( \frac{q(T)}{T} \right) \right],$$

as $T \to \infty$. 

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(b) If \( q(T) \sim T \), then

\[
\sum_{t=g+2}^{T} \sum_{s=g+1}^{s-g} \sum_{j=1}^{s-g} \exp \left\{ -\frac{b(t-g-j)}{q(T)} \right\} \exp \left\{ -\frac{b(s-g-j)}{q(T)} \right\} \\
\times \sum_{k=1}^{s-g} \exp \left\{ -\frac{b(t-g-k)}{q(T)} \right\} \exp \left\{ -\frac{b(s-g-k)}{q(T)} \right\} \\
= \left\{ \frac{q(T)^4}{32b^4} \left[ \exp \left\{ -\frac{4bT}{q(T)} \right\} + 4(g-1) \exp \left\{ -\frac{2bT}{q(T)} \right\} - 5 \right] + \frac{Tq(T)^3}{8b^3} \left[ 2 \exp \left\{ -\frac{2bT}{q(T)} \right\} + 1 \right] \right\} \\
\times \left[ 1 + O \left( \frac{1}{T} \right) \right],
\]

as \( T \to \infty \).

**Proof of Lemma SE-6:**

To proceed, note first that

\[
\sum_{t=g+2}^{T} \sum_{s=g+1}^{s-g} \sum_{j=1}^{s-g} \exp \left\{ -\frac{b(t-g-j)}{q(T)} \right\} \exp \left\{ -\frac{b(s-g-j)}{q(T)} \right\} \\
\times \sum_{k=1}^{s-g} \exp \left\{ -\frac{b(t-g-k)}{q(T)} \right\} \exp \left\{ -\frac{b(s-g-k)}{q(T)} \right\} \\
= \sum_{t=g+2}^{T} \exp \left\{ -\frac{2b(t-g)}{q(T)} \right\} \exp \left\{ \frac{2b}{q(T)} \right\} \sum_{j=1}^{s-g} \exp \left\{ \frac{2b}{q(T)} (j-1) \right\} \\
\times \sum_{s=g+1}^{t-1} \exp \left\{ -\frac{2b(s-g)}{q(T)} \right\} \exp \left\{ \frac{2b}{q(T)} \right\} \sum_{k=1}^{s-g} \exp \left\{ \frac{2b}{q(T)} (k-1) \right\} \\
= \exp \left\{ \frac{4b}{q(T)} \right\} \sum_{t=g+2}^{T} \exp \left\{ -\frac{2b(t-g)}{q(T)} \right\} \sum_{j=1}^{s-g} \exp \left\{ \frac{2b}{q(T)} (j-1) \right\} \\
\times \sum_{s=g+1}^{t-1} \exp \left\{ -\frac{2b(s-g)}{q(T)} \right\} \sum_{k=1}^{s-g} \exp \left\{ \frac{2b}{q(T)} (k-1) \right\}
\]
where the last equality above follows from part (a) of Lemma SE-5. It follows by further calculation
that

\[
\sum_{t=g+2}^{T} \sum_{s=g+1}^{t-1} \sum_{j=1}^{s-g} \exp \left\{ -b \frac{(t-g-j)}{q(T)} \right\} \exp \left\{ -b \frac{(s-g-j)}{q(T)} \right\} \\
\times \sum_{k=1}^{s-g} \exp \left\{ -b \frac{(t-g-k)}{q(T)} \right\} \exp \left\{ -b \frac{(s-g-k)}{q(T)} \right\}
\]

\[
= \exp \left\{ -\frac{2b}{q(T)} \right\} \left[ 1 - \exp \left\{ \frac{2b}{q(T)} \right\} \right]^{-2} \left[ 1 - \exp \left\{ -\frac{2b}{q(T)} \right\} \right]^{-2} \left[ 1 - \exp \left\{ -\frac{2b(T-g-1)}{q(T)} \right\} \right]^{-1} \\
- \exp \left\{ -\frac{4b}{q(T)} \right\} \left[ 1 - \exp \left\{ \frac{2b}{q(T)} \right\} \right]^{-2} \left[ 1 - \exp \left\{ -\frac{2b}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{ -\frac{4b}{q(T)} \right\} \right]^{-1} \\
\times \left[ 1 - \exp \left\{ -\frac{4b(T-g-1)}{q(T)} \right\} \right] \\
-2 \exp \left\{ \frac{4b}{q(T)} \right\} \left[ 1 - \exp \left\{ \frac{2b}{q(T)} \right\} \right]^{-2} \left[ 1 - \exp \left\{ -\frac{2b}{q(T)} \right\} \right]^{-2} \exp \left\{ -\frac{4b}{q(T)} \right\} \\
+2 \exp \left\{ \frac{4b}{q(T)} \right\} \left[ 1 - \exp \left\{ \frac{2b}{q(T)} \right\} \right]^{-2} \left[ 1 - \exp \left\{ -\frac{2b}{q(T)} \right\} \right]^{-2} \\
\times (T-g) \exp \left\{ -\frac{2b}{q(T)} (T-g+1) \right\} \\
-2 \exp \left\{ \frac{4b}{q(T)} \right\} \left[ 1 - \exp \left\{ \frac{2b}{q(T)} \right\} \right]^{-2} \left[ 1 - \exp \left\{ -\frac{2b}{q(T)} \right\} \right]^{-2} \\
\times (T-g-1) \exp \left\{ -\frac{2b(T-g+2)}{q(T)} \right\} \\
+ \exp \left\{ \frac{2b}{q(T)} \right\} \left[ 1 - \exp \left\{ \frac{2b}{q(T)} \right\} \right]^{-3} \left[ 1 - \exp \left\{ -\frac{2b}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{ -\frac{2b(T-g-1)}{q(T)} \right\} \right] \\
- \exp \left\{ \frac{4b}{q(T)} \right\} \left[ 1 - \exp \left\{ \frac{2b}{q(T)} \right\} \right]^{-3} (T-g-1)
\]

Now, consider part (a), where we take \( q(T) \to \infty \) such that \( q(T)/T \to 0 \). In this case, we have
from the above calculations that

\[
\sum_{t=g+2}^{T} \sum_{s=g+1}^{t-1} \sum_{j=1}^{s-g} \exp \left\{ -\frac{b(t-g-j)}{q(T)} \right\} \exp \left\{ -\frac{b(s-g-j)}{q(T)} \right\} \\
\times \sum_{k=1}^{s-g} \exp \left\{ -\frac{b(t-g-k)}{q(T)} \right\} \exp \left\{ -\frac{b(s-g-k)}{q(T)} \right\} \\
= \exp \left\{ -\frac{2b}{q(T)} \right\} \left[ 1 - \exp \left\{ -\frac{2b}{q(T)} \right\} \right]^{-2} \left[ 1 - \exp \left\{ -\frac{2b}{q(T)} \right\} \right]^{-2} \left[ 1 - \exp \left\{ -\frac{2b(T-g-1)}{q(T)} \right\} \right] \\
\times \left[ 1 - \exp \left\{ -\frac{4b(T-g-1)}{q(T)} \right\} \right]^{-2} \\
-2 \left[ 1 - \exp \left\{ -\frac{2b}{q(T)} \right\} \right]^{-2} \left[ 1 - \exp \left\{ -\frac{2b}{q(T)} \right\} \right]^{-2} \\
-2 \exp \left\{ -\frac{4b}{q(T)} \right\} \left[ 1 - \exp \left\{ -\frac{2b}{q(T)} \right\} \right]^{-2} \left[ 1 - \exp \left\{ -\frac{2b}{q(T)} \right\} \right]^{-2} \\
\times (T-g) \exp \left\{ -\frac{2b(T-g+1)}{q(T)} \right\} \left[ 1 - \exp \left\{ -\frac{2b}{q(T)} \right\} \right]^{-1} \\
+2 \exp \left\{ -\frac{4b}{q(T)} \right\} \left[ 1 - \exp \left\{ -\frac{2b}{q(T)} \right\} \right]^{-2} \left[ 1 - \exp \left\{ -\frac{2b}{q(T)} \right\} \right]^{-2} \exp \left\{ -\frac{2b(T-g+2)}{q(T)} \right\} \\
+ \exp \left\{ -\frac{2b}{q(T)} \right\} \left[ 1 - \exp \left\{ -\frac{2b}{q(T)} \right\} \right]^{-3} \left[ 1 - \exp \left\{ -\frac{2b}{q(T)} \right\} \right]^{-3} \exp \left\{ -\frac{2b(T-g-1)}{q(T)} \right\} \\
- \exp \left\{ -\frac{4b}{q(T)} \right\} \left[ 1 - \exp \left\{ -\frac{2b}{q(T)} \right\} \right]^{-3} (T-g-1) \\
= \frac{1}{16} q(T)^4 \left[ 1 + O \left( \frac{1}{q(T)} \right) \right] \left[ 1 + O \left( \exp \left\{ -\frac{2b}{q(T)} \right\} \right) \right] \\
- \frac{1}{32} q(T)^4 \left[ 1 + O \left( \frac{1}{q(T)} \right) \right] \left[ 1 + O \left( \exp \left\{ -\frac{4b}{q(T)} \right\} \right) \right] \\
- \frac{1}{8} q(T)^4 \left[ 1 + O \left( \frac{1}{q(T)} \right) \right] + O \left( Tq(T)^3 \exp \left\{ -2b \frac{T}{q(T)} \right\} \right) + O \left( q(T)^4 \exp \left\{ -2b \frac{T}{q(T)} \right\} \right) \\
- \frac{1}{16} q(T)^4 \left[ 1 + O \left( \frac{1}{q(T)} \right) \right] \left[ 1 + O \left( \exp \left\{ -\frac{2b}{q(T)} \right\} \right) \right] \\
+ \frac{1}{8} Tq(T)^3 \left[ 1 + O \left( \frac{1}{q(T)} \right) \right] \left[ 1 + O \left( \frac{1}{T} \right) \right] \\
= \frac{1}{8} Tq(T)^3 \left[ 1 + O \left( \frac{1}{q(T)} \right) + O \left( \frac{q(T)}{T} \right) + O \left( \exp \left\{ -\frac{2b}{T} \frac{T}{q(T)} \right\} \right) \right] \\
= \frac{1}{8} Tq(T)^3 \left[ 1 + O \left( \frac{1}{q(T)} \right) + O \left( \frac{q(T)}{T} \right) \right],
\]
as required.
Next, we turn our attention to part (b), where we take \( q(T) \sim T \). In this case, from previous calculations we have

\[
\sum_{t=g+2}^{T} \sum_{s=g+1}^{T-1} \sum_{j=1}^{s-g} \exp \left\{ -b \frac{(t-g-j)}{q(T)} \right\} \exp \left\{ -b \frac{(s-g-j)}{q(T)} \right\} \\
\times \sum_{k=1}^{s-g} \exp \left\{ -b \frac{(t-g-k)}{q(T)} \right\} \exp \left\{ -b \frac{(s-g-k)}{q(T)} \right\} \\
= \exp \left\{ \frac{-2b}{q(T)} \right\} \left[ 1 - \exp \left\{ \frac{2b}{q(T)} \right\} \right]^{-2} \left[ 1 - \exp \left\{ -\frac{2b}{q(T)} \right\} \right]^{-2} \left[ 1 - \exp \left\{ -\frac{2b(T-g-1)}{q(T)} \right\} \right]^{-1} \\
- \exp \left\{ \frac{-4b}{q(T)} \right\} \left[ 1 - \exp \left\{ \frac{2b}{q(T)} \right\} \right]^{-2} \left[ 1 - \exp \left\{ -\frac{2b}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{ -\frac{4b}{q(T)} \right\} \right]^{-1} \\
\times \left[ 1 - \exp \left\{ -\frac{4b(T-g-1)}{q(T)} \right\} \right] \\
-2 \left[ 1 - \exp \left\{ \frac{2b}{q(T)} \right\} \right]^{-2} \left[ 1 - \exp \left\{ -\frac{2b}{q(T)} \right\} \right]^{-2} \\
+ 2 \exp \left\{ \frac{4b}{q(T)} \right\} \left[ 1 - \exp \left\{ \frac{2b}{q(T)} \right\} \right]^{-2} \left[ 1 - \exp \left\{ -\frac{2b}{q(T)} \right\} \right]^{-2} (T-g) \exp \left\{ -\frac{2b(T-g+1)}{q(T)} \right\} \\
- 2 \exp \left\{ \frac{4b}{q(T)} \right\} \left[ 1 - \exp \left\{ \frac{2b}{q(T)} \right\} \right]^{-2} \left[ 1 - \exp \left\{ -\frac{2b}{q(T)} \right\} \right]^{-2} (T-g-1) \exp \left\{ -\frac{2b(T-g+2)}{q(T)} \right\} \\
+ \exp \left\{ \frac{2b}{q(T)} \right\} \left[ 1 - \exp \left\{ \frac{2b}{q(T)} \right\} \right]^{-3} \left[ 1 - \exp \left\{ -\frac{2b}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{ -\frac{2b(T-g-1)}{q(T)} \right\} \right]^{-1} \\
- \exp \left\{ \frac{4b}{q(T)} \right\} \left[ 1 - \exp \left\{ \frac{2b}{q(T)} \right\} \right]^{-3} (T-g-1),
\]

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so that

\[
\sum_{t=g+2}^{T} \sum_{s=g+1}^{t-1} \sum_{j=1}^{s-g} \exp \left\{ -\frac{b(t - g - j)}{q(T)} \right\} \exp \left\{ -\frac{b(s - g - j)}{q(T)} \right\} \\
\times \sum_{k=1}^{s-g} \exp \left\{ -\frac{b(t - g - k)}{q(T)} \right\} \exp \left\{ -\frac{b(s - g - k)}{q(T)} \right\}
\]

\[
= \exp \left\{ -\frac{2b}{q(T)} \right\} \left[ 1 - \exp \left\{ \frac{2b}{q(T)} \right\} \right]^{-2} \left[ 1 - \exp \left\{ -\frac{2b}{q(T)} \right\} \right]^{-2} \\
\times \left[ 1 - \exp \left\{ -\frac{2bT}{q(T)} \right\} \exp \left\{ \frac{2b(g + 1)}{q(T)} \right\} \right] \\
- \exp \left\{ -\frac{2b}{q(T)} \right\} \left[ 1 - \exp \left\{ \frac{2b}{q(T)} \right\} \right]^{-2} \left[ 1 - \exp \left\{ -\frac{2b}{q(T)} \right\} \right]^{-2} \\
\times \left[ 1 - \exp \left\{ -\frac{2bT}{q(T)} \right\} \exp \left\{ \frac{4b(g + 1)}{q(T)} \right\} \right] \\
-2 \left[ 1 - \exp \left\{ \frac{2b}{q(T)} \right\} \right]^{-2} \left[ 1 - \exp \left\{ -\frac{2b}{q(T)} \right\} \right]^{-2} \\
\times (T - g) \exp \left\{ -\frac{2bT}{q(T)} \right\} \exp \left\{ \frac{2b(g - 1)}{q(T)} \right\} \\
-2 \exp \left\{ \frac{4b}{q(T)} \right\} \left[ 1 - \exp \left\{ \frac{2b}{q(T)} \right\} \right]^{-2} \left[ 1 - \exp \left\{ -\frac{2b}{q(T)} \right\} \right]^{-2} (T - g - 1) \exp \left\{ -\frac{2bT}{q(T)} \right\} \\
\times \exp \left\{ \frac{2b(g - 2)}{q(T)} \right\} \\
+ \exp \left\{ \frac{2b}{q(T)} \right\} \left[ 1 - \exp \left\{ \frac{2b}{q(T)} \right\} \right]^{-3} \left[ 1 - \exp \left\{ -\frac{2b}{q(T)} \right\} \right]^{-1} \\
\times \left[ 1 - \exp \left\{ -\frac{2bT}{q(T)} \right\} \exp \left\{ \frac{2b(g + 1)}{q(T)} \right\} \right] \\
- \exp \left\{ \frac{4b}{q(T)} \right\} \left[ 1 - \exp \left\{ \frac{2b}{q(T)} \right\} \right]^{-3} (T - g - 1)
\]
= \exp \left\{ -\frac{2b}{q(T)} \right\} \left[ 1 - \exp \left\{ \frac{2b}{q(T)} \right\} \right]^{-2} \left[ 1 - \exp \left\{ -\frac{2b}{q(T)} \right\} \right]^{-2} \\
\times \left[ 1 - \exp \left\{ -\frac{2bT}{q(T)} \right\} \exp \left\{ \frac{2b(g+1)}{q(T)} \right\} \right] \\
- \exp \left\{ -\frac{4b}{q(T)} \right\} \left[ 1 - \exp \left\{ \frac{2b}{q(T)} \right\} \right]^{-2} \left[ 1 - \exp \left\{ -\frac{2b}{q(T)} \right\} \right]^{-1} \\
\times \left[ 1 - \exp \left\{ -\frac{4b}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{ -\frac{4bT}{q(T)} \right\} \exp \left\{ \frac{4b(g+1)}{q(T)} \right\} \right] \\
- 2 \left[ 1 - \exp \left\{ \frac{2b}{q(T)} \right\} \right]^{-2} \left[ 1 - \exp \left\{ -\frac{2b}{q(T)} \right\} \right]^{-2} \\
+ 2T \left[ 1 + \frac{4b}{q(T)} + \frac{8b^2}{q(T)^2} + O \left( \frac{1}{T^3} \right) \right] \left[ -\frac{2b}{q(T)} - 2 \frac{b^2}{q(T)^2} + O \left( \frac{1}{T^3} \right) \right]^{-2} \\
\times \left[ \frac{2b}{q(T)} - \frac{2b^2}{q(T)^2} + O \left( \frac{1}{T^3} \right) \right]^{-2} \exp \left\{ -\frac{2bT}{q(T)} \right\} \\
\times \left[ 1 + \frac{2b(g-1)}{q(T)} + \frac{2b^2(g-1)^2}{q(T)^2} + O \left( \frac{1}{T^3} \right) \right] \\
- 2g \exp \left\{ \frac{4b}{q(T)} \right\} \left[ 1 - \exp \left\{ \frac{2b}{q(T)} \right\} \right]^{-2} \left[ 1 - \exp \left\{ -\frac{2b}{q(T)} \right\} \right]^{-2} \exp \left\{ -\frac{2bT}{q(T)} \right\} \\
\times \exp \left\{ \frac{2b(g-1)}{q(T)} \right\} \\
- 2T \left[ 1 + 4 \frac{b}{q(T)} + 8 \frac{b^2}{q(T)^2} + O \left( \frac{1}{T^3} \right) \right] \left[ -\frac{2b}{q(T)} - 2 \frac{b^2}{q(T)^2} + O \left( \frac{1}{T^3} \right) \right]^{-2} \\
\times \left[ \frac{2b}{q(T)} - \frac{2b^2}{q(T)^2} + O \left( \frac{1}{T^3} \right) \right]^{-2} \exp \left\{ -\frac{2bT}{q(T)} \right\} \exp \left\{ \frac{2b(g-2)}{q(T)} \right\} \\
+ 2(g+1) \exp \left\{ \frac{4b}{q(T)} \right\} \left[ 1 - \exp \left\{ \frac{2b}{q(T)} \right\} \right]^{-2} \left[ 1 - \exp \left\{ -\frac{2b}{q(T)} \right\} \right]^{-2} \\
\times \exp \left\{ -\frac{2bT}{q(T)} \right\} \exp \left\{ \frac{2b(g-2)}{q(T)} \right\} \\
+ \exp \left\{ \frac{2b}{q(T)} \right\} \left[ 1 - \exp \left\{ \frac{2b}{q(T)} \right\} \right]^{-3} \left[ 1 - \exp \left\{ -\frac{2b}{q(T)} \right\} \right]^{-1} \\
\times \left[ 1 - \exp \left\{ -\frac{2bT}{q(T)} \right\} \exp \left\{ \frac{2b(g+1)}{q(T)} \right\} \right] \\
- \exp \left\{ \frac{4b}{q(T)} \right\} \left[ 1 - \exp \left\{ \frac{2b}{q(T)} \right\} \right]^{-3} (T-g-1)
\[
\begin{align*}
&= \frac{1}{16} \frac{q(T)^4}{b^4} \left[ 1 - \exp \left\{ \frac{-2bT}{q(T)} \right\} \right] \left[ 1 + O \left( \frac{1}{T} \right) \right] \\
&\quad - \frac{1}{32} \frac{q(T)^4}{b^4} \left[ 1 - \exp \left\{ \frac{-4bT}{q(T)} \right\} \right] \left[ 1 + O \left( \frac{1}{T} \right) \right] - \frac{1}{8} \frac{q(T)^4}{b^4} \left[ 1 + O \left( \frac{1}{T} \right) \right] \\
&\quad + 2T \left[ 1 + \frac{4b}{q(T)} + \frac{8b^2}{q(T)^2} + O \left( \frac{1}{T^3} \right) \right] \left[ -\frac{2b}{q(T)} - \frac{2b^2}{q(T)^2} + O \left( \frac{1}{T^3} \right) \right]^{-2} \\
&\quad \times \left[ \frac{2b}{q(T)} - \frac{2b^2}{q(T)^2} + O \left( \frac{1}{T^3} \right) \right]^{-2} \exp \left\{ -\frac{2bT}{q(T)} \right\} \\
&\quad \times \left[ 1 + \frac{2b(g-1)}{q(T)} + \frac{2b^2(g-1)^2}{q(T)^2} + O \left( \frac{1}{T^3} \right) \right] \\
&\quad - \frac{g}{8} \frac{q(T)^4}{b^4} \exp \left\{ \frac{-2bT}{q(T)} \right\} \left[ 1 + O \left( \frac{1}{T} \right) \right] \\
&\quad - 2T \left[ 1 + 4 \frac{b}{q(T)} + 8 \frac{b^2}{q(T)^2} + O \left( \frac{1}{T^3} \right) \right] \left[ -\frac{2b}{q(T)} - 2 \frac{b^2}{q(T)^2} + O \left( \frac{1}{T^3} \right) \right]^{-2} \\
&\quad \times \left[ \frac{2b}{q(T)} - \frac{2b^2}{q(T)^2} + O \left( \frac{1}{T^3} \right) \right]^{-2} \exp \left\{ -\frac{2bT}{q(T)} \right\} \\
&\quad \times \left[ 1 + \frac{2b(g-2)}{q(T)} + \frac{2b^2(g-2)^2}{q(T)^2} + O \left( \frac{1}{T^3} \right) \right] \\
&\quad + \frac{(g+1)}{8} \frac{q(T)^4}{b^4} \exp \left\{ \frac{-2bT}{q(T)} \right\} \left[ 1 + O \left( \frac{1}{T} \right) \right] \\
&\quad - \frac{1}{16} \frac{q(T)^4}{b^4} \left[ 1 - \exp \left\{ \frac{2bT}{q(T)} \right\} \right] \left[ 1 + O \left( \frac{1}{T} \right) \right] \\
&\quad + \frac{1}{8} \frac{Tq(T)^3}{b^3} \left[ 1 + O \left( \frac{1}{T} \right) \right] - \frac{(g+1)}{8} \frac{q(T)^3}{b^3} \left[ 1 + O \left( \frac{1}{T} \right) \right]
\end{align*}
\]
\begin{align*}
&= \frac{1}{16} \frac{q(T)^4}{b^4} \left[ 1 - \exp \left\{ -\frac{2bT}{q(T)} \right\} \right] \left[ 1 + O \left( \frac{1}{T} \right) \right] \\
&\quad - \frac{1}{32} \frac{q(T)^4}{b^4} \left[ 1 - \exp \left\{ -\frac{4bT}{q(T)} \right\} \right] \left[ 1 + O \left( \frac{1}{T} \right) \right] - \frac{1}{8} \frac{q(T)^4}{b^4} \left[ 1 + O \left( \frac{1}{T} \right) \right] \\
&\quad + 2T \left[ 1 + \frac{4b}{q(T)} + \frac{8b^2}{q(T)^2} + O \left( \frac{1}{T^3} \right) \right] \frac{q(T)^2}{4b^2} \left[ 1 + \frac{b}{q(T)} + O \left( \frac{1}{T^2} \right) \right]^{-2} \\
&\quad \times \frac{q(T)^2}{4b^2} \left[ 1 - \frac{b}{q(T)} + O \left( \frac{1}{T^2} \right) \right]^{-2} \exp \left\{ -\frac{2bT}{q(T)} \right\} \\
&\quad \times \left[ 1 + \frac{2b(g-1)}{q(T)} + \frac{2b^2(g-1)^2}{q(T)^2} + O \left( \frac{1}{T^3} \right) \right] \\
&\quad - \frac{g q(T)^4}{8} \frac{1}{b^4} \exp \left\{ -\frac{2bT}{q(T)} \right\} \left[ 1 + O \left( \frac{1}{T} \right) \right] \\
&\quad - 2T \left[ 1 + \frac{4b}{q(T)} + \frac{8b^2}{q(T)^2} + O \left( \frac{1}{T^3} \right) \right] \frac{q(T)^2}{4b^2} \left[ 1 + \frac{b}{q(T)} + O \left( \frac{1}{T^2} \right) \right]^{-2} \\
&\quad \times \frac{q(T)^2}{4b^2} \left[ 1 - \frac{b}{q(T)} + O \left( \frac{1}{T^2} \right) \right]^{-2} \exp \left\{ -\frac{2bT}{q(T)} \right\} \left[ 1 + \frac{2b(g-2)}{q(T)} + \frac{2b^2(g-2)^2}{q(T)^2} + O \left( \frac{1}{T^3} \right) \right] \\
&\quad + \frac{(g+1) q(T)^4}{8} \frac{1}{b^4} \exp \left\{ -\frac{2bT}{q(T)} \right\} \left[ 1 + O \left( \frac{1}{T} \right) \right] \frac{1}{16} \frac{q(T)^4}{b^4} \left[ 1 - \exp \left\{ -\frac{2bT}{q(T)} \right\} \right] \left[ 1 + O \left( \frac{1}{T} \right) \right] \\
&\quad + \frac{1}{8} \frac{T q(T)^3}{b^3} \left[ 1 + O \left( \frac{1}{T} \right) \right] - \frac{(g+1) q(T)^3}{8} \frac{1}{b^3} \left[ 1 + O \left( \frac{1}{T} \right) \right]
\end{align*}
\[
\begin{align*}
&= \frac{1}{16} \frac{q(T)^4}{b^4} \left[ 1 - \exp \left\{ -\frac{2bT}{q(T)} \right\} \right] \left[ 1 + O \left( \frac{1}{T} \right) \right] \\
&\quad - \frac{1}{32} \frac{q(T)^4}{b^4} \left[ 1 - \exp \left\{ -\frac{4bT}{q(T)} \right\} \right] \left[ 1 + O \left( \frac{1}{T} \right) \right] - \frac{1}{8} \frac{q(T)^4}{b^4} \left[ 1 + O \left( \frac{1}{T} \right) \right] \\
&\quad + \frac{1}{8} \frac{Tq(T)^4}{b^4} \left[ 1 + \frac{4b}{q(T)} + \frac{8b^2}{q(T)^2} + O \left( \frac{1}{T^3} \right) \right] \left[ 1 - \frac{2b}{q(T)} + O \left( \frac{1}{T^2} \right) \right] \\
&\quad \times \left[ 1 + \frac{2b}{q(T)} + O \left( \frac{1}{T^2} \right) \right] \left[ 1 + \frac{2b(g-1)}{q(T)} + \frac{2b^2(g-1)^2}{q(T)^2} + O \left( \frac{1}{T^3} \right) \right] \\
&\quad \times \frac{g q(T)^4}{8 b^4} \exp \left\{ -\frac{2bT}{q(T)} \right\} \left[ 1 + O \left( \frac{1}{T} \right) \right] \\
&\quad - \frac{1}{8} \frac{Tq(T)^4}{b^4} \left[ 1 + 1 \frac{4b}{q(T)} + \frac{8b^2}{q(T)^2} + O \left( \frac{1}{T^3} \right) \right] \left[ 1 - \frac{2b}{q(T)} + O \left( \frac{1}{T^2} \right) \right] \\
&\quad \times \left[ 1 + \frac{2b}{q(T)} + O \left( \frac{1}{T^2} \right) \right] \left[ 1 + \frac{2b(g-2)}{q(T)} + \frac{2b^2(g-2)^2}{q(T)^2} + O \left( \frac{1}{T^3} \right) \right] \\
&\quad \times \frac{(g+1) q(T)^4}{8 b^4} \exp \left\{ -\frac{2bT}{q(T)} \right\} \left[ 1 + O \left( \frac{1}{T} \right) \right] \\
&\quad - \frac{1}{16} \frac{q(T)^4}{b^4} \left[ 1 - \exp \left\{ -\frac{2bT}{q(T)} \right\} \right] \left[ 1 + O \left( \frac{1}{T} \right) \right] \\
&\quad - \frac{1}{32} \frac{q(T)^4}{b^4} \left[ 1 - \exp \left\{ -\frac{4bT}{q(T)} \right\} \right] \left[ 1 + O \left( \frac{1}{T} \right) \right] - \frac{1}{8} \frac{q(T)^4}{b^4} \left[ 1 + O \left( \frac{1}{T} \right) \right] \\
&\quad + \frac{1}{8} \frac{Tq(T)^4}{b^4} \left[ 1 + \frac{4b}{q(T)} + O \left( \frac{1}{T^2} \right) \right] \left[ 1 + \frac{2bg}{q(T)} + O \left( \frac{1}{T^2} \right) \right] \\
&\quad \exp \left\{ -\frac{2bT}{q(T)} \right\} \\
&\quad - \frac{g q(T)^4}{8 b^4} \exp \left\{ -\frac{2bT}{q(T)} \right\} \left[ 1 + O \left( \frac{1}{T} \right) \right] \\
&\quad - \frac{1}{8} \frac{Tq(T)^4}{b^4} \left[ 1 + \frac{4b}{q(T)} + O \left( \frac{1}{T^2} \right) \right] \left[ 1 + \frac{2bg}{q(T)} + O \left( \frac{1}{T^2} \right) \right] \\
&\quad \exp \left\{ -\frac{2bT}{q(T)} \right\} \\
&\quad + \frac{1}{8} \frac{Tq(T)^3}{b^3} \left[ 1 + O \left( \frac{1}{T} \right) \right] - \frac{(g+1) q(T)^3}{8 b^3} \left[ 1 + O \left( \frac{1}{T} \right) \right]
\end{align*}
\]
Let $b$ and $g$ be a positive integer. Then, the following statements are true as $T \to \infty$
(a) If $T/q(T) \to 0$, then
\[
\sum_{t=g+2}^{T} \sum_{s=g+1}^{t-1} \sum_{j=1}^{s-g} \exp \left\{ -b \frac{(t-g-j)}{q(T)} \right\} \exp \left\{ -b \frac{(s-g-j)}{q(T)} \right\} = \frac{1}{6} T^3 \left[ 1 + O \left( \max \left\{ \frac{T}{q(T)}, \frac{1}{T} \right\} \right) \right].
\]

(b) If $q(T) \sim T$, then
\[
\sum_{t=g+2}^{T} \sum_{s=g+1}^{t-1} \sum_{j=1}^{s-g} \exp \left\{ -b \frac{(t-g-j)}{q(T)} \right\} \exp \left\{ -b \frac{(s-g-j)}{q(T)} \right\} = \frac{Tq(T)^2}{2b^2} \left[ 1 - \frac{3q(T)}{2b} + \frac{2q(T)}{b} \exp \left\{ -\frac{bT}{q(T)} \right\} - \frac{1}{2b} \frac{q(T)}{T} \exp \left\{ -\frac{2bT}{q(T)} \right\} \right] \left[ 1 + O \left( \frac{1}{T} \right) \right].
\]

(c) If $q(T) \to \infty$ such that $q(T)/T \to 0$, then
\[
\sum_{t=g+2}^{T} \sum_{s=g+1}^{t-1} \sum_{j=1}^{s-g} \exp \left\{ -b \frac{(t-g-j)}{q(T)} \right\} \exp \left\{ -b \frac{(s-g-j)}{q(T)} \right\} = \frac{Tq(T)^2}{2b^2} \left[ 1 + O \left( \frac{q(T)}{T} \right) \right].
\]

**Proof of Lemma SE-7:**

To proceed, note first that
\[
\sum_{t=g+2}^{T} \sum_{s=g+1}^{t-1} \sum_{j=1}^{s-g} \exp \left\{ -b \frac{(t-g-j)}{q(T)} \right\} \exp \left\{ -b \frac{(s-g-j)}{q(T)} \right\} = \sum_{t=g+2}^{T} \sum_{s=g+1}^{t-1} \exp \left\{ -\frac{2b}{q(T)} \right\} \exp \left\{ -\frac{b(s-g)}{q(T)} \right\} \sum_{j=1}^{s-g} \exp \left\{ 2b \left( \frac{j-1}{q(T)} \right) \right\}
\]
\[
= \left[ 1 - \exp \left\{ \frac{2b}{q(T)} \right\} \right]^{-1} \sum_{t=g+2}^{T} \sum_{s=g+1}^{t-1} \left[ \exp \left\{ -\frac{b(t-g)}{q(T)} \right\} \exp \left\{ -\frac{b(s-g)}{q(T)} \right\} \right] \times \exp \left\{ \frac{2b}{q(T)} \right\} \left( 1 - \exp \left\{ 2b \frac{s-g}{q(T)} \right\} \right) \right]^{-1} \sum_{t=g+2}^{T} \exp \left\{ -\frac{b(t-g)}{q(T)} \right\}
\]
\[
\times \sum_{s=g+1}^{t-1} \exp \left\{ -\frac{b(s-g-1)}{q(T)} \right\} - \exp \left\{ \frac{2b}{q(T)} \right\} \exp \left\{ \frac{b}{q(T)} \right\} \left[ 1 - \exp \left\{ \frac{2b}{q(T)} \right\} \right]^{-1} \sum_{t=g+2}^{T} \exp \left\{ -\frac{b(t-g)}{q(T)} \right\}
\]
\[
\times \sum_{s=g+1}^{t-1} \exp \left\{ \frac{b(s-g-1)}{q(T)} \right\}
\]
\[
\begin{align*}
&= \exp \left\{ \frac{b}{q(T)} \right\} \left[ 1 - \exp \left\{ \frac{2b}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{ -\frac{b}{q(T)} \right\} \right]^{-1} \\
&\quad \times \sum_{t=g+2}^{T} \exp \left\{ -\frac{b(t-g)}{q(T)} \right\} \left[ 1 - \exp \left\{ -\frac{b(t-g-1)}{q(T)} \right\} \right]^{-1} \\
&\quad - \exp \left\{ \frac{3b}{q(T)} \right\} \left[ 1 - \exp \left\{ \frac{2b}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{ \frac{b}{q(T)} \right\} \right]^{-1} \\
&\quad \times \sum_{t=g+2}^{T} \exp \left\{ -\frac{b(t-g)}{q(T)} \right\} \left[ 1 - \exp \left\{ \frac{b(t-g-1)}{q(T)} \right\} \right]^{-1} \\
&\quad - \exp \left\{ \frac{3b}{q(T)} \right\} \left[ 1 - \exp \left\{ \frac{2b}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{ \frac{b}{q(T)} \right\} \right]^{-1} \\
&\quad \times \left[ \exp \left\{ -\frac{2b}{q(T)} \right\} \sum_{t=g+2}^{T} \exp \left\{ -\frac{b(t-g-2)}{q(T)} \right\} \right] - \exp \left\{ -\frac{b}{q(T)} \right\} (T-g-1)
\end{align*}
\]
\[
\begin{align*}
&= \left[ 1 - \exp \left\{ \frac{2b}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{ -\frac{b}{q(T)} \right\} \right]^{-2} \exp \left\{ -\frac{b}{q(T)} \right\} \\
&\quad - \left[ 1 - \exp \left\{ \frac{2b}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{ -\frac{b}{q(T)} \right\} \right]^{-2} \exp \left\{ -\frac{b(T - g)}{q(T)} \right\} \\
&\quad - \left[ 1 - \exp \left\{ \frac{2b}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{ -\frac{b}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{ -\frac{2b}{q(T)} \right\} \right]^{-1} \exp \left\{ -\frac{2b}{q(T)} \right\} \\
&\quad + \left[ 1 - \left\{ \exp \frac{2b}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{ -\frac{b}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{ -\frac{2b}{q(T)} \right\} \right]^{-1} \\
&\quad \times \exp \left\{ -\frac{2b(T - g)}{q(T)} \right\} \\
&\quad - \left[ 1 - \exp \left\{ \frac{2b}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{ \frac{b}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{ -\frac{b}{q(T)} \right\} \right]^{-1} \exp \left\{ \frac{b}{q(T)} \right\} \\
&\quad + \left[ 1 - \exp \left\{ \frac{2b}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{ \frac{b}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{ -\frac{b}{q(T)} \right\} \right]^{-1} \\
&\quad \times \exp \left\{ -\frac{b(T - g - 2)}{q(T)} \right\} \\
&\quad + \exp \left\{ \frac{2b}{q(T)} \right\} \left[ 1 - \exp \left\{ \frac{2b}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{ \frac{b}{q(T)} \right\} \right]^{-1} (T - g - 1)
\end{align*}
\]
Next, note that
\[
\begin{align*}
&\left[1 - \exp\left(\frac{2b}{q(T)}\right)\right]^{-1} \left[1 - \exp\left(-\frac{b}{q(T)}\right)\right]^{-2} \\
&= \left[-\frac{2b}{q(T)} - \frac{4b^2}{2q(T)^2} - \frac{8}{6} \frac{b^3}{q(T)^3} - \frac{16}{24} \frac{b^4}{q(T)^4} + O\left(\frac{1}{q(T)^5}\right)\right]^{-1} \\
&\times \left[-\frac{b}{q(T)} + \frac{b^2}{2q(T)^2} - \frac{1}{6} \frac{b^3}{q(T)^3} + \frac{1}{24} \frac{b^4}{q(T)^4} + O\left(\frac{1}{q(T)^5}\right)\right]^{-2} \\
&= -\frac{q(T)^3}{2b^3} \left[1 - \frac{b}{q(T)} - \frac{2}{3} \frac{b^2}{q(T)^2} - \frac{3}{4} \frac{b^3}{q(T)^3} - \frac{3}{2} \frac{b^4}{q(T)^4} + O\left(\frac{1}{q(T)^5}\right)\right] \\
&\times \left[1 - \frac{b}{2q(T)} - \frac{3}{6} \frac{b^2}{q(T)^2} - \frac{1}{24} \frac{b^3}{q(T)^3} + O\left(\frac{1}{q(T)^4}\right)\right]^{-2} \\
&= -\frac{q(T)^3}{2b^3} \left[1 - \frac{b}{q(T)} + \frac{1}{3} \frac{b^2}{q(T)^2} + O\left(\frac{1}{q(T)^4}\right)\right] \\
&\times \left[1 + \frac{b}{q(T)} + \frac{5}{12} \frac{b^2}{q(T)^2} + \frac{1}{24} \frac{b^3}{q(T)^3} + O\left(\frac{1}{q(T)^4}\right)\right] \\
&= \frac{q(T)^3}{2b^3c^3} \left[1 + \frac{b}{q(T)} + \frac{b}{q(T)} + \frac{5}{12} \frac{b^2}{q(T)^2} - \frac{b^2}{q(T)^2} + \frac{1}{3} \frac{b^3}{q(T)^3} + O\left(\frac{1}{q(T)^4}\right)\right] \\
&\times -\frac{5}{12} \frac{b^3}{q(T)^3} + \frac{1}{3} \frac{b^3}{q(T)^3} + O\left(\frac{1}{q(T)^4}\right) \\
&= -\frac{q(T)^3}{2b^3} \left[1 - \frac{1}{4} \frac{b^2}{q(T)^2} + O\left(\frac{1}{q(T)^4}\right)\right]
\end{align*}
\]
\[
\left[1 - \exp \left\{ \frac{2b}{q(T)} \right\} \right]^{-1} \left[1 - \exp \left\{ -\frac{b}{q(T)} \right\} \right]^{-1} \left[1 - \exp \left\{ -\frac{2b}{q(T)} \right\} \right]^{-1}
\]
\[
= \left[ -\frac{2b}{q(T)} - \frac{4b^2}{2q(T)^2} - \frac{8b^3}{6q(T)^3} - \frac{16b^4}{24q(T)^4} + O \left( \frac{1}{q(T)^5} \right) \right]^{-1}
\times \left[ -\frac{b}{q(T)} + \frac{b^2}{2q(T)^2} - \frac{b^3}{6q(T)^3} + \frac{1}{24q(T)^4} + O \left( \frac{1}{q(T)^5} \right) \right]^{-1}
\times \left[ -\frac{2b}{q(T)} + \frac{4b^2}{2q(T)^2} - \frac{8b^3}{6q(T)^3} + \frac{16b^4}{24q(T)^4} + O \left( \frac{1}{q(T)^5} \right) \right]^{-1}
\]
\[
= -\frac{q(T)^3}{4b^3} \left[ 1 + \frac{b}{q(T)} + \frac{2b^2}{3q(T)^2} + \frac{1}{3q(T)^3} + O \left( \frac{1}{q(T)^4} \right) \right]^{-1}
\times \left[ 1 - \frac{b}{2q(T)} + \frac{b^2}{6q(T)^2} - \frac{1}{24q(T)^3} + O \left( \frac{1}{q(T)^4} \right) \right]^{-1}
\times \left[ 1 - \frac{b}{q(T)} + \frac{2b^2}{3q(T)^2} - \frac{1}{3q(T)^3} + O \left( \frac{1}{q(T)^4} \right) \right]^{-1}
\]
\[
= -\frac{q(T)^3}{4b^3} \left[ 1 - \frac{b}{q(T)} - \frac{2b^2}{3q(T)^2} + \frac{b^3}{3q(T)^3} - \frac{1}{3q(T)^4} + \frac{4}{q(T)^5} + O \left( \frac{1}{q(T)^6} \right) \right]
\times \left[ 1 + \frac{b}{2q(T)} - \frac{1}{6q(T)^2} + \frac{2}{q(T)^3} + \frac{1}{24q(T)^4} + \frac{8}{q(T)^5} - \frac{1}{6q(T)^6} + \frac{1}{q(T)^7} + O \left( \frac{1}{q(T)^8} \right) \right]
\times \left[ 1 + \frac{b}{q(T)} - \frac{2b^2}{3q(T)^2} + \frac{1}{3q(T)^3} - \frac{4}{3q(T)^4} + \frac{b^3}{q(T)^5} + O \left( \frac{1}{q(T)^6} \right) \right]^{-1}
\]
\[
= -\frac{q(T)^3}{4b^3} \left[ 1 - \frac{b}{q(T)} + \frac{1}{3q(T)^2} + O \left( \frac{1}{q(T)^3} \right) \right] \left[ 1 + \frac{b}{2q(T)} + \frac{1}{12q(T)^2} + O \left( \frac{1}{q(T)^3} \right) \right]^{-1}
\times \left[ 1 + \frac{b}{q(T)} + \frac{1}{3q(T)^2} + O \left( \frac{1}{q(T)^3} \right) \right]^{-1}
\]
\[
= -\frac{q(T)^3}{4b^3} \left[ 1 - \frac{b}{q(T)} + \frac{b}{2q(T)} + \frac{1}{12q(T)^2} + \frac{b^2}{3q(T)^2} + \frac{b^3}{3q(T)^3} - \frac{1}{2q(T)^2} - \frac{1}{12q(T)^3} + \frac{1}{6q(T)^3} + O \left( \frac{1}{q(T)^4} \right) \right]
\times \left[ 1 + \frac{b}{q(T)} + \frac{b}{12q(T)^2} + \frac{1}{12q(T)^3} + O \left( \frac{1}{q(T)^4} \right) \right]^{-1}
\]
\[
= -\frac{q(T)^3}{4b^3} \left[ 1 - \frac{b}{2q(T)} + \frac{1}{12q(T)^2} + \frac{b^2}{3q(T)^2} + \frac{b^3}{3q(T)^3} - \frac{1}{2q(T)^2} - \frac{1}{12q(T)^3} + \frac{1}{6q(T)^3} + O \left( \frac{1}{q(T)^4} \right) \right]
\times \left[ 1 + \frac{b}{q(T)} + \frac{b}{12q(T)^2} + \frac{1}{12q(T)^3} + O \left( \frac{1}{q(T)^4} \right) \right]^{-1}
\]
\[
= -\frac{q(T)^3}{4b^3} \left[ 1 + \frac{b}{q(T)} - \frac{1}{4q(T)^2} - \frac{1}{6q(T)^3} + O \left( \frac{1}{q(T)^4} \right) \right]
\]
\[
\left[ 1 - \exp \left\{ \frac{2b}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{ \frac{b}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{ -\frac{b}{q(T)} \right\} \right]^{-1}
\]
\[
\times \left[ -\frac{2b}{q(T)} - \frac{4b^2}{2q(T)^2} - \frac{8b^3}{6q(T)^3} - \frac{16b^4}{24q(T)^4} + O \left( \frac{1}{q(T)^2} \right) \right]^{-1}
\]
\[
\times \left[ -\frac{b}{q(T)} - \frac{b^2}{2q(T)^2} - \frac{b^3}{6q(T)^3} - \frac{1}{24q(T)^3} + O \left( \frac{1}{q(T)^4} \right) \right]^{-1}
\]
\[
\times \left[ -\frac{b}{q(T)} + \frac{b^2}{2q(T)^2} - \frac{b^3}{6q(T)^3} + \frac{1}{24q(T)^3} + O \left( \frac{1}{q(T)^5} \right) \right]^{-1}
\]
\[
= \frac{q(T)^3}{2b^3} \left[ 1 + \frac{b}{q(T)} + \frac{2b^2}{3q(T)^2} + \frac{1}{3q(T)^3} + O \left( \frac{1}{q(T)^4} \right) \right]^{-1}
\]
\[
\times \left[ 1 - \frac{b}{2q(T)} + \frac{b^2}{6q(T)^2} + \frac{1}{24q(T)^3} + O \left( \frac{1}{q(T)^4} \right) \right]^{-1}
\]
\[
\times \left[ 1 - \frac{b}{2q(T)} + \frac{b^2}{6q(T)^2} - \frac{1}{24q(T)^3} + O \left( \frac{1}{q(T)^4} \right) \right]^{-1}
\]
\[
= \frac{q(T)^3}{2b^3} \left[ 1 - \frac{b}{q(T)} - \frac{2b^2}{3q(T)^2} + \frac{b^2}{q(T)^2} - \frac{1}{3q(T)^3} + \frac{4}{3q(T)^3} - \frac{b^3}{q(T)^3} + O \left( \frac{1}{q(T)^4} \right) \right]
\]
\[
\times \left[ 1 - \frac{b}{2q(T)} - \frac{1}{6q(T)^2} + \frac{1}{4q(T)^2} - \frac{1}{24q(T)^3} + \frac{1}{8q(T)^3} + \frac{1}{6q(T)^3} + \frac{1}{8q(T)^3} + O \left( \frac{1}{q(T)^4} \right) \right]
\]
\[
\times \left[ 1 + \frac{b}{2q(T)} - \frac{1}{6q(T)^2} + \frac{1}{4q(T)^2} + \frac{1}{24q(T)^3} - \frac{1}{6q(T)^3} + \frac{1}{8q(T)^3} + O \left( \frac{1}{q(T)^4} \right) \right]
\]
\[
= \frac{q(T)^3}{2b^3} \left[ 1 - \frac{b}{q(T)} + \frac{1}{3q(T)^2} + O \left( \frac{1}{q(T)^4} \right) \right] \left[ 1 - \frac{b}{2q(T)} + \frac{1}{12q(T)^2} + O \left( \frac{1}{q(T)^4} \right) \right]
\]
\[
\times \left[ 1 + \frac{b}{2q(T)} + \frac{1}{12q(T)^2} + O \left( \frac{1}{q(T)^4} \right) \right]
\]
\[
= \frac{q(T)^3}{2b^3} \left[ 1 - \frac{b}{q(T)} - \frac{b}{2q(T)} + \frac{1}{12q(T)^2} + \frac{1}{3q(T)^2} + \frac{1}{2q(T)^2} - \frac{1}{12q(T)^3} \right.
\]
\[
- \frac{1}{6q(T)^3} + O \left( \frac{1}{q(T)^4} \right) \left[ 1 + \frac{b}{2q(T)} + \frac{1}{12q(T)^2} + O \left( \frac{1}{q(T)^4} \right) \right]
\]
\[ \frac{q(T)^3}{2b^3} \left[ 1 - \frac{3}{2q(T)} + \frac{b^2}{12q(T)^2} - \frac{1}{4q(T)^3} + O \left( \frac{1}{q(T)^4} \right) \right] \]
\[ \times \left[ 1 + \frac{b}{2q(T)} + \frac{b^2}{12q(T)^2} + O \left( \frac{1}{q(T)^4} \right) \right] \]
\[ = \frac{q(T)^3}{2b^3} \left[ 1 + \frac{1}{2q(T)} - \frac{3}{2q(T)} + \frac{11}{12q(T)^2} - \frac{3}{4q(T)^2} + \frac{1}{12q(T)^3} - \frac{1}{8q(T)^3} \right. \]
\[ + \frac{11}{24q(T)^3} \left. - \frac{1}{4q(T)^3} + O \left( \frac{1}{q(T)^4} \right) \right] \]
\[ = \frac{q(T)^3}{2b^3} \left[ 1 - \frac{b}{q(T)} + \frac{1}{4q(T)^2} + \frac{1}{12q(T)^3} + O \left( \frac{1}{q(T)^4} \right) \right], \]
\[
\begin{align*}
1 - \exp \left\{ \frac{2b}{q(T)} \right\} & \quad 1 - \exp \left\{ \frac{b}{q(T)} \right\} \\
= \left[ -\frac{2b}{q(T)} - \frac{4b^2}{2q(T)^2} - \frac{8}{6q(T)^3} - \frac{16}{24q(T)^4} + O \left( \frac{1}{q(T)^5} \right) \right]^{-1} \\
\times \left[ -\frac{b}{q(T)} - \frac{b^2}{2q(T)^2} - \frac{b^3}{6q(T)^3} - \frac{1}{24q(T)^4} + O \left( \frac{1}{q(T)^5} \right) \right]^{-1} \\
= \frac{q(T)^2}{2b^2} \left[ 1 + \frac{b}{q(T)} + \frac{2}{3q(T)^2} + \frac{1}{3q(T)^3} + O \left( \frac{1}{q(T)^4} \right) \right]^{-1} \\
\times \left[ 1 + \frac{b}{2q(T)} + \frac{b^2}{6q(T)^2} + \frac{1}{24q(T)^3} + O \left( \frac{1}{q(T)^4} \right) \right]^{-1} \\
= \frac{q(T)^2}{2b^2} \left[ 1 - \frac{b}{q(T)} - \frac{2}{3q(T)^2} + \frac{b^2}{q(T)^2} - \frac{1}{3q(T)^3} + \frac{4}{3q(T)^3} - \frac{3}{8q(T)^3} + \frac{1}{6q(T)^3} + O \left( \frac{1}{q(T)^4} \right) \right] \\
\times \left[ 1 - \frac{b}{2q(T)} - \frac{1}{6q(T)^2} + \frac{b^2}{4q(T)^2} - \frac{1}{24q(T)^3} + \frac{3}{8q(T)^3} + \frac{1}{6q(T)^3} + O \left( \frac{1}{q(T)^4} \right) \right] \\
= \frac{q(T)^2}{2b^2} \left[ 1 - \frac{b}{q(T)} + \frac{1}{3q(T)^2} + O \left( \frac{1}{q(T)^3} \right) \right] \left[ 1 - \frac{b}{2q(T)} + \frac{1}{12q(T)^2} + O \left( \frac{1}{q(T)^3} \right) \right] \\
= \frac{q(T)^2}{2b^2} \left[ 1 - \frac{3}{2q(T)} + \frac{11}{12q(T)^2} - \frac{1}{4q(T)^3} + O \left( \frac{1}{q(T)^4} \right) \right],
\end{align*}
\]
\[
\exp \left\{ -\frac{b(T - g)}{q(T)} \right\} \\
= \exp \left\{ \frac{gb}{q(T)} \right\} \exp \left\{ -\frac{bT}{q(T)} \right\} \\
= \left[ 1 + \frac{gb}{q(T)} + \frac{1}{2} \frac{g^2b^2}{q(T)^2} + \frac{1}{3} \frac{g^3b^3}{q(T)^3} + O \left( \frac{1}{q(T)^4} \right) \right] \\
\times \left[ 1 - \frac{bT}{q(T)} + \frac{b^2T^2}{2q(T)^2} - \frac{1}{6} \frac{b^3T^3}{q(T)^3} + O \left( \frac{T^4}{q(T)^4} \right) \right] \\
= 1 - \frac{bT}{q(T)} + \frac{gb}{q(T)} + \frac{b^2T^2}{2q(T)^2} - \frac{gb^2T}{q(T)^2} - \frac{1}{6} \frac{b^3T^3}{q(T)^3} + \frac{1}{2} \frac{g^2b^2}{q(T)^2} + \frac{g}{2} \frac{b^3T^2}{q(T)^3} \\
- \frac{1}{2} \frac{g^2b^2}{q(T)^3} + \frac{1}{6} \frac{g^3b^3}{q(T)^3} + O \left( \frac{T^4}{q(T)^4} \right) \\
\exp \left\{ -\frac{2b(T - g)}{q(T)} \right\} \\
= \exp \left\{ \frac{2gb}{q(T)} \right\} \exp \left\{ -\frac{2bT}{q(T)} \right\} \\
= \left[ 1 + \frac{2gb}{q(T)} + \frac{4g^2}{2} \frac{b^2}{q(T)^2} + \frac{4g^3}{3} \frac{b^3}{q(T)^3} + O \left( \frac{1}{q(T)^4} \right) \right] \\
\times \left[ 1 - \frac{2bT}{q(T)} + \frac{2b^2T^2}{q(T)^2} - \frac{4}{3} \frac{b^3T^3}{q(T)^3} + O \left( \frac{T^4}{q(T)^4} \right) \right] \\
= 1 - \frac{2bT}{q(T)} + \frac{2gb}{q(T)} + \frac{2b^2T^2}{q(T)^2} - \frac{4g}{3} \frac{b^2T}{q(T)^2} - \frac{4}{3} \frac{b^3T^3}{q(T)^3} + \frac{2g^2b^2}{q(T)^2} + \frac{4g}{3} \frac{b^3T^2}{q(T)^3} \\
- \frac{4g^2}{3} \frac{b^3T}{q(T)^3} + \frac{4g^3}{3} \frac{b^3}{q(T)^3} + O \left( \frac{T^4}{q(T)^4} \right) \\
\exp \left\{ -\frac{b(T - g - 2)}{q(T)} \right\} \\
= \exp \left\{ \frac{(g + 2)b}{q(T)} \right\} \exp \left\{ -\frac{bT}{q(T)} \right\} \\
= \left[ 1 + (g + 2) \frac{b}{q(T)} + \frac{(g + 2)^2}{2} \frac{b^2}{q(T)^2} + \frac{(g + 2)^3}{6} \frac{b^3}{q(T)^3} + O \left( \frac{1}{q(T)^4} \right) \right] \\
\times \left[ 1 - \frac{bT}{q(T)} + \frac{b^2T^2}{2q(T)^2} - \frac{1}{6} \frac{b^3T^3}{q(T)^3} + O \left( \frac{T^4}{q(T)^4} \right) \right] \\
= 1 - \frac{bT}{q(T)} + \frac{(g + 2)b}{q(T)} + \frac{1}{2} \frac{b^2T^2}{q(T)^2} - \frac{(g + 2)}{2} \frac{b^2T}{q(T)^2} - \frac{1}{6} \frac{b^3T^3}{q(T)^3} + \frac{(g + 2)^2}{2} \frac{b^2}{q(T)^2} \\
+ \frac{g + 2}{2} \frac{b^3T^2}{q(T)^3} - \frac{(g + 2)^2}{2} \frac{b^3T}{q(T)^3} + \frac{(g + 2)^3}{6} \frac{b^3}{q(T)^3} + O \left( \frac{T^4}{q(T)^4} \right).
It follows that

\[
\sum_{t=g+2}^{T} \sum_{j=1}^{s-g} \exp \left\{ -b \left( t - g - j \right) \right\} \exp \left\{ -b \left( s - g - j \right) \right\} =
\]

\[
- q(T)^3 \left[ 1 - \frac{1}{4} \frac{b^2}{q(T)^2} + O \left( \frac{1}{q(T)^4} \right) \right] \left[ 1 - \frac{b T}{q(T)} + \frac{b^2}{2 q(T)^2} - \frac{b^3}{6 q(T)^3} + O \left( \frac{1}{q(T)^4} \right) \right]
\]

\[
+ q(T)^3 \left[ 1 - \frac{1}{4} \frac{b^2}{q(T)^2} + O \left( \frac{1}{q(T)^4} \right) \right] \left[ 1 - \frac{2 b T}{q(T)} + \frac{2 g b}{q(T)} + \frac{b^2 T^2}{2 q(T)^2} - \frac{gb^2 T}{q(T)^2} - \frac{1}{6 q(T)^3} \right]
\]

\[
+ 1 - \frac{q(T)^3}{4 b^2} \left[ 1 + \frac{b}{2 q(T)} - \frac{1}{4} \frac{b^2}{q(T)^2} - \frac{1}{6} \frac{b^3}{q(T)^3} + O \left( \frac{1}{q(T)^4} \right) \right] \left[ 1 - \frac{2 b T}{q(T)} + \frac{2 g b}{q(T)} + \frac{b^2 T^2}{2 q(T)^2} \right]
\]

\[
- \frac{4 g b^2 T}{q(T)^2} - \frac{4 b^3 T^3}{3 q(T)^3} + \frac{2 g^2 b^2}{q(T)^2} + \frac{4 g b^3 T^2}{q(T)^3} - 4 g^2 b^3 T + \frac{4 g^3 b^3}{3 q(T)^3} + O \left( \frac{T^4}{q(T)^4} \right)
\]

\[
- \frac{q(T)^3}{2 b^2} \left[ 1 - \frac{b}{q(T)} + \frac{1}{4} \frac{b^2}{q(T)^2} + \frac{1}{12} \frac{b^3}{q(T)^3} + O \left( \frac{1}{q(T)^4} \right) \right] \left[ 1 + \frac{b}{2 q(T)} + \frac{1}{6} \frac{b^3}{q(T)^3} + O \left( \frac{1}{q(T)^4} \right) \right]
\]

\[
+ \frac{q(T)^3}{2 b^2} \left[ 1 - \frac{b}{q(T)} + \frac{1}{4} \frac{b^2}{q(T)^2} + \frac{1}{12} \frac{b^3}{q(T)^3} + O \left( \frac{1}{q(T)^4} \right) \right] \left[ 1 - \frac{b T}{q(T)} + \frac{(g + 2) b}{q(T)} + \frac{b^2 T^2}{2 q(T)^2} - \frac{(g + 2) b^2 T}{q(T)^2} - \frac{1}{6} \frac{b^3 T^3}{q(T)^3} + \frac{(g + 2)^2 b}{2 q(T)^2} \right]
\]

\[
+ \frac{g + 2}{2} \frac{b^3 T^2}{q(T)^3} - \frac{(g + 2)^2}{2} \frac{b^3 T}{q(T)^3} + \frac{(g + 2)^3}{6} \frac{b^3}{q(T)^3} + O \left( \frac{T^4}{q(T)^4} \right)
\]

\[
+ \frac{q(T)^2}{2 b^2} \left[ 1 - \frac{3}{2} \frac{b}{q(T)} + \frac{11}{12} \frac{b^2}{q(T)^2} - \frac{1}{4} \frac{b^3}{q(T)^3} + O \left( \frac{1}{q(T)^4} \right) \right] (T - g - 1)
\]

\[
\times \left[ 1 + \frac{2 b}{q(T)} + \frac{2 b^2}{q(T)^2} + \frac{4 b^3}{3 q(T)^3} + O \left( \frac{1}{q(T)^4} \right) \right],
\]

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so that

\[
\sum_{t=0}^{T} \sum_{s=g}^{t-1} \sum_{j=1}^{s-g} \exp \left\{ -\frac{b(t-g-j)}{q(T)} \right\} \exp \left\{ -\frac{b(s-g-j)}{q(T)} \right\} = -\frac{q(T)^3}{2b^3} \left[ 1 - \frac{b}{2q(T)} + \frac{b^2}{4q(T)^2} - \frac{b^3}{6q(T)^3} \right] + O \left( \frac{1}{q(T)^4} \right)
\]

\[
+ \frac{q(T)^3}{2b^3} \left[ 1 - \frac{b}{2q(T)} + \frac{b^2}{4q(T)^2} + \frac{b^2 T}{2q(T)^{1/2}} - \frac{b T}{q(T)} - 4 \frac{b^2 T}{2q(T)^{1/2}} + 2 \frac{b^2 T}{q(T)^{1/2}} - 4 \frac{b^2 T}{q(T)^2} - \frac{b^2 T}{q(T)^2} - 3 \frac{b^3 T}{q(T)^3} \right]
\]

\[
- \frac{q(T)^3}{2b^3} \left[ 1 - \frac{b}{2q(T)} + \frac{b^2}{4q(T)^2} - \frac{b^3}{6q(T)^3} \right] + O \left( \frac{1}{q(T)^4} \right)
\]

\[
+ \frac{q(T)^3}{2b^3} \left[ 1 - \frac{b}{2q(T)} + \frac{b^2}{4q(T)^2} - \frac{b^3}{6q(T)^3} \right] + O \left( \frac{1}{q(T)^4} \right)
\]

\[
+ \frac{Tq(T)^2}{2b^2} \left[ 1 + \frac{2b}{q(T)} - \frac{3b}{2q(T)} + 2 \frac{b^2}{q(T)^2} - 3 \frac{b^2}{q(T)^2} + \frac{11b^2}{2q(T)^2} + \frac{4b^3}{3q(T)^3} - \frac{3b^3}{q(T)^3} \right]
\]

\[
- \frac{g}{2} \frac{q(T)^2}{b^2} \left[ 1 + \frac{2b}{q(T)} - \frac{3b}{2q(T)} + 2 \frac{b^2}{q(T)^2} - 3 \frac{b^2}{q(T)^2} + \frac{11b^2}{2q(T)^2} + \frac{4b^3}{3q(T)^3} - \frac{3b^3}{q(T)^3} \right]
\]
\[
\sum_{t=g+2}^{T} \sum_{s=g+1}^{T-1} \sum_{j=1}^{s-g} \exp \left\{ -b \frac{(t-g-j)}{q(T)} \right\} \exp \left\{ -b \frac{(s-g-j)}{q(T)} \right\}
= -\frac{q(T)^3}{2b^3} \left[ 1 - \frac{b}{q(T)} + \frac{1}{4} \frac{b^2}{q(T)^2} + \frac{1}{12} \frac{b^3}{q(T)^3} + O \left( \frac{1}{q(T)^4} \right) \right] + \frac{q(T)^3}{2b^3} \left[ 1 - \frac{bT}{q(T)} + \frac{gb}{q(T)} + \frac{b^2 T^2}{2q(T)^2} - \frac{gb^2 T}{q(T)^2} - \frac{b^3 T^3}{6q(T)^3} + \frac{2g^2 - 1}{4} \frac{b^2}{q(T)^2} + \frac{g}{2} \frac{b^3 T^2}{q(T)^3} - \left( \frac{2g^2 - 1}{4} \right) \frac{b^3 T}{q(T)^3} + \frac{g}{12} \frac{b^3}{q(T)^3} + O \left( \frac{T^4}{q(T)^4} \right) \right] + \frac{q(T)^3}{4b^3} \left[ 1 - \frac{3}{2} \frac{b}{q(T)} + \frac{3}{4} \frac{b^2}{q(T)^2} + O \left( \frac{1}{q(T)^4} \right) \right] - \frac{q(T)^3}{4b^3} \left[ 1 - \frac{2bT}{q(T)} + \frac{4g + 1}{2} \frac{b}{q(T)} + \frac{2}{q(T)^2} - \frac{(4g + 1) b^2 T}{q(T)^2} - \frac{4}{3} \frac{b^3 T^3}{q(T)^3} + \frac{8g^2 + 4g - 1}{4} \frac{b^2}{q(T)^2} + \frac{(4g + 1) b^3 T^2}{q(T)^3} - \frac{8g^2 + 4g - 1}{2} \frac{b^3 T}{q(T)^3} + \frac{8g^3 + 6g^2 - 3g - 1}{6} \frac{b^3}{q(T)^3} + O \left( \frac{T^4}{q(T)^4} \right) \right] - \frac{q(T)^3}{2b^3} \left[ 1 - \frac{1}{4} \frac{b^2}{q(T)^2} + O \left( \frac{1}{q(T)^4} \right) \right] + \frac{q(T)^3}{2b^3} \left[ 1 - \frac{bT}{q(T)} + \frac{(g + 1) b}{q(T)} + \frac{1}{2} \frac{b^2 T^2}{q(T)^2} - \frac{(g + 1) b^2 T}{q(T)^2} - \frac{1}{6} \frac{b^3 T^3}{q(T)^3} + \frac{2(g + 2)^2 - 4(g + 2) + 1}{4} \frac{b^2}{q(T)^2} + \frac{g + 1}{2} \frac{b^3 T^2}{q(T)^3} - \frac{2(g + 2)^2 - 4(g + 2) + 1}{4} \frac{b^3 T}{q(T)^3} + \frac{2(g + 2)^3 - 6(g + 2)^2 + 3(g + 2) + 1}{12} \frac{b^3}{q(T)^3} + O \left( \frac{T^4}{q(T)^4} \right) \right] + \frac{Tq(T)^2}{2b^3} \left[ 1 + \frac{1}{2} \frac{b}{q(T)} - \frac{1}{12} \frac{b^2}{q(T)^2} - \frac{1}{12} \frac{b^3}{q(T)^3} + O \left( \frac{1}{q(T)^4} \right) \right] - \frac{g + 1}{2} \frac{q(T)^2}{b^3} \left[ 1 + \frac{1}{2} \frac{b}{q(T)} - \frac{1}{12} \frac{b^2}{q(T)^2} - \frac{1}{12} \frac{b^3}{q(T)^3} + O \left( \frac{1}{q(T)^4} \right) \right]
\]
\[
\sum_{t=g+2}^{T} \sum_{s=g+1}^{t-1} \sum_{j=1}^{s-g} \exp \left\{-\frac{b(t-g-j)}{q(T)}\right\} \exp \left\{-\frac{b(s-g-j)}{q(T)}\right\}
\]

\[
= \left[-\frac{q(T)^3}{2b^3} + \frac{1}{2} \frac{q(T)^2}{b^2} - \frac{1}{8} \frac{q(T)}{b} - \frac{1}{24} + O \left(\frac{1}{q(T)}\right)\right]
\]

\[
+ \left[\frac{q(T)^3}{2b^3} - \frac{1}{2} \frac{Tq(T)^2}{b^2} + \frac{g}{2} \frac{q(T)^2}{b^2} + \frac{T^2q(T)}{2b} - \frac{gTq(T)}{2b} - \frac{1}{12} T^3\right]
\]

\[
\quad + \left(\frac{2g^2-1}{8} \right) \frac{q(T)}{b} + \frac{g}{4} - \frac{\left(\frac{2g^2-1}{8}\right)T}{5} + O \left(\frac{T^4}{q(T)}\right)\]
\]

\[
+ \left[\frac{q(T)^3}{4b^3} - \frac{3}{8} \frac{q(T)^2}{b^2} + \frac{3}{16} \frac{q(T)}{b} + O \left(\frac{1}{q(T)}\right)\right]
\]

\[
- \left[\frac{q(T)^3}{4b^3} - \frac{1}{8} \frac{Tq(T)^2}{b^2} + \frac{4g}{8} \frac{q(T)^2}{b^2} + \frac{T^2q(T)}{2b} - \frac{4g + Tq(T)}{4b} - \frac{1}{3} T^3\right]
\]

\[
+ \frac{8g^2 + 4g - 1}{16} \frac{q(T)}{b} + \frac{4g + 1}{4} T^2 - \frac{8g^2 + 4g - 1}{8} T
\]

\[
+ \frac{8g^3 + 6g^2 - 3g - 1}{24} + O \left(\frac{T^4}{q(T)}\right)\]
\]

\[
- \left[\frac{q(T)^3}{8b^3} - \frac{1}{8} \frac{q(T)}{b} + O \left(\frac{1}{q(T)}\right)\right]
\]

\[
+ \left[\frac{q(T)^3}{8b^3} - \frac{1}{2} \frac{Tq(T)^2}{b^2} + \frac{g + 1}{2} \frac{q(T)^2}{b^2} + \frac{1}{4} \frac{T^2q(T)}{b} - \frac{g + 1}{2} \frac{Tq(T)}{b}\right]
\]

\[
- \frac{1}{12} T^3 + \frac{2(g + 2)^2 - 4(g + 2) + 1}{8} \frac{q(T)}{b} + \frac{g + 1}{4} T^2 - \frac{2(g + 2)^2 - 4(g + 2) + 1}{8} T
\]

\[
+ \frac{2(g + 2)^3 - 6(g + 2)^2 + 3(g + 2) + 1}{24} + O \left(\frac{T^4}{q(T)}\right)\]
\]

\[
+ \left[\frac{Tq(T)^2}{2b^2} + \frac{1}{4} \frac{Tq(T)}{b} - \frac{1}{24} T - \frac{1}{24} bT + O \left(\frac{T}{q(T)^2}\right)\right]
\]

\[
- \left[\frac{g + 1}{2} \frac{q(T)^2}{b^2} + \frac{g + 1}{4} \frac{q(T)}{b} - \frac{g + 1}{24} - \frac{g + 1}{24} \frac{b}{q(T)} + O \left(\frac{1}{q(T)^2}\right)\right]
\]

\[
= \frac{1}{6} T^3 \left[ 1 + O \left(\max \left\{\frac{T}{q(T)}, \frac{1}{T}\right\}\right) \right]
\]

as required for part (a).

Next, consider part (b), where we take \( q(T) \sim T \). Specializing the calculations given in the proof
of part (a), we have that

\[
\sum_{t=g+2}^{T} \sum_{s=g+1}^{t-1} \sum_{j=1}^{s-2} \exp \left\{ -b \frac{(t - g - j)}{q(T)} \right\} \exp \left\{ -b \frac{(s - g - j)}{q(T)} \right\} =
\]

\[
= \left[ 1 - \exp \left\{ \frac{2b}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{ -\frac{b}{q(T)} \right\} \right]^{-2} \exp \left\{-\frac{b}{q(T)} \right\}
\]

\[
- \left[ 1 - \exp \left\{ \frac{2b}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{-\frac{b}{q(T)} \right\} \right]^{-2} \exp \left\{ -\frac{bT}{q(T)} \right\}
\]

\[
- \left[ 1 - \exp \left\{ \frac{2b}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{ -\frac{b}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{-\frac{2b}{q(T)} \right\} \right]^{-1} \exp \left\{ \frac{b}{q(T)} \right\}
\]

\[
+ \left[ 1 - \exp \left\{ \frac{2b}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{ -\frac{b}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{-\frac{2b}{q(T)} \right\} \right]^{-1}
\]

\[
\times \exp \left\{ \frac{b(g + 2)}{q(T)} \right\} \exp \left\{-\frac{bT}{q(T)} \right\}
\]

\[
+ \exp \left\{ \frac{2b}{q(T)} \right\} \left[ 1 - \exp \left\{ \frac{2b}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{ -\frac{b}{q(T)} \right\} \right]^{-1} (T - g - 1)
\]

\[
= - \frac{q(T)^3}{2b^2} \left[ 1 + O \left( \frac{1}{T} \right) \right] + \frac{q(T)^3}{2b^2} \exp \left\{-\frac{bT}{q(T)} \right\} \left[ 1 + O \left( \frac{1}{T} \right) \right] + O \left( \frac{1}{T} \right)
\]

\[
- \frac{q(T)^3}{4b^2} \exp \left\{-\frac{2bT}{q(T)} \right\} \left[ 1 + O \left( \frac{1}{T} \right) \right] - \frac{q(T)^3}{2b^2} \left[ 1 + O \left( \frac{1}{T} \right) \right] + O \left( \frac{1}{T} \right)
\]

\[
+ \frac{Tq(T)^2}{2b^2} \left[ 1 + O \left( \frac{1}{T} \right) \right]
\]

\[
= \frac{Tq(T)^2}{2b^2} \left[ 1 - \frac{3q(T)}{2bT} + \frac{2q(T)}{bT^2} \exp \left\{-\frac{2bT}{q(T)} \right\} - \frac{1}{2b} q(T) \exp \left\{-\frac{2bT}{q(T)} \right\} \right] \left[ 1 + O \left( \frac{1}{T} \right) \right],
\]

as required to show part (b).
Finally, consider part (c), where we take $q(T) \to \infty$ such that $q(T)/T \to 0$. In this case, we have
\[
\sum_{t=g+2}^{T} \sum_{s=g+1}^{t-1} \sum_{j=1}^{s-2} \exp \left\{ -b \frac{(t-g-j)}{q(T)} \right\} \exp \left\{ -b \frac{(s-g-j)}{q(T)} \right\} \\
= \left[ 1 - \exp \left\{ \frac{2b}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{ - \frac{b}{q(T)} \right\} \right]^{-2} \exp \left\{ - \frac{b}{q(T)} \right\} \\
- \left[ 1 - \exp \left\{ \frac{2b}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{ - \frac{b}{q(T)} \right\} \right]^{-2} \exp \left\{ \frac{bg}{q(T)} \right\} \exp \left\{ - \frac{bT}{q(T)} \right\} \\
- \left[ 1 - \exp \left\{ \frac{2b}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{ - \frac{b}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{ - \frac{2b}{q(T)} \right\} \right]^{-1} \exp \left\{ - \frac{2b}{q(T)} \right\} \\
+ \left[ 1 - \exp \left\{ \frac{2b}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{ - \frac{b}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{ - \frac{2b}{q(T)} \right\} \right]^{-1} \\
\times \exp \left\{ \frac{2bg}{q(T)} \right\} \exp \left\{ - \frac{2bT}{q(T)} \right\} \\
- \left[ 1 - \exp \left\{ \frac{2b}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{ \frac{b}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{ - \frac{b}{q(T)} \right\} \right]^{-1} \exp \left\{ \frac{b}{q(T)} \right\} \\
+ \left[ 1 - \exp \left\{ \frac{2b}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{ \frac{b}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{ - \frac{b}{q(T)} \right\} \right]^{-1} \\
\times \exp \left\{ \frac{b(g+2)}{q(T)} \right\} \exp \left\{ - \frac{bT}{q(T)} \right\} \\
+ \exp \left\{ \frac{2b}{q(T)} \right\} \left[ 1 - \exp \left\{ \frac{2b}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{ \frac{b}{q(T)} \right\} \right]^{-1} (T-g-1) \\
= -\frac{q(T)^3}{2b^4} \left[ 1 + O \left( \frac{1}{T} \right) \right] + \frac{q(T)^3}{2b^3} \exp \left\{ - \frac{bT}{q(T)} \right\} \left[ 1 + O \left( \frac{1}{T} \right) \right] + \frac{q(T)^3}{4b^4} \left[ 1 + O \left( \frac{1}{T} \right) \right] \\
- \frac{q(T)^3}{4b^4} \exp \left\{ - \frac{2bT}{q(T)} \right\} \left[ 1 + O \left( \frac{1}{T} \right) \right] - \frac{q(T)^3}{2b^2} \left[ 1 + O \left( \frac{1}{T} \right) \right] + \frac{q(T)^3}{2b^2} \exp \left\{ - \frac{bT}{q(T)} \right\} \left[ 1 + O \left( \frac{1}{T} \right) \right] \\
+ \frac{Tq(T)^2}{2b^2} \left[ 1 + O \left( \frac{1}{T} \right) \right] \\
= \frac{Tq(T)^2}{2b^2} \left[ 1 + O \left( \frac{q(T)}{T} \right) \right] \left[ 1 + O \left( \frac{1}{T} \right) \right] \\
= \frac{Tq(T)^2}{2b^2} \left[ 1 + O \left( \frac{q(T)}{T} \right) \right],
\]
as required to show part (c). □.

Lemma SE-8:

Let $b$ be a positive integer.
(a) Suppose that \( q(T) \sim T \). Then,

\[
\sum_{t=3}^{T} \sum_{s=2}^{t-1} \sum_{j=1}^{s-1} \exp \left( -b \frac{t-1-s}{q(T)} \right) \exp \left( -b \frac{s-1-j}{q(T)} \right) = \left( \frac{Tq(T)^2}{b^2} \left[ 1 + \exp \left( -\frac{Tb}{q(T)} \right) \right] - \frac{2q(T)^3}{b^2} \left[ 1 - \exp \left( -\frac{Tb}{q(T)} \right) \right] \right) \left[ 1 + O \left( \frac{1}{T} \right) \right],
\]

as \( T \to \infty \).

(b) Suppose that \( q(T) \to \infty \) such that \( q(T)/T \to 0 \). Then,

\[
\sum_{t=3}^{T} \sum_{s=2}^{t-1} \sum_{j=1}^{s-1} \exp \left( -b \frac{t-1-s}{q(T)} \right) \exp \left( -b \frac{s-1-j}{q(T)} \right) = \frac{Tq(T)^2}{b^2} \left[ 1 + O \left( \frac{1}{q(T)} \right) + O \left( \frac{q(T)}{T} \right) \right],
\]

as \( T \to \infty \).

**Proof of Lemma SE-8:**

To proceed, note first that

\[
\sum_{t=3}^{T} \sum_{s=2}^{t-1} \sum_{j=1}^{s-1} \exp \left( -b \frac{t-1-s}{q(T)} \right) \exp \left( -b \frac{s-1-j}{q(T)} \right)
\]

\[= \sum_{t=3}^{T} \sum_{s=2}^{t-1} \exp \left( -b \frac{t-1-s}{q(T)} \right) \exp \left( -b \frac{s-1-j}{q(T)} \right) \exp \left( \frac{b}{q(T)} \right) \sum_{j=1}^{s-1} \exp \left( \frac{b}{q(T)} \right)
\]

\[= \sum_{t=3}^{T} \exp \left( \frac{2b}{q(T)} \right) \sum_{s=2}^{t-1} \exp \left( \frac{2b}{q(T)} \right) \left[ 1 - \exp \left( \frac{b}{q(T)} \right) \right]^{t-2} \left[ 1 - \exp \left( \frac{b}{q(T)} \right) \right]^{s-1} \left[ 1 - \exp \left( \frac{b}{q(T)} \right) \right] \exp \left( -b \frac{t-1-s}{q(T)} \right) \exp \left( -b \frac{s-1-j}{q(T)} \right)
\]

\[= \exp \left( \frac{2b}{q(T)} \right) \left[ 1 - \exp \left( \frac{b}{q(T)} \right) \right]^{t-2} \exp \left( \frac{b}{q(T)} \right) \sum_{t=3}^{T} \exp \left( -b \frac{t-1-s}{q(T)} \right) \sum_{s=2}^{t-1} \exp \left( -b \frac{s-1-j}{q(T)} \right). \]
Next, we apply part (a) of Lemma SE-5; and, after further calculation, we obtain

\[
\sum_{t=3}^{T} \sum_{s=2}^{t-1} \sum_{j=1}^{s-1} \exp \left\{ -b \frac{t-1-s}{q(T)} \right\} \exp \left\{ -b \frac{s-1-j}{q(T)} \right\}
\]

\[
= \exp \left\{ \frac{2b}{q(T)} \right\} \left[ 1 - \exp \left\{ \frac{b}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{ -\frac{b}{q(T)} \right\} \right]^{-2} \times \left[ \exp \left\{ -\frac{2b}{q(T)} \right\} - (T-1) \exp \left\{ -\frac{T}{q(T)} \right\} + (T-2) \exp \left\{ -\frac{T+1}{q(T)} \right\} \right] \\
- \exp \left\{ \frac{3b}{q(T)} \right\} \left[ 1 - \exp \left\{ \frac{b}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{ \frac{b}{q(T)} \right\} \right]^{-1} \sum_{t=3}^{T} \exp \left\{ -\frac{t-1}{q(T)} \right\} \left[ 1 - \exp \left\{ \frac{t-2}{q(T)} \right\} \right] \\
= \exp \left\{ \frac{2b}{q(T)} \right\} \left[ 1 - \exp \left\{ \frac{b}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{ -\frac{b}{q(T)} \right\} \right]^{-2} \times \left[ \exp \left\{ -\frac{2b}{q(T)} \right\} - (T-1) \exp \left\{ -\frac{T}{q(T)} \right\} + (T-2) \exp \left\{ -\frac{T+1}{q(T)} \right\} \right] \\
- \exp \left\{ \frac{3b}{q(T)} \right\} \left[ 1 - \exp \left\{ \frac{b}{q(T)} \right\} \right]^{-2} \left[ 1 - \exp \left\{ -\frac{b}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{ -\frac{T-2}{q(T)} \right\} \right] \\
- \exp \left\{ \frac{2b}{q(T)} \right\} \left[ 1 - \exp \left\{ \frac{b}{q(T)} \right\} \right]^{-2} \left( T-2 \right). 
\]

Now, consider part (a), where we take \( q(T) \sim T \). From the calculations we have performed above, we have
\[ \sum_{t=3}^{T} \sum_{s=2}^{t-1} \sum_{j=1}^{s-1} \exp \left\{ -b \frac{t-1-s}{q(T)} \right\} \exp \left\{ -b \frac{s-1-j}{q(T)} \right\} \]

\[ \times \left[ \exp \left\{ -\frac{2b}{q(T)} \right\} - (T-1) \exp \left\{ -\frac{T}{q(T)} \right\} + (T-2) \exp \left\{ -\frac{T}{q(T)} \right\} \exp \left\{ -\frac{b}{q(T)} \right\} \right] \]

\[ - \exp \left\{ \frac{b}{q(T)} \right\} \left[ 1 - \exp \left\{ \frac{b}{q(T)} \right\} \right]^{-2} \left[ 1 - \exp \left\{ -\frac{b}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{ -\frac{T-2}{q(T)} \right\} \right] \]

\[ + \exp \left\{ \frac{2b}{q(T)} \right\} \left[ 1 - \exp \left\{ \frac{b}{q(T)} \right\} \right]^{-2} (T-2) \]

\[ = \exp \left\{ \frac{2b}{q(T)} \right\} \left[ 1 - \exp \left\{ \frac{b}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{ -\frac{b}{q(T)} \right\} \right]^{-2} \]

\[ \times \left[ \exp \left\{ -\frac{2b}{q(T)} \right\} - (T-1) \exp \left\{ -\frac{T}{q(T)} \right\} \right] \left( 1 - \frac{b}{q(T)} + O \left( \frac{1}{T^2} \right) \right) \]

\[ - \exp \left\{ \frac{b}{q(T)} \right\} \left[ 1 - \exp \left\{ \frac{b}{q(T)} \right\} \right]^{-2} \left[ 1 - \exp \left\{ -\frac{b}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{ -\frac{T-2}{q(T)} \right\} \right] \]

\[ + \exp \left\{ \frac{2b}{q(T)} \right\} \left[ 1 - \exp \left\{ \frac{b}{q(T)} \right\} \right]^{-2} (T-2) \]

\[ = \exp \left\{ \frac{2b}{q(T)} \right\} \left[ 1 - \exp \left\{ \frac{b}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{ -\frac{b}{q(T)} \right\} \right]^{-2} \]

\[ \times \left[ \exp \left\{ -\frac{2b}{q(T)} \right\} - T \exp \left\{ -\frac{bT}{q(T)} \right\} + \exp \left\{ -\frac{T}{q(T)} \right\} \right] \]

\[ + T \exp \left\{ -\frac{T}{q(T)} \right\} - \frac{bT}{q(T)} \exp \left\{ -\frac{bT}{q(T)} \right\} - 2 \exp \left\{ -\frac{T}{q(T)} \right\} \exp \left\{ -\frac{bT}{q(T)} \right\} + O \left( \frac{1}{T} \right) \]

\[ - \exp \left\{ \frac{b}{q(T)} \right\} \left[ 1 - \exp \left\{ \frac{b}{q(T)} \right\} \right]^{-2} \left[ 1 - \exp \left\{ -\frac{b}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{ -\frac{T-2}{q(T)} \right\} \right] \]

\[ + \exp \left\{ \frac{2b}{q(T)} \right\} \left[ 1 - \exp \left\{ \frac{b}{q(T)} \right\} \right]^{-2} (T-2) \]
\[
\begin{align*}
&= \left(-\frac{q(T)}{b}\right) \left(\frac{q(T)}{b}\right)^2 \left[1 - \exp\left\{-\frac{Tb}{q(T)}\right\}\right] \left[1 + O\left(\frac{1}{T}\right)\right] \\
&- \left(-\frac{q(T)}{b}\right) \left(\frac{q(T)}{b}\right)^2 \frac{T}{q(T)} \exp\left\{-\frac{Tb}{q(T)}\right\} \left[1 + O\left(\frac{1}{T}\right)\right] \\
&- \left(-\frac{q(T)}{b}\right) \left(\frac{q(T)}{b}\right)^2 \left[1 - \exp\left\{-\frac{Tb}{q(T)}\right\}\right] \left[1 + O\left(\frac{1}{T}\right)\right] \\
&+ \left(-\frac{q(T)}{b}\right)^2 T \left[1 + O\left(\frac{1}{T}\right)\right] \\
&= \left(-\frac{q(T)}{b}\right) \left(\frac{q(T)}{b}\right)^2 \left[1 + O\left(\frac{1}{T}\right)\right] \\
&- \left(-\frac{q(T)}{b}\right) \left(\frac{q(T)}{b}\right)^2 \frac{T}{q(T)} \exp\left\{-\frac{Tb}{q(T)}\right\} \left[1 + O\left(\frac{1}{T}\right)\right] \\
&+ T \left(-\frac{q(T)}{b}\right)^2 T \left[1 + O\left(\frac{1}{T}\right)\right]
\end{align*}
\]

which is the required result.

Next, we turn our attention to part (b), where we take \(q(T) \to \infty\) such that \(q(T)/T \to 0\). Here, from previous calculations, we have

\[
\begin{align*}
&\sum_{t=3}^{T} \sum_{s=2}^{t-1} \sum_{j=1}^{s-1} \exp\left\{-\frac{t-1-s}{q(T)}\right\} \exp\left\{-\frac{j}{q(T)}\right\} \\
&= \exp\left\{-\frac{2b}{q(T)}\right\} \left[1 - \exp\left\{-\frac{b}{q(T)}\right\}\right]^{-1} \left[1 - \exp\left\{-\frac{b}{q(T)}\right\}\right]^{-2} \\
&\times \left[1 - \exp\left\{-\frac{2b}{q(T)}\right\} - (T - 1) \exp\left\{-\frac{T}{q(T)}\right\} + (T - 2) \exp\left\{-\frac{T + 1}{q(T)}\right\}\right] \\
&- \exp\left\{-\frac{b}{q(T)}\right\} \left[1 - \exp\left\{-\frac{b}{q(T)}\right\}\right]^{-2} \left[1 - \exp\left\{-\frac{b}{q(T)}\right\}\right]^{-1} \left[1 - \exp\left\{-\frac{T - 2}{q(T)}\right\}\right] \\
&+ \exp\left\{-\frac{2b}{q(T)}\right\} \left[1 - \exp\left\{-\frac{b}{q(T)}\right\}\right]^{-2} (T - 2)
\end{align*}
\]

which is the required result. \(\square\)

**Lemma SE-9:**
Let $b$ and $c$ be positive integers.

(a) If $q(T) \sim T$, then

$$
\sum_{t=3}^{T} \sum_{s=2}^{t-1} \sum_{j=1}^{s-1} \sum_{k=1}^{c} \exp \left\{-b \frac{t - 1 - j}{q(T)} \right\} \exp \left\{-c \frac{s - 1 - k}{q(T)} \right\} \\
= \left( \frac{T^2 q(T)^2}{2bc} - \frac{q(T)^4}{b^3 c} \right) \left[ 1 - \exp \left\{-b \frac{T}{q(T)} \right\} \right] + \frac{T q(T)^3}{b^2 c} \exp \left\{-\frac{b T}{q(T)} \right\} - \frac{T q(T)^3}{bc^2} \\
+ \frac{q(T)^4}{bc^2} \left[ 1 - \exp \left\{-c \frac{T}{q(T)} \right\} \right] - \frac{q(T)^4}{b^2 c^2} \left[ 1 - \exp \left\{-c \frac{T}{q(T)} \right\} \right] \\
- \frac{q(T)^4}{bc^2 (b+c)} \left[ 1 - \exp \left\{-c (b+c) \frac{T}{q(T)} \right\} \right] \left[ 1 + O \left( \frac{1}{T} \right) \right],
$$

as $T \to \infty$.

(b) If $q(T) \to \infty$ such that $q(T)/T \to 0$, then

$$
\sum_{t=3}^{T} \sum_{s=2}^{t-1} \sum_{j=1}^{s-1} \sum_{k=1}^{c} \exp \left\{-b \frac{t - 1 - j}{q(T)} \right\} \exp \left\{-c \frac{s - 1 - k}{q(T)} \right\} \\
= \frac{T^2 q(T)^2}{2bc} \left[ 1 + O \left( \frac{1}{q(T)} \right) \right] + O \left( \frac{q(T)}{T} \right),
$$

as $T \to \infty$.

Proof of Lemma SE-9:

To proceed, note first that

$$
\sum_{t=3}^{T} \sum_{s=2}^{t-1} \sum_{j=1}^{s-1} \sum_{k=1}^{c} \exp \left\{-b \frac{t - 1 - j}{q(T)} \right\} \sum_{k=1}^{c} \exp \left\{-c \frac{s - 1 - k}{q(T)} \right\} \\
= \left[ 1 - \exp \left\{-\frac{b}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{-\frac{c}{q(T)} \right\} \right]^{-1} \\
\times \sum_{t=3}^{T} \sum_{s=2}^{t-1} \left[ 1 - \exp \left\{-\frac{t - 1}{q(T)} \right\} \right] \left[ 1 - \exp \left\{-\frac{s - 1}{q(T)} \right\} \right] \\
= \left[ 1 - \exp \left\{-\frac{b}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{-\frac{c}{q(T)} \right\} \right]^{-1} \\
\times \sum_{t=3}^{T} \sum_{s=2}^{t-1} \left[ 1 - \exp \left\{-\frac{t - 1}{q(T)} \right\} \right] - \exp \left\{-\frac{s - 1}{q(T)} \right\} + \exp \left\{-\frac{t - 1}{q(T)} \right\} \exp \left\{-\frac{s - 1}{q(T)} \right\} \\
= \left[ 1 - \exp \left\{-\frac{b}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{-\frac{c}{q(T)} \right\} \right]^{-1} \sum_{t=3}^{T} \sum_{s=2}^{t-1} \left[ (t-2) - (t-2) \exp \left\{-\frac{t - 1}{q(T)} \right\} \right] \\
- \exp \left\{-\frac{c}{q(T)} \right\} \sum_{s=2}^{t-1} \exp \left\{-\frac{s - 2}{q(T)} \right\} \exp \left\{-\frac{t - 1}{q(T)} \right\} \exp \left\{-\frac{c}{q(T)} \right\} \sum_{s=2}^{t-1} \exp \left\{-\frac{s - 2}{q(T)} \right\}.\]
Applying part (a) of Lemma SE-5 and performing additional calculation, we get

\[
\sum_{t=3}^{T} \sum_{s=2}^{t-1} \sum_{j=1}^{t-1} \exp \left\{ -\frac{b}{q(T)} \frac{t-1-j}{s} \right\} \sum_{k=1}^{t-1} \exp \left\{ -\frac{c}{q(T)} \frac{s-1-k}{t} \right\} = \\
\left[ 1 - \exp \left\{ -\frac{b}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{ -\frac{c}{q(T)} \right\} \right]^{-1} \frac{(T-2)(T-1)}{2} \\
- \left[ 1 - \exp \left\{ -\frac{b}{q(T)} \right\} \right]^{-3} \left[ 1 - \exp \left\{ -\frac{c}{q(T)} \right\} \right]^{-1} \exp \left\{ -\frac{2b}{q(T)} \right\} \\
-(T-1) \exp \left\{ -\frac{bT}{q(T)} \right\} + (T-2) \exp \left\{ -\frac{T+1}{q(T)} \right\} \\
- \left[ 1 - \exp \left\{ -\frac{b}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{ -\frac{c}{q(T)} \right\} \right]^{-2} \exp \left\{ -\frac{c}{q(T)} \right\} \\
\times \sum_{t=3}^{T} \left[ 1 - \exp \left\{ -\frac{c}{q(T)} \right\} \right] \\
+ \left[ 1 - \exp \left\{ -\frac{b}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{ -\frac{c}{q(T)} \right\} \right]^{-2} \exp \left\{ -\frac{c}{q(T)} \right\} \\
\times \sum_{t=3}^{T} \exp \left\{ -\frac{t-1}{q(T)} \right\} \left[ 1 - \exp \left\{ -\frac{c}{q(T)} \right\} \right] \\
= \left[ 1 - \exp \left\{ -\frac{b}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{ -\frac{c}{q(T)} \right\} \right]^{-1} \frac{(T-2)(T-1)}{2} \\
- \left[ 1 - \exp \left\{ -\frac{b}{q(T)} \right\} \right]^{-3} \left[ 1 - \exp \left\{ -\frac{c}{q(T)} \right\} \right]^{-1} \exp \left\{ -\frac{2b}{q(T)} \right\} \\
-(T-1) \exp \left\{ -\frac{bT}{q(T)} \right\} + (T-2) \exp \left\{ -\frac{T+1}{q(T)} \right\} \\
- \left[ 1 - \exp \left\{ -\frac{b}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{ -\frac{c}{q(T)} \right\} \right]^{-2} \exp \left\{ -\frac{c}{q(T)} \right\} \\
\times \left\{ (T-2) - \exp \left\{ -\frac{c}{q(T)} \right\} \left[ 1 - \exp \left\{ -\frac{c}{q(T)} \right\} \right]^{-1} \right\} \\
+ \left[ 1 - \exp \left\{ -\frac{b}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{ -\frac{c}{q(T)} \right\} \right]^{-1} \exp \left\{ -\frac{c}{q(T)} \right\} \\
\times \exp \left\{ -\frac{2b}{q(T)} \right\} \left[ 1 - \exp \left\{ -\frac{b}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{ -\frac{T-2}{q(T)} \right\} \right] \\
- \left[ 1 - \exp \left\{ -\frac{b}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{ -\frac{c}{q(T)} \right\} \right]^{-2} \exp \left\{ -\frac{c}{q(T)} \right\} \\
\times \exp \left\{ -\frac{2b}{q(T)} \right\} \exp \left\{ -\frac{c}{q(T)} \right\} \sum_{t=3}^{T} \exp \left\{ -\left( b + c \right) \frac{t-3}{q(T)} \right\}
\]
For this case, we have

\[
\begin{aligned}
\text{Now, consider part (a), where we take } q(T) &\sim T. \text{ In this case,} \\
\sum_{t=3}^{T} \sum_{s=2}^{t-1} \sum_{j=1}^{t-1} \exp \left\{ -\frac{t-1-j}{q(T)} \right\} \sum_{k=1}^{s-1} \exp \left\{ -\frac{s-1-k}{q(T)} \right\} \\
= \frac{T^2 q(T)^2}{2bc} + \frac{q(T)^3}{b^2c} \left[ 1 + O \left( \frac{1}{T} \right) \right] + \frac{Tq(T)^3}{b^2c} \exp \left\{ -\frac{bT}{q(T)} \right\} \left[ 1 + O \left( \frac{1}{T} \right) \right] \\
+ \frac{Tq(T)^3}{bc^2} \left[ 1 + O \left( \frac{1}{T} \right) \right] + \frac{Tq(T)^4}{bc^2} \left[ 1 - \exp \left\{ -\frac{T}{q(T)} \right\} \right] \left[ 1 + O \left( \frac{1}{T} \right) \right] \\
= \left( \frac{T^2 q(T)^2}{2bc} - \frac{q(T)^4}{b^3c} \left[ 1 - \exp \left\{ -\frac{T}{q(T)} \right\} \right] \right) + \frac{Tq(T)^3}{b^2c} \exp \left\{ -\frac{bT}{q(T)} \right\} - \frac{Tq(T)^3}{bc^2} \\
+ \frac{q(T)^4}{bc^2} \left[ 1 - \exp \left\{ -\frac{T}{q(T)} \right\} \right] + \frac{q(T)^4}{b^2c^2} \left[ 1 - \exp \left\{ -\frac{T}{q(T)} \right\} \right] \\
- \frac{q(T)^4}{bc^2 (b + c)} \left[ 1 - \exp \left\{ -\frac{T}{q(T)} \right\} \right] \left[ 1 + O \left( \frac{1}{T} \right) \right] \\
\end{aligned}
\]

Next, we turn our attention to part (b), where we take \( q(T) \to \infty \) such that \( q(T)/T \to 0 \). For this
Let
\[ E \left( Y_{i,T}^2 \right) = \omega_{i,T}^2, \]
where \( \omega_{N,T}^2 = \sum_{i=1}^{N} \omega_{i,T}^2. \) Suppose that for all \( \epsilon > 0 \)
\[ \lim_{N,T \to \infty} \sum_{i=1}^{N} E \left[ \xi_{i,N,T}^2 \left\{ \left| \xi_{i,N,T} \right| > \epsilon \right\} \right] = 0 \quad (13) \]
Then, as \( N, T \to \infty \) jointly \( \sum_{i=1}^{N} \xi_{i,N,T} \Rightarrow N (0, 1). \)

**Lemma SE-11:**

Given Assumptions 1-3, the following results hold.

(a) \( N^{-1} (T - 1)^{-1} \sum_{i=1}^{N} \sum_{t=2}^{T} a_{it}^2 = \mu_a^2 + \sigma_a^2 + O_p (N^{-1/2}); \)

(b) \( N^{-1} \sum_{i=1}^{N} a_i = \mu_a + O_p (N^{-1/2}); \)

(c) \( \sum_{i=1}^{N} \sum_{t=2}^{T} a_i \varepsilon_{it} - g = O_p \left( \sqrt{NT} \right), \) for \( g \in \{0, 1\}; \)

(d) \( \varepsilon_{NT} = N^{-1} (T - 1)^{-1} \sum_{i=1}^{N} \sum_{t=2}^{T} \varepsilon_{it} = O_p \left( N^{-1/2} T^{-1/2} \right); \)

(e) \( N^{-1} \sum_{i=1}^{N} \varepsilon_{it}^2 = \frac{\sigma^2}{T-1} + O_p \left( \frac{1}{T \sqrt{N}} \right) = O_p \left( \frac{1}{T} \right), \) where \( \varepsilon_i = (T - 1)^{-1} \sum_{s=2}^{T} \varepsilon_{is}; \)
(f) $N^{-1} (T - 1)^{-1} \sum_{i=1}^{N} \sum_{t=g+2}^{T} \varepsilon_{it-g}^2 = \sigma^2 + O_p \left( \max \left\{ N^{-1/2} T^{-1/2}, T^{-1} \right\} \right)$ for (fixed) non-negative integer $g$;

(g) $\sum_{i=1}^{N} \sum_{t=1}^{T} a_i \Delta \varepsilon_{it} = O_p \left( \sqrt{N} \right)$;

(h) $N^{-1/2} T^{-1/2} \sum_{i=1}^{N} \sum_{t=1}^{T} \varepsilon_{it-g-1} \varepsilon_{it-g} \Rightarrow N \left( 0, \sigma^4 \right)$, for (fixed) non-negative integer $g$;

(i) $\sum_{i=1}^{N} \varepsilon_i \varepsilon_i = O_p \left( \sqrt{N} \right)$.

**Proof of Lemma SE-11:**

To show part (a), note that, by Assumption 2,

$$E \left[ \frac{1}{N (T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} a_i^2 \right] = \frac{1}{N (T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} E \left[ a_i^2 \right] = \frac{1}{N (T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} \left( \mu_a^2 + \sigma_a^2 \right)$$

Moreover, by part (b) of Assumption 2,

$$Var \left( \frac{1}{N (T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} a_i^2 \right) = E \left( \frac{1}{N (T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} \left( a_i^2 - \left( \mu_a^2 + \sigma_a^2 \right) \right)^2 \right)$$

$$= \frac{1}{N^2 (T-1)^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{t=2}^{T} \sum_{s=2}^{T} E \left[ \left( a_i^2 - \left( \mu_a^2 + \sigma_a^2 \right) \right) \left( a_j^2 - \left( \mu_a^2 + \sigma_a^2 \right) \right) \right]$$

$$= \frac{1}{N^2} \sum_{i=1}^{N} E \left[ \left( a_i^2 - \left( \mu_a^2 + \sigma_a^2 \right) \right)^2 \right] = \frac{E \left[ a_i^4 \right] - \left( \mu_a^2 + \sigma_a^2 \right)^2}{N} = O \left( \frac{1}{N} \right).$$

Using Markov’s inequality, we deduce that

$$\frac{1}{N (T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} a_i^2 = \mu_a^2 + \sigma_a^2 + O_p \left( \frac{1}{\sqrt{N}} \right).$$

To show part (b), note that

$$E \left[ \frac{1}{N} \sum_{i=1}^{N} a_i \right] = \frac{1}{N} \sum_{i=1}^{N} E \left[ a_i \right] = \mu_a,$$

and

$$Var \left( \frac{1}{N} \sum_{i=1}^{N} a_i \right) = E \left( \frac{1}{N} \sum_{i=1}^{N} \left( a_i - \mu_a \right)^2 \right) = \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} E \left[ \left( a_i - \mu_a \right) \left( a_j - \mu_a \right) \right]$$

$$= \frac{1}{N^2} \sum_{i=1}^{N} E \left[ \left( a_i - \mu_a \right)^2 \right] = \frac{\sigma_a^2}{N}.$$
Using Markov’s inequality, we deduce that

\[
\frac{1}{N} \sum_{i=1}^{N} a_i = \mu_a + O_p \left( \frac{1}{\sqrt{N}} \right).
\]

To show (c), note that

\[
E \left( \sum_{i=1}^{N} \sum_{t=2}^{T} a_i \varepsilon_{it-g} \right)^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{t=2}^{T} \sum_{s=2}^{T} E[a_i a_j] E[\varepsilon_{i,t-g} \varepsilon_{j,s-g}] = \sum_{i=1}^{N} \sum_{t=2}^{T} a_i^2 E[\varepsilon_{i,t-g}^2] = \sigma^2 (\mu_a^2 + \sigma^2_a) N (T-1) = O (NT),
\]

from which it follows by applying the Markov’s inequality that

\[
\sum_{i=1}^{N} \sum_{t=2}^{T} a_i \varepsilon_{it-g} = O_p \left( \sqrt{NT} \right),
\]

as required.

To show part (d), note that

\[
E \left( \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} \varepsilon_{it} \right)^2 = \frac{1}{N^2 (T-1)^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{t=2}^{T} \sum_{s=2}^{T} E[\varepsilon_{it} \varepsilon_{js}] = \frac{1}{N^2 (T-1)^2} \sum_{i=1}^{N} \sum_{t=2}^{T} E[\varepsilon_{it}^2] = \frac{\sigma^2}{N(T-1)} = O \left( \frac{1}{NT} \right).
\]

It follows from the Markov’s inequality that

\[
\frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} \varepsilon_{it} = O_p \left( \frac{1}{\sqrt{NT}} \right),
\]

as required.

To show (e), note first that

\[
\frac{1}{N} \sum_{i=1}^{N} E[\varepsilon_{it}^2] = \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} \sum_{s=2}^{T} E[\varepsilon_{it} \varepsilon_{is}] = \frac{\sigma^2}{(T-1)} = O (T^{-1}),
\]

as required.
and
\[
E \left[ \frac{1}{N} \sum_{i=1}^{N} \varepsilon_i^2 - \frac{\sigma^2}{(T - 1)} \right]^2
= E \left[ \frac{1}{N (T - 1)^2} \sum_{i=1}^{N} \sum_{t=2}^{T} \sum_{s=2}^{T} \varepsilon_{it} \varepsilon_{is} \right]^2 - 2 \frac{1}{N} \sum_{i=1}^{N} E \left[ \varepsilon_i^2 \right] \frac{\sigma^2}{(T - 1)} + \frac{\sigma^4}{(T - 1)^2}
= \frac{1}{N^2 (T - 1)^4} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{t=2}^{T} \sum_{s=2}^{T} \sum_{g=2}^{T} \sum_{v=2}^{T} E \left[ \varepsilon_{it} \varepsilon_{is} \varepsilon_{jg} \varepsilon_{jv} \right] - \frac{\sigma^4}{(T - 1)^2}
= \frac{1}{N^2 (T - 1)^4} \sum_{i=1}^{N} \sum_{t=2}^{T} E \left[ \varepsilon_{it}^4 \right] + \frac{1}{N^2 (T - 1)^4} \sum_{i \neq j}^{N} \sum_{t=2}^{T} \sum_{g=2}^{T} \sigma^4 + \frac{3}{N^2 (T - 1)^4} \sum_{i=1}^{N} \sum_{t \neq s}^{T} \sigma^4
- \frac{\sigma^4}{(T - 1)^2}
= \frac{E \left[ \varepsilon_{it}^4 \right]}{N (T - 1)^3} + \frac{\sigma^4}{N^2 (T - 1)^2} N (N - 1) + \frac{3\sigma^4}{N^2 (T - 1)^4} (T - 2) (T - 1) - \frac{\sigma^4}{(T - 1)^2}
= - \frac{\sigma^4}{N^2 (T - 1)^4} \frac{3\sigma^4 (T - 2)}{N (T - 1)^3} + \frac{E \left[ \varepsilon_{it}^4 \right]}{N (T - 1)^3} = O \left( \frac{1}{NT^2} \right).
\]

It follows by Markov’s inequality that
\[
\frac{1}{N} \sum_{i=1}^{N} \varepsilon_i^2 = \frac{\sigma^2}{(T - 1)} + O_p \left( \frac{1}{T \sqrt{N}} \right).
\]

To show part (f), note that, for all positive integer \( T > g + 2 \)
\[
E \left[ \frac{1}{N (T - 1)} \sum_{i=1}^{N} \sum_{t=g+2}^{T} \varepsilon_{it}^2 \right] = \frac{1}{N (T - 1)} \sum_{i=1}^{N} \sum_{t=g+2}^{T} E \left[ \varepsilon_{it}^2 \right] = \frac{T - g - 1}{T - 1} \sigma^2 = \sigma^2 + O \left( \frac{1}{T} \right).
\]
Moreover, for all $T > g + 2$, we have

\[
E \left[ \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=g+2}^{T} \varepsilon_{it-g}^2 - \frac{T-g-1}{T-1} \sigma^2 \right]^2
\]

\[
= \frac{1}{N^2(T-1)^2} \sum_{i \neq j} \sum_{s \neq t} \sum_{s=g+2}^{T} \sum_{t=g+2}^{T} E \left[ \varepsilon_{is-g}^2 \varepsilon_{jt-g}^2 \right] - 2 \frac{\sigma^2}{N(T-1)} \sum_{i=1}^{N} \sum_{t=g+2}^{T} E \left[ \varepsilon_{it-g}^2 \right]
\]

\[+ \left( \frac{T-g-1}{T-1} \right)^2 \sigma^4\]

\[
= \frac{1}{N^2(T-1)^2} \sum_{i \neq j} \sum_{s \neq t} E \left[ \varepsilon_{is-g}^2 \right] E \left[ \varepsilon_{jt-g}^2 \right] + \frac{1}{N^2(T-1)^2} \sum_{i \neq j} \sum_{t=g+2}^{T} E \left[ \varepsilon_{is-g}^2 \right] E \left[ \varepsilon_{it-g}^2 \right]
\]

\[+ \frac{1}{N^2(T-1)^2} \sum_{i=1}^{N} \sum_{s \neq t} E \left[ \varepsilon_{is-g}^2 \right] E \left[ \varepsilon_{jt-g}^2 \right] + \frac{(T-g-1) E \left[ \varepsilon_{it}^4 \right]}{N(T-1)^2} - \left( \frac{T-g-1}{T-1} \right)^2 \sigma^4\]

\[
= \frac{(N-1)(T-g-1)(T-g-2)}{N(T-1)^2} \sigma^4 - \left( \frac{T-g-1}{T-1} \right)^2 \sigma^4 + \frac{(N-1)(T-g-1)}{N(T-1)^2} \sigma^4
\]

\[+ \frac{(T-g-1)(T-g-2)}{N(T-1)^2} \sigma^4 + \frac{(T-g-1) E \left[ \varepsilon_{it}^4 \right]}{N(T-1)^2} \sigma^4\]

\[
= \frac{(T-g-1)(T-g-2)}{(T-1)^2} \sigma^4 - \frac{(T-g-1)(T-g-2)}{N(T-1)^2} \sigma^4 - \frac{(T-1)^2 - 2g(T-1) + g^2}{(T-1)^2} \sigma^4
\]

\[+ \frac{(T-g-1)(T-g-1)(T-g-2)}{N(T-1)^2} \sigma^4 + \frac{(T-g-1) E \left[ \varepsilon_{it}^4 \right]}{N(T-1)^2} \sigma^4\]

\[
= \frac{(T-g-1)(T-g-2)}{(T-1)^2} \sigma^4 - \frac{(T-1)^2 - 2g(T-1) + g^2}{(T-1)^2} \sigma^4 + \frac{(T-g-1) E \left[ \varepsilon_{it}^4 \right]}{N(T-1)^2} \sigma^4 - \frac{(T-g-1)}{N(T-1)^2} \sigma^4
\]

\[+ \frac{(T-g-1) E \left[ \varepsilon_{it}^4 \right]}{N(T-1)^2} \sigma^4\]

\[
= \frac{(T-g-1)^2 - g(T-1) - (g+1)(T-1) + g(g+1)}{(T-1)^2} \sigma^4 - \frac{(T-1)^2 - 2g(T-1) + g^2}{(T-1)^2} \sigma^4
\]

\[+ \frac{\sigma^4}{(T-1)^2} - \frac{g \sigma^4}{N(T-1)} + \frac{g \sigma^4}{N(T-1)^2} + \frac{(T-g) E \left[ \varepsilon_{it}^4 \right]}{N(T-1)^2} - \frac{g E \left[ \varepsilon_{it}^4 \right]}{N(T-1)^2}\]

\[
= - \frac{\sigma^4}{(T-1)^2} + \frac{g \sigma^4}{N(T-1)} + \frac{(T-g) E \left[ \varepsilon_{it}^4 \right]}{N(T-1)^2} - \frac{g E \left[ \varepsilon_{it}^4 \right]}{N(T-1)^2}\]

\[+ \frac{E \left[ \varepsilon_{it}^4 \right]}{N(T-1)^2} \sigma^4 - \frac{g E \left[ \varepsilon_{it}^4 \right]}{N(T-1)^2}\]

\[
= - \frac{\sigma^4}{N(T-1)} + \frac{E \left[ \varepsilon_{it}^4 \right]}{N(T-1)} + O \left( \frac{1}{NT^2} \right)
\]

\[= O \left( \frac{1}{NT} \right)\]
It follows by Markov’s inequality that as $N,T \to \infty$

$$ \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=g+2}^{T} \varepsilon_{it-g}^2 = \sigma^2 + O \left( \frac{1}{T} \right) + O_p \left( \frac{1}{\sqrt{NT}} \right) = \sigma^2 + O_p \left( \max \left\{ \frac{1}{\sqrt{NT}}, \frac{1}{T} \right\} \right). $$

To show part (g), write

$$ \sum_{i=1}^{N} \sum_{t=2}^{T} a_i \Delta \varepsilon_{it} = \sum_{i=1}^{N} a_i \sum_{t=2}^{T} \Delta \varepsilon_{it} = \sum_{i=1}^{N} a_i (\varepsilon_{iT} - \varepsilon_{i1}). $$

Note that

$$ E \left[ \sum_{i=1}^{N} \sum_{t=2}^{T} a_i \Delta \varepsilon_{it} \right]^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} E [a_i a_j] E [(\varepsilon_{iT} - \varepsilon_{i1})(\varepsilon_{jT} - \varepsilon_{j1})] $$

$$ = \sum_{i=1}^{N} E [a_i^2] E [(\varepsilon_{iT} - \varepsilon_{i1})^2] = 2\sigma^2 (\mu_a^2 + \sigma_a^2) N$$

from which it follows by Markov’s inequality that

$$ \sum_{i=1}^{N} \sum_{t=2}^{T} a_i \Delta \varepsilon_{it} = O_p \left( \sqrt{N} \right). $$

To show part (h), let $Y_{i,T} = \frac{1}{\sqrt{T}} \sum_{t=g+3}^{T} \varepsilon_{it-g-1} \varepsilon_{it-g}$ and note that

$$ E [Y_{i,T}] = \frac{1}{\sqrt{T}} \sum_{t=g+3}^{T} E [\varepsilon_{it-g-1}] E [\varepsilon_{it-g}] = 0. $$

Moreover,

$$ \omega_{N,T}^2 = \sum_{i=1}^{N} E [Y_{i,T}^2] = \frac{1}{T} \sum_{i=1}^{N} \sum_{t=g+3}^{T} \sum_{s=g+3}^{T} E [\varepsilon_{it-g-1} \varepsilon_{it-g} \varepsilon_{is-g-1} \varepsilon_{is-g}] $$

$$ = \sigma^4 N(T-g-2) = \sigma^4 N \left[ 1 + O \left( \frac{1}{T} \right) \right], $$

so that

$$ \frac{\omega_{N,T}}{\sqrt{N}} \to \sigma^2 \in (0,\infty) \quad \text{as} \ N,T \to \infty. $$

Hence, to show the asymptotic normality of

$$ U_{N,T} = \frac{1}{\omega_{N,T}/\sqrt{N}} \sum_{i=1}^{N} Y_{i,T}, $$

it suffices to verify a Liapounov-type condition

$$ \lim_{N,T \to \infty} \frac{1}{N^2} \sum_{i=1}^{N} E [Y_{i,T}^4] = 0. \quad (14) $$
By direct calculation,
\[
\frac{1}{N^2} \sum_{i=1}^{N} E [Y_i^4] = \frac{1}{N^2T^2} \sum_{i=1}^{N} E \left[ \left( \sum_{t=g+3}^{T} \varepsilon_{it-g-1} \varepsilon_{it-g} \right)^4 \right]
\]
\[
= \frac{1}{N^2T^2} \sum_{i=1}^{N} \sum_{g=4}^{T} \sum_{s=4}^{T} \sum_{t=4}^{T} \sum_{u=4}^{T} E [\varepsilon_{ig-2} \varepsilon_{is-2} \varepsilon_{iu-2} \varepsilon_{igt} \varepsilon_{is-1} \varepsilon_{it-1} \varepsilon_{iu-1}]
\]
\[
= \frac{1}{N^2T^2} \sum_{i=1}^{N} \sum_{t=4}^{T} E [\varepsilon_{it-2}^4] E [\varepsilon_{it-1}^4] + \frac{6}{N^2T^2} \sum_{i=1}^{N} \sum_{s=6}^{T} \sum_{t=4}^{T} E [\varepsilon_{is-2}^2] E [\varepsilon_{is-1}^2] E [\varepsilon_{it-2}^2] E [\varepsilon_{it-1}^2]
\]
\[
+ \frac{6}{N^2T^2} \sum_{i=1}^{N} \sum_{t=4}^{T} E [\varepsilon_{it-1}^2] E [\varepsilon_{it-2}^2]
\]
\[
= \left( E [\varepsilon_{it-1}^4] \right)^2 \frac{(T-3)}{NT^2} + 6\sigma^8 \frac{1}{NT^2} \sum_{s=6}^{T} (s-5) + 6E [\varepsilon_{it-1}^4] \frac{\sigma^4 (T-4)}{NT^2}
\]
\[
= O \left( \frac{1}{NT} \right).
\]

Since the Lyapounov-type condition (14) implies Lindeberg-type condition (13) given in Lemma SE-10 above, it follows from Lemma SE-10 that
\[
U_{N,T} = \frac{1}{\omega_{N,T}} \sum_{i=1}^{N} Y_{i,T} = \frac{1}{\omega_{N,T} \sqrt{T}} \sum_{i=1}^{N} \sum_{t=g+3}^{T} \varepsilon_{it-g-1} \varepsilon_{it-g}
\]
\[
= \frac{1}{\sigma^2 \sqrt{NT}} \sum_{i=1}^{N} \sum_{t=g+3}^{T} \varepsilon_{it-g-1} \varepsilon_{it-g} + o_p(1) \Rightarrow N(0,1).
\]

Furthermore, we deduce from the Cramér convergence theorem that
\[
\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=g+3}^{T} \varepsilon_{it-g-1} \varepsilon_{it-g} \Rightarrow N(0,\sigma^4),
\]
as required.

Finally, to show part (i), note that
\[
E \left[ \sum_{i=1}^{N} \varepsilon_{i1} \varepsilon_{i2} \right]^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} E [\varepsilon_{i1} \varepsilon_{i2} \varepsilon_{j1} \varepsilon_{j2}] = \sum_{i=1}^{N} E [\varepsilon_{i1}^2] E [\varepsilon_{i2}^2] = \sigma^4 N,
\]
so that
\[
\sum_{i=1}^{N} \varepsilon_{i1} \varepsilon_{i2} = O_p \left( \sqrt{N} \right),
\]
as desired for part (i).

**Lemma SE-12:**
Suppose that Assumptions 1 and 4 hold. Then, for $T/q(T) \to 0$ and $r \in (0,1]$ as $T \to \infty$, where

$$w_{i,[Tr]} = \sum_{j=1}^{[Tr]} \exp \left( - \frac{([Tr] - j)}{q(T)} \right) \varepsilon_{ij} + \exp \left\{ - \frac{[Tr]}{q(T)} \right\} w_{i0}.$$  

Proof of Lemma SE-12: 
To proceed, for $(j-1)/T \leq s < j/T$, set

$$\int_{(j-1)/T}^{j/T} dX_{i,T}(s) = \frac{1}{\sigma \sqrt{T}} \varepsilon_{ij}$$
and note that, since $w_{i0} = O_p(1)$ for all $i$ in light of Assumption 4 and the Markov’s inequality, we have that

$$\frac{1}{\sqrt{T}} w_{i,[Tr]} = \frac{1}{\sqrt{T}} \sum_{j=1}^{[Tr]} \exp \left( - \frac{([Tr] - j)}{q(T)} \right) \varepsilon_{ij} + \frac{1}{\sqrt{T}} \exp \left\{ - \frac{[Tr]}{q(T)} \right\} w_{i0}$$

$$= \sigma \sum_{j=1}^{[Tr]} \exp \left( - \frac{([Tr] - j)}{q(T)} \right) \int_{(j-1)/T}^{j/T} dX_{i,T}(s) + o_p(1)$$

$$= \sigma \sum_{j=1}^{[Tr]} \int_{(j-1)/T}^{j/T} \exp \left\{ - \frac{T}{q(T)} \frac{[Tr] - j}{T} \right\} \int_{(j-1)/T}^{j/T} dX_{i,T}(s) + o_p(1)$$

$$= \sigma \sum_{j=1}^{[Tr]} \int_{(j-1)/T}^{j/T} \exp \left\{ - \frac{T}{q(T)} \left( s - \frac{j}{T} \right) \right\} \exp \left\{ - \frac{T}{q(T)} \frac{[Tr]}{T} - r \right\}$$

$$\times \exp \left\{ - \frac{T}{q(T)} \left( s - \frac{j}{T} \right) \right\} dX_{i,T}(s) + o_p(1).$$

Now, for $(j-1)/T \leq s < j/T$ and $j = 1,...,[Tr]$

$$\int_{(j-1)/T}^{j/T} \exp \left\{ - \frac{T}{q(T)} \left( r - s \right) \right\} \mathbb{I} \{ \varepsilon_{ij} \geq 0 \} \ dX_{i,T}(s)$$

$$\leq \int_{(j-1)/T}^{j/T} \exp \left\{ - \frac{T}{q(T)} \left( r - s \right) \right\} \exp \left\{ - \frac{T}{q(T)} \frac{[Tr]}{T} - r \right\}$$

$$\times \exp \left\{ - \frac{T}{q(T)} \left( s - \frac{j}{T} \right) \right\} \mathbb{I} \{ \varepsilon_{ij} \geq 0 \} \ dX_{i,T}(s)$$

$$\leq \exp \left\{ \frac{2}{q(T)} \right\} \int_{(j-1)/T}^{j/T} \exp \left\{ - \frac{T}{q(T)} \left( r - s \right) \right\} \mathbb{I} \{ \varepsilon_{ij} \geq 0 \} \ dX_{i,T}(s)$$

$$= \int_{(j-1)/T}^{j/T} \exp \left\{ - \frac{T}{q(T)} \left( r - s \right) \right\} \mathbb{I} \{ \varepsilon_{ij} \geq 0 \} \ dX_{i,T}(s) \left[ 1 + O \left( \frac{1}{q(T)} \right) \right],$$
so that

\[
\int_{(j-1)/T}^{j/T} \exp\left\{ -\frac{T}{T} \frac{1}{q(T)} (r - s) \right\} \exp\left\{ -\frac{T}{T} \frac{[Tr]}{T} - r \right\} \
\times \exp\left\{ -\frac{T}{T} \frac{1}{q(T)} \left[ s - \frac{j}{T} \right] \right\} \mathbb{1} \{ \varepsilon_{ij} \geq 0 \} \, dX_{i,T}(s)
\]

\[
= \int_{(j-1)/T}^{j/T} \exp\left\{ -\frac{T}{T} \frac{1}{q(T)} (r - s) \right\} \mathbb{1} \{ \varepsilon_{ij} \geq 0 \} \, dX_{i,T}(s) \left[ 1 + O\left( \frac{1}{q(T)} \right) \right]
\]

Moreover,

\[
\int_{(j-1)/T}^{j/T} \exp\left\{ -\frac{T}{T} \frac{1}{q(T)} (r - s) \right\} \mathbb{1} \{ \varepsilon_{ij} < 0 \} \, dX_{i,T}(s)
\]

\[
\geq \int_{(j-1)/T}^{j/T} \exp\left\{ -\frac{T}{T} \frac{1}{q(T)} (r - s) \right\} \exp\left\{ -\frac{T}{T} \frac{[Tr]}{T} - r \right\} \
\times \exp\left\{ -\frac{T}{T} \frac{1}{q(T)} \left[ s - \frac{j}{T} \right] \right\} \mathbb{1} \{ \varepsilon_{ij} < 0 \} \, dX_{i,T}(s)
\]

\[
\geq \exp\left\{ -\frac{T}{q(T)} \right\} \int_{(j-1)/T}^{j/T} \exp\left\{ -\frac{T}{T} \frac{1}{q(T)} (r - s) \right\} \mathbb{1} \{ \varepsilon_{ij} < 0 \} \, dX_{i,T}(s)
\]

\[
= \int_{(j-1)/T}^{j/T} \exp\left\{ -\frac{T}{T} \frac{1}{q(T)} (r - s) \right\} \mathbb{1} \{ \varepsilon_{ij} < 0 \} \, dX_{i,T}(s) \left[ 1 + O\left( \frac{1}{q(T)} \right) \right],
\]

so that

\[
\int_{(j-1)/T}^{j/T} \exp\left\{ -\frac{T}{T} \frac{1}{q(T)} (r - s) \right\} \exp\left\{ -\frac{T}{T} \frac{[Tr]}{T} - r \right\} \
\times \exp\left\{ -\frac{T}{T} \frac{1}{q(T)} \left[ s - \frac{j}{T} \right] \right\} \mathbb{1} \{ \varepsilon_{ij} < 0 \} \, dX_{i,T}(s)
\]

\[
= \int_{(j-1)/T}^{j/T} \exp\left\{ -\frac{T}{T} \frac{1}{q(T)} (r - s) \right\} \mathbb{1} \{ \varepsilon_{ij} < 0 \} \, dX_{i,T}(s) \left[ 1 + O\left( \frac{1}{q(T)} \right) \right].
\]
It follows that

\[
\int_{(j-1)/T}^{j/T} \exp \left\{ -\frac{T}{q(T)} (r - s) \right\} \exp \left\{ -\frac{T}{q(T)} \left[ \frac{[Tr]}{T} - r \right] \right\} dX_{i,T}(s)
\]

\[
= \int_{(j-1)/T}^{j/T} \exp \left\{ -\frac{T}{q(T)} (r - s) \right\} \exp \left\{ -\frac{T}{q(T)} \left[ \frac{[Tr]}{T} - r \right] \right\} dX_{i,T}(s)
\]

\[
\times \exp \left\{ -\frac{T}{q(T)} \left[ s - \frac{j}{T} \right] \right\} \mathbb{I} \{ \varepsilon_{ij} < 0 \} dX_{i,T}(s)
\]

\[
+ \int_{(j-1)/T}^{j/T} \exp \left\{ -\frac{T}{q(T)} (r - s) \right\} \exp \left\{ -\frac{T}{q(T)} \left[ \frac{[Tr]}{T} - r \right] \right\} dX_{i,T}(s)
\]

\[
\times \exp \left\{ -\frac{T}{q(T)} \left[ s - \frac{j}{T} \right] \right\} \mathbb{I} \{ \varepsilon_{ij} \geq 0 \} dX_{i,T}(s)
\]

\[
= \int_{(j-1)/T}^{j/T} \exp \left\{ -\frac{T}{q(T)} (r - s) \right\} \mathbb{I} \{ \varepsilon_{ij} < 0 \} dX_{i,T}(s) \left[ 1 + O \left( \frac{1}{q(T)} \right) \right]
\]

\[
+ \int_{(j-1)/T}^{j/T} \exp \left\{ -\frac{T}{q(T)} (r - s) \right\} \mathbb{I} \{ \varepsilon_{ij} \geq 0 \} dX_{i,T}(s) \left[ 1 + O \left( \frac{1}{q(T)} \right) \right]
\]

\[
= \int_{(j-1)/T}^{j/T} \exp \left\{ -\frac{T}{q(T)} (r - s) \right\} dX_{i,T}(s) \left[ 1 + O \left( \frac{1}{q(T)} \right) \right].
\]

Hence,

\[
\frac{1}{\sqrt{T}} W_{i,[Tr]}
\]

\[
= \sigma \sum_{j=1}^{[Tr]} \int_{(j-1)/T}^{j/T} \exp \left\{ -\frac{T}{q(T)} (r - s) \right\} \exp \left\{ -\frac{T}{q(T)} \left[ \frac{[Tr]}{T} - r \right] \right\} dX_{i,T}(s) + o_p(1)
\]

\[
= \sigma \sum_{j=1}^{[Tr]} \int_{(j-1)/T}^{j/T} \exp \left\{ -\frac{T}{q(T)} (r - s) \right\} dX_{i,T}(s) \left[ 1 + O \left( \frac{1}{q(T)} \right) \right] + o_p(1)
\]

\[
= \sigma \int_0^r \exp \left\{ -\frac{T}{q(T)} (r - s) \right\} dX_{i,T}(s) \left[ 1 + O \left( \frac{1}{q(T)} \right) \right] + o_p(1)
\]

\[
= \sigma X_{i,T}(r) - \frac{T}{q(T)} \int_0^r \sigma \exp \left( -\frac{T}{q(T)} (r - s) \right) X_{i,T}(s) ds + o_p(1)
\]

\[
= \sigma X_{i,T}(r) + O_p \left( \frac{T}{q(T)} \right) + o_p(1)
\]

\[
\Rightarrow \sigma W_{i}(r),
\]

as \( T \to \infty \), which is the required result. □

**Lemma SE-13:** (Phillips and Moon, 1999, Corollary 1): Suppose that \( Y_{i,T} = C_i Q_{i,T} \), where \( Q_{i,T} \) are i.i.d. across \( i \) for all \( T \), and the \( C_i \) are \((m \times m)\) nonrandom matrices for all \( i \). Assume that
\( Q_{i,T} \) are integrable for all \( T \) and \( Q_{i,T} \Rightarrow Q_i \) as \( T \rightarrow \infty \). Assume that \( C = \lim_N \frac{1}{N} \sum_{i=1}^{N} C_i \) exists. If \( \|Q_{i,T}\| \) is uniformly integrable in \( T \) for all \( i \), and if \( \sup_i \|C_i\| < \infty \), then
\[
\frac{1}{N} \sum_{i=1}^{N} Y_{i,T} \overset{p}{\rightarrow} CE [Q_i]
\]
as \( N, T \rightarrow \infty \).

In the subsequent lemmas, we find it useful to decompose \( w_{it} \) as
\[
w_{it} = w_{it} + \rho_T w_{i0}
\]
where \( w_{it} = \sum_{j=1}^{t} \rho_T^{(t-j)} \varepsilon_{ij} \).

**Lemma SE-14:**

Suppose that Assumptions 1 and 4 hold. If \( \rho_T = 1 \) for all \( T \) sufficiently large, then, as \( N, T \rightarrow \infty \),
\[
\frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{it-g,T}^2 \overset{p}{\rightarrow} \frac{\sigma^2}{2} \text{ for } g \in \{1, 2\}.
\]

**Proof of Lemma SE-14:**

To proceed, note first that, under the assumption here, there exists a positive integer \( I_\rho \) such that for all \( T \geq I_\rho \), the triangular array process \( \{w_{it-g,T}\} \) has the partial sum representation \( w_{it-1,T} = \sum_{j=1}^{t-g} \varepsilon_{ij} \). Thus, for all \( T \) sufficiently large, we can write
\[
\frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{it-g,T}^2 = \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{it-g,T}^2 + 2 \frac{1}{NT^{3/2}} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{it-g,T} w_{i0} + \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{i0}^2
\]
\[
= \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} \left( \frac{1}{\sqrt{T}} \sum_{j=1}^{t-g} \varepsilon_{ij} \right)^2 + 2 \frac{1}{NT^{3/2}} \sum_{i=1}^{N} \sum_{t=2}^{T} \left( \frac{1}{\sqrt{T}} \sum_{j=1}^{t-g} \varepsilon_{ij} \right) w_{i0} + \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{i0}^2.
\]

Consider first the case where \( g = 1 \). Define
\[
Q_{i,T} = \frac{1}{T} \sum_{t=2}^{T} \left( \frac{1}{\sqrt{T}} \sum_{j=1}^{t-1} \varepsilon_{ij} \right)^2.
\]

Note that we can apply arguments similar to that given in the proof of Theorem 3.1 part (a) in Phillips (1987) to obtain
\[
Q_{i,T} \Rightarrow \sigma^2 \int_0^{1} \left[ W_i (r) \right]^2 dr = Q_i \quad (say).
\]

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Hence, all the conditions of Lemma SE-13 are satisfied, and we deduce from this lemma that

\[
E[|Q_{i,T}|] = E[Q_{i,T}] = \frac{1}{T^2} \sum_{t=2}^{T} \sum_{i=1}^{N-1} \sum_{k=1}^{T-1} E[\varepsilon_{ij}\varepsilon_{ik}]
\]

so that

\[
\sigma^2 \frac{1}{T^2} \sum_{t=2}^{T} (t-1) = \sigma^2 \frac{1}{T^2} \frac{T(T-1)}{2} \to \frac{\sigma^2}{2},
\]

so that

\[
\lim_{T \to \infty} E[|Q_{i,T}|] = \frac{\sigma^2}{2} = \sigma^2 \int_{0}^{1} E[W_i(g)]^2 \, dg = E[Q_i] \text{ for all } i.
\]

It follows from Theorem 5.4 of Billingsley (1968) that \{|Q_{i,T}|\} is uniformly integrable in \(T\) for all \(i\). Hence, all the conditions of Lemma SE-13 are satisfied, and we deduce from this lemma that

\[
\frac{1}{N} \sum_{i=1}^{N} Q_{i,T} = \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1} \overset{p}{\to} CE[Q_i] = \frac{\sigma^2}{2},
\]

as \(N, T \to \infty\).

Now, by Assumption 4, there exists a positive constant \(C\) such that

\[
E\left[\frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it}^2\right] = \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} E[w_{it}^2] \leq \sup_{i} E[w_{i0}^2] \frac{(T-1)}{T^2} \leq C\frac{(T-1)}{T^2} = O(T^{-1})
\]

so that, applying Markov’s inequality, we obtain

\[
\frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it}^2 = O_p\left(\frac{1}{T}\right).
\]

It follows from the Cauchy-Schwarz inequality that

\[
\left|\frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1,T} w_{i0}\right| \leq \sqrt{\frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it}^2} \sqrt{\frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{i0}^2} = O_p(1) \times O_p\left(T^{-1/2}\right)
\]

so that by the Cramér convergence theorem, we deduce that

\[
\frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{it-1,T} \overset{p}{\to} \frac{\sigma^2}{2}.
\]

Next, consider the case \(g = 2\). Here, note that, for \(T\) sufficiently large,

\[
\frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-2,T}^2 = \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1,T}^2 - \frac{1}{NT^2} \sum_{i=1}^{N} w_{iT-1,T}^2
\]

\[
= \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1,T}^2 - \frac{1}{NT^2} \sum_{i=1}^{N} \left(\sum_{j=1}^{T-1} \varepsilon_{ij}\right)^2 - 2 \frac{1}{NT^2} \sum_{i=1}^{N} \left(\sum_{j=1}^{T-1} \varepsilon_{ij}\right) w_{i0} - \frac{1}{NT^2} \sum_{i=1}^{N} w_{i0}^2.
\]
Now, using Assumption 4
\[ E \left( \frac{1}{NT^2} \sum_{i=1}^{N} w_{i0}^2 \right) = \frac{1}{NT^2} \sum_{i=1}^{N} E[w_{i0}^2] \leq \frac{C}{T^2} = O_p \left( \frac{1}{T^2} \right). \]

Moreover,
\[ E \left( \frac{1}{NT^2} \sum_{i=1}^{N} \left( \sum_{j=1}^{T-1} \varepsilon_{ij} \right)^2 \right) = \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{j=1}^{T-1} \sum_{k=1}^{T-1} E[\varepsilon_{ij}\varepsilon_{ik}] = \sigma^2 \frac{T-1}{T^2} = O \left( \frac{1}{T} \right), \]
so that by Markov’s inequality, we deduce that
\[ \frac{1}{NT^2} \sum_{i=1}^{N} w_{i0}^2 = O_p \left( \frac{1}{T^2} \right), \quad \frac{1}{NT^2} \sum_{i=1}^{N} \left( \sum_{j=1}^{T-1} \varepsilon_{ij} \right)^2 = O \left( \frac{1}{T} \right). \]
The Cauchy-Schwarz inequality further implies that
\[ \left| \frac{1}{NT^2} \sum_{i=1}^{N} \left( \sum_{j=1}^{T-1} \varepsilon_{ij} \right) w_{i0} \right| \leq \frac{1}{NT^2} \sum_{i=1}^{N} \left( \sum_{j=1}^{T-1} \varepsilon_{ij} \right)^2 \left( \sum_{i=1}^{N} w_{i0}^2 \right) = O_p \left( \frac{1}{\sqrt{T}} \right) O_p \left( \frac{1}{T} \right) = O_p \left( \frac{1}{T^{3/2}} \right). \]
It follows that
\[ \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=3}^{T} w_{it-2, T}^2 = \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1, T}^2 + O_p \left( \frac{1}{T} \right) = \frac{\sigma^2}{2} + o_p (1), \]
as required. \( \square \)

**Lemma SE-15:**

Suppose that Assumptions 1 and 4 hold. If \( \rho_T = \exp \{-1/q(T)\} \) such that \( T/q(T) \to 0 \), then, as \( N,T \to \infty \),
\[ \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{it-g, T}^2 \to \frac{\sigma^2}{2} \text{ for } g \in \{1,2\}. \]

**Proof of Lemma SE-15:**

To proceed, first write
\[ w_{it-g, T}^2 = w_{it-g, T}^2 + 2w_{it-g} \rho_T^{t-g} w_{i0} + \rho_T^{2(t-g)} w_{i0}^2 \]
\[ = \left( \sum_{j=1}^{t-g} \exp \left\{ -\frac{(t-g-j)}{q(T)} \right\} \varepsilon_{ij} \right)^2 + 2 \left( \sum_{j=1}^{t-g} \exp \left\{ -\frac{(t-g-j)}{q(T)} \right\} \varepsilon_{ij} \right) \exp \left\{ -\frac{(t-g)}{q(T)} \right\} w_{i0} \]
\[ + \exp \left\{ -\frac{2(t-g)}{q(T)} \right\} w_{i0}^2 \]
Now, consider the case where \( g = 1 \). In this case, note that

\[
\frac{1}{T^2} \sum_{t=2}^{T} w_{t-1,T}^2 = \frac{1}{T^2} \sum_{t=1}^{T} \left( \sum_{j=1}^{t-1} \exp \left\{ -\frac{(t-1-j)}{q(T)} \right\} \varepsilon_{ij} \right)^2
\]

\[
= \sigma^2 \frac{1}{T} \sum_{t=1}^{T} \left[ \sum_{j=1}^{t-1} \exp \left\{ -\frac{T}{q(T)} \left( \frac{t-1-j}{T} \right) \right\} \varepsilon_{ij} \right]^2
\]

\[
= \sigma^2 \sum_{t=1}^{T} \exp \left\{ -\frac{2T}{q(T)} \left( \frac{t-1}{T} \right) \right\} \frac{1}{\sigma \sqrt{T}} \left[ \sum_{j=1}^{t-1} \exp \left\{ \frac{T}{q(T)} \frac{j}{\sigma \sqrt{T}} \right\} \varepsilon_{ij} \right]^2.
\]

Now, for \( 0 < r < \frac{T}{T-1} \), we define

\[
X_{i,T}(r) = \frac{1}{\sigma \sqrt{T}} \sum_{j=1}^{[Tr]} \varepsilon_{ij} = \frac{1}{\sigma \sqrt{T}} \sum_{j=1}^{t-1} \varepsilon_{ij}, \quad (15)
\]

\[
\tilde{X}_{i,T}(r) = \sum_{j=1}^{t-1} \exp \left\{ \frac{T}{q(T)} \frac{j}{\sigma \sqrt{T}} \right\} \varepsilon_{ij}. \quad (16)
\]

Next, write

\[
\frac{1}{T^2} \sum_{t=1}^{T} w_{t-1,T}^2 = \sigma^2 \sum_{t=1}^{T} \exp \left\{ -\frac{2T}{q(T)} \left( \frac{t-1}{T} \right) \right\} \int_{(t-1)/T}^{t/T} \tilde{X}_{i,T}^2(r) \, dr
\]

\[
= \sigma^2 \int_{(t-1)/T}^{t/T} \exp \left\{ -\frac{2T}{q(T)} \left[ \left( \frac{t-1}{T} \right) - r + \frac{T}{T-1} \right] \right\} \tilde{X}_{i,T}^2(r) \, dr
\]

\[
= \sigma^2 \int_{0}^{1} \exp \left\{ -\frac{2T}{q(T)} r \right\} \exp \left\{ \frac{2T}{q(T)} \left[r - \left( \frac{t-1}{T} \right) \right] \right\} \tilde{X}_{i,T}^2(r) \, dr,
\]

and note that for \( t = 1, ..., T \),

\[
\int_{(t-1)/T}^{t/T} \exp \left\{ -\frac{2Tr}{q(T)} \right\} \tilde{X}_{i,T}^2(r) \, dr \leq \int_{(t-1)/T}^{t/T} \exp \left\{ -\frac{2Tr}{q(T)} \right\} \exp \left\{ \frac{2T}{q(T)} \left[r - \left( \frac{t-1}{T} \right) \right] \right\} \tilde{X}_{i,T}^2(r) \, dr
\]

\[
\leq \exp \left\{ \frac{2}{q(T)} \right\} \int_{(t-1)/T}^{t/T} \exp \left\{ -\frac{2T}{q(T)} r \right\} \tilde{X}_{i,T}^2(r) \, dr,
\]

from which it follows that

\[
\sigma^2 \int_{0}^{1} \exp \left\{ -\frac{2Tr}{q(T)} \right\} \tilde{X}_{i,T}^2(r) \, dr \leq \sigma^2 \int_{0}^{1} \exp \left\{ -\frac{2Tr}{q(T)} \right\} \exp \left\{ \frac{2T}{q(T)} \left[r - \left( \frac{t-1}{T} \right) \right] \right\} \tilde{X}_{i,T}^2(r) \, dr
\]

\[
\leq \sigma^2 \exp \left\{ \frac{2}{q(T)} \right\} \int_{0}^{1} \exp \left\{ -\frac{2T}{q(T)} r \right\} \tilde{X}_{i,T}^2(r) \, dr
\]

\[
= \sigma^2 \int_{0}^{1} \exp \left\{ -\frac{2Tr}{q(T)} \right\} \tilde{X}_{i,T}^2(r) \, dr \left[1 + O_p \left( \frac{1}{q(T)} \right) \right],
\]
so that

\[
\sigma^2 \int_0^1 \exp \left\{ -\frac{2T}{q(T)} r \right\} \exp \left\{ \frac{2T}{q(T)} \left[ r - \left( \frac{t-1}{T} \right) \right] \right\} \tilde{X}_{r,T}^2(r) \, dr = \sigma^2 \int_0^1 \exp \left\{ -\frac{2T}{q(T)} r \right\} \tilde{X}_{r,T}^2(r) \, dr \left[ 1 + O_p \left( \frac{1}{q(T)} \right) \right].
\]

By similar argument, we have

\[
\int_{(j-1)/T}^{j/T} \exp \left\{ \frac{T}{q(T)} s \right\} \exp \left\{ \frac{T}{q(T)} \left[ \frac{j}{T} - s \right] \right\} dX_{i,T}(s) = \int_{(j-1)/T}^{j/T} \exp \left\{ \frac{T}{q(T)} s \right\} dX_{i,T}(s) \left[ 1 + O_p \left( \frac{1}{q(T)} \right) \right],
\]

(17)

for \( j = 1, \ldots, T \) and \( (j-1)/T \leq s < j/T \).

Moreover, for \( j = 1, \ldots, t \) we have

\[
\int_{(j-1)/T}^{j/T} dX_{i,T}(s) = X_{i,T} \left( \frac{j}{T} \right) - X_{i,T} \left( \frac{j-1}{T} \right) = \frac{1}{\sigma \sqrt{T}} \left\{ \sum_{k=1}^{j} \varepsilon_{ik} - \sum_{k=1}^{j-1} \varepsilon_{ik} \right\} = \frac{1}{\sigma \sqrt{T}} \varepsilon_{ij}.
\]

(18)

Using (18), we can define for \( (t-1)/T \leq r < t/T \),

\[
\tilde{X}_r = \sum_{j=1}^{t-1} \exp \left\{ \frac{T}{q(T)} \frac{j}{T} \right\} \varepsilon_{ij} \sigma \sqrt{T} \exp \left\{ \frac{T}{q(T)} \left[ \frac{j}{T} - \frac{r}{T} \right] \right\} dX_{i,T}(s)
\]

(19)

\[
= \sum_{j=1}^{t-1} \int_{(j-1)/T}^{j/T} \exp \left\{ \frac{T}{q(T)} s \right\} \exp \left\{ \frac{T}{q(T)} \left[ \frac{j}{T} - s \right] \right\} dX_{i,T}(s)
\]

\[
= \sum_{j=1}^{t-1} \int_{(j-1)/T}^{j/T} \exp \left\{ \frac{T}{q(T)} s \right\} \exp \left\{ \frac{T}{q(T)} \left[ \frac{j}{T} - s \right] \right\} dX_{i,T}(s)
\]

\[
= \sum_{j=1}^{t-1} \int_{(j-1)/T}^{j/T} \exp \left\{ \frac{T}{q(T)} s \right\} dX_{i,T}(s) \left[ 1 + O_p \left( \frac{1}{q(T)} \right) \right] \quad \text{(by (17))}
\]

\[
= \int_0^{\frac{T}{T}} \exp \left\{ \frac{T}{q(T)} s \right\} dX_{i,T}(s) \left[ 1 + O_p \left( \frac{1}{q(T)} \right) \right]
\]

\[
= \int_0^{\frac{T}{T}} \exp \left\{ \frac{T}{q(T)} s \right\} dX_{i,T}(s) \left[ 1 + O_p \left( \frac{1}{T} \right) \right]
\]

\[
= \left( \exp \left\{ \frac{T}{q(T)} s \right\} X_{i,T}(s) \right)_{s=0}^{r} + \int_0^{r} -\frac{T}{q(T)} \exp \left\{ \frac{T}{q(T)} s \right\} X_{i,T}(s) \, ds \left[ 1 + O_p \left( \frac{1}{q(T)} \right) \right]
\]

\[
= \left[ \exp \left\{ \frac{T}{q(T)} r \right\} X_{i,T}(r) - \frac{T}{q(T)} \int_0^{r} \exp \left\{ \frac{T}{q(T)} s \right\} X_{i,T}(s) \, ds \right] \left[ 1 + O_p \left( \frac{1}{q(T)} \right) \right].
\]
so that
\[
\frac{1}{T^2} \sum_{t=1}^{T} w_{it-1,T}^2
\]
\[
= \sigma^2 \int_0^1 \exp \left\{ -\frac{2T}{q(T)} r \right\} \bar{X}_{i,T}^2 (r) \, dr \left[ 1 + O_p \left( \frac{1}{q(T)} \right) \right]
\]
\[
= \sigma^2 \int_0^1 \exp \left\{ -\frac{2T}{q(T)} r \right\} \left\{ \exp \left\{ \frac{T}{q(T)} r \right\} X_{i,T} (r) - \frac{T}{q(T)} \int_0^r \exp \left\{ \frac{T}{q(T)} s \right\} X_{i,T} (s) \, ds \right\}^2 \, dr
\]
\[
\times \left[ 1 + O_p \left( \frac{1}{q(T)} \right) \right]
\]
\[
= \sigma^2 \int_0^1 \exp \left\{ -\frac{2T}{q(T)} r \right\} \exp \left\{ \frac{2T}{q(T)} r \right\} \left\{ X_{i,T} (r) - \frac{T}{q(T)} \int_0^r \exp \left\{ \frac{T}{q(T)} [s - r] \right\} X_{i,T} (s) \, ds \right\}^2 \, dr
\]
\[
\times \left[ 1 + O_p \left( \frac{1}{q(T)} \right) \right]
\]
\[
= \sigma^2 \int_0^1 \left[ X_{i,T} (r) - \frac{T}{q(T)} \int_0^r \exp \left\{ \frac{T}{q(T)} [s - r] \right\} X_{i,T} (s) \, ds \right]^2 \, dr \left[ 1 + O_p \left( \frac{1}{q(T)} \right) \right]
\]
\[
= \sigma^2 \int_0^1 \left[ X_{i,T} (g) \right]^2 \, dr \left[ 1 + O_p \left( \frac{T}{q(T)} \right) + O_p \left( \frac{1}{q(T)} \right) \right]
\]
\[
= \sigma^2 \int_0^1 \left[ X_{i,T} (g) \right]^2 \, dr \left[ 1 + O_p \left( \frac{T}{q(T)} \right) \right].
\]

Hence, by the continuous mapping theorem,
\[
\frac{1}{T^2} \sum_{t=1}^{T} w_{it-1,T}^2 \Rightarrow \sigma^2 \int_0^1 [W_i (r)]^2 \, dr
\]
as \( T \to \infty \).

Next, we verify the conditions of Corollary 1 of Phillips and Moon (1999), given here as Lemma SE-13. To proceed, define
\[
Q_{i,T} = \frac{1}{T^2} \sum_{t=2}^{T} w_{it-1}^2.
\]
Note first that \( Q_{i,T} \) is integrable in light of Assumption 1; and, by the argument given previously, we have that, as \( T \to \infty \),
\[
Q_{i,T} \Rightarrow \sigma^2 \int_0^1 [W_i (r)]^2 \, dr = Q_i \quad (\text{say}).
\]
Moreover, in this case, \( C_i = 1 \) for all \( i \), so that trivially, \( C = \lim_N (1/N) \sum_{i=1}^{N} C_i = 1 < \infty \) and...
Hence, all the conditions of Lemma SE-13 are satisfied, and we deduce from this lemma that

$$
E \left[ |Q_{i,T}| \right] = \frac{1}{T^2} \sum_{t=1}^{T} E \left[ w_{it-1,T}^2 \right]
$$

so that

$$
\lim_{T \to \infty} E \left[ |Q_{i,T}| \right] = \frac{\sigma^2}{2} = \sigma^2 \int_{0}^{1} E \left[ W_i(g) \right]^2 dg = E \left[ Q_i \right] \text{ for all } i.
$$

It follows from Theorem 5.4 of Billingsley (1968) that \{\{Q_{i,T}\}\} is uniformly integrable in $T$ for all $i$. Hence, all the conditions of Lemma SE-13 are satisfied, and we deduce from this lemma that

$$
\frac{1}{N} \sum_{i=1}^{N} Q_{i,T} = \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1}^2 \overset{p}{\rightarrow} CE \left[ Q_i \right] = \frac{\sigma^2}{2},
$$

as $N, T \to \infty$.

In addition, note that, by Assumption 4, there exists a positive constant $C$ such that

$$
E \left[ \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} \rho_T^{2(t-1)} \sigma_{i0}^2 \right]
$$

$$
\leq \sup_{i} E \left[ \sigma_{i0}^2 \right] \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} \rho_T^{2(t-1)}
$$

$$
\leq \frac{C}{T^2} \exp \left\{ -\frac{2}{q(T)} \right\} \left[ 1 - \exp \left\{ -\frac{2}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{ -\frac{2(T-1)}{q(T)} \right\} \right]
$$

$$
= O \left( T^{-2} \right) O \left( q(T) \right) O \left( T/q(T) \right) = O \left( T^{-1} \right),
$$

so that, applying Markov’s inequality, we obtain

$$
\frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} \rho_T^{2(t-1)} w_{i0}^2 = O_p \left( \frac{1}{T} \right).
$$

It follows from the Cauchy-Schwarz inequality that

$$
\left| \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1}^2 \rho_T^{t-1} \sigma_{i0}^2 \right| \leq \sqrt{ \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} \rho_T^{2(t-1)} w_{i0}^2 } \sqrt{ \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} \rho_T^{2(t-1)} \sigma_{i0}^2 } = O_p \left( T^{-1/2} \right) = O_p \left( T^{-1/2} \right),
$$

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so that, using Slutsky’s Theorem, we further deduce that

\[
\frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1,T}^2 = \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1,T}^2 + \frac{2}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1,T} \rho_t^{-1} w_{i0} + \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} \rho_t^{2(i-1)} w_{i0}^2 \rho_t^{-1} \sigma^2.
\]

Next, consider the case \( g = 2 \). Here, note that in this case

\[
\frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=3}^{T} w_{it-2,T}^2 = \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1,T}^2 - \frac{1}{NT^2} \sum_{i=1}^{N} w_{it-1,T}^2
\]

\[
= \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1,T}^2 - \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=1}^{T-1} \exp \left( -\frac{(T-1-j)}{q(T)} \right) \varepsilon_{ij}^2.
\]

Now,

\[
E \left( \frac{1}{T^2} \sum_{j=1}^{T-1} \exp \left\{ -\frac{(T-1-j)}{q(T)} \right\} \varepsilon_{ij}^2 \right)^2
\]

\[
= \frac{1}{T^2} \sum_{j=1}^{T-1} \sum_{k=1}^{T-1} \exp \left\{ -\frac{(T-1-j)}{q(T)} \right\} \exp \left\{ -\frac{(T-1-k)}{q(T)} \right\} E[\varepsilon_{ij} \varepsilon_{ik}]
\]

\[
= \sigma^2 \frac{1}{T^2} \sum_{j=1}^{T-1} \exp \left\{ -\frac{2(T-1-j)}{q(T)} \right\}.
\]

Applying part (a) of Lemma SE-3 with \( b = 2 \) and \( g = 1 \), we obtain

\[
\sigma^2 \frac{1}{T^2} \sum_{j=1}^{T-1} \exp \left\{ -2 \left( \frac{T-1-j}{q(T)} \right) \right\}
\]

\[
= \sigma^2 \frac{1}{T} \left[ 1 - \frac{1}{T} - \frac{T}{q(T)} + O \left( \max \left\{ \frac{T^2}{q(T)^2}, \frac{1}{q(T)} \right\} \right) \right]
\]

\[
= O \left( T^{-1} \right).
\]

It follows from Markov’s inequality that

\[
\frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=3}^{T} w_{it-2,T}^2 = \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1,T}^2 + O_p \left( \frac{1}{\sqrt{T}} \right) = \frac{\sigma^2}{2} + o_p \left( 1 \right).
\]
Moreover, note that, again by Assumption 4, we obtain

\[
E \left[ \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=3}^{T} \rho_T^{2(t-2)} w_{i0}^2 \right] 
\leq \frac{C}{T^2} \exp \left\{ -\frac{2}{q(T)} \right\} \left[ 1 - \exp \left\{ -\frac{2}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{ -\frac{2(T-2)}{q(T)} \right\} \right] = O(T^{-2}) O(1) O(q(T)) O(T/q(T)) = O(T^{-1}),
\]
from which we deduce, using Markov’s inequality, that

\[
\frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=3}^{T} \rho_T^{2(t-2)} w_{i0}^2 = O_p \left( \frac{1}{T} \right)
\]
The Cauchy-Schwarz inequality then further implies that

\[
\left| \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=3}^{T} w_{it-2} \rho_T^{t-1} w_{i0} \right| \leq \left( \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=3}^{T} w_{it-2}^2 \right)^{1/2} \left( \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=3}^{T} \rho_T^{2(t-2)} w_{i0}^2 \right)^{1/2} = O_p(1) O_p \left( T^{-1/2} \right) = O_p \left( T^{-1/2} \right),
\]
so that, by making use of the Slutsky’s Theorem, we get

\[
\frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=3}^{T} w_{it-2}^2 = \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=3}^{T} w_{it-2}^2 + \frac{2}{NT^2} \sum_{i=1}^{N} \sum_{t=3}^{T} \rho_T^{t-2} w_{i0}^2 + \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=3}^{T} \rho_T^{2(t-2)} w_{i0}^2 \rho_T^2 \frac{\sigma^2}{2},
\]
as required. \( \square \)

Lemma SE-16:

Let \( g \) be a non-negative integer. Under Assumptions 1 and 4, the following results hold as \( N, T \to \infty \).

(a) If \( \rho_T = 1 \) for all \( T \) sufficiently large, then

\[
\sum_{i=1}^{N} \sum_{t=g+1}^{T} E \left[ w_{it-g,T}^2 \right] = NT^2 \frac{\sigma^2}{2} \left[ 1 + O \left( \frac{1}{T} \right) \right],
\]

\[
\sum_{i=1}^{N} \sum_{t=g+1}^{T} \sum_{s=g+1}^{t-1} E \left[ w_{it-g,T} w_{is-g,T} \right] = NT^3 \frac{\sigma^2}{6} \left[ 1 + O \left( \frac{1}{T} \right) \right].
\]

(b) If \( \rho_T = \exp \{-1/q(T)\} \) such that \( T/q(T) \to 0 \), then

\[
\sum_{i=1}^{N} \sum_{t=g+1}^{T} E \left[ w_{it-g,T}^2 \right] = NT^2 \frac{\sigma^2}{2} \left[ 1 + O \left( \max \left\{ \frac{T}{q(T)}, \frac{1}{T} \right\} \right) \right],
\]

\[
\sum_{i=1}^{N} \sum_{t=g+2}^{T} \sum_{s=g+1}^{t-1} E \left[ w_{it-g,T} w_{is-g,T} \right] = NT^3 \frac{\sigma^2}{6} \left[ 1 + O \left( \max \left\{ \frac{T}{q(T)}, \frac{1}{T} \right\} \right) \right].
\]
(c) If \( \rho_T = \exp \{-1/q(T)\} \) such that \( q(T) \sim T \), then

\[
\sum_{i=1}^{N} \sum_{t=g+1}^{T} E[w_{it-g,T}^2] = \sigma^2 N\frac{q(T)^2}{4} \left[ \exp \left\{ -\frac{2T}{q(T)} \right\} + \frac{2T}{q(T)} - 1 \right] \left[ 1 + O \left( \frac{1}{T} \right) \right],
\]

and

\[
\sum_{i=1}^{N} \sum_{t=g+2}^{T} \sum_{s=g+1}^{t-1} E[w_{it-g,T} w_{is-g,T}]
= NTq(T)^2 \frac{\sigma^2}{2} \left[ 1 - \frac{3q(T)}{2T} + \frac{2q(T)}{T} \exp \left\{ -\frac{T}{q(T)} \right\} - \frac{1}{T} \frac{q(T)}{q(T)} \exp \left\{ -\frac{2T}{q(T)} \right\} \right] \left[ 1 + O \left( \frac{1}{T} \right) \right]
\]

(d) If \( \rho_T = \exp \{-1/q(T)\} \) such that \( q(T) \to \infty \) but \( q(T)/T \to 0 \), then

\[
\sum_{i=1}^{N} \sum_{t=g+1}^{T} E[w_{it-g,T}^2] = NTq(T) \frac{\sigma^2}{2} \left[ 1 + O \left( \frac{1}{T} \right) \right] + O \left( \frac{q(T)}{T} \right),
\]

and

\[
\sum_{i=1}^{N} \sum_{t=g+2}^{T} \sum_{s=g+1}^{t-1} E[w_{it-g,T} w_{is-g,T}]
= NTq(T)^2 \frac{\sigma^2}{2} \left[ 1 + O \left( \frac{q(T)}{T} \right) \right].
\]

**Proof of Lemma SE-16:**

First consider part (a). Note that, under the assumption here, there exists a positive integer \( I_p \) such that for all \( T \geq \max \{ I_p, g+2 \} \), the triangular array process \( \{w_{it-g,T}\} \) has the partial sum representation \( w_{it-g,T} = \sum_{j=1}^{t-g} \varepsilon_{ij} \). Thus, by direct calculation, we have that, for all \( T \geq \max \{ I_p, g+2 \} \),

\[
\sum_{i=1}^{N} \sum_{t=g+1}^{T} E[w_{it-g,T}^2] = \sum_{i=1}^{N} \sum_{t=g+1}^{T} \sum_{t-g}^{t} \sum_{k=1}^{t-g} E[\varepsilon_{ij} \varepsilon_{ik}] = \sigma^2 N \sum_{t=g+1}^{T} (t-g)
\]

\[
= \frac{\sigma^2}{2} N(T-g)(T-g+1) = NT^2 \frac{\sigma^2}{2} \left[ 1 + O \left( \frac{1}{T} \right) \right],
\]

and

\[
\sum_{i=1}^{N} \sum_{t=g+2}^{T} \sum_{s=g+1}^{t-1} E[w_{it-g,T} w_{is-g,T}]
= \sum_{i=1}^{N} \sum_{t=g+2}^{T} \sum_{s=g+1}^{t} \sum_{j=1}^{t-g} \sum_{k=1}^{t-g} E[\varepsilon_{ij} \varepsilon_{ik}] = \sigma^2 N \sum_{t=g+2}^{T} \sum_{s=g+1}^{t-1} (s-g)
\]

\[
= N \frac{\sigma^2}{2} \sum_{t=g+2}^{T} (t-g)(t-g-1) = N \frac{\sigma^2}{2} \left[ \frac{(T-g)(T-g+1)(2T-2g+1) - (T-g)(T-g+1)}{6} \right]
\]

\[
= NT^3 \frac{\sigma^2}{6} \left[ 1 + O \left( \frac{1}{T} \right) \right],
\]

which completes the proof for part (a).
Now, to show parts (b)-(d), note first that, for all $T \geq g + 2$, we have
\[
\sum_{i=1}^{N} \sum_{t=g+1}^{T} E \left[ w_{it-g,T}^2 \right] = \sum_{i=1}^{N} \sum_{t=g+1}^{T} \sum_{j=1}^{t-g} \sum_{k=1}^{T-t} \exp \left\{ -\frac{(t-g-j)}{q(T)} \right\} \exp \left\{ -\frac{(t-g-k)}{q(T)} \right\} E [\varepsilon_{ij}\varepsilon_{ik}] \\
= \sigma^2 N \sum_{t=g+1}^{T} \sum_{j=1}^{t-g} \exp \left\{ -2\frac{(t-g-j)}{q(T)} \right\},
\]
and
\[
\sum_{i=1}^{N} \sum_{t=g+1}^{T} \sum_{s=g+1}^{T} E \left[ w_{it-g,T}^2 w_{is-g,T} \right] = \sum_{i=1}^{N} \sum_{t=g+1}^{T} \sum_{s=g+1}^{T} \sum_{j=1}^{t-g} \sum_{k=1}^{s-g} \exp \left\{ -\frac{(t-g-j)}{q(T)} \right\} \exp \left\{ -\frac{(s-g-k)}{q(T)} \right\} E [\varepsilon_{ij}\varepsilon_{ik}] \\
= \sigma^2 N \sum_{t=g+1}^{T} \sum_{s=g+1}^{T} \sum_{j=1}^{t-g} \sum_{k=1}^{s-g} \exp \left\{ -\frac{(t-g-k)}{q(T)} \right\} \exp \left\{ -\frac{(s-g-k)}{q(T)} \right\}.
\]

Consider part (b), where we take $\rho_T = \exp \{ -1/q(T) \}$ such that $T/q(T) \to 0$. Applying part (a) of Lemma SE-1 with $b = g + 1$ and $d = 2$, we obtain
\[
\sum_{i=1}^{N} \sum_{t=g+1}^{T} E \left[ w_{it-g,T}^2 \right] = \sigma^2 N \sum_{t=g+1}^{T} \sum_{j=1}^{t-g} \exp \left\{ -2\frac{(t-g-j)}{q(T)} \right\} \\
= NT^2 \sigma^2 \left[ 1 + O \left( \frac{T}{q(T)} \frac{1}{T} \right) \right] \left( \frac{1}{T} \right) + O \left( \frac{T}{q(T)} \right),
\]
while, applying part (a) of Lemma SE-7, we get
\[
\sum_{i=1}^{N} \sum_{t=g+1}^{T} \sum_{s=g+1}^{T} E \left[ w_{it-g,T} w_{is-g,T} \right] = \sigma^2 N \sum_{t=g+1}^{T} \sum_{s=g+1}^{T} \sum_{j=1}^{t-g} \sum_{k=1}^{s-g} \exp \left\{ -\frac{(t-g-k)}{q(T)} \right\} \exp \left\{ -\frac{(s-g-k)}{q(T)} \right\} \\
= N T^3 \sigma^2 \left[ 1 + O \left( \frac{T}{q(T)} \frac{1}{T} \right) \right],
\]
as required for part (b).

We now turn our attention to part (c), where we take $\rho_T = \exp \{ -1/q(T) \}$ such that $q(T) \sim T$. Applying part (b) of Lemma SE-1 with $b = g + 1$ and $d = 2$, we get in this case
\[
\sum_{i=1}^{N} \sum_{t=g+1}^{T} E \left[ w_{it-g,T}^2 \right] = \sigma^2 N \sum_{t=g+1}^{T} \sum_{j=1}^{t-g} \exp \left\{ -2\frac{(t-g-j)}{q(T)} \right\} \\
= \sigma^2 N \frac{q(T)^2}{4} \left[ \exp \left\{ -\frac{2T}{q(T)} \right\} + \frac{2T}{q(T)} - 1 \right] \left[ 1 + O \left( \frac{1}{T} \right) \right].
\]
Moreover, applying part (b) of Lemma SE-7 with $b = 1$, we obtain

$$
\sum_{i=1}^{N} \sum_{t=g+2}^{T} \sum_{s=g+1}^{t-1} E [w_{i-t-g,T} w_{s-g,T}]
$$

$$
= \sigma^2 N \sum_{t=g+2}^{T} \sum_{s=g+1}^{t-1} \sum_{k=1}^{s-g} \exp \left\{ - \left( \frac{t-g-k}{q(T)} \right) \right\} \exp \left\{ - \left( \frac{s-g-k}{q(T)} \right) \right\}
$$

$$
= NT q(T)^2 \frac{\sigma^2}{2} \left[ 1 - \frac{3 q(T)}{2 T} + \frac{2 q(T)}{T} \exp \left\{ - \frac{T}{q(T)} \right\} - \frac{1 q(T)}{2 T} \exp \left\{ - \frac{2 T}{q(T)} \right\} \right] \left[ 1 + O \left( \frac{1}{T} \right) \right].
$$

Together, these two results show part (c).

Finally, consider part (d), where we take $\rho_T = \exp \{-1/q(T)\}$ such that $q(T) \to \infty$ but $q(T)/T \to 0$. Applying part (c) of Lemma SE-1 with $b = g + 1$ and $d = 2$, we get

$$
\sum_{i=1}^{N} \sum_{t=g+2}^{T} \sum_{s=g+1}^{t-1} E [w_{i-t-g,T} w_{i-s,T}]
$$

$$
= \sigma^2 N \sum_{t=g+2}^{T} \sum_{s=g+1}^{t-1} \sum_{j=1}^{t-g} \exp \left\{ - \frac{2(t-g-j)}{q(T)} \right\}
$$

$$
= NT q(T) \frac{\sigma^2}{2} \left[ 1 + O \left( \frac{1}{q(T)} \right) + O \left( \frac{q(T)}{T} \right) \right].
$$

Moreover, applying part (c) of Lemma SE-7 with $b = 1$, we obtain

$$
\sum_{i=1}^{N} \sum_{t=g+2}^{T} \sum_{s=g+1}^{t-1} E [w_{i-t-g,T} w_{i-s,T}]
$$

$$
= \sigma^2 N \sum_{t=g+2}^{T} \sum_{s=g+1}^{t-1} \sum_{k=1}^{s-g} \exp \left\{ - \left( \frac{t-g-k}{q(T)} \right) \right\} \exp \left\{ - \left( \frac{s-g-k}{q(T)} \right) \right\}
$$

$$
= NT q(T)^2 \frac{\sigma^2}{2} \left[ 1 + O \left( \frac{q(T)}{T} \right) \right].
$$

Together, these two results show part (d). \qed

**Lemma SE-17:**

Let $g$ be a positive integer, and suppose that Assumptions 1 and 4 hold. Then, the following statements are true as $N, T \to \infty$.

(a) If $q(T) \sim T$

$$
\frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{i-t-g,T}^2 = \frac{q(T)^2 \sigma^2}{2 T^2} \left[ \exp \left\{ - \frac{2T}{q(T)} \right\} + \frac{2T}{q(T)} - 1 \right]
$$

$$
+ O_p \left( \frac{1}{\sqrt{N}} \right) + O_p \left( \frac{1}{\sqrt{T}} \right).
$$
(b) If $q(T) \to \infty$ such that $q(T)/T \to 0$, then

$$
\frac{1}{NTq(T)} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{it-g,T}^2 = \frac{\sigma^2}{2} + O_p \left( \max \left\{ \frac{1}{q(T)}, \frac{q(T)}{T}, \sqrt{\frac{q(T)}{NT}}, \frac{1}{\sqrt{T}} \right\} \right).
$$

(c) If $\rho_T \in G_{St} = \{|\rho_T| = \exp \left\{ -\frac{1}{q(T)} \right\} : q(T) \geq 0 \text{ and } q(T) = O(1) \text{ as } T \to \infty \}$, then

$$
\frac{1-\rho_T^2}{NT} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{it-g,T}^2 = \sigma^2 + O_p \left( T^{-1/2} \right).
$$

Proof of Lemma SE-17:

To proceed, again write

$$
\frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{it-g,T}^2 = \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{it-g,T}^2 + 2 \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{it-g} \rho_T^{t-g} w_{i0}
$$

$$
+ \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=g+1}^{T} \rho_T^{2(t-g)} w_{i0}^2.
$$

To show part (a), note first that from part (c) of Lemma SE-16 above, we have that

$$
\frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=g+1}^{T} E \left[ w_{it-g,T}^2 \right] - \frac{q(T)^2 \sigma^2}{2} \left[ \exp \left\{ -\frac{2T}{q(T)} \right\} + \frac{2T}{q(T)} - 1 \right] = O \left( \frac{1}{T} \right),
$$

as $N, T \to \infty$. Thus, to show the desired result, it suffices to show that

$$
\frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{it-g,T}^2 - \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=g+1}^{T} E \left[ w_{it-g,T}^2 \right] = O_p \left( \frac{1}{\sqrt{N}} \right),
$$

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as $N, T \to \infty$. To proceed, note that for all $T \geq g + 1$, we have

$$E \left( \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{i,g,T}^2 - \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=g+1}^{T} E \left[ w_{i,g,T}^2 \right] \right)^2$$

$$= E \left( \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{i,g,T}^2 - \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=g+1}^{T} \sum_{k=1}^{t-g} \exp \left\{ -2 \left( t - g - k \right) \right\} \right)^2$$

$$= \frac{1}{N^2 T^4} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{t=g+1}^{T} \sum_{s=g+1}^{T} E \left[ \left( w_{i,g,T}^2 - \sigma^2 \sum_{j=1}^{t-g} \exp \left\{ -2 \left( t - g - j \right) \right\} \right) \left( w_{j,g,T}^2 - \sigma^2 \sum_{k=1}^{s-g} \exp \left\{ -2 \left( s - g - k \right) \right\} \right) \right]$$

$$+ \frac{2}{N^2 T^4} \sum_{i=1}^{N} \sum_{t=g+1}^{T} \sum_{s=g+1}^{T} \sum_{j=1}^{t-1} E \left[ \left( w_{i,g,T}^2 - \sigma^2 \sum_{j=1}^{t-g} \exp \left\{ -2 \left( t - g - j \right) \right\} \right) \left( w_{j,g,T}^2 - \sigma^2 \sum_{k=1}^{s-g} \exp \left\{ -2 \left( s - g - k \right) \right\} \right) \right]. \quad (19)$$

Now, taking each of the two terms on the right-hand side of (19) in turn, we have upon applying part
(b) of Lemma SE-1 with \( b = g + 1 \) and \( d = 4 \) and part (b) of Lemma SE-2 with \( b = 2 \)

\[
\frac{1}{N^2 T^4} \sum_{i=1}^{N} \sum_{t=g+1}^{T} E \left[ w_{it-g,T}^2 - \sigma^2 \sum_{j=1}^{t-g} \exp \left\{ -2 \frac{(t - g - j)}{q(T)} \right\} \right]^2
\]

\[
= \frac{1}{N^2 T^4} \sum_{i=1}^{N} \sum_{t=g+1}^{T} E \left[ w_{iT}^2 \right] - \frac{1}{N^2 T^4} \sum_{i=1}^{N} \sum_{t=g+1}^{T} \left( \sigma^2 \sum_{j=1}^{t-g} \exp \left\{ -2 \frac{(t - g - j)}{q(T)} \right\} \right)^2
\]

\[
= \frac{1}{N^2 T^4} \sum_{i=1}^{N} \sum_{t=g+1}^{T} \sum_{j=1}^{T-g} \sum_{k=1}^{t-g} \sum_{l=1}^{t-g} \exp \left\{ - \frac{(t - g - k)}{q(T)} \right\} \exp \left\{ - \frac{(t - g - l)}{q(T)} \right\} E \left[ \varepsilon_i \varepsilon_j \varepsilon_k \varepsilon_l \right]
\]

\[
- \sigma^4 \frac{N}{N^2 T^4} \sum_{i=1}^{N} \sum_{t=g+1}^{T} \left( \sum_{j=1}^{t-g} \exp \left\{ -2 \frac{(t - g - j)}{q(T)} \right\} \right)^2
\]

\[
\leq \frac{1}{N^2 T^4} \sum_{i=1}^{N} \sum_{t=g+1}^{T} \sum_{j=1}^{t-g} \exp \left\{ -4 \frac{(t - g - j)}{q(T)} \right\} E \left[ \varepsilon_i \varepsilon_j \right]
\]

\[
+ \frac{1}{N^2 T^4} \sum_{i=1}^{N} \sum_{t=g+1}^{T} \left[ \sum_{j=1}^{t-g} \sum_{k=1}^{t-g} \sum_{l=1}^{t-g} \exp \left\{ - \frac{(t - g - j)}{q(T)} \right\} \exp \left\{ - \frac{(t - g - k)}{q(T)} \right\} \exp \left\{ - \frac{(t - g - l)}{q(T)} \right\} \right]^2
\]

\[
- \sigma^4 \frac{N}{N^2 T^4} \sum_{i=1}^{N} \sum_{t=g+1}^{T} \left( \sum_{j=1}^{t-g} \exp \left\{ -2 \frac{(t - g - j)}{q(T)} \right\} \right)^2
\]

\[
= \frac{1}{N^2 T^4} \sum_{i=1}^{N} \sum_{t=g+1}^{T} \sum_{j=1}^{t-g} \exp \left\{ -4 \frac{(t - g - j)}{q(T)} \right\} E \left[ \varepsilon_i \varepsilon_j \right]
\]

\[
+ 2 \sigma^4 \frac{N}{N^2 T^4} \sum_{i=1}^{N} \sum_{t=g+1}^{T} \left[ \sum_{j=1}^{t-g} \sum_{k=1}^{t-g} \sum_{l=1}^{t-g} \exp \left\{ - \frac{(t - g - j)}{q(T)} \right\} \exp \left\{ - \frac{(t - g - k)}{q(T)} \right\} \exp \left\{ - \frac{(t - g - l)}{q(T)} \right\} \right]^2
\]

\[
= \frac{E \left[ \varepsilon_i \varepsilon_j \right] N q(T)^2}{N^2 T^4} \left[ \exp \left\{ - \frac{4T}{q(T)} \right\} + \frac{4T}{q(T)} - 1 \right] \left[ 1 + O \left( \frac{1}{T} \right) \right]
\]

\[
- 2 \sigma^4 \frac{N q(T)^3}{N^2 T^4} \left[ 3 - \frac{4T}{q(T)} - 4 \exp \left\{ - \frac{2T}{q(T)} \right\} + \exp \left\{ - \frac{4T}{q(T)} \right\} \right] \left[ 1 + O \left( \frac{1}{T} \right) \right]
\]

\[
= O \left( \frac{1}{NT^2} \right) + O \left( \frac{1}{NT} \right)
\]

\[
= O \left( \frac{1}{NT} \right).
\]
Moreover, note that

\[
\frac{1}{N^2T^4} \sum_{i=1}^{N} \sum_{t=g+2}^{T} \sum_{s=g}^{T-1} E \left\{ \left[ w_{i-g,T}^2 - \sigma^2 \sum_{j=1}^{t-g} \exp \left\{ -2 \left( \frac{t - g - j}{q(T)} \right) \right\} \right] \times \left[ w_{i-s-g,T}^2 - \sigma^2 \sum_{k=1}^{s-g} \exp \left\{ -2 \left( \frac{s - g - k}{q(T)} \right) \right\} \right] \right\}
\]

\[
= \frac{1}{N^2T^4} \sum_{i=1}^{N} \sum_{t=g+2}^{T} \sum_{s=g+1}^{T-1} E \left[ w_{i-g,T}^2 w_{i-s-g,T}^2 \right]
\]

\[
- \frac{2\sigma^4}{N^2T^4} \sum_{i=1}^{N} \sum_{t=g+2}^{T} \sum_{s=g+1}^{T-1} \sum_{j=1}^{t} \sum_{k=1}^{s-g} \exp \left\{ -2 \left( \frac{t - g - j}{q(T)} \right) \right\} \sum_{k=1}^{s-g} \exp \left\{ -2 \left( \frac{s - g - k}{q(T)} \right) \right\}
\]

\[
+ \frac{\sigma^4}{N^2T^4} \sum_{i=1}^{N} \sum_{t=g+2}^{T} \sum_{s=g+1}^{T-1} \sum_{j=1}^{t} \sum_{k=1}^{s-g} \exp \left\{ -2 \left( \frac{t - g - j}{q(T)} \right) \right\} \sum_{k=1}^{s-g} \exp \left\{ -2 \left( \frac{s - g - k}{q(T)} \right) \right\}
\]

\[
= \frac{1}{N^2T^4} \sum_{i=1}^{N} \sum_{t=g+2}^{T} \sum_{s=g+1}^{T-1} \sum_{h=1}^{t} \sum_{j=1}^{s-g} \exp \left\{ - \left( \frac{t - g - h}{q(T)} \right) \right\} \exp \left\{ - \left( \frac{t - g - j}{q(T)} \right) \right\}
\]

\[
\times \sum_{k=1}^{s-g} \sum_{l=1}^{t} \sum_{j=1}^{s-g} \exp \left\{ - \left( \frac{s - g - k}{q(T)} \right) \right\} \exp \left\{ - \left( \frac{s - g - l}{q(T)} \right) \right\} E \left[ \varepsilon_{ih} \varepsilon_{ij} \varepsilon_{ik} \varepsilon_{il} \right]
\]

\[
- \sigma^4 \frac{1}{N^2T^4} \sum_{i=1}^{N} \sum_{t=g+2}^{T} \sum_{s=g+1}^{T-1} \sum_{j=1}^{t} \sum_{k=1}^{s-g} \exp \left\{ - \left( \frac{t - g - j}{q(T)} \right) \right\} \sum_{k=1}^{s-g} \exp \left\{ - \left( \frac{s - g - k}{q(T)} \right) \right\}
\]

\[
+ \sigma^4 \frac{1}{N^2T^4} \sum_{i=1}^{N} \sum_{t=g+2}^{T} \sum_{s=g+1}^{T-1} \sum_{j=1}^{t} \sum_{k=1}^{s-g} \exp \left\{ - \left( \frac{t - g - j}{q(T)} \right) \right\} \sum_{k=1}^{s-g} \exp \left\{ - \left( \frac{s - g - k}{q(T)} \right) \right\}
\]

\[
- \sigma^4 \frac{1}{N^2T^4} \sum_{i=1}^{N} \sum_{t=g+2}^{T} \sum_{s=g+1}^{T-1} \sum_{j=1}^{t} \sum_{k=1}^{s-g} \exp \left\{ - \left( \frac{t - g - j}{q(T)} \right) \right\} \sum_{k=1}^{s-g} \exp \left\{ - \left( \frac{s - g - k}{q(T)} \right) \right\}
\]

\[
+ 2\sigma^4 \frac{1}{N^2T^4} \sum_{i=1}^{N} \sum_{t=g+2}^{T} \sum_{s=g+1}^{T-1} \sum_{j=1}^{t} \sum_{k=1}^{s-g} \exp \left\{ - \left( \frac{t - g - j}{q(T)} \right) \right\} \sum_{k=1}^{s-g} \exp \left\{ - \left( \frac{s - g - k}{q(T)} \right) \right\}
\]

\[
\times \sum_{k=1}^{s-g} \exp \left\{ - \left( \frac{t - g - k}{q(T)} \right) \right\} \sum_{k=1}^{s-g} \exp \left\{ - \left( \frac{s - g - k}{q(T)} \right) \right\}
\]

\[
- 2\sigma^4 \frac{1}{N^2T^4} \sum_{i=1}^{N} \sum_{t=g+2}^{T} \sum_{s=g+1}^{T-1} \sum_{j=1}^{t} \sum_{k=1}^{s-g} \exp \left\{ - \left( \frac{t - g - j}{q(T)} \right) \right\} \sum_{k=1}^{s-g} \exp \left\{ - \left( \frac{s - g - j}{q(T)} \right) \right\}
\]

\[
- \sigma^4 \frac{1}{N^2T^4} \sum_{i=1}^{N} \sum_{t=g+2}^{T} \sum_{s=g+1}^{T-1} \sum_{j=1}^{t} \sum_{k=1}^{s-g} \exp \left\{ - \left( \frac{t - g - j}{q(T)} \right) \right\} \sum_{k=1}^{s-g} \exp \left\{ - \left( \frac{s - g - k}{q(T)} \right) \right\}
\]
Hence,

$$\frac{1}{N^{2T^4}} \sum_{i=1}^{N} \sum_{t=g+2}^{T} \sum_{s=g+1}^{t-1} E \left\{ \left[ \frac{w_{it-g,T}^2 - \sigma^2 \sum_{j=1}^{t-g} \exp \left\{ -2 \frac{(t - g - j)}{q(T)} \right\}}{w_{is-g,T}^2 - \sigma^2 \sum_{k=1}^{s-g} \exp \left\{ -2 \frac{(s - g - k)}{q(T)} \right\}} \right] \right\}$$

$$\leq \left( \frac{E \left[ \epsilon_{ij}^4 \right] + 3\sigma^4}{N^{2T^4}} \right) \sum_{i=1}^{N} \sum_{t=g+2}^{T} \sum_{s=g+1}^{t-1} \sum_{j=1}^{t-g} \exp \left\{ -2 \frac{(t - g - j)}{q(T)} \right\} \exp \left\{ -2 \frac{(s - g - j)}{q(T)} \right\}$$

$$+ \frac{2\sigma^4}{N^{2T^4}} \sum_{i=1}^{N} \sum_{t=g+2}^{T} \sum_{s=g+1}^{t-1} \sum_{j=1}^{t-g} \exp \left\{ - \frac{(t - g - j)}{q(T)} \right\} \exp \left\{ - \frac{(s - g - j)}{q(T)} \right\}$$

$$\times \sum_{k=1}^{s-g} \exp \left\{ - \frac{(t - g - k)}{q(T)} \right\} \exp \left\{ - \frac{(s - g - k)}{q(T)} \right\}$$

$$= \left( \frac{E \left[ \epsilon_{ij}^4 \right] + 3\sigma^4}{NT^4} \right) \frac{Tq(T)^2}{8} \times \left[ 1 - \frac{3q(T)}{4T} + \frac{q(T)}{T} \exp \left\{ - \frac{2T}{q(T)} \right\} - \frac{1}{4} \frac{q(T)}{T} \exp \left\{ - \frac{4T}{q(T)} \right\} \right] \left[ 1 + O \left( \frac{1}{T} \right) \right]$$

$$+ \frac{2\sigma^4}{NT^4} \left( \frac{q(T)^4}{32} \right) \left[ \exp \left\{ - \frac{4T}{q(T)} \right\} + 4(g - 1) \exp \left\{ - \frac{2T}{q(T)} \right\} - 5 \right]$$

$$+ \frac{Tq(T)^3}{8} \left[ 2 \exp \left\{ - \frac{2T}{q(T)} \right\} + 1 \right] \left[ 1 + O \left( \frac{1}{T} \right) \right]$$

$$= O \left( \frac{1}{NT} \right) + O \left( \frac{1}{N} \right)$$

$$= O \left( \frac{1}{N} \right).$$  \hspace{1cm} (21)$$

where the third-to-last equality is justified by applying part (b) of Lemma SE-7 with $b = 2$ and part (b) of Lemma SE-6 with $b = 1$.  

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It follows from (19), (20), and (21), that

\[
E \left( \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=g+1}^{T} \frac{w^2_{it-g, T}}{w^2_{i0}} - \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=g+1}^{T} E \left[ \frac{w^2_{it-g}}{w^2_{i0}} \right] \right)^2
\]

\[
= \frac{1}{N^2T^4} \sum_{i=1}^{N} \sum_{t=g+1}^{T} E \left[ w^2_{it-g, T} - \sigma^2 \sum_{j=1}^{t-g} \exp \left\{ -2 \frac{(t-g-j)}{q(T)} \right\} \right]^2
\]

\[
+ \frac{2}{N^2T^4} \sum_{i=1}^{N} \sum_{t=4}^{T} \sum_{s=3}^{t-1} E \left\{ \left[ w^2_{it-g, T} - \sigma^2 \sum_{j=1}^{t-2} \exp \left\{ -2 \frac{(t-2-j)}{q(T)} \right\} \right] \right.
\]

\[
\times \left. \left[ w^2_{is-g, T} - \sigma^2 \sum_{k=1}^{s-2} \exp \left\{ -2 \frac{(s-2-k)}{q(T)} \right\} \right] \right\}
\]

\[
= O \left( \frac{1}{NT} \right) + O \left( \frac{1}{N} \right)
\]

\[
= O \left( \frac{1}{N} \right) = o(1),
\]

from which it follows by Markov’s inequality that

\[
\frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=3}^{T} w^2_{it-g, T} = \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=3}^{T} E \left[ \frac{w^2_{it-g, T}}{w^2_{i0}} \right] + O_p \left( \frac{1}{\sqrt{N}} \right)
\]

\[
= \frac{q(T)^2 \sigma^2}{T^2} \frac{\exp \left\{ - \frac{2T}{q(T)} \right\}}{4} + \frac{2T}{q(T)} - 1 + O_p \left( \frac{1}{T} \right) + O_p \left( \frac{1}{\sqrt{N}} \right)
\]

\[
= \frac{q(T)^2 \sigma^2}{T^2} \frac{\exp \left\{ - \frac{2T}{q(T)} \right\}}{4} + \frac{2T}{q(T)} - 1 + O_p \left( \frac{1}{\sqrt{N}} \right),
\]

as \( N, T \to \infty. \)

Moreover, note that, by Assumption 4, we have

\[
E \left[ \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=g+1}^{T} \rho_T^{2(t-g)} w^2_{i0} \right]
\]

\[
\leq \sup_i E \left[ \frac{w^2_{i0}}{T^2} \right] \exp \left\{ - \frac{2}{q(T)} \right\} \left[ 1 - \exp \left\{ - \frac{2}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{ - \frac{2(T-g)}{q(T)} \right\} \right]
\]

\[
= O \left( T^{-2} \right) O \left( 1 \right) O \left( 1 \right) = O \left( T^{-1} \right),
\]

so that, applying Markov’s inequality, we further deduce that

\[
\frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=g+1}^{T} \rho_T^{2(t-g)} w^2_{i0} = O_p \left( \frac{1}{T} \right),
\]
The Cauchy-Schwarz inequality then implies that

\[
\left| \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{it-g} \rho_{T}^{t-g} w_{i0} \right| \leq \sqrt{\frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{it-g}^2} \sqrt{\frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=g+1}^{T} \rho_{T}^{2(t-g)} w_{i0}^2}
\]

\[
= O_p \left( T^{-1/2} \right) = O_p \left( T^{-1/2} \right)
\]

from which it follows that

\[
\frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{it-g,T}^2 = \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{it-g,T}^2 + \frac{2}{NT^2} \sum_{i=1}^{N} \sum_{t=g+1}^{T} \rho_{T}^{t-g} w_{i0} + \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=g+1}^{T} \rho_{T}^{2(t-g)} w_{i0}^2
\]

\[
= \frac{q(T)^2 \sigma^2}{4} \left[ \exp \left\{ -\frac{2T}{q(T)} \right\} + \frac{2T}{q(T)} - 1 \right] + O_p \left( \frac{1}{\sqrt{N}} \right) + O_p \left( \frac{1}{\sqrt{T}} \right),
\]

as required.

To show part (b), write

\[
\frac{1}{NTq(T)} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{it-g,T}^2 - \frac{\sigma^2}{2}
\]

\[
= \frac{1}{NTq(T)} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{it-g,T}^2 - \frac{1}{NTq(T)} \sum_{i=1}^{N} \sum_{t=g+1}^{T} E \left[ w_{it-g,T}^2 \right] + \frac{1}{NTq(T)} \sum_{i=1}^{N} \sum_{t=g+1}^{T} E \left[ w_{it-g,T}^2 \right] - \frac{\sigma^2}{2}.
\]

Now, from part (d) of Lemma SE-16 above, we have that, as \( N,T \to \infty \),

\[
\frac{1}{NTq(T)} \sum_{i=1}^{N} \sum_{t=g+1}^{T} E \left[ w_{it-g,T}^2 \right] = \frac{\sigma^2}{2} + O \left( \frac{1}{q(T)} \right) + O \left( \frac{q(T)}{T} \right).
\]

Thus, to show part (b), it suffices to show that

\[
\frac{1}{NTq(T)} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{it-g,T}^2 = \frac{1}{NTq(T)} \sum_{i=1}^{N} \sum_{t=g+1}^{T} E \left[ w_{it-g}^2 \right] + O_p \left( \frac{\sqrt{T}}{NT} \right),
\]
as $N, T \to \infty$. Similar to the proof of part (a) above, we have

$$E \left( \frac{1}{NTq(T)} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{i,t-g,T}^2 - \frac{1}{NTq(T)} \sum_{i=1}^{N} \sum_{t=g+1}^{T} E \left[ w_{i,t-g}^2 \right] \right)^2 \leq \frac{1}{N^2T^2q(T)^2} \sum_{i=1}^{N} \sum_{t=g+1}^{T} \sum_{t'=t}^{t+g} \exp \left( -2 \frac{(t-g-j)}{q(T)} \right) \exp \left( -2 \frac{(s-g-k)}{q(T)} \right)$$

$$+ \frac{2 \sigma^4}{N^2T^2q(T)^2} \sum_{i=1}^{N} \sum_{t=g+1}^{T} \sum_{t'=t}^{t+g} \sum_{s=g+1}^{t+g} \exp \left( -2 \frac{(t-g-j)}{q(T)} \right) \exp \left( -2 \frac{(s-g-k)}{q(T)} \right)$$

$$+ \frac{\sigma^4}{N^2T^2q(T)^2} \sum_{i=1}^{N} \sum_{t=g+1}^{T} \sum_{t'=t}^{t+g} \sum_{s=g+1}^{t+g} \exp \left( -2 \frac{(t-g-j)}{q(T)} \right) \exp \left( -2 \frac{(s-g-k)}{q(T)} \right)$$

$$= \frac{E \left[ \varepsilon_{ij}^4 \right]}{N^2T^2q(T)^2} \frac{NTq(T)}{4} \left[ 1 + O \left( \frac{1}{q(T)} \right) + O \left( \frac{q(T)}{T} \right) \right]$$

$$+ \frac{2 \sigma^4}{N^2T^2q(T)^2} \frac{NTq(T)^2}{4} \left[ 1 + O \left( \frac{q(T)^2}{T} \right) \right]$$

$$+ \frac{\sigma^4}{N^2T^2q(T)^2} \frac{NTq(T)}{8} \left[ 1 + O \left( \max \left\{ \frac{q(T)}{T}, \frac{1}{q(T)} \right\} \right) \right]$$

$$+ \frac{2 \sigma^4}{N^2T^2q(T)^2} \frac{NTq(T)^3}{8} \left[ 1 + O \left( \frac{1}{q(T)} \right) + O \left( \frac{q(T)^2}{T} \right) \right]$$

$$= O \left( \frac{1}{NTq(T)} \right) + O \left( \frac{1}{NT} \right) + O \left( \frac{1}{NT} \right) + O \left( \frac{q(T)}{NT} \right)$$

$$= O \left( \frac{q(T)}{NT} \right),$$

where we have applied part (c) of Lemma SE-1, part (c) of Lemma SE-2, part (c) of Lemma SE-7 with
\[ b = 2, \text{ and part (a) of Lemma SE-6 with } b = 1 \text{ in calculating the order of magnitudes given above. It follows by Markov's inequality that, as } N, T \to \infty, \]
\[
\frac{1}{NTq(T)} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{it-g,T}^2 = \frac{1}{NTq(T)} \sum_{i=1}^{N} \sum_{t=g+1}^{T} E \left[ w_{it-g,T}^2 \right] + O_p \left( \sqrt{\frac{q(T)}{NT}} \right) = \frac{\sigma^2}{2} + O_p \left( \max \left\{ \frac{1}{q(T)}; \frac{q(T)}{T}; \sqrt{\frac{q(T)}{NT}} \right\} \right)
\]
as required.

Moreover, note that, using Assumption 4, we have
\[
E \left[ \frac{1}{NTq(T)} \sum_{i=1}^{N} \sum_{t=g+1}^{T} \rho_T^{2(t-g)} w_{i0}^2 \right] \leq \sup_{t} \left( \frac{2}{Tq(T)} \right) \exp \left\{ -2 \frac{2}{q(T)} \right\} \left[ 1 - \exp \left\{ -2 \frac{2}{q(T)} \right\} \right] \left[ 1 - \exp \left\{ -2 \frac{(T-g)}{q(T)} \right\} \right] = O \left( T^{-1} q(T)^{-1} \right) O \left( 1 \right) O \left( q(T) \right) O \left( 1 \right) = O \left( \frac{1}{T} \right),
\]
so that
\[
\frac{1}{NTq(T)} \sum_{i=1}^{N} \sum_{t=g+1}^{T} \rho_T^{2(t-g)} w_{i0}^2 = O_p \left( \frac{1}{T} \right),
\]
by Markov's inequality. The Cauchy-Schwarz inequality further implies
\[
\left| \frac{1}{NTq(T)} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{it-g,T}^2 \rho_T^{1-t-g} w_{i0} \right| \leq \sqrt{\frac{1}{NTq(T)} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{it-g}^2} \sqrt{\frac{1}{NTq(T)} \sum_{i=1}^{N} \sum_{t=g+1}^{T} \rho_T^{2(t-g)} w_{i0}^2} = O_p \left( T^{-1/2} \right) = O_p \left( T^{-1/2} \right),
\]
from which it follows that
\[
\frac{1}{NTq(T)} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{it-g,T}^2 = \frac{1}{NTq(T)} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{it-g,T}^2 + \frac{2}{NTq(T)} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{it-g} \rho_T^{1-t-g} w_{i0} + \frac{1}{NTq(T)} \sum_{i=1}^{N} \sum_{t=g+1}^{T} \rho_T^{2(t-g)} w_{i0}^2 = \frac{\sigma^2}{2} + O_p \left( \max \left\{ \frac{1}{q(T)}; \frac{q(T)}{T}; \sqrt{\frac{q(T)}{NT}} \right\} \right) + O_p \left( \frac{1}{T} \right) + O_p \left( \frac{1}{\sqrt{T}} \right) = \frac{\sigma^2}{2} + O_p \left( \max \left\{ \frac{1}{q(T)}; \frac{q(T)}{T}; \sqrt{\frac{q(T)}{NT}}; \frac{1}{\sqrt{T}} \right\} \right),
\]
which is the desired result for part (b).
To show part (c), write

\[
\frac{1 - \rho_T^2}{NT} \sum_{i=1}^{N} \sum_{t=g+1}^{T} \frac{w_{i-t,g,T}^2 - \sigma^2}{2} \quad \text{vs.} \quad \frac{1 - \rho_T^2}{NT} \sum_{i=1}^{N} \sum_{t=g+1}^{T} \frac{w_{i-t,g,T}^2}{2} \quad \text{and} \quad \frac{1 - \rho_T^2}{NT} \sum_{i=1}^{N} \sum_{t=g+1}^{T} \frac{E[w_{i-t,g,T}^2]}{2} + \frac{1 - \rho_T^2}{NT} \sum_{i=1}^{N} \sum_{t=g+1}^{T} \frac{E[w_{i-t,g,T}^2] - \sigma^2}{2}.
\]

Note that for all \( T \geq g + 1 \)

\[
\frac{1 - \rho_T^2}{NT} \sum_{i=1}^{N} \sum_{t=g+1}^{T} \frac{w_{i-t,g,T}^2}{2} = \frac{1 - \rho_T^2}{NT} \sum_{i=1}^{N} \sum_{t=g+1}^{T} \frac{\sum_{t=g+1}^{T} \rho_T^{2(t-g)-j} E[\varepsilon_{ij}\varepsilon_{ik}]}{2} = \frac{\sigma^2}{T} \frac{1 - \rho_T^2}{T} \sum_{i=1}^{N} \sum_{t=g+1}^{T} \frac{1 - \rho_T^{2(t-g)}}{1 - \rho_T^2} = \frac{\sigma^2}{T} \left[ 1 - \frac{\rho_T^{2(T-g)}}{1 - \rho_T^2} \right] = \frac{\sigma^2}{T} \left[ 1 - \frac{T}{\rho_T^2 (1 - \rho_T^2)} \right]
\]

Since we assume here that \( \rho_T^2 = \exp \{ -2/q(T) \} \) with \( q(T) = O(1) \), it follows that there exist a positive constant \( C_q \) and a positive integer \( T^* \) such that for all \( T \geq \max \{ T^*, g + 1 \} \),

\[
0 \leq \frac{\rho_T^{2(T-g)}}{T (1 - \rho_T^2)} \leq \frac{1}{T (1 - \exp \{ -2/C_q \})} = O(\frac{1}{T}),
\]

so that, in this case,

\[
\frac{1 - \rho_T^2}{NT} \sum_{i=1}^{N} \sum_{t=g+1}^{T} E[w_{i-t,g,T}^2] = \sigma^2 + O(\frac{1}{T}).
\]

Thus, to complete the proof of this part, it suffices to show that

\[
\frac{1 - \rho_T^2}{NT} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{i-t,g,T}^2 = \frac{1 - \rho_T^2}{NT} \sum_{i=1}^{N} \sum_{t=g+1}^{T} E[w_{i-t,g,T}^2] + O_p \left( \frac{1}{\sqrt{NT}} \right).
\]

To proceed, write

\[
E \left( \frac{1 - \rho_T^2}{NT} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{i-t,g,T}^2 - \frac{1 - \rho_T^2}{NT} \sum_{i=1}^{N} \sum_{t=g+1}^{T} E[w_{i-t,g,T}^2] \right)^2 = E \left( \frac{1 - \rho_T^2}{NT} \sum_{i=1}^{N} \sum_{t=g+1}^{T} \left[ w_{i-t,g,T}^2 - \sigma^2 \sum_{j=1}^{t-g} \rho_T^{2(t-g-j)} \right] \right)^2
\]

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\[
\frac{(1 - \rho_T^2)^2}{N^2T^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{t=g+1}^{T} \sum_{s=g+1}^{T} E \left[ \left( w_{t-g,T}^2 - \sigma^2 \sum_{k=1}^{t-g} \rho_T^{2(t-g-k)} \right) \left( w_{s-g,T}^2 - \sigma^2 \sum_{\ell=1}^{t-g} \rho_T^{2(t-g-\ell)} \right) \right] 
\]

Next, note that there exist a positive constant \( C_q \) and a positive integer \( T^* \) such that for all \( T \geq T^* \)

\[
\frac{(1 - \rho_T^2)^2}{N^2T^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{t=g+1}^{T} \sum_{s=g+1}^{T} E \left[ w_{t-g,T}^4 \right] - \frac{(1 - \rho_T^2)^2}{N^2T^2} \sum_{i=1}^{N} \sum_{t=g+1}^{T} \left( \sigma^2 \sum_{k=1}^{t-g} \rho_T^{2(t-g-k)} \right)^2 
\]

\[
\leq E \left[ A_{ij}^4 \right] \frac{(1 - \rho_T^2)^2}{N^2T^2} \sum_{i=1}^{N} \sum_{t=g+1}^{T} \sum_{s=g+1}^{T} \rho_T^{4(t-g-k)} + \frac{3\sigma^4 (1 - \rho_T^2)^2}{N^2T^2} \sum_{i=1}^{N} \sum_{t=g+1}^{T} \left( \sum_{k=1}^{t-g} \rho_T^{2(t-g-k)} \right)^2 
\]

\[
\leq E \left[ A_{ij}^4 \right] \frac{(1 - \rho_T^2)^2}{N^2T^2} \sum_{i=1}^{N} \sum_{t=g+1}^{T} \sum_{s=g+1}^{T} \frac{1 - \rho_T^{4(T-g)}}{1 - \rho_T^4} 
\]

\[
\leq \frac{E \left[ A_{ij}^4 \right]}{NT} \left[ 1 + \frac{g}{T} + \frac{1}{T \left( 1 - \exp \{-4/C_q \} \right)} \right] + \frac{2\sigma^4}{NT} \left[ 1 + \frac{g}{T} + \frac{2}{T \left( 1 - \exp \{-2/C_q \} \right)} + \frac{1}{T \left( 1 - \exp \{-4/C_q \} \right)} \right] = O \left( \frac{1}{NT} \right), 
\]
and

\[
\frac{(1 - \rho_T^2)^2}{N^2 T^2} \sum_{i=1}^{N} \sum_{t=g+2}^{T} \sum_{s=g+1}^{t-1} \left[ \left( \frac{u^2_{t-g,T}}{\sigma^2} \rho_T^{t-g} \sum_{k=1}^{t-g} \rho_T^{2(t-g-k)} \right) \left( \frac{u^2_{s-g,T}}{\sigma^2} \rho_T^{2(t-g-\ell)} \right) \right] \\
-2\sigma^4 \frac{(1 - \rho_T^2)^2}{N^2 T^2} \sum_{i=1}^{N} \sum_{t=g+2}^{T} \sum_{s=g+1}^{t-1} \sum_{k=1}^{t-g} \rho_T^{2(t-g-k)} \sum_{\ell=1}^{s-g} \rho_T^{2(t-g-\ell)} \\
+ \frac{(1 - \rho_T^2)^2}{N^2 T^2} \sum_{i=1}^{N} \sum_{t=g+2}^{T} \sum_{s=g+1}^{t-1} \sum_{k=1}^{t-g} \rho_T^{2(t-g-k)} \sum_{\ell=1}^{s-g} \rho_T^{2(t-g-\ell)} \\
= \frac{(1 - \rho_T^2)^2}{N^2 T^2} \sum_{i=1}^{N} \sum_{t=g+2}^{T} \sum_{s=g+1}^{t-1} \left[ \sum_{h=1}^{t-g} \rho_T^{(t-g-h)} \rho_T^{(t-g-\ell)} \sum_{k=1}^{s-g} \rho_T^{(s-g-k)} \rho_T^{(s-g-\ell)} E[\varepsilon_i \varepsilon_j \varepsilon_k \varepsilon_l] \right] \\
- \sigma^4 \frac{(1 - \rho_T^2)^2}{N^2 T^2} \sum_{i=1}^{N} \sum_{t=g+2}^{T} \sum_{s=g+1}^{t-1} \sum_{k=1}^{t-g} \rho_T^{2(t-g-k)} \sum_{\ell=1}^{s-g} \rho_T^{2(t-g-\ell)}
\]
\[
\begin{align*}
&= E \left[ \varepsilon_{ij}^4 \right] \frac{(1 - \rho_T^2)^2}{N^2 T^2} \sum_{t=1}^N \sum_{g=2}^T \sum_{s=g+1}^{t-1} \sum_{j=1}^{s-g} \rho_T^{2(t-g-j)} \rho_T^{2(s-g-j)} \\
+ &\sigma^4 \frac{(1 - \rho_T^2)^2}{N^2 T^2} \sum_{i=1}^N \sum_{g=2}^T \sum_{s=g+1}^{t-1} \sum_{j=1}^{s-g} \rho_T^{2(t-g-k)} \rho_T^{2(s-g-\ell)} \\
- &\sigma^4 \frac{(1 - \rho_T^2)^2}{N^2 T^2} \sum_{i=1}^N \sum_{g=2}^T \sum_{s=g+1}^{t-1} \sum_{j=1}^{s-g} \rho_T^{2(t-g-j)} \rho_T^{2(s-g-j)} \\
+ &2 \sigma^4 \frac{(1 - \rho_T^2)^2}{N^2 T^2} \sum_{i=1}^N \sum_{g=2}^T \sum_{s=g+1}^{t-1} \sum_{j=1}^{s-g} \rho_T^{2(t-g-j)} \rho_T^{2(s-g-j)} \\
- &\sigma^4 \frac{(1 - \rho_T^2)^2}{N^2 T^2} \sum_{i=1}^N \sum_{g=2}^T \sum_{s=g+1}^{t-1} \sum_{j=1}^{s-g} \rho_T^{2(t-g-j)} \rho_T^{2(s-g-\ell)} \\
- &\sigma^4 \frac{(1 - \rho_T^2)^2}{N^2 T^2} \sum_{i=1}^N \sum_{g=2}^T \sum_{s=g+1}^{t-1} \sum_{j=1}^{s-g} \rho_T^{2(t-g-j)} \rho_T^{2(s-g-j)} \\
= & \left( E \left[ \varepsilon_{ij}^4 \right] - 3 \sigma^4 \right) \frac{(1 - \rho_T^2)^2}{N^2 T^2} \sum_{i=1}^N \sum_{g=2}^T \sum_{s=g+1}^{t-1} \sum_{j=1}^{s-g} \rho_T^{2(t-s)} \frac{1 - \rho_T^{4(s-g)}}{1 - \rho_T^4} \\
+ &2 \sigma^4 \frac{(1 - \rho_T^2)^2}{N^2 T^2} \sum_{i=1}^N \sum_{g=2}^T \sum_{s=g+1}^{t-1} \rho_T^{2(t-s)} \frac{1 - \rho_T^{4(s-g)}}{1 - \rho_T^4} \\
= & \frac{E \left[ \varepsilon_{ij}^4 \right] - 3 \sigma^4}{N T^2} \left[ 1 + \rho_T^2 \sum_{t=g+2}^T \frac{\rho_T^2 (1 - \rho_T^{2(t-g-1)})}{(1 - \rho_T^2)} \right] - \sum_{t=g+2}^T \rho_T^{2(t-g+1)} \frac{1 - \rho_T^{2(t-g-1)}}{(1 - \rho_T^2)} \\
+ &2 \sigma^4 \frac{T}{N T^2} \sum_{t=g+2}^T \rho_T^2 \frac{1 - \rho_T^{2(t-g-1)}}{(1 - \rho_T^2)} - \frac{4 \sigma^4 \rho_T^2}{N T^2} \sum_{t=g+2}^T \rho_T^{2(t-g-1)} (t - g - 1) \\
+ &2 \sigma^4 \frac{T}{N T^2} \sum_{t=g+2}^T \rho_T^{2(t-g+1)} \frac{1 - \rho_T^{2(t-g-1)}}{(1 - \rho_T^2)} \right]
\end{align*}
\]
Applying part (b) of Lemma SE-5 and performing additional calculation, we get

\[
\frac{(1 - \rho_T^2)^2}{N^2 T^2} \sum_{i=1}^{N} \sum_{t=g+2}^{T} \sum_{s=g+1}^{t-1} E \left[ \left( \frac{w_{i,t-g,t}^2}{N^2 T^2} - \sigma^2 \sum_{k=1}^{t-g} \rho_T^2 (t-g-k) \right) \left( \frac{w_{i,s-g,t}^2}{N^2 T^2} - \sigma^2 \sum_{l=1}^{t-g} \rho_T^2 (t-g-l) \right) \right]
\]

\[
= \frac{E \left[ \varepsilon_{ij}^2 \right] - 3\sigma^4}{NT^2} \frac{\rho_T^2}{1 + \rho_T^2} \left[ (T - g - 1) - \frac{\rho_T^2}{\left(1 - \rho_T^2\right)} \left(1 - \rho_T^2\right)\right]
\]

\[
- \frac{E \left[ \varepsilon_{ij}^2 \right] - 3\sigma^4}{NT^2} \frac{\rho_T^2}{1 + \rho_T^2} \left(1 - \rho_T^2\right)
\]

\[
+ \frac{E \left[ \varepsilon_{ij}^2 \right] - 3\sigma^4}{NT^2} \frac{\rho_T^2}{1 + \rho_T^2} \left(1 - \rho_T^2\right)
\]

\[
+ \frac{2\sigma^4}{NT^2} \frac{\rho_T^2}{1 + \rho_T^2} \left[ (T - g - 1) - \frac{\rho_T^2}{\left(1 - \rho_T^2\right)} \left(1 - \rho_T^2\right)\right]
\]

\[
- \frac{4\sigma^4 \rho_T^4}{NT^2} \frac{\rho_T^2}{1 + \rho_T^2} \left(1 - \rho_T^2\right)^2
\]

\[
+ \frac{2\sigma^4}{NT^2} \frac{\rho_T^2}{1 + \rho_T^2} \frac{1}{(1 - \rho_T^2)} - \frac{2\sigma^4}{NT^2} \frac{\rho_T^2}{1 + \rho_T^2} \frac{1}{(1 - \rho_T^2)}
\]

\[
\leq \frac{\left[ E \left[ \varepsilon_{ij}^2 \right] + 3\sigma^4 \right]}{NT} \left[ 1 + \frac{g + 1}{T} + \frac{1}{1 - \exp \{-2/C_q\}} \right]
\]

\[
+ \frac{2\sigma^4}{NT} \left[ 1 + \frac{g + 1}{T} + \frac{1}{1 - \exp \{-2/C_q\}} \right] + \frac{4\sigma^4}{NT} \frac{1}{1 - \exp \{-2/C_q\}}
\]

\[
+ \frac{\left[ E \left[ \varepsilon_{ij}^2 \right] + 3\sigma^4 \right]}{NT^2} \left[ \frac{1}{1 - \exp \{-2/C_q\}} + \frac{1}{1 - \exp \{-4/C_q\}} \right]
\]

\[
+ \frac{8\sigma^4}{NT^2} \frac{(g + 1)}{(1 - \exp \{-2/C_q\})^2}
\]

\[
+ \frac{2\sigma^4}{NT^2} \left[ \frac{1}{(1 - \exp \{-2/C_q\})^2} + \frac{1}{(1 - \exp \{-2/C_q\}) (1 - \exp \{-4/C_q\})} \right]
\]

\[
= O \left( \frac{1}{NT} \right).
\]

It follows that

\[
E \left( \frac{1 - \rho_T^2}{NT} \sum_{i=1}^{N} \sum_{t=g+1}^{T} \frac{w_{i,t-g,t}^2}{NT} - \frac{1 - \rho_T^2}{NT} \sum_{i=1}^{N} \sum_{t=g+1}^{T} E \left[ \frac{w_{i,t-g,t}^2}{NT} \right] \right)^2 = O_p \left( \frac{1}{NT} \right),
\]
and by Markov’s inequality, we deduce that

\[
\frac{1 - \rho_T^2}{NT} \sum_{i=1}^N \sum_{t=g+1}^T w_{it-g,T}^2 = \frac{1 - \rho_T^2}{NT} \sum_{i=1}^N \sum_{t=g+1}^T E[w_{it-g,T}^2] + O\left(\frac{1}{\sqrt{NT}}\right) \\
= \sigma^2 + O_p\left(\max\left\{\frac{1}{\sqrt{NT}}, \frac{1}{T}\right\}\right).
\]

Moreover, note that, using Assumption 4, we have

\[
E\left[\frac{1 - \rho_T^2}{NT} \sum_{i=1}^N \sum_{t=g+1}^T \rho_T^{2(t-g)} w_{i0}^2\right] \leq \frac{(1 - \rho_T^2)}{T} \sup_i E[w_{i0}^2] \rho_T \frac{1 - \rho_T^{2(T-g)}}{1 - \rho_T^2} = O(T^{-1}),
\]

so that

\[
\frac{1 - \rho_T^2}{NT} \sum_{i=1}^N \sum_{t=g+1}^T \rho_T^{2(t-g)} w_{i0}^2 = O_p\left(\frac{1}{T}\right).
\]

The Cauchy-Schwarz inequality then implies that

\[
\left|\frac{1 - \rho_T^2}{NT} \sum_{i=1}^N \sum_{t=g+1}^T w_{it-g,T} w_{i0}\right| \leq \sqrt{\frac{1 - \rho_T^2}{NT} \sum_{i=1}^N \sum_{t=g+1}^T w_{it-g,T}^2} \sqrt{\frac{1 - \rho_T^2}{NT} \sum_{i=1}^N \sum_{t=g+1}^T \rho_T^{2(t-g)} w_{i0}^2} \\
= O_p (1) O_p \left(T^{-1/2}\right) = O_p \left(T^{-1/2}\right),
\]

from which it follows that

\[
\frac{1 - \rho_T^2}{NT^2} \sum_{i=1}^N \sum_{t=g+1}^T w_{it-g,T}^2 \\
= \frac{1 - \rho_T^2}{NT} \sum_{i=1}^N \sum_{t=g+1}^T w_{it-g,T}^2 + \frac{2(1 - \rho_T^2)}{NT} \sum_{i=1}^N \sum_{t=g+1}^T w_{it-g,T} w_{i0} + \frac{1 - \rho_T^2}{NT} \sum_{i=1}^N \sum_{t=g+1}^T \rho_T^{2(t-g)} w_{i0}^2 \\
= \sigma^2 + O_p\left(\max\left\{\frac{1}{\sqrt{NT}}, \frac{1}{T}\right\}\right) + O_p(T^{-1}) + O_p\left(T^{-1/2}\right) \\
= \sigma^2 + O_p\left(T^{-1/2}\right),
\]

as required for part (c).  \(\square\)

**Lemma SE-18:**

Let \(g\) be a positive integer. Under Assumptions 1-4, the following results hold as \(N, T \to \infty\)

(a) If \(\rho_T = 1\) for all \(T\) sufficiently large, then

\[
\sum_{i=1}^N \sum_{t=g+1}^T a_i w_{it-g,T} = O_p\left(\max\left\{\sqrt{NT^3/2}, NT\right\}\right).
\]

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(b) If $\rho_T = \exp \{-1/q(T)\}$ such that $T/q(T) \to 0$, then
\[
\sum_{i=1}^{N} \sum_{t=g+1}^{T} a_i w_{it-g,T} = O_p \left( \max \left\{ \sqrt{NT^{3/2}} , NT \right\} \right).
\]

(c) If $\rho_T = \exp \{-1/q(T)\}$ such that $q(T) \sim T$, then
\[
\sum_{i=1}^{N} \sum_{t=g+1}^{T} a_i w_{it-g,T} = O_p \left( \max \left\{ \sqrt{NT^3} , Nq(T) \right\} \right).
\]

(d) If $\rho_T = \exp \{-1/q(T)\}$ such that $q(T) \to \infty$ but $q(T)/T \to 0$, then
\[
\sum_{i=1}^{N} \sum_{t=g+1}^{T} a_i w_{it-g,T} = O_p \left( \max \left\{ \sqrt{Nq(T)} , Nq(T) \right\} \right).
\]

(e) If $\rho_T \in \mathcal{G} = \left\{ |\rho_T| = \exp \{-1/q(T)\} : q(T) \geq 0 \text{ and } q(T) = O(1) \text{ as } T \to \infty \right\}$, then
\[
\sum_{i=1}^{N} \sum_{t=g+1}^{T} a_i w_{it-g,T} = O_p \left( \max \left\{ \sqrt{NT} , N \right\} \right).
\]

Proof of Lemma SE-18:
To proceed, write
\[
\sum_{i=1}^{N} \sum_{t=g+1}^{T} a_i w_{it-g,T} = \sum_{i=1}^{N} \sum_{t=g+1}^{T} a_i w_{it-g,T} + \sum_{i=1}^{N} \sum_{t=g+1}^{T} a_i \rho_{T-g} w_{i0},
\]
where $w_{t-g} = \sum_{j=1}^{t-g} \rho_{T-g-j} \varepsilon_{ij}$. Note that for $T \geq g + 1$
\[
E \left[ \sum_{i=1}^{N} \sum_{t=g+1}^{T} a_i w_{it-g,T} \right]^2 = \sum_{i=1}^{N} \sum_{j=1}^{T} \sum_{t=g+1}^{T} \sum_{s=g+1}^{T} E \left[ a_i a_j w_{it-g,T} w_{js-g,T} \right]
\]
\[
= (\mu_a^2 + \sigma_a^2) \sum_{i=1}^{N} \sum_{j=1}^{T} \sum_{t=g+1}^{T} \sum_{s=g+1}^{T} E \left[ w_{it-g,T} w_{js-g,T} \right]
\]
\[
= (\mu_a^2 + \sigma_a^2) \sum_{i=1}^{N} \sum_{j=1}^{T} \sum_{t=g+1}^{T} \sum_{s=g+1}^{T} E \left[ w_{it-g,T}^2 \right] + 2 (\mu_a^2 + \sigma_a^2) \sum_{i=1}^{N} \sum_{s=g+2}^{T} \sum_{t=g+1}^{T} E \left[ w_{it-g,T} w_{is-g,T} \right].
\]
Now, consider part (a), where we take $\rho_T = 1$ for all $T$ sufficiently large. In this case, we can apply
the results of part (a) of Lemma SE-16 to get

\[
E \left[ \sum_{i=1}^{N} \sum_{t=g+1}^{T} a_i w_{it-g,T} \right]^2
= \left( \mu_a^2 + \sigma_a^2 \right) \sum_{i=1}^{N} \sum_{t=g+1}^{T} E \left[ w_{t-g,T}^2 \right] + 2 \left( \mu_a^2 + \sigma_a^2 \right) \sum_{i=1}^{N} \sum_{t=g+1}^{T} \sum_{s=g+2}^{s-1} E \left[ w_{t-g,T} w_{s-g,T} \right]
= \sigma^2 \left( \mu_a^2 + \sigma_a^2 \right) \frac{N T^2}{2} \left[ 1 + O \left( \frac{1}{T} \right) \right] + 2 \sigma^2 \left( \mu_a^2 + \sigma_a^2 \right) \frac{N T^3}{6} \left[ 1 + O \left( \frac{1}{T} \right) \right]
= N T^3 \frac{\sigma^2 \left( \mu_a^2 + \sigma_a^2 \right)}{3} \left[ 1 + O \left( \frac{1}{T} \right) \right] = O \left( N T^3 \right),
\]
so that by Markov’s inequality, we obtain

\[
\sum_{i=1}^{N} \sum_{t=g+1}^{T} a_i w_{it-g,T} = O_p \left( \sqrt{N T^3/2} \right).
\]

Moreover, using Assumptions 2 and 4, we have

\[
\sum_{i=1}^{N} E \left[ a_i^2 \right] = N \left( \mu_a^2 + \sigma_a^2 \right) = O \left( N \right),
\]
\[
\sum_{i=1}^{N} \left( \sum_{t=g+1}^{T} \rho_T^{t-g} w_{i0} \right)^2 = \sum_{i=1}^{N} \left( \sum_{t=g+1}^{T} w_{i0} \right)^2 \quad \text{(for all } T \text{ sufficiently large)}
\leq \sup_i E \left[ w_{i0}^2 \right] N (T - g)^2 = O \left( N T^2 \right),
\]
from which it follows by Markov’s inequality that

\[
\sum_{i=1}^{N} a_i^2 = O_p \left( N \right), \quad \sum_{i=1}^{N} \left( \sum_{t=g+1}^{T} \rho_T^{t-g} w_{i0} \right)^2 = O_p \left( N T^2 \right).
\]

Applying the Cauchy-Schwarz inequality, we further obtain that

\[
\left| \sum_{i=1}^{N} \sum_{t=g+1}^{T} a_i \rho_T^{t-g} w_{i0} \right|^2 \leq \sum_{i=1}^{N} a_i^2 \left( \sum_{t=g+1}^{T} \rho_T^{t-g} w_{i0} \right)^2 = O_p \left( \sqrt{N} \right) O_p \left( \sqrt{N T} \right) = O_p \left( N T \right),
\]
so that

\[
\sum_{i=1}^{N} \sum_{t=g+1}^{T} a_i w_{it-g,T} = \sum_{i=1}^{N} \sum_{t=g+1}^{T} a_i w_{it-g,T} + \sum_{i=1}^{N} \sum_{t=g+1}^{T} a_i \rho_T^{t-g} w_{i0}
= O_p \left( \sqrt{N T^3/2} \right) + O_p \left( N T \right)
= O_p \left( \max \left\{ \sqrt{N T^3/2}, N T \right\} \right),
\]
as required for part (a).
Next, consider part (b), where we take $\rho_T = \exp\{-1/q(T)\}$ such that $T/q(T) \to 0$. In this case, we apply the results of parts (b) of Lemma SE-16 to get

$$E \left[ \sum_{i=1}^{N} \sum_{t=g+1}^{T} a_i w_{i,t-g,T} \right]^2$$

$$= (\mu_a^2 + \sigma_a^2) \sum_{i=1}^{N} \sum_{t=g+1}^{T} E [w_{i,t-g,T}^2] + 2 (\mu_a^2 + \sigma_a^2) \sum_{i=1}^{N} \sum_{s=g+2}^{T} \sum_{t=g+1}^{s-1} E [w_{i,t-g,T} w_{i,s-g,T}]$$

$$= \sigma^2 (\mu_a^2 + \sigma_a^2) N T^2 \left[ 1 + O \left( \max \left\{ \frac{T}{q(T)}, \frac{1}{T} \right\} \right) \right] + 2\sigma^2 (\mu_a^2 + \sigma_a^2) N T^3 \left[ 1 + O \left( \max \left\{ \frac{T}{q(T)}, \frac{1}{T} \right\} \right) \right]$$

$$= N T^3 \frac{\sigma^2 (\mu_a^2 + \sigma_a^2)}{3} \left[ 1 + O \left( \max \left\{ \frac{T}{q(T)}, \frac{1}{T} \right\} \right) \right] = O(NT^3).$$

It follows from Markov’s inequality that we obtain $\sum_{i=1}^{N} \sum_{t=g+1}^{T} a_i w_{i,t-g,T} = O_p \left( \sqrt{NT^{3/2}} \right)$.

Moreover, using Assumptions 2 and 4, we have

$$\sum_{i=1}^{N} E \left[ a_i^2 \right] = N (\mu_a^2 + \sigma_a^2) = O(N),$$

$$\sum_{i=1}^{N} E \left( \sum_{t=g+1}^{T} \rho_T^{t-g} w_{i0} \right)^2 \leq \sup E \left[ w_{i0}^2 \right] N \exp \left\{ -\frac{2}{q(T)} \right\} \left[ 1 - \exp \left\{ -\frac{1}{q(T)} \right\} \right]^{-2} \left[ 1 - \exp \left\{ -(T-g) \right\} \right]$$

$$= O(N) \times O(1) \times O \left( \frac{T^2}{q(T)^2} \right) \times O \left( \frac{T^2}{q(T)^2} \right) = O(NT^2),$$

from which it follows again by Markov’s inequality that

$$\sum_{i=1}^{N} a_i^2 = O_p \left( N \right), \quad \sum_{i=1}^{N} \left( \sum_{t=g+1}^{T} \rho_T^{t-g} w_{i0} \right)^2 = O_p \left( NT^2 \right).$$

Applying the Cauchy-Schwarz inequality, we further obtain that

$$\left| \sum_{i=1}^{N} \sum_{t=g+1}^{T} a_i \rho_T^{t-g} w_{i0} \right| \leq \sqrt{\sum_{i=1}^{N} a_i^2} \sqrt{\sum_{i=1}^{N} \left( \sum_{t=g+1}^{T} \rho_T^{t-g} w_{i0} \right)^2} = O_p \left( \sqrt{N} \right) O_p \left( \sqrt{N} T \right) = O_p \left( NT \right),$$

so that

$$\sum_{i=1}^{N} \sum_{t=g+1}^{T} a_i w_{i,t-g,T} = \sum_{i=1}^{N} \sum_{t=g+1}^{T} a_i w_{i,t-g,T} + \sum_{i=1}^{N} \sum_{t=g+1}^{T} a_i \rho_T^{t-g} w_{i0}$$

$$= O_p \left( \sqrt{NT^{3/2}} \right) + O_p \left( NT \right)$$

$$= O_p \left( \max \left\{ \sqrt{NT^{3/2}}, NT \right\} \right),$$

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as required for part (b).

Next, consider part (c), where we take \( \rho_T = \exp \{ -1/q(T) \} \) such that \( q(T) \sim T \). Here, we apply part (c) of Lemma SE-16 to obtain

\[
E \left[ \sum_{i=1}^{N} \sum_{t=g+1}^{T} a_i w_{i-t-g,T} \right]^2
= (\mu^2_a + \sigma^2_a) \sum_{i=1}^{N} \sum_{t=g+1}^{T} E \left[ w_{i-t-g,T}^2 \right] + 2 (\mu^2_a + \sigma^2_a) \sum_{i=1}^{N} \sum_{s=g+2}^{T} \sum_{t=g+1}^{s-1} E \left[ w_{i-t-g,T} w_{i-s-g,T} \right]
\]

\[
= \sigma^2 (\mu^2_a + \sigma^2_a) \frac{N q(T)^2}{4} \left[ \exp \left\{ -\frac{2T}{q(T)} \right\} + \frac{2T}{q(T)} - 1 \right] \left[ 1 + O \left( \frac{1}{T} \right) \right]
\]

\[
+ 2 (\mu^2_a + \sigma^2_a) N T q(T)^2 \frac{\sigma^2}{2} \times \left[ 1 - \frac{3 q(T)}{2T} + \frac{2g(T)}{T} \exp \left\{ -\frac{T}{q(T)} \right\} - \frac{1}{2} \exp \left\{ -\frac{T}{q(T)} \right\} \right] \left[ 1 + O \left( \frac{1}{T} \right) \right]
\]

\[
= \sigma^2 (\mu^2_a + \sigma^2_a) N T q(T)^2 \left[ 1 - \frac{3 q(T)}{2T} + \frac{2g(T)}{T} \exp \left\{ -\frac{T}{q(T)} \right\} - \frac{1}{2} \exp \left\{ -\frac{T}{q(T)} \right\} \right] \times \left[ 1 + O \left( \frac{1}{T} \right) \right]
\]

\[
= O \left( NT^3 \right).
\]

It follows from Markov’s inequality that we obtain

\[
\sum_{i=1}^{N} \sum_{t=g+1}^{T} a_i w_{i-t,g,T} = O_p \left( \sqrt{NT^3/2} \right).
\]

Moreover, using Assumptions 2 and 4, we obtain

\[
\sum_{i=1}^{N} E \left[ a_i^2 \right] = N (\mu^2_a + \sigma^2_a) = O(N),
\]

and

\[
\sum_{i=1}^{N} E \left( \sum_{t=g+1}^{T} \rho_{T}^{t-g} w_{i0} \right)^2
\leq \sup_i E \left[ w_{i0}^2 \right] N \exp \left\{ -\frac{2}{q(T)} \right\} \left[ 1 - \exp \left\{ -\frac{1}{q(T)} \right\} \right]^{-2} \left[ 1 - \exp \left\{ -\frac{(T-g)}{q(T)} \right\} \right]^2
\]

\[
= O(N) \times O(1) \times O(T^2) \times O(1) = O \left( NT^2 \right),
\]

so that similar to the proof of part (b) above, we have

\[
\left| \sum_{i=1}^{N} \sum_{t=g+1}^{T} a_i \rho_{T}^{t-g} w_{i0} \right| \leq \sqrt{\sum_{i=1}^{N} a_i^2} \sqrt{\sum_{i=1}^{N} \left( \sum_{t=g+1}^{T} \rho_{T}^{t-g} w_{i0} \right)^2} = O_p \left( \sqrt{N} \right) \times O_p \left( \sqrt{NT} \right) = O_p \left( NT \right)
\]

upon application of the Markov and the Cauchy-Schwarz inequalities. It follows that
\[
\sum_{i=1}^{N} \sum_{t=g+1}^{T} a_i w_{it-g,T} = \sum_{i=1}^{N} \sum_{t=g+1}^{T} a_i w_{it-g,T} + \sum_{i=1}^{N} \sum_{t=g+1}^{T} a_i \rho_t^{t-g} w_{i0} \\
= O_p \left( \sqrt{NT^3/2} \right) + O_p \left( NT \right) \\
= O_p \left( \max \left\{ \sqrt{NT^3/2}, NT \right\} \right),
\]

as required for part (c).

For part (d), we consider the case where \( \rho_T = \exp \left\{ -1/q(T) \right\} \) such that \( q(T) \to \infty \) but \( q(T)/T \to 0 \).

Here, we apply part (d) of Lemma SE-16 to obtain

\[
E \left[ \sum_{i=1}^{N} \sum_{t=g+1}^{T} a_i w_{it-g,T} \right]^2 \\
= (\mu_a^2 + \sigma_a^2) \sum_{i=1}^{N} \sum_{t=g+1}^{T} E \left[ w_{it-g,T}^2 \right] + 2 (\mu_a^2 + \sigma_a^2) \sum_{i=1}^{N} \sum_{s=g+1}^{T} \sum_{t=g+1}^{T} E \left[ w_{it-g,T} w_{is-g,T} \right] \\
= \sigma^2 (\mu_a^2 + \sigma_a^2) \frac{NTq(T)}{2} \left[ 1 + O \left( \frac{1}{q(T)} \right) + O \left( \frac{q(T)}{T} \right) \right] \\
+ 2\sigma^2 (\mu_a^2 + \sigma_a^2) \frac{NTq(T)}{2} \left[ 1 + O \left( \frac{q(T)}{T} \right) \right] \\
= NTq(T)^2 \sigma^2 (\mu_a^2 + \sigma_a^2) \left[ 1 + O \left( \max \left\{ \frac{q(T)}{T}, \frac{1}{q(T)} \right\} \right) \right] \\
= O \left( NTq(T)^2 \right),
\]

from which it follows, by Markov’s inequality, that \( \sum_{i=1}^{N} \sum_{t=g+1}^{T} a_i w_{it-g,T} = O_p \left( \sqrt{NTq(T)} \right) \).

Moreover, using Assumptions 2 and 4, we obtain

\[
\sum_{i=1}^{N} E \left[ a_i^2 \right] = N (\mu_a^2 + \sigma_a^2) = O(N),
\]

and

\[
\sum_{i=1}^{N} E \left( \sum_{t=g+1}^{T} \rho_t^{t-g} w_{i0} \right)^2 \\
\leq \sup_i E \left[ w_{i0}^2 \right] N \exp \left\{ -\frac{2}{q(T)} \right\} \left[ 1 - \exp \left\{ -\frac{1}{q(T)} \right\} \right]^{-2} \left[ 1 - \exp \left\{ -\frac{(T-g)}{q(T)} \right\} \right]^2 \\
= O(N) \times O(1) \times O \left( q(T)^2 \right) \times O(1) = O \left( Nq(T)^2 \right),
\]

from which it follows by Markov’s inequality that

\[
\sum_{i=1}^{N} a_i^2 = O_p(N), \quad \sum_{i=1}^{N} \left( \sum_{t=g+1}^{T} \rho_t^{t-g} w_{i0} \right)^2 = O_p \left( Nq(T)^2 \right).
\]

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Applying the Cauchy-Schwarz inequality, we further obtain that

\[
\left| \sum_{i=1}^{N} \sum_{t=g+1}^{T} a_i \rho_T^{t-g} w_{i0} \right| \leq \sqrt{\sum_{i=1}^{N} a_i^2} \sqrt{\sum_{t=g+1}^{T} \left( \sum_{i=1}^{N} \rho_T^{t-g} w_{i0} \right)^2} = O_p \left( \sqrt{N} \right) \times O_p \left( \sqrt{Nq(T)} \right) = O_p \left( Nq(T) \right),
\]

so that

\[
\sum_{i=1}^{N} \sum_{t=g+1}^{T} a_i w_{it-g,T} = \sum_{i=1}^{N} \sum_{t=g+1}^{T} a_i w_{it-g,T} + \sum_{i=1}^{N} \sum_{t=g+1}^{T} a_i \rho_T^{t-g} w_{i0}
\]

\[
= O_p \left( \sqrt{NTq(T)} \right) + O_p \left( Nq(T) \right)
\]

\[
= O_p \left( \max \left\{ \sqrt{NTq(T)}, Nq(T) \right\} \right),
\]

as required for part (d).

Finally, to show part (e), note that in this case

\[
E \left[ \sum_{i=1}^{N} \sum_{t=g+1}^{T} a_i w_{it-g,T} \right]^2
\]

\[
= (\mu_a^2 + \sigma_a^2) N \sum_{t=g+1}^{T} E \left[ w_{it-g,T}^2 \right] + 2 (\mu_a^2 + \sigma_a^2) N \sum_{s=g+2}^{T} \sum_{t=g+1}^{s-1} E \left[ w_{it-g,T} w_{s-t,g,T} \right]
\]

\[
= (\mu_a^2 + \sigma_a^2) N \sum_{t=g+1}^{T} \sum_{t'=g+1}^{t} \sum_{t''=g+1}^{t-t'} \rho_T^{2(t'-g)} E [\varepsilon_{ij}] E [\varepsilon_{ik}] + 2 (\mu_a^2 + \sigma_a^2) N \sum_{s=g+2}^{T} \sum_{t=g+1}^{s-1} \sum_{t'=g+1}^{t} \sum_{t''=g+1}^{t-t'} \rho_T^{2(t-s)} E [\varepsilon_{ij}] E [\varepsilon_{ik}]
\]

\[
= (\mu_a^2 + \sigma_a^2) \sigma^2 N \sum_{t=g+1}^{T} \sum_{t'=g+1}^{t} \rho_T^{2(t'-g)} + 2 (\mu_a^2 + \sigma_a^2) \sigma^2 N \sum_{s=g+2}^{T} \sum_{t=g+1}^{s-1} \sum_{t'=g+1}^{t} \rho_T^{2(t-s)}
\]

\[
= \frac{(\mu_a^2 + \sigma_a^2) \sigma^2}{1 - \rho_T^2} N (T-g) - \frac{(\mu_a^2 + \sigma_a^2) \sigma^2 \rho_T^2}{(1 - \rho_T^2)} N \sum_{t=g+1}^{T} \rho_T^{2(t-g-1)}
\]

\[
+ \frac{2 (\mu_a^2 + \sigma_a^2) \sigma^2}{1 - \rho_T^2} N \sum_{t=g+2}^{T} \sum_{s=g+1}^{t-1} \rho_T^{2(t-s)} (1 - \rho_T^{2(s-g)})
\]

\[
= \frac{(\mu_a^2 + \sigma_a^2) \sigma^2}{1 - \rho_T^2} N (T-g) - \frac{(\mu_a^2 + \sigma_a^2) \sigma^2 \rho_T^2}{(1 - \rho_T^2)^2} N \sum_{t=g+1}^{T} \rho_T^{2(t-g-1)}
\]

\[
+ \frac{2 (\mu_a^2 + \sigma_a^2) \sigma^2 \rho_T^2}{(1 - \rho_T^2)(1 - \rho_T)} N \sum_{t=g+2}^{T} (1 - \rho_T^{2(g-1)}) - \frac{2 (\mu_a^2 + \sigma_a^2) \sigma^2}{1 - \rho_T^2} N \sum_{t=g+2}^{T} \rho_T^{2(t-g-1)} \sum_{s=g+1}^{t-1} \rho_T^{2(s-g-1)}
\]

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Since we assume that $q(T) = O(1)$ in this case, it follows that there exist a positive constant $C_q$ and a positive integer $T^*$ such that for all $T \geq T^*$

$$|\rho_T| \leq \exp\left\{-\frac{1}{C_q}\right\} < 1$$

Applying this upper bound we obtain for all $T \geq T^*$

$$
E \left[ \sum_{i=1}^{N} \sum_{t=g+1}^{T} a_i w_{it-g,T} \right]^2
\leq \frac{(\mu_a^2 + \sigma_a^2) \sigma^2}{1 - \rho_T^2} N (T - g) + \frac{(\mu_a^2 + \sigma_a^2) \sigma^2}{1 - \rho_T^2} N (T - g - 1) - \frac{2 (\mu_a^2 + \sigma_a^2) \sigma^2 \rho_T^2 (1 - \rho_T^{T-g-1})}{(1 - \rho_T^2) (1 - \rho_T)^2} N
$$

$$+ \frac{2 (\mu_a^2 + \sigma_a^2) \sigma^2 |\rho_T|}{(1 - \rho_T^2) (1 - |\rho_T|)} N (T - g - 1) + \frac{2 (\mu_a^2 + \sigma_a^2) \sigma^2 \rho_T^2 (1 - \rho_T^{T-g-1})}{(1 - \rho_T^2) (1 - |\rho_T|)^2} N
$$

$$+ 4 \frac{(\mu_a^2 + \sigma_a^2) \sigma^2 N |\rho_T|^3}{(1 - \rho_T^2) (1 - \rho_T)^2} + 2 \frac{(\mu_a^2 + \sigma_a^2) \sigma^2 N \rho_T^4 (1 - \rho_T^{2(T-g-1)})}{(1 - \rho_T^2)^2 (1 - \rho_T)}$$
Applying the Cauchy-Schwarz inequality, we further obtain that

\[
\begin{align*}
&\leq (\mu_n^2 + \sigma_n^2) \sigma^2 \left[ \frac{N (T-g)}{1 - \exp \{-2/C_q\}} + \frac{N}{(1 - \exp \{-2/C_q\})^2} + \frac{2N (T-g-1)}{(1 - \exp \{-2/C_q\}) (1 - \exp \{-1/C_q\})} \right] \\
&\quad + \frac{2N}{(1 - \exp \{-2/C_q\}) (1 - \exp \{-1/C_q\})^2} \\
&\quad + \frac{2N}{(1 - \exp \{-2/C_q\})^2 (1 - \exp \{-1/C_q\})} \\
&= O(NT),
\end{align*}
\]

so that, using Markov’s inequality, we deduce that \(\sum_{i=1}^N \sum_{t=g+1}^T a_i w_{it-g,T} = O_p\left(\sqrt{NT}\right)\).

Moreover, using Assumptions 2 and 4, we obtain

\[
\sum_{i=1}^N E \left[ a_i^2 \right] = N (\mu_n^2 + \sigma_n^2) = O(N),
\]

\[
\sum_{i=1}^N \left( \sum_{t=g+1}^T \rho_{T}^{t-g} w_{i0} \right)^2 \leq N \sup_i E \left[ w_{i0}^2 \right] \rho_{T}^2 \left( \frac{1 - \rho_{T}^{T-g}}{1 - \rho_{T}} \right)^2 = O(N) O(1) = O(N),
\]

from which it follows by Markov’s inequality that

\[
\sum_{i=1}^N a_i^2 = O_p(N), \quad \sum_{i=1}^N \left( \sum_{t=g+1}^T \rho_{T}^{t-g} w_{i0} \right)^2 = O_p(N)
\]

Applying the Cauchy-Schwarz inequality, we further obtain that

\[
\left| \sum_{i=1}^N \sum_{t=g+1}^T a_i \rho_{T}^{t-g} w_{i0} \right| \leq \sqrt{\sum_{i=1}^N a_i^2} \sqrt{\sum_{i=1}^N \left( \sum_{t=g+1}^T \rho_{T}^{t-g} w_{i0} \right)^2} = O_p(\sqrt{N}) O_p(\sqrt{N}) = O_p(N),
\]

so that

\[
\sum_{i=1}^N \sum_{t=g+1}^T a_i w_{it-g,T} = \sum_{i=1}^N \sum_{t=g+1}^T a_i w_{it-g,T} + \sum_{i=1}^T \sum_{t=g+1}^T a_i \rho_{T}^{t-g} w_{i0} \]

\[
= O_p\left(\sqrt{NT}\right) + O_p(N) = O_p\left(\max\left\{\sqrt{NT}, N\right\}\right),
\]

as required for part (e). \(\square\)

**Lemma SE-19 (Phillips and Moon, 1999, Theorem 3):** Suppose that \(Y_i,T = C_i Q_i,T\), where the \((m \times 1)\) random vectors \(Q_i,T\) are i.i.d. \((0, \Sigma_T)\) across \(i\) for all \(T\) and the \(C_i\) are \((m \times m)\) nonzero and nonrandom matrices. Assume the following conditions hold.

1. Let \(\sigma_T^2 = \lambda_{\min}(\Sigma_T)\) and \(\lim \inf_T \sigma_T^2 > 0\);
2. \(\max_{1 \leq i \leq n} \|C_i\|^2 / \lambda_{\min} \left( \sum_{i=1}^N C_i C_i' \right) = O(1/N)\) as \(N \to \infty\).
(iii) \( ||Q_{i,T}||^2 \) are uniformly integrable in \( T \);
(iv) \( \lim_{N,T} (1/N) \sum_{i=1}^N C_i \Sigma_T C_i' = \Omega > 0. \)

Then,
\[
X_{N,T} = \frac{1}{\sqrt{N}} \sum_{i=1}^N Y_{i,T} \Rightarrow N (0, \Omega), \text{ as } N,T \to \infty.
\]

**Lemma SE-20:**

Let \( g \) be a non-negative integer. Under Assumptions 1 and 4, the following results hold as \( N,T \to \infty \).

(a) If \( \rho_T = \exp \{-1/q(T)\} \) such that \( T/q(T) \to 0 \), then
\[
\frac{1}{\sqrt{NT}} \sum_{i=1}^N \sum_{t=g+2}^T w_{it-g-1,T} \epsilon_{it-g} \Rightarrow N \left( 0, \frac{\sigma^4}{2} \right).
\]

(b) If \( \rho_T = 1 \) for all \( T \) sufficiently large, then
\[
\frac{1}{\sqrt{NT}} \sum_{i=1}^N \sum_{t=g+2}^T w_{it-g-1,T} \epsilon_{it-g} \Rightarrow N \left( 0, \frac{\sigma^4}{2} \right).
\]

**Proof of Lemma SE-20:**

Consider first part (a), where we take \( \rho_T = \exp \{-1/q(T)\} \) such that \( T/q(T) \to 0 \). Write

\[
\frac{1}{\sqrt{NT}} \sum_{i=1}^N \sum_{t=g+2}^T w_{it-g-1,T} \epsilon_{it-g} = \frac{1}{\sqrt{NT}} \sum_{i=1}^N \sum_{t=g+2}^T w_{it-g-1,T} \epsilon_{it-g} + \frac{1}{\sqrt{NT}} \sum_{i=1}^N \sum_{t=g+2}^T \epsilon_{it-g} \rho_T^{t-g-1} w_{i0},
\]

where \( w_{it-g-1,T} = \sum_{j=1}^{t-g-1} \rho_T^{t-g-1-j} \epsilon_{ij} \). Note that

\[
\frac{1}{T} \sum_{t=2}^T w_{it,T}^2 = \frac{1}{T} \sum_{t=2}^T \left( \exp \left( -\frac{1}{q(T)} \right) w_{it-1,T} + \epsilon_{it} \right)^2\]
\[
= \frac{1}{T} \sum_{t=2}^T \epsilon_{it}^2 + 2 \exp \left\{ -\frac{1}{q(T)} \right\} \frac{1}{T} \sum_{t=2}^T w_{it-1,T} \epsilon_{it} + \exp \left\{ -\frac{2}{q(T)} \right\} \frac{1}{T} \sum_{t=g+2}^T w_{it-1,T}^2
\]

so that

\[
\frac{1}{T} \sum_{t=2}^T w_{it-1,T} \epsilon_{it} \]
\[
= \frac{1}{2} \exp \left\{ \frac{1}{q(T)} \right\} \left[ \frac{1}{T} \sum_{t=2}^T w_{it,T}^2 - \exp \left\{ -\frac{1}{q(T)} \right\} \frac{1}{T} \sum_{t=1}^T \sum_{t=1}^T w_{it-1,T}^2 - \frac{1}{T} \sum_{t=1}^T \epsilon_{it}^2 \right].
\]
Now,

\[
\frac{1}{T} \sum_{t=2}^{T} w_{it,T}^2 - \exp \left\{ -\frac{2}{q(T)} \right\} \frac{1}{T} \sum_{t=2}^{T} w_{it-1,T}^2
\]

\[
= \frac{1}{T} \sum_{t=2}^{T} w_{it,T}^2 - \left[ 1 - \frac{2}{q(T)} + O \left( \frac{1}{q(T)^2} \right) \right] \frac{1}{T} \sum_{t=2}^{T} w_{it-1,T}^2
\]

\[
= \frac{1}{T} \sum_{t=2}^{T} w_{it,T}^2 - \frac{1}{T} \sum_{t=2}^{T} w_{it-1,T}^2 + \frac{2T}{q(T)} \left( \frac{1}{T^2} \sum_{t=2}^{T} w_{it-1,T}^2 \right) + O_p \left( \frac{T}{q(T)^2} \right)
\]

\[
= \frac{1}{T} w_{T,T}^2 - \frac{1}{T} w_{1,T}^2 + \frac{2T}{q(T)} \left( \frac{1}{T^2} \sum_{t=1}^{T} w_{it-1,T}^2 \right) + O_p \left( \frac{T}{q(T)^2} \right)
\]

\[
= \frac{1}{T} w_{T,T}^2 - \frac{1}{T} w_{t,T}^2 + \frac{2T}{q(T)} \left( \frac{1}{T^2} \sum_{t=2}^{T} w_{it-1,T}^2 \right) + O_p \left( \frac{T}{q(T)^2} \right)
\]

\[
= \frac{1}{T} w_{T,T}^2 + O_p \left( \max \left\{ \frac{T}{q(T)}, \frac{1}{T} \right\} \right)
\]

Next, note that

\[
\frac{1}{T} \sum_{t=g+2}^{T} w_{it-g-1,T} w_{it-g} = \frac{1}{T} \sum_{t=2}^{T} w_{t-1,T} w_{it} - \frac{1}{T} \sum_{s=1}^{g-1} w_{T-g+s,T} w_{T-g+1+s}
\]
Observe that

\[
E \left[ \frac{1}{T} \sum_{s=1}^{g-1} \sum_{v=1}^{T-g+s} \sum_{k=1}^{T-g+g+s} \sum_{\ell=1}^{T-g-v} \exp \left\{ -\frac{T-g+s-k}{q(T)} \right\} \exp \left\{ -\frac{T-g+v-\ell}{q(T)} \right\} \right] = \frac{1}{T^2} \sum_{s=1}^{g-1} \sum_{k=1}^{T-g+s} \exp \left\{ -\frac{T-g+s-k}{q(T)} \right\}
\]

so that applying Markov’s inequality, we obtain

\[
\frac{\sigma^4}{T^2} \sum_{s=1}^{g-1} \sum_{k=1}^{T-g+s} \exp \left\{ -\frac{T-g+s-k}{q(T)} \right\} \leq O_p \left( \frac{1}{T} \right)
\]

so that applying Markov’s inequality, we obtain

\[
\frac{1}{T} \sum_{s=1}^{g-1} \sum_{v=1}^{T-g+s} \sum_{k=1}^{T-g+g+s} \sum_{\ell=1}^{T-g-v} \exp \left\{ -\frac{T-g+s-k}{q(T)} \right\} \exp \left\{ -\frac{T-g+v-\ell}{q(T)} \right\} \leq O_p \left( \frac{1}{\sqrt{T}} \right)
\]
Hence,

\[ \frac{1}{T} \sum_{t=g+2}^{T} w_{it-g-1,T} \varepsilon_{it-g} \]

\[ = \frac{1}{T} \sum_{t=2}^{T} w_{it-1,T} \varepsilon_{it} - \frac{1}{T} \sum_{s=1}^{g-1} w_{iT-g+s,T} \varepsilon_{ iT-g+1+s} \]

\[ = \frac{1}{2} \exp \left\{ \frac{1}{q(T)} \right\} \left[ \frac{1}{T} \sum_{t=1}^{T} w_{it}^2 - \exp \left\{ - \frac{2}{q(T)} \right\} \frac{1}{T} \sum_{t=1}^{T} w_{it-1}^2 - \frac{1}{T} \sum_{t=1}^{T} \varepsilon_{it}^2 \right] + O_p \left( \frac{1}{\sqrt{T}} \right) \]

\[ = \left[ \frac{1}{2T} w_{iT}^2 \right] - \frac{1}{2} \sigma^2 + O_p \left( \frac{T}{q(T)} \right) + O_p \left( \frac{1}{\sqrt{T}} \right) \]

\[ \Rightarrow \frac{1}{2} \left[ \sigma W_i(1) \right] - \frac{1}{2} \sigma^2 = \frac{\sigma^2}{2} \left( \left[ W_i(1) \right]^2 - 1 \right), \tag{22} \]

where the last line follows from applying Lemma SE-12 with \( r = 1 \) and the Cramér convergence theorem. Moreover, note that

\[ d \left[ W_i(g) \right]^2 = (W_i(g) + dW_i(g))^2 - [W_i(g)]^2 \]

\[ = [W_i(g)]^2 + 2W_i(g) dW_i(g) + [dW_i(g)]^2 - [W_i(g)]^2 \]

\[ = 2W_i(g) dW_i(g) + [dW_i(g)]^2 \]

\[ = 2W_i(g) dW_i(g) + dg \text{ a.s.} \]

It follows that

\[ \int_{0}^{1} W_i(g) dW_i(g) = \frac{1}{2} \left( \left[ W_i(1) \right]^2 - 1 \right). \tag{23} \]

Substituting (23) into (22), we obtain

\[ \frac{1}{T} \sum_{t=g+2}^{T} w_{it-g-1,T} \varepsilon_{it-g} \Rightarrow \sigma^2 \int_{0}^{1} W_i(g) dW_i(g), \]

as \( T \to \infty \).

Next, we verify the conditions of Theorem 3 of Phillips and Moon (1999), given above as Lemma SE-19. First define

\[ Q_{i,T} = \frac{1}{T} \sum_{t=g+2}^{T} w_{it-g-1,T} \varepsilon_{it-g}, \]

and note that by direct calculation and by applying the results given parts (a) and (b) of Lemma SE-1
with \( d = 2 \) and \( b = g + 2 \), we have

\[
\lim_{T \to \infty} \inf T \to \infty \sigma_T^2 = \lim_{T \to \infty} E \left[ Q_{i,T}^2 \right] = \lim_{T \to \infty} E \left[ \sum_{i=1}^{N} \sum_{t=g+2}^{T} w_{it-g-1,T\varepsilon_{it-g}} \right]^2
\]

\[
= \lim_{T \to \infty} \frac{1}{T^2} \sum_{t=g+2}^{T} \sum_{s=g+2}^{T} \sum_{k=1}^{t-g-1} \sum_{t=1}^{s-g-1} \exp \left\{ -\frac{t-g-1-k}{q(T)} \right\} \exp \left\{ -\frac{s-g-1-\ell}{q(T)} \right\} E \left[ \varepsilon_{it-g-1,T\varepsilon_{it-g}} \right]
\]

\[
= \sigma^4 \lim_{T \to \infty} \frac{1}{T^2} \sum_{t=g+2}^{T} \sum_{s=g+2}^{T} \sum_{k=1}^{t-g-1} \sum_{t=1}^{s-g-1} \exp \left\{ -2t-g-1-k \right\} \frac{1}{q(T)} = \sigma^4 > 0,
\]

so that condition (i) of Lemma SE-19 is satisfied. Moreover, in this case, we have \( C_i = 1 \) for all \( i \), so that \( \max_{1 \leq i \leq n} \| C_i \|^2 / \lambda_{\min} \left( \sum_{i=1}^{N} C_i C_i' \right) = 1/N = O(1/N) \) as required by condition (ii). Next, note that by (22) above,

\[
Q_{i,T} = \frac{1}{T} \sum_{t=g+2}^{T} w_{it-g-1,T\varepsilon_{it-g}} \Rightarrow \sigma^2 \left[ \chi^2_1 - 1 \right] \equiv Q,
\]

as \( T \to \infty \), so that by the continuous mapping theorem

\[
Q_{i,T}^2 \Rightarrow Q^2 = \frac{\sigma^4}{4} \left[ \chi^2_1 - 1 \right]^2, \text{ as } T \to \infty.
\]

In addition,

\[
E \left[ Q^2 \right] = \frac{\sigma^4}{4} \left[ E \left[ \chi^2_1 \right]^2 - 2E \left[ \chi^2_1 \right] + 1 \right] = \frac{\sigma^4}{4} \{ 3 - 2 + 1 \} = \frac{\sigma^4}{2},
\]

and note that, as \( T \to \infty \),

\[
\lim_{T \to \infty} E \left[ Q_{i,T}^2 \right] = \sigma^4 \lim_{T \to \infty} \frac{1}{T^2} \sum_{t=g+2}^{T} \sum_{s=g+2}^{T} \sum_{k=1}^{t-g-1} \sum_{t=1}^{s-g-1} \exp \left\{ -2t+g-1-k \right\} \frac{1}{q(T)} = \frac{\sigma^4}{2} = E \left[ Q^2 \right].
\]

It follows again from Theorem 5.4 of Billingsley (1968) that \( \{ Q_{i,T}^2 \} \) is uniformly integrable, so that condition (iii) of Lemma SE-19 is satisfied. Finally, note that in this case

\[
\lim_{N,T} (1/N) \sum_{i=1}^{N} C_i \Sigma_T C_i' = \lim_{N,T} (1/N) \sum_{i=1}^{N} E \left[ Q_{i,T}^2 \right] = \frac{\sigma^4}{2} > 0,
\]

so that condition (iv) is satisfied as well. It follows then from Lemma SE-19 that

\[
\frac{1}{\sqrt{N}} \sum_{i=1}^{N} Q_{i,T} = \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=g+2}^{T} w_{it-g-1,T\varepsilon_{it-g}} \Rightarrow N \left( 0, \frac{\sigma^4}{2} \right),
\]

as \( N, T \to \infty \).
Moreover, by Assumptions 1 and 4,
\[
E \left[ \left( \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=g+2}^{T} \varepsilon_{it-g} \rho_{T}^{t-g-1} w_{i0} \right)^{2} \right]
\]
\[= \frac{1}{NT^{2}} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{t=g+2}^{T} \sum_{t=g+2}^{T} E[w_{i0}w_{j0}] \rho_{T}^{t-g-1} \rho_{T}^{s-g-1} E[\varepsilon_{it-g}\varepsilon_{js-g}]
\]
\[= \frac{\sigma^{2}}{NT^{2}} \sum_{i=1}^{N} E[w_{i0}^{2}] \sum_{t=g+2}^{T} \rho_{T}^{2(t-g-1)}
\]
\[= \frac{\sigma^{2}}{T^{2}} \left( \sup_{i} E[w_{i0}^{2}] \right) \exp \left\{ -\frac{2}{q(T)} \right\} \left[ 1 - \exp \left\{ -\frac{2}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{ -\frac{2(T-g-1)}{q(T)} \right\} \right] = O(T^{-2}) \times O(1) \times O(1) \times O(q(T)) \times O(T/q(T)) = O(T^{-1}).
\]
It follows from Markov’s inequality that
\[
\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=g+2}^{T} \varepsilon_{it-g} \rho_{T}^{t-g-1} w_{i0} = O_{p} \left( \frac{1}{\sqrt{T}} \right).
\]
Hence,
\[
\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=g+2}^{T} w_{it-g-1,T} \varepsilon_{it-g} = \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=g+2}^{T} w_{it-g-1,T} \varepsilon_{it-g} + \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=g+2}^{T} \varepsilon_{it-g} \rho_{T}^{t-g-1} w_{i0}
\]
\[= \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=g+2}^{T} w_{it-g-1,T} \varepsilon_{it-g} + O_{p} \left( \frac{1}{\sqrt{T}} \right) \Rightarrow N \left( 0, \frac{\sigma^{4}}{2} \right),
\]
which is the desired result.

To show part (b), note that, in this case,
\[
w_{it,T} = w_{it,T} + \rho_{T}^{t-g-1} w_{i0} = \sum_{j=1}^{t} \varepsilon_{ij} + w_{i0}
\]
where the second equality holds for all \( T \) sufficiently large. Now, by direct calculation,
\[
E \left[ \frac{1}{T} \sum_{s=1}^{g-1} w_{T-g+s,T} \varepsilon_{it-T+g+1+s,T} \right]^{2} = \frac{1}{T^{2}} \sum_{s=1}^{g-1} \sum_{v=1}^{T-g+s} \sum_{k=1}^{T-g+v} \sum_{\ell=1}^{T} E[\varepsilon_{i\ell} \varepsilon_{it-T+g+1+s} \varepsilon_{iT-T+g+1+v}]
\]
\[= \frac{\sigma^{4}}{T^{2}} \sum_{s=1}^{g-1} (T-g+s) = \frac{\sigma^{4}}{T^{2}} \left[ (T-g)(g-1) + \frac{g(g-1)}{2} \right] = O \left( \frac{1}{T} \right),
\]
so that we can apply Markov’s inequality to obtain
\[
\frac{1}{T} \sum_{s=1}^{g-1} w_{T-g+s,T} \varepsilon_{iT-T+g+1+s,T} = O_{p} \left( \frac{1}{\sqrt{T}} \right).
\]
Using this result and following the arguments of part (b) of Theorem 3.1 in Phillips (1987), we further obtain

\[
\frac{1}{T} \sum_{t=2}^{T} w_{t-1,T \epsilon_{it-g}} = \frac{1}{T} \sum_{t=2}^{T} w_{t-1,T \epsilon_{it}} - \frac{1}{T} \sum_{s=1}^{g-1} w_{T-g+s,T \epsilon_{it-g+1+s}}
\]

\[
= \frac{1}{T} \sum_{t=2}^{T} w_{t-1,T \epsilon_{it}} + O_p \left( \frac{1}{\sqrt{T}} \right) \Rightarrow \frac{\sigma^2}{2} \left( \left| W_i(1) \right|^2 - 1 \right) .
\]

Next, we verify the conditions of Theorem 3 of Phillips and Moon (1999), given above as Lemma SE-19. To proceed, define \( Q_{i,T} = T^{-1} \sum_{i=g+2}^{T} w_{i-T-1,T \epsilon_{it}-g} \) as before, and note that by direct calculation, we have

\[
\lim_{T \to \infty} \frac{\sigma^2_T}{T} = \lim_{T \to \infty} E \left[ Q_{i,T}^2 \right] = \lim_{T \to \infty} \frac{1}{T^2} \sum_{t=g+2}^{T} \sum_{s=g+2}^{T} \sum_{k=1}^{T} \sum_{\ell=1}^{T} E \left[ \xi_{ik} \epsilon_{j \ell} \epsilon_{it-g} \epsilon_{js-g} \right]
\]

\[
= \sigma^4 \lim_{T \to \infty} \frac{1}{T^2} \sum_{t=g+2}^{T} (t - g - 1) = \sigma^4 \lim_{T \to \infty} \frac{1}{T^2} \frac{(T - g)(T - g - 1)}{2} = \frac{\sigma^4}{2} > 0 .
\]

The rest of the steps for verifying the conditions of Lemma SE-19 follows in a manner similar to that given in part (a) above. Hence, by applying Lemma SE-19, we deduce that

\[
\frac{1}{\sqrt{N}} \sum_{i=1}^{N} Q_{i,T} = \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=g+2}^{T} w_{t-g-1,T \epsilon_{it}-g} \Rightarrow N \left( 0, \frac{\sigma^4}{2} \right) ,
\]

as \( N, T \to \infty \).

Moreover, by Assumptions 1 and 4,

\[
E \left[ \left( \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=g+2}^{T} \epsilon_{it-g} w_{i0} \right)^2 \right] = \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{t=g+2}^{T} \sum_{s=g+2}^{T} E \left[ w_{i0} w_{j0} \right] E \left[ \epsilon_{it-g} \epsilon_{js-g} \right]
\]

\[
= \frac{\sigma^2}{NT^2} (T - g - 1) \sum_{i=1}^{N} E \left[ w_{i0}^2 \right] = \frac{\sigma^2}{T} \left( \sup_{i} E \left[ w_{i0}^2 \right] \right) \left[ 1 + O \left( \frac{1}{T} \right) \right] = O \left( T^{-1} \right) .
\]

It follows from Markov’s inequality that

\[
\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=g+2}^{T} \epsilon_{it-g} w_{i0} = O_p \left( \frac{1}{\sqrt{T}} \right) .
\]

Hence, in this case

\[
\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=g+2}^{T} w_{it-g-1,T \epsilon_{it-g}} = \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=g+2}^{T} w_{it-T-1,T \epsilon_{it}} + \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=g+2}^{T} \epsilon_{it-g} w_{i0}
\]

(for all \( T \) sufficiently large)

\[
= \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=g+2}^{T} w_{it-g-1,T \epsilon_{it-g}} + O_p \left( \frac{1}{\sqrt{T}} \right) \Rightarrow N \left( 0, \frac{\sigma^4}{2} \right) ,
\]

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which is the desired result. □

Lemma SE-21:

Let \( w_{iT-2} = \sum_{j=1}^{T-2} \rho_T^{(T-2-j)} \varepsilon_{ij} \). Under Assumptions 1 and 4, the following statements are true, as \( N, T \to \infty \).

(a) If \( \rho_T = 1 \) for all \( T \) sufficiently large, then

\[
\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} w_{iT-2} T \varepsilon_{iT} = \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} w_{iT-2} T \varepsilon_{iT} + O_p \left( \frac{1}{\sqrt{T}} \right) = O_p \left( 1 \right).
\]

(b) If \( \rho_T = \exp \{ -1/q(T) \} \) such that \( T/q(T) \to 0 \), then

\[
\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} w_{iT-2} T \varepsilon_{iT} = \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} w_{iT-2} T \varepsilon_{iT} + O_p \left( \frac{1}{\sqrt{T}} \right) = O_p \left( 1 \right).
\]

(c) If \( \rho_T = \exp \{ -1/q(T) \} \) such that \( q(T) \sim T \), then

\[
\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} w_{iT-2} T \varepsilon_{iT} = \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} w_{iT-2} T \varepsilon_{iT} + O_p \left( \frac{1}{\sqrt{T}} \right) = O_p \left( 1 \right).
\]

(d) If \( \rho_T = \exp \{ -1/q(T) \} \) such that \( q(T) \to \infty \) but \( q(T)/T \to 0 \), then

\[
\frac{1}{\sqrt{Nq(T)}} \sum_{i=1}^{N} w_{iT-2} T \varepsilon_{iT} = \frac{1}{\sqrt{Nq(T)}} \sum_{i=1}^{N} w_{iT-2} T \varepsilon_{iT} + O_p \left( \frac{1}{\sqrt{q(T)}} \exp \left\{ - \frac{T}{q(T)} \right\} \right) = O_p \left( 1 \right).
\]

(e) If \( \rho_T \in \mathcal{G}_{St} = \left\{ |\rho_T| = \exp \{ -1/q(T) \} : q(T) \geq 0 \text{ and } q(T) \to O(1) \text{ as } T \to \infty \right\} \), then

\[
\frac{1}{\sqrt{N}} \sum_{i=1}^{N} w_{iT-2} T \varepsilon_{iT} = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} w_{iT-2} T \varepsilon_{iT} + O_p \left( \frac{1}{\sqrt{q(T)}} \right) = O_p \left( 1 \right).
\]

Proof of Lemma SE-21:

To proceed, write

\[
\sum_{i=1}^{N} w_{iT-2} T \varepsilon_{iT} = \sum_{i=1}^{N} w_{iT-2} T \varepsilon_{iT} + \sum_{i=1}^{N} \varepsilon_{iT} \rho_T^{(T-2)} w_{i0},
\]

where \( w_{iT-2} = \sum_{j=1}^{T-2} \rho_T^{(T-2-j)} \varepsilon_{ij} \).
Now, consider part (a) where, by assumption, there exists a positive integer \( I_\rho \) such that for all \( T \geq I_\rho \), the sequence \( \{w_{i,T-2,T}\} \) has the partial sum representation \( w_{i,T-2,T} = \sum_{j=1}^{T-2} \varepsilon_{ij} \). Hence, for all \( T \geq I_\rho \), we have by direct calculation,

\[
E \left[ \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} w_{i,T-2,T}\varepsilon_{iT} \right]^2 = \frac{1}{NT} \sum_{i=1}^{N} \sum_{h=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{T-2} E[\varepsilon_{ij}\varepsilon_{iT}\varepsilon_{hk}\varepsilon_{hT}]
\]

\[
= \frac{1}{NT} \sum_{i=1}^{N} \sum_{j=1}^{T-2} E[\varepsilon_{ij}^2] E[\varepsilon_{iT}^2] = \sigma^4 \frac{N(T-2)}{NT} = O(1).
\]

It follows from Markov’s inequality that \((NT)^{-1/2} \sum_{i=1}^{N} w_{i,T-2,T}\varepsilon_{iT} = O_p(1)\).

Moreover, by Assumptions 1 and 4,

\[
E \left[ \left( \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \varepsilon_{iT}w_{i0} \right)^2 \right] = \frac{1}{NT} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{T-2} E[w_{i0}w_{j0}] E[\varepsilon_{iT}\varepsilon_{jT}]
\]

\[
\leq \sigma^2 \frac{N}{NT} \left( \sup_i \sum_{k=1}^{T-2} \sum_{j=1}^{N} E[w_{i0}^2] \right) = O(T^{-1}),
\]

from which, it follows by Markov’s inequality

\[
\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \varepsilon_{iT}w_{i0} = O_p \left( \frac{1}{\sqrt{T}} \right). \tag{24}
\]

Hence, in this case,

\[
\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} w_{i,T-2,T}\varepsilon_{iT} = \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} w_{i,T-2,T}\varepsilon_{iT} + \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \varepsilon_{iT}w_{i0}
\]

(for all \( T \) sufficiently large)

\[
= \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} w_{i,T-2,T}\varepsilon_{iT} + O_p \left( \frac{1}{\sqrt{T}} \right) = O_p(1),
\]

as required for part (a).

Next, to show parts (b)-(d), note first that

\[
E \left[ \sum_{i=1}^{N} w_{i,T-2,T}\varepsilon_{iT} \right]^2
\]

\[
= \sum_{i=1}^{N} \sum_{h=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{T-2} \exp \left\{ -\left( \frac{T-2-j}{q(T)} \right) \right\} \exp \left\{ -\left( \frac{T-2-k}{q(T)} \right) \right\} E[\varepsilon_{ij}\varepsilon_{iT}\varepsilon_{hk}\varepsilon_{hT}]
\]

\[
= \sum_{i=1}^{N} \sum_{j=1}^{T-2} \exp \left\{ -2 \left( \frac{T-2-j}{q(T)} \right) \right\} E[\varepsilon_{ij}^2] E[\varepsilon_{iT}^2]
\]

\[
= \sigma^4 N \sum_{j=1}^{T-2} \exp \left\{ -2 \left( \frac{T-2-j}{q(T)} \right) \right\}.
\]

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Now, consider part (b), where we take \( \rho_T = \exp\{-1/q(T)\} \) such that \( T/q(T) \to 0 \). Making use of part (a) of Lemma SE-3 with \( b = 2 \) and \( g = 2 \), we have

\[
\sum_{j=1}^{T-2} \exp \left\{ -2 \left( \frac{T - 2 - j}{q(T)} \right) \right\} = T \left[ 1 + O \left( \max \left\{ \frac{T}{q(T)}, \frac{1}{T} \right\} \right) \right] = O(T).
\]

It follows that in this case

\[
E \left[ \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} w_{i,T-2,T} \varepsilon_{iT} \right]^2 = \sigma^4 \frac{N}{NT} \sum_{j=1}^{T-2} \exp \left\{ -2 \left( \frac{T - 2 - j}{T} \right) \right\} = O(1),
\]

from which we deduce, using Markov’s inequality, that \( (NT)^{-1/2} \sum_{i=1}^{N} w_{i,T-2,T} \varepsilon_{iT} = O_p(1) \).

Moreover, by Assumptions 1 and 4,

\[
E \left[ \left( \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \varepsilon_{iT} \rho_T^{T-2} w_{i0} \right)^2 \right] = \frac{1}{NT} \sum_{i=1}^{N} \sum_{j=1}^{N} \rho_T^{2(T-2)} E [w_{i0} w_{j0}] E [\varepsilon_{iT} \varepsilon_{jT}] \leq \sigma^2 \frac{N}{NT} \left( \sup_i E [w_{i0}^2] \right) \exp \left\{ - \frac{2(T - 2)}{q(T)} \right\} = O(T^{-1}),
\]

from which it follows by Markov’s inequality that

\[
\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \varepsilon_{iT} \rho_T^{T-2} w_{i0} = O_p \left( \frac{1}{\sqrt{T}} \right).
\]

Hence,

\[
\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} w_{i,T-2,T} \varepsilon_{iT} = \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} w_{i,T-2,T} \varepsilon_{iT} + \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \varepsilon_{iT} \rho_T^{T-2} w_{i0} = \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} w_{i,T-2,T} \varepsilon_{iT} + O_p \left( \frac{1}{\sqrt{T}} \right) = O_p(1),
\]

as required for part (b).

We now turn our attention to part (c), where we take \( \rho_T = \exp\{-1/q(T)\} \) such that \( q(T) \sim T \). Using part (b) of Lemma SE-3 with \( b = 2 \) and \( g = 2 \), we have that

\[
E \left[ \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} w_{i,T-2,T} \varepsilon_{iT} \right]^2 = \sigma^4 \frac{N}{NT} \sum_{j=1}^{T-2} \exp \left\{ -2 \left( \frac{T - 2 - j}{q(T)} \right) \right\} \leq \frac{\sigma^4}{2} \frac{N q(T)}{NT} \left[ 1 - \exp \left\{ - \frac{2T}{q(T)} \right\} \right] \left[ 1 + O \left( \frac{1}{T} \right) \right] = O(1),
\]

so that again we deduce that \( (NT)^{-1/2} \sum_{i=1}^{N} w_{i,T-2,T} \varepsilon_{iT} = O_p(1) \).
Moreover, by Assumptions 1 and 4,

\[
E \left[ \left( \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \varepsilon_i T \rho_T^{T-2} w_{i0} \right)^2 \right] = \frac{1}{NT} \sum_{i=1}^{N} \sum_{j=1}^{N} \rho_T^{2(T-2)} E \left[ w_{i0} w_{j0} \right] E \left[ \varepsilon_i T \varepsilon_j T \right] \\
\leq \frac{\sigma^2}{T} \left( \sup_i E \left[ w_{i0}^2 \right] \right) \exp \left\{ -\frac{2(T-2)}{q(T)} \right\} = O(T^{-1}),
\]

from which it follows by Markov’s inequality that

\[
\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \varepsilon_i T \rho_T^{T-2} w_{i0} = O_p \left( \frac{1}{\sqrt{T}} \right). \tag{26}
\]

Hence,

\[
\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} w_{iT-2,T} \varepsilon_{iT} = \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} w_{iT-2,T} \varepsilon_{iT} + \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \varepsilon_i T \rho_T^{T-2} w_{i0} \\
= \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} w_{iT-2,T} \varepsilon_{iT} + O_p \left( \frac{1}{\sqrt{T}} \right) = O_p(1),
\]

which shows part (c).

For part (d), we consider the case where \( \rho_T = \exp \{ -1/q(T) \} \) such that \( q(T) \to \infty \) but \( q(T)/T \to 0 \). In this case, we use part (c) of Lemma SE-3 with \( b = 2 \) and \( g = 2 \) to obtain

\[
E \left[ \frac{1}{\sqrt{Nq(T)}} \sum_{i=1}^{N} w_{iT-2,T} \varepsilon_{iT} \right]^2 = \sigma^4 \frac{N}{Nq(T)} \sum_{j=1}^{T-2} \exp \left\{ -2 \left( \frac{T-2-j}{q(T)} \right) \right\} \\
= \sigma^4 \frac{Nq(T)}{2} \left[ 1 + O \left( \frac{1}{q(T)} \right) \right] = O(1),
\]

from which it follows, by Markov’s inequality, that \( (Nq(T))^{-1/2} \sum_{i=1}^{N} w_{iT-2,T} \varepsilon_{iT} = O_p(1) \).

Moreover, by Assumptions 1 and 4,

\[
E \left[ \left( \frac{1}{\sqrt{Nq(T)}} \sum_{i=1}^{N} \varepsilon_i T \rho_T^{T-2} w_{i0} \right)^2 \right] = \frac{1}{Nq(T)} \sum_{i=1}^{N} \sum_{j=1}^{N} \rho_T^{2(T-2)} E \left[ w_{i0} w_{j0} \right] E \left[ \varepsilon_i T \varepsilon_j T \right] \\
\leq \frac{\sigma^2}{Nq(T)} \left( \sup_i E \left[ w_{i0}^2 \right] \right) \exp \left\{ -\frac{2(T-2)}{q(T)} \right\} \\
= \frac{\sigma^2}{q(T)} \exp \left\{ -\frac{2T}{q(T)} \right\},
\]

from which it follows by Markov’s inequality that

\[
\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \varepsilon_i T \rho_T^{T-2} w_{i0} = O_p \left( \frac{1}{\sqrt{q(T)}} \exp \left\{ -\frac{T}{q(T)} \right\} \right). \tag{27}
\]
Hence,

\[
\frac{1}{\sqrt{N q(T)}} \sum_{i=1}^{N} w_{iT-2,T} \varepsilon_{iT} = \frac{1}{\sqrt{N q(T)}} \sum_{i=1}^{N} w_{iT-2,T} \varepsilon_{iT} + \frac{1}{\sqrt{N q(T)}} \sum_{i=1}^{N} \varepsilon_{iT} \rho_{T} T - 2 w_{i0} = \frac{1}{\sqrt{N q(T)}} \sum_{i=1}^{N} w_{iT-2,T} \varepsilon_{iT} + O_p \left( \frac{1}{\sqrt{q(T)}} \exp \left\{ - \frac{T}{q(T)} \right\} \right) = O_p(1),
\]

as required for part (d).

Finally, to show part (e), note that since, in this case, \( q(T) = O(1) \), there is some positive constant \( C_q \) and some positive integer \( T^* \) such that for all \( T \geq T^* \)

\[
0 \leq \rho_T^2 = \exp \left\{ - \frac{2}{q(T)} \right\} \leq \exp \left\{ - \frac{2}{C_q} \right\} < 1,
\]

from which it follows that

\[
E \left[ \frac{1}{\sqrt{N}} \sum_{i=1}^{N} w_{iT-2,T} \varepsilon_{iT} \right]^2 = \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N} \sum_{\ell=1}^{T-2} \rho_{T-2, j} \rho_{T-2, \ell} E [\varepsilon_{ij} \varepsilon_{iT} \varepsilon_{k\ell} \varepsilon_{iT}] = \sigma^4 \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} \rho_{T}^{2(T-2)} \leq \frac{\sigma^4}{1 - \exp \left\{ -2/C_q \right\}} \quad \text{(for all } T \geq T^*)
\]

\[
= O(1).
\]

It follows from applying Markov’s inequality that \( N^{-1/2} \sum_{i=1}^{N} w_{iT-2,T} \varepsilon_{iT} = O_p(1) \)

Moreover, by Assumptions 1 and 4,

\[
E \left[ \left( \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \varepsilon_{iT} \rho_{T} T - 2 w_{i0} \right)^2 \right] = \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} E \left[ w_{i0} w_{j0} \right] \rho_{T}^{2(T-2)} E [\varepsilon_{iT} \varepsilon_{iT}] = \sigma^2 \frac{1}{N} \sum_{i=1}^{N} \left[ w_{i0}^2 \right] \rho_{T}^{2(T-2)} \leq \sigma^2 \left( \sup_i E \left[ w_{i0}^2 \right] \right) \rho_{T}^{2(T-2)} = O \left( \rho_{T}^{2(T-2)} \right),
\]

from which it follows by Markov’s inequality that

\[
\frac{1}{\sqrt{N}} \sum_{i=1}^{N} \varepsilon_{iT} \rho_{T} T - 2 w_{i0} = O_p \left( \rho_{T}^{(T-2)} \right).
\]
Hence,

\[
\frac{1}{\sqrt{N}} \sum_{i=1}^{N} w_{iT-2} T \epsilon_{iT} = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} w_{iT-2} T \epsilon_{iT} + \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \epsilon_{iT} \rho_T^{T-2} w_{i0} \\
= \frac{1}{\sqrt{N}} \sum_{i=1}^{N} w_{iT-2} T \epsilon_{iT} + O_p \left( \rho_T^{(T-2)} \right) = O_p (1),
\]

as required for part (e). □.

To facilitate stating the next lemma, we introduce the following notations

\[
X_{i,T} = -\frac{1}{\sqrt{T}} \sum_{t=4}^{T} \epsilon_{it-2} \epsilon_{it-1}, \quad (28)
\]

\[
Y_{i,T} = \frac{1}{\sqrt{T}} w_{iT-2} \epsilon_{iT}. \quad (29)
\]

Lemma SE-22:

Under Assumptions 1 and 4, the following statements are true as \( N, T \to \infty \).

(a) If \( \rho_T = \exp \{-1/q(T)\} \) such that \( T/q(T) \to 0 \), then

\[
\frac{1}{\sqrt{2\sigma^2 N}} \sum_{i=1}^{N} (X_{i,T} + Y_{i,T}) \Rightarrow N (0, 1).
\]

(b) If \( \rho_T = \exp \{-1/q(T)\} \) such that \( q(T) \sim T \), then

\[
\frac{1}{\overline{\omega}_T \sqrt{N}} \sum_{i=1}^{N} (X_{i,T} + Y_{i,T}) \Rightarrow N (0, 1),
\]

where

\[
\overline{\omega}_T = \sigma^2 \sqrt{1 + \frac{q(T)}{T} \left[ 1 - \exp \{-2T/q(T)\} \right]/2}.
\]

(c) If \( \rho_T = \exp \{-1/q(T)\} \) such that \( q(T) \to \infty \) but \( q(T)/T \to 0 \), then

\[
\frac{1}{\sigma^2 \sqrt{N}} \sum_{i=1}^{N} (X_{i,T} + Y_{i,T}) = \frac{1}{\sigma^2 \sqrt{N}} \sum_{i=1}^{N} X_{i,T} + O_p \left( \sqrt{q(T)/T} \right) \Rightarrow N (0, 1).
\]

Proof of Lemma SE-22:

To proceed, first decompose \( Y_{i,T} \) as follows:

\[
Y_{i,T} = \frac{1}{\sqrt{T}} w_{iT-2} T \epsilon_{iT} = \frac{1}{\sqrt{T}} w_{iT-2} T \epsilon_{iT} + \frac{1}{\sqrt{T}} \epsilon_{iT} \rho_T^{T-2} w_{i0} \\
= Y_{i,T} + \frac{1}{\sqrt{T}} \epsilon_{iT} \rho_T^{T-2} w_{i0},
\]

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where \( w_{T-2} = \sum_{j=1}^{T-2} \rho_T^{(T-2-j)} \varepsilon_{ij} \) and \( Y_{i,T} = \frac{1}{\sqrt{T}} w_{T-2} \varepsilon_{iT} \), and we perform some preliminary moment calculations. Let

\[
U_{N,T} = \sum_{i=1}^{N} (X_{i,T} + Y_{i,T}).
\]

Next, note that

\[
E [X_{i,T}] = -\frac{1}{\sqrt{T}} \sum_{t=4}^{T} E [\varepsilon_{it-2} \varepsilon_{it-1}] = 0
\]

\[
E [Y_{i,T}] = \frac{1}{\sqrt{T}} \sum_{j=1}^{T-2} \exp \left\{ -\frac{T - 2 - j}{q(T)} \right\} E [\varepsilon_{ij} \varepsilon_{iT}] = 0,
\]

and, thus,

\[
E [U_{N,T}] = \sum_{i=1}^{N} (E [X_{i,T}] + E [Y_{i,T}]) = 0.
\]

In addition, note that

\[
E [X_{i,T}^2] = \frac{1}{T} \sum_{t=4}^{T} \sum_{s=4}^{T} E [\varepsilon_{it-2} \varepsilon_{it-1} \varepsilon_{is-2} \varepsilon_{is-1}] = \frac{1}{T} \sum_{t=4}^{T} E [\varepsilon_{it-2}^2] E [\varepsilon_{it-1}^2] = \sigma^4 \left( \frac{T - 3}{T} \right) = \sigma^4 + O \left( \frac{1}{T} \right),
\]

\[
E [Y_{i,T}^2] = \frac{1}{T} \sum_{j=1}^{T-2} \sum_{k=1}^{T-2} \exp \left\{ -\frac{T - 2 - j}{q(T)} \right\} \exp \left\{ -\frac{T - 2 - k}{q(T)} \right\} E [\varepsilon_{ij} \varepsilon_{ik} \varepsilon_{iT} \varepsilon_{iT}] = \sigma^4 \left( \frac{T - 3}{T} \right)
\]

\[
E [X_{i,T} Y_{i,T}] = -\frac{1}{T} \sum_{t=4}^{T} \sum_{\ell=1}^{T-2} \exp \left\{ -\frac{T - 2 - \ell}{q(T)} \right\} E [\varepsilon_{it-2} \varepsilon_{it-1} \varepsilon_{i\ell} \varepsilon_{iT}] = 0.
\]

It follows that

\[
\omega^2_{i,T} = E \left[ (X_{i,T} + Y_{i,T})^2 \right] = E [X_{i,T}^2] + E [Y_{i,T}^2] + 2E [X_{i,T} Y_{i,T}] = \sigma^4 \left( \frac{T - 3}{T} \right) + \sigma^4 \left( \frac{T - 3}{T} \right) = 0.
\]
so that
\[
\omega_{N,T}^2 = \sum_{i=1}^{N} \omega_{i,T}^2 = \sigma^4 N \left[ \left( \frac{T - 3}{T} \right) + \frac{1}{T} \sum_{\ell=1}^{T-2} \exp \left\{ -2 \frac{T - 2 - \ell}{q(T)} \right\} \right].
\]

Now, consider part (a), where we take \( T/q(T) \to 0 \). In this case, we apply part (a) of Lemma SE-3 with \( b = 2 \) and \( g = 2 \) to obtain
\[
\frac{\omega_{N,T}^2}{N} = \frac{1}{N} \sum_{i=1}^{N} \omega_{i,T}^2
\]
\[
= \sigma^4 \frac{1}{N} \left\{ \left( \frac{T - 3}{T} \right) + \frac{1}{T} \left[ 1 + O \left( \max \left\{ \frac{T}{q(T)}, \frac{1}{T} \right\} \right) \right] \right\}
\]
\[
= 2\sigma^4 + O \left( \max \left\{ \frac{T}{q(T)}, \frac{1}{T} \right\} \right).
\]

Expression (30) and Assumption 1 then imply that there exists a positive constant \( C \) such that
\[
0 < \frac{1}{C} \leq \frac{\omega_{N,T}}{\sqrt{N}} \leq C < \infty \text{ eventually as } N, T \to \infty.
\]

Next, note that, by the result given in expression (25)
\[
\frac{1}{\omega_{N,T}/\sqrt{N}} \frac{1}{\sqrt{N}} \sum_{i=1}^{N} (X_{i,T} + Y_{i,T})
\]
\[
= \frac{1}{\omega_{N,T}/\sqrt{N}} \frac{1}{\sqrt{N}} \sum_{i=1}^{N} (X_{i,T} + Y_{i,T}) + \frac{1}{\omega_{N,T}/\sqrt{N}} \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \varepsilon_{i,T} \rho_T^{T-2} \omega_{i,0}
\]
\[
= \frac{1}{\omega_{N,T}/\sqrt{N}} \frac{1}{\sqrt{N}} \sum_{i=1}^{N} (X_{i,T} + Y_{i,T}) + o_p(1).
\]

Hence, showing the desired result is equivalent to showing the asymptotic normality of
\[
U_{N,T} = \frac{1}{\omega_{N,T}/\sqrt{N}} \frac{1}{\sqrt{N}} \sum_{i=1}^{N} (X_{i,T} + Y_{i,T})
\]

To show the asymptotic normality of \( U_{N,T} \), it suffices to verify a Liapounov-type condition of the form
\[
\lim_{N,T \to \infty} \frac{1}{N^2} \sum_{i=1}^{N} E \left[ (X_{i,T} + Y_{i,T})^4 \right] = 0.
\]

To show this, note first that by Loève’s \( c_r \) inequality, we have that
\[
\frac{1}{N^2} \sum_{i=1}^{N} E \left[ (X_{i,T} + Y_{i,T})^4 \right]
\]
\[
\leq \frac{1}{N^2} \sum_{i=1}^{N} \left\{ E \left[ \left( -\frac{1}{\sqrt{T}} \sum_{\ell=4}^{T} \varepsilon_{i,\ell-2} \varepsilon_{i,\ell-1} \right)^4 \right] + E \left[ \left( \frac{1}{\sqrt{T}} \sum_{j=1}^{T-2} \exp \left\{ -2 \frac{T - 2 - j}{q(T)} \right\} \varepsilon_{ij} \varepsilon_{iT} \right)^4 \right] \right\}
\]
\[
= 8 \frac{1}{N^2 T^2} \sum_{i=1}^{N} E \left[ \left( \sum_{\ell=1}^{T} \varepsilon_{i,\ell-2} \varepsilon_{i,\ell-1} \right)^4 \right] + 8 \frac{1}{N^2 T^2} \sum_{i=1}^{N} E \left[ \left( \sum_{j=1}^{T-2} \exp \left\{ -2 \frac{T - 2 - j}{q(T)} \right\} \varepsilon_{ij} \varepsilon_{iT} \right)^4 \right].
\]
Next, note that

\[
E \left[ \left( \sum_{t=4}^{T} \varepsilon_{it-2} \varepsilon_{it-1} \right)^4 \right] = \sum_{g=4}^{T} \sum_{s=4}^{T} \sum_{t=4}^{T} \sum_{u=4}^{T} E \left[ \varepsilon_{ig-2} \varepsilon_{is-2} \varepsilon_{it-2} \varepsilon_{iu-2} \varepsilon_{ig-1} \varepsilon_{is-1} \varepsilon_{it-1} \varepsilon_{iu-1} \right]
\]

\[
= \sum_{t=4}^{T} E \left[ \varepsilon_{it-2}^2 \right] E \left[ \varepsilon_{it-1}^2 \right] + 6 \sum_{s=6}^{T-2} \sum_{t=4}^{T} E \left[ \varepsilon_{is-2}^2 \right] E \left[ \varepsilon_{it-1}^2 \right] E \left[ \varepsilon_{it-2}^2 \right] E \left[ \varepsilon_{it-1}^2 \right] \\
+ 6 \sum_{t=4}^{T-1} E \left[ \varepsilon_{it-2}^2 \right] E \left[ \varepsilon_{it-1}^2 \right] E \left[ \varepsilon_{it-2}^2 \right] \\
= (E \left[ \varepsilon_{it-1}^4 \right])^2 (T - 3) + 6\sigma^8 \sum_{s=6}^{T} (s - 5) + 6E \left[ \varepsilon_{it-1}^4 \right] \sigma^4 (T - 4)
\]

\[
= 6\sigma^8 \left( \frac{T - 5}{2} \right) \left( \frac{T - 4}{2} \right) + (E \left[ \varepsilon_{it-1}^4 \right])^2 (T - 3) + 6E \left[ \varepsilon_{it-1}^4 \right] \sigma^4 (T - 4)
\]

\[
= 3\sigma^8 T^2 \left[ 1 + O \left( \frac{1}{T} \right) \right],
\]

(33)

and

\[
E \left[ \left( \sum_{j=1}^{T-2} \exp \left\{ -\frac{T - 2 - j}{q(T)} \right\} \varepsilon_{ij} \varepsilon_{iT} \right)^4 \right]
\]

\[
= \sum_{g=1}^{T-2} \sum_{s=1}^{T-2} \sum_{t=1}^{T-2} \sum_{u=1}^{T-2} \left\{ \exp \left\{ -\frac{T - 2 - g}{q(T)} \right\} \exp \left\{ -\frac{T - 2 - s}{q(T)} \right\} \exp \left\{ -\frac{T - 2 - t}{q(T)} \right\} \right\} \times \exp \left\{ -\frac{T - 2 - u}{q(T)} \right\} E \left[ \varepsilon_{ig} \varepsilon_{is} \varepsilon_{it} \varepsilon_{iu} \right] E \left[ \varepsilon_{iT}^4 \right]
\]

\[
= \sum_{s=1}^{T-2} \exp \left\{ -4\frac{T - 2 - s}{q(T)} \right\} E \left[ \varepsilon_{is}^4 \right] E \left[ \varepsilon_{iT}^4 \right] \\
+ 6 \sum_{g=2}^{T-2} \sum_{s=1}^{g-1} \exp \left\{ -2\frac{T - 2 - g}{q(T)} \right\} \exp \left\{ -2\frac{T - 2 - s}{q(T)} \right\} \exp \left\{ -2\frac{T - 2 - s}{q(T)} \right\} \exp \left\{ -2\frac{T - 2 - s}{q(T)} \right\} E \left[ \varepsilon_{ig}^2 \right] E \left[ \varepsilon_{is}^2 \right] E \left[ \varepsilon_{iT}^4 \right]
\]

\[
= E \left[ \varepsilon_{is}^4 \right] E \left[ \varepsilon_{iT}^4 \right] \sum_{s=1}^{T-2} \exp \left\{ -4\frac{T - 2 - s}{q(T)} \right\}
\]

\[
+ 6\sigma^4 E \left[ \varepsilon_{iT}^4 \right] \sum_{g=2}^{T-2} \sum_{s=1}^{g-1} \exp \left\{ -2\frac{T - 2 - g}{q(T)} \right\} \exp \left\{ -2\frac{T - 2 - s}{q(T)} \right\}
\]

\[
= E \left[ \varepsilon_{is}^4 \right] E \left[ \varepsilon_{iT}^4 \right] T \left[ 1 + O \left( \max \left\{ \frac{T}{T}, \frac{1}{T} \right\} \right) \right]
\]

\[
+ 6\sigma^4 E \left[ \varepsilon_{iT}^4 \right] \frac{T^2}{2} \left[ 1 + O \left( \max \left\{ \frac{T}{T}, \frac{1}{T} \right\} \right) \right],
\]

(34)

where the last equality follows from applying part (a) of Lemma SE-3 with \( b = 4 \) and \( g = 2 \) and by applying part (a) of Lemma SE-4 with \( d = 2 \). Now, applying (33) and (34) to the upper bound in (32),
we get

\[
\frac{1}{N^2} \sum_{i=1}^{N} E \left[ (X_{i,T} + Y_{i,T})^4 \right] \leq 8 \frac{1}{N^2 T^2} \sum_{i=1}^{N} 3 \sigma^8 T^2 \left[ 1 + O \left( \frac{1}{T} \right) \right]
\]

\[
+ 8 E \left[ \varepsilon_{i,T}^4 \right] E \left[ \varepsilon_{i,T}^4 \right] \frac{1}{N^2 T^2} \sum_{i=1}^{N} T \left[ 1 + O \left( \max \left\{ \frac{T}{q(T)}, \frac{1}{T} \right\} \right) \right]
\]

\[
+ 8 \sigma^4 E \left[ \varepsilon_{i,T}^4 \right] \frac{1}{N^2 T^2} \sum_{i=1}^{N} 6 \frac{T^2}{2} \left[ 1 + O \left( \max \left\{ \frac{T}{q(T)}, \frac{1}{T} \right\} \right) \right]
\]

\[
= 24 \sigma^8 \frac{1}{N} + 8 E \left[ \varepsilon_{i,T}^4 \right] E \left[ \varepsilon_{i,T}^4 \right] \frac{1}{N T} + 24 \sigma^4 E \left[ \varepsilon_{i,T}^4 \right] \frac{1}{N} + O \left( \max \left\{ \frac{T}{N q(T)}, \frac{1}{N T} \right\} \right)
\]

\[
= O \left( N^{-1} \right).
\]

Since the Liapounov-type condition (31) implies the Lindeberg-type condition (13) given in Lemma SE-10 above, it follows from Lemma SE-10 that

\[
U_{N,T} = \frac{1}{\omega_{N,T}} \sum_{i=1}^{N} (X_{i,T} + Y_{i,T})
\]

\[
= \frac{1}{\sqrt{2 \sigma^2}} \frac{1}{\sqrt{N}} \sum_{i=1}^{N} (X_{i,T} + Y_{i,T}) + O \left( \max \left\{ \frac{T}{q(T)}, \frac{1}{T} \right\} \right)
\]

\[
\Rightarrow N \left( 0, 1 \right).
\]

Next, consider part (b), where we take \( q(T) \sim T \). Here, we apply part (b) of Lemma SE-3 with \( b = 2 \) and \( g = 2 \) to obtain

\[
\frac{\omega^2_{N,T}}{N} = \frac{1}{N} \sum_{i=1}^{N} \omega^2_{i,T}
\]

\[
= \sigma^4 \frac{1}{N} \left[ \left( \frac{T-3}{T} \right) + \frac{1}{T} \sum_{\ell=1}^{T-2} \exp \left\{ -2 \frac{T-2-\ell}{q(T)} \right\} \right]
\]

\[
= \sigma^4 \left\{ \left( \frac{T-3}{T} \right) + \frac{1}{T} \frac{q(T)}{2} \left[ 1 - \exp \left\{ -\frac{2T}{q(T)} \right\} \right] \left[ 1 + O \left( \frac{1}{T} \right) \right] \right\}
\]

\[
= \sigma^4 \left[ 1 + \frac{q(T)}{T} \frac{1}{2} \left[ 1 - \exp \left\{ -\frac{2T}{q(T)} \right\} \right] \right] + O \left( \frac{1}{T} \right).
\]

Note that, in light of Assumption 1, there exists a positive constant \( C \) such that

\[
0 < \frac{1}{C} \leq \frac{\omega_{N,T}}{\sqrt{N}}
\]

\[
= \sigma^2 \sqrt{1 + \frac{q(T)}{T} \left[ 1 - \exp \left\{ -2T/q(T) \right\} \right]} \left[ 1 + O \left( \frac{1}{T} \right) \right]
\]

\[
\leq C < \infty \text{ eventually as } N,T \to \infty.
\]
Next, note that, by the result given in expression (26)
\[
\frac{1}{\omega_{N,T}/\sqrt{N}} \frac{1}{\sqrt{N}} \sum_{i=1}^{N} (X_{i,T} + Y_{i,T}) = \frac{1}{\omega_{N,T}/\sqrt{N}} \frac{1}{\sqrt{N}} \sum_{i=1}^{N} (X_{i,T} + Y_{i,T}) + \frac{1}{\omega_{N,T}/\sqrt{N}} \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \varepsilon_{iT} \nu_{T}^{T-2} w_{i0}
\]
\[
= \frac{1}{\omega_{N,T}/\sqrt{N}} \frac{1}{\sqrt{N}} \sum_{i=1}^{N} (X_{i,T} + Y_{i,T}) + o_p(1).
\]
Hence, showing the desired result is equivalent to showing the asymptotic normality of
\[
U_{N,T} = \frac{1}{\omega_{N,T}} U_{N,T} = \frac{1}{\omega_{N,T}/\sqrt{N}} \sqrt{N} \sum_{i=1}^{N} (X_{i,T} + Y_{i,T}).
\]
To show the asymptotic normality of \(U_{N,T}\), it suffices to verify a Liapounov-type condition of the form
\[
\lim_{N,T \to \infty} \sum_{i=1}^{N} E \left[ \left( \frac{1}{\sqrt{N}} \sum_{i=1}^{N} (X_{i,T} + Y_{i,T}) \right)^4 \right] = 0.
\]
To proceed, note that using calculations similar to that given for (34), we have
\[
E \left[ \left( \sum_{j=1}^{T-2} \exp \left\{ - \frac{T - 2 - j}{q(T)} \varepsilon_{ij} \varepsilon_{iT} \right\} \right)^4 \right]
\]
\[
= E \left[ \varepsilon_{is}^4 \right] E \left[ \varepsilon_{iT}^4 \right] \sum_{s=1}^{T-2} \exp \left\{ -4 \frac{T - 2 - s}{q(T)} \right\}
\]
\[
+ 6 \sigma^4 E \left[ \varepsilon_{is}^4 \right] \sum_{g=2}^{T-2} \sum_{s=1}^{g-1} \exp \left\{ -2 \frac{T - 2 - g}{q(T)} \right\} \exp \left\{ -2 \frac{T - 2 - s}{q(T)} \right\}
\]
\[
= E \left[ \varepsilon_{is}^4 \right] E \left[ \varepsilon_{iT}^4 \right] q(T) \left[ 1 - \exp \left\{ -4 \frac{T}{q(T)} \right\} \right] \left[ 1 + O \left( \frac{1}{T} \right) \right]
\]
\[
+ 6 \sigma^4 E \left[ \varepsilon_{iT}^4 \right] q(T)^2 \left[ 1 - \exp \left\{ -2 \frac{T}{q(T)} \right\} \right] \left[ 1 + O \left( \frac{1}{T} \right) \right],
\]
where the last equality follows from applying part (b) of Lemma SE-3 with \(b = 4\) and \(g = 2\) and by applying part (b) of Lemma SE-4 with \(d = 2\). Now, applying (33) and (36) to the upper bound in (32),
we get

\[
\frac{1}{N^2} \sum_{i=1}^{N} E \left[ (X_{i,T} + Y_{i,T})^4 \right] \\
\leq 8 \frac{1}{N^2 T^2} \sum_{i=1}^{N} 3\sigma^8 T^2 \left[ 1 + O \left( \frac{1}{T} \right) \right] \\
+ 8E \left[ \epsilon_{is}^4 \right] E \left[ \epsilon_{iT}^4 \right] \frac{1}{N^2 T^2} \sum_{i=1}^{N} q(T) \left[ \frac{1 - \exp \left\{ -\frac{4T}{q(T)} \right\}}{4} \right] \left[ 1 + O \left( \frac{1}{T} \right) \right] \\
+ 8\sigma^4 E \left[ \epsilon_{iT}^4 \right] \frac{1}{N^2 T^2} \sum_{i=1}^{N} 6q(T)^2 \left[ \frac{1 - \exp \left\{ -\frac{2T}{q(T)} \right\}}{8} \right] \left[ 1 + O \left( \frac{1}{T} \right) \right] \right) \\
= 24\sigma^8 \frac{1}{N} + 2E \left[ \epsilon_{is}^4 \right] E \left[ \epsilon_{iT}^4 \right] \frac{q(T)}{NT} \left[ 1 - \exp \left\{ -\frac{4T}{q(T)} \right\} \right] \\
+ 6\sigma^4 E \left[ \epsilon_{iT}^4 \right] \frac{q(T)^2}{NT^2} \left[ 1 - \exp \left\{ -\frac{2T}{q(T)} \right\} \right]^2 + O \left( \frac{1}{NT} \right) \\
= O \left( \frac{1}{N} \right),
\]

which verifies the required Liapounov-type condition. Since this condition implies the Lindeberg-type condition (13) stated in Lemma SE-10 above, it follows from Lemma SE-10 that

\[
U_{N,T} = \frac{1}{\omega_{N,T}} \sum_{i=1}^{N} (X_{i,T} + Y_{i,T}) \\
= \frac{1}{\omega_{N,T}/\sqrt{N}} \frac{1}{\sqrt{N}} \sum_{i=1}^{N} (X_{i,T} + Y_{i,T}) \\
= \frac{1}{\omega_{T}/\sqrt{N}} \sum_{i=1}^{N} (X_{i,T} + Y_{i,T}) + O \left( \frac{1}{T} \right) \\
\Rightarrow N \left( 0, 1 \right),
\]

where

\[
\omega_T = \sigma^2 \sqrt{1 + \frac{q(T)}{T} \left[ \frac{1 - \exp \left\{ -\frac{2T}{q(T)} \right\}}{2} \right]}.
\]

Finally, we turn our attention to part (c). Here, we consider the case \( q(T) \to \infty \) such that \( q(T)/T \to 0 \). Applying part (c) of Lemma SE-3 with \( b = 2 \) and \( g = 2 \), we obtain

\[
\frac{\omega_{N,T}^2}{N} = \frac{1}{N} \sum_{i=1}^{N} \omega_{i,T}^2 = \sigma^4 \frac{1}{N} \left\{ \left( \frac{T-3}{T} \right) + \frac{q(T)}{T} \left[ 1 + O \left( \frac{1}{q(T)} \right) \right] \right\} \\
= \sigma^4 + O \left( \frac{q(T)}{T} \right). \tag{37}
\]

Hence, in light of Assumption 1, there exists positive constant \( C \) such that

\[
0 < \frac{1}{C} \leq \frac{\omega_{N,T}^2}{N} \leq C < \infty \quad \text{eventually as } N, T \to \infty. \tag{38}
\]
Next, applying the result given in expression (27), part (d) of Lemma SE-21 and (38) above, we obtain

\[
U_{N,T} = 1 \frac{\omega_{N,T}}{N} \sum_{i=1}^{N} (X_{i,T} + Y_{i,T})
\]

\[
= 1 \frac{\omega_{N,T} / \sqrt{N}}{N} \sum_{i=1}^{N} X_{i,T} + 1 \frac{\omega_{N,T} / \sqrt{N}}{N} \sum_{i=1}^{N} Y_{i,T}
\]

\[
+ 1 \frac{1}{\omega_{N,T} / \sqrt{N}} \sqrt{q(T)} \frac{1}{\sqrt{N} q(T)} \sum_{i=1}^{N} \varepsilon_{iT} \rho_{T}^{-2} w_{i0}
\]

\[
= 1 \frac{\omega_{N,T}}{N} \sum_{i=1}^{N} X_{i,T} + O_{p} \left( \frac{\sqrt{q(T)}}{T} \right) + o_{p} \left( \frac{\sqrt{q(T)}}{T} \right).
\]

Hence, to prove the result in part (c), we need to show the asymptotic normality of

\[
U_{N,T}^{(1)} = 1 \frac{\omega_{N,T}}{N} \sum_{i=1}^{N} X_{i,T} = 1 \frac{1}{\omega_{N,T} / \sqrt{N}} \sum_{i=1}^{N} X_{i,T}.
\]

To do so, we verify the Liapounov-type condition

\[
\lim_{N,T \to \infty} \sum_{i=1}^{N} E \left[ \frac{1}{\sqrt{N}} \sum_{i=1}^{N} X_{i,T} \right]^{4} = \lim_{N,T \to \infty} \frac{1}{N^{2}} \sum_{i=1}^{N} E \left[ X_{i,T}^{4} \right] = 0.
\]

Applying (33), we get

\[
\frac{1}{N^{2}} \sum_{i=1}^{N} E \left[ X_{i,T}^{4} \right] \leq \frac{1}{N^{2} T^{2}} \sum_{i=1}^{N} 3 \sigma^{8} T^{2} \left[ 1 + O \left( \frac{1}{T} \right) \right]
\]

\[
= 3 \sigma^{8} \frac{1}{N} + O \left( \frac{1}{NT} \right)
\]

\[
= O \left( N^{-1} \right),
\]

which verifies the required Liapounov-type condition. Since this condition implies the Lindeberg-type condition (13) stated in Lemma SE-10 above, it follows from Lemma SE-10 that

\[
U_{N,T}^{(1)} = 1 \frac{\omega_{N,T}}{N} \sum_{i=1}^{N} X_{i,T}
\]

\[
= 1 \frac{1}{\sigma^{2}} \sqrt{N} \sum_{i=1}^{N} X_{i,T} + O \left( \frac{q(T)}{T} \right)
\]

\[
\Rightarrow N \left( 0, 1 \right),
\]

as required. □

**Lemma SE-23:** Suppose that Assumptions 1 and 4 hold. If
\( \rho_T \in \mathcal{G}_S = \left\{ |q_T| = \exp \left\{ -\frac{1}{q(T)} \right\} : q(T) \geq 0 \text{ and } q(T) = O(1) \text{ as } T \to \infty \right\} \), then as \( N, T \to \infty \)

\[
\frac{1}{\omega_{N,T}} \sum_{i=1}^{N} (X_{i,T} + Y_{i,T}) = \sqrt{\frac{1}{N} \sum_{i=1}^{N} (X_{i,T} + Y_{i,T}) + o_p(1)} \Rightarrow N(0, 1).
\]

where in this case

\[
X_{i,T} = -\frac{1}{\sqrt{T}} \sum_{t=4}^{T} \varepsilon_{it-2} \varepsilon_{it-1}, \quad Y_{i,T} = (1 - \rho_T) \frac{1}{\sqrt{T}} \sum_{t=4}^{T} w_{it-3,T} \varepsilon_{it-1},
\]

\[
Y_{i,T} = (1 - \rho_T) \frac{1}{\sqrt{T}} \sum_{t=4}^{T} w_{it-3,T} \varepsilon_{it-1}, \quad w_{it-3,T} = \sum_{j=1}^{t-3} \rho_T^{(t-3-j)} \varepsilon_{ij},
\]

and

\[
\omega_{i,T}^2 = \sigma^4 \left\{ \left( 1 - \frac{3}{T} \right) + \left( 1 - \rho_T \right) \frac{1}{1 + \rho_T} \left[ 1 - \frac{3}{T} - \frac{\rho_T^{2(T-3)}}{T(1 - \rho_T^2)} \right] \right\}
\]

\[
= \frac{2\sigma^4}{1 + \rho_T} + O \left( \frac{1}{T} \right).
\]

Proof of Lemma SE-23:

To proceed, first decompose \( Y_{i,T} \) as follows:

\[
Y_{i,T} = (1 - \rho_T) \frac{1}{\sqrt{T}} \sum_{t=4}^{T} w_{it-3,T} \varepsilon_{it-1}
\]

\[
= (1 - \rho_T) \frac{1}{\sqrt{T}} \sum_{t=4}^{T} w_{it-3,T} \varepsilon_{it-1} + (1 - \rho_T) \frac{1}{\sqrt{T}} \sum_{t=4}^{T} \varepsilon_{it-1} \rho_T^{t-3} w_{0i}
\]

\[
= Y_{i,T} + (1 - \rho_T) \frac{1}{\sqrt{T}} \sum_{t=4}^{T} \varepsilon_{it-1} \rho_T^{t-3} w_{0i}.
\]

where \( Y_{i,T} \) and \( w_{it-3,T} = \sum_{j=1}^{t-3} \rho_T^{(t-3-j)} \varepsilon_{ij} \) are defined in the statement of the lemma. Note that under the assumption of this lemma, \( q(T) = O(1) \), so that there exist some positive constant \( C_q \) and some positive integer \( T^* \) such that for all \( T \geq T^* \)

\[
0 \leq |\rho_T| = \exp \left\{ -\frac{1}{q(T)} \right\} \leq \exp \left\{ -\frac{1}{C_q} \right\} < 1.
\] (39)

We should use this bound in various places in the argument given below. Now, let

\[
U_{N,T} = \sum_{i=1}^{N} (X_{i,T} + Y_{i,T}),
\]
where \( X_{i,T} = -T^{-1/2} \sum_{t=4}^{T} \varepsilon_{it-2} \varepsilon_{it-1} \) and \( Y_{i,T} \) is as defined above. Next, note that

\[
E[X_{i,T}] = -\frac{1}{\sqrt{T}} \sum_{t=4}^{T} E[\varepsilon_{it-2} \varepsilon_{it-1}] = 0,
\]

\[
E[Y_{i,T}] = (1 - \rho_{T}) \frac{1}{\sqrt{T}} \sum_{t=4}^{T} \sum_{j=1}^{T-3} \rho_{T}^{(t-3-j)} E[\varepsilon_{ij} \varepsilon_{it-1}] = 0,
\]

and, thus,

\[
E[U_{N,T}] = \sum_{i=1}^{N} (E[X_{i,T}] + E[Y_{i,T}]) = 0.
\]

In addition, note that

\[
E[X_{i,T}^2] = \frac{1}{T} \sum_{t=4}^{T} \sum_{s=4}^{T} E[\varepsilon_{it-2} \varepsilon_{it-1} \varepsilon_{is-2} \varepsilon_{is-1}] = \frac{1}{T} \sum_{t=4}^{T} E[\varepsilon_{it-2}^2] E[\varepsilon_{it-1}^2] = \sigma^4 \left( \frac{T-3}{T} \right) = \sigma^4 + O \left( \frac{1}{T} \right),
\]

\[
E[Y_{i,T}^2] = (1 - \rho_{T})^2 \frac{1}{T} \sum_{t=4}^{T} \sum_{s=4}^{T} \sum_{j=1}^{T-3} \sum_{k=1}^{T-s-3} \rho_{T}^{(t-3-j)} \rho_{T}^{(s-3-k)} E[\varepsilon_{ij} \varepsilon_{it-1} \varepsilon_{ik} \varepsilon_{is-1}] = \sigma^4 \left( \frac{1 - \rho_{T}}{1 + \rho_{T}} \right) + O \left( \frac{1}{T} \right),
\]

\[
E[X_{i,T} Y_{i,T}] = -(1 - \rho_{T}) \frac{1}{T} \sum_{t=4}^{T} \sum_{s=4}^{T} \sum_{j=1}^{T-3} \rho_{T}^{(t-3-j)} E[\varepsilon_{ij} \varepsilon_{it-1} \varepsilon_{is-2} \varepsilon_{is-1}] = 0.
\]

It follows that

\[
\omega_{i,T}^2 = E \left[ (X_{i,T} + Y_{i,T})^2 \right] = E[X_{i,T}^2] + E[Y_{i,T}^2] + 2E[X_{i,T} Y_{i,T}] = \sigma^4 \left( \frac{T-3}{T} \right) + \sigma^4 \left( \frac{1 - \rho_{T}}{1 + \rho_{T}} \right) \left[ 1 - \frac{3}{T} - \frac{\rho_{T}^2 \left( 1 - \rho_{T}^2 \right)}{T \left( 1 - \rho_{T}^2 \right)} \right] = \frac{2\sigma^4}{1 + \rho_{T}} - \sigma^4 \left[ \frac{6}{1 + \rho_{T}} + \frac{\rho_{T}^2 \left( 1 - \rho_{T}^2 \right)}{\left( 1 + \rho_{T} \right)^2} \right] = \frac{2\sigma^4}{1 + \rho_{T}} - \sigma^4 \left[ \frac{6}{1 + \rho_{T}} + \frac{\rho_{T}^2 \left( 1 - \rho_{T}^2 \right)}{(1 + \rho_{T})^2} \right],
\]
and, thus,

\[
\frac{\omega_{N,T}^2}{N} = \frac{1}{N} \sum_{i=1}^{N} \omega_{i,T}^2
\]

\[
= \frac{1}{N} \left\{ \frac{2N\sigma^4}{1 + \rho_T} - \frac{\sigma^4 N}{T} \left[ \frac{6}{1 + \rho_T} + \frac{\rho_T^2 \left(1 - \rho_T^{2(T-3)}\right)}{(1 + \rho_T)^2} \right] \right\}
\]

\[
= \frac{2\sigma^4}{1 + \rho_T} - \frac{\sigma^4}{T} \left[ \frac{6}{1 + \rho_T} + \frac{\rho_T^2 \left(1 - \rho_T^{2(T-3)}\right)}{(1 + \rho_T)^2} \right].
\]

Next, note that

\[
\frac{\sigma^4}{T} \left[ \frac{6}{1 + \rho_T} + \frac{\rho_T^2 \left(1 - \rho_T^{2(T-3)}\right)}{(1 + \rho_T)^2} \right] \leq \frac{1}{T} \exp \{-1/C_q\} \left[ 6 + \frac{1}{(1 - \exp \{-1/C_q\})} \right],
\]

for all \(T \geq T^*\) and for all \(N\), so that, in light of Assumption 1,

\[
\frac{\sigma^4}{T} \left[ \frac{6}{1 + \rho_T} + \frac{\rho_T^2 \left(1 - \rho_T^{2(T-3)}\right)}{(1 + \rho_T)^2} \right] = O \left( \frac{1}{T} \right),
\]

and

\[
\frac{\omega_{N,T}^2}{N} = \frac{1}{N} \sum_{i=1}^{N} \omega_{i,T}^2 = \frac{2\sigma^4}{1 + \rho_T} + O \left( \frac{1}{T} \right).
\]

Moreover, \(0 < \sigma^4 < \infty\) by Assumption 1, so that

\[
0 < \sigma^4 \leq \frac{2\sigma^4}{1 + \rho_T} \leq \frac{2\sigma^4}{(1 - \exp \{-1/C_q\})} < \infty,
\]

from which it follows that there exists a positive constant \(C\) such that

\[
0 < \frac{1}{C} \leq \omega_{N,T} \sqrt{N} \leq C < \infty \text{ eventually as } N,T \to \infty.
\]

Moreover, by Assumptions 1 and 4,

\[
E \left[ \left( 1 - \rho_T \right) \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=4}^{T} \varepsilon_{it-1} \rho_T^{t-3} w_i \right]^2 \right]
\]

\[
= \frac{(1 - \rho_T)^2}{NT} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{t=4}^{T} E \left[ w_{ij} w_{ij} \right] \rho_T^{t-3} \rho_T^{s-3} E \left[ \varepsilon_{it-1} \varepsilon_{is-1} \right]
\]

\[
= \frac{\sigma^2 (1 - \rho_T)^2}{NT} \sum_{i=1}^{N} \left[ E \left[ w_{ij}^2 \right] \sum_{t=4}^{T} \rho_T^{2(t-3)} \right]
\]

\[
= \frac{\sigma^2}{T} \left( \sup_i E \left[ w_{i0}^2 \right] \right) \rho_T^2 \frac{1 - \rho_T^{2(T-3)}}{1 - \rho_T^2} = O \left( T^{-1} \right),
\]

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from which it follows by Markov’s inequality that

\[
(1 - \rho_T) \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=4}^{T} \varepsilon_{it-1} \rho_T^{t-3} w_{i0} = O_p \left( \frac{1}{\sqrt{T}} \right).
\]

Applying this result, we have

\[
\frac{1}{\omega_{N,T}/\sqrt{N}} \frac{1}{\sqrt{N}} \sum_{i=1}^{N} (X_{i,T} + Y_{i,T}) = \frac{1}{\omega_{N,T}/\sqrt{N}} \sum_{i=1}^{N} (X_{i,T} + Y_{i,T})
\]

\[
+ \frac{1}{\omega_{N,T}/\sqrt{N}} (1 - \rho_T) \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=4}^{T} \varepsilon_{it-1} \rho_T^{t-3} w_{i0}
\]

\[
= \frac{1}{\omega_{N,T}/\sqrt{N}} \frac{1}{\sqrt{N}} \sum_{i=1}^{N} (X_{i,T} + Y_{i,T}) + O_p \left( \frac{1}{\sqrt{T}} \right).
\]

Hence, showing the desired result is equivalent to showing the asymptotic normality of

\[
U_{N,T} = \frac{1}{\omega_{N,T}/\sqrt{N}} \frac{1}{\sqrt{N}} \sum_{i=1}^{N} (X_{i,T} + Y_{i,T}).
\]

To proceed, we again verify the Liapounov-type condition

\[
\lim_{N,T \to \infty} \frac{1}{N^2} \sum_{i=1}^{N} E \left[ (X_{i,T} + Y_{i,T})^4 \right] = 0. \tag{40}
\]

To show this, note first that by Loève’s \( c_r \) inequality, we have

\[
\frac{1}{N^2} \sum_{i=1}^{N} E \left[ (X_{i,T} + Y_{i,T})^4 \right]
\]

\[
\leq \frac{1}{N^2} \sum_{i=1}^{N} 8 \left\{ E \left[ \left( \frac{-1}{\sqrt{T}} \sum_{t=4}^{T} \varepsilon_{it-2} \varepsilon_{it-1} \right)^4 \right] + E \left[ \left( (1 - \rho_T) \frac{1}{\sqrt{T}} \sum_{t=4}^{T} \sum_{j=1}^{t-3} \rho_T^{t-3-j} \varepsilon_{ij} \varepsilon_{it-1} \right)^4 \right] \right\}
\]

\[
= 8 \frac{1}{N^2 T^2} \sum_{i=1}^{N} E \left[ \left( \sum_{t=4}^{T} \varepsilon_{it-2} \varepsilon_{it-1} \right)^4 \right] + 8 \frac{(1 - \rho_T)^4}{N^2 T^2} \sum_{i=1}^{N} E \left[ \left( \sum_{t=4}^{T} w_{it-3} \varepsilon_{it-1} \right)^4 \right].
\]

From previous calculations, we have that

\[
E \left[ \left( \sum_{t=4}^{T} \varepsilon_{it-2} \varepsilon_{it-1} \right)^4 \right] = 6 \sigma^8 \frac{(T-5)(T-4)}{2} + (E [\varepsilon_{it-1}^4])^2 (T-3) + 6E [\varepsilon_{it-1}^4] \sigma^4 (T-4)
\]

\[
= 3 \sigma^8 T^2 \left[ 1 + O \left( \frac{1}{T} \right) \right].
\]
Moreover, in light of Assumption 1, there exists a positive constant \( C \) such that

\[
E \left[ \left( \sum_{t=4}^{T} w_{t-3}T \varepsilon_{it-1} \right)^4 \right] 
\leq C \left( \sum_{t=4}^{T} \sum_{j=1}^{T} \rho_T^{4(t-3-j)} + \sum_{t=6}^{T} \sum_{s=4}^{T} \sum_{j=1}^{T} \rho_T^{2(t-3-[s-1])} \rho_T^{(t-3-j)} \rho_T^{(s-3-j)} \right) 
+ \sum_{t=6}^{T} \sum_{s=4}^{T} \sum_{j=1}^{T} \rho_T^{(t-3-[s-1])} \rho_T^{2(t-3-j)} \rho_T^{(s-3-j)} \right)
\]

\[
\leq C \left( \sum_{t=4}^{T} \sum_{j=1}^{T} \rho_T^{4(t-3-j)} + \sum_{t=6}^{T} \sum_{s=4}^{T} \sum_{j=1}^{T} \rho_T^{2(t-3-[s-1])} \rho_T^{(t-3-j)} \rho_T^{(s-3-j)} \right)
+ \sum_{t=6}^{T} \sum_{s=4}^{T} \sum_{j=1}^{T} \rho_T^{(t-3-[s-1])} \rho_T^{2(t-3-j)} \rho_T^{(s-3-j)} \right)
\]

\[
\leq C \left( \sum_{t=4}^{T} \sum_{j=1}^{T} \rho_T^{4(t-3-j)} + \sum_{t=6}^{T} \sum_{s=4}^{T} \sum_{j=1}^{T} \rho_T^{2(t-3-[s-1])} \rho_T^{(t-3-j)} \rho_T^{(s-3-j)} \right)
+ \sum_{t=6}^{T} \sum_{s=4}^{T} \sum_{j=1}^{T} \rho_T^{(t-3-[s-1])} \rho_T^{2(t-3-j)} \rho_T^{(s-3-j)} \right)
\]

\[
\leq C \left( \sum_{t=4}^{T} \sum_{j=1}^{T} \rho_T^{4(t-3-j)} + \sum_{t=6}^{T} \sum_{s=4}^{T} \sum_{j=1}^{T} \rho_T^{2(t-3-[s-1])} \rho_T^{(t-3-j)} \rho_T^{(s-3-j)} \right)
+ \sum_{t=6}^{T} \sum_{s=4}^{T} \sum_{j=1}^{T} \rho_T^{(t-3-[s-1])} \rho_T^{2(t-3-j)} \rho_T^{(s-3-j)} \right)
\]

\[
\leq C \left( \sum_{t=4}^{T} \sum_{j=1}^{T} \rho_T^{4(t-3-j)} + \sum_{t=6}^{T} \sum_{s=4}^{T} \sum_{j=1}^{T} \rho_T^{2(t-3-[s-1])} \rho_T^{(t-3-j)} \rho_T^{(s-3-j)} \right)
+ \sum_{t=6}^{T} \sum_{s=4}^{T} \sum_{j=1}^{T} \rho_T^{(t-3-[s-1])} \rho_T^{2(t-3-j)} \rho_T^{(s-3-j)} \right)
\]

\[
\leq C \left( \sum_{t=4}^{T} \sum_{j=1}^{T} \rho_T^{4(t-3-j)} + \sum_{t=6}^{T} \sum_{s=4}^{T} \sum_{j=1}^{T} \rho_T^{2(t-3-[s-1])} \rho_T^{(t-3-j)} \rho_T^{(s-3-j)} \right)
+ \sum_{t=6}^{T} \sum_{s=4}^{T} \sum_{j=1}^{T} \rho_T^{(t-3-[s-1])} \rho_T^{2(t-3-j)} \rho_T^{(s-3-j)} \right)
\]

\[
= C \left( I + II + III \right), \text{ (say)}. 
\]

Now, using the bound (39), we obtain for all \( T \geq T^* \)
Applying part (b) of Lemma SE-5 and performing additional calculation, we get

\[
I = \sum_{t=4}^{T} \sum_{j=1}^{T-t-3} \rho_T^{4(t-3-j)} = \sum_{t=4}^{T} \frac{1 - \rho_T^{4(t-3)}}{1 - \rho_T^{4}} = \frac{1}{1 - \rho_T^{4}} \left[ T - 3 - \frac{\rho_T^{4} \left( 1 - \rho_T^{4(T-3)} \right)}{1 - \rho_T^{4}} \right]
\]

\[
\leq \frac{T}{1 - \exp \{-4/C_q\}} \left[ 1 + \frac{3}{T} + \frac{1}{T (1 - \exp \{-4/C_q\})} \right] = O(T).
\]

(ii)

\[
II = 4 \sum_{t=6}^{T} \sum_{s=4}^{T-t-3} \rho_T^{2(t-3-[s-1])} \frac{4(s-3-j)}{(s-3-j)}
\]

\[
= 4 \sum_{t=6}^{T} \sum_{s=4}^{T-t-3} \rho_T^{2(t-3-[s-1])} \frac{1 - \rho_T^{2(s-3)}}{1 - \rho_T^{2}}
\]

\[
= \frac{4}{1 - \rho_T^{2}} \sum_{t=6}^{T} \sum_{s=4}^{T-t-3} \rho_T^{2(t-2-s)} - 4 \frac{\rho_T^{2}}{1 - \rho_T^{2}} \sum_{t=6}^{T} \sum_{s=4}^{T-t-3} \rho_T^{2(t-6)}
\]

\[
= 4 \frac{\rho_T^{2}(T-5)}{(1 - \rho_T^{2})^2} - 4 \rho_T^{2} \frac{(1 - \rho_T^{2(T-5)})}{(1 - \rho_T^{2})^3} - 4 \frac{\rho_T^{2}}{1 - \rho_T^{2}} \sum_{t=6}^{T} \rho_T^{2(t-6)} - 4 \frac{\rho_T^{2}}{1 - \rho_T^{2}} \sum_{t=6}^{T} \rho_T^{2(t-6)} (t - 6).
\]

Applying part (b) of Lemma SE-5 and performing additional calculation, we get

\[
II = 4 \frac{\rho_T^{2(T-5)}}{(1 - \rho_T^{2})^2} - 4 \rho_T^{2} \frac{(1 - \rho_T^{2(T-5)})}{(1 - \rho_T^{2})^3} + 4 \rho_T^{2} \frac{(1 - \rho_T^{2(T-5)})}{(1 - \rho_T^{2})^2}
\]

\[
-4 \rho_T^{2} \frac{(\rho_T^{2} - (T-5) \rho_T^{2(T-5)} + (T-6) \rho_T^{2(T-4)})}{(1 - \rho_T^{2})^3}
\]

\[
= 4 \rho_T^{2} \frac{\rho_T^{2(T-5)}}{(1 - \rho_T^{2})^2} \left[ 1 + \frac{\rho_T^{2(T-5)}}{(1 - \rho_T^{2})} - \rho_T^{2(T-4)} \right] - \frac{20\rho_T^{2}}{(1 - \rho_T^{2})^2} \rho_T^{2} \left[ (1 - \rho_T^{2}) - 4 \frac{(1 - \rho_T^{2})^{2}}{3} \right]
\]

\[
+ 4 \frac{\rho_T^{2}}{(1 - \rho_T^{2})^2} \rho_T^{2(T-5)} - 4 \rho_T^{2} \frac{(1 - \rho_T^{2})^3}{(1 - \rho_T^{2})^3} - 4 \rho_T^{2} \left[ 1 + 5 \rho_T^{2(T-6)} - 6 \rho_T^{2(T-5)} \right]
\]

\[
\leq \frac{4T}{(1 - \exp \{-2/C_q\})^2} \left[ 1 + \frac{2}{1 - \exp \{-2/C_q\}} \right] + \frac{20}{(1 - \exp \{-2/C_q\})^2}
\]

\[
+ \frac{4}{(1 - \exp \{-2/C_q\})^3} + \frac{20}{(1 - \exp \{-2/C_q\})^2} + \frac{48}{(1 - \exp \{-2/C_q\})^3}
\]

\[
= O(T).
\]
(iii)

\[
III = 2 \sum_{t=5}^{T} \sum_{s=4}^{t-3} \sum_{j=1}^{s-3} \rho_T^{2(t-3-j)} \rho_T^{2(s-3-k)} \\
= \frac{2}{(1 - \rho_T^2)^2} \sum_{t=5}^{T} \left(1 - \rho_T^{2(t-3)}\right) \sum_{s=4}^{t-1} \left(1 - \rho_T^{2(s-3)}\right) \\
= \frac{2}{(1 - \rho_T^2)^2} \sum_{t=5}^{T} \left(1 - \rho_T^{2(t-3)}\right) \left[ (t - 4) - \frac{\rho_T^2}{(1 - \rho_T^2)} \right] \\
= \frac{2}{(1 - \rho_T^2)^2} \sum_{t=5}^{T} \left(1 - \rho_T^{2(t-3)}\right) \left[ (t - 4) - \frac{\rho_T^2}{(1 - \rho_T^2)} \right] \left[ \rho_T^4 (T - 4) - \rho_T^6 (T - 4) \right] \\
= \frac{2}{(1 - \rho_T^2)^2} \left[ \frac{2T^2}{1 - \rho_T^2} \right] \left[ \frac{1 - \frac{7}{2T}}{T (1 - \rho_T^2)} + \frac{\rho_T^2}{T (1 - \rho_T^2)} + \frac{\rho_T^6}{T^2 (1 - \rho_T^2)} \right] \\
\leq \frac{T^2}{(1 - \exp \{-2/C_q\})^2} \left[ 1 + \frac{9}{T} + \frac{2}{T (1 - \exp \{-2/C_q\})} + \frac{12}{T^2} + \frac{10}{T^2 (1 - \exp \{-2/C_q\})} + \frac{24}{T^2 (1 - \exp \{-2/C_q\}) (1 - \exp \{-4/C_q\})} \right] \\
= O(T^2),
\]

where the fifth equality above follows in part from applying part (b) of Lemma SE-5.
It follows from the results given for expressions $I − III$ that
\[ \sum_{i=1}^{N} E \left[ \left( \sum_{t=4}^{T} w_{it-3,T} \varepsilon_{it-1} \right)^4 \right] = O(NT^2). \]

Hence,
\[
\frac{1}{N^2} \sum_{i=1}^{N} E \left[ (X_{i,T} + Y_{i,T})^4 \right] = 8 \frac{1}{N^2 T^2} \sum_{i=1}^{N} E \left[ \left( \sum_{t=4}^{T} \varepsilon_{it-2} \varepsilon_{it-1} \right)^4 \right] + 8 \frac{(1 - \rho_T)^4}{N^2 T^2} \sum_{i=1}^{N} E \left[ \left( \sum_{t=4}^{T} \frac{w_{it-3} \varepsilon_{it-1}}{\sqrt{N}} \right)^4 \right] \\
= O \left( \frac{1}{N^2 T^2} \right) O(NT^2) + O \left( \frac{1}{N^2 T^2} \right) O(NT^2) \\
= O \left( \frac{1}{N} \right),
\]

so the Liapunov-type condition (40) is satisfied. Now, since the Liapunov-type condition (40) implies the Lindeberg-type condition (13) given in Lemma SE-10 above, it follows from Lemma SE-10 that
\[ U_{N,T} = \frac{1}{\omega_{N,T}} \sum_{i=1}^{N} \left( X_{i,T} + Y_{i,T} \right) \Rightarrow N(0,1). \]

Moreover, note that
\[ \frac{1}{\omega_{N,T}} \sum_{i=1}^{N} \left( X_{i,T} + Y_{i,T} \right) = \sqrt{\frac{1 + \rho_T}{2 \sigma^4}} \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left( X_{i,T} + Y_{i,T} \right) + O \left( \frac{1}{T} \right). \]

This implies that
\[ \sqrt{\frac{1 + \rho_T}{2 \sigma^4}} \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left( X_{i,T} + Y_{i,T} \right) = \frac{1}{\omega_{N,T}} \sum_{i=1}^{N} \left( X_{i,T} + Y_{i,T} \right) + O \left( \frac{1}{T} \right) \Rightarrow N(0,1), \]
as required. \( \square \)

**Lemma SE-24:**
Suppose that Assumptions 1 and 4 hold. If \( \rho_T = 1 \) for all \( T \) sufficiently large, then, as \( N, T \to \infty \),
\[ \frac{1}{\sqrt{2 \sigma^2 \sqrt{N}}} \sum_{i=1}^{N} \left( X_{i,T} + Y_{i,T} \right) \Rightarrow N(0,1), \]
where
\[ X_{i,T} = \frac{1}{\sqrt{T}} \sum_{t=4}^{T} \varepsilon_{it-2} \varepsilon_{it-1}, \quad Y_{i,T} = \frac{1}{\sqrt{T}} w_{iT-2,T} \varepsilon_{iT}. \]

**Proof of Lemma SE-24:**

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To proceed, first decompose $Y_{i,T}$ as follows:

\[
Y_{i,T} = \frac{1}{\sqrt{T}} w_{iT-2,T} \varepsilon_{iT} = \frac{1}{\sqrt{T}} w_{iT-2,T} \varepsilon_{iT} + \frac{1}{\sqrt{T}} \varepsilon_{iT} \rho_T^{T-2} w_{i0}
\]

\[= Y_{i,T} + \frac{1}{\sqrt{T}} \varepsilon_{iT} \rho_T^{T-2} w_{i0},\]

where $w_{iT-2} = \sum_{j=1}^{T-2} \rho_T^{(T-2-j)} \varepsilon_{ij}$ and $Y_{i,T} = \frac{1}{\sqrt{T}} w_{iT-2,T} \varepsilon_{iT}$. Let $U_{N,T} = \sum_{i=1}^{N} (X_{i,T} + Y_{i,T})$, as before. Now, as shown in the proof of Lemma SE-22, $E[X_{i,T}] = 0$ and

\[E[X_{i,T}^2] = \sigma^4 \left( \frac{T - 3}{T} \right) = \sigma^4 + O \left( \frac{1}{T} \right).\]

Moreover, given the assumption here, there exists a positive integer $I_\rho$ such that for all $T \geq I_\rho$, the triangular array process $\{w_{it,T}\}$ has the partial sum representation $w_{it,T} = \sum_{j=1}^t \varepsilon_{ij}$. Hence, for all $T \geq I_\rho$, we can make the following moment calculations:

\[E[\varepsilon_{ij} \varepsilon_{iT}] = 0,\]

\[E[\varepsilon_{ij}^2] = \frac{1}{T} \sum_{j=1}^{T-2} \sum_{k=1}^{T-2} E[\varepsilon_{ij} \varepsilon_{ik} \varepsilon_{iT} \varepsilon_{iT}] = \frac{1}{T} \sum_{j=1}^{T-2} E[\varepsilon_{ij}^2] E[\varepsilon_{iT}^2] = \sigma^4 \frac{T-2}{T},\]

and

\[E[X_{i,T} Y_{i,T}] = \frac{1}{T} \sum_{t=4}^{T} \sum_{\ell=1}^{T-2} E[\varepsilon_{it-2} \varepsilon_{it-1} \varepsilon_{it} \varepsilon_{iT}] = -\frac{1}{T} \sum_{t=4}^{T} \sum_{\ell=1}^{T-2} E[\varepsilon_{it-2} \varepsilon_{it-1} \varepsilon_{it}] E[\varepsilon_{iT}] = 0.\]

It follows from these calculations that

\[E[U_{N,T}] = \sum_{i=1}^{N} (E[X_{i,T}] + E[Y_{i,T}]) = 0,\]

and

\[\frac{\omega_{N,T}^2}{N} = \frac{1}{N} \sum_{i=1}^{N} \omega_{i,T}^2 = \frac{1}{N} \sum_{i=1}^{N} E[(X_{i,T} + Y_{i,T})^2] = \frac{1}{N} \sum_{i=1}^{N} E[X_{i,T}^2] + \frac{1}{N} \sum_{i=1}^{N} E[Y_{i,T}^2] + 2 \frac{1}{N} \sum_{i=1}^{N} E[X_{i,T} Y_{i,T}] = \frac{\sigma^4}{N} \left[ \left( \frac{T - 3}{T} \right) + \frac{T - 2}{T} \right] = 2 \sigma^4 + O \left( \frac{1}{T} \right).\]

Given Assumption 1, it follows that, in this case, there exists a positive constant $C$ such that

\[0 < \frac{1}{C} \leq \frac{\omega_{N,T}}{\sqrt{N}} \leq C < \infty \text{ eventually as } N,T \to \infty.\]
Next, note that, for all $T$ sufficiently large,

\[
\frac{1}{\omega_{N,T}/\sqrt{N}} \frac{1}{\sqrt{N}} \sum_{i=1}^{N} (X_{i,T} + Y_{i,T}) = \frac{1}{\omega_{N,T}/\sqrt{N}} \frac{1}{\sqrt{N}} \sum_{i=1}^{N} (X_{i,T} + Y_{i,T}) + \frac{1}{\omega_{N,T}/\sqrt{N} \sqrt{NT}} \sum_{i=1}^{N} \varepsilon_{i,T}w_{i0}
\]

where the second equality follows from the result given in expression (24) above. Hence, showing the desired result is equivalent to showing the asymptotic normality of

\[
U_{N,T} = \frac{1}{\omega_{N,T}/\sqrt{N}} \frac{1}{\sqrt{N}} \sum_{i=1}^{N} (X_{i,T} + Y_{i,T}).
\]

To show the asymptotic normality of $U_{N,T}$, it suffices to verify a Liapounov-type condition of the form

\[
\lim_{N,T \to \infty} \frac{1}{N^2} \sum_{i=1}^{N} E \left( (X_{i,T} + Y_{i,T})^4 \right) = 0. \tag{41}
\]

To show this, note first that by Loève’s $c_r$ inequality, we have

\[
\frac{1}{N^2} \sum_{i=1}^{N} E \left( (X_{i,T} + Y_{i,T})^4 \right) \leq 8 \frac{1}{N^2T^2} \sum_{i=1}^{N} E \left[ \left( \sum_{t=4}^{T} \varepsilon_{i,2t-2}^t \varepsilon_{i,2t-1}^t \right)^4 \right] + 8 \frac{1}{N^2T^2} \sum_{i=1}^{N} E \left[ \left( \sum_{j=1}^{T-2} \varepsilon_{ij}^T \varepsilon_{iT} \right)^4 \right]. \tag{42}
\]

From the proof of Lemma SE-22, we have

\[
E \left[ \left( \sum_{t=4}^{T} \varepsilon_{i,2t-2}^t \varepsilon_{i,2t-1}^t \right)^4 \right] = 6\sigma^8 \frac{(T-5)(T-4)}{2} + \left( E \left[ \varepsilon_{i,t-1}^4 \right] \right)^2 (T-3) + 6 E \left[ \varepsilon_{i,t-1}^4 \right] \sigma^4 (T-4) = 3\sigma^8 T^2 \left[ 1 + O \left( \frac{1}{T} \right) \right]. \tag{43}
\]

In addition, note that, under Assumption 1,

\[
E \left[ \sum_{j=1}^{T-2} \varepsilon_{ij}^T \varepsilon_{iT}^4 \right] = \sum_{g=1}^{T-2T-2} \sum_{s=1}^{T-2T-2} \sum_{t=1}^{T-2T-2} \sum_{u=1}^{T-2T-2} E \left[ \varepsilon_{is}^4 \varepsilon_{it}^4 \varepsilon_{iu}^4 \right] E \left[ \varepsilon_{iT}^4 \right]
\]

\[
= E \left[ \varepsilon_{is}^4 \right] E \left[ \varepsilon_{iT}^4 \right] (T-2) + 6\sigma^4 E \left[ \varepsilon_{iT}^4 \right] \sum_{g=2}^{T-2} (g-1)
\]

\[
= E \left[ \varepsilon_{is}^4 \right] E \left[ \varepsilon_{iT}^4 \right] (T-2) + 3\sigma^4 E \left[ \varepsilon_{iT}^4 \right] (T-3)(T-2) = 3\sigma^4 E \left[ \varepsilon_{iT}^4 \right] T^2 \left[ 1 + O \left( \frac{1}{T} \right) \right]. \tag{44}
\]
Now, applying (43) and (44) to the upper bound in (42), we get
\[ \frac{1}{N^2} \sum_{i=1}^{N} E \left[ (X_{i,T} + Y_{i,T})^4 \right] \leq 8 \frac{1}{N^2 T^2} \sum_{i=1}^{N} 3\sigma^8 T^2 \left[ 1 + O \left( \frac{1}{T} \right) \right] \]
\[ + 8 \frac{1}{N^2 T^2} \sum_{i=1}^{N} 3\sigma^4 E \left[ \varepsilon_{iT}^4 \right] T^2 \left[ 1 + O \left( \frac{1}{T} \right) \right] \]
\[ = 24\sigma^8 \frac{1}{N} + 24\sigma^4 E \left[ \varepsilon_{iT}^4 \right] \frac{1}{N} + O \left( \frac{1}{NT} \right). \]
\[ = O \left( N^{-1} \right). \]

Since the Liapounov-type condition (41) implies the Lindeberg-type condition (13) given in Lemma SE-10 above, it follows from Lemma SE-10 that
\[ U_{N,T} = \frac{1}{\omega_{N,T}} \sum_{i=1}^{N} (X_{i,T} + Y_{i,T}) \]
\[ = \frac{1}{\sqrt{2\sigma^2}} \frac{1}{\sqrt{N}} \sum_{i=1}^{N} (X_{i,T} + Y_{i,T}) + O \left( \frac{1}{T} \right) \]
\[ \Rightarrow N (0,1), \]
as required. □

**Lemma SE-25:**

Let \( d \) be a non-negative integer and let \( g \) be an integer such that \( g \geq 2 \). Suppose further that \( g > d \). Under Assumptions 1 and 4, the following statements are true as \( N, T \to \infty \).

(a) If \( \rho_T = 1 \) for all \( T \) sufficiently large, then
\[ \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{it-g,T \varepsilon_{it-d}} = \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{it-g,T \varepsilon_{it-d}} + O_p \left( \frac{1}{\sqrt{T}} \right) = O_p (1). \]

(b) If \( \rho_T = \exp \{-1/q(T)\} \) such that \( T/q(T) \to 0 \), then
\[ \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{it-g,T \varepsilon_{it-d}} = \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{it-g,T \varepsilon_{it-d}} + O_p \left( \frac{1}{\sqrt{T}} \right) = O_p (1). \]
and
\[ \frac{(1 - \rho_T)}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{it-g,T \varepsilon_{it-d}} = O_p \left( \frac{1}{q(T)} \right). \]

(c) If \( \rho_T = \exp \{-1/q(T)\} \) such that \( q(T) \sim T \), then
\[ \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{it-g,T \varepsilon_{it-d}} = \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{it-g,T \varepsilon_{it-d}} + O_p \left( \frac{1}{\sqrt{T}} \right) = O_p (1). \]
Proof of Lemma SE-25:

To proceed, first write

\[ \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{t-g, T} \varepsilon_{it-d} = \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{t-g, T} \varepsilon_{it-d} + \sum_{i=1}^{N} \sum_{t=g+1}^{T} \varepsilon_{it-d} \rho_{T}^{t-g} w_{0}, \]

where \( w_{t-g, T} = \sum_{j=1}^{t-g} \rho_{T}^{t-g-j} \varepsilon_{ij} \).

Consider first part (a). Note that, under the assumption here, there exists a positive integer \( I_{\rho} \) such that for all \( T \geq I_{\rho} \), the triangular array process \( \{w_{t-g, T}\} \) has the partial sum representation \( w_{t-g, T} = \sum_{j=1}^{t} \varepsilon_{ij} \). Hence, for all \( T \geq \max \{I_{\rho}, g+1\} \), we obtain by direct calculation

\[ E \left[ \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{t-g, T} \varepsilon_{it-d} \right]^2 = \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{j=g+1}^{T} \varepsilon_{ij} \sum_{s=g+1}^{T} \sum_{k=1}^{T} \sum_{\ell=1}^{T} E[\varepsilon_{ik} \varepsilon_{j\ell} \varepsilon_{it-d} \varepsilon_{js-d}] \]

\[ = \frac{\sigma^4}{NT^2} \sum_{i=1}^{N} \sum_{t=g+1}^{T} (t-g) \]

\[ = \frac{\sigma^4 N(T-g)(T-g+1)}{2} \]

\[ = \frac{\sigma^4 NT^2}{2} \left[ 1 + O\left(\frac{1}{T}\right) \right] = O(1), \]
so that by Markov’s inequality \( N^{-1/2}T^{-1} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{it-g,T} \varepsilon_{it-d} w_{i0} = O_p(1) \).

Moreover, by Assumptions 1 and 4,

\[
E \left[ \left( \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=g+1}^{T} \varepsilon_{it-d} w_{i0} \right)^2 \right] = \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{t=g+1}^{T} \sum_{s=g+1}^{T} E \left[ w_{i0} w_{j0} \right] E \left[ \varepsilon_{it-d} \varepsilon_{js-d} \right]
\]

\[
= \frac{\sigma^2}{NT^2} \sum_{i=1}^{N} E \left[ w_{i0}^2 \right] (T - g)
\]

\[
\leq \frac{\sigma^2}{T} \left( \sup_{i} E \left[ w_{i0}^2 \right] \right) \frac{(T - g)}{T} = O \left( T^{-1} \right),
\]

from which it follows, by Markov’s inequality, that

\[
\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=g+1}^{T} \varepsilon_{it-d} w_{i0} = O_p \left( \frac{1}{\sqrt{T}} \right).
\]

Hence,

\[
\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{it-g,T} \varepsilon_{it-d} = \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{it-g,T} \varepsilon_{it-d} + \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=g+1}^{T} \varepsilon_{it-d} w_{i0}
\]

(for all \( T \) sufficiently large)

\[
= \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{it-g,T} \varepsilon_{it-d} + O_p \left( \frac{1}{\sqrt{T}} \right) = O_p \left( 1 \right).
\]

Now, to show parts (b)-(d), we first write

\[
E \left[ \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{it-g,T} \varepsilon_{it-d} \right]^2 = \frac{N \sigma^4}{NT^2} \sum_{i=1}^{N} \sum_{t=g+1}^{T} \sum_{s=g+1}^{T} \sum_{k=1}^{t-g} \exp \left\{ -\frac{t - g - k}{q(T)} \right\} \exp \left\{ -\frac{s - g - \ell}{q(T)} \right\} E \left[ \varepsilon_{ik} \varepsilon_{j\ell} \varepsilon_{it-d} \varepsilon_{js-d} \right]
\]

\[
= \frac{N \sigma^4}{NT^2} \sum_{i=1}^{N} \sum_{t=g+1}^{T} \sum_{k=1}^{t-g} \exp \left\{ -\frac{t - g - k}{q(T)} \right\}
\]

for \( T \geq g + 1 \).

Next, consider part (b), where we take \( \rho_T = \exp \{-1/q(T)\} \) such that \( T/q(T) \to 0 \). In this case, we apply part (a) of Lemma SE-1 with \( b = g + 1 \) and \( d = 2 \) to obtain

\[
E \left[ \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{it-g,T} \varepsilon_{it-d} \right]^2 = \frac{N \sigma^4}{NT^2} \sum_{t=g+1}^{T} \sum_{k=1}^{t-g} \exp \left\{ -\frac{t - g - k}{q(T)} \right\}
\]

\[
= \frac{NT^2 \sigma^4}{2} \left[ 1 + O_p \left( \max \left\{ \frac{T}{q(T)}, \frac{1}{T} \right\} \right) \right] = O(1),
\]
so that, by Markov’s inequality, we obtain \( N^{-1/2}T^{-1} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{it-g,T} \varepsilon_{it-d} = O_p(1) \).
Moreover, by Assumptions 1 and 4,

\[
E \left[ \left( \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=g+1}^{T} \varepsilon_{it-d} \rho_{T}^{t-g} w_{i0} \right)^2 \right] = \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{t=g+1}^{T} \sum_{s=g+1}^{T} \rho_{T}^{l-g} \rho_{T}^{s-g} E[w_{i0}w_{j0}] E[\varepsilon_{it-d} \varepsilon_{js-d}] \\
= \frac{\sigma^2}{T^2} \sum_{i=1}^{N} E[w_{i0}^2] \sum_{t=g+1}^{T} \rho_{T}^{2(t-g)} \\
= \frac{\sigma^2}{T^2} \sup_{i} E[w_{i0}^2] \exp\left\{-\frac{2}{q(T)}\right\} \left[1 - \exp\left\{-\frac{2}{q(T)}\right\}\right]^{-1} \left[1 - \exp\left\{-\frac{2(T-g)}{q(T)}\right\}\right] \\
= O(T^{-2}) \times O(1) \times O(1) \times O(q(T)) \times O(T/q(T)) = O(T^{-1}),
\]
from which it follows by Markov’s inequality that

\[
\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=g+1}^{T} \varepsilon_{it-d} \rho_{T}^{t-g} w_{i0} = O_p\left(\frac{1}{\sqrt{T}}\right).
\]

Hence,

\[
\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{it-g,T} \varepsilon_{it-d} = \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{it-g,T} \varepsilon_{it-d} + \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=g+1}^{T} \varepsilon_{it-d} \rho_{T}^{t-g} w_{i0} \\
= \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{it-g,T} \varepsilon_{it-d} + O_p\left(\frac{1}{\sqrt{T}}\right) = O_p(1).
\]

Furthermore,

\[
\frac{1 - \rho_{T}}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{it-g,T} \varepsilon_{it-d} = \left(1 - \exp\left\{-\frac{1}{q(T)}\right\}\right) \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{it-g,T} \varepsilon_{it-d} \\
= \frac{1}{q(T)} \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{it-g,T} \varepsilon_{it-d} \left[1 + O_p\left(\frac{1}{q(T)}\right)\right] \\
= O_p\left(\frac{1}{q(T)}\right).
\]

Next, consider part (c), where we take \( \rho_{T} = \exp\{-1/q(T)\} \) such that \( q(T) \sim T \). Here, applying part (b) of Lemma SE-1 with \( b = g + 1 \) and \( d = 2 \), we get

\[
E \left[ \left( \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{it-g,T} \varepsilon_{it-d} \right)^2 \right]^2 = \sigma^4 N \frac{1}{NT^2} \sum_{t=g+1}^{T} \sum_{k=1}^{t-g} \exp\left\{-\frac{2(t-g-k)}{q(T)}\right\} \\
= \frac{N q(T)^2 \sigma^4}{NT^2} \left[ \exp\left\{-\frac{2T}{q(T)}\right\} + \frac{2T}{q(T)} - 1 \right] \left[1 + O_p\left(\frac{1}{T}\right)\right] \\
= O(1),
\]
so that, using Markov’s inequality, we again obtain $N^{-1/2}T^{-1} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{i-t}T \varepsilon_{it-d} = O_p(1)$.

Moreover, by Assumptions 1 and 4,

$$
E \left[ \left( \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=g+1}^{T} \varepsilon_{it-d} \rho_T^{-T} w_{i0} \right)^2 \right] = \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{t=g+1}^{T} \sum_{s=g+1}^{T} \rho_T^{-g} \rho_T^{s-g} E \left[ w_{i0} w_{j0} \right] E \left[ \varepsilon_{it-d} \varepsilon_{jt-d} \right]
$$

$$
= \frac{\sigma^2}{NT^2} \sum_{i=1}^{N} \sum_{t=g+1}^{T} \rho_T^{2(t-g)} 
$$

$$
= \frac{\sigma^2}{T^2} \left( \sup_i E \left[ w_{i0}^2 \right] \right) \exp \left\{ - \frac{2}{q(T)} \right\} \left[ 1 - \exp \left\{ - \frac{2}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{ - \frac{2(T-g)}{q(T)} \right\} \right] \exp \left\{ - \frac{2(T-g)}{q(T)} \right\} 
$$

$$
= O \left( T^{-2} \right) O \left( 1 \right) O \left( 1 \right) O \left( T \right) O \left( 1 \right) = O \left( T^{-1} \right),
$$

from which it follows, by Markov’s inequality, that

$$
\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=g+1}^{T} \varepsilon_{it-d} \rho_T^{-T} w_{i0} = O_p \left( \frac{1}{\sqrt{T}} \right).
$$

Hence,

$$
\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{i-t}T \varepsilon_{it-d} = \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{i-t}T \varepsilon_{it-d} + \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=g+1}^{T} \varepsilon_{it-d} \rho_T^{-T} w_{i0} 
$$

$$
= \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{i-t}T \varepsilon_{it-d} + O_p \left( \frac{1}{T} \right) = O_p \left( 1 \right).
$$

Furthermore, note that

$$
\frac{(1 - \rho_T) \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{i-t}T \varepsilon_{it-d}}{\sqrt{NT}} = \left( 1 - \exp \left\{ - \frac{1}{q(T)} \right\} \right) \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{i-t}T \varepsilon_{it-d} 
$$

$$
= \frac{1}{q(T)} \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{i-t}T \varepsilon_{it-d} \left[ 1 + O_p \left( \frac{1}{q(T)} \right) \right] 
$$

$$
= O_p \left( \frac{1}{T} \right).
$$

Now, consider part (d), where we take $\rho_T = \exp \left\{ -1/q(T) \right\}$ such that $q(T) \to \infty$ but $q(T)/T \to 0$.

Here, applying part (c) of Lemma SE-1 with $b = g + 1$ and $d = 2$, we get

$$
E \left[ \frac{1}{\sqrt{NTq(T)}} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{i-t}T \varepsilon_{it-d} \right]^2 = \sigma^4 \frac{N}{NTq(T)} \sum_{t=g+1}^{T} \sum_{k=1}^{t-g} \exp \left\{ -2T - g - k \right\} \frac{q(T)}{q(T)} 
$$

$$
= \frac{NTq(T)}{NTq(T)} \sigma^4 \left[ 1 + O_p \left( \frac{q(T)}{T} \right) + O_p \left( \frac{1}{q(T)} \right) \right] 
$$

$$
= O \left( 1 \right),
$$
so that, applying Markov’s inequality, we obtain
\[ N^{-1/2}T^{-1/2}q(T)^{-1/2} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{it-g,T}^{2} = O_p(1). \]
Moreover, by Assumptions 1 and 4,
\[
E \left[ \left( \frac{1}{\sqrt{NTq(T)}} \sum_{i=1}^{N} \sum_{t=g+1}^{T} \varepsilon_{it-d}^{T-g} w_{i0} \right)^{2} \right]
\] 
\[
= \frac{1}{NTq(T)} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{t=g+1}^{T} \sum_{s=g+1}^{T} \rho_{T}^{t-g} \rho_{T}^{s-g} E[w_{i0}w_{j0}] E[\varepsilon_{it-d}\varepsilon_{js-d}]
\] 
\[
= \frac{\sigma^{2}}{NTq(T)} \sum_{i=1}^{N} E[w_{i0}^{2}] \sum_{t=g+1}^{T} \rho_{T}^{2(t-g)}
\] 
\[
= \frac{\sigma^{2}}{Tq(T)} \sup_{i} E[w_{i0}^{2}] \exp \left\{ -\frac{2}{q(T)} \right\} \left[ 1 - \exp \left\{ -\frac{2}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{ -\frac{2(T-g)}{q(T)} \right\} \right]
\] 
\[
= O \left( T^{-1}q(T)^{-1} \right) O(1) O(1) O(q(T)) O(1)
\] 
\[
= O \left( T^{-1} \right),
\]
from which it follows by Markov’s inequality that
\[
\frac{1}{\sqrt{NTq(T)}} \sum_{i=1}^{N} \sum_{t=g+1}^{T} \varepsilon_{it-d}^{T-g} w_{i0} = O_p \left( \frac{1}{\sqrt{T}} \right).
\]
Hence,
\[
\frac{1}{\sqrt{NTq(T)}} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{it-g,T} \varepsilon_{it-d}
\] 
\[
= \frac{1}{\sqrt{NTq(T)}} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{it-g,T} \varepsilon_{it-d} + \frac{1}{\sqrt{NTq(T)}} \sum_{i=1}^{N} \sum_{t=g+1}^{T} \varepsilon_{it-d}^{T-g} w_{i0}
\] 
\[
= \frac{1}{\sqrt{NTq(T)}} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{it-g,T} \varepsilon_{it-d} + O_p \left( \frac{1}{\sqrt{T}} \right) = O_p \left( 1 \right)
\]
Furthermore, note that
\[
\frac{(1 - \rho_{T})}{\sqrt{NTq(T)}} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{it-g,T} \varepsilon_{it-d} = \left( 1 - \exp \left\{ -\frac{1}{q(T)} \right\} \right) \frac{1}{\sqrt{NTq(T)}} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{it-g,T} \varepsilon_{it-d}
\] 
\[
= \frac{1}{q(T)} \sqrt{NTq(T)} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{it-g,T} \varepsilon_{it-d} \left[ 1 + O_p \left( \frac{1}{q(T)} \right) \right]
\] 
\[
= O_p \left( \frac{1}{q(T)} \right).
\]
Finally, for part (e), note that for all $T \geq g + 1$, we have

$$E \left[ \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{it-g,T} \varepsilon_{it-d} \right]^2$$

$$= \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=g+1}^{T} \sum_{s=g+1}^{T} \sum_{j=1}^{T} \rho_T^{(t-g-j)} \rho_T^{(s-g-j)} E \left[ \varepsilon_{ij} \varepsilon_{hk} \varepsilon_{it-d} \varepsilon_{hs-d} \right]$$

$$= \frac{\sigma^4}{NT} \sum_{t=g+1}^{T} \sum_{j=1}^{T} \rho_T^{2(t-g-j)}$$

$$= \frac{\sigma^4}{1} \frac{1}{1 - \rho_T^2} T \left( T - g \right) - \frac{\rho_T^2 \left( 1 - \rho_T^2 \right)}{1 - \rho_T^2}.$$

Since here we assume that $|\rho_T| = \exp \left\{ -\frac{1}{q(T)} \right\}$ with $q(T) = O(1)$, it follows that there exist a positive constant $C_q$ and a positive integer $T^*$ such that for all $T \geq T^*$, we have

$$|\rho_T| \leq \exp \left\{ -\frac{1}{C_q} \right\} < 1.$$

Using this bound, we have $T \geq \max \{ T^*, g + 1 \}$

$$E \left[ \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{it-g,T} \varepsilon_{it-d} \right]^2 = \frac{\sigma^4}{1 - \rho_T^2} T \left( T - g \right) - \frac{\rho_T^2 \left( 1 - \rho_T^2 \right)}{1 - \rho_T^2}$$

$$\leq \frac{\sigma^4}{1 - \exp \{-2/C_q\}} \left[ 1 + \frac{1}{1 - \exp \{-2/C_q\}} \right] = O(1).$$

Hence, by applying Markov’s inequality, we further obtain $N^{-1/2} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{it-g,T} \varepsilon_{it-d} = O_p(1)$.
Moreover, by Assumptions 1 and 4,

\[
E \left[ \left( \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=g+1}^{T} \varepsilon_{it-d} r_{t-g}^{i} \epsilon_{t} \right)^{2} \right] = \frac{1}{NT} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{t=g+1}^{T} \sum_{s=g+1}^{T} \rho_{T}^{t-g} \rho_{T}^{s-g} E \left[ \epsilon_{it} \epsilon_{j,t} \right] E \left[ \varepsilon_{it-d} \varepsilon_{js-d} \right]
\]

\[
= \sigma^{2} \sum_{i=1}^{N} E \left[ w_{i0}^{2} \right] \sum_{t=g+1}^{T} \sum_{s=g+1}^{T} \rho_{T}^{2(t-g)}
\]

\[
= \sigma^{2} \left( \sup_{i} E \left[ w_{i0}^{2} \right] \right) \rho_{T}^{2} \frac{1 - \rho_{T}^{2(T-g)}}{1 - \rho_{T}^{2}}
\]

\[
= O \left( T^{-1} \right) O \left( 1 \right) O \left( 1 \right)
\]

\[
= O \left( T^{-1} \right),
\]

from which it follows, by Markov’s inequality, that

\[
\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=g+1}^{T} \varepsilon_{it-d} r_{t-g}^{i} \epsilon_{t} = O_{\mathbb{P}} \left( \frac{1}{\sqrt{T}} \right).
\]

Hence,

\[
\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{it-g,T} \varepsilon_{it-d} = \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{it-g,T} \varepsilon_{it-d} + \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=g+1}^{T} \varepsilon_{it-d} r_{t-g}^{i} \epsilon_{t} = O_{\mathbb{P}} \left( \frac{1}{\sqrt{T}} \right) = O_{\mathbb{P}} \left( 1 \right).
\]

Finally, note that

\[
\left| \frac{(1 - \rho_{T})}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{it-g,T} \varepsilon_{it-d} \right| \leq 2 \left| \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{it-g,T} \varepsilon_{it-d} \right| = O_{\mathbb{P}} \left( 1 \right),
\]

which completes the proof for part (e). \( \square \)

**Lemma SE-26:**

Suppose that Assumptions 1 and 4 hold. Then, the following statements are true as \( N, T \to \infty \)

(a) If \( \rho_{T} = 1 \) for all \( T \) sufficiently large, then

\[
\bar{w}_{-1,NT} = O_{\mathbb{P}} \left( \max \left\{ \frac{T}{N}, 1 \right\} \right).
\]
(b) If \( \rho_T = \exp\{-1/q(T)\} \) such that \( T/q(T) \to 0 \), then
\[
\bar{w}_{-1,N,T} = O_p \left( \max \left\{ \frac{T}{N}, 1 \right\} \right),
\]
as \( N, T \to \infty \).

(c) If \( \rho_T = \exp\{-1/q(T)\} \) such that \( q(T) \sim T \), then
\[
\bar{w}_{-1,N,T} = O_p \left( \max \left\{ \frac{q(T)}{\sqrt{NT}}, 1 \right\} \right),
\]
as \( N, T \to \infty \).

(d) If \( \rho_T = \exp\{-1/q(T)\} \) such that \( q(T) \to \infty \) but \( q(T)/T \to 0 \), then
\[
\bar{w}_{-1,N,T} = O_p \left( \max \left\{ \frac{1}{\sqrt{NT}}, \frac{1}{T} \right\} \right),
\]
as \( N, T \to \infty \).

(e) If \( \rho_T \in G_{St} = \{|\rho_T| = \exp\{-1/q(T)\} : q(T) \geq 0 \text{ and } q(T) = O(1) \text{ as } T \to \infty \} \), then
\[
\bar{w}_{-1,N,T} = O_p \left( \max \left\{ \frac{1}{\sqrt{NT}}, \frac{1}{T} \right\} \right),
\]
as \( N, T \to \infty \).

**Proof of Lemma SE-26:**
To proceed, note that
\[
\bar{w}_{-1,N,T} = \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1} = \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1} + \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} \rho_{t-1}^{i-1} w_{i0},
\]
where \( w_{t-1,T} = \sum_{j=1}^{t-1} \rho_{t-1-j}^{i} \varepsilon_{ij} \).

Consider first part (a). Note that, under the assumption here, there exists a positive integer \( I_{\rho} \) such that for all \( T \geq I_{\rho} \), the triangular array process \( \{w_{it,T}\} \) has the partial sum representation
$w_{it,T} = \sum_{j=1}^{t} \varepsilon_{ij}$. Hence, for all $T \geq \max \{ I_\rho, g + 1 \}$, we obtain by direct calculation

\[
E \left[ \left( \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1} \right)^2 \right] = \frac{1}{N^2(T-1)^2} \sum_{i=1}^{N} \sum_{t=2}^{T} \sum_{s=2}^{T} E \left[ w_{it-1,T} w_{is-1,T} \right] = \frac{1}{N^2(T-1)^2} \sum_{i=1}^{N} \sum_{t=2}^{T} \sum_{s=2}^{T} \sum_{j=1}^{t-1} \sum_{k=1}^{s-1} E[\varepsilon_{ij}\varepsilon_{tk}] = \frac{\sigma^2}{N^2(T-1)^2} \sum_{i=1}^{N} \sum_{t=2}^{T} (t-1) + \frac{2\sigma^2}{N^2(T-1)^2} \sum_{i=1}^{N} \sum_{t=3}^{T} \sum_{s=2}^{t-1} (s-1) = \frac{\sigma^2}{N(T-1)^2} \frac{T(T-1)}{2} + \frac{2\sigma^2}{N(T-1)^2} \sum_{t=3}^{T} \sum_{s=2}^{t-1} (t-1)(t-2) = \frac{T\sigma^2}{2N(T-1)^2} + \frac{\sigma^2}{N(T-1)^2} \frac{(T-2)(T-1)(2T-3)}{6} + \frac{\sigma^2}{N(T-1)^2} \frac{(T-2)(T-1)}{2} = \frac{\sigma^2}{3N} \left[ 1 + O \left( \frac{1}{T} \right) \right].
\]

It follows by Markov’s inequality that

\[
\frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1} = O_p \left( \frac{\sqrt{T}}{N} \right).
\]

Moreover, by Assumption 4 and Liapounov’s inequality, we have

\[
E \left[ \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{i0} \right] \leq \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} E |w_{i0}| \leq \left( \sup_i E \left[ w_{i0}^2 \right] \right)^{1/2} \leq C < \infty,
\]

for some (positive) constant $C$. It follows, by Markov’s inequality, that

\[
\frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{i0} = O_p \left( 1 \right).
\]

Hence,

\[
\bar{w}_{-1,N,T} = \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1} + \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{i0} \quad \text{(for all } T \text{ sufficiently large)}
\]

\[
= O_p \left( \sqrt{\frac{T}{N}} \right) + O_p \left( 1 \right) = O_p \left( \max \left\{ \sqrt{\frac{T}{N}}, 1 \right\} \right),
\]

which shows part (a).

Now, to show parts (b)-(d), first write
we see that in this case

It follows by the Markov’s inequality that

Next, consider part (b), where we assume that $\rho_T = \exp\{-1/q(T)\}$ such that $T/q(T) \to 0$. Applying part (a) of Lemma SE-1 with $b = 2$ and $d = 2$ and part (a) of Lemma SE-7 with $b = 1$ and $g = 1$, we see that in this case

\[
E \left[ \left( \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1} \right)^2 \right]
\]

\[
= \frac{1}{N^2(T-1)^2} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{s=2}^{T} \sum_{j=1}^{T-1} \sum_{s-1}^{T} \exp \left\{ -\frac{t-1-j}{q(T)} \right\} \exp \left\{ -\frac{s-1-k}{q(T)} \right\} E[\varepsilon_{ij}\varepsilon_{lk}]
\]

\[
= \frac{\sigma^2}{N^2(T-1)^2} \sum_{i=1}^{N} \sum_{t=2}^{T} \sum_{j=1}^{t-1} \exp \left\{ -2\frac{t-1-j}{q(T)} \right\}
\]

\[
+ 2\sigma^2 \frac{1}{N^2(T-1)^2} \sum_{i=1}^{N} \sum_{t=2}^{T} \sum_{j=1}^{t-1} \sum_{s=2}^{T} \sum_{j=1}^{s-1} \exp \left\{ -\frac{t-1-j}{q(T)} \right\} \exp \left\{ -\frac{s-1-j}{q(T)} \right\}
\]

\[
= \frac{\sigma^2}{N(T-1)^2} \sum_{t=2}^{T} \sum_{j=1}^{t-1} \exp \left\{ -2\frac{t-1-j}{q(T)} \right\}
\]

\[
+ 2\sigma^2 \frac{1}{N(T-1)^2} \sum_{t=3}^{T} \sum_{s=2}^{t-1} \sum_{j=1}^{s-1} \exp \left\{ -\frac{t-1-j}{q(T)} \right\} \exp \left\{ -\frac{s-1-j}{q(T)} \right\}
\]

\[
\Rightarrow E \left[ \left( \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1} \right)^2 \right]
\]

\[
= \frac{\sigma^2}{N(T-1)^2} \sum_{t=2}^{T} \sum_{j=1}^{t-1} \exp \left\{ -2\frac{t-1-j}{q(T)} \right\}
\]

\[
+ 2\sigma^2 \frac{1}{N(T-1)^2} \sum_{t=3}^{T} \sum_{s=2}^{t-1} \sum_{j=1}^{s-1} \exp \left\{ -\frac{t-1-j}{q(T)} \right\} \exp \left\{ -\frac{s-1-j}{q(T)} \right\}
\]

\[
\Rightarrow E \left[ \left( \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1} \right)^2 \right]
\]

\[
= \frac{\sigma^2}{N(T-1)^2} \left[ 1 + O \left( \max \left\{ \frac{T}{q(T), \frac{1}{T}} \right\} \right) \right]
\]

\[
+ 2\sigma^2 \frac{1}{N(T-1)^2} \left[ 1 + O \left( \max \left\{ \frac{T}{q(T), \frac{1}{T}} \right\} \right) \right]
\]

\[
= O \left( \frac{T}{N} \right).
\]

It follows by the Markov’s inequality that

\[
\frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1} = O_p \left( \sqrt{\frac{T}{N}} \right)
\]
Moreover, by Assumption 4 and Liapounov’s inequality, we have

\[
E \left| \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} \rho_{T}^{t-1} w_{i0} \right|
\]

\[
\leq \frac{1}{N(T-1)} \sum_{i=1}^{N} E |w_{i0}| \sum_{t=2}^{T} \rho_{T}^{t-1}
\]

\[
\leq \frac{1}{T-1} \left( \sup_{i} E \left[ |w_{i0}|^2 \right] \right)^{1/2} \exp \left\{ -\frac{1}{q(T)} \right\} \left[ 1 - \exp \left\{ -\frac{1}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{ -\frac{T-1}{q(T)} \right\} \right]
\]

\[
\leq O(T^{-1}) O(1) O(1) O(q(T)) O(T/q(T)) = O(1)
\]

from which it follows, by Markov’s inequality, that

\[
\frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} \rho_{T}^{t-1} w_{i0} = O_p(1).
\]

Hence,

\[
\bar{w}_{-1,N,T} = \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1} + \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} \rho_{T}^{t-1} w_{i0}
\]

\[
= O_p \left( \sqrt{\frac{T}{N}} \right) + O_p(1) = O_p \left( \max \left\{ \sqrt{\frac{T}{N}}, 1 \right\} \right),
\]

which shows part (b).

Consider part (c), where we assume that \( \rho_{T} = \exp \{-1/q(T)\} \) such that \( q(T) \sim T \). Applying part (b) of Lemma SE-1 with \( b = 2 \) and \( d = 2 \) and part (b) of Lemma SE-7 with \( b = 1 \), we see that in this case

\[
E \left[ \left( \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1} \right)^2 \right]
\]

\[
= \sigma^2 \frac{1}{N(T-1)^2} \sum_{t=2}^{T} \sum_{j=1}^{t-1} \exp \left\{ -\frac{t-1-j}{q(T)} \right\}
\]

\[
+ 2 \sigma^2 \frac{1}{N(T-1)^2} \sum_{t=3}^{T} \sum_{s=2}^{t-1} \sum_{j=1}^{s-1} \exp \left\{ -\frac{t-1-j}{q(T)} \right\} \exp \left\{ -\frac{s-1-j}{q(T)} \right\}
\]

\[
= \sigma^2 \frac{1}{N(T-1)^2} q(T)^2 \left[ \exp \left\{ -\frac{2T}{q(T)} \right\} + \frac{2T}{q(T)} - 1 \right] \left[ 1 + O(1/T) \right]
\]

\[
+ 2 \sigma^2 \frac{T q(T)^2}{2} \left[ 1 - \frac{3 q(T)}{2T} + \frac{q(T)}{T} \exp \left\{ -\frac{T}{q(T)} \right\} - \frac{1}{2T} q(T) \exp \left\{ -\frac{2T}{q(T)} \right\} \right]
\]

\[
\times \left[ 1 + O \left( \frac{1}{T} \right) \right]
\]

\[
= O \left( \frac{q(T)^2}{NT} \right) = O \left( \frac{q(T)}{N} \right).
\]
It follows by the Markov’s inequality that
\[ \frac{1}{N(T-1)} \sum_{t=1}^{N} \sum_{i=1}^{T} w_{it-1} = O_p \left( \sqrt{\frac{q(T)}{N}} \right) = O_p \left( \sqrt{\frac{T}{N}} \right). \]

Moreover, by Assumption 4 and Liapounov’s inequality, we have
\[ E \left[ \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} \rho_{it-1}^{t-1} w_{it0} \right] \]
\[ \leq \frac{1}{T-1} \left( \sup_i E \left[ w_{i0}^2 \right] \right)^{1/2} \exp \left\{ -\frac{1}{q(T)} \right\} \left[ 1 - \exp \left\{ -\frac{1}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{ -\frac{T-1}{q(T)} \right\} \right] \]
\[ \leq O(T^{-1}) \times O(1) \times O(1) \times O(T) \times O(1) = O(1), \]

from which it follows, by Markov’s inequality, that
\[ \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} \rho_{it-1}^{t-1} w_{it0} = O_p \left( 1 \right). \]

Hence,
\[ \overline{\overline{w}}_{-1,N,T} = \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1} + \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} \rho_{it-1}^{t-1} w_{it0} \]
\[ = O_p \left( \sqrt{\frac{T}{N}} \right) + O_p \left( 1 \right) = O_p \left( \max \left\{ \sqrt{\frac{T}{N}}, 1 \right\} \right), \]

as required for part (c).

We turn our attention now to part (d) where we assume that \( \rho_T = \exp \{-1/q(T)\} \) such that \( q(T) \to \infty \) but \( q(T)/T \to 0 \). In this case, applying part (c) of Lemma SE-1 with \( b = 2 \) and \( d = 2 \) as well as part (c) of Lemma SE-7 with \( b = 1 \), we get
\[ E \left[ \left( \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1} \right)^2 \right] \]
\[ = \sigma^2 \frac{1}{N(T-1)^2} \sum_{t=2}^{T} \sum_{j=1}^{t-1} \exp \left\{ -\frac{2(t-1-j)}{q(T)} \right\} \]
\[ + 2\sigma^2 \frac{1}{N(T-1)^2} \sum_{t=3}^{T} \sum_{s=2}^{t-1} \sum_{j=1}^{s-1} \exp \left\{ -\frac{t-1-j}{q(T)} \right\} \exp \left\{ -\frac{s-1-j}{q(T)} \right\} \]
\[ = \sigma^2 \frac{1}{N(T-1)^2} \frac{Tq(T)}{2} \left[ 1 + O \left( \frac{1}{q(T)} \right) + O \left( \frac{q(T)}{T} \right) \right] \]
\[ + 2\sigma^2 \frac{1}{N(T-1)^2} \frac{Tq(T)^2}{2} \left[ 1 + O \left( \frac{1}{q(T)} \right) + O \left( \frac{q(T)}{T} \right) \right] \]
\[ = O \left( \frac{q(T)^2}{NT} \right). \]
It follows by the Markov’s inequality that

\[
\frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1} = O_p \left( \frac{q(T)}{\sqrt{NT}} \right) = o_p(1).
\]

Moreover, by Assumption 4 and Liapounov’s inequality, we have

\[
E \left| \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} \rho_{t}^{i-1} w_{i0} \right| \leq \frac{1}{T-1} \left( \sup_{i} E \left[ w_{i0}^2 \right] \right)^{1/2} \exp \left\{ -\frac{1}{q(T)} \right\} \left[ 1 - \exp \left\{ -\frac{1}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{ -\frac{T-1}{q(T)} \right\} \right] \leq O(T^{-1}) \times O(1) \times O(1) \times O(q(T)) \times O(1) = O(q(T)/T),
\]

from which it follows, by Markov’s inequality, that

\[
\frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} \rho_{t}^{i-1} w_{i0} = O_p \left( \frac{q(T)}{T} \right).
\]

Hence,

\[
\bar{w}_{-1,N,T} = \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1} + \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} \rho_{t}^{i-1} w_{i0} = O_p \left( \frac{q(T)}{\sqrt{NT}} \right) + O_p \left( \frac{q(T)}{T} \right) = O_p \left( \max \left\{ \frac{q(T)}{\sqrt{NT}}, \frac{q(T)}{T} \right\} \right),
\]

which shows part (d).

Finally, to show part (e), write
Now, since \( q(T) = O(1) \) in this case, there exist some positive constant \( C_q \) and some positive integer \( T^* \) such that for all \( T \geq T^* \)

\[
0 \leq |\rho_T| = \exp \left\{ \frac{1}{q(T)} \right\} \leq \exp \left\{ -\frac{1}{C_q} \right\} < 1
\]
so that

\[
E \left[ \bar{w}_{-1,N,T} \right] 
\leq \frac{\sigma^2}{N(T-1)} \frac{1}{1 - \rho_T^2} + \frac{\sigma^2}{N(T-1)^2} \frac{\rho_T^2 (1 - \rho_T^{2(T-1)})}{(1 - \rho_T^2)^2} 
\]

\[
+ \frac{2 \sigma^2}{N(T-1)^2} \frac{|\rho_T|}{(1 - \rho_T^2) (1 - \rho_T)} + \frac{\rho_T^2 (1 - \rho_T^{(T-2)})}{N(T-1)^2} \frac{1}{(1 - \rho_T^2)^2 (1 - \rho_T)} 
\]

\[
+ \frac{2 \sigma^2}{N(T-1)^2} \frac{\rho_T^3 (1 - \rho_T^{(T-2)})}{(1 - \rho_T^2)^2 (1 - \rho_T)} + \frac{\rho_T^3 (1 - \rho_T^{2(T-2)})}{N(T-1)^2} \quad \sigma^2 
\]

\[
\leq \frac{1}{N(T-1)} \left(1 - \exp \{-2/C_q\}\right) + \frac{\sigma^2}{N(T-1)^2} \left(1 - \exp \{-2/C_q\}\right)^2 
\]

\[
+ \frac{1}{N(T-1)^2} \frac{|\rho_T|}{(1 - \rho_T) (1 - \exp \{-1/C_q\})} + \frac{2 \sigma^2}{N(T-1)^2} \frac{\rho_T^2 (1 - \rho_T^{(T-2)})}{(1 - \rho_T^2)^2 (1 - \rho_T)} 
\]

\[
= O \left( \frac{1}{NT} \right) 
\]

It follows by the Markov’s inequality that

\[
\frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1} = O_p \left( \frac{1}{\sqrt{NT}} \right). 
\]

Moreover, by Assumption 4 and Liapounov’s inequality, we have

\[
E \left| \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} \rho_T^{t-1} w_{i0} \right| \leq \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} \rho_T^{t-1} E |w_{i0}| 
\]

\[
\leq \frac{1}{T-1} \left( \sup_i E \left[ w_{i0}^2 \right] \right)^{1/2} \frac{|\rho_T|}{1 - |\rho_T|} \left(1 - |\rho_T^{T-1}| \right) 
\]

\[
= O \left( \frac{1}{T} \right), 
\]

from which it follows, by Markov’s inequality, that

\[
\frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} \rho_T^{t-1} w_{i0} = O_p \left( \frac{1}{T} \right). 
\]

Hence,

\[
\bar{w}_{-1,N,T} = \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1} + \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} \rho_T^{t-1} w_{i0} 
\]

\[
= O_p \left( \frac{1}{\sqrt{NT}} \right) + O_p \left( \frac{1}{T} \right) = O_p \left( \max \left\{ \frac{1}{\sqrt{NT}} \frac{1}{T} \right\} \right), 
\]

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which completes the proof of part (e). □

Lemma SE-27:

Suppose that Assumptions 1 and 4 hold. Then, the following statements are true as $N, T \to \infty$

(a) If $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \sim T$, then

$$\frac{1}{w_{Z,N,T}} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1,T} \varepsilon_{it} \Rightarrow Z \equiv N(0,1),$$
as $N, T \to \infty$, where

$$w_{Z,N,T}^2 = \sigma^4 \sum_{i=1}^{N} \sum_{t=2}^{T} \sum_{j=1}^{t-1} \exp\left\{-2 \frac{t-1-j}{q(T)}\right\}.$$  

(b) If $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \to \infty$ but $q(T)/T \to 0$, then

$$\frac{1}{\sqrt{NTq(T)}} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1,T} \varepsilon_{it} \Rightarrow N\left(0, \frac{\sigma^4}{2}\right),$$
as $N, T \to \infty$.

(c) If $\rho_T \in \mathcal{G}_{St} = \left\{ |\rho_T| = \exp\left\{-\frac{1}{q(T)}\right\} : q(T) \geq 0 \text{ and } q(T) = O(1) \text{ as } T \to \infty \right\}$, then

$$\sqrt{\frac{1-\rho_T^2}{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1,T} \varepsilon_{it} \Rightarrow N\left(0, \sigma^4\right)$$

Proof of Lemma SE-27:

Before embarking on the proof of the individual parts of this lemma, we first introduce some additional notation and perform some preliminary moment calculations. To proceed, first write

$$\sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1,T} \varepsilon_{it} = \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1,T} \varepsilon_{it} + \sum_{i=1}^{N} \sum_{t=2}^{T} \varepsilon_{it} \rho_T^{t-1} w_{i0},$$

where $w_{it-1,T} = \sum_{j=1}^{t-1} \rho_T^{t-1-j} \varepsilon_{ij}$. Let

$$Z_{i,T} = \frac{1}{T} \sum_{t=2}^{T} w_{it-1,T} \varepsilon_{it},$$
and note that
\[
E[Z_{i,T}] = \frac{1}{T} \sum_{t=2}^{T} E[w_{it-1,T}e_{it}] = \frac{1}{T} \sum_{t=2}^{T} \sum_{j=1}^{t-1} \exp \left\{ -\frac{t-1-j}{q(T)} \right\} E[e_{ij}] = 0,
\]
\[
E[Z_{i,T}^2] = \frac{1}{T^2} \sum_{t=2}^{T} \sum_{s=2}^{T} E[w_{it-1,T}w_{is-1,T}e_{it}e_{is}] = \frac{1}{T^2} \sum_{t=2}^{T} \sum_{s=2}^{T} \sum_{j=1}^{t-1} \sum_{k=1}^{s-1} \exp \left\{ -\frac{t-1-j}{q(T)} \right\} \exp \left\{ -\frac{s-1-k}{q(T)} \right\} E[e_{ij}e_{ik}e_{it}e_{is}]
\]
\[
= \sigma^4 \frac{1}{T^2} \sum_{t=2}^{T} \sum_{j=1}^{t-1} \exp \left\{ -\frac{2(t-1-j)}{q(T)} \right\}.
\]
It follows that
\[
\omega^2_{Z,N,T} = \sum_{i=1}^{N} E[Z_{i,T}^2] = \sigma^4 N \frac{1}{T^2} \sum_{t=2}^{T} \sum_{j=1}^{t-1} \exp \left\{ -\frac{2(t-1-j)}{q(T)} \right\}.
\]
Now, consider part (a), where we assume that \( q(T) \sim T \). Using part (b) of Lemma SE-1 with \( b = 2 \) and \( d = 2 \), we see that in this case
\[
\frac{\omega^2_{Z,N,T}}{T^2} = \frac{\sigma^4 N}{T^2} \frac{1}{T^2} \sum_{t=2}^{T} \sum_{j=1}^{t-1} \exp \left\{ -\frac{2(t-1-j)}{q(T)} \right\}
\]
\[
= \sigma^4 N \frac{q(T)^2}{4T^2} \left[ \exp \left\{ -\frac{2T}{q(T)} \right\} + \frac{2T}{q(T)} - 1 \right] \left[ 1 + O \left( \frac{1}{T} \right) \right]
\]
so that
\[
\frac{\omega^2_{Z,N,T}}{N} = \sigma^4 \frac{q(T)^2}{4T^2} \left[ \exp \left\{ -\frac{2T}{q(T)} \right\} + \frac{2T}{q(T)} - 1 \right] \left[ 1 + O \left( \frac{1}{T} \right) \right].
\]
Moreover, the assumption \( q(T) \sim T \) implies that there exists a constant \( \overline{C} > 1 \) such that
\[
0 < \frac{1}{\overline{C}} \leq \frac{T}{q(T)} \leq \overline{C} < \infty \text{ eventually as } T \to \infty.
\]
In addition, let \( f(x) = \exp \{-2x\} + 2x - 1 \). Since \( f(0) = 0 \) and \( f'(x) = -2 \exp \{-2x\} + 2 > 0 \) for all \( x > 0 \), it follows that by the mean value theorem that for all \( a \in (0, \infty) \) there exists some \( b \in (0, a) \) such that
\[
\exp \{-2a\} + 2a - 1 = f(a) = f'(b) a > 0.
\]
Using these facts, we deduce that
\[
0 < \exp \left\{ -\frac{2}{\overline{C}} \right\} + \frac{2}{\overline{C}} - 1 < \exp \left\{ -2\overline{C} \right\} + 2\overline{C} - 1 < \infty.
\]
It then follows from Assumption 1 that there exists a positive constant \( C \) such that
\[
0 < \frac{1}{C} \leq \frac{\omega_{Z,N,T}}{\sqrt{N}} \leq C < \infty \text{ eventually as } N, T \to \infty.
\]
Moreover, by Assumptions 1 and 4,

\[
E \left[ \left( \frac{1}{\sqrt{N T}} \sum_{i=1}^{N} \sum_{t=2}^{T} \epsilon_{i t} \rho_{T}^{t-1} w_{i0} \right)^{2} \right] = \frac{1}{N T^{2}} \sum_{i_1=1}^{N} \sum_{j_1=1}^{N} \sum_{t_2=2}^{T} \rho_{T}^{t_2-1} \rho_{T}^{t_1-1} E [w_{i_0} w_{j_0}] E [\epsilon_{i t} \epsilon_{j s}] = \sigma^{2} \frac{N^{2}}{T^{2}} \sum_{i=1}^{N} E [w_{i0}^{2}] \sum_{t=2}^{T} \rho_{T}^{2(t-1)}
\]

\[
= \frac{\sigma^{2}}{T^{2}} \left( \sup_{i} E [w_{i0}^{2}] \right) \exp \left\{ -\frac{2}{q(T)} \right\} \left[ 1 - \exp \left\{ -\frac{2}{q(T)} \right\} \right]^{-1} \left[ 1 - \exp \left\{ -\frac{2(T-1)}{q(T)} \right\} \right] = O \left( T^{-2} \right) \times O \left( 1 \right) \times O \left( 1 \right) \times O \left( T \right) \times O \left( 1 \right) = O \left( T^{-1} \right),
\]

from which it follows, by Markov’s inequality, that

\[
\frac{1}{\sqrt{N T}} \sum_{i=1}^{N} \sum_{t=2}^{T} \epsilon_{i t} \rho_{T}^{t-1} w_{i0} = O_{P} \left( \frac{1}{\sqrt{T}} \right).
\]

Hence,

\[
\frac{1}{\omega_{Z,N,T}/\sqrt{N}} \frac{1}{\sqrt{N T}} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{i t-1,T} \epsilon_{i t} = \frac{1}{\omega_{Z,N,T}/\sqrt{N}} \frac{1}{\sqrt{N T}} \sum_{i=1}^{N} \sum_{t=g+2}^{T} w_{i t-1,T} \epsilon_{i t} + \frac{1}{\omega_{Z,N,T}/\sqrt{N}} \frac{1}{\sqrt{N T}} \sum_{i=1}^{N} \sum_{t=g+2}^{T} \epsilon_{i t} \rho_{T}^{t-1} w_{i0}
\]

\[
= \frac{1}{\omega_{Z,N,T}/\sqrt{N}} \frac{1}{\sqrt{N}} \sum_{i=1}^{N} Z_{i,T} + O_{P} \left( \frac{1}{\sqrt{T}} \right).
\]

Thus, showing the desired result is equivalent to showing the asymptotic normality of

\[
U_{Z,N,T} = \frac{1}{\omega_{Z,N,T}/\sqrt{N}} \frac{1}{\sqrt{N}} \sum_{i=1}^{N} Z_{i,T},
\]

To show the asymptotic normality of \( U_{N,T} \), it suffices to verify a Liapounov-type condition of the form

\[
\lim_{N,T \to \infty} \frac{1}{N^{2}} \sum_{i=1}^{N} E \left[ Z_{i,T}^{4} \right] = 0.
\]

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Now, given Assumption 1, there exists a positive constant $C$ such that

$$E \left[ Z_{i,T}^t \right]$$

$$= \frac{1}{T^4} \sum_{t=2}^{T} \sum_{s=2}^{T} \sum_{g=2}^{T} \sum_{u=2}^{T} \sum_{j}^{T} \sum_{k=1}^{T} \exp \left\{ -\frac{t-1-j}{q(T)} \right\} \exp \left\{ -\frac{s-1-k}{q(T)} \right\} E \left[ w_{it-1,T} w_{is-1,T} w_{ig-1,T} w_{iu-1,T} \xi_is \xi_ig \right]$$

$$= \frac{1}{T^4} \sum_{t=2}^{T} \sum_{s=2}^{T} \sum_{g=2}^{T} \sum_{u=2}^{T} \sum_{j}^{T} \sum_{k=1}^{T} \exp \left\{ -\frac{t-1-j}{q(T)} \right\} \exp \left\{ -\frac{s-1-k}{q(T)} \right\}$$

$$\exp \left\{ -\frac{g-1-\ell}{q(T)} \right\} \exp \left\{ -\frac{u-1-h}{q(T)} \right\} E \left[ \xi_{ij} \xi_{ik} \xi_{it} \xi_{ih} \xi_{is} \xi_{ig} \xi_{iu} \right]$$

$$\leq C \left( \frac{1}{T^4} \sum_{t=2}^{T} \sum_{j=1}^{T-1} \exp \left\{ -4\frac{t-1-j}{q(T)} \right\} \right)$$

$$+ \frac{1}{T^4} \sum_{t=3}^{T} \sum_{s=2}^{T} \sum_{j=1}^{T-1} \exp \left\{ -2\frac{t-1-s}{q(T)} \right\} \exp \left\{ -\frac{t-1-j}{q(T)} \right\} \exp \left\{ -\frac{s-1-j}{q(T)} \right\}$$

$$+ \frac{1}{T^4} \sum_{t=3}^{T} \sum_{s=2}^{T} \sum_{j=1}^{T-1} \exp \left\{ -\frac{t-1-s}{q(T)} \right\} \exp \left\{ -2\frac{t-1-j}{q(T)} \right\} \exp \left\{ -\frac{s-1-j}{q(T)} \right\}$$

$$+ \frac{1}{T^4} \sum_{t=3}^{T} \sum_{s=2}^{T} \sum_{j=1}^{T-1} \exp \left\{ -2\frac{t-1-s}{q(T)} \right\} \exp \left\{ -2\frac{s-1-j}{q(T)} \right\}$$

$$+ \frac{1}{T^4} \sum_{t=3}^{T} \sum_{s=2}^{T} \sum_{j=1}^{T-1} \sum_{k=1}^{T} \exp \left\{ -2\frac{t-1-j}{q(T)} \right\} \exp \left\{ -2\frac{s-1-k}{q(T)} \right\}$$

$$+ \frac{1}{T^4} \sum_{t=3}^{T} \sum_{s=2}^{T} \sum_{j=1}^{T-1} \exp \left\{ -\frac{t-1-j}{q(T)} \right\} \exp \left\{ -s-1-j \right\}$$

$$+ \frac{1}{T^4} \sum_{t=3}^{T} \sum_{s=2}^{T} \sum_{j=1}^{T-1} \exp \left\{ -\frac{t-1-s}{q(T)} \right\} \exp \left\{ -2\frac{t-1-j}{q(T)} \right\} \exp \left\{ -\frac{s-1-j}{q(T)} \right\} \exp \left\{ -2\frac{s-1-j}{q(T)} \right\}$$

$$+ \frac{1}{T^4} \sum_{t=3}^{T} \sum_{s=2}^{T} \sum_{j=1}^{T-1} \exp \left\{ -\frac{t-1-j}{q(T)} \right\} \exp \left\{ -s-1-j \right\}$$

$$+ \frac{1}{T^4} \sum_{t=3}^{T} \sum_{s=2}^{T} \sum_{j=1}^{T-1} \exp \left\{ -\frac{t-1-s}{q(T)} \right\} \exp \left\{ -2\frac{t-1-j}{q(T)} \right\} \exp \left\{ -\frac{s-1-j}{q(T)} \right\} \exp \left\{ -2\frac{s-1-j}{q(T)} \right\}$$

$$+ \frac{1}{T^4} \sum_{t=3}^{T} \sum_{s=2}^{T} \sum_{j=1}^{T-1} \exp \left\{ -\frac{t-1-j}{q(T)} \right\} \exp \left\{ -s-1-j \right\}$$

$$+ \frac{1}{T^4} \sum_{t=3}^{T} \sum_{s=2}^{T} \sum_{j=1}^{T-1} \exp \left\{ -\frac{t-1-s}{q(T)} \right\} \exp \left\{ -2\frac{t-1-j}{q(T)} \right\} \exp \left\{ -\frac{s-1-j}{q(T)} \right\} \exp \left\{ -2\frac{s-1-j}{q(T)} \right\}$$
\[
\begin{align*}
\leq C \left( \frac{1}{T^4} \sum_{t=2}^{T} \sum_{j=1}^{t-1} \exp \left\{ -4 \frac{t-1-j}{q(T)} \right\} \\
+ \frac{1}{T^4} \sum_{t=3}^{T} \sum_{s=2}^{t-1} \sum_{j=1}^{s-1} \exp \left\{ -4 \frac{t-1-s}{q(T)} \right\} \exp \left\{ -\frac{t-s}{q(T)} \right\} \exp \left\{ -2 \frac{s-1-j}{q(T)} \right\} \\
+ \frac{1}{T^4} \sum_{t=3}^{T} \sum_{s=2}^{t-1} \sum_{j=1}^{s-1} \exp \left\{ -2 \frac{t-1-s}{q(T)} \right\} \exp \left\{ -2 \frac{s-1-j}{q(T)} \right\} \\
+ \frac{1}{T^4} \sum_{t=3}^{T} \sum_{s=2}^{t-1} \sum_{j=1}^{s-1} \exp \left\{ -\frac{t-1-s}{q(T)} \right\} \exp \left\{ -\frac{t-1-j}{q(T)} \right\} \exp \left\{ -2 \frac{s-1-j}{q(T)} \right\} \\
+ \frac{1}{T^4} \sum_{t=3}^{T} \sum_{s=2}^{t-1} \sum_{j=1}^{s-1} \exp \left\{ -2 \frac{t-1-j}{q(T)} \right\} \exp \left\{ -2 \frac{s-1-k}{q(T)} \right\} \\
+ \frac{1}{T^4} \sum_{t=3}^{T} \sum_{s=2}^{t-1} \sum_{j=1}^{s-1} \sum_{k=1}^{s-1} \exp \left\{ -2 \frac{t-1-j}{q(T)} \right\} \sum_{k=1}^{s-1} \exp \left\{ -2 \frac{t-1-k}{q(T)} \right\} \sum_{s-1}^{1} \exp \left\{ -2 \frac{t-1-k}{q(T)} \right\} \right)
\leq C \left( \frac{1}{T^4} \sum_{t=2}^{T} \sum_{j=1}^{t-1} \exp \left\{ -4 \frac{t-1-j}{q(T)} \right\} + \frac{2}{T^4} \sum_{t=3}^{T} \sum_{s=2}^{t-1} \sum_{j=1}^{s-1} \exp \left\{ -2 \frac{t-1-s}{q(T)} \right\} \exp \left\{ -2 \frac{s-1-j}{q(T)} \right\} \\
+ \frac{2}{T^4} \sum_{t=3}^{T} \sum_{s=2}^{t-1} \sum_{j=1}^{s-1} \exp \left\{ -2 \frac{t-1-s}{q(T)} \right\} \exp \left\{ -s-j \right\} \exp \left\{ -2 \frac{s-1-j}{q(T)} \right\} \\
+ \frac{2}{T^4} \sum_{t=3}^{T} \sum_{s=2}^{t-1} \sum_{j=1}^{s-1} \sum_{k=1}^{s-1} \exp \left\{ -2 \frac{t-1-j}{q(T)} \right\} \exp \left\{ -2 \frac{s-1-k}{q(T)} \right\} \right)
\leq C \left( \frac{1}{T^4} \sum_{t=2}^{T} \sum_{j=1}^{t-1} \exp \left\{ -4 \frac{t-1-j}{q(T)} \right\} + \frac{4}{T^4} \sum_{t=3}^{T} \sum_{s=2}^{t-1} \sum_{j=1}^{s-1} \exp \left\{ -2 \frac{t-1-s}{q(T)} \right\} \exp \left\{ -2 \frac{s-1-j}{q(T)} \right\} \\
+ \frac{2}{T^4} \sum_{t=3}^{T} \sum_{s=2}^{t-1} \sum_{j=1}^{s-1} \sum_{k=1}^{s-1} \exp \left\{ -2 \frac{t-1-j}{q(T)} \right\} \exp \left\{ -2 \frac{s-1-k}{q(T)} \right\} \right)
\end{align*}
\]
\( b = 2 \), and part (a) of Lemma SE-9 with \( b = c = 2 \) that

\[
E[Z_{i,T}^4] \
\leq C \left( \frac{1}{T^4} \frac{q(T)^2}{16} \left[ \exp \left\{ -\frac{4T}{q(T)} \right\} + \frac{4T}{q(T)} - 1 \right] \left[ 1 + O \left( \frac{1}{T} \right) \right] \\
+ \frac{4}{T^4} \left( \frac{Tq(T)^2}{4} \left[ 1 + \exp \left\{ -\frac{2T}{q(T)} \right\} \right] - \frac{2q(T)^3}{8} \left[ 1 - \exp \left\{ -\frac{2T}{q(T)} \right\} \right] \right) \left[ 1 + O \left( \frac{1}{T} \right) \right] \\
+ \frac{2}{T^4} \left( \frac{T^2q(T)^2}{8} - \frac{q(T)^4}{16} \left[ 1 - \exp \left\{ -\frac{2T}{q(T)} \right\} \right] - \frac{Tq(T)^3}{8} \left[ 1 - \exp \left\{ -\frac{2T}{q(T)} \right\} \right] \right) \left[ 1 + O \left( \frac{1}{T} \right) \right] \\
+ \frac{q(T)^4}{16} \left[ 1 - \exp \left\{ -\frac{2T}{q(T)} \right\} \right] + \frac{q(T)^4}{16} \left[ 1 - \exp \left\{ -\frac{2T}{q(T)} \right\} \right] \\
- \frac{q(T)^4}{32} \left[ 1 - \exp \left\{ -\frac{4T}{q(T)} \right\} \right] \left[ 1 + O \left( \frac{1}{T} \right) \right] \\
\right) \left[ 1 + O \left( \frac{1}{T} \right) \right],
\]

so that

\[
\frac{1}{N^2} \sum_{i=1}^{N} E[Z_{i,T}^4] \\
= C \frac{1}{4N} \left( \frac{q(T)^2}{T^2} - \frac{q(T)^3}{T^3} \left[ 1 - \exp \left\{ -\frac{2T}{q(T)} \right\} \right] + \frac{1}{4} \frac{q(T)^4}{T^4} \left[ 1 - \exp \left\{ -\frac{2T}{q(T)} \right\} \right]^2 \right) \\
\times \left[ 1 + O \left( \frac{1}{T} \right) \right] \\
= O \left( \frac{1}{N} \right).
\]

Since the Liapounov condition implies the Lindeberg-type condition (13), we deduce from Lemma SE-10 that

\[
U_{Z,N,T} = \frac{1}{\omega_{Z,N,T}} \sum_{i=1}^{N} Z_{i,T} \Rightarrow Z \equiv N(0,1),
\]

as \( N,T \to \infty \).

Next, consider part (b), where we assume that \( \rho_T = \exp \{ -1/q(T) \} \) such that \( q(T) \to \infty \) but
\(q(T)/T \to 0\). In this case, using part (c) of Lemma SE-1 with \(b = 2\) and \(d = 2\), we see that

\[
\omega^2_{Z,N,T} = \sigma^4 N \frac{1}{T^2} \sum_{t=2}^{T} \sum_{j=1}^{t-1} \exp \left\{ -\frac{t - 1 - j}{q(T)} \right\}
\]

\[
= \sigma^4 N \frac{T q(T)}{2 T^2} \left[1 + O \left( \frac{1}{q(T)} \right) + O \left( \frac{q(T)}{T} \right) \right]
\]

\[
= \frac{\sigma^4 N q(T)}{2} \left[1 + O \left( \frac{1}{q(T)} \right) + O \left( \frac{q(T)}{T} \right) \right],
\]

so that

\[
\frac{T}{N q(T)} \omega^2_{Z,N,T} = \frac{\sigma^4}{2} \left[1 + O \left( \frac{1}{q(T)} \right) + O \left( \frac{q(T)}{T} \right) \right].
\]

It then follows from Assumption 1 that there exists a positive constant \(C\) such that

\[
0 < \frac{1}{C} \leq \sqrt{\frac{T}{N q(T)} \omega_{Z,N,T}} \leq C < \infty \quad \text{eventually as } N, T \to \infty.
\]

Moreover, by Assumptions 1 and 4,

\[
E \left[ \left( \frac{1}{\sqrt{N T q(T)}} \sum_{i=1}^{N} \sum_{t=2}^{T} \varepsilon_i t \rho_{T-1} w_{i0} \right)^2 \right] = \frac{1}{N T q(T)} \sum_{i=1}^{N} \sum_{j=1}^{T} \sum_{t=2}^{T} \rho_{t-1} \rho_{T-t} E \left[ w_{i0} w_{j0} \right] E \left[ \varepsilon_i \varepsilon_j \right]
\]

\[
= \frac{\sigma^2}{N T q(T)} \sum_{i=1}^{N} E \left[ w_{i0}^2 \right] \sum_{t=2}^{T} \rho_{t}^{2(t-1)}
\]

\[
= \frac{\sigma^2}{q(T) T} \left( \sup_i E \left[ w_{i0}^2 \right] \right) \exp \left\{ -\frac{2}{q(T)} \right\} \left[1 - \exp \left\{ -\frac{2}{q(T)} \right\} \right]^{-1} \left[1 - \exp \left\{ -\frac{2(T - 1)}{q(T)} \right\} \right]
\]

\[
= O \left( q(T)^{-1} T^{-1} \right) \times O(1) \times O(1) \times O(q(T)) \times O(1) = O \left( T^{-1} \right),
\]

from which it follows, by Markov’s inequality, that

\[
\frac{1}{N T q(T)} \sum_{i=1}^{N} \sum_{t=2}^{T} \varepsilon_i t \rho_{t}^{-1} w_{i0} = O_p \left( \frac{1}{\sqrt{T}} \right).
\]

Hence,

\[
\frac{1}{\omega_{Z,N,T}/\sqrt{N q(T)}} \frac{1}{\sqrt{N T q(T)}} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{i0} \varepsilon_i t \rho_{t-1} w_{i0}
\]

\[
= \frac{1}{\omega_{Z,N,T}/\sqrt{N q(T)}} \frac{1}{\sqrt{N T q(T)}} \sum_{i=1}^{N} \sum_{t=g+2}^{T} w_{i0} \varepsilon_i t \rho_{t-1} w_{i0}
\]

\[
+ \frac{1}{\omega_{Z,N,T}/\sqrt{N q(T)}} \frac{1}{\sqrt{N T q(T)}} \sum_{i=1}^{N} \sum_{t=g+2}^{T} \varepsilon_i t \rho_{t-1} w_{i0}
\]

\[
= \frac{1}{\omega_{Z,N,T}/\sqrt{N q(T)}} \sqrt{\frac{T}{N q(T)}} \sum_{i=1}^{N} Z_i \varepsilon + O_p \left( \frac{1}{\sqrt{T}} \right).
\]
Thus, showing the desired result is equivalent to showing the asymptotic normality of

\[ U_{Z,N,T} = \frac{1}{\omega_{Z,N,T}} \frac{T}{\sqrt{Nq(T)/T}} \sqrt{T Nq(T)} \sum_{i=1}^{N} Z_{i,T}, \]

To show the asymptotic normality of \( U_{N,T} \), it suffices to verify a Liapounov-type condition of the form

\[ \lim_{N,T \to \infty} \left( \frac{T}{Nq(T)} \right)^2 \sum_{i=1}^{N} E \left[ Z_{i,T}^4 \right] = 0. \]

Now, similar to part (a) above, we have that, given Assumption 1, there exists a positive constant \( C \) such that

\[
E \left[ Z_{i,T}^4 \right] \leq C \left( \frac{1}{T^4} \frac{Tq(T)}{4} \left[ 1 + O \left( \frac{1}{q(T)} \right) + O \left( \frac{q(T)}{T} \right) \right] + \frac{4}{T^4} \frac{Tq(T)^2}{4} \left[ 1 + O \left( \frac{1}{q(T)} \right) + O \left( \frac{q(T)}{T} \right) \right] \right)
\]

\[
+ \frac{2}{T^4} \frac{T^2q(T)^2}{8} \left[ 1 + O \left( \frac{1}{q(T)} \right) + O \left( \frac{q(T)}{T} \right) \right] \]

\[
= C \frac{1}{4} \frac{q(T)^2}{T^2} \left[ 1 + \frac{4}{T} + \frac{1}{Tq(T)} \right] \left[ 1 + O \left( \frac{1}{q(T)} \right) + O \left( \frac{q(T)}{T} \right) \right]
\]

\[
= C \frac{1}{4} \frac{q(T)^2}{T} \left[ 1 + O \left( \frac{1}{q(T)} \right) + O \left( \frac{q(T)}{T} \right) \right].
\]

Hence,

\[
\left( \frac{T}{Nq(T)} \right)^2 \sum_{i=1}^{N} E \left[ Z_{i,T}^4 \right] \leq C \frac{1}{4N} \left[ 1 + O \left( \frac{1}{q(T)} \right) + O \left( \frac{q(T)}{T} \right) \right] = O \left( \frac{1}{N} \right).
\]

Since the Liapounov condition implies the Lindeberg-type condition (13), we deduce from Lemma SE-10 that

\[ U_{Z,N,T} = \frac{1}{\omega_{Z,N,T}} \frac{T}{\sqrt{Nq(T)/T}} \sqrt{T Nq(T)} \sum_{i=1}^{N} Z_{i,T} \Rightarrow Z \equiv N (0, 1), \]

as \( N, T \to \infty \). By the Cramér convergence theorem, we then have that

\[
\sqrt{T} \sum_{i=1}^{N} Z_{i,T} = \frac{1}{\sqrt{NTq(T)}} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1,t} \varepsilon_{it} \Rightarrow N \left( 0, \frac{\sigma^4}{2} \right),
\]

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as \( N, T \to \infty \).

To show part (d), we first define

\[
Q_{i,T} = \sqrt{1 - \frac{\rho_T^2}{T}} \sum_{t=2}^{T} w_{it-1,T} \varepsilon_{it},
\]

and note that

\[
E[Q_{i,T}] = \sqrt{1 - \frac{\rho_T^2}{T}} \sum_{t=2}^{T} E[w_{it-1,T} \varepsilon_{it}] = \sqrt{1 - \frac{\rho_T^2}{T}} \sum_{t=2}^{T} \sum_{j=1}^{t-1} \rho_T^{(t-1-j)} E[\varepsilon_{ij} \varepsilon_{it}] = 0,
\]

\[
E[Q_{i,T}^2] = \frac{1 - \rho_T^2}{T} \sum_{t=2}^{T} \sum_{s=2}^{T} E[w_{it-1,T} w_{is-1,T} \varepsilon_{it} \varepsilon_{is}]
\]

\[
= \frac{1 - \rho_T^2}{T} \sum_{t=2}^{T} \sum_{s=2}^{T} \sum_{j=1}^{t-1} \sum_{k=1}^{s-1} \rho_T^{(t-1-j)} \rho_T^{(s-1-k)} E[\varepsilon_{ij} \varepsilon_{ik} \varepsilon_{it} \varepsilon_{is}]
\]

\[
= \sigma^4 \frac{1 - \rho_T^2}{T} \sum_{t=2}^{T} \sum_{j=1}^{t-1} \rho_T^{2(t-1-j)}.
\]

It follows that

\[
\omega_{Q,N,T}^2 = \sum_{i=1}^{N} E[Q_{i,T}^2] = \sigma^4 N \frac{1 - \rho_T^2}{T} \sum_{t=2}^{T} \sum_{j=1}^{t-1} \rho_T^{2(t-1-j)}
\]

\[
= \sigma^4 N \left( \frac{1 - \rho_T^2}{T} \sum_{t=2}^{T} \left[ \frac{1 - \rho_T^{2(t-1)}}{1 - \rho_T^2} \right] \right)
\]

\[
= \sigma^4 N \left( 1 - \frac{1}{T} - \frac{\rho_T^2 (1 - \rho_T^{2(T-1)})}{T (1 - \rho_T^2)} \right).
\]

Since, in this case, we take \( \rho_T^2 = \exp\{-2/q(T)\} \) with \( q(T) = O(1) \), it follows that there exist a positive constant \( C_q \) and a positive integer \( T^* \) such that for all \( T \geq T^* \)

\[
\rho_T^2 \leq \exp\left\{-\frac{2}{C_q}\right\} < 1.
\]

Using this bound, we further deduce that all \( T \geq T^* \)

\[
0 \leq \frac{\rho_T^2 (1 - \rho_T^{2(T-1)})}{T (1 - \rho_T^2)} \leq \frac{1}{T (1 - \exp\{-2/C_q\})} = O\left(\frac{1}{T}\right),
\]

so that

\[
\frac{\omega_{Q,N,T}^2}{N} = \sigma^4 \left[ 1 + O\left(\frac{1}{T}\right) \right].
\]

Thus, it follows from Assumption 1 that there exists a positive constant \( C \) such that

\[
0 < \frac{1}{C} \leq \frac{\omega_{Q,N,T}}{\sqrt{N}} \leq C < \infty \quad \text{eventually as } N, T \to \infty.
\]

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Moreover, by Assumptions 1 and 4,

\[
E \left[ \left( \sqrt{\frac{1 - \rho_T^2}{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} \varepsilon_{it} \rho_T^{t-1} w_{i0} \right)^2 \right] \\
= \frac{1 - \rho_T^2}{NT} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{t=2}^{T} \rho_T^{t-1} \rho_T^{s-1} E [w_{i0} w_{j0}] E [\varepsilon_{it} \varepsilon_{js}] \\
= \frac{\sigma^2 (1 - \rho_T^2)}{NT} \sum_{i=1}^{N} E [w_{i0}^2] \sum_{t=2}^{T} \rho_T^{2(t-1)} \\
= \frac{\sigma^2 (1 - \rho_T^2)}{T} \left( \sup_i E [w_{i0}^2] \right) \frac{\rho_T^2 (1 - \rho_T^{2(T-1)})}{1 - \rho_T^2} \\
= O \left( T^{-1} \right) \times O (1) \times O (1) = O \left( T^{-1} \right) ,
\]

from which it follows, by Markov’s inequality, that

\[
\sqrt{\frac{1 - \rho_T^2}{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} \varepsilon_{it} \rho_T^{t-1} w_{i0} = O_p \left( \frac{1}{\sqrt{T}} \right) .
\]

Hence,

\[
\frac{1}{\omega_{Q,N,T}/\sqrt{N}} \sqrt{\frac{1 - \rho_T^2}{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1,T} \varepsilon_{it} \\
= \frac{1}{\omega_{Q,N,T}/\sqrt{N}} \sqrt{\frac{1 - \rho_T^2}{NT}} \sum_{i=1}^{N} \sum_{t=g+2}^{T} w_{it-1,T} \varepsilon_{it} + \frac{1}{\omega_{Q,N,T}/\sqrt{N}} \sqrt{\frac{1 - \rho_T^2}{NT}} \sum_{i=1}^{N} \sum_{t=g+2}^{T} \varepsilon_{it} \rho_T^{t-1} w_{i0} \\
= \frac{1}{\omega_{Q,N,T}/\sqrt{N}} \sqrt{\frac{1}{N}} \sum_{i=1}^{N} Q_{i,T} + O_p \left( \frac{1}{\sqrt{T}} \right) .
\]

Thus, showing the desired result is equivalent to showing the asymptotic normality of

\[
U_{Q,N,T} = \frac{1}{\omega_{Q,N,T}/\sqrt{N}} \sqrt{\frac{1}{N}} \sum_{i=1}^{N} Q_{i,T} = \frac{1}{\omega_{Q,N,T}/\sqrt{N}} \sqrt{\frac{1 - \rho_T^2}{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1} \varepsilon_{it} .
\]

To show the asymptotic normality of \( U_{N,T} \), it suffices to verify a Liapounov-type condition of the form

\[
\lim_{N,T \to \infty} \frac{1}{N^2} \sum_{i=1}^{N} E [Q_{i,T}^4] = 0.
\]

To proceed, note that following calculations similar to that given in the proof of Lemma SE-23, we have
that, in light of Assumption 1, there exists a positive constant $C$ such that

$$
E[Q^4_{i,T}] = E \left[ \left( \frac{1 - \rho_T^2}{T} \sum_{t=2}^{T} w_{i,t-1,T} \epsilon_{it} \right)^4 \right] 
$$

$$
= \frac{1 - \rho_T^2}{T^2} \sum_{t=2}^{T} \sum_{j=1}^{t-1} \rho_T^{4(t-1-j)} + 4 \frac{1 - \rho_T^2}{T^2} \sum_{t=2}^{T} \sum_{s=2}^{t-1} \sum_{j=1}^{s-1} \rho_T^{2(t-1-s)} \rho_T^{2(s-1-j)} 
$$

$$
+ 2 \frac{1 - \rho_T^2}{T^2} \sum_{t=2}^{T} \sum_{s=2}^{t-1} \sum_{j=1}^{s-1} \sum_{k=1}^{s-k} \rho_T^{2(t-1-j)} \rho_T^{2(s-1-k)} 
$$

$$
= C (I + II + III), \text{ (say).}
$$

Moreover, using the upper bound $\rho_T^2 \leq \exp \{-2/C_q\} < 1$ for all $T \geq T^*$ as above, we have

$$
I = \frac{(1 - \rho_T^2)^2}{T^2} \sum_{t=2}^{T} \sum_{j=1}^{t-1} \rho_T^{4(t-1-j)} = \frac{(1 - \rho_T^2)^2}{T^2} \sum_{t=2}^{T} \frac{1 - \rho_T^{4(t-1)}}{1 - \rho_T^4} 
$$

$$
= \frac{(1 - \rho_T^2)^2}{T^2} \frac{1}{1 - \rho_T^4} \left[ T - \frac{\rho_T^4 \left( 1 - \rho_T^{4(T-1)} \right)}{1 - \rho_T^4} \right] 
$$

$$
= \frac{1}{T^2} \frac{1}{1 + \rho_T^2} \left[ (1 - \rho_T^2) (T - 1) - \frac{\rho_T^4 \left( 1 - \rho_T^{4(T-1)} \right)}{1 + \rho_T^2} \right] 
$$

$$
\leq \frac{1}{T} \left[ 1 + \frac{1}{2T} \right] = O \left( \frac{1}{T} \right),
$$
$$II = 4 \left(1 - \frac{\rho_T^2}{T^2}\right)^2 \sum_{t=3}^{T} \sum_{s=2}^{t-1} \sum_{j=1}^{s-1} \frac{\rho_T^{2(t-1-s)}}{\rho_T^{2(s-1-j)}}$$

$$= 4 \left(1 - \frac{\rho_T^2}{T^2}\right)^2 \sum_{t=3}^{T} \sum_{s=2}^{t-1} \rho_T^{2(t-1-s)} \frac{1 - \rho_T^{2(s-1)}}{1 - \rho_T^2}$$

$$= 4 \left(1 - \frac{\rho_T^2}{T^2}\right)^2 \frac{1}{1 - \rho_T^2} \sum_{t=3}^{T} \sum_{s=2}^{t-1} \rho_T^{2(t-1-s)} - 4 \left(1 - \frac{\rho_T^2}{T^2}\right)^2 \frac{1}{1 - \rho_T^2} \sum_{t=3}^{T} \sum_{s=2}^{t-1} \rho_T^{2(t-2)}$$

$$= 4 \left(1 - \frac{\rho_T^2}{T^2}\right)^2 \frac{1}{(1 - \rho_T^2)^2} \sum_{t=3}^{T} \left(1 - \rho_T^{2(t-2)}\right) - 4 \left(1 - \frac{\rho_T^2}{T^2}\right)^2 \frac{1}{1 - \rho_T^2} \sum_{t=3}^{T} \rho_T^{2(t-2)} (t - 2)$$

(by part (b) of Lemma SE-5)

$$= 4 \frac{T - 2}{T^2} - 4 \frac{1}{T^2} \rho_T^2 \left(1 - \rho_T^{2(T-2)}\right) - 4 \left(1 - \frac{\rho_T^2}{T^2}\right) \rho_T^2 - (T - 1) \rho_T^{2(T-1)} + (T - 2) \rho_T^{2T} \left(1 - \rho_T^2\right)^2$$

$$= \frac{4}{T} \left[1 + \frac{2}{1 - \exp\{-2/C_q\}}\right] + \frac{4}{T^2} \left[2 + \frac{5}{1 - \exp\{-2/C_q\}}\right]$$

$$= O \left(\frac{1}{T}\right),$$
and

\[ III = 2 \frac{(1 - \rho_T^2)^2}{T^2} \sum_{s=2}^{\infty} \sum_{j=1}^{s-1} \sum_{k=1}^{t-1} \rho_T^{2(t-1-j)} \rho_T^{2(s-1-k)} \]

\[ = \frac{(1 - \rho_T^2)^2}{T^2} \cdot \frac{2}{1 - \rho_T^2} \sum_{s=2}^{T} \left( 1 - \rho_T^{2(t-1)} \right) \sum_{s=2}^{t-1} \left( 1 - \rho_T^{2(s-1)} \right) \]

\[ = \frac{2}{T^2} \sum_{t=3}^{T} \left( 1 - \rho_T^{2(t-1)} \right) \left[ (t - 2) - \frac{\rho_T^2 (1 - \rho_T^{2(t-2)})}{(1 - \rho_T^2)^2} \right] \]

\[ + \frac{2}{T^2} \rho_T^4 \left( 1 - \rho_T^{2(T-2)} \right) \frac{(1 - \rho_T)^2}{(1 - \rho_T^2)^2} \]

\[ = \frac{(T - 2) (T - 1)}{T^2} \cdot \frac{2 \rho_T^2 (T - 1)}{T^2} \left( 1 - \rho_T^{2(T-1)} \right) + (T - 2) \left( 1 - \rho_T^2 \right) \left( 1 - \rho_T^4 \right) \]

\[ = 1 - \frac{3}{T} \cdot \frac{2 \rho_T^2}{T (1 - \rho_T^2)} + \frac{2 \rho_T^2}{T (1 - \rho_T^2)} - \frac{2 \rho_T^2}{T (1 - \rho_T^2)^2} \]

\[ = 1 + \frac{1}{T} \left[ 3 + \frac{2}{1 - \exp \{-2/C_q\}} \right] + \frac{4}{1 - \exp \{-2/C_q\}^2} \]

\[ + \frac{2}{T^2} \left[ 1 + \frac{2}{1 - \exp \{-2/C_q\}} + \frac{6}{(1 - \exp \{-2/C_q\})^2} + \frac{2}{(1 - \exp \{-2/C_q\})(1 - \exp \{-4/C_q\})} \right] \]

\[ = O(1), \]

from which we deduce that

\[ \frac{1}{N^2} \sum_{i=1}^{N} E \left[ Q_i^4 \right] = O \left( \frac{1}{N} \right) = o(1). \]

Since the Liapounov condition implies the Lindeberg-type condition (13), we deduce from Lemma SE-10 that

\[ U_{Q,N,T} = \frac{1}{\omega_{Q,N,T} / \sqrt{N}} \sqrt{\frac{1 - \rho_T^2}{N^T}} \sum_{i=1}^{T} \sum_{j=1}^{T} w_{ij} \xi _{ij} \Rightarrow Z \equiv N(0,1) \]
as \( N, T \to \infty \). By the Cramér convergence theorem, we then have that

\[
\sqrt{\frac{1 - \rho_T^2}{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1} \varepsilon_{it} = \frac{\omega_{Q,N,T}}{\sqrt{N}} U_{Q,N,T} = \sigma^2 U_{Q,N,T} \left[ 1 + O \left( \frac{1}{T} \right) \right] \Rightarrow N \left( 0, \sigma^4 \right)
\]
as required. \( \Box \)

**Lemma SE-28:**

Under Assumptions 1 and 4, the following statements are hold as \( N, T \to \infty \).

(a) If \( \rho_T = 1 \) for all \( T \) sufficiently large, then

\[
\sum_{i=1}^{N} w_{i,-1}^2 = O_p \left( NT \right).
\]

(b) If \( \rho_T = \exp \left\{ -1/(q(T)) \right\} \) such that \( T/q(T) \to 0 \), then

\[
\sum_{i=1}^{N} w_{i,-1}^2 = O_p \left( NT \right).
\]

(c) If \( \rho_T = \exp \left\{ -1/(q(T)) \right\} \) such that \( q(T) \sim T \), then

\[
\sum_{i=1}^{N} w_{i,-1}^2 = O_p \left( Nq(T) \right) = O_p \left( NT \right).
\]

(d) If \( \rho_T = \exp \left\{ -1/(q(T)) \right\} \) such that \( q(T) \to \infty \) but \( q(T)/T \to 0 \), then

\[
\sum_{i=1}^{N} w_{i,-1}^2 = O_p \left( \frac{Nq(T)^2}{T} \right).
\]

(e) If \( \rho_T \in \mathcal{G}_{\dagger} = \left\{ |\rho_T| = \exp \left\{ -1/q(T) \right\} : q(T) \geq 0 \text{ and } q(T) = O(1) \text{ as } T \to \infty \right\} \), then

\[
\sum_{i=1}^{N} w_{i,-1}^2 = O_p \left( \frac{N}{T} \right).
\]

**Proof of Lemma SE-28:**

To proceed, note that

\[
\sum_{i=1}^{N} w_{i,-1}^2 = \sum_{i=1}^{N} \left( \frac{1}{T-1} \sum_{t=2}^{T} w_{it-1,T} \right)^2
\]

\[
= \sum_{i=1}^{N} \left( \frac{1}{T-1} \sum_{t=2}^{T} w_{it-1,T} + \frac{1}{T-1} \sum_{t=2}^{T} \rho_{t-1} w_{i0} \right)^2
\]

\[
= \sum_{i=1}^{N} \left( \frac{1}{T-1} \sum_{t=2}^{T} w_{it-1,T} \right)^2 + 2 \sum_{i=1}^{N} \left( \frac{1}{T-1} \sum_{t=2}^{T} w_{it-1,T} \right) \left( \frac{1}{T-1} \sum_{t=2}^{T} \rho_{t-1} w_{i0} \right)
\]

\[
+ \sum_{i=1}^{N} \left( \frac{1}{T-1} \sum_{t=2}^{T} \rho_{t-1} w_{i0} \right)^2,
\]

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where \( w_{t-1,T} = \sum_{j=1}^{t-1} \rho_t^{t-1-j} \varepsilon_{ij} \).

Consider first part (a). Note that, under the assumption here, there exists a positive integer \( I_\rho \) such that for all \( T \geq I_\rho \), the triangular array process \( \{ w_{t,T} \} \) has the partial sum representation \( w_{t,T} = \sum_{j=1}^{t} \varepsilon_{ij} \). Hence, for all \( T \geq \max \{ I_\rho, 3 \} \), we obtain by direct calculation

\[
E \left[ \sum_{i=1}^{N} \left( \frac{1}{T-1} \sum_{t=2}^{T} w_{t-1,T} \right)^2 \right] = \frac{1}{(T-1)^2} \sum_{i=1}^{N} \sum_{s=2}^{T} \sum_{t=2}^{T} \sum_{s=1}^{T} \sum_{v=1}^{T} E[\varepsilon_{ig}\varepsilon_{iv}] \\
= \frac{\sigma^2 N (T-1)^2}{(T-1)^2} \sum_{i=1}^{N} \sum_{s=2}^{T} \sum_{t=2}^{T} \sum_{s=1}^{T} \sum_{v=1}^{T} E[\varepsilon_{ig}\varepsilon_{iv}] \\
= \frac{\sigma^2 N (T-1)^2}{2} + \frac{2\sigma^2 N (T-1)^2}{2} \sum_{i=1}^{N} \sum_{s=2}^{T} \sum_{t=2}^{T} \sum_{s=1}^{T} \sum_{v=1}^{T} E[\varepsilon_{ig}\varepsilon_{iv}] \\
= \frac{\sigma^2}{3} NT \left[ 1 + O \left( \frac{1}{T} \right) \right].
\]

Applying Markov’s inequality, we deduce that

\[
\sum_{i=1}^{N} \left( \frac{1}{T-1} \sum_{t=2}^{T} w_{t-1,T} \right)^2 = O_p (NT).
\]

Moreover, by Assumption 4,

\[
E \left[ \sum_{i=1}^{N} \left( \frac{1}{T-1} \sum_{t=2}^{T} w_{t,0} \right)^2 \right] \leq N \left( \sup_{i} E \left[ w_{i0}^2 \right] \right) = O (N)
\]

from which it follows, by Markov’s inequality, that

\[
\sum_{i=1}^{N} \left( \frac{1}{T-1} \sum_{t=2}^{T} w_{t,0} \right)^2 = O_p (N).
\]

Also, by the Cauchy-Schwarz inequality,

\[
\left| \sum_{i=1}^{N} \left( \frac{1}{T-1} \sum_{t=2}^{T} w_{t-1,T} \right) \left( \frac{1}{T-1} \sum_{t=2}^{T} w_{t,0} \right) \right| \leq \left( \sum_{i=1}^{N} \left( \frac{1}{T-1} \sum_{t=2}^{T} w_{t-1,T} \right)^2 \right)^{1/2} \left( \sum_{i=1}^{N} \left( \frac{1}{T-1} \sum_{t=2}^{T} w_{t,0} \right)^2 \right)^{1/2} = O_p (\sqrt{NT}) O_p (\sqrt{N}) = O_p \left( N \sqrt{T} \right).
\]
Hence,

$$\sum_{i=1}^{N} \frac{w_{i,-1}^2}{w_i} = \sum_{i=1}^{N} \left( \frac{1}{T-1} \sum_{t=2}^{T} w_{it-1,T} \right)^2 + 2 \sum_{i=1}^{N} \left( \frac{1}{T-1} \sum_{t=2}^{T} w_{it-1,T} \right) \left( \frac{1}{T-1} \sum_{t=2}^{T} w_{i0} \right)$$

$$+ \sum_{i=1}^{N} \left( \frac{1}{T-1} \sum_{t=2}^{T} w_{i0} \right)^2 \quad \text{(for all } T \text{ sufficiently large)}$$

$$= O_p(NT) + O_p\left( N\sqrt{T} \right) + O_p(N) = O_p(NT)$$

as required for part (a).

Now, to show parts (b)-(d), we first make some preliminary moment calculations. Note that

$$E \left[ \sum_{i=1}^{N} \left( \frac{1}{T-1} \sum_{t=2}^{T} w_{is-1,T} \right)^2 \right]$$

$$= \frac{1}{(T-1)^2} \sum_{i=1}^{N} \sum_{t=2}^{T} \sum_{s=2}^{T} \sum_{g=1}^{s-1} \sum_{v=1}^{t-1} \exp \left\{ -s - 1 - g \over q(T) \right\} \exp \left\{ -t - 1 - v \over q(T) \right\} E[\xi_i \xi_v]$$

$$= \frac{\sigma^2}{(T-1)^2} \sum_{i=1}^{N} \sum_{t=2}^{T} \sum_{s=2}^{T} \sum_{g=1}^{s-1} \exp \left\{ -2t - 1 - v \over q(T) \right\}$$

$$+ 2 \frac{(T-1)^2}{(T-1)^2} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{s=2}^{T} \sum_{g=1}^{s-1} \exp \left\{ -s - 1 - g \over q(T) \right\} \exp \left\{ -t - 1 - g \over q(T) \right\}$$

$$= \frac{\sigma^2 N}{(T-1)^2} \sum_{t=2}^{T} \sum_{s=2}^{s-1} \exp \left\{ -2t - 1 - v \over q(T) \right\}$$

$$+ 2 \frac{\sigma^2 N}{(T-1)^2} \sum_{t=3}^{T} \sum_{s=2}^{s-1} \sum_{g=1}^{s-1} \exp \left\{ -s - 1 - g \over q(T) \right\} \exp \left\{ -t - 1 - g \over q(T) \right\}.$$

Next, consider part (b), where we take $\rho_T = \exp \{-1/q(T)\}$ such that $T/q(T) \to 0$. Applying part (a) of Lemma SE-1 with $b = 2$ and $d = 2$ and part (a) of Lemma SE-7 with $b = 1$ and $g = 1$, we obtain

$$E \left[ \sum_{i=1}^{N} \left( \frac{1}{T-1} \sum_{s=2}^{T} w_{is-1,T} \right)^2 \right]$$

$$= \frac{\sigma^2 N}{(T-1)^2} \sum_{t=2}^{T} \sum_{s=2}^{s-1} \exp \left\{ -2t - 1 - v \over q(T) \right\}$$

$$+ 2 \frac{\sigma^2 N}{(T-1)^2} \sum_{t=3}^{T} \sum_{s=2}^{s-1} \sum_{g=1}^{s-1} \exp \left\{ -s - 1 - g \over q(T) \right\} \exp \left\{ -t - 1 - g \over q(T) \right\}$$

$$= \frac{\sigma^2 N}{(T-1)^2} T^2 \left[ 1 + O\left( \frac{1}{T} \right) \right] + 2 \frac{\sigma^2 N}{(T-1)^2} T^3 \left[ 1 + O\left( \frac{T}{q(T)} \right) + O\left( \frac{1}{T} \right) \right]$$

$$= \frac{\sigma^2}{3NT} \left[ 1 + O\left( \frac{T}{q(T)} \right) + O\left( \frac{1}{T} \right) \right] = O(NT).$$
Using Markov's inequality, we deduce that
\[
\sum_{i=1}^{N} \left( \frac{1}{T-1} \sum_{s=2}^{T} w_{i,s-1,T} \right)^2 = O_p(NT) .
\]

Moreover, by Assumption 4,
\[
E \left[ \sum_{i=1}^{N} \left( \frac{1}{T-1} \sum_{t=2}^{T} \rho_{T-1}^{t-1} w_{i0} \right)^2 \right] \\
\leq \frac{N}{(T-1)^2} \left( \sup_{i} E \left[ w_{i0}^2 \right] \right) \exp \left\{ -\frac{2}{q(T)} \right\} \left[ 1 - \exp \left\{ -\frac{1}{q(T)} \right\} \right]^2 \left[ 1 - \exp \left\{ -\frac{T-1}{q(T)} \right\} \right]^2 \\
= O(NT^{-2}) O(1) O(1) O(q(T)^2) O(T^2/q(T)^2) = O(N)
\]
from which it follows by Markov’s inequality that
\[
\sum_{i=1}^{N} \left( \frac{1}{T-1} \sum_{t=2}^{T} \rho_{T-1}^{t-1} w_{i0} \right)^2 = O_p(N) .
\]

Also, by the Cauchy-Schwarz inequality,
\[
\left| \sum_{i=1}^{N} \left( \frac{1}{T-1} \sum_{t=2}^{T} w_{i,t-1,T} \right) \left( \frac{1}{T-1} \sum_{t=2}^{T} \rho_{T-1}^{t-1} w_{i0} \right) \right| \\
\leq \sqrt{\sum_{i=1}^{N} \left( \frac{1}{T-1} \sum_{s=2}^{T} w_{i,s-1,T} \right)^2} \sqrt{\sum_{i=1}^{N} \left( \frac{1}{T-1} \sum_{t=2}^{T} \rho_{T-1}^{t-1} w_{i0} \right)^2} \\
= O_p\left( \sqrt{NT} \right) O_p\left( \sqrt{N} \right) = O_p\left( N\sqrt{T} \right)
\]
so that
\[
\sum_{i=1}^{N} w_{i,-1}^2 = \sum_{i=1}^{N} \left( \frac{1}{T-1} \sum_{t=2}^{T} w_{i,t-1,T} \right)^2 + 2 \sum_{i=1}^{N} \left( \frac{1}{T-1} \sum_{t=2}^{T} w_{i,t-1,T} \right) \left( \frac{1}{T-1} \sum_{t=2}^{T} \rho_{T-1}^{t-1} w_{i0} \right) \\
+ \sum_{i=1}^{N} \left( \frac{1}{T-1} \sum_{t=2}^{T} \rho_{T-1}^{t-1} w_{i0} \right)^2 \\
= O_p(NT) + O_p\left( N\sqrt{T} \right) + O_p\left( N \right) = O_p(NT) ,
\]
as required for part (b).

To show part (c), where we take \( \rho_T = \exp \{ -1/q(T) \} \) such that \( q(T) \sim T \), we apply part (b) of
Lemma SE-1 with $b = 2$ and $d = 2$ and part (b) of Lemma SE-7 with $b = 1$ to obtain

\[
E \left[ \sum_{i=1}^{N} \left( \frac{1}{T-1} \sum_{s=2}^{T} w_{is-1,t} \right)^2 \right] = \frac{\sigma^2 N}{(T-1)^2} \sum_{t=2}^{T} \sum_{v=1}^{T-1} \exp \left\{ -\frac{t - 1 - v}{q(T)} \right\} + 2 \frac{\sigma^2 N}{(T-1)^2} \sum_{t=3}^{T} \sum_{s=2}^{t-1} \sum_{g=1}^{T-1} \exp \left\{ -\frac{s - 1 - g}{q(T)} \right\} \exp \left\{ -\frac{t - 1 - g}{q(T)} \right\}
\]

\[
= \frac{\sigma^2 N}{(T-1)^2} \frac{q(T)^2}{4} \left[ \exp \left\{ -\frac{2T}{q(T)} \right\} + \frac{2T}{q(T)} - 1 \right] \left[ 1 + O \left( \frac{1}{T} \right) \right] + 2 \frac{\sigma^2 N}{(T-1)^2} \frac{Tq(T)^2}{2} \left[ 1 - \frac{3q(T)}{2T} + 2 \frac{q(T)}{T} \exp \left\{ -\frac{T}{q(T)} \right\} - \frac{1}{2} \frac{q(T)}{T} \exp \left\{ -\frac{2T}{q(T)} \right\} \right] \left[ 1 + O \left( \frac{1}{T} \right) \right]
\]

\[
= \sigma^2 N q(T) \frac{q(T)}{T} \left[ 1 - \frac{3q(T)}{2T} + 2 \frac{q(T)}{T} \exp \left\{ -\frac{T}{q(T)} \right\} - \frac{1}{2} \frac{q(T)}{T} \exp \left\{ -\frac{2T}{q(T)} \right\} \right] \left[ 1 + O \left( \frac{1}{T} \right) \right]
\]

\[
= O \left( Nq(T) \right).
\]

Using Markov’s inequality, we deduce that

\[
\sum_{i=1}^{N} \left( \frac{1}{T-1} \sum_{s=2}^{T} w_{is-1,t} \right)^2 = O_p \left( Nq(T) \right) = O_p \left( NT \right).
\]

Moreover, by Assumption 4,

\[
E \left[ \sum_{i=1}^{N} \left( \frac{1}{T-1} \sum_{t=2}^{T} \rho_T^{t-1} w_{i0} \right)^2 \right]
\]

\[
\leq \frac{N}{(T-1)^2} \left( \sup_i E \left[ w_{i0}^2 \right] \right) \exp \left\{ -\frac{2}{q(T)} \right\} \left[ 1 - \exp \left\{ -\frac{1}{q(T)} \right\} \right] \leq \left[ 1 - \exp \left\{ -\frac{1}{q(T)} \right\} \right]^2 \leq O \left( NT^{-2} \times O(1) \times O(1) \times O(T^2) \times O(1) = O \left( N \right) \right).
\]

from which it follows, by Markov’s inequality, that

\[
\sum_{i=1}^{N} \left( \frac{1}{T-1} \sum_{t=2}^{T} \rho_T^{t-1} w_{i0} \right)^2 = O_p \left( N \right).
\]

Also, by the Cauchy-Schwarz inequality,

\[
\left| \sum_{i=1}^{N} \left( \frac{1}{T-1} \sum_{t=2}^{T} w_{it-1,t} \right) \left( \frac{1}{T-1} \sum_{t=2}^{T} \rho_T^{t-1} w_{i0} \right) \right| \leq \sqrt{\sum_{i=1}^{N} \left( \frac{1}{T-1} \sum_{s=2}^{T} w_{is-1,t} \right)^2} \sqrt{\sum_{i=1}^{N} \left( \frac{1}{T-1} \sum_{t=2}^{T} \rho_T^{t-1} w_{i0} \right)^2}
\]

\[
= O_p \left( \sqrt{NT} \right) O_p \left( \sqrt{N} \right) = O_p \left( N \sqrt{T} \right).
\]

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so that
\[
\sum_{i=1}^{N} w_{i,-1}^2 = \sum_{i=1}^{N} \left( \frac{1}{T-1} \sum_{t=2}^{T} w_{i,t-1,t} \right)^2 + 2 \sum_{i=1}^{N} \left( \frac{1}{T-1} \sum_{t=2}^{T} w_{i,t-1,t} \right) \left( \frac{1}{T-1} \sum_{t=2}^{T} \rho_{t}^{-1} w_{i0} \right) \\
+ \sum_{i=1}^{N} \left( \frac{1}{T-1} \sum_{t=2}^{T} \rho_{t}^{-1} w_{i0} \right)^2 = O_p(NT) + O_p\left( N \sqrt{T} \right) + O_p(N) = O_p(NT).
\]

We turn our attention now to part (d), where we take \( \rho_T = \exp\{-1/q(T)\} \) such that \( q(T) \to \infty \) but \( q(T)/T \to 0 \). Applying part (c) of Lemma SE-1 with \( b = 2 \) and \( d = 2 \) and part (c) of Lemma SE-7 with \( b = 1 \), we obtain
\[
E \left[ \sum_{i=1}^{N} \left( \frac{1}{T-1} \sum_{s=2}^{T} w_{is-1,T} \right)^2 \right] = \frac{\sigma^2 N}{(T-1)^2} \sum_{t=2}^{T-1} \sum_{v=1}^{t-1} \exp \left\{ -2 t - 1 - v \right\} + 2 \frac{\sigma^2 N}{(T-1)^2} \sum_{s=2}^{T} \sum_{t=1}^{s-1} \sum_{g=1}^{s-1} \exp \left\{ -\frac{s - 1 - g}{q(T)} \right\} \exp \left\{ -\frac{t - 1 - g}{q(T)} \right\} = \frac{\sigma^2 N}{(T-1)^2} T q(T) \left[ 1 + O \left( \frac{1}{q(T)} \right) + O \left( \frac{q(T)}{T} \right) \right] + 2 \frac{\sigma^2 N}{(T-1)^2} T q(T)^2 \left[ 1 + O \left( \max \left\{ \frac{q(T)}{T}, \frac{1}{q(T)} \right\} \right) \right] = O \left( \frac{N q(T)^2}{T} \right).
\]

Using Markov’s inequality, we deduce that
\[
\sum_{i=1}^{N} \left( \frac{1}{T-1} \sum_{s=2}^{T} w_{is-1,T} \right)^2 = O_p \left( \frac{N q(T)^2}{T} \right).
\]

Moreover, note that, using Assumption 4, we obtain
\[
E \left[ \sum_{i=1}^{N} \left( \frac{1}{T-1} \sum_{t=2}^{T} \rho_{t}^{-1} w_{i0} \right)^2 \right] \leq \frac{N}{(T-1)^2} \left( \sup_{i} E \left[ w_{i0}^2 \right] \right) \exp \left\{ -\frac{2}{q(T)} \right\} \left[ 1 - \exp \left\{ -\frac{1}{q(T)} \right\} \right]^{-2} \left[ 1 - \exp \left\{ -\frac{T - 1}{q(T)} \right\} \right]^2 = O \left( N T^{-2} \right) \times O(1) \times O(1) \times O \left( \frac{q(T)^2}{T} \right) \times O(1) = O \left( N q(T)^2 / T^2 \right),
\]
so that it follows, by Markov’s inequality, that
\[
\sum_{i=1}^{N} \left( \frac{1}{T-1} \sum_{t=2}^{T} \rho_{t}^{-1} w_{i0} \right)^2 = O_p \left( \frac{N q(T)^2}{T^2} \right).
\]
Also, by the Cauchy-Schwarz inequality,

\[
\sum_{i=1}^{N} \left( \frac{1}{T-1} \sum_{t=2}^{T} w_{i,t-1,T} \right) \left( \frac{1}{T-1} \sum_{t=2}^{T} \rho_{t-1} w_{i0} \right) \leq \sqrt{\sum_{i=1}^{N} \left( \frac{1}{T-1} \sum_{t=2}^{T} w_{i,t-1,T} \right)^2} \sqrt{\sum_{i=1}^{N} \left( \frac{1}{T-1} \sum_{t=2}^{T} \rho_{t-1} w_{i0} \right)^2} = O_p \left( \sqrt{Nq(T)}T \right) O_p \left( \sqrt{Nq(T)}T \right) = O_p \left( \frac{Nq(T)^2}{T^{3/2}} \right).
\]

Hence,

\[
\sum_{i=1}^{N} w_{i,-1}^2 = \sum_{i=1}^{N} \left( \frac{1}{T-1} \sum_{t=2}^{T} w_{i,t-1,T} \right)^2 + 2 \sum_{i=1}^{N} \left( \frac{1}{T-1} \sum_{t=2}^{T} w_{i,t-1,T} \right) \left( \frac{1}{T-1} \sum_{t=2}^{T} \rho_{t-1} w_{i0} \right) + \sum_{i=1}^{N} \left( \frac{1}{T-1} \sum_{t=2}^{T} \rho_{t-1} w_{i0} \right)^2 = O_p \left( \frac{Nq(T)^2}{T} \right) + O_p \left( \frac{Nq(T)^2}{T^{3/2}} \right) + O_a.s. \left( \frac{Nq(T)^2}{T^{2}} \right) = O_p \left( \frac{Nq(T)^2}{T} \right).
\]

Finally, we consider part (e), where we take \( \rho_T \in G_{St} \). Since we assume here that \( q(T) = O(1) \), there exist some positive constant \( C_q \) and some positive integer \( T^* \) such that for all \( T \geq T^* \)

\[
0 \leq |\rho_T| = \exp \left\{ -\frac{1}{q(T)} \right\} \leq \exp \left\{ -\frac{1}{C_q} \right\} < 1.
\]
Using this bound, we obtain for all $T \geq T^*$

$$
E \left[ \frac{N}{T-1} \sum_{s=2}^{T} \frac{w_{s-1,T}}{T} \right]^2
= \frac{\sigma^2 N}{(T-1)^2} \sum_{t=2}^{T} \sum_{u=1}^{t-1} \frac{2(1-\rho_{T}^2)}{(1-\rho_{T}^2)^2} + \frac{\sigma^2 N}{(T-1)^2} \sum_{t=3}^{T} \sum_{s=2}^{t-1} \frac{t-1-(s-1)}{(1-\rho_{T}^2)^2} \sum_{g=1}^{s-1} \rho_{T}^{(s-1-g)} \rho_{T}^{(t-1-g)}
= \frac{\sigma^2 N}{(T-1)^2} \sum_{t=2}^{T} \frac{1-\rho_{T}^{2(t-1)}}{(1-\rho_{T}^2)^2} + \frac{\sigma^2 N}{(T-1)^2} \sum_{t=3}^{T} \sum_{s=2}^{t-1} \frac{t-1-(s-1)}{(1-\rho_{T}^2)^2} \sum_{g=1}^{s-1} \rho_{T}^{2(s-1-g)}
= \frac{N}{(T-1)^2} \frac{\sigma^2}{1-\rho_{T}^2} - \frac{N\sigma^2}{(T-1)^2} \frac{\rho_{T}^2}{(1-\rho_{T}^2)^2} + \frac{\sigma^2 N}{(T-1)^2} \sum_{t=3}^{T} \frac{\rho_{T}^{(t-2)}}{1-\rho_{T}^2} - \frac{\sigma^2 N}{(T-1)^2} \sum_{t=3}^{T} \frac{\rho_{T}^{(t-2)}}{1-\rho_{T}}
\leq \frac{N}{(T-1)^2} \frac{\sigma^2}{1-\exp\left\{-\frac{2}{C_q}\right\}} + \frac{N}{(T-1)^2} \frac{\sigma^2}{(1-\exp\left\{-\frac{2}{C_q}\right\})^2} + \frac{N}{(T-1)^2} \frac{2\sigma^2}{(1-\exp\left\{-\frac{2}{C_q}\right\})^2}
+ \frac{N}{(T-1)^2} \frac{\sigma^2}{(1-\exp\left\{-\frac{2}{C_q}\right\})^2} + \frac{N}{(T-1)^2} \frac{\sigma^2}{(1-\exp\left\{-\frac{2}{C_q}\right\})^3} + \frac{N}{(T-1)^2} \frac{2\sigma^2}{(1-\exp\left\{-\frac{2}{C_q}\right\})^3}
\leq O \left( \frac{N}{T} \right).
$$

Using Markov's inequality, we deduce that

$$
\sum_{i=1}^{N} \left( \frac{1}{T-1} \sum_{s=2}^{T} w_{s-1,T} \right)^2 = O_p \left( \frac{N}{T} \right).
$$
Moreover, note that, by Assumption 4,

\[
E \left[ \sum_{i=1}^{N} \left( \frac{1}{T-1} \sum_{t=2}^{T} \rho_{t-1}^{T} w_{i0} \right)^2 \right] \leq \frac{N}{(T-1)^2} \left( \sup_{i} E \left[ w_{i0}^2 \right] \right) \left( \sum_{t=2}^{T} |\rho_T|^{t-1} \right)^2
\]

\[
\leq \frac{N}{(T-1)^2} \left( \sup_{i} E \left[ w_{i0}^2 \right] \right) \left( |\rho_T|^2 \left( 1 - |\rho_T|^{T-1} \right)^2 \right)
\]

\[
= O \left( NT^{-2} \right) \times O \left( 1 \right) \times O \left( 1 \right) = O \left( N/T^2 \right),
\]

so that it follows, by Markov’s inequality, that

\[
\sum_{i=1}^{N} \left( \frac{1}{T-1} \sum_{t=2}^{T} \rho_{t-1}^{T} w_{i0} \right)^2 = O_p \left( \frac{N}{T^2} \right).
\]

Also, by the Cauchy-Schwarz inequality,

\[
\left| \sum_{i=1}^{N} \left( \frac{1}{T-1} \sum_{t=2}^{T} w_{it-1,T} \right) \left( \frac{1}{T-1} \sum_{t=2}^{T} \rho_{t-1}^{T} w_{i0} \right) \right| \leq \sqrt{\sum_{i=1}^{N} \left( \frac{1}{T-1} \sum_{s=2}^{T} w_{is-1,T} \right)^2} \sqrt{\sum_{i=1}^{N} \left( \frac{1}{T-1} \sum_{t=2}^{T} \rho_{t-1}^{T} w_{i0} \right)^2}
\]

\[
= O_p \left( \sqrt{\frac{N}{T}} \right) O_p \left( \sqrt{\frac{N}{T}} \right) = O_p \left( \frac{N}{T^{3/2}} \right),
\]

so that

\[
\sum_{i=1}^{N} w_{i,-1}^2 = \sum_{i=1}^{N} \left( \frac{1}{T-1} \sum_{t=2}^{T} w_{it-1,T} \right)^2 + 2 \sum_{i=1}^{N} \left( \frac{1}{T-1} \sum_{t=2}^{T} w_{it-1,T} \right) \left( \frac{1}{T-1} \sum_{t=2}^{T} \rho_{t-1}^{T} w_{i0} \right)
\]

\[
+ \sum_{i=1}^{N} \left( \frac{1}{T-1} \sum_{t=2}^{T} \rho_{t-1}^{T} w_{i0} \right)^2
\]

\[
= O_p \left( \frac{N}{T} \right) + O_p \left( \frac{N}{T^{3/2}} \right) + O_p \left( \frac{N}{T^2} \right) = O_p \left( \frac{N}{T} \right),
\]

as required for part (e). □

**Lemma SE-29:**

Under Assumptions 1 and 4, the following statements hold as \( N, T \to \infty \).

(a) If \( \rho_T = 1 \) for all \( T \) sufficiently large, then

\[
\frac{1}{N} \sum_{i=1}^{N} \bar{z}_i \bar{w}_{i,-1} = O_p \left( 1 \right).
\]
(b) If $\rho_T = \exp \{-1/q(T)\}$ such that $T/q(T) \to 0$, then

$$\frac{1}{N} \sum_{i=1}^{N} \varepsilon_i w_{i,-1} = O_p(1).$$

(c) If $\rho_T = \exp \{-1/q(T)\}$ such that $q(T) \sim T$, then

$$\frac{1}{N} \sum_{i=1}^{N} \varepsilon_i w_{i,-1} = O_p(1).$$

(d) If $\rho_T = \exp \{-1/q(T)\}$ such that $q(T) \to \infty$ but $q(T)/T \to 0$, then

$$\frac{1}{N} \sum_{i=1}^{N} \varepsilon_i w_{i,-1} = O_p\left( \frac{1}{T} \right).$$

(e) If $\rho_T \in G_{St} = \{ |\rho_T| = \exp \{-1/q(T)\} : q(T) \geq 0 \text{ and } q(T) = O(1) \text{ as } T \to \infty \}$, then

$$\frac{1}{N} \sum_{i=1}^{N} \varepsilon_i w_{i,-1} = O_p\left( \frac{1}{T} \right).$$

Proof of Lemma SE-29:

By the Cauchy-Schwarz inequality, we have

$$\left| \frac{1}{N} \sum_{i=1}^{N} \varepsilon_i w_{i,-1} \right| \leq \frac{1}{N} \sum_{i=1}^{N} |\varepsilon_i w_{i,-1}| \leq \sqrt{\frac{1}{N} \sum_{i=1}^{N} \varepsilon_i^2} \sqrt{\frac{1}{N} \sum_{i=1}^{N} w_{i,-1}^2}. $$

The results for parts (a)-(d) then follow immediately from applying, respectively, the results from parts (a)-(d) of Lemma SE-28 as well as part (e) of Lemma SE-11. $\Box$

Lemma SE-30:

Under Assumptions 1-4, the following statements hold as $N, T \to \infty$.

(a) If $\rho_T = 1$ for all $T$ sufficiently large, then

$$\sum_{i=1}^{N} a_i w_{iT-2,T} = O_p\left( \max \left\{ \sqrt{NT}, N \right\} \right).$$

(b) If $\rho_T = \exp \{-1/q(T)\}$ such that $T/q(T) \to 0$, then

$$\sum_{i=1}^{N} a_i w_{iT-2,T} = O_p\left( \max \left\{ \sqrt{NT}, N \right\} \right).$$
(c) If $\rho_T = \exp \{-1/q(T)\}$ such that $q(T) \sim T$, then
\[
\sum_{i=1}^{N} a_i w_{iT-2,T} = O_p \left( \max \left\{ \sqrt{NT}, N \right\} \right).
\]

(d) If $\rho_T = \exp \{-1/q(T)\}$ such that $q(T) \to \infty$ but $q(T)/T \to 0$, then
\[
\sum_{i=1}^{N} a_i w_{iT-2,T} = O_p \left( \sqrt{Nq(T)} \right).
\]

(e) If $\rho_T \in \mathcal{G}_{St} = \{ |\rho_T| = \exp \left\{ -\frac{1}{q(T)} \right\} : q(T) \geq 0 \text{ and } q(T) = O(1) \text{ as } T \to \infty \}$, then
\[
\sum_{i=1}^{N} a_i w_{iT-2,T} = O_p \left( \sqrt{N} \right).
\]

**Proof of Lemma SE-30:**

To proceed, first write
\[
\sum_{i=1}^{N} a_i w_{iT-2,T} = \sum_{i=1}^{N} a_i w_{iT-2,T} + \sum_{i=1}^{N} a_i \rho_T^{T-2} w_i 0,
\]
where $w_{T-2} = \sum_{j=1}^{T-2} \rho_T^{(T-2-j)} \varepsilon_{ij}$.

Consider part (a). Note that, under the assumption here, there exists a positive integer $I_\rho$ such that for all $T \geq I_\rho$, $\{ w_{T-2,T} \}$ has the partial sum representation $w_{T-2,T} = \sum_{j=1}^{T-2} \varepsilon_{ij}$. Hence, for all $T \geq I_\rho$, we obtain by direct calculation
\[
E \left( \sum_{i=1}^{N} a_i w_{iT-2,T} \right)^2 \leq \sum_{i=1}^{N} \sum_{j=1}^{N} E [a_i^2] E [w_{iT-2,T} w_{iT-2,T}] \leq \sum_{i=1}^{N} E [a_i^2] E [w_{iT-2,T} w_{iT-2,T}] + \sum_{i \neq j} E [a_i] E [a_j] E [w_{iT-2,T} w_{iT-2,T}] = (\mu_a^2 + \sigma_a^2) \sum_{i=1}^{N} \sum_{j=1}^{T-2} E [\varepsilon_{ij} \varepsilon_{is}] = (\mu_a^2 + \sigma_a^2) \sigma^2 N (T - 2) = O(NT).
\]

It follows by Markov’s inequality that $\sum_{i=1}^{N} a_i w_{iT-2,T} = O_p \left( \sqrt{NT} \right)$.

Moreover, by Assumptions 2 and 4, we have
\[
E \left( \sum_{i=1}^{N} a_i^2 \right) = N (\mu_a^2 + \sigma_a^2) = O(N), \quad E \left( \sum_{i=1}^{N} w_{0T}^2 \right) \leq \sup_i E \left( w_{0T}^2 \right) N = O(N),
\]

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from which it follows, by Markov’s inequality, that
\[ \sum_{i=1}^{N} a_i^2 = O_p(N), \quad \sum_{i=1}^{N} w_{i0}^2 = O_p(N). \]
Applying the Cauchy-Schwarz inequality, we further obtain
\[ \left| \sum_{i=1}^{N} a_i w_{i0} \right| \leq \sqrt{\sum_{i=1}^{N} a_i^2} \sqrt{\sum_{i=1}^{N} w_{i0}^2} = O_p(\sqrt{N}) O_p(\sqrt{N}) = O_p(N). \]
Hence,
\[ \sum_{i=1}^{N} a_i w_{iT-2,T} = \sum_{i=1}^{N} a_i w_{iT-2,T} + \sum_{i=1}^{N} a_i w_{i0} \]
for all \( T \) sufficiently large,
\[ = O_p(\sqrt{NT}) + O_p(N) = O_p\left(\max\left\{\sqrt{NT}, N\right\}\right), \]
as required for part (a).
Now, to show parts (b)-(d), note first that
\[ E \left( \sum_{i=1}^{N} a_i w_{iT-2,T} \right)^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} E [a_i a_j w_{iT-2,T} w_{iT-2,T}] \]
\[ = \sum_{i=1}^{N} E [a_i^2] E \left[ w_{iT-2,T}^2 \right] + \sum_{i \neq j} E [a_i] E [a_j] E \left[ w_{iT-2,T} \right] E \left[ w_{iT-2,T} \right] \]
\[ = (\mu_a^2 + \sigma_a^2) \sum_{i=1}^{N} \sum_{g=1}^{T-2} \sum_{s=1}^{T-2} \exp \left\{ -\frac{(T-2-g)}{q(T)} \right\} \exp \left\{ -\frac{(T-2-s)}{q(T)} \right\} E [\varepsilon_{ig}\varepsilon_{is}] \]
\[ = (\mu_a^2 + \sigma_a^2) \sigma_a^2 \sum_{i=1}^{N} \sum_{s=1}^{T-2} \exp \left\{ -\frac{(T-2-s)}{q(T)} \right\} \]
\[ = (\mu_a^2 + \sigma_a^2) \sigma_a^2 N \sum_{s=1}^{T-2} \exp \left\{ -\frac{(T-2-s)}{q(T)} \right\}. \]
Next, consider part (b), where we take \( \rho_T = \exp \{-1/q(T)\} \) such that \( T/q(T) \to 0 \). In this case, we apply Lemma SE-3 part (a) to obtain
\[ E \left( \sum_{i=1}^{N} a_i w_{iT-2,T} \right)^2 = (\mu_a^2 + \sigma_a^2) \sigma_a^2 N T \left[ 1 + O \left( \frac{1}{T} \right) + O \left( \frac{T}{q(T)} \right) \right], \]
so that we deduce via Markov’s inequality \( \sum_{i=1}^{N} a_i w_{iT-2,T} = O_p(\sqrt{NT}) \).
Moreover, by Assumptions 2 and 4, we have
\[ E \left[ \sum_{i=1}^{N} a_i^2 \right] = N (\mu_a^2 + \sigma_a^2) = O(N), \]
\[ E \left[ \sum_{i=1}^{N} \rho_T^{2(T-2)} w_{i0}^2 \right] \leq \sup_i E \left[ w_{i0}^2 \right] N \exp \left\{ -\frac{2(T-2)}{q(T)} \right\} = O(N), \]
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from which it follows, by Markov’s inequality, that

$$\sum_{i=1}^{N} a_i^2 = O_p(N), \quad \sum_{i=1}^{N} \rho_T^{2(T-2)} w_{i0}^2 = O_p(N).$$

Applying the Cauchy-Schwarz inequality, we further obtain

$$\left| \sum_{i=1}^{N} a_i \rho_T^{T-2} w_{i0} \right| \leq \sqrt{\sum_{i=1}^{N} a_i^2 \sum_{i=1}^{N} \rho_T^{2(T-2)} w_{i0}^2} = O_p(\sqrt{N}) O_p(\sqrt{N}) = O_p(N).$$

Hence,

$$\sum_{i=1}^{N} a_i w_{iT-2,T} = \sum_{i=1}^{N} a_i w_{iT-2,T} + \sum_{i=1}^{N} a_i \rho_T^{T-2} w_{i0}
= O_p(\sqrt{NT}) + O_p(N)
= O_p(\sqrt{NT}) + O_p(N) = O_p(\max\{\sqrt{NT}, N\})$$

as required for part (b).

Consider part (c), where we take $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \sim T$. Here, we apply Lemma SE-3 part (b) to obtain

$$E \left( \sum_{i=1}^{N} a_i w_{iT-2,T} \right)^2 = (\mu_a^2 + \sigma_a^2) \sigma^2 N q(T) \left[ 1 - \exp\left\{-\frac{2T}{q(T)}\right\} \right] \left[ 1 + O\left(\frac{1}{T}\right)\right],$$

so again we deduce via Markov’s inequality

$$\sum_{i=1}^{N} a_i w_{iT-2,T} = O_p(\sqrt{NT}).$$

Moreover, by Assumptions 2 and 4, we have

$$E \left[ \sum_{i=1}^{N} a_i^2 \right] = N \left( \mu_a^2 + \sigma_a^2 \right) = O(N),$$

$$E \left[ \sum_{i=1}^{N} \rho_T^{2(T-2)} w_{i0}^2 \right] \leq \sup_i E \left[ w_{i0}^2 \right] N \exp \left\{-\frac{2(T-2)}{q(T)}\right\} = O(N),$$

from which it follows, by Markov’s inequality, that

$$\sum_{i=1}^{N} a_i^2 = O_p(N), \quad \sum_{i=1}^{N} \rho_T^{2(T-2)} w_{i0}^2 = O_p(N).$$

Applying the Cauchy-Schwarz inequality, we further obtain

$$\left| \sum_{i=1}^{N} a_i \rho_T^{T-2} w_{i0} \right| \leq \sqrt{\sum_{i=1}^{N} a_i^2 \sum_{i=1}^{N} \rho_T^{2(T-2)} w_{i0}^2} = O_p(\sqrt{N}) O_p(\sqrt{N}) = O_p(N).$$
Hence,

\[
\sum_{i=1}^{N} a_i w_i T^{-2} = \sum_{i=1}^{N} a_i w_{i-2, T} + \sum_{i=1}^{N} a_i \rho_T^{T-2} w_{i0}
\]

\[
= O_p \left( \sqrt{N}T \right) + O_p (N)
\]

\[
= O_p \left( \sqrt{N}T \right) + O_p (N) = O_p \left( \max \left\{ \sqrt{N}T, N \right\} \right),
\]

as required for part (c).

Turning our attention to part (d), where we take \( \rho_T = \exp \left\{ -1/q(T) \right\} \) such that \( q(T) \to \infty \) but \( q(T)/T \to 0 \). Applying part (c) of Lemma SE-3, we obtain

\[
E \left( \sum_{i=1}^{N} a_i w_i T^{-2}T_0 \right)^2 = \left( \mu^2 + \sigma^2_T \right) \sigma^2 \left( \frac{q(T)}{N} \right) \left[ 1 + O \left( \frac{1}{q(T)} \right) \right],
\]

so that the use of Markov's inequality yields

\[
\sum_{i=1}^{N} a_i w_i T^{-2, T} = O_p \left( \sqrt{N q(T)} \right).
\]

Moreover, by Assumptions 2 and 4, we have

\[
E \left[ \sum_{i=1}^{N} a_i^2 \right] = N \left( \mu^2 + \sigma^2_T \right) = O (N),
\]

\[
E \left[ \sum_{i=1}^{N} \rho_T^{2(T-2)} w_{i0}^2 \right] \leq \sup_i E \left[ w_{i0}^2 \right] N \exp \left\{ -2 \left( T - \frac{2}{q(T)} \right) \right\} = O \left( N \exp \left\{ - \frac{2T}{q(T)} \right\} \right),
\]

from which it follows, by Markov's inequality, that

\[
\sum_{i=1}^{N} a_i^2 = O_p (N), \quad \sum_{i=1}^{N} \rho_T^{2(T-2)} w_{i0}^2 = O_p \left( N \exp \left\{ - \frac{2T}{q(T)} \right\} \right).
\]

Applying the Cauchy-Schwarz inequality, we further obtain

\[
\left| \sum_{i=1}^{N} a_i \rho_T^{T-2} w_{i0} \right| \leq \sqrt{\sum_{i=1}^{N} a_i^2} \sqrt{\sum_{i=1}^{N} \rho_T^{2(T-2)} w_{i0}^2}
\]

\[
= O_p \left( \sqrt{N} \right) O_p \left( \sqrt{N} \exp \left\{ - \frac{T}{q(T)} \right\} \right) = O_p \left( N \exp \left\{ - \frac{T}{q(T)} \right\} \right).
\]

Hence,

\[
\sum_{i=1}^{N} a_i w_i T^{-2} = \sum_{i=1}^{N} a_i w_{i-2, T} + \sum_{i=1}^{N} a_i \rho_T^{T-2} w_{i0}
\]

\[
= O_p \left( \sqrt{N q(T)} \right) + O_p \left( N \exp \left\{ - \frac{T}{q(T)} \right\} \right) = O_p \left( \sqrt{N q(T)} \right),
\]

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as required for part (d).

Finally, consider part (e), where we take $\rho_T \in \mathcal{G}_{St}$. In this case,

$$
E \left( \sum_{i=1}^{N} a_i w_{T-2,T} \right)^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} E \left[ a_i a_j w_{T-2,T} w_{jT-2,T} \right] = (\mu_a^2 + \sigma_a^2) \sigma^2 N \sum_{s=1}^{T-2} \rho_T^{2(T-2-s)} = (\mu_a^2 + \sigma_a^2) \sigma^2 N \frac{1 - \rho_T^{2(T-2)}}{1 - \rho_T^2}.
$$

Since we assume that $\rho_T^2 = \exp \left\{ -\frac{2}{q(T)} \right\}$ with $q(T) = O(1)$ here, it follows that there exist a positive constant $C_q$ and a positive integer $T^*$ such that for all $T \geq T^*$

$$
\rho_T^2 = \exp \left\{ -\frac{2}{q(T)} \right\} \leq \exp \left\{ -\frac{2}{C_q} \right\} < 1,
$$

from which we further deduce, in light of Assumptions 1 and 2, that

$$
E \left( \sum_{i=1}^{N} a_i w_{T-2,T} \right)^2 = (\mu_a^2 + \sigma_a^2) \sigma^2 N \frac{1 - \rho_T^{2(T-2)}}{1 - \rho_T^2} \leq N \frac{(\mu_a^2 + \sigma_a^2) \sigma^2}{1 - \exp \left\{ -2/C_q \right\}} = O(N).
$$

Hence, by applying Markov’s inequality, we obtain $\sum_{i=1}^{N} a_i w_{T-2,T} = O_p \left( \sqrt{N} \right)$.

Moreover, by Assumptions 2 and 4, we have

$$
E \left[ \sum_{i=1}^{N} a_i^2 \right] = N (\mu_a^2 + \sigma_a^2) = O(N),
$$

$$
E \left[ \sum_{i=1}^{N} \rho_T^{2(T-2)} w_{i0}^2 \right] \leq \sup_i E \left[ w_{i0}^2 \right] N \rho_T^{2(T-2)} = O(N \rho_T^{2T}),
$$

from which it follows, by Markov’s inequality, that

$$
\sum_{i=1}^{N} a_i^2 = O_p(N), \quad \sum_{i=1}^{N} \rho_T^{2(T-2)} w_{i0}^2 = O_p(N \rho_T^{2T}).
$$

Applying the Cauchy-Schwarz inequality, we further obtain

$$
\left| \sum_{i=1}^{N} a_i \rho_T^{T-2} w_{i0} \right| \leq \left( \sum_{i=1}^{N} a_i^2 \right)^{1/2} \left( \sum_{i=1}^{N} \rho_T^{2(T-2)} w_{i0}^2 \right)^{1/2} = O_p \left( \sqrt{N} \right) O_p \left( \sqrt{N \rho_T^T} \right) = O_p \left( N \rho_T^T \right).
$$

Hence,

$$
\sum_{i=1}^{N} a_i w_{T-2,T} = \sum_{i=1}^{N} a_i w_{T-2,T} + \sum_{i=1}^{N} a_i \rho_T^{T-2} w_{i0} = O_p \left( \sqrt{N} \right) + O_p \left( N \rho_T^T \right) = O_p \left( \sqrt{N} \right),
$$

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as required for part (e). □

**Lemma SE-31:**

Let \( w_{T-2} = \sum_{j=1}^{T-2} \rho_T^{(T-2-j)} \xi_{ij} \). Under Assumptions 1 and 4, the following statements hold as \( N, T \to \infty \).

(a) If \( \rho_T = 1 \) for all \( T \) sufficiently large, then

\[
\frac{1}{NT} \sum_{i=1}^{N} E \left[ w_{iT-2,T}^2 \right] = \sigma^2 + O \left( \frac{1}{T} \right).
\]

(b) If \( \rho_T = \exp \{-1/q(T)\} \) such that \( T/q(T) \to 0 \),

\[
\frac{1}{NT} \sum_{i=1}^{N} E \left[ w_{iT-2,T}^2 \right] = \sigma^2 + O \left( \frac{1}{T} \right) + O \left( \frac{T}{q(T)} \right).
\]

(c) If \( \rho_T = \exp \{-1/q(T)\} \) such that \( q(T) \sim T \),

\[
\frac{1}{NT} \sum_{i=1}^{N} E \left[ w_{iT-2,T}^2 \right] = \frac{\sigma^2 q(T)}{2} \left[ 1 - \exp \left\{ -\frac{2T}{q(T)} \right\} \right] + O \left( \frac{1}{T} \right).
\]

(d) If \( \rho_T = \exp \{-1/q(T)\} \) such that \( q(T) \to \infty \) but \( q(T)/T \to 0 \),

\[
\frac{1}{NT} \sum_{i=1}^{N} E \left[ w_{iT-2,T}^2 \right] = O \left( \frac{q(T)}{T} \right) = o(1).
\]

(e) If \( \rho_T \in \mathcal{G}_{St} = \left\{ |\rho_T| = \exp \left\{ -\frac{1}{q(T)} \right\} : q(T) \geq 0 \text{ and } q(T) = O(1) \text{ as } T \to \infty \right\} \), then

\[
\frac{1}{NT} \sum_{i=1}^{N} E \left[ w_{iT-2,T}^2 \right] = O \left( \frac{1}{T} \right) = o(1).
\]

**Proof of Lemma SE-31:**

Consider first part (a). Note that, under the assumption here, there exists a positive integer \( I_\rho \) such that, for all \( T \geq I_\rho \), \( \{w_{iT-2,T}\} \) has the partial sum representation \( w_{iT-2,T} = \sum_{j=1}^{T-2} \xi_{ij} \). Hence, for all \( T \geq I_\rho \), we obtain by direct calculation

\[
\frac{1}{NT} \sum_{i=1}^{N} E \left[ w_{iT-2,T}^2 \right] = \frac{1}{NT} \sum_{i=1}^{N} \sum_{g=1}^{T-2} \sum_{s=1}^{T-2} E [\xi_{ig}\xi_{is}] = \sigma^2 \frac{N(T-2)}{NT} = \sigma^2 + O \left( \frac{1}{T} \right).
\]
Now, to show parts (b)-(d), note first that

\[
\sum_{i=1}^{N} E \left[ w_{T-2,T}^2 \right] \\
= \sum_{i=1}^{N} \sum_{g=1}^{T-2} \sum_{s=1}^{T-2} \exp \left\{ \frac{-(T - 2 - g)}{q(T)} \right\} \exp \left\{ \frac{-(T - 2 - s)}{q(T)} \right\} E [\varepsilon_{ig}\varepsilon_{is}] \\
= \sigma^2 N \sum_{s=1}^{T-2} \exp \left\{ -\frac{2(T - 2 - s)}{q(T)} \right\}. 
\]

Next, consider part (b), where we take \( \rho_T = \exp \{-1/q(T)\} \) such that \( T/q(T) \to 0 \). In this case, we apply Lemma SE-3 part (a) to obtain

\[
E \left( \sum_{i=1}^{N} w_{T-2,T}^2 \right) = \sigma^2 NT \left[ 1 + O \left( \frac{1}{T} \right) + O \left( \frac{T}{q(T)} \right) \right],
\]

so that

\[
\frac{1}{NT} \sum_{i=1}^{N} E \left[ w_{T-2,T}^2 \right] = \sigma^2 + O \left( \frac{1}{T} \right) + O \left( \frac{T}{q(T)} \right). 
\]

Consider part (c), where we take \( \rho_T = \exp \{-1/q(T)\} \) such that \( q(T) \sim T \). Here, we apply Lemma SE-3 part (b) to obtain

\[
E \left( \sum_{i=1}^{N} w_{T-2,T}^2 \right) = \sigma^2 N \frac{q(T)}{2} \left[ 1 - \exp \left\{ -\frac{2T}{q(T)} \right\} \right] \left[ 1 + O \left( \frac{1}{T} \right) \right],
\]

so that

\[
\frac{1}{NT} \sum_{i=1}^{N} E \left[ w_{T-2,T}^2 \right] = \frac{\sigma^2}{2} \frac{q(T)}{T} \left[ 1 - \exp \left\{ -\frac{2T}{q(T)} \right\} \right] + O \left( \frac{1}{T} \right).
\]

We turn our attention now to part (d), where we take \( \rho_T = \exp \{-1/q(T)\} \) such that \( q(T) \to \infty \) but \( q(T)/T \to 0 \). Applying part (c) of Lemma SE-3, we obtain

\[
E \left( \sum_{i=1}^{N} w_{T-2,T}^2 \right) = \sigma^2 N \frac{q(T)}{2} \left[ 1 + O \left( \frac{1}{q(T)} \right) \right],
\]

so that

\[
\frac{1}{NT} \sum_{i=1}^{N} E \left[ w_{T-2,T}^2 \right] = \frac{\sigma^2}{2} \frac{q(T)}{T} + O \left( \frac{1}{T} \right) = O \left( \frac{q(T)}{T} \right).
\]

Finally, to show part (e),

\[
\sum_{i=1}^{N} E \left[ w_{T-2,T}^2 \right] = \sigma^2 N \sum_{s=1}^{T-2} \rho_T^{2(T-2-s)} = \sigma^2 N \frac{1 - \rho_T^{2(T-2)}}{1 - \rho_T^2}.
\]
Here, we take $\rho_2^T = \exp\left\{-\frac{2}{q(T)}\right\}$ with $q(T) = O(1)$, so that there exist a positive constant $C_q$ and a positive integer $T^*$ such that for all $T \geq T^*$

$$\rho_2^T = \exp\left\{-\frac{2}{q(T)}\right\} \leq \exp\left\{-\frac{2}{C_q}\right\} < 1,$$

from which we deduce that

$$\frac{1}{NT} \sum_{i=1}^{N} E\left[w_{i,T-2,T}^2\right] \leq \frac{\sigma^2}{T} \frac{1}{1 - \exp\left\{-2/C_q\right\}} = O\left(\frac{1}{T}\right) = o(1). \quad \square$$

**Lemma SE-32:**

Under Assumptions 1 and 4, the following statements hold as $N,T \to \infty$.

(a) If $\rho_T = 1$ for all $T$ sufficiently large, then

$$\frac{1}{NT} \sum_{i=1}^{N} w_{i,T-2,T}^2 = \sigma^2 + O_p\left(\frac{1}{\sqrt{N}}, \frac{1}{\sqrt{T}}\right).$$

(b) If $\rho_T = \exp\left\{-1/q(T)\right\}$ such that $T/q(T) \to 0$, then

$$\frac{1}{NT} \sum_{i=1}^{N} w_{i,T-2,T}^2 = \sigma^2 + O_p\left(\max\left\{\frac{1}{\sqrt{N}}, \frac{1}{\sqrt{T}}, \frac{T}{q(T)}\right\}\right).$$

(c) If $\rho_T = \exp\left\{-1/q(T)\right\}$ such that $q(T) \sim T$, then

$$\frac{1}{NT} \sum_{i=1}^{N} w_{i,T-2,T}^2 = \frac{\sigma^2 q(T)}{2T} \left[1 - \exp\left\{-\frac{2T}{q(T)}\right\}\right] + O_p\left(\max\left\{\frac{1}{\sqrt{N}}, \frac{1}{\sqrt{T}}\right\}\right).$$

(d) If $\rho_T = \exp\left\{-1/q(T)\right\}$ such that $q(T) \to \infty$ but $q(T)/T \to 0$, then

$$\frac{1}{NT} \sum_{i=1}^{N} w_{i,T-2,T}^2 = O_p\left(\frac{q(T)}{T}\right).$$

(e) If $\rho_T \in \mathcal{G}_\text{St} = \left\{ |\rho_T| = \exp\left\{-\frac{1}{q(T)}\right\} : q(T) \geq 0 \text{ and } q(T) = O(1) \text{ as } T \to \infty \right\}$, then

$$\frac{1}{NT} \sum_{i=1}^{N} w_{i,T-2,T}^2 = O_p\left(\frac{1}{T}\right).$$

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Proof of Lemma SE-32:

To proceed, first write

\[
\frac{1}{NT} \sum_{i=1}^{N} w^2_{iT-2,T} = \frac{1}{NT} \sum_{i=1}^{N} w^2_{iT-2,T} + 2 \frac{1}{NT} \sum_{i=1}^{N} w_{iT-2} T^{-2} w_{i0} + \frac{1}{NT} \sum_{i=1}^{N} 2(T-2) w^2_{i0},
\]

\[
w_{iT-2} = \sum_{j=1}^{T-2} \rho_T^{(T-2-j)} \varepsilon_{ij}.
\]

Consider first part (a). Note that, under the assumption here, there exists a positive integer \(I_\rho\) such that for all \(T \geq I_\rho\), \(\{w_{iT-2,T}\}\) has the partial sum representation \(w_{iT-2,T} = \sum_{j=1}^{T-2} \varepsilon_{ij}\). Hence, for all \(T \geq I_\rho\), we obtain by direct calculation

\[
E \left( \frac{1}{NT} \sum_{i=1}^{N} \left( w^2_{iT-2,T} - E \left( w^2_{iT-2,T} \right) \right) \right)^2
\]

\[
= \frac{1}{N^2 T^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \left( E \left( w^2_{iT-2,T} - E \left( w^2_{iT-2,T} \right) \right) \right)^2
\]

\[
= \frac{1}{N^2 T^2} \sum_{i=1}^{N} \left\{ E \left( w^4_{iT-2,T} \right) - (E \left( w^2_{iT-2,T} \right))^2 \right\}
\]

\[
= \frac{1}{NT} \sum_{g=1}^{T-2} \sum_{s=1}^{T-2} \sum_{t=1}^{T-2} \sum_{u=1}^{T-2} E \left[ \varepsilon_{ig} \varepsilon_{is} \varepsilon_{it} \varepsilon_{iu} \right] - \frac{1}{NT^2} \left( \sum_{g=1}^{T-2} \sum_{s=1}^{T-2} E \left[ \varepsilon_{ig} \varepsilon_{is} \right] \right)^2
\]

\[
\leq E \left[ \varepsilon^4_{it} \right] \frac{T-2}{NT^2} + 3 \sigma^4 \left( \frac{T-2)^2}{NT^2} \right) - \sigma^4 \left( \frac{T-2)^2}{NT^2} \right) \]

\[
= E \left[ \varepsilon^4_{it} \right] \frac{T-2}{NT^2} + 2 \sigma^4 \left( \frac{T-2)^2}{NT^2} \right) = O \left( N^{-1} \right).
\]

It follows from Markov’s inequality and part (a) of Lemma SE-31 that

\[
\frac{1}{NT} \sum_{i=1}^{N} w^2_{iT-2,T} = \frac{1}{NT} \sum_{i=1}^{N} E \left[ w^2_{iT-2,T} \right] + O_p \left( \frac{1}{\sqrt{N}} \right) = \sigma^2 + O_p \left( \frac{1}{\sqrt{N}} \right) + O_p \left( \frac{1}{T} \right)
\]

\[
= \sigma^2 + O_p \left( \frac{1}{\sqrt{N}} \right).
\]

Moreover, using Assumption 4, we have

\[
E \left[ \frac{1}{NT} \sum_{i=1}^{N} w^2_{i0} \right] \leq \frac{1}{T} \sup_i E \left[ w^2_{i0} \right] = O \left( T^{-1} \right),
\]

from which, it follows, by Markov’s inequality, that

\[
\frac{1}{NT} \sum_{i=1}^{N} w^2_{i0} = O_p \left( \frac{1}{\sqrt{T}} \right).
\]
Also, by the Cauchy-Schwarz inequality,

\[
\left| \frac{1}{NT} \sum_{i=1}^{N} w_{iT-2}w_{0} \right| \leq \sqrt{\frac{1}{NT} \sum_{i=1}^{N} w_{iT-2}^2} \sqrt{\frac{1}{NT} \sum_{i=1}^{N} w_{0}^2} = o_p \left( \frac{1}{NT} \right) = o_p \left( \frac{1}{T} \right).
\]

Hence,

\[
\frac{1}{NT} \sum_{i=1}^{N} w_{iT-2,T}^2 = \frac{1}{NT} \sum_{i=1}^{N} w_{iT-2,T}^2 + 2 \frac{1}{NT} \sum_{i=1}^{N} w_{iT-2}w_{0} + \frac{1}{NT} \sum_{i=1}^{N} w_{0}^2
\]

(for all $T$ sufficiently large)

\[
= \sigma^2 + o_p \left( \max \left\{ \frac{1}{\sqrt{N}}, \frac{1}{\sqrt{T}} \right\} \right),
\]

as required for part (a).

Now, to show parts (b)-(d), note first that

\[
E \left( \frac{1}{NT} \sum_{i=1}^{N} \left( w_{iT-2,T}^2 - E \left[ w_{iT-2,T}^2 \right] \right) \right)^2 = \frac{1}{N^2T^2} \sum_{i=1}^{N} \left\{ E \left[ w_{iT-2,T}^4 \right] - (E \left[ w_{iT-2,T}^2 \right])^2 \right\}
\]

\[
= \frac{1}{NT^2} \sum_{g=1}^{T-2} \sum_{s=1}^{T-2} \sum_{t=1}^{T-2} \sum_{v=1}^{T-2} \exp \left\{ -\left( \frac{T-2-g}{q(T)} \right) \right\} \exp \left\{ -\left( \frac{T-2-s}{q(T)} \right) \right\} \exp \left\{ -\left( \frac{T-2-t}{q(T)} \right) \right\} \times \exp \left\{ -\left( \frac{T-2-v}{q(T)} \right) \right\} E \left[ \varepsilon_{igisitiv} \right]
\]

\[
\leq \frac{1}{NT^2} \sum_{s=1}^{T-2} \exp \left\{ -\frac{4(T-2-s)}{q(T)} \right\} E \left[ \varepsilon_{is}^4 \right] + \frac{3\sigma^4}{N} \left( \frac{1}{T} \sum_{s=1}^{T-2} \exp \left\{ -\frac{2(T-2-s)}{q(T)} \right\} \right)^2
\]

\[
- \frac{\sigma^4}{N} \left( \frac{1}{T} \sum_{s=1}^{T-2} \exp \left\{ -\frac{2(T-2-s)}{q(T)} \right\} \right)^2
\]

\[
= \frac{1}{NT^2} \sum_{s=1}^{T-2} \exp \left\{ -\frac{4(T-2-s)}{q(T)} \right\} E \left[ \varepsilon_{is}^4 \right] + \frac{2\sigma^4}{N} \left( \frac{1}{T} \sum_{s=1}^{T-2} \exp \left\{ -\frac{2(T-2-s)}{q(T)} \right\} \right)^2.
\]

Next, consider part (b), where we take $\rho_T = \exp \left\{ -1/q(T) \right\}$ such that $T/q(T) \to 0$. In this case,
we apply Lemma SE-3 part (a) to obtain
\[
E \left( \frac{1}{NT} \sum_{i=1}^{N} (w_{T-2,T}^2 - E[w_{T-2,T}^2]) \right)^2
\]
\[
= \frac{1}{NT^2} \sum_{s=1}^{T-2} \exp \left\{ -4 \frac{(T-2-s)}{q(T)} \right\} E[\varepsilon_{i,s}^4] + \frac{2\sigma^4}{N} \left( \frac{1}{T} \sum_{s=1}^{T-2} \exp \left\{ -2 \frac{(T-2-s)}{q(T)} \right\} \right)^2
\]
\[
= \frac{E[\varepsilon_{i,s}^4]}{NT^2} T \left[ 1 + O \left( \frac{1}{T} \right) + O \left( \frac{T}{q(T)} \right) \right] + \frac{2\sigma^4}{NT^2} T^2 \left[ 1 + O \left( \frac{1}{T} \right) + O \left( \frac{T}{q(T)} \right) \right]
\]
\[
= O \left( \frac{1}{N} \right).
\]

It follows from Markov’s inequality and part (b) of Lemma SE-31 above that
\[
\frac{1}{NT} \sum_{i=1}^{N} w_{T-2,T}^2 = \frac{1}{NT} \sum_{i=1}^{N} E[w_{T-2,T}^2] + O_p \left( \frac{1}{\sqrt{N}} \right)
\]
\[
= \sigma^2 + O_p \left( \frac{1}{\sqrt{N}} \right) + O_p \left( \frac{T}{q(T)} \right).
\]

Moreover, using Assumption 4, we have
\[
E \left[ \frac{1}{NT} \sum_{i=1}^{N} \rho_T^{2(T-2)} w_{i0}^2 \right] \leq \frac{1}{T} \left( \sup_i E [w_{i0}^2] \right) \exp \left\{ -2 \frac{(T-2)}{q(T)} \right\} = O \left( \frac{1}{T} \right) O(1) O(1) = O \left( \frac{1}{T} \right)
\]

from which it follows, by Markov’s inequality, that
\[
\frac{1}{NT} \sum_{i=1}^{N} \rho_T^{2(T-2)} w_{i0}^2 = O_p \left( \frac{1}{T} \right).
\]

Also, by the Cauchy-Schwarz inequality,
\[
\left| \frac{1}{NT} \sum_{i=1}^{N} w_{T-2,T} \rho_T^{T-2} w_{i0} \right| \leq \sqrt{ \frac{1}{NT} \sum_{i=1}^{N} w_{T-2,T}^2 } \sqrt{ \frac{1}{NT} \sum_{i=1}^{N} \rho_T^{2(T-2)} w_{i0}^2 } = O_p \left( \frac{1}{\sqrt{T}} \right) O_p \left( \frac{1}{\sqrt{T}} \right) = O_p \left( \frac{1}{\sqrt{T}} \right).
\]

Hence,
\[
\frac{1}{NT} \sum_{i=1}^{N} w_{T-2,T}^2 = \frac{1}{NT} \sum_{i=1}^{N} w_{T-2,T}^2 + 2 \frac{1}{NT} \sum_{i=1}^{N} w_{T-2,T} \rho_T^{T-2} w_{i0} + \frac{1}{NT} \sum_{i=1}^{N} \rho_T^{2(T-2)} w_{i0}^2
\]
\[
= \sigma^2 + O_p \left( \max \left\{ \frac{1}{\sqrt{N}}, \frac{1}{\sqrt{T}}, \frac{T}{q(T)} \right\} \right),
\]

as required for part (b).
Consider part (c), where we take $\rho_T = \exp \{-1/q(T)\}$ such that $q(T) \sim T$. Here, we apply Lemma SE-3 part (b) to obtain
\[
E \left( \frac{1}{NT} \sum_{i=1}^{N} \left( w_{iT-2,T}^2 - E[w_{iT-2,T}^2] \right) \right)^2
\]
\[
= \frac{1}{NT^2} \sum_{s=1}^{T-2} \exp \left\{ -\frac{4(T-2-s)}{q(T)} \right\} E \left[ \frac{\epsilon_s^4}{N} \right] + \frac{2\sigma^4}{N} \left( \frac{1}{T} \sum_{s=1}^{T-2} \exp \left\{ -\frac{2(T-2-s)}{q(T)} \right\} \right)^2
\]
\[
= \frac{E[\epsilon_i^4]}{NT^2} \frac{q(T)}{4} \left[ 1 - \exp \left\{ -\frac{4T}{q(T)} \right\} \right] \left[ 1 + O \left( \frac{1}{T} \right) \right]
+ \frac{2\sigma^4}{NT^2} \frac{q(T)^2}{4} \left[ 1 - \exp \left\{ -\frac{2T}{q(T)} \right\} \right]^2 \left[ 1 + O \left( \frac{1}{T} \right) \right]
= O \left( \frac{1}{N} \right).
\]

It follows from Markov’s inequality and part (c) of Lemma SE-31 that
\[
\frac{1}{NT} \sum_{i=1}^{N} w_{iT-2,T}^2 = \frac{1}{NT} \sum_{i=1}^{N} E[w_{iT-2,T}^2] + O_p \left( \frac{1}{\sqrt{N}} \right)
= \frac{\sigma^2 q(T)}{2T} \left[ 1 - \exp \left\{ -\frac{2T}{q(T)} \right\} \right] + O_p \left( \frac{1}{\sqrt{N}} \right) + O_p \left( \frac{1}{T} \right)
= \frac{\sigma^2 q(T)}{2T} \left[ 1 - \exp \left\{ -\frac{2T}{q(T)} \right\} \right] + O_p \left( \frac{1}{\sqrt{N}} \right).
\]

Moreover, using Assumption 4, we have
\[
E \left[ \frac{1}{NT} \sum_{i=1}^{N} \rho_T^{2(T-2)} w_{i0}^2 \right] \leq \frac{1}{T} \left( \sup_i E[w_{i0}^2] \right) \exp \left\{ -\frac{2(T-2)}{q(T)} \right\} = O \left( \frac{1}{T} \right) O(1) O(1) = O \left( \frac{1}{T} \right),
\]
from which it follows, by Markov’s inequality, that
\[
\frac{1}{NT} \sum_{i=1}^{N} \rho_T^{2(T-2)} w_{i0}^2 = O_p \left( \frac{1}{T} \right).
\]

Also, by the Cauchy-Schwarz inequality,
\[
\left| \frac{1}{NT} \sum_{i=1}^{N} w_{iT-2,T}^2 \rho_T^{T-2} w_{i0} \right| \leq \sqrt{\frac{1}{NT} \sum_{i=1}^{N} w_{iT-2,T}^2} \sqrt{\frac{1}{NT} \sum_{i=1}^{N} \rho_T^{2(T-2)} w_{i0}^2} = O_p(1) O_p \left( \frac{1}{\sqrt{T}} \right) = O_p \left( \frac{1}{\sqrt{T}} \right).
\]

Hence,
\[
\frac{1}{NT} \sum_{i=1}^{N} w_{iT-2,T}^2 = \frac{1}{NT} \sum_{i=1}^{N} w_{iT-2,T}^2 + 2 \frac{1}{NT} \sum_{i=1}^{N} w_{iT-2,T} \rho_T^{T-2} w_{i0} + \frac{1}{NT} \sum_{i=1}^{N} \rho_T^{2(T-2)} w_{i0}^2
= \frac{\sigma^2 q(T)}{2T} \left[ 1 - \exp \left\{ -\frac{2T}{q(T)} \right\} \right] + O_p \left( \max \left\{ \frac{1}{\sqrt{N}}, \frac{1}{\sqrt{T}} \right\} \right),
\]

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as required for part (c).

We now turn our attention to part (d), where we take \( \rho_T = \exp\{-1/q(T)\} \) such that \( q(T) \rightarrow \infty \) but \( q(T)/T \rightarrow 0 \). Applying part (c) of Lemma SE-3, we obtain

\[
E\left( \frac{1}{NT} \sum_{i=1}^{N} (w_{T-2,T}^i - E[w_{T-2,T}]) \right)^2
\]

\[
= \frac{1}{NT^2} \sum_{s=1}^{T-2} \exp\left\{ -4\frac{(T-2-s)}{q(T)} \right\} E\left[ \varepsilon_{is}^4 \right] + \frac{2\sigma^4}{N} \left( \frac{1}{T} \sum_{s=1}^{T-2} \exp\left\{ -2\frac{(T-2-s)}{q(T)} \right\} \right)^2
\]

\[
= \frac{E\left[ \varepsilon_{is}^4 \right]}{NT^2} q(T) \left[ 1 + O \left( \frac{1}{q(T)} \right) \right] + \frac{2\sigma^4}{NT^2} \frac{q(T)^2}{4} \left[ 1 + O \left( \frac{1}{q(T)} \right) \right]
\]

\[
= O \left( \frac{q(T)^2}{NT^2} \right).
\]

It follows from Markov’s inequality and part (d) of Lemma SE-31 that

\[
\frac{1}{NT} \sum_{i=1}^{N} w_{T-2,T}^2 = \frac{1}{NT} \sum_{i=1}^{N} E[w_{T-2,T}^2] + O_p \left( \frac{q(T)}{\sqrt{NT}} \right)
\]

\[
= O \left( \frac{q(T)}{T} \right) + O_p \left( \frac{q(T)}{\sqrt{NT}} \right) = O \left( \frac{q(T)}{T} \right).
\]

Moreover, using Assumption 4, we have

\[
E \left[ \frac{1}{NT} \sum_{i=1}^{N} \rho_T^{2(T-2)} w_{i0}^2 \right] \leq \frac{1}{T} \left( \sup_i E[w_{i0}^2] \right) \exp \left\{ -2\frac{(T-2)}{q(T)} \right\}
\]

\[
= O \left( \frac{1}{T} \right) O(1) O \left( \exp \left\{ -2\frac{T}{q(T)} \right\} \right) = O \left( \frac{1}{T} \exp \left\{ -\frac{2T}{q(T)} \right\} \right),
\]

from which it follows, by Markov’s inequality, that

\[
\frac{1}{NT} \sum_{i=1}^{N} \rho_T^{2(T-2)} w_{i0}^2 = O_p \left( \frac{1}{T} \exp \left\{ -\frac{2T}{q(T)} \right\} \right).
\]

Also, by the Cauchy-Schwarz inequality,

\[
\left| \frac{1}{NT} \sum_{i=1}^{N} w_{T-2,T}^i \rho_T^{T-2} w_{i0} \right| \leq \sqrt{\frac{1}{NT} \sum_{i=1}^{N} w_{T-2,T}^2} \sqrt{\frac{1}{NT} \sum_{i=1}^{N} \rho_T^{2(T-2)} w_{i0}^2}
\]

\[
= O_p \left( \sqrt{\frac{q(T)}{T}} \right) O_p \left( \frac{1}{\sqrt{T}} \exp \left\{ -\frac{T}{q(T)} \right\} \right) = O_p \left( \frac{\sqrt{q(T)}}{T} \exp \left\{ -\frac{T}{q(T)} \right\} \right).
\]

Hence,
\[
\frac{1}{NT} \sum_{i=1}^{N} w_{iT-2,T}^2 = \frac{1}{NT} \sum_{i=1}^{N} w_{iT-2,T}^2 + 2 \frac{1}{NT} \sum_{i=1}^{N} w_{iT-2}\rho_T^{T-2} w_{i0} + \frac{1}{NT} \sum_{i=1}^{N} \rho_T^{2(T-2)} w_{i0}^2 \\
= O_p \left( \frac{q(T)}{T} \right) + O_p \left( \frac{\sqrt{q(T)} T}{T} \exp \left\{ - \frac{T}{q(T)} \right\} \right) + O_p \left( \frac{1}{T} \exp \left\{ - \frac{2T}{q(T)} \right\} \right) \\
= O_p \left( \frac{q(T)}{T} \right),
\]

as required for part (d).

Finally, to show part (e), note that
\[
E \left( \frac{1}{NT} \sum_{i=1}^{N} (w_{iT-2,T}^2 - E [w_{iT-2,T}^2]) \right)^2 \\
= \frac{1}{NT^2} \sum_{s=1}^{T-2} \rho_T^{4(T-2-s)} E [\varepsilon_{is}^4] + \frac{2\sigma^4}{N} \left( \frac{1}{T} \sum_{s=1}^{T-2} \rho_T^{2(T-2-s)} \right)^2 \\
= \frac{E [\varepsilon_{is}^4]}{NT^2} \left( \frac{1 - \rho_T^4(T-2)}{1 - \rho_T^4} \right) + \frac{2\sigma^4}{NT^2} \left( \frac{1 - \rho_T^2(T-2)}{1 - \rho_T^2} \right)^2.
\]

Here, we take \( q(T) = O(1) \), so that there exist a positive constant \( C_q \) and a positive integer \( T^* \) such that for all \( T \geq T^* \)
\[
\rho_T^g = \exp \left\{ - \frac{g}{q(T)} \right\} \leq \exp \left\{ - \frac{g}{C_q} \right\} < 1 \quad \text{with} \quad g \in \{2, 4\},
\]
from which we further deduce that
\[
E \left( \frac{1}{NT} \sum_{i=1}^{N} (w_{iT-2,T}^2 - E [w_{iT-2,T}^2]) \right)^2 \\
\leq \frac{1}{NT^2} \left[ \frac{E [\varepsilon_{is}^4]}{1 - \exp \{-4/C_q\}} + \frac{2\sigma^4}{(1 - \exp \{-2/C_q\})^2} \right] = O \left( \frac{1}{NT^2} \right).
\]

It follows from Markov’s inequality and part (e) of Lemma SE-31 that
\[
\frac{1}{NT} \sum_{i=1}^{N} w_{iT-2,T}^2 = \frac{1}{NT} \sum_{i=1}^{N} E [w_{iT-2,T}^2] + O_p \left( \frac{1}{\sqrt{NT}} \right) \\
= O_p \left( \frac{1}{T} \right) + O_p \left( \frac{1}{\sqrt{NT}} \right) = O_p \left( \frac{1}{T} \right).
\]

Moreover, using Assumption 4, we have
\[
E \left[ \frac{1}{NT} \sum_{i=1}^{N} \rho_T^{2(T-2)} w_{i0}^2 \right] \leq \frac{1}{T} \left( \sup_i E [w_{i0}^2] \right) \rho_T^{2(T-2)} = O \left( \frac{1}{T} \right) O(1) O \left( \rho_T^{2(T-2)} \right) = O \left( \frac{\rho_T^{2T}}{T} \right),
\]

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from which it follows, by Markov’s inequality, that
\[
\frac{1}{NT} \sum_{i=1}^{N} \rho_T^{2(T-2)} w_{i0}^2 = O_p \left( \frac{\rho_T^2}{T} \right).
\]
Also, by the Cauchy-Schwarz inequality,
\[
\left| \frac{1}{NT} \sum_{i=1}^{N} w_{iT-2} \rho_T^{T-2} w_{i0} \right| \leq \sqrt{\frac{1}{NT} \sum_{i=1}^{N} w_{iT-2}^2} \sqrt{\frac{1}{NT} \sum_{i=1}^{N} \rho_T^{2(T-2)} w_{i0}^2}
\]
\[
= O_p \left( \frac{1}{\sqrt{T}} \right) O_p \left( \frac{\rho_T^T}{\sqrt{T}} \right) = O_p \left( \frac{\rho_T^T}{T} \right).
\]
Hence,
\[
\frac{1}{NT} \sum_{i=1}^{N} w_{iT-2}^2 = \frac{1}{NT} \sum_{i=1}^{N} w_{iT-2}^2 + 2 \frac{1}{NT} \sum_{i=1}^{N} w_{iT-2} \rho_T^{T-2} w_{i0} + \frac{1}{NT} \sum_{i=1}^{N} \rho_T^{2(T-2)} w_{i0}^2
\]
\[
= O_p \left( \frac{1}{T} \right) + O_p \left( T^{-1} \rho_T^T \right) + O_p \left( T^{-1} \rho_T^{2T} \right) = O_p \left( \frac{1}{T} \right),
\]
as required for part (e). □

Lemma SE-33 below is a well-known lemma concerning strictly increasing functions which characterize subsequences.

**Lemma SE-33:** Let \( f : \mathbb{N} \to \mathbb{N} \) be a strictly increasing function in its argument, where \( \mathbb{N} \) denotes the set of natural numbers, i.e., \( \{1, 2, ..., \} \). Then, \( f(T) \geq T \) for all \( T \in \mathbb{N} \).

**Proof:**
The proof follows trivially by mathematical induction after first noting that \( f(1) \geq 1 \) since \( f(1) \in \mathbb{N} \). □

**Lemma SE-34:**
Let \( \varphi(x) = 1 - \frac{1}{4x} \exp \{ -2x \} + 2x - 1 \). Then, \( \varphi(x) \geq 1/2 \), for \( 0 < x < \infty \).

**Proof:**
Note that
\[
\varphi'(x) = \frac{1}{4x^2} \exp \{ -2x \} + 2x - 1 - \frac{1}{4x} [-2 \exp \{ -2x \} + 2]
\]
\[
= \frac{1}{4x^2} \exp \{ -2x \} + 2x - 1 + 2x \exp \{ -2x \} - 2x
\]
\[
= \frac{1}{4x^2} [(1 + 2x) \exp \{ -2x \} - 1]
\]
\[
< 0, \text{ for all } x \text{ such that } 0 < x < \infty,
\]
where the last inequality follows from the inequality $1 + 2x < \exp \{2x\}$ for all $x \in (0, \infty)$. Next, observe that

$$
\lim_{x \to \infty} \varphi(x) = \lim_{x \to \infty} \left[ 1 + \frac{1}{4x} + \frac{1}{4x^2e^{2x}} - \frac{1}{2} \right] = \frac{1}{2}
$$

Also, let $f(x) = \frac{1}{4x} \left[ \exp \{-2x\} + 2x - 1 \right]$, and note that, by L'Hôpital's rule,

$$
\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{-2\exp\{-2x\} + 2}{4} = 0.
$$

From these calculations, we see that $\varphi(x)$ is a function which approaches $1$ as $x \to 0$, which approaches $1/2$ as $x \to \infty$, and which is monotonically decreasing in between. The required result, thus, follows. □

**Lemma SE-35:**

Let

$$
\begin{align*}
\mathcal{G}_1^0 &= \{\rho_T : \rho_T = 1 \text{ for all } T \text{ sufficiently large}\}, \\
\mathcal{G}_2^0 &= \left\{ \rho_T = \exp \left\{ -\frac{1}{q(T)} \right\} : T/q(T) \to 0 \right\}, \\
\mathcal{G}_3^0 &= \left\{ \rho_T = \exp \left\{ -\frac{1}{q(T)} \right\} : q(T) \sim T \right\}, \\
\mathcal{G}_4^0 &= \left\{ \rho_T = \exp \left\{ -\frac{1}{q(T)} \right\} : q(T) \to \infty \text{ but } q(T)/T \to 0 \right\}, \\
\mathcal{G}_5^0 &= \mathcal{G}_S = \left\{ |\rho_T| = \exp \left\{ -\frac{1}{q(T)} \right\} : q(T) \geq 0 \text{ and } q(T) = O(1) \text{ as } T \to \infty \right\}.
\end{align*}
$$

Suppose that $\rho_T \in \bigcup_{j=1}^5 \mathcal{G}_j^0$. Then, under Assumptions 1 and 4,

$$
\frac{\rho_T}{\sqrt{NT}} \sum_{i=1}^N w_{i0} \varepsilon_{i2} = O_p \left( \frac{1}{\sqrt{T}} \right),
$$

as $N, T \to \infty$.

**Proof of Lemma SE-35:**

Note that, by Assumptions 1 and 4,

$$
E \left( \frac{1}{\sqrt{NT}} \sum_{i=1}^N w_{i0} \varepsilon_{i2} \right)^2 = \frac{1}{NT} \sum_{i=1}^N \sum_{j=1}^N E [w_{i0}w_{j0}] E [\varepsilon_{i2}\varepsilon_{j2}]
$$

$$
= \frac{\sigma^2}{NT} \sum_{i=1}^N E [w_{i0}^2]
$$

$$
\leq \frac{\sigma^2}{T} \sup_i E [w_{i0}^2] = O \left( \frac{1}{T} \right),
$$

from which we deduce, using Markov's inequality, that

$$
\frac{1}{\sqrt{NT}} \sum_{i=1}^N w_{i0} \varepsilon_{i2} = O_p \left( \frac{1}{\sqrt{T}} \right).
$$
Moreover, since $|\rho_T| \leq 1$ for all $T$ sufficiently large for $\rho_T \in \bigcup_{j=1}^{5} G_j^0$, it further follows that

$$\frac{\rho_T}{\sqrt{NT}} \sum_{i=1}^{N} w_{io} \varepsilon_{i2} = O_p \left( \frac{1}{\sqrt{T}} \right),$$

as required. $\square$

Appendix SF: Additional Monte Carlo Results

This section reports additional Monte Carlo results focusing on comparing finite sample performance of alternative point estimators. The simulation results reported here therefore complement those reported in the main text of the paper, which focused on comparing finite sample performance of confidence procedures. The data generating processes we consider for the purpose of this study are similar to those considered in the paper. In particular, data are generated by the process

$$y_{it} = a_i + w_{it},$$

$$w_{it} = \rho w_{i\text{t-1}} + \varepsilon_{it}, \text{ for } i = 1, \ldots, N \text{ and } t = 1, \ldots, T;$$

where $\{\varepsilon_{it}\} \equiv i.i.d. N(0,1)$ and $\{a_i\} \equiv i.i.d. N(2,1)$. We vary $w_{i0} = 0, 2$ and $\rho = 1.00, 0.99, 0.95, 0.90, 0.80, \text{ and } 0.60$. In addition, we let $N = 100, 200$. When $N = 100$, we take $T = 50, 100$; and when $N = 200$, we consider $T = 100, 200$.

We compare the point estimation properties of the AIP estimator

$$\hat{\rho}_{\text{AIP}} = w_{IC} \hat{\rho}_{\text{IVD}} + (1 - w_{IC}) \hat{\rho}_{\text{pols}},$$

where

$$w_{IC} = \left[ 1 + \exp \left\{ \frac{1}{2} \left( T_{NT} + \sqrt{N \left( 1 + \ln \left( \ln T \right)^{1/2} \right)} \right) \right\} \right]^{-1},$$

with that of the the bias-corrected within-group (BCWG) estimator of Hahn and Kuersteiner (2002), the POLS estimator, the Anderson-Hsiao IV estimator, the X-differencing estimator of Han, Phillips, and Sul (2014), and the Arellano-Bover IV estimator proposed in Arellano and Bover (1995) and further analyzed in Blundell and Bond (1998). Here, we differentiate between the Anderson-Hsiao IV estimator and the Arellano-Bover IV estimator by denoting the former as $\hat{\rho}_{\text{IVD}}$, since it is based on performing IV on the first-differenced equation, and denoting the Arellano-Bover IV estimator as $\hat{\rho}_{\text{IVL}}$, since it is based on performing IV on the equation in levels. Tables SF-1 through SF-4 report median bias of the estimators included in the comparison for different configurations of $N, T, \rho_0$, and $w_{i0}$, whereas Tables SF-5 through SF-8 report results for the range between the 0.05 and the 0.95 quantiles for the same set of experiments. Looking at the results reported in these tables, it seems that the only general conclusion we can make is that no particular estimator dominates all others either in terms of median bias or in terms of 0.05-0.95 quantile range. Different estimators perform better or worse for different subclasses of experiments characterized by $N, T, \rho_0$, and $w_{i0}$. However, there are some patterns in the simulation data which we will summarize below.

---

1We have chosen to use median bias and 0.05-0.95 quantile range to measure the centrality and the dispersion of the point estimators because these measures are robust to the possible non-existence of finite sample moments of estimators.
First, the results of the first four tables show that, in the cases where \(w_{i0} = 0\) which is consistent with the assumption of mean stationarity of the initial condition, \(\hat{\rho}_{IVL}\) tends to be the best performer in terms of median bias overall. On the other hand, \(\hat{\rho}_{IVL}\) is also the estimator whose performance is most sensitive to the specification of the initial condition, so that, under the specification that \(w_{i0} = 2\) which violates the assumption of mean stationarity, the performance of \(\hat{\rho}_{IVL}\) deteriorates considerably, particularly in the cases where the underlying process is stable. More specifically, for experiments where \(w_{i0} = 2\) and \(\rho_0 \leq 0.9\), the performance of \(\hat{\rho}_{IVL}\) in terms of median bias is only fifth amongst the six estimators. On the other hand, relative to the other estimators, the performance of the AIP estimator \(\hat{\rho}_{AIP}\) is the most robust across different specifications of \(N, T, \rho_0, \) and \(w_{i0}\) in the sense that, across all experiments, it is the only estimator which never ranked in the bottom two in terms of median bias. Moreover, compared to the other five estimators, \(\hat{\rho}_{AIP}\) seems to perform particularly well in terms of median bias in the experiments reported in Table SF-4, where the sample sizes are relatively large (\(N = 200\) and \(T = 100\) or \(200\)) and where mean stationarity of the initial condition is not assumed, i.e., \(w_{i0} = 2\). In this set of experiments, \(\hat{\rho}_{AIP}\) ranks first or second in ten of the twelve experiments.

With respect to the other estimators, note that the results on the median bias for the Anderson-Hsiao IV estimator \(\hat{\rho}_{IVD}\) and the POLS estimator \(\hat{\rho}_{POLS}\) are very much in agreement with what is predicted by our large sample theory. POLS does poorly in terms of median bias when the underlying process is stable but much better when \(\rho_0 = 1\), while IV is just the opposite. The performance of the X-differencing estimator of Han, Phillips, and Sul (2014) is similar to that of POLS in the sense that it is better when \(\rho_0\) is in the “more persistent” region of the parameter space than when it is in the “more stable” region; but, overall, the median bias of the X-differencing estimator is smaller than that of POLS, and its performance across the parameter space is more uniform than that of POLS. Finally, the median biases of the BCWG estimator are also smaller when \(\rho_0\) is in the \(0.6 - 0.9\) range than in cases where \(0.95 \leq \rho_0 \leq 1\). These results are consistent with results given in Hahn and Kuersteiner (2002), as their bias-correction procedure is specifically designed to remove second-order biases in the stable case, not the unit root case.

Next, we turn our attention to Tables SF-5 through SF-8, which report results on the dispersion of various estimators as measured by the 0.05-0.95 quantile range. Here, note that the POLS estimator tends to be the best performer, particularly in the unit root and near unit root cases. On the other hand, AIP does well in the persistent region of the parameter space (e.g., when \(\rho_0 = 1\) or \(\rho_0 = 0.99\)) but tends to do less well relative to all the “OLS-type” estimators (i.e., \(\hat{\rho}_{BCWG}, \hat{\rho}_{POLS},\) and \(\hat{\rho}_{XD}\)) in terms of dispersion when the underlying process is stable. The degree of dispersion exhibited by AIP very much reflects that of the constituent estimators, \(\hat{\rho}_{IVD}\) and \(\hat{\rho}_{POLS}\), from which it is constructed. Hence, when the underlying process is very persistent, the weights on AIP shift toward POLS, thus, taking advantage of the efficiency of the latter estimator when the true autoregressive coefficient is unity or very close to unity. On the other hand, when the underlying process is stable, the weights on AIP shift toward the Anderson-Hsiao IV estimator, which, although well-centered in terms of having a small median bias, tends to be less efficient relative to the “OLS-type” estimators; and our simulation results show that the AIP estimator inherits both the virtue and the vice of this estimator.
## Table SF-1: Median Bias

\(N = 100, \ w_{i0} = 0\)

<table>
<thead>
<tr>
<th>(\rho_0)</th>
<th>(T)</th>
<th>(\hat{\rho}_{BCWG})</th>
<th>(\hat{\rho}_{POLS})</th>
<th>(\hat{\rho}_{IVD})</th>
<th>(\hat{\rho}_{XD})</th>
<th>(\hat{\rho}_{IVL})</th>
<th>(\hat{\rho}_{AIP})</th>
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Results are based on 10,000 simulations.

## Table SF-2: Median Bias

\(N = 100, \ w_{i0} = 2\)

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<tr>
<th>(\rho_0)</th>
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Results are based on 10,000 simulations.
### Table SF-3: Median Bias

\( N = 200, \ w_{i0} = 0 \)

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<tr>
<th>( \rho_0 )</th>
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</table>

Results are based on 10,000 simulations.

### Table SF-4: Median Bias

\( N = 200, \ w_{i0} = 2 \)

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<th>( \rho_0 )</th>
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<td>-2.60 \times 10^{-6}</td>
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</table>

Results are based on 10,000 simulations.
Table SF-5: Nine Decile Range 0.05 to 0.95
\[ N = 100, \, w_{i0} = 0 \]

<table>
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<tr>
<th>( \rho_0 )</th>
<th>( T )</th>
<th>( \hat{\rho}_{BCWG} )</th>
<th>( \hat{\rho}_{POLS} )</th>
<th>( \hat{\rho}_{IVD} )</th>
<th>( \hat{\rho}_{XD} )</th>
<th>( \hat{\rho}_{IVL} )</th>
<th>( \hat{\rho}_{AIP} )</th>
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Results are based on 10,000 simulations.

Table SF-6: Nine Decile Range 0.05 to 0.95
\[ N = 100, \, w_{i0} = 2 \]

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<tr>
<th>( \rho_0 )</th>
<th>( T )</th>
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<th>( \hat{\rho}_{POLS} )</th>
<th>( \hat{\rho}_{IVD} )</th>
<th>( \hat{\rho}_{XD} )</th>
<th>( \hat{\rho}_{IVL} )</th>
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</table>

Results are based on 10,000 simulations.
Table SF-7: Nine Decile Range 0.05 to 0.95

\[ N = 200, w_{i0} = 0 \]

<table>
<thead>
<tr>
<th>( \rho )</th>
<th>( T )</th>
<th>( \hat{\rho}_{BCWG} )</th>
<th>( \hat{\rho}_{POLS} )</th>
<th>( \hat{\rho}_{IVD} )</th>
<th>( \hat{\rho}_{XD} )</th>
<th>( \hat{\rho}_{IVL} )</th>
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<td>0.0324</td>
<td>0.0043</td>
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<td>0.0026</td>
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</table>

Results are based on 10,000 simulations.

Table SF-8: Nine Decile Range 0.05 to 0.95

\[ N = 200, w_{i0} = 2 \]

<table>
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<tr>
<th>( \rho )</th>
<th>( T )</th>
<th>( \hat{\rho}_{BCWG} )</th>
<th>( \hat{\rho}_{POLS} )</th>
<th>( \hat{\rho}_{IVD} )</th>
<th>( \hat{\rho}_{XD} )</th>
<th>( \hat{\rho}_{IVL} )</th>
<th>( \hat{\rho}_{AIP} )</th>
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<tr>
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Results are based on 10,000 simulations.

References


