Political Economy of Sovereign Debt: Cycles of Debt Crisis and Inequality Overhang

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Abstract

We study optimal fiscal and redistributive policy in an open economy without commitment. We show that high level of inequality in the past creates strong incentives for default on external debt; what we refer to as inequality overhang. Our main result is that optimal repayment policies for economies with high levels of debt involve large reductions in external debt positions, resembling extreme austerity measures. Such repayment policies lead to cyclical dynamics in external debt and fiscal policies such as transfers and pensions. Finally, we show that large external debt reductions coincide with increases in fraction of government debt held by the domestic citizens.

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1 Introduction

Recent events in the Euro zone have highlighted the political determinants of countries’ external debt policies. Following the crisis in 2009, many European countries have implemented highly unpopular fiscal consolidation programs, the so-called austerity measures, in order to reduce their external debt burden. More recently, this lack of popularity has resulted in resurgence of political parties that advocate anti-austerity measures suggesting a possible cyclical pattern in debt and more broadly fiscal policies. More systematically, as shown by Voth (2012) and Ponticelli and Voth (2012), large austerity measures typically lead to social unrest and in many cases reversal of these policies. The main purpose of this paper is to provide a political economy model of determination of fiscal policy where policies are determined under lack of commitment and in presence of redistributive motives. We show that such cycles which feature austerity arise naturally as a result of the interaction between lack of commitment and redistributive motives.

The cyclical movements in external debt and tax policy are of the following form: when external public debt is low, the government runs large redistributive programs, large deficits and accumulation of public foreign debt. Eventually the maximal amount of external debt that can be issued is reached and the government must undergo a large fiscal consolidation associated with a drastic adjustment of external public debt position, large inequality and increase in the share of government debt held by domestic residents. The burden of adjustment is on current generations. The dynamics is cyclical because the large fiscal consolidation allows future governments to pursue lax fiscal policies.

We illustrate this mechanism in an overlapping generations economy where in each period a continuum of households with heterogeneous labor productivities are born. Households work, consume and save for retirement when they are old. Government uses distortionary taxes on earnings and assets as well as external and domestic debt to raise revenues to spend on transfers to the young and to the old and to finance some exogenously given government consumption.

We consider two scenarios for the determination of policies. First, we consider the case in which policies are chosen by a planner that lacks commitment and has motives to redistribute resources within and across generations. Second, we consider the case in which policies are the outcome of an explicit probabilistic voting game along the lines of Lindbeck and Weibull (1987) (See also Song et al. (2012), Farhi et al. (2012), and Scheuer and Wolitzky (2014) for similar applications to dynamic settings.) As it is standard in probabilistic voting models, the equilibrium policies are the result of a game between fictitious planner in each period that chooses policies to maximize a weighted average of
the welfare of the generations alive in the current period. Relative to the first setup, there are some elements of strategic interactions among the fictitious planners. Our main result about cyclicity of fiscal policies apply to both of the setups.

In our setup, governments have two motives for the governments to renege on previous promises, e.g., repayments of government debt, pension payments and wealth taxes. First, the government has a foreign motive to default in that it is tempted to repudiate its external debt to reduce the payments it makes to foreign lenders. Second, the government has a domestic motive to renege on past promises due to the presence of wealth inequality (formed in the past) and redistributive motives. The government has an incentive to expropriate wealth and redistribute the resulting resources across the population, thereby reducing inequality. The force that balances these motives and prevents default - defined broadly as reneging on promises made by previous governments - is the disruptions in private credit markets associated with it.

The interplay between the two incentives to default is the critical mechanism that drives our novel results. This can be seen from the effect of an increase in wealth inequality on incentives to default on external debt. In particular, when wealth inequality is high, the domestic motive to renege on promises is high and so the amount of foreign debt that can be credibly supported is low. Hence, when current income inequality is high (and so future wealth inequality is high) the current government can credibly issue a lower amount of debt than when inequality is low. We call this mechanism inequality overhang.

The cyclical nature of policies follows from the inequality overhang. To understand the forces operating, consider a situation in which inherited foreign debt is sufficiently high so that the current government cannot roll it over in a credible way. The current government is then forced to reduce the debt it passes to future governments. We call this episodes fiscal consolidations. Because of lack of commitment, the government is unable to equally share the burden of foreign debt repayments across generations, it must necessarily cut back from transfers and tolerate high level of income and consumption inequality within a generation. High level of inequality coincides with stronger incentives for default in the future and as a result, future government should be compensated with low level of external debt in order not to default. This implies that the reduction of foreign debt must be substantial to make the arrangement credible because when inequality is high, low foreign debt can be issued. This large reduction in debt leaves the future governments with few resources committed to repaying foreign lenders. This allows the government to implement large redistributive program, to issue a lot of foreign debt forcing future governments into fiscal consolidations.

Additionally, we show that fiscal consolidations are accompanied by an increase in the
share of government debt held domestically. This is mainly due to the fact that making consolidations credible requires a future compensations. Hence it is optimal for the current government to promise high consumption to the current young households when they reach the old age. This can be achieved through a combination of high pension promises and by issuing large amount of domestic debt. To incentivize young households to save, the government must subsidize the interest rate they receive relative to the interest rate prevailing on international credit markets.

The cyclical pattern of fiscal policies and inequality overhang is consistent with several observations from Latin American countries. In particular, some studies have pointed to the a negative relationship between inequality (typically income inequality measured by the Gini index) and a country’s likelihood of default – See Berg and Sachs (1988) and more recently Aizenman and Jinjarak (2012). Furthermore, that fiscal policies are cyclical have long been observed in various Latin American economies. Of special interest is the studies done by Dornbusch and Edwards (Dornbusch and Edwards (1991), Dornbusch and Edwards (1989)) and Sachs (1989). These authors observe that fiscal policies in Latin America seem to be cyclical and driven by populism. As we will discuss, some features of these policies are consistent with the dynamics in our mode. Finally, the implication of our model that during fiscal consolidation episodes government debt’s clientele become more concentrated among domestic households is consistent with the recent experience of the European countries as well some of the Latin American economies experiencing debt crisis.

Related Literature. In this paper, we build on several strands of literature in public finance, macroeconomics and international economics.

Our work is most closely related to the recent dynamic public finance literature on redistributive taxes where either policy makers cannot commit to future policies or policies are determined via a political equilibrium. Notable examples include Acemoglu et al. (2008), Farhi et al. (2012), and Scheuer and Wolitzky (2014). This literature focuses on closed economy models and draws implications regarding taxation of capital. In our model, we abstract from capital accumulation and instead focus on interactions between sovereign default and wealth inequality. While for simplicity, we focus on linear tax functions, we conjecture that our results will go through even when non-linear taxes are allowed. In this regard, we extend the model in Werning (2007) to an open economy with overlapping generations.

Our paper is also related to the large literature in international economics on determinants of sovereign debt and their interactions with domestic policies. Examples include
Eaton and Gersovitz (1981), Arellano (2008), Aguiar et al. (2009), Dovis (2013), among many others. Most of these models abstract from distributional issues within the country by considering representative agent economies. The models thus become resemblant of a contracting problem between borrower and lender; either with exogenously imposed contractual incompleteness as in Eaton and Gersovitz (1981) and Arellano (2008) or endogenously incomplete contracts as in Aguiar et al. (2009) and Dovis (2013). Focusing on distributional issues within a country allows us to study interactions between inequality and foreign debt. The cyclicality of debt repayment is a direct consequence of this interaction.

Two recent papers address some of the issues related to this interaction: First, D’Erasmo and Mendoza (2013) consider a closed economy with heterogeneous agents in which government does not have access to asset taxes while it can issue debt and default on it. They show that in order to explain the level of domestic debt observed in Europe, government objective must feature a political bias towards government’s creditors. Second, Ferriere (2014) argues that when the government can change the progressivity of the tax code, progressive taxes can be used to mitigate cost of default and cost of borrowing. As in these papers, we show that wealth inequality negatively affects government’s ability to borrow. However, our novel result is that due to the inequality overhang effect, it is optimal to make large repayment when highly indebted and thus policies resemble potentially extreme austerity.

The rest of the paper is organized as follows. Section 2 describes the model. Section 3 characterizes the equilibrium where policies are chosen by a planner without commitment. Section 4 characterizes the political economy environment. Section 5 discusses generalizations of the model. Section 6 provides empirical support for our results.

## 2 Model Setup

In this section, we describe the basic framework in which we analyze the determinants of policies. We consider an overlapping generations economy populated by a continuum of heterogeneous households together with the government.

**Households.** Time, $t$, is discrete and $t = 0, 1, \cdots$. At the beginning of each period, a generation of households are born and live for two periods. Each generation born at $t$ is consisted of a continuum of households who draw a productivity type $\theta_t$ when they are born. They work, consume and save when young and consume when old. We assume that $\theta \in \{\theta^1 < \cdots < \theta^N\}$ and that $\Pr(\theta_t = \theta^i) = \mu^i$ where the average value of $\theta^i$ is normalized to 1, i.e., $\sum_{i=1}^{N} \mu^i \theta^i = 1$. We refer to households with productivity type $\theta^i$,
as households of type $i$. The labor productivity of an agent of type $i$ is given by $Zθ^i$, where $Z$ is a common factor affecting productivity of all households. Households value consumption and labor supply according to the following preferences:

$$U\left(c_{t,0}^i, 1 - \frac{y_t^i}{θ^i}\right) + \beta u\left(c_{t,1}^i\right)$$  \hspace{1cm} (1)

where $c_{t,0}^i, c_{t,1}^i$ are consumption when young and old, respectively while $y_t^i$ is the effective hours worked by the agent when young – $Zy_t^i$ is income. In most of the paper we focus on a case where the utility function is given by

$$\log c_{t,0}^i + ψ \log \left(1 - \frac{y_t^i}{θ^i}\right) + β \log c_{t,1}^i$$  \hspace{1cm} (2)

In section 5.1, we discuss this choice of utility function and its role in the results as well as extension our results to other preferences.

A young individual of type $i$ has $a_{i,t}$ in wealth at the end of the period. Finally, there is an initial old generation at $t = 0$ whose wealth distribution are given by $\{a_{i-1}\}_{i=1,\ldots,N}$ while their consumption is given by $c_{t-1,1}^i$.

**Technology.** Labor is the only input for production and total output in the economy is given by $Z\sum_i μ_i y_{i,t}$.

**Government.** Government provides transfers to agents when young, $T_t$, pensions to households when old $P_t$ and pays for government purchases $G$. It finances these expenses with revenues raised from taxing labor income, taxing assets saved for retirement as well as borrowing from the rest of the world. Given the structure of government finances, the government budget constraint is given by

$$(1 + r_t) B_t + (1 + r^*) B^*_t + T_t + P_t + G = τ_{t,t} Z \sum_i μ_i y_{i,t} + τ_{a,t} \sum_i μ_i a_{i,t-1} + B^*_{t+1} + B_{t+1}$$  \hspace{1cm} (3)

where $B_t$ and $B^*_t$ are, respectively, face value of government’s domestic and foreign debt, $T_t$ is transfers to the young, $P_t$ is public pensions for the old, $τ_{t,t}$ is the tax rate on earnings, $τ_{a,t}$ is the tax rate on assets (private pensions), $r^*$ is the interest rate on government’s international debt while $r_t$ is the interest rate on government’s domestic debt. In addition, $δ_t ∈ [0, 1]$, is the fraction of the foreign debt that is repaid in each period. Note that since the government can impose asset taxes on domestic holdings of its debt, it is without loss of generality to assume that all domestic is repaid.
Given the above government policies, we can write households’ budget constraint as

\[ \begin{align*}
    c_{i,t,0}^i + a_{i,t}^i & \leq Z y_{i,t}^i (1 - \tau_{a,t}) + T_t^i \\
    c_{i,t,1}^i & \leq (1 + \hat{r}_{t+1}) a_{i,t}^i (1 - \tau_{t+1,a}) + P_{t+1}^i
\end{align*} \tag{4} \tag{5} \]

In addition, the initial old simply consume their after-tax asset income and pensions, or

\[ c_{-1,1}^i = a_{-1}^i (1 - \tau_{a,0}) + P_0. \]

Note that we have assumed that government are the only domestic entity that can borrow and lend in international credit markets. While this is an extreme assumption, when the government has the ability to impose capital controls (taxes on foreign transactions) it is without loss of generality. This is because when government can impose capital controls, taxes on foreign transactions can be imposed in such a way that domestic households have no incentive to trade with the rest of the world and only hold domestic debt.

**Markets.** Feasibility of allocations requires that domestic credit market, as well as goods markets clear. That is,

\[ \sum_i \mu_i a_{i,t} = B_{t+1} \]

\[ \sum_i \mu_i c_{i,t,0}^i + \sum_i \mu_i c_{i,t-1,1}^i = Z \sum_i \mu_i y_{i,t}^i + B_{t+1}^* - (1 + r^*) \delta t B_t^* \]

**Competitive Equilibrium with Taxes.** As it is standard in the Ramsey approach, the above market structure as well as households optimization puts a constraint on the set of allocations that can be achieved by the government. Before describing determination of policies, our notion of competitive equilibrium makes precise the type of restriction the above market structure puts on government’s choice.

**Definition 1** Given a sequence of government policies \( \{B_t, B_t^*, \tau_{l,t}, \tau_{a,t}, T_t, P_t\}_{t=0}^\infty \) as well as international interest rate, \( r^* \), a competitive equilibrium with taxes is given by \( \{c_{i,t,0}^i, c_{i,t,1}^i, y_{i,t}^i, a_{i,t}^i\}_{t=0}^\infty \) together with prices \( \{\hat{r}_t\}_{t=0}^\infty \) where (i) \( c_{i,t,0}^i, c_{i,t,1}^i, y_{i,t}^i, a_{i,t}^i \) maximizes (1) subject to (4) and (5), (ii) domestic bond interest rate, \( \hat{r}_t \), adjust so that domestic credit markets clear, i.e., (6) holds.

### 2.1 Characterizing the set of Competitive Equilibria

Our approach for characterizing optimal policy in this paper is the primal approach as in Lucas and Stokey (1983). That is, we characterize the set of allocations that can be supported by competitive equilibria with taxes and subsequently drive properties of policies from allocations. To do so, we first characterize the set of (interior) allocations that can be
implemented as a competitive equilibrium. Namely, we resort to a result developed by Werning (2007) and leave the derivation to the appendix.

**Lemma 2** Suppose that households’ preferences are given by (2). Given initial asset distribution for the initial old, \( \{a^i_0\}_{i=1,...,N'} \) and an initial foreign government debt, \( B^*_0 \), an allocation \( \left\{ \{c^i_{t,0}, c^i_{t-1,1}, y^i_t, a^i_t\}_{i=1,...,N'} \right\}_{t=0}^\infty \) can be supported as a competitive equilibrium with taxes if and only if it satisfies the following:

1. there exist a vector of market weights \( \{\phi_t^i\}_{i=1,...,N'} \) where \( \sum_i \mu^i \phi_t^i = 1 \) such that

   \[
   c^i_{t,0} = \phi_t^i C^i_{t,0}, c^i_{t,1} = \phi_t^i C^i_{t,1}, \theta^i - y^i_t = \phi_t^i (1 - Y_t) \tag{7}
   \]

   where \( C^i_{t,0}, Y_t, \) and \( C^i_{t,1} \) are the aggregate consumption of the young, aggregate effective hours worked, and aggregate consumption of the old households that are born at \( t \), respectively.

2. The market weights satisfy

   \[
   \phi_t^i = 1 + \frac{\psi}{1 + \psi + \beta} \frac{\theta^i - 1}{1 - Y_t} \tag{8}
   \]

3. Aggregate allocations satisfy the feasibility condition:

   \[
   C^i_{t,0} + C^i_{t-1,1} = Z Y_t + B^*_{t+1} - (1 + r^*) \delta_t B^*_t \tag{9}
   \]

Proof can be found in the Appendix.

The above lemma, characterizes the set of allocations that can be supported in a competitive equilibrium with taxes. The first condition in Lemma 2 is a direct result of linearity of the tax function. Linearity of the tax function that the marginal rate of substitution between consumption when young and old and earnings should be equated across households.\(^1\) Log preferences then imply that (7) must hold. The second property in Lemma 2 is the so-called implementability constraint and is derived from the consolidated budget constraint associated with (4) and (5).

As the above lemma establishes, unlike the standard Ramsey approach in representative agent models (for examples as in Lucas and Stokey (1983) or Chari and Kehoe (1999)), the competitive equilibrium imposes no constraint on aggregate allocations besides feasibility. In other words, any aggregate allocation that satisfies feasibility for which market

\(^1\)The market weights \( \phi_t^i \) are the Negishi weights associated with the competitive equilibrium with taxes.
weights exists to satisfy the above can be supported as a competitive equilibrium with taxes. It does, however, impose restrictions on distribution of consumption and earnings as described by the above lemma.

### 3 A Planning Problem

In this section, we establish the main results of our paper as the outcome of a planning problem without commitment. In particular, we show three main results. First, higher inequality lowers the capacity of the country to borrow internationally. Second, allocations exhibit cyclical dynamics, i.e., convergence of allocations to their steady state value is cyclical. Finally, downward adjustment in external debt are coincident with an increase in domestic holdings of government debt.

Our approach in this section is to study a planning problem without commitment where the objective of the planner is a weighted average of households’ utility within and across generations. In particular, at each point in time, the planner’s objectives are given by

\[
\sum_{s=t}^{\infty} \hat{\beta}^{s-t} \sum_{i} \alpha_i \mu_i u_i^s + \hat{\beta}^{-1} \sum_{i} \alpha_i \mu_i u_i^{t-1}. 
\]

(10)

where \(u_i^s\) is the utility of a young household of type \(i\) born at \(t\) while \(u_i^{t-1}\) is that of an old household of type \(i\) at \(t\). The coefficient \(\alpha_i\) is the welfare weight assigned to a household of type \(i\) within a generation while \(\hat{\beta}^t\) is the welfare weight assigned to generation \(t\). We normalize \(\alpha_i\)’s so that \(\sum_i \mu_i \alpha_i = 1\). Thus the planner sets policies described in section 2 in order to maximize (10).

In section 4, we study a probabilistic voting model and discuss how it can lead to similar welfare functions.

We make two assumptions with respect to this objective that are crucial to our analysis:

**Assumption 3** \(\hat{\beta} (1 + r^*) \leq 1\).

The first assumption states that the price of borrowing funds from international credit markets is at most equal to the intergenerational rate of time preferences for the domestic planner.

**Assumption 4 (Redistributive Motives)** \(\alpha_i\) is weakly decreasing in \(i\).

The second assumption states that the government uses higher welfare weights on individuals with lower labor productivity. One implication of this assumption is that
optimal labor income taxes are always positive. One social welfare function that satisfies this assumption is utilitarian objective for which \( \alpha^1 = 1 \). Another example is the so-called Rawlsian or max-min welfare function where \( \alpha^1 = 1 \) and \( \alpha^i = 0 \) for values of \( i > 1 \).

**Optimal policy with commitment.** Before discussing the key properties of optimal policy without commitment, it is helpful to mention what happens when the planner can commit. With commitment, independent of the initial level of external debt, \( B^*_0 \), the government is able to use international markets to equate consumption and leisure across generations. As a result, consumption of the young, consumption of the old, output and labor income taxes are constant. Note that labor income taxes are chosen to achieve certain level of redistribution within generations while the allocation of consumption across generations are chosen to achieve redistribution across generations. In other words, commitment enables the government to distribute the burden of the initial debt on all future generations while achieve its desired cross-generation consumption and tax smoothing.

**Incentives to retract from promised policies.** There are two main reasons that the optimal policies described above are not time consistent. First, as it is typical in open economy models, if the government is a net debtor, it has strong incentives to default on its promised repayment policies. Second, at each period there is a non-degenerate distribution of wealth among the old households and since the government is inequality-averse it has strong incentive to expropriate all the wealth via a 100% tax on wealth and redistribute it equally among the old. As it becomes clear, the interaction between these two incentives for reneging on promised policies drives most of our results.

**Optimal policy without commitment.** We now describe the constraints imposed by the planner’s lack of ability to future policies. We start our analysis by a rather informal approach to modelling lack of commitment by the government. We assume that lack of commitment, simply implies that at each point in time, the value of the planner’s objective cannot fall below a certain level \( W \). The value of \( W \) is associated with an allocation where the young do not save, the old only consume the pensions received from the government, and the economy is in financial autarky with respect to the rest of the world. Given these constraints, the government optimally chooses the labor income taxes and lump-sum transfers to the old and young. Formally, an allocation is said to be sustainable if it satisfies

\[
\frac{\beta}{\hat{\beta}} \sum_i \alpha^i \mu^i u^i_{t-1,1} + \sum_{s=t}^{\infty} \beta^{s-t} \sum_i \alpha^i \mu^i u^i_s \geq W. \tag{11}
\]
where

\[ W = \frac{1}{1 - \beta} \max_{c_i^0, c_i^1, y^i, \tau^i, T} \sum_i \alpha^i \mu^i u \left( c_i^0, y^i; \theta^i \right) + \frac{\beta}{\hat{\beta}} \sum_i \alpha^i \mu^i u \left( c_1^0 \right) \]  

s.t. \ \ \ \sum_i \mu^i c_i^0 + c_1 + G = \mu^i Z y^i

\[ \theta^i \left( 1 - \tau^i \right) u_{c,0}^i + u_y^i = 0 \]

\[ c_0^i = (1 - \tau^i) Z y^i + T \]

We refer to (11) as the sustainability constraint.

While our analysis here is rather informal, in Appendix, we formalize our modeling of commitment by developing a formal game between the planner, households, and foreign lenders and show that an allocation can be the equilibrium in such a game if and only if it satisfies the characterization in Lemma 2 and satisfies (11). The basic idea is that upon any reneging on policies by the planner (for example, default on external debt), domestic households as well foreign lenders react by excluding the government from international and domestic credit markets. Therefore, in this game, it is without loss of generality to assume that default does not occur in equilibrium. Because of this, in what follows, we set \( \delta^t = 1 \).

Using the above restrictions, we can thus define the best sustainable equilibrium is an allocation that satisfies the conditions in Lemma 2 and (11) and it maximizes 10 at time 0. Formally, the best sustainable equilibrium is the solution to the following optimization problem:

\[ \max \sum_{t=0}^{\infty} \beta^t \sum_i \alpha^i \mu^i \left[ \log c_i^{t,0} + \psi \log \left( 1 - y_i^t / \theta^i \right) + \beta \log c_i^{t,1} \right] + \frac{\beta}{\hat{\beta}} \sum_i \alpha^i \mu^i \log c_{-1,1}^i \]
subject to

\[ c_{i,0}^{t} = \varphi_{i}^{t} C_{t,0}, \quad c_{i,1}^{t} = \varphi_{i}^{t} C_{t,1}, \quad y_{i}^{t} = 1 - \varphi_{i}^{t} (1 - Y_{t}), \quad \forall t \geq 0 \]

\[ \varphi_{i}^{t} = \varphi_{i}^{t} (Y) = 1 + \frac{\psi^{t} \theta^{t - 1}}{1 + \psi + \beta^{t} (1 - Y_{t})}, \quad \forall t \geq 0 \]

\[ B_{0}^{*} = \sum_{t=0}^{\infty} \frac{1}{(1 + r^{*})^{t}} [ZY_{t} - C_{t,0} - C_{t-1,1}] \]

\[ \sum_{s=t}^{\infty} \beta^{s-t} \sum_{i} \alpha_{i} \mu_{i} \left[ \log c_{i,0}^{s} + \psi \log \left( 1 - y_{i}^{s} / \theta^{s} \right) + \beta \log c_{i,1}^{s} \right] + \frac{\beta}{\beta} \sum_{i} \alpha_{i} \mu_{i} \log c_{i-1,1}^{t} \geq W \]

\[ c_{i-1,1} = a_{i-1}^{t} + p_{0} \]

\[ C_{i-1,1} = \sum_{i} \mu_{i} a_{i-1}^{t} + p_{0} \]

\[ \left\{ a_{i-1}^{t} \right\}_{i=1}^{1}, B_{0}^{*} : \text{given} \]

Note that in the above optimization problem, we have imposed that first period asset holdings are not taxed. Note that in our environment, since there is no capital, taxes on asset holdings can be set to 0 without loss of generality – domestic debt issuance by the government can affect domestic interest rate and thus government can affect the intertemporal margin by issuing domestic debt. This is not true in period 0 as the planner can without cost impose a hundred percent tax on initial assets and wipe out the undesired initial inequality. Instead, we assume away this initial motive for taxation of assets in order to keep the problem stationary. This modification is purely for stationarity and does not change our main results.\(^2\)

**A Recursive Formulation.** Problem (P) has a multi-dimensional state variable given by the distribution of debt in the economy - domestic and foreign. To make this problem tractable, we solve it in two stages. First, we solve the problem for all generations except the initial old. Second, we solve the problem for the initial old for a given distribution of debt in the economy. Solving the problem in two stages has the benefit that the planning problem in the first stage can be reduced to a recursive problem where there is single state variable; the promised utility to the initial young going forward.

Formally, we define the promised utility for the planner by

\(^2\)This is in contrast with Chari and Kehoe (1990) and the subsequent literature. In their paper, first period is always different than the subsequent periods in that the government can expropriate all initial capital.
\[ V_t = \sum_{s=t}^{\infty} \beta^{s-t} \sum_i \alpha^i \mu^i \left[ \log c^i_{s,0} + \psi \log \left( 1 - y^i_s / \theta^i_s \right) + \beta \log c^i_{s,1} \right] \]

This is the value to the planner from generations that are born in period \( t \) onwards. Given this value, we can write the recursive formulation of the problem of efficiently allocating resources to generation that are born from period 0 onward. This problem in its dual form is represented by the following Bellman equation:

\[ \hat{B} (V) = \max_{C_0, C_1, Y} ZY - C_0 - \frac{1}{1 + r^*} C_1 + \frac{1}{1 + r^*} \hat{B} (V') \quad (P1) \]

subject to

\[ \frac{\beta}{\hat{B}^{\alpha}} U_p (C_0, C_1, Y) + \hat{B} V' = V \]
\[ \frac{\beta}{\hat{B}} U_p (C_1, Y) + V' \geq W \quad (13) \]

where \( U_p (C_0, C_1, Y) \) is the value to the planner from a generation whose aggregate consumption when young and old are given by \( C_0 \) and \( C_1 \) while the output produced is \( Y \). Note that given these aggregates, distribution of consumption and leisure and consequently social welfare is determined by the constraint (8). The function \( U_p (C_1, Y) \) is similarly defined for the old. Use log preferences, we can write

\[ U_p (C_0, C_1, Y) = \log C_0 + \beta \log C_1 + \psi \log (1 - Y) + (1 + \psi + \beta) H (Y) \]
\[ U_p (C_1, Y) = \log C_1 + H (Y) \]

where the function \( H (Y) \) captures the inequality aversion motives that are present in the planner’s social welfare function

\[ H (Y) = \sum \alpha^i \mu^i \log \varphi^i (Y) \]

Given the value function \( \hat{B} (V) \), the first stage problem is given by

\[ W \left( \left\{ a^i_{-1} \right\}, B_0^* \right) = \max_{P_0, Y_0} \frac{\beta}{\beta} \sum \alpha^i \mu^i \log (a^i_{-1} + P_0) + V_0 \quad (P2) \]
subject to

\[-P_0 - \sum_i \mu_i a_{i-1} + \hat{B}(V_0) = (1 + r^\ast) B_0^\ast \]  \hspace{1cm} (14)

\[\frac{\beta}{\hat{\beta}} \sum \alpha^i \mu^i \log (a^i_{i-1} + P_0) + V_0 \geq W \] \hspace{1cm} (15)

Note that given the stationarity of the planning problem in (P), the problem of the planner at date \( t \) who inherited external debt \( B_t^\ast \) and asset holding distributions given by \( \{a_i^t\}_{i=1}^1 \) is identical to the problem in stage 2. Thus the forces identified in the above problem are also present over time. The following proposition connects the solution to (P) to that of (P1) and (P2).

**Proposition 5** If an allocation solves the planning problem in (P) and the associated promised utility to the government at time 0 is given by \( V_0 \), then the allocations must be generated from the policy functions in (P1) while \( V_0 \) and initial pensions \( P_0 \) are the solution to (P2). Conversely, if an allocation generated from the policy function in (P1) and (P2) satisfies

\[\limsup_{t \to \infty} \frac{\beta^t}{\hat{\beta}} V_t = 0\]

then it must be the solution to (P)

Proposition 5 which is similar to the standard principle of optimality (see Stokey et al. (1989) and is therefore omitted.

The following lemma establishes the basic properties of the value function \( \hat{B}(V) \):

**Lemma 6** The value function \( \hat{B}(V) \) is strictly decreasing, strictly concave and differentiable.

The proof uses standard dynamic programming techniques and is therefore left for the Appendix.

### 3.1 Inequality Overhang

We can use program (P2) to show our first main result; higher level of inequality lowers external debt capacity. We refer to this result as *inequality overhang*.

Consider the unconstrained version of the program (P2) that maximizes the objective subject to the resource constraint (14). When the value of this objective, \( W^u (\{a_{i-1}\}, B_0^\ast) \), is greater than or equal to \( W \), the solution to this unconstrained problem coincides with that of (P2). However, when \( W^u (\{a_{i-1}\}, B_0^\ast) < W \), the constraint set in (P2) is an empty
set – the maximized value is lower than \( W \) and thus no feasible allocations exist that deliver the value \( W \). In words, the planner cannot avoid default on external debt and tolerate asset inequality for values of \( \{\{a_{i-1}\}, B_0^*\} \) such that \( W^{\alpha} (\{a_{i-1}\}, B_0) < W \). This implies that for any distribution of initial asset holdings, \( \{a_{i-1}\} \), we can define external debt capacity \( \bar{B}^* (\{a_{i-1}\}) \) by the highest value \( B^* \) that satisfies

\[
W (\{a_{i-1}\}, B^*) = W.
\]

The following proposition contains our main result about external debt capacity changes with inequality:

**Proposition 7** The value function \( W (\{a_{i-1}\}, B^*) \) is decreasing in \( B^* \). Furthermore, an increase in \( \{a_{i-1}\} \) in the sense of second order stochastic dominance increases the value function \( W (\{a_{i-1}\}, B^*) \). As a result, debt capacity \( B^* (\{a_{i-1}\}) \) increases in response to an increase in the sense of second order stochastic dominance.

The above result is very intuitive; when initial asset inequality increases, the value to the planner from the initial old decreases. This increases the government’s incentive to default. As a result, foreign lenders are less willing to lend and therefore, debt capacity is lower. Although, this result holds with respect to inequality in asset holding among the old, since in our model asset distribution among the old is closely tied to income inequality in the period before, the same effect holds with respect to income inequality. In other words, if due to changes in government policies, income inequality increases, external debt capacity must decrease. Next, we describe how this effect can cause cyclical dynamics in external debt.

### 3.2 Cyclicality of Debt

In this section, we study dynamics of debt over time. Our main result is that due to the inequality overhang, convergence of external debt to its steady state, is cyclical. We show this by showing that the policy function for continuation utility in (P1) is decreasing whenever the sustainability constraint binds while it is increasing when the sustainability constraint is slack. The following theorem states this result:

**Theorem 8** \( V' (V) \) and \( C_1(V) \) are U-shaped: There exists \( V^* \) such that for all \( V < V^* \), \( V' (V) \) and \( C_1(V) \) are decreasing and for all \( V \geq V^* \), \( V' (V) \) and \( C_1(V) \) are increasing. Additionally, for all \( V \leq V^* \), the sustainability constraint is binding while for all \( V > V^* \) the sustainability constraint is slack.
The proof is deferred to the appendix. Mechanically, for low values of $V$, the country needs to do lots of repayment which leads to a high value of output. A high value of output implies a high level of within generation inequality. Due to inequality overhang, high level of inequality creates strong motives for default and as a result, the future old and future generations should be promised a high level of utility, i.e., low level of debt.

Mathematically, the non-monotonicity of the policy function is a consequence of the fact that current and future promised utility, $V$ and $V'$ respectively, are substitute. An increase in $V$, when accompanied by a decrease in $V'$ allows the planner to lower inequality among the future old and as a result current young. Therefore, the change in consumption required to deliver $V$ is smaller. In other words, when the sustainability constraint is binding, the objective function in (P1) is sub-modular in $V$ and $V'$. This is in contrast to many economic applications as studied by Hopenhayn and Prescott (1992).

The cyclicality of the allocations implied by the policy functions in (P1) also translate into external debt. In particular, in period $t$, the value of external debt is given by

$$b_t^* = \hat{B}(V_t) - C_1(V_{t-1}) = \hat{B}(V'(V_{t-1})) - C_1(V_{t-1})$$

When the sustainability constraint is binding at $t$, an increase in $V_{t-1}$ leads to a decrease in $C_1(V_{t-1})$ as well as $V'(V_{t-1})$. Since $\hat{B}(\cdot)$ is a decreasing function, an increase in $V_{t-1}$ must lead to an increase $b_t^*$; the movements in $b_t^*$ and $V_t$ are in opposite directions. Thus $b_t^*$ follows the same cyclical behavior as $V_t$.

### 3.3 Repatriation of Government Debt

In this section, we describe the behavior debt holdings of domestic households and its relation with respect to external debt. Note that in our model a version of Ricardian equivalence holds. This can be seen via a perturbation argument in a competitive equilibrium with taxes. Consider a change in transfers to a particular generation that leaves the present value of the transfers unchanged together with its associated adjustment in domestic debt that keeps government budget constraint hold. Since the present value of transfers is unchanged, the consumption and production behavior of all the generations remains unchanged. This implies that domestic debt and transfers are indeterminate. This result is reminiscent of a similar result in Bhandari et al. (2013) an infinite horizon model.

Given this indeterminacy, we focus on a decentralization where, in each generation, the households with lowest productivity do not have access to credit markets. In other
words, transfers to the young and the old are so that

\[ c_{t,0}^1 = Z y_t^1 (1 - \tau_t^1) + T_t \]
\[ c_{t,1}^1 = P_{t+1} \]

Knowing the above, we can write

\[ B_{t+1} = \sum_i \mu^i a^i_t \]

where \( a^i_t \) is the asset holdings by the young households of type \( i \) that are born at \( t \). Using the budget constraint in (5) and the assumption that asset taxes are not used in equilibrium, we can write

\[
B_{t+1} = \sum_i \mu^i a^i_t = \sum_i \mu^i (c^1_{t,1} - P_{t+1})
= C_{t,1} - c^1_{t,1} = (1 - \phi^1_t)C_{t,1}
\]

In the above formulation, we have used the property that individuals with lowest productivity are hand-to-mouth. The above implies that we can define domestic debt as a function of the state of the economy given by the function \( B(V_t) \) where \( V_t \) is the government’s payoff at \( t \) as

\[
B(V) = \frac{\psi}{1 + \psi + \beta} \frac{1 - \theta^1}{1 - Y(V)} C_1(V)
\]  

(16)

where \( C_1(\cdot) \) and \( Y(\cdot) \) are the policy functions in (P1) and we have used the definition of \( \phi^1_t \) from (8).

The above relationship points towards the evolution of domestic debt over time. Mechanically, when \( V \) decreases and sustainability constraint is binding, \( C_1(\cdot) \) and \( Y(\cdot) \) both increase. This implies that \( B(V) \) is a decreasing function of \( V \). Therefore, for low values of \( V_t \), or episodes that involve large reduction in value of external debt, aggregate consumption of the old, \( C_{t,1} \) together with share of consumption going to the more productive young households, \( 1 - \phi^1_t \), must be high. Hence, total domestic debt must be high during episodes in which foreign debt is repaid.

Intuitively, as established above for episodes of large repayment of foreign debt, inequality increases and so does after tax income of highly productive households. This implies that the young productive households’ demand for saving increases. The government in return increases its’ domestic debt issuance, in response to high demand for saving from the young households.
We summarize this discussion in the following Theorem:

**Theorem 9** Let \( B(V) \) be defined by (16). Then, \( B(V) \) is U-shaped in \( V \). Furthermore, \( \arg\min_V B(V) = \arg\min_V V'(V) \).

### 4 Political Economy Model

So far, we have considered an environment in which policies are chosen by a planner that lacks commitment. In this section, we explicitly consider a political process through which policies are determined. To do so, we follow the probabilistic voting model of Lindbeck and Weibull (1987). We show that our main results extend to this environment. In particular, there are cycles and fiscal consolidations coincides with an increase in the share of domestic holdings of government debt.

#### 4.1 Political Economy Game

The political economy game described here is akin to that of Lindbeck and Weibull (1987) (See also Song et al. (2012), Farhi et al. (2012) and Scheuer and Wolitzky (2014) for applications to dynamic settings). We assume that in each period, political competition occurs between two parties \( j = A \) and \( B \) and the timing of the game is as follows:

**Stage 1.** At the beginning of the period, each party \( j \in \{A, B\} \) proposes a policy given by
\[
\sigma_{t,j} = (\delta_{t,j}, \tau_{a,t,j}, P_{t,j}, T_{t,j}, \tau_{l,t,j}, B_{t+1,j}, B_{t+1,j}^*) .
\]

**Stage 2.** Households vote for their preferred policy and the party with majority of the votes wins the election. In the case of a tie, each party wins the election with probability 1/2.

**Stage 3.** Policies are implemented.

As in Lindbeck and Weibull (1987), we assume that households’ decision to vote is affected by an additive utility shock that is independent of their labor income or asset holdings. In particular, we assume that an old household of type \( i \), drives a utility from the election of party \( j \in \{A, B\} \) which is given by
\[
u_{t-1,i,j} + \varepsilon_{t,j}^{i,o} \]
where \( \nu_{t-1,i,j} \) is the utility of the old household under \( \sigma_{t,j} \) while \( \varepsilon_{t,j}^{i,o} \) is a continuous random variable. Similarly, for a young household of type \( i \), the payoff from election of party \( j \in \{A, B\} \) is given by
\[
u_{t,j} + \varepsilon_{t,j}^i \]
where \( \nu_{t,j} \) is the young household’s utility under policy \( \sigma_{t,j} \) and \( \varepsilon_{t,j}^i \) is a continuous random variable. We assume that \( \varepsilon_{t,j}^{i,o}, \varepsilon_{t,j}^i \) are random variables that are
identically and independently distributed across young (old) households of type \( i \). Note that the utility of the young household depends on policies implemented at \( t + 1 \) and thus, \( u^{i}{_{t,1}} \) should be calculated using expectation of future policies. This can potentially create strategic interaction between policy decisions over time.

We assume that each party’s objective is to maximize the probability of winning an election. For a given policy choice by each party \( (\sigma^A_t, \sigma^B_t) \), the probability of winning the election by party \( A \) is given by

\[
\frac{1}{2} + \omega \sum_i \mu^i \alpha^i \left( u^{i-1,1}_t - u^{i-1,1}_B \right) + \sum_i \mu^i \alpha^i \left( u^{i}_A - u^{i}_B \right)
\]

where \( F^i \) is the cumulative distribution for the random variable \( \epsilon^{i}_t - \epsilon^{i}_B \) and \( F^0 \) is defined accordingly for the old. We assume that \( \epsilon^{i}_t - \epsilon^{i}_B \)'s are distributed such that \( \epsilon^{i-1}_t - \epsilon^{i-1}_B \) and \( \epsilon^{i}_t - \epsilon^{i}_A \) are distributed uniformly over the intervals \( \left[-\frac{1}{2\alpha^i}, \frac{1}{2\alpha^i}\right] \) and \( \left[-\frac{1}{2\omega^i\alpha^i}, \frac{1}{2\omega^i\alpha^i}\right] \), respectively. Additionally, we assume that \( \alpha^i \) is small enough so that \( u^{i}_A - u^{i}_B \) is always interior to the support of \( F^i \) – a similar property holds for \( u^{i-1,1}_A - u^{i-1,1}_B \). These assumptions imply that probability of winning the election for party \( A \) is given by

We make the following assumption regarding the distribution of shocks to political preferences:

**Assumption 10** The distribution of \( \epsilon^{i}_t \) and \( \epsilon^{i}_0 \) are such that \( \alpha^1 \geq \cdots \geq \alpha^N \).

This assumption is the analog of Assumption 4 in section 3. It implies that policy outcomes are redistributive towards households with low income and low assets.

To complete the description of the political economy game, we need to specify the behavior of foreign creditors. As in the previous section, we assume that there is a competitive pool of foreign creditors who price the arrow securities depending on the repayment policies of the sovereign. We assume that foreign creditors discount rate is given by \((1 + r^*)^{-1}\).

We can thus define the equilibrium of the political economy game between foreign creditors, domestic households and the politicians. Let \( h^t \) represent the history of actions by all players up to period \( t \). Then, a subgame perfect equilibrium (SPE, henceforth) of the policy game is described by the following:

1. A sequence of policies optimally chosen by each party at each period \( t \), \( \sigma^i_{t,j} \). This strategy is a function that maps a history of the game, \( h^t \), to the set of policy choices
in period $t$. We refer to the the implemented policy sequence determined by the outcome of the elections described above as $\rho_t^+$. We refer to the sequence of policy proposals, $\sigma_{t,i}$, by $\sigma$.

2. Allocations and prices $x = \left\{ c_{t-1,1}^i, c_{t,0}^i, y_t^i, a_t^i, r_t, \hat{r}_t \right\}_{t=0}^\infty$ constitute a competitive equilibrium from any period onwards.

A pair $(\sigma, x)$ is said to be an SPE if it satisfies the above conditions.

Note that, although we have assumed away aggregate shocks to the state of the economy, due to the political uncertainty, agents involved in the policy game need to calculate expectation of future outcomes. Thus, the notion of competitive equilibrium should be adjusted for the said expectations.

4.2 Characterization of SPE Outcomes

In this section, we start our analysis by characterizing the set of equilibria of the political economy game. We focus on the symmetric equilibria and derive necessary and sufficient conditions that allocations resulting from the equilibria of the political economy game must satisfy.

The probabilistic voting game between the two parties is symmetric in that when the parties use the same strategies, they win the election with probability $1/2$. In general, the party which chooses policies that deliver a higher weighted average of the utilities given by

$$\omega \sum_i \alpha_i \mu_i u_{t-1,1}^i + \sum_i \alpha_i \mu_i u_{t,0}^i$$

will be the winner of the election. This implies that given future strategies by future politicians, any pure strategy Nash equilibrium of the election game must involve symmetric policy proposal by the two parties. Furthermore, the outcome of the election game in each period must maximize the objective in (17). We can thus think of a fictitious planner in each period that chooses policies as to maximize (17). We refer to this fictitious planner as government. Thus, any equilibrium outcome of the policy game is the outcome of a game between the governments in each period who each maximize (17). Note that this does not rule out strategic interaction among the governments due to the fact that each planner is biased toward the living generation.

The following lemma characterizes the set of allocations that can be implemented as a SPE of the political economy game.
Lemma 11 Given the international interest rate $r^*$, initial foreign debt, $B^*_0$, and an initial distribution of assets for the initial old, $\{a_i\}$, allocations $\{c_{i-1,t}, c_{i,0}, y_i, a_i\}^t_{t=0}$, prices $\{\tilde{r}_t, r_t\}$, and government policies $\{B_t, B^*_t, \tau_{l,t}, \tau_{a,t}, T_t, P_t\}^\infty_{t=0}$ are a SPE outcome if and only if:

1. The allocations, prices and government policies are a competitive equilibrium with taxes;
2. The allocations satisfy the following political sustainability constraint for all $t \geq 0$

$$\omega \sum_i \alpha^i \mu^i u^i_{t-1,1} + \sum_i \alpha^i \mu^i u^i_{t,0} \geq V,$$

where $V$ is the value of the worst equilibrium and it is given by the following programming problem:

$$V = \max_{\tilde{c}_0, c_1, y, \theta, T} \omega \sum_i \alpha^i \mu^i u^i (c_1) + \sum_i \alpha^i \mu^i u^i (c_1, y, \theta)$$

s.t.

$$\sum_i \mu^i c^i_0 + c_1 + G = Z \sum_i \mu^i y^i,$$

$$\theta^i (1 - \tau_l) u^i_{c,0} + u^i_z = 0,$$

$$c^i_0 = (1 - \tau_l) Z y^i + T$$

The proof for this lemma can be found in the appendix. The idea is similar to Chari and Kehoe (1990) and Abreu (1988). We can think of the value of $V$ as being the analog of $W$ in section 3. That is, the subgame perfect equilibrium that provides the lowest value to every fictitious planner in each period.

To derive the implications for policies and allocations associated with the political economy game we need to specify a criterion to select among the set of equilibrium outcomes in Lemma 11. We consider two alternatives. First, similarly to what we did in section 3, we consider the SPE outcome that maximizes the welfare (10) introduced in section 3. We refer to such outcome as best SPE outcome. Second, we consider a selection criterion in the spirit of the canonical Eaton and Gersovitz (1981) model of sovereign default. We will show that in both cases - with some minor qualifications - the main conclusions derived in section 3 remain valid.

### 4.3 Best SPE Outcome

We now turn to characterizing the best SPE outcome with log-log utility. We rank outcomes according to the welfare function that we introduced in the section 3 that attaches
a Pareto weight of $\hat{\beta}^t \alpha^i$ to an agent of type $i$ born at $t \geq 0$. Note that the objective is time-consistent due to geometric discounting.

Under our preference specification (2), the welfare function (10) can be written solely in terms of aggregate allocations and initial pension payments as

$$
\beta/\hat{\beta} \sum_i \mu^i \alpha^i \log \left( (1 - \tau_{a,0}) a_0^i + p_0 \right) + \sum_{t=0}^\infty \hat{\beta}^t \left[ \log C_{0t} + \Omega (Y_t) + \beta \log C_{1t+1} \right],
$$

where we define $\Omega (Y) = \psi \log (1 - Y) + (1 + \psi + \beta)H (Y)$. The problem for the planner at $t = 0$ is then to choose an aggregate allocation $\{C_{0t}, C_{1t}, Y_t\}_{t=0}^\infty$ and time zero policies $\{p_0, \tau_{a,0}, \delta_0\}$ that maximize (19) subject to the present value version of the consolidated budget constraint for the country

$$
\sum_t \left( \frac{1}{1 + r^*} \right)^t [ZY_t - C_{0t} - C_{1t} - G] \geq B_0^* \delta_0
$$

and the sequence of political sustainability constraints from $t = 1$ and onward,

$$
\omega \left[ \log C_{1t} + H (Y_{t-1}) \right] + \left[ \log C_{0t} + \Omega (Y_t) + \beta \log C_{1t+1} \right] \geq v_t
$$

and for $t = 0$,

$$
\omega \sum_i \mu^i \alpha^i \log \left( (1 - \tau_{a,0}) a_0^i + p_0 \right) + \left[ \log C_{0t} + \Omega (Y_t) + \beta \log C_{1t+1} \right] \geq v_0.
$$

The main result of this section is that the best SPE outcome has cycles similar to the ones in section 3. To show this, we employ a somewhat different approach from the one used in section 3. We assume that $\hat{\beta} (1 + r^*) = 1$. Using this assumption, greatly simplifies the planning program associated with the SPE. In particular, it can be recursively with the state variable $v_t$ which the average utility for a generation,

$$
v_t = \log C_{0t} + \Omega (Y_t) + \beta \log C_{1t+1}.
$$

The Bellman equation associated with this program in its dual form is given by

$$
P (v) = \max ZY - C_0 - \frac{1}{1 + r^*} C_1 + \eta v + \frac{1}{1 + r^*} P (v')
$$

This assumption is mainly made for simplicity of the analysis. Our numerical simulations show that the main insights holds even when $\hat{\beta} (1 + r^*) < 1$.\footnote{This assumption is mainly made for simplicity of the analysis. Our numerical simulations show that the main insights holds even when $\hat{\beta} (1 + r^*) < 1$.}
subject to

\[
\begin{align*}
\omega \left[ \log C_1 + H(Y) \right] + v' & \geq \mathcal{V} \\
\log C_0 + \Omega(Y) + \beta \log C_1 & = v
\end{align*}
\]

The parameter \( \eta \) is the multiplier associated with delivering a certain level of utility at time 0 to the planner. Note that this problem can be thought of as a game between three agents: fictitious planner associated with the government in each period, foreign lenders and the planner whose payoff is given by (19). The value \( P(v) \) is the payoff to the foreign creditors plus the value to the planner weighted by \( \eta \). Under the assumption that foreign creditors and the planner discount future payoffs at the same rate, the welfare weight \( \eta \) is time independent and so is the value function \( P(v) \).\(^4\) One feature of the value function stated above is that \( P(v) \) is hump-shaped. That is, for low values of \( v \), an increase in \( v \) leads to an increase in the value to the planner and since the marginal cost of this increase in \( v \) for foreign creditors is low, \( P(v) \) must increase. On the other hand, for high values of \( v \), the decline in the payoff to the foreign creditors is large enough so that \( P(v) \) is decreasing.

Our result about the dynamics of the model are concerning the policy function \( v'(v) \). Naturally, these dynamics depends on the value of the multiplier \( \eta \) which in turn is determined by the level of initial debt/inequality.

When the multiplier is low, i.e., when initial debt/inequality are low, achieving efficient allocations in the steady state is feasible. Thus for low values of \( \eta \), steady state is characterized by an allocation for which the political sustainability constraint is slack. This value also satisfies \( P'(v_{ss}) = 0 \). For such a specification, convergence occurs at most in two periods. When \( v \) is high enough, \( v' = v^* \) while for low values of \( v \), \( v'(v) > v^* \).

When multiplier is high, i.e., for high initial debt/inequality, efficient allocations are not sustainable. This implies that steady state \( v_{ss} \), must satisfy \( P'(v_{ss}) < 0 \) and the political sustainability constraint must bind. In this case, we can show that convergence to steady state is cyclical. We thus have the following proposition:

**Proposition 12** There exists a value of \( \eta^* > 0 \) such that the solution to ((P3)) is characterized as follows:

i. If \( \eta \leq \eta^* \), there exists a unique steady state value of \( v \), \( v_{ss} \), such that \( P'(v_{ss}) = 0 \). The policy function \( v'(v) \) is weakly decreasing. For any initial value of \( v_0 \), \( v_2 = v_{ss} \).

\(^4\)When \( \hat{\beta} (1 + r^*) < 1 \), the value function becomes a function of time as the welfare weight associated with the planner’s utility declines with time. As we show in the appendix, the main argument in proposition 12 holds when \( \hat{\beta} (1 + r^*) < 1 \).
ii. If \( \eta > \eta^* \), there exists a unique steady state value of \( v, v_{ss} \), such that \( P'(v_{ss}) < 0 \). The policy function \( v'(v) \) is weakly decreasing. If \( v_0 < v_{ss} \), then \( v_t \neq v_{ss} \) and \( v_t > v_{ss} \) when \( t \) is even while \( v_t < v_{ss} \) when \( t \) is odd.

The proof can be found in the appendix.

The dynamics of the best SPE outcome is then consistent with the cycles we described in section 3. When total indebtedness is low and utility for current generation is high, we observe an increase in government debt, especially foreign debt, and income inequality is low. This increase in indebtedness translates into low utility for future generations and high income inequality. To satisfy the political sustainability constraint, (21), the current government must reduce the resources next period that are committed to foreign lenders, \( B^* \), and future old, \( C_1 \), so that future government does not have an incentive to default given the large inequality.

4.4 Markov Equilibrium

We now consider a different selection in the spirit of the canonical Eaton and Gersovitz (1981) model of sovereign default. We focus on equilibria that have the following properties: i) strategies are the same after two histories with no default if the inherited foreign debt, the distribution of domestic assets and pension payments made by the previous government are the same, ii) a default by the government triggers reversion to worst equilibrium. We say that the current government defaults on promises made by previous governments if it does not repay inherited foreign debt, it taxes assets of the initial old above a certain level exogenously specified or it lowers pension payments below what was promised by the previous government. Without loss of generality we are going to set the maximal tax on assets to zero.

Formally, we make our selection operational by setting up the equilibrium recursively. The state variable is \( (B, z) \) where \( B = (B^*, \{a^i\}, P^e) \), \( P^e \) are pension payments promised by the previous government, and \( z \in \{0, 1\} \) is a variable that records if previous governments defaulted in the past. We adopt the convention that \( z = 1 \) if previous governments defaulted in the past and zero otherwise. Given our selection, if \( z = 1 \) then value for the current government is \( V \) for all \( B \), \( V(B, 0) = V \). Taking as given the decision of the future governments, denoted by a bar, the value function for the government that inherited a state \( (B, 0) \) is given by

\[
V(B, 0) = \max \{v(B), V\}
\]

(23)

where the value \( v \) is the value associated with honoring all debt obligations and it can be
expressed in primal form as

\[
\nu(B) = \max_{\pi=(P,C_0,Y,B')} \omega \sum \mu_i \alpha_i \log (a_i + P) + \log C_0 + \hat{\Omega}(Y) + \beta \sum \mu_i \alpha_i \log (a_i' + \bar{P}(B'))
\]

subject to

\[
\sum \mu_i \left( a_i + P \right) + C_0 + B^* + G \leq ZY + \frac{1}{1 + r^s} B^s
\]

(25)

\[
P \geq P_e
\]

(26)

\[
\bar{v}(B') \geq V
\]

(27)

\[
a_i' + \bar{P}(B') = \varphi_i(Y) \left( \sum \mu_i a_i' + \bar{P}(B') \right)
\]

(28)

where \( \hat{\Omega}(Y) = \Omega(Y) - \beta H(Y) \), \( \bar{P} \) and \( \bar{v} \) are the policy rule and the value function used by future government, and recall that \( \varphi_i(Y) \) is the consumption share for type \( i \) household given by (8).

The constraint (25) is the consolidated budget constraint for the country in sequential form. Constraint (26) requires that the pension payments by the current government, \( P \), are at least what promised by the previous government, \( P_e \). The constraint (27), guarantees that next period government has no incentive to default. The current government leaves a state to next period government such that it finds not optimal to default. Finally, constraint (28) ensures that assets holdings \( \{a_i\} \) are consistent with optimality of the households saving decisions. With log-preferences, this amount to assume that next period consumption share of type \( i \) agent is equal to current consumption share \( \varphi_i(Y) \).

Notice that requiring that the current government passes a state to the future government such that it does not have a strict incentive to default is without loss of generality. In fact, suppose to the contrary that the current government chooses \( B' \) such that \( \nu(B') < V \). Then it follows that the price of foreign debt equals zero and so no resources can be raised from foreign lenders. Moreover, since there is a default next period, private agents optimality implies that \( a_i' = 0 \) for all \( i \). Hence, the current government is better off defaulting today, strictly if foreign debt is strictly positive or there is domestic asset inequality, contradicting optimality for the current government.

A Markov equilibrium are policy rules \( \bar{P}, C_0, \bar{Y}, B' \) and value functions \( \bar{V} \) and \( \bar{v} \) such they
satisfy

\begin{align*}
P (B|\bar{P}, \bar{v}) &= P (B), \quad C_0 (B|\bar{P}, \bar{v}) = C_0 (B), \quad Y (B|\bar{P}, \bar{v}) = Y (B), \\
B' (B|\bar{P}, \bar{v}) &= B' (B), \quad \text{and } v (B|\bar{P}, \bar{v}) = \bar{v} (B)
\end{align*}

for all \( B \) such that \( \bar{v} (B) \geq V \) where \( P, C_0, Y, B', V, \) and \( v \) are the policy rule and the value associated with the problem in (23) and (24) given \( \bar{P} \) and \( \bar{v} \).

**Characterization.** The large dimensionality of the state \( B \) makes the characterization of the Markov equilibrium as stated in (24) cumbersome. We next show that (other than for period zero), we can characterize the Markov equilibrium outcome recursively using a single state variable, namely the amount of resources that have been committed by previous government to foreign lenders and old households. We will denote such variable \( \hat{B} = B^* + C_1 \).

To this end, notice that it is without loss of generality for the current government to choose \( P^e' = \bar{P} (B') \) whenever \( P^e' < \bar{P} (B') \) as it does not change the value in (24). It then follows that consumption of old agents along any Markov equilibrium outcome path is solely determined by policies of the previous period government. The first term in the objective function of (24), \( \omega \sum_i \mu_i \log (a^i + P) \), is predetermined. We can then rewrite the value of the current government as choosing policies to maximize the value for the generation born in the current period given that resources \( \hat{B} = B^* + C_1 \) have been committed by previous government to foreign lenders and old households subject to budget feasibility, the political sustainability constraint for the government next period.

Formally, consider the following auxiliary problem

\begin{equation}
\begin{aligned}
w (\hat{B}) &= \max_{C_0, Y, C_1, B^*, t} \log C_0 + \Omega (Y) + \beta \log (C_1') \\
\text{subject to } \quad \hat{B} + C_0 + G &\leq ZY + \frac{B^*}{1+r^*} \\
\omega \left[ H (Y) + \log (C_1') \right] + w (B^* + C_1') &\geq V
\end{aligned}
\end{equation}

and denote the policy rules associated with it with a superscript \( w \). The problem at \( t = 0 \) is only slightly different because initial pension promise does not necessarily optimal from
the perspective of the current government. We can then write:

\[ \hat{v}(\hat{B}) = \max_p \omega \sum_i \mu^i \alpha^i \log (a^i + P) + w \left( \sum_i \mu^i a^i + P + B^* \right) \]  

subject to \( P \geq P^e \).

In the appendix, we show that outcomes obtained from the policy rules associated with (29) and (32) are a Markov equilibrium outcome:

**Lemma 13** Given an initial condition \((\hat{B}, 0)\) such that it is not optimal to default in period zero, the outcome obtained from \(C_{1,0} = \sum_i \mu^i \log (a^i + \hat{P}(\hat{B}))\) and for all \(t \geq 0\)

\[ (C_{0t}, Y_t, C_{1t+1}, B_{t+1}^*) = (C_{0w}, Y^w, C_{1w}^w, B_{t+1}^{s*w}) (C_{1t} + B_t^*) \]

is a Markov equilibrium outcome of the political economy game.

**Inequality Overhang and Debt Cycles.** Lemma 13 implies that we can then consider (29) to characterize the Markov equilibrium outcome. The main result for this section is that the Markov equilibrium outcome features cycles in total government debt, \(\hat{B}_t\). The result is driven by inequality overhang that is present in the Markov equilibrium.

**Proposition 14** The policy rules associated with the problem in (29) are such that: i) \(C_{0}(\hat{B})\) is decreasing and \(Y(\hat{B})\) is increasing, ii) \(\hat{B}'(\hat{B})\) is decreasing.

Part i) states that average consumption of current young is decreasing in the promises made by previous government to current old and foreign lenders. Aggregate output and therefore income inequality is instead increasing in \(\hat{B}\). Part ii) states that total debt obligations for the next period, \(\hat{B}'\), are decreasing in current obligations, \(\hat{B}\). This immediately implies that the equilibrium outcome path has cycles. When inherited obligations, \(\hat{B}_t\), are above their stationary value, \(\hat{B}_{ss}\), next period are going to be below \(\hat{B}_{ss}\). The opposite happens when \(\hat{B}_t < \hat{B}_{ss}\).

The intuition for these results is similar to the one in section 3. When the current government inherits large obligations, \(\hat{B}\), it must go through a period of fiscal consolidation by reducing total obligations passed to the future government then current government and it must tolerate high income inequality. To avoid future default - and hence collapse of domestic credit markets and ability to borrow from abroad at all - the reduction in total debt obligations (including pensions) cannot be small because high income inequality in the current period translates into high wealth inequality in the future period and so it tightens the political sustainability constraint for future government.
Austerity-like episodes happen along the equilibrium outcome path when policies are chosen by governments that care only about the welfare of current generations. Even in this case it is optimal for the government to reduce its debt and tolerate large inequality. The current government prefers to run austerity-like measures than defaulting on the obligations chosen by previous government. Reneging on these promises will result in a collapse in domestic credit market which will further lower the utility of the current government.

While we are able to show that there are cycles in total government debt (including pension promises), the dynamics of the domestic and foreign components depend on parameters. In particular, solving the model numerically, we have that if $\beta$ is sufficiently small then $B''(\hat{B})$ is decreasing and the share of debt held domestically is increasing in total government debt. This implies that during periods of consolidation foreign debt decreases and the share of sovereign debt held by domestic resident is increasing. The outcome hence displays the repatriation of government debt that was documented for Southern European countries during the recent crisis. This dynamics is illustrated in Figure 1. Such result is not present if $\beta$ is sufficiently high. In such case the bulk of reduction in total indebtedness in driven by a reduction in domestic government debt and pension promises as illustrated in Figure 2.

**Reneging on Pensions does not trigger default.** So far we assumed that if the current government lowers pension payments below what was promised by the previous government triggers a default. This essentially gives past government full control of the consumption for the old. This assumption is important for our result about cycles. To see this, we consider the case in which the current government can freely choose $P$ without triggering default our we do not. Mechanically, this amount to drop the constraint (26) from the current government problem (24). We show that in this case there are no cycles.

In the appendix we show that we can characterize the Markov equilibrium outcome in this scenario using a programming problem similar to (29). In fact, for $t \geq 1$ the equilibrium outcome solves a modified version of (29) in which we add the constraint

$$\Theta(Y) \frac{w}{C_1} = -w'(\hat{B}')$$

(33)

where $\Theta(Y) \equiv \sum_i \mu^i \alpha^i 1/\varphi^i(Y)$. Constraint (33) guarantees that pension payments promised by the current government are optimal from the perspective of the future government.

We show in the appendix that there are no cycles in government debt. The motive is that in addition to the inequality overhang effect, there is another channel through which inequality affects the political sustainability constraint. The more dispersed are
Figure 1: Policy rules associated with Markov equilibrium, low discount factor

asset holdings of future old, the more future government wants to provide them with pension transfers. This effect can be seen by inspecting (33): as inequality and hence $Y$ increases, $\Theta(Y)$ increases and so $C_1$ must increase for a fixed $\hat{B}$. Hence leaving a dispersed wealth distribution is a way that the current government has to guarantee a large (average) consumption level of the future old. Under our preferences this effect dominates the inequality overhang effect and so higher inequality (higher $Y$) actually relaxes the political sustainability constraint everything else equal. Then $\hat{B}'(\hat{B})$ is increasing and there are no cycles.

5 Extensions and Discussion

5.1 The Role of Assumptions

In this section, we discuss the role of various assumption underlying our results.
The Role of Imperfect Redistribution. A crucial assumption in our analysis is the inability of the government to perfectly redistribute resources among households. To see this, suppose that instead of an affine tax schedule, the government had access to individual dependent taxes and transfers. That is, suppose that transfers to the young and old households can depend on households’ productivities. This would imply that the government can achieve its desired level of redistribution, namely that consumption among households with different productivities are allocated so that $\alpha^i u_{c,t,a} = \alpha^j u_{c,t,a}$ for $a = 0, 1$. In other words, distribution of consumption is independent of output and hence constant over time. As a result, the sustainability constraint in (11) can be written as

$$\frac{\beta}{\bar{\beta}} [\log C_1 + \bar{H}] + V' \geq W$$

where $\bar{H}$ is the payoff to the government of having the desired level of inequality and since distribution of consumption is constant over time, is constant. This, in turn, implies
that once the sustainability constraint binds, the value of consumption for the old and continuation value, \( V' \) is independent of the state of the economy. This result holds in most models without commitment; see for example Thomas and Worrall (1988) among many others. This implies that imperfect redistribution plays a crucial role in our main result.

**The Role of Preferences.** One possibility is that our result is due to strong income effects implied by log-log preferences. It is possible that the decreasing pattern of inequality as a function of promised value, \( V \), is because as \( V \) increases, income effect implies that output should decline which in turn implies lower inequality.

While the problem with general preferences cannot be characterized analytically, here, we consider GHH preferences of the form \( u(c, l) = \log(c - v(l)) \) where \( v(l) = \psi l^{1+1/\epsilon}/(1 + 1/\epsilon) \), where \( \epsilon > 0 \) is the Frisch elasticity of labor supply. With GHH preferences, the income effect is not present at the individual level. As we will show, however, the same result holds. In the Appendix, we show via numerical simulations that similar results hold for more general balanced-growth-preferences.

With GHH preferences specified above, a result similar to the log-log case holds and government objective is given by\(^5\)

\[
U^p(C_0, C_1, Y) = \log \left( C_0 - v \left( \frac{Y}{\hat{\theta}} \right) \right) + \beta \log C_1 + (1 + \beta) \sum_{i=1}^{N} \alpha^i \mu^i \log \varphi^i
\]

where \( \hat{\theta} \) is a weighted average productivity measure given by

\[
\hat{\theta} = \left[ \sum_{j=1}^{N} \mu^j \left( \theta^j \right)^{1+\epsilon} \right]^{1/(1+\epsilon)}
\]

Additionally, budget constraint for households imply that

\[
\varphi^i = 1 + \frac{1}{\epsilon(1+\beta)} \frac{v \left( Y/\hat{\theta} \right)}{C_0 - v \left( Y/\hat{\theta} \right)} \frac{(\theta^i)^{1+\epsilon} - \hat{\theta}^{1+\epsilon}}{\hat{\theta}^{1+\epsilon}}
\]

As it can be seen, due to non-separability of leisure and consumption, inequality\(^6\), as represented by \( \{ \varphi^i \}_{i=1}^{N} \), depends not only on output, \( Y \), but also on consumption in period 0. As the formula shows, when \( C_0 \) increases, inequality declines. This is because keeping

\(^5\)We establish these formulas in the Appendix E.

\(^6\)With GHH preferences, \( \varphi^i \) is given by the ratio of \( c^i_0 - v \left( y^i/\theta^i \right) \) to \( C_0 - v \left( Y/\hat{\theta} \right) \), which represents the distribution of period-0 utility as opposed to consumption.
Y constant, marginal tax rate does not change (due to GHH) and an increase in aggregate consumption must be accompanied by an increase in transfers and hence, a decline in inequality. As before, an increase in Y leads to a decrease in inequality. As the above formula establishes, inequality is captured by the variable \( C_0 Y^{-1-1/\epsilon} \).

In the appendix E, we show that in the recursive formulation of the problem, inequality declines with promised utility. This would imply that when the sustainability constraint is binding, an increase in promised utility must be accompanied by a decrease in continuation values and inequality overhang has the same effect as in case of log-log preferences. The idea behind this result is that as promised utility increases, consumption for the young must increase as well which in turn leads to lower inequality. One implication of the decline in inequality due to increase in consumption is that the government can lower marginal tax rates and hence output increases with promised utility. Not surprisingly, this is the opposite of the log-log case.

### 5.2 Extension with Shocks

In this section, we extend the model to allow for shocks to government spending.\(^7\) We show that all of our results in the deterministic model hold in this extension. Much of the model setup is similar to that presented in section 2. Therefore, we skip many of the details as they remain unchanged and describe the changes made to the model. In each period \( t \geq 0 \) there is a realization of a stochastic event \( s_t \in S = \{s(1) < \cdots < s(K)\} \subset \mathbb{R} \) which is independently and identically distributed according to the distribution \( \pi(s) \). History of shocks are denoted by \( s^t = (s_0, \cdots, s_t) \in S^{t+1} \). With a slight abuse of notation, we represent the probability distribution of \( s^t \) by \( \pi(s^t) = \prod_{t=0}^{t} \pi(s_t) \). The random variable \( s_t \) directly affects government purchases given by \( G(s_t) \) where \( G \) is an decreasing function of \( s_t \); that is, \( s(1) \) is the state with highest government spending while \( s(K) \) is the state with lowest value of government spending. Given this representation of histories, allocations are represented by functions of the history of shocks, \( s^t \).

**Market Structure.** In line with Lucas and Stokey (1983) and Werning (2007), we assume that asset markets are complete and young households as well as the government can trade a complete set of arrow securities in each state \( s^t \) at prices given by \( \{q_{t+1}(s^{t+1})\}_{s^{t+1} \succeq s^t} \). In addition, government can issue full state contingent debt in international markets at prices given by \( \{q^*_{t+1}(s^{t+1})\}_{s^{t+1} \succeq s^t} \). As a result, a household’s budget

\(^7\) Alternatively, as in Aguiar et al. (2009), we can reinterpret these as terms of trade shocks.
constraints are given by
\[ c^i_{t,0} (s^t) + \sum_{s^{t+1} \succ s^t} q_{t+1} (s^{t+1}) a^i_{t+1} (s^{t+1}) \leq Z y^i_t (s^t) (1 - \tau_{l,t} (s^t)) + T_t (s^t) , \]  
\[ c^i_{t,1} (s^t) \leq P^*_{t+1} (s^{t+1}) + q^*_{t+1} (s^{t+1}) \left( 1 - \tau_{a,t+1} (s^{t+1}) \right) \]  
\[ , \forall t, s^t \in S^{t+1} \]  
while government budget constraint is given by
\[ P_t (s^t) + T_t (s^t) + G_t (s^t) + B_t (s^t) + B^*_t (s^t) = \sum_{s^{t+1} \succ s^t} q^*_t (s^{t+1}) B^*_t (s^{t+1}) \]  
\[ + \sum_{s^{t+1} \succ s^t} q_{t+1} (s^{t+1}) B_{t+1} (s^{t+1}) + \pi_t (s^t) Z \sum_{i=1}^N y^i_t (s^t) + \tau_{a,t} (s^t) \sum_{i=1}^N a^i_t (s^t) \]  
\[ (35) \]

Note that our notion of debt is slightly different from the deterministic model in section 3 as \( B_t \) and \( B^*_t \) is the total face value of domestic and foreign debt due to be paid at the beginning of period \( t \). We make the following assumptions about price of government debt in international markets:

**Assumption 15** Prices \( \{ q^*_t (s^{t+1}) \} \), \( s^{t+1} \succ s^t \) satisfy the following properties:

1. **(Stationarity)** Interest rate \( r^* \) must exist such that \( q^*_t (s^{t+1}) = \pi_t (s^{t+1}|s^t) \frac{1}{1+r^*} = \pi (s^t) \frac{1}{1+r^*} \), where \( \pi_t (s^{t+1}|s^t) \) is the probability of \( s^{t+1} \) subject to \( s^t \).

2. **(Impatience)** \( \hat{\beta} (1 + r^*) < 1 \).

In the appendix D, we show that when the world economy is consisted of identical countries subject to government spending shocks as described and in an economy à la Kehoe and Levine (1993) and Alvarez and Jermann (2000), stationarity implies that there must exist a constant interest rate. Furthermore, since an unconstrained country prices a state contingent security, the first relationship must be satisfied. Finally, since countries would like to avoid binding borrowing constraints (sustainability constraints) in the future, supply of saving is higher than that of unconstrained steady state and as a result interest rates must be lower than the discount rate.

As in section 3, we focus on a planning problem without commitment. An allocation
is, then, said to be sustainable if it satisfies

\[
\frac{\beta}{\tilde{\beta}} U^p_1 \left( C_{t-1,1} \left( s^t \right) ; Y_{t-1} \left( s^{t-1} \right) \right) + \sum_{\tau=t}^{\infty} \beta^\tau \sum_{s^\tau \succ s^t} \pi_{s^\tau \succ s^t} \ U^p \left( C_{\tau,0} \left( s^\tau \right), Y_{\tau} \left( s^\tau \right), C_{\tau,1} \left( s^\tau \right) \right) \geq W \left( s_t \right)
\]  

(36)

where as before, \( W \left( s_t \right) \), is the highest value to the government when government and households are in financial autarky. The same two stage procedure can be used in order to find the best sustainable allocations. The associated Bellman equation, with shocks can be written as

\[
\hat{B} \left( V \right) = \max ZY - C_0 + \frac{1}{1+r^*} \sum_{s' \in S} \pi \left( s' \right) \left[ -C_1 \left( s' \right) + \hat{B} \left( V' \left( s' \right) \right) \right]
\]

(P3)

subject to

\[
\log C_0 + \psi \log (1-Y) + (1+\psi+\beta) H(Y) + \beta \sum_{s' \in S} \pi \left( s' \right) \left[ \frac{\beta}{\hat{\beta}} \log C_1 \left( s' \right) + V' \left( s' \right) \right] = V
\]

\[
\frac{\beta}{\hat{\beta}} \left[ \log C_1 \left( s' \right) + H(Y) \right] + V' \left( s' \right) \geq W \left( s' \right)
\]

Given a solution to the above functional equation, \( B(V) \) and its associated policy function for continuation values, \( V'(V,s) \), the ergodic set associated with this dynamic system is given by \( V \subset \mathbb{R} \) where

\[
V' \left( V, S \right) = V.
\]

(37)

Furthermore, the invariant distribution associated with \( V'(V,s) \) is given by \( \mu \) which is a probability measure on \( (V,V) \) where \( V \) is the set of all Borel subsets of \( V \) and must satisfy

\[
\mu \left( A \right) = \int_V \sum_{s \in S} \pi \left( s \right) \mathbf{1} \left[ V' \left( V, s \right) \in A \right] d\mu \left( V \right)
\]

(38)

The following proposition is the equivalent of theorem 8 for the economy with shocks:

**Proposition 16** The policy functions in (P3) satisfy the following properties:

1. \( Y(V) \) is decreasing in \( V \),

2. \( V'(V,s) \) and \( C_1 \left( V, s \right) \) are U-shaped in \( V \); there exists \( V^* \left( s \right) \), such that \( V' \left( V, s \right) \left( C_1 \left( V, s \right) \right) \) for all \( V < V^* \left( s \right) \) while it is increasing for all \( V > V^* \left( s \right) \);
3. $V^*(s)$ is increasing in $s$.

The intuition here is the same as in the deterministic case. As promised value $V$ declines, the economy must produce more and consume less. This increase in output, in turn, leads to an increase in inequality for the current young and future old. As a result, aggregate consumption of the future old as well as promised value of future generations must increase.

Figure 3: U-shaped policy function

Figure 3 depicts the policy functions implied by (P3) for an example with two values of shocks ($K = 2$). In line with proposition 16, policy functions are U-shaped in promised value $V$ and decreasing with respect to government spending. This however has stark implication about long-run behavior of external debt. In particular, given these policy functions the ergodic set of values of $V$ in the long run, $\mathcal{V}$ is a subset of the interval $[\underline{V}, \overline{V}]$, where $\overline{V} = V'(V^*(s(1)), s(1))$ – recall that $s(1)$ represents the state with a high government spending. The dynamics of debt in the long-run, as implied by policy functions depicted in Figure 3, points toward a stark notion of large debt repayment: A country that has the highest value of foreign debt sustained in the long-run, upon recovery, must adjust to its
highest long-run level. The following proposition summarizes this discussion:

**Proposition 17** Consider the stationary distribution $\mu$ and the stochastic process $V_t^*(s^\dagger)$ it imposes on promised utility. Let $\underline{V} = \inf V$ and $\overline{V} = \sup V$. Then for every $\epsilon > 0$, there exists $\epsilon' > 0$ such that $\Pr (V_{t+1}^* > \overline{V} - \epsilon' | V^*_t < \underline{V} + \epsilon) \geq \pi (s(K))$.

Proof can be found in the Appendix.

At the heart of the above result is the U-shape of the continuation value policy function and inequality overhang. As the country experiences a sequence of high government spending shocks and increases its foreign borrowing, its debt capacity upon recovery declines. This is because as debt increases, inequality among the young increases and as a result, upon recovery (experiencing a high value of government spending) the country cannot borrow as much.

### 6 Empirical Support for the Model

As we have shown, our model has important implications for the joint dynamics of debt (external and domestic), redistributive policies and inequality. Here, we provide evidence for the main implications of the model.

The main implication of our model is the cyclical patterns of external debt and repayment in which when initial debt values are low, the government borrows heavily externally and expands its redistributive programs while for high enough values of external debt, transfers are reduced and external debt is repaid.

Various cyclical patterns of sovereign debt and redistributive policies have been identified by Sachs (1989) as well as a series of papers in the volume by Dornbusch and Edwards (1991). Dornbusch and Edwards refer to these cyclical patterns as populist cycles in macroeconomic policies. Such cycles, as they assert, typically involve redistributive policies through various measures of supporting low and middle-income workers; a worsening of government finances as well as the country’s net foreign asset position (typically involving deterioration of foreign exchange reserves); and finally a crisis episodes followed by austerity measures and intervention by foreign organization such as the IMF. The recurrence of these observations are, thus, posed as a puzzle. Sachs (1989) writes:

*And yet, a major puzzle remains. The populist episodes we have reviewed ended in collapse, sometimes even in tragedy. Perón was forced into exile, leaving a weakened...*

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8 Although part of their focus is on the real appreciation of the currency and its effect on import prices and wages in the non-tradable sector, many of the episodes identified also involve labor policies involving low and middle-income workers.
economy and a society both politicized and deeply divided ... Why did these leaders opt for such a dangerous strategy? At least Brazil and Peru should have had the benefit of the lessons of the other two experiences.

Dornbusch and Edwards also refer to this apparent lack of memory in policy making and find it puzzling. Our model is a potential explanation for the cyclical behavior observed in policy making in Latin America.

Sachs (1989) as well as Dornbusch and Edwards (1991) discuss various episodes in Latin American countries that are akin to the cycles arising from our model. Examples include: Argentina under Perón, Chile under Allende, Peru under Garcia, Brazil under Sarney, Mexico under Echeveria – see Kaufman and Stallings (1991) for further discussions.

Policies undertaken by Juan Perón in Argentina and Salvador Allende in Chile are of particular interest. As secretary of labor, Perón introduced various pro labor policies including an increase in minimum wages, substantial increase pension benefits, as well as establishing a close relationship with the labor unions with these policies continuing when he came to power in 1946. During the same episode Argentina experienced a decline in current account (from 1.99 billion Pesos in 1946 to -610 millions Pesos in 1949; see Taylor (1998)) as well as a decline in foreign reserve holdings (from 1.1 billion dollars in 1946 to 258 millions dollars in 1948). By the early 1950’s there are signs of moves towards austerity measures. In a speech in 1952 Perón stated – see Mallon and Sourrouille (1975):

The justicialista [i.e. Peronist] economy asserts that the production of the economy should first satisfy the needs of its inhabitants and only export the surplus; the surplus, nothing more. With this theory the boys here, of course, eat more every day and consume more, so that the surplus is smaller. But these poor guys have been submerged for fifty years; for this reason I have let them spend and eat and waste everything they wanted to for five years . . . but now we undoubtedly must begin to reorder things so as not to waste any more.

In 1952, wage freezes were introduced and exports are subsidized and import restrictions were implemented. As a result current account increased to 7.761 billion pesos in 1953 (2.1 billions pesos in 1954).

Similarities can be drawn with Allende’s term in Chile. Prior to Allende, mainly thanks to an increase in price of copper (its main export) in the three years leading to 1970, Chilean economy’s stock of foreign exchange reserves increased from 125 million dollars in 1968 to 394 million dollars in 1970 while trade surplus remained positive and
stable (see Larraín and Meller (1991)). Upon winning the election in 1971, the Allende government started various redistributive programs through redistribution of land, nationalization of industries, and increasing minimum wages – real minimum wages for blue collar workers were increased by 56% in the first quarter of 1971. The decline of price of copper during Allende’s government was a contributing factor to a decline in trade balance as well as a budget deficit of about 24% of GDP. This followed by a collapse of foreign exchange reserve. Following the collapse of the Allende’s government, trade deficit and government finances improved.

A key mechanism in our model that leads to the cyclical behavior is what we refer to as inequality overhang. The result that countries with high (current wealth or previous income) inequality can support a lower amount of public foreign debt. Some evidence that the mechanism is operating can be found in Berg and Sachs (1988). They document that higher income inequality is a significant predictor of the probability that a country will reschedule its foreign debt and of the interest rate spreads of the bond price in secondary market. They consider a sample of emerging economies and find that income distribution - measured as the ratio of household income of top 20 percentiles over bottom 20 percentiles - is a significant predictor of the probability of rescheduling once controlling for other variables such as outward orientation of trade policies, share of agriculture in GNP, level of per capita GDP, changes in terms of trade. They find similar result when considering the effect of income inequality on the spreads.

Finally, another key prediction of our model is that during fiscal consolidations the share of government debt held domestically increases. This feature is well documented for the recent European debt crisis. For instance, Broner et al. (2014) document that the share of government debt held by residents increases in Italy, Spain, Portugal, Ireland, and Greece (and to a less extent in France) while in Germany the share of debt held domestically continues a long-run downward trend. (see Figure 4 in their paper). Additionally Brutti and Sauré (2014) and Merler and Pisani-Ferry (2012) provide further evidence on repatriation of government debt in the Euro zone.

The same dynamic can be seen in several crises in Latin America. In the figure below we plot the share of domestic and foreign government debt over the period 1990-2007 for a sample of Latin American countries that run into a debt crisis. The data are from the

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9Since most of the countries that rescheduled and have high spreads are in Latin America and since such economies are characterized by extreme income inequality, one may think that high income inequality is just picking up a “Latin American” fixed effect. Interestingly enough, Berg and Sachs (1988) show that introducing a dummy variable for Latin America does not change the result: income inequality is still an important predictor of rescheduling.

10More recently, Aizenman and Jinjarak (2012) find a positive association between sovereign default risk (measured by the CDS spread) and inequality.
dataset assembled in Panizza (2008) Figure 6 illustrates, clearly, that following the Tequila crisis Mexico observed a drastic increase in debt issued domestically. Such behavior is also present - albeit to a less extent - for the crisis in Argentina in 2002 and in Ecuador in 1999. (A notable exception is Uruguay in 2003.) The same result also holds for Thailand in 1998.

![Graphs of debt over time for various countries](image)

Figure 4: Share of domestic government debt over total government debt

7 Conclusion

References


—— (1991): *The macroeconomics of populism*, University of Chicago Press. 4, 36, 37


SACHS, J. D. (1989): “Social conflict and populist policies in Latin America,”. 4, 36, 37


A Proofs

A.1 Proof of Lemma 2

Note that with log-log preferences, we must have

\[
\frac{1}{c_{t,0}^i} = \beta (1 + \hat{r}_t) (1 - \tau_{a,t+1}) \frac{1}{c_{t,1}^i}
\]

\[
Z \frac{1 - \tau_{t,t}}{c_{t,0}^i} = \psi \frac{1}{\theta^i - y_t^i}
\]

Therefore

\[
\frac{c_{i,0}^i}{c_{t,0}^i} = \frac{c_{i,1}^i}{c_{t,1}^i} = \frac{\theta^i - y_t^i}{\theta^l - y_t^l}
\] (39)

If we let \(C_{t,0}\) be aggregate consumption for the young at \(t\), and define \(\varphi_t^i = c_{t,0}^i/C_{t,0}\), then the above equations imply that

\[
c_{0,t}^i = \varphi_t^i C_{0,t}
\] (40)

\[
c_{1,t}^i = \varphi_t^i C_{1,t}
\] (41)

\[
\theta^i - y_t^i = \varphi_t^i (1 - Y_t)
\] (42)

where we have used the fact that \(\sum \mu^i \theta^i = 1\) and assumed that all allocations are interior.\(^\text{11}\) Note that the above together with (39) imply that

\[
Z \frac{1 - \tau_{t,t}}{C_{t,0}} = \psi \frac{1}{1 - Y_t}, \quad \frac{1}{C_{t,0}} = \beta (1 + \hat{r}_t) (1 - \tau_{a,t+1}) \frac{1}{C_{t,1}}
\] (43)

We can write the budget constraint (4) and (5) as

\[
c_{t,0}^i + \frac{1}{(1 + \hat{r}_t) (1 - \tau_{a,t+1})} c_{t,1}^i = Y_t^i (1 - \tau_{t,t}) + T_t + \frac{1}{(1 + \hat{r}_t) (1 - \tau_{a,t+1})} P_{t+1}
\]

Using (43) we have

\[
c_{t,0}^i + \frac{\beta C_{t,0}}{C_{t,1}} c_{t,1}^i = y_t^i \psi C_{t,0} / (1 - Y_t) + T_t + \frac{1}{(1 + \hat{r}_t) (1 - \tau_{a,t+1})} P_{t+1}
\]

\(^\text{11}\)Since our preferences do not satisfy Inada condition for hours worked at 0 hours, we need to make assumption about the dispersion of productivities so that everyone works positive hours. When we allow for hours worked to be zero, all of our results will go through while they involve more extensive and cumbersome algebra and are available upon request.
This can be written as
\[
\frac{1}{C_{t,0}} c_{t,0}^{i} - \frac{\psi}{1 - Y_t} y_{t}^{i} + \beta \frac{1}{C_{t,1}} c_{t,1}^{i} = \hat{T}_t.
\]
for some constant \( \hat{T}_t \). Replacing from (40)–(42), we have
\[
\frac{1}{C_{t,0}} \phi_{t,0}^{i} C_{t,0}^{i} - \frac{\psi}{1 - Y_t} \left[ \theta_{t}^{i} - \phi_{t,1}^{i} (1 - Y_t) \right] + \beta \frac{1}{C_{t,1}} \phi_{t,1}^{i} C_{t,1}^{i} = \hat{T}.
\]
\[
(1 + \psi + \beta) \frac{\psi \theta_{t}^{i}}{1 - Y_t} = \hat{T}
\]
Taking averages across \( i \)’s, we have
\[
(1 + \psi + \beta) \sum_{i} \mu_{t}^{i} \phi_{t}^{i} - \sum_{i} \frac{\psi \mu_{t}^{i} \theta_{t}^{i}}{1 - Y_t} = \hat{T}
\]
\[
(1 + \psi + \beta) - \frac{\psi}{1 - Y_t} = \hat{T}
\]
Hence,
\[
(1 + \psi + \beta) \phi_{t}^{i} - \frac{\psi \theta_{t}^{i}}{1 - Y_t} = (1 + \psi + \beta) - \frac{\psi}{1 - Y_t}
\]
or
\[
\phi_{t}^{i} = 1 + \frac{\psi}{1 + \psi + \beta} \frac{\theta_{t}^{i} - 1}{1 - Y_t}
\]
This completes the proof.

Q.E.D.

A.2 Proof of Lemma 6

We prove that the value function is concave. The proof of differentiability and monotonicity is standard and follows the same techniques from Stokey et al. (1989). In order to show concavity of the value function, it is sufficient to show that the constraint set is convex and the rest of the proof follows from Stokey et al. (1989). Note that given lemma 2, the constraint set is given by
\[
\log C_0 + \psi \log (1 - Y) + (1 + \psi + \beta) H(Y) + \beta \log C_1 + \hat{\beta} V' = V
\]
\[
\frac{\beta}{\hat{\beta}} [\log C_1 + H(Y)] + \hat{\beta} V' \geq W
\]
where

\[ H(Y) = \sum_i \alpha_i \mu_i \log \left( 1 + \frac{\psi \theta_i - 1}{1 + \psi + \beta (1 - Y)} \right). \] (44)

Since \( \log \) is a concave function, in order to prove the convexity of the constraint set, it is sufficient to show that \( H(Y) \) is concave. The following lemma, proves this result:

**Lemma 18** Let \( H(Y) \) be the function defined by (44). Then \( H(Y) \) is strictly decreasing and strictly concave.

**Proof.** We have

\[
H(Y) = \sum \mu_i \alpha_i \log \left[ 1 + \frac{\psi \theta_i - 1}{1 + \psi + \beta (1 - Y)} \right]
\]

\[
H'(Y) = \sum \mu_i \alpha_i \frac{\psi \theta_i - 1}{1 + \psi + \beta (1 - Y)} \frac{1}{1 - Y} \theta_i - 1
\]

From assumption 4, \( \alpha_i \) is decreasing in \( i \) and so is \( \frac{1}{1 + \psi + \beta (1 - Y)} \). Hence, \( \frac{\alpha_i}{1 + \psi + \beta (1 - Y)} \) is decreasing in \( i \) while \( \frac{\psi \theta_i - 1}{1 - Y} \) is increasing in \( i \). Hence from Chebyshev’s sum inequality (see Hardy et al. (1952)), we must have

\[
\sum_{i=1}^{N} \mu_i \frac{\alpha_i}{1 + \psi + \beta (1 - Y)} \frac{\psi \theta_i - 1}{1 - Y} < \sum_{i=1}^{N} \mu_i \frac{\alpha_i}{1 + \psi + \beta (1 - Y)} \frac{\psi \theta_i - 1}{1 - Y} \sum_{i=1}^{N} \mu_i \frac{\psi \theta_i - 1}{1 - Y} = 0
\]

This establishes that \( H(Y) \) is strictly decreasing in \( Y \).
Furthermore

\[
H'(Y) = \frac{1}{1-Y} \sum \mu_i \alpha_i \frac{\psi_i}{1+\psi+\beta} (\theta_i - 1) \\
H''(Y) = \frac{1}{(1-Y)^2} \sum \mu_i \alpha_i \frac{\psi_i}{1+\psi+\beta} (\theta_i - 1) \\
+ \frac{1}{1-Y} \sum \mu_i \alpha_i \frac{\psi_i}{1+\psi+\beta} (\theta_i - 1) \left(1-Y+\frac{\psi_i}{1+\psi+\beta} (\theta_i - 1)\right)^2
\]

Note that in the final expression above, the first term is negative while a similar application of Chebyshev’s sum inequality implies that the second term is negative as well and hence \(H(Y)\) must be strictly concave. ■

This concludes the proof.

Q.E.D.

A.3 Proof of Theorem 8

Note that since \(U^p\) and \(U_1^p\) are both logarithmic in terms of \(C_1\) and are separable, the margin between \(C_1\) and \(V'\) in (P1) is not distorted, i.e., we have

\[
\hat{B}'(V') = -\frac{\hat{\beta}}{\beta} C_1
\]

Since \(\hat{B}\) is concave, \(\hat{B}'\) is decreasing and therefore the changes in \(C_1\) and \(V'\) in response to changes in \(V\) is in the same direction (both decreasing or both increasing). Therefore, if the sustainability constraint is binding, it is sufficient to show that \(Y\) is decreasing in \(V\).

Since, we have shown that the value function is differentiable, we can take first order conditions from (P1). Note that there are two possibilities:

1. The sustainability constraint is slack.

In this case, we have the following FOC

\[
Z + \lambda \left[ -\frac{\psi}{1-Y} + (1+\psi+\beta) H'(Y) \right] = 0
\]

where \(\lambda\) is the lagrange multiplier associated with the promise keeping constraint. Fur-
thermore, from envelope condition for (P1), we must have

\[ B'(V) = -\lambda. \]

Since \( B'(V) \) is concave, the above condition implies that \( \lambda \) is increasing in \( V \). We can then write the FOC associated with \( Y \) as

\[ \frac{Z}{\lambda} = \frac{\psi}{1-Y} - (1+\psi+\beta)H'(Y) \]

Since \( H \) is concave, the right hand side of the above equation is increasing in \( Y \) and the left hand side is decreasing in \( \lambda \), it must be that \( Y \) decreases as \( \lambda \) increases. In other words, \( Y \) is a decreasing function of \( V \).

Furthermore, when the sustainability constraint is not binding, the first order condition associated with \( V' \) together with the envelope condition implies that

\[ \frac{1}{1+r^*}B'(V) + \lambda \hat{\beta} = 0 \rightarrow \frac{1}{1+r^*}B'(V) - \hat{\beta}B'(V) = 0 \]

Since \( \hat{B} \) is concave, the above implies that \( V' \) is an increasing function of \( V \). Similarly, the FOC associated with \( C_1 \) is given by

\[ -1 + \frac{\lambda \beta}{C_1} = 0 \rightarrow C_1 = \beta \lambda = -\beta B'(V) \]

Hence \( C_1 \) must be an increasing function of \( V \). Hence, when the constraint is slack, the left hand side of (13) is increasing in \( V \). So there must exists a \( V^* \) so that for values of \( V > V^* \) the sustainability constraint is slack while for values of \( V \leq V^* \), the sustainability constraint must be binding. Hence, we have established that for values of \( V > V^* \), \( Y \) is a decreasing function of \( V \).

2. Sustainability constraint is binding.

As we show above \( V^* \) must exist such that sustainability constraint is binding only if \( V < V^* \). When \( V \) is below \( V^* \), sustainability constraint binds and hence, we must have

\[ V' = W - \frac{\beta}{\hat{\beta}} [\log C_1 + H(Y)] \]

Using this relationship the promise keeping constraint becomes

\[ \log C_0 + \psi \log (1-Y) + (1+\psi)H(Y) + \hat{\beta}W = V \]
while the objective becomes
\[ ZY - C_0 + \frac{1}{1+r^*} \left[ -C_1 + B \left( W - \frac{\beta}{\hat{\beta}} (\log C_1 + H(Y)) \right) \right] \]

Using this objective and taking first order conditions with respect to \( Y \) and \( C_1 \), we have
\[ Z - \frac{\beta}{\hat{\beta} (1+r^*)} B' (V') H' (Y) + \lambda \left[ \frac{-\psi}{1-Y} + (1+\psi) H' (Y) \right] = 0 \quad (46) \]
\[ 1 + \frac{\beta}{\hat{\beta} C_1} B' (V') = 0 \quad (47) \]

where \( V' \) is given by (43). As before, the envelope condition implies that \( B' (V) = -\lambda \) and hence, we need to show that \( Y \) is a decreasing function of \( \lambda \). Consider a small change in \( \lambda \), given by \( d\lambda \). Let the change in \( C_1 \) and \( V' \) be defined by \( dC_1 \) and \( dV' \), respectively. Then, we must have the following relationships – based on (43) and (47)
\[ dV' = \frac{1}{B' (V')} (dC_1 + H' (Y) dY) \]
\[ dC_1 = -\frac{\beta}{\hat{\beta}} B'' (V') dV' \]

which implies that
\[ dV' = \frac{H' (Y)}{B' (V') + \frac{\beta}{\hat{\beta}} B'' (V')} dY \]

Using the above equation and taking a total derivative from (46), we have
\[ -\frac{\beta}{\hat{\beta} (1+r^*)} \left[ B'' (V') H' (Y) dV' + B' (V') H'' (Y) dY \right] + \lambda \left[ \frac{-\psi}{(1-Y)^2} + (1+\psi) H'' (Y) \right] dY \]
\[ + d\lambda \left[ -\frac{\psi}{1-Y} + (1+\psi) H' (Y) \right] = 0 \]
\[ -\frac{\beta}{\hat{\beta} (1+r^*)} \left[ B'' (V') \frac{(H' (Y))^2}{B' (V') + \frac{\beta}{\hat{\beta}} B'' (V')} dY + B' (V') H'' (Y) dY \right] + \lambda \left[ \frac{-\psi}{(1-Y)^2} + (1+\psi) H'' (Y) \right] dY \]
\[ = d\lambda \left[ \frac{\psi}{1-Y} - (1+\psi) H' (Y) \right] \]
Hence,
\[
\frac{dY}{d\lambda} = \frac{\psi - Y - (1 + \psi) H'(Y)}{-\frac{\beta}{\beta(1+r^*)}\left( B''(V') \frac{(H'(Y))^2}{B'(V') + \frac{2}{\beta} B''(V')} + B'(V') H''(Y) \right) + \lambda \left[ -\frac{\psi}{(1-Y)^2} + (1 + \psi) H''(Y) \right]}
\]

Note that \( H' > 0 \) and hence the numerator is a positive number. Furthermore, \( B'' < 0, B' < 0, H'' < 0 \) together with \( \lambda > 0 \) imply that the denominator is negative. Hence, \( \frac{dY}{d\lambda} < 0. \)

This completes the proof.

Q.E.D.

B  Proof for the Political Economy Model

First we prove some properties of \( w \) in (29) that we will use in the proofs below. We cannot prove that the operator implicitly defined in (29) is a contraction. We will then focus on the largest fixed point of (29).

Claim 1  \( w(\hat{B}) \) is strictly decreasing, concave and differentiable.

Proof. We are going to use techniques similar to APS. The operator implicitly defined by the right hand side of (29) is monotone. Then, we can show along the lines of Theorem 5 in APS that if we start iterating on it from \( w_0 \geq w \) then the sequence \( \{w_n\} \) converges to \( w \).

Note that of \( w_n \) is concave then \( w_{n+1} \) is concave as well because the constraint set in (29) is convex and the objective function is convex. Then the limit must be convex as well. A similar argument can be applied to show that \( w \) is decreasing. Differentiability follows from Benveniste-Scheinkman theorem.

Claim 2  The political sustainability constraint is always binding.

Proof. Suppose by way of contradiction that the current government chooses a policy \( (P, C_0, Y, C_1', B'^*) \) such that (27) is slack. Then the government can increase the value of current generation by increasing foreing debt by a small amount (still respecting incentive of next period government not to default) and with debt issuance increase consumption of young agents. This variation clearly increases the value. Hence the sustainability is always binding.

Letting \( \lambda \) and \( \eta \) be the multiplier on the consolidated budget constraint and the political sustainability constraint, the first order necessary condition for an optimum in (29)
\[ \frac{1}{C_0} - \lambda = 0 \]  \( (48) \)
\[ \Omega' (Y) - \eta \omega H' (Y) - \lambda Z = 0 \]  \( (49) \)
\[ \beta \frac{1}{C_1} + \eta \left[ \frac{\omega}{C_1} + w' (B') \right] = 0 \]  \( (50) \)
\[ \frac{\lambda}{1 + r^*} + \eta w' (B') = 0 \]  \( (51) \)

and the envelope condition is \( w' (B) = -\lambda \). Notice for later reference that the multiplier on the political sustainability constraint \( \eta \) can be expressed in terms of allocations as
\[ \eta = \frac{\Omega' (Y) - \frac{1}{C_0} \omega' (Y)}{\omega H' (Y)} = \frac{1}{C_0} Z \]  \( (52) \)

Note that \( \eta > 0 \) since the political sustainability constraint is always binding.

**B.1 Proof of Lemma 13**

Consider \( w \) that solves the functional equation defined by the right hand side of \( (29) \). For all \( B \) such that \( v (B) \geq V_{AUT} \), let \( \hat{P} (B) \) be the decision rule associated with problem \( (32) \). We can construct the other equilibrium objects for a Markov equilibrium as follows:

\[ \hat{P} (B) = \hat{P} (B) \]
\[ \bar{C}_0 (B) = C^w_0 (x (B)), \quad \bar{Y} (B) = Y^w (x (B)), \quad \bar{B}^w (B) = B^{*w} (x (B)), \]
\[ \bar{P}^e (B) = \varphi ^1 (Y^w (x (B))) C^w_1 (x (B)), \quad \bar{a}^1 (B) = \varphi ^1 (Y^w (x (B))) C^w_1 (x (B)) - \bar{P}^e (B), \]

and \( \hat{v} (B) = \hat{v} (B) \) where
\[ x (B) \equiv B^* + \sum \mu^i a^i + \hat{P} (B). \]

To show that our constructed policy rules and values constitute a Markov equilibrium, we have to show that for all \( B \) such that \( v (B) \geq V \) we have that \( P (B | \hat{P} (\cdot), \hat{v} (\cdot)) = \hat{P} (B) \) and \( v (B | \hat{P} (\cdot), \hat{v} (\cdot)) = \hat{v} (B) \).

First, notice that we can combine problems \( (29) \) and \( (32) \) to obtain
\[ \hat{v} (B) = \max _{P, C_0, Y, C^1_1, B^*'} \omega \sum _i \mu^i a^i \log (b^i + P) + \log C_0 - \Omega (Y) + \beta \left[ H (Y) + \log (C^1_1) \right] \]  \( (53) \)
subject to

\[
\sum_{i} \mu_i \left( b^i + p \right) + c_0 + b^* + g \leq ZY + \frac{b^{*'}}{1+r^*} \tag{54}
\]

\[
\omega \left[ H(Y) + \log c_1 \right] + w \left( b^{*'} + c_1 \right) \geq V \tag{55}
\]

Second, since the sustainability is always binding, it is without loss of generality to replace the constraint (55) with

\[
\hat{v}(B') \geq V
\]

where \( B' = (b^{*''}, \{ \varphi^i(Y) c_1 - p^e \}, p^e) \). In fact, we know that in general

\[
\hat{v}(B') \geq \omega \left[ H(Y) + \log c_1 \right] + w \left( b^{*''} + c_1 \right)
\]

but at the optimum in (29) the above is an equality. To see this, suppose by way of contradiction that

\[
\hat{v}(B') > \omega \left[ H(Y) + \log c_1 \right] + w \left( b^{*''} + c_1 \right) = V
\]

Note that for this to be the case we must have that \( \hat{p}' > p^e \) otherwise we have an equality. So the next period government wants to redistribute resources toward the old. So we can equivalently write

\[
\omega \left[ H(Y) + \log c_1 \right] + w \left( b^{*''} + c_1 \right) = \omega \sum_{i} \mu_i \alpha_i \log \left( \varphi^i(Y) c_1 + \Delta \right) + w \left( b^{*''} + c_1 - \Delta \right)
\]

where \( \Delta \geq 0 \) are the additional pension payments provided by the future government. Given concavity of \( w \), the optimal \( \Delta^* \) is greater than zero if and only if

\[
\Theta \frac{\omega}{c_1} > -w'(\hat{B}') \tag{56}
\]

where \( \Theta \equiv \sum \mu_i \alpha_i / \varphi^i(Y) > 1 \). Consider now the following variation that increases pensions by \( \varepsilon \), decreases output by \( \varepsilon_y \) and increases foreign debt by \( \varepsilon^* \). The variation is
feasible in (29) if
\[-\omega H' \varepsilon_y + \frac{\omega}{C_1} \varepsilon = \Theta \frac{\omega}{C_1} \varepsilon \Rightarrow \varepsilon_y = \frac{(\Theta - 1) \frac{\omega}{C_1} \varepsilon}{-\omega H'} > 0\]
\[\left[ \Theta \frac{\omega}{C_1} + w' (\hat{B}') \right] \varepsilon + w' (\hat{B}') \varepsilon^* = 0 \Rightarrow \varepsilon^* = \frac{\left[ \Theta \frac{\omega}{C_1} + w' (\hat{B}') \right]}{-w' (\hat{B}')} \varepsilon\]

where \( \varepsilon^* > 0 \) given (56). If the original allocation is optimal it must be that this variation does not increase the objective value so
\[\Delta \text{obj} = \frac{1}{C_0} \left( \frac{\varepsilon^*}{1+r^*} - Z \varepsilon_y \right) + \frac{\beta}{C_1} \varepsilon + \left[ \frac{\psi}{1-Y} - (1+\psi+\beta) H'(Y) \right] \varepsilon_y \leq 0\]
\[\Rightarrow \frac{\Delta \text{obj}}{\varepsilon} = \frac{1}{C_0 - \omega w' (\hat{B}')(1+r^*)} + \frac{\beta}{C_1 \left[ 1-Y - (1+\psi+\beta) H'(Y) - Z \frac{1}{C_0} \right]} \left( \Theta - 1 \right) \frac{\omega}{C_1} \leq 0\]

Note that under (56), the first term is positive, the second is positive. For the right hand side to be negative it is necessary that the last term is negative but from (52) we know that also the last term is positive. Hence at the optimum (56) cannot hold. Then it must be that future government does not want to redistribute toward the old by increasing pension payments beyond what promised by the previous government. Hence we can rewrite (53) as
\[\hat{v}(\mathcal{B}) = \max_{\{p, C_0, Y, C_1, B^*, p^e\}} \omega \sum_i \mu_i \alpha_i \log (b^i + p) + \log C_0 - \Omega (Y) + \beta \left[ H(Y) + \log (C_1') \right] \quad (57)\]

subject to
\[\sum_i \mu_i (b^i + p) + C_0 + B^* + G \leq ZY + \frac{B^{*,e}}{1+r^*} \quad (58)\]
\[\hat{v}(\mathcal{B}') \geq V \quad (59)\]
\[\mathcal{B}' = B^*, \{\varphi^i (Y) C_1 - p^e \}, p^e \quad (60)\]

It is then evident that (57) is equivalent to (24) and so \( P (\mathcal{B} | \hat{P} (\cdot), \hat{v} (\cdot)) = \hat{P} (\mathcal{B}) \) and \( v (\mathcal{B} | \hat{P} (\cdot), \hat{v} (\cdot)) = \hat{v} (\mathcal{B}) \).
B.2 Proof of Proposition 14

Part i): The fact that $C_0$ is decreasing in $\hat{B}$ follows from combining (48) with the envelope condition and using concavity of $w$.

Part ii): Suppose by way of contradiction that $\hat{B}'$ is increasing in $\hat{B}$. From (50) and concavity of $v$ it follows immediately that $C_1$ is then decreasing in $\hat{B}$. Hence from the political sustainability constraint it follows that $Y$ must be decreasing in $\hat{B}'$. But totally differentiating the first order condition for output (49) we obtain

$$\left[ \Omega'' + \frac{\lambda H''}{(1 + r^*)v'} \right] \frac{dY}{d\hat{B}} = \frac{\lambda H'v''}{(1 + r^*)^2} \frac{dB'}{d\hat{B}} + \left[ Z + \frac{H'(Y)}{(1 + r^*) v'} \right] \frac{d\lambda}{d\hat{B}}$$

implying that $Y$ is increasing in $\hat{B}$ since $\lambda$ is also increasing in $\hat{B}$ and all coefficients are positive. This is a contradiction. Hence it must be that $\hat{B}'$ is decreasing.

C Proof for the Extensions

C.1 Proof of Proposition 16

We first show the following Lemma:

**Lemma 19** The solution to the functional equation (P3) satisfies the following:

1. A unique $B(V)$ exists that satisfies (P3). Furthermore, $B(V)$ is strictly decreasing, strictly concave and differentiable.

2. The policy function $V'(V, s)$ is increasing in $s$.

**Proof.** Properties of the Value Function. The proof follows closely that of Lemma 6. Specifically since the function $H(Y)$ is concave in $Y$, the constraint set is convex and as a result the value function is strictly concave. The rest of the claims can be proven using standard techniques from Stokey et al. (1989).

Properties of the Policy Function. Here, we show that $V'(V, s)$ is increasing in $s$. Note that for any $s$, we must have that

$$\frac{\beta}{\hat{B} C_1(s)} = -\frac{1}{B'(V'(s))}$$
This implies that if \( C_1(s) > C_1(s') \), then \( V'(s) > V'(s') \), since \( B(\cdot) \) is decreasing and concave. Consider the set of states \( S_1 \subset S \) for which the sustainability constraint is slack. Then, the first order conditions associated with \((P1)\) imply that

\[
\forall s, s' \in S_1, C_1(s) = C_1(s'), V'(s) = V'(s').
\]

Consider a state \( s \in S - S_1 \). We show that \( \min S - S_1 > \max S_1 \). Suppose not. Let \( \hat{s} = \max S_1 > \bar{s} = \min S - S_1 \). Then, it must be that

\[
\frac{\beta}{\bar{\beta}} \log C_1(\hat{s}) + V'(\hat{s}) > \mathcal{W}(\hat{s}) - \frac{\beta}{\bar{\beta}} H(Y) \tag{62}
\]

Since \( \mathcal{W}(s) \) is increasing in \( s \), simply because government spending decreases with \( s \), we must have \( \mathcal{W}(\hat{s}) > \mathcal{W}(\bar{s}) \). Hence, the above inequality and equality imply that \( V'(\hat{s}) > V'(\bar{s}) \). We show that if this is the case, there is a perturbation that increases the objective and hence we have a contradiction. Consider a small increase in \( V'(\hat{s}) \) by \( \epsilon > 0 \) and a small decrease in \( V'(\bar{s}) \) by \( \frac{\pi(\hat{s})}{\pi(\bar{s})} \epsilon \). This perturbation does not change the value that the current government receives from the allocation and hence the promise keeping constraint is satisfied. Furthermore, since \( \epsilon \) is small and \((62)\) is strict inequality, sustainability constraints are satisfied. However, the difference between the value of the objective for this perturbation to the original objective is given by

\[
\pi(\bar{s}) B'(V'(\bar{s})) - \pi(\hat{s}) B'(V'(\hat{s})) = \pi(\bar{s}) [B'(V'(\bar{s})) - B'(V'(\hat{s}))] > 0
\]

The derivative of the above expression at \( \epsilon = 0 \) is given by

\[
-\pi(\bar{s}) B'(V'(\bar{s})) - \pi(\hat{s}) B'(V'(\hat{s})) = \pi(\hat{s}) [B'(V'(\hat{s})) - B'(V'(\hat{s}))] > 0
\]

where the last inequality follows from strict concavity of \( B \). This means that for \( \epsilon > 0 \) and small enough this perturbation increases the value of the objective in \((P1)\) and hence we have a contradiction. Therefore, it must be that \( \min S - S_1 > \max S_1 \). Hence, if we let \( s^* = \max S_1 \), we must have

\[
\frac{\beta}{\bar{\beta}} \log C_1(s) + V'(s) = \mathcal{W}(s) - \frac{\beta}{\bar{\beta}} H(Y), \forall s > s^*,
\]

\[
\frac{\beta}{\bar{\beta}} \log C_1(s) + V'(s) > \mathcal{W}(s) - \frac{\beta}{\bar{\beta}} H(Y), \forall s \leq s^*
\]
The first set of equalities above imply that $V'(s)$ must be increasing in $s$. Furthermore, a similar argument to the perturbation above can be used to show that $V'(s') = V'(s^*) < V'(s)$, for all $s > s^* \geq s'$. This implies that $V'(s)$ is weakly decreasing.

Next, we prove the claims in Proposition 16.

**Proof of 1.**

As the proof in C.1 shows, for each $V$ there is a state $s^*$ so that for all $s \leq s^*$, the sustainability constraint is slack while for all $s > s^*$, the sustainability constraint is binding. By continuity of the policy function, it must be that this property holds for an interval including $V$. We can then use the binding sustainability constraints to solve for $V'(s)$ and write the promise keeping constraint as

$$\log C_0 + \psi \log (1 - Y) + (1 + \psi) H(Y) + \sum_{s \leq s^*} \pi(s) \left[ \beta H(Y) + \beta \log C_1(s) + \hat{\beta} V'(s) \right] + \hat{\beta} \sum_{s > s^*} \pi(s) W(s) = V$$

while the value of the objective is given by

$$Z Y - C_0 + \frac{1}{1 + r^*} \sum_{s \leq s^*} \left[ -C_1(s) + B(V'(s)) \right] + \frac{1}{1 + r^*} \sum_{s > s^*} \pi(s) \left[ -C_1(s) + B \left( W(s) - \frac{\beta}{\hat{\beta}} (\log C_1(s) + H(Y)) \right) \right]$$

Hence, the FOC associated with $Y$ as well as that of $\{C_1(s)\}_{s > s^*}$ are given by

$$Z = \sum_{s > s^*} \pi(s) \frac{\beta}{\hat{\beta}(1 + r^*)} B'(V'(s)) H'(Y) + \lambda \left[ -\frac{\psi}{1 - Y} + \left( 1 + \psi + \beta \sum_{s \leq s^*} \pi(s) \right) H'(Y) \right] = 0$$

$$1 + B'(V'(s)) \frac{\beta}{\hat{\beta} C_1(s)} = 0$$

Note that as before, the envelope condition implies that $B'(V) = -\lambda$ and in order to show that $Y(V)$ is decreasing in $V$, it is sufficient to show that as $\lambda$ rises, $Y$ decreases. Taking total derivative from the above equations:

$$dC_1(s) = -\frac{\beta}{\hat{\beta}} B''(V'(s)) dV'(s)$$

$$dV'(s) = -\frac{\beta}{\hat{\beta}} \left[ \frac{1}{C_1(s)} dC_1(s) + H'(Y) dY \right]$$
The above imply that
\[
dV' (s) = -\frac{\beta}{\hat{\beta}} \left[ -\frac{\hat{\beta}}{\beta B'(V'(s))} dC_1 (s) + H'(Y) dY \right]
= -\frac{\beta}{\hat{\beta}} \left[ -\frac{\hat{\beta}}{\beta B'(V'(s))} \beta B'' (V'(s)) dV'(s) + H'(Y) dY \right]
\]
\[
dV' (s) = \frac{-\beta}{\hat{\beta}} H'(Y) dY \frac{1}{1 + \hat{\beta} B''(V'(s)) / \beta B'(V'(s))}
\]

Taking a total derivative from (63), we have
\[
-\frac{\beta}{\hat{\beta} (1 + r^*)} \sum_{s > s^*} \pi (s) \left[ B'' (V'(s)) H'(Y) dV'(s) + B'(V'(s)) H'' (Y) dY \right] + d\lambda \left[ -\frac{\psi}{1 - Y} + \left( 1 + \psi + \beta \sum_{s \leq s^*} \pi (s) \right) H'(Y) \right] + \lambda \left[ -\frac{\psi}{(1 - Y)^2} + \left( 1 + \psi + \beta \sum_{s \leq s^*} \pi (s) \right) H'' (Y) \right] dY = 0
\]
or
\[
-\frac{\beta}{\hat{\beta} (1 + r^*)} dY \sum_{s > s^*} \pi (s) \left[ B'' (V'(s)) \frac{-\beta}{\hat{\beta}} (H'(Y))^2}{1 + \frac{\beta}{\hat{\beta} B''(V'(s))} + B'(V'(s)) H'' (Y)} \right] + d\lambda \left[ -\frac{\psi}{1 - Y} + \left( 1 + \psi + \beta \sum_{s \leq s^*} \pi (s) \right) H'(Y) \right] + \lambda \left[ -\frac{\psi}{(1 - Y)^2} + \left( 1 + \psi + \beta \sum_{s \leq s^*} \pi (s) \right) H'' (Y) \right] dY = 0
\]

which with more concise notation can be written as
\[
\frac{dY}{d\lambda} = \frac{-\frac{\psi}{(1 - Y)^2} + \left( 1 + \psi + \beta \sum_{s \leq s^*} \pi (s) \right) H'}{\lambda \left[ \frac{\psi}{(1 - Y)^2} - \left( 1 + \psi + \beta \sum_{s \leq s^*} \pi (s) \right) H'' \right] + \frac{\beta}{\hat{\beta} (1 + r^*)} \sum_{s > s^*} \pi (s) \left[ B'' (s) \frac{-\beta}{\hat{\beta}} (H')^2}{1 + \frac{\beta}{\hat{\beta} B''(V'(s))} + B'(s) H''} \right]}
\]

The denominator of the above expression is positive while the denominator is negative. This is because \( H' < 0, H'' < 0, B' < 0, \) and \( B'' < 0. \) Therefore, \( \frac{dY}{d\lambda} < 0. \) This proves the first claim.
Proof of 2.
Given the first claim that $Y(V)$ is decreasing, whenever the sustainability constraint is binding, we must have

$$\frac{\beta}{\hat{\beta}} \left[ \log C_1(s) + H(Y) \right] + V'(s) = W(s)$$

Since $Y$ is decreasing in $V$, we must have that

$$\frac{\beta}{\hat{\beta}} \log C_1(s) + V'(s) = W(s) - \frac{\beta}{\hat{\beta}} H(Y)$$

and since $H$ is decreasing, $\frac{\beta}{\hat{\beta}} \log C_1(s) + V'(s)$ must be decreasing in $V$ as well. Since

$$\frac{\beta}{\hat{\beta} C_1(s)} = -\frac{1}{B'(V'(s))}$$

$C_1(s)$ and $V'(s)$ must move together and hence, $C_1(s)$ and $V'(s)$ are decreasing whenever the sustainability constraint is binding. It remains to show that there must exist $V^*(s)$ so that for all values of $V < V^*(s)$, the sustainability constraint for state $s$ is binding and vice versa. Suppose this does not hold. That is, suppose that $V_1$ exists such that for a neighborhood of values of $V$ below $V_1$, the sustainability constraint is slack while for a neighborhood of values of $V$ above $V_1$, the sustainability constraint is binding. Then by continuity of the policy function from theorem of the maximum, it must be that at $V_1$ the sustainability constraint is binding. Note that since for values of $V \in (V_1 - \varepsilon, V_1)$ for a small $\varepsilon > 0$, the sustainability constraint is slack, and we must have

$$\frac{1}{1 + r^*} B'(V'(V, s)) - \hat{\beta} B'(V) = 0$$

$$-\frac{1}{1 + r^*} - \beta B'(V) \frac{1}{C_1(V, s)} = 0$$

where we have used the FOCs associated with $C_1$ and $V'$ and the envelope condition. From concavity of the value function, the above imply that $V'(V, s)$ and $C_1(V, s)$ are both increasing in $V$. Hence, it must be that

$$V'(V_1, s) \geq V'(V, s), \forall V \in (V_1 - \varepsilon, V_1)$$

$$C_1(V_1, s) \geq C_1(V, s), \forall V \in (V_1 - \varepsilon, V_1)$$
From part 1, we know that $Y(V_1) \leq Y(V), \forall V \in (V_1 - \varepsilon, V_1)$. Therefore, for $V \in (V_1 - \varepsilon, V_1)$

$$\frac{\beta}{\beta} [\log C_1(V_1, s) + H(Y(V_1))] + V'(V_1, s) \geq \frac{\beta}{\beta} [\log C_1(V, s) + H(Y(V))] + V'(V, s)$$

$$\geq W(s)$$

Which is a contradiction. This completes the proof.

*Proof of 3.*

Suppose not. That is, there exists $i$, $i + 1$ such that $V^*(s(i)) > V^*(s(i + 1))$. This implies that for values of $V \in (V^*(s(i + 1)), V^*(s(i)))$, the sustainability constraint for $s(i)$ is binding but that of $s(i + 1)$ is slack. We know from lemma 19 that $V'(s(i + 1)) > V'(s)$. Hence a perturbation of the form $\hat{V}'(s(i + 1)) = V'(s(i + 1)) - \varepsilon$ and $\hat{V}'(s(i)) = V'(s(i)) + \varepsilon \frac{\pi(s(i + 1))}{\pi(s(i))}$ increases the objective and satisfies the all the constraints. This concludes the proof.

### C.2 Proof of Proposition 17

We first show the following Lemma:

**Lemma 20** The dynamic system implied by the policy function $V'(V, s)$ has a compact ergodic set $V \subset \mathbb{R}$. Furthermore, a probability measure $\mu$ exists that satisfies (38).

**Proof.** We first show that there exists an ergodic set $V$ that is a subset of a bounded interval. Namely, let $V = V'(V^*(s(1)), s(1))$ and $V = V'(V, s(K))$. We show that for any $V \in [\underline{V}, \overline{V}]$ and $V \in S$, then $V'(V, s) \in [\underline{V}, \overline{V}]$. Note that for any $V > V^*(s(1))$, $V'(V, s(1)) \geq V'(V^*(s(1)), s(1)) = \underline{V}$. Furthermore, from lemma 19, we have that

$$V'(V, s(i)) > V'(V, s(1)) \geq \underline{V}, \forall 1 \leq i \leq K$$

Now suppose that $V < V^*(s(1))$, then from proposition 16 we have that

$$V'(V, s(i)) \geq V'(V, s(1)) \geq V'(V^*(s(1)), s(1)) = \underline{V}$$

Hence, for all $V$, we must have that $V'(V, s) \geq \underline{V}$.

Now, suppose that $V \in [\underline{V}, \overline{V}]$. Then either sustainability constraint for $s(K)$ is binding or slack. If it is slack, then we must have

$$B'(V'(V, s(K))) = (1 + r^*)\hat{B}'(V)$$
since $\hat{\beta}(1 + r^*) < 1$ and $B' < 0$,

$$B'(V) < B'(V'(V, s(K)))$$

and concavity of $B(\cdot)$ implies that $V > V'(V, s(K))$. Hence,

$$\nabla \geq V > V'(V, s(K)) \geq V'(V, s(i)), \forall i$$

If the sustainability constraint is binding,

$$V < V \rightarrow V'(V, s(K)) < V'(V', s(K)) = \nabla$$

and furthermore

$$V'(V, s(i)) < V'(V, s(K)) = \nabla.$$ 

This establishes that any ergodic set $V$ must be a subset of $[\nabla, \overline{V}]$.

Next, we show that a stationary distribution $\mu$ exists. Since $[\nabla, \overline{V}]$ is a compact subset of $\mathbb{R}$, then by Riesz Representation Theorem (Dunford and Schwartz (1958), IV.6.3), the space of regular measures on $[\nabla, \overline{V}]$ is isomorphic to the space $C^*(\nabla, \overline{V})$, the dual of the space of bounded continuous functions on $[\nabla, \overline{V}]$. Moreover, by Banach-Alaoglu Theorem (Rudin (1991), Theorem 3.15), the set $\{f \in C^*(\nabla, \overline{V}) ; \|f\| \leq k\}$ is a compact set in the weak-* topology for any $k > 0$. Equivalently, the set of regular measures, $\psi$, with $\|\psi\| \leq 1$, is compact. Since non-negativity and full measure on $[\nabla, \overline{V}]$ are closed properties, we must have the set

$$\mathcal{M}([\nabla, \overline{V}]) = \{\psi ; \psi \text{ a regular measure on } [\nabla, \overline{V}], \psi([\nabla, \overline{V}]) = 1, \psi \geq 0\}$$

is compact in weak-* topology. Since any ergodic set $V$ is a subset of $[\nabla, \overline{V}]$, the transformation $T$ on $\mathcal{M}([\nabla, \overline{V}])$ given by

$$T(\psi)(A) = \int_{[\nabla, \overline{V}]} \sum_{s \in S} \pi(s) \mathbf{1}[V'(V, s) \in A] \, d\psi(V)$$

maps $\mathcal{M}([\nabla, \overline{V}])$ into itself. Since from theorem of maximum, $V'(V, s)$ is continuous in $V$, the above mapping is continuous. Thus by Schauder-Tychonoff Theorem (Dunford and Schwartz (1958), V.10.5), $T$ has a fixed point $\mu$. This concludes the proof. ■

To prove the claim in proposition 17, it is sufficient to show that $V'(V, s(K)) = \nabla$. Suppose not. That is, suppose that $V'(V, s(K)) < \nabla$. Since $\nabla = \sup V$, it must be that for some $V \leq \nabla$, $V'(V, s(K)) = \nabla$ – since $V'(V, s)$ is increasing in $s$. Since $\nabla = \inf V$, it must
be that $V > \bar{V}$. Now, if the sustainability constraint for $s(K)$ is slack at $V$, it must be that $V > \bar{V}$ which cannot be. So it must be that the sustainability constraint for $s(K)$ at $V$ is binding. As a result, $V'(V, s(K)) > V'(V, s(K)) = \bar{V}$ which is a contradiction. Hence, we must have $V'(V, s(K)) = \bar{V}$. This concludes the proof.

### D A Model of The World Economy

In this section, we describe a world economy consisted of identical countries who experience i.i.d. shocks to government spending. We show that in the steady state of such economy, the world interest rate is constant and satisfies the relationship $\hat{\beta}(1 + r^*) < 1$.

The world is consisted of a continuum of countries that are as described in section 5.2. The initial state of each country is defined by its initial inequality $\Phi - 1$, foreign debt $B$ and government spending represented by $s_0$. We let $x = (\Phi - 1, B, s_0)$ and $\psi_0$ be the initial distribution of $x$ across countries. In particular, let $X$ be the set of all possible $x$. Then, from market clearing $\psi_0$ must satisfy

$$
\int_X B(x) \, d\psi_0(x) = 0
$$

Allocations for each country can be identified by its history of shocks and its initial state $x$. These allocations are given by $\{ C_{t,0}(s^t, x), C_{t-1,1}(s^t, x), Y_t(s^t, x), \Phi_t(s^t, x) \}$, where $s^t = (s_1, \ldots, s_t)$.

Hence, the world planning problem that treats each country symmetrically can be written as

$$
\max \int_X \left\{ \frac{\beta}{\beta} U^p(s^{t+1}) C_{t,0}(s^t, x), C_{t,1}(s^t, x), Y_t(s^t, x), \Phi_t(s^t, x) \right\} \, d\psi_0(x)
$$

subject to

$$
\int_X \sum_{s^t} \pi(s^t) \left[ C_{t,0}(s^t, x) + C_{t-1}(s^t, x) + G(s_t) - ZY_t(s^t, x) \right] \, d\psi_0(x) = 0
$$

$$
\frac{\beta}{\beta} U^p(s^{t+1}) C_{t,1}(s^t, x), \Phi_{t-1}(s^t, x)
$$

$$
\sum_{\tau = t}^{\infty} \sum_{s^\tau \geq s^t} \pi(s^\tau|s^t) \hat{\beta}^{\tau-t} U^p \left( C_{\tau,0}(s^\tau, x), C_{\tau,1}(s^{\tau+1}, x), Y_\tau(s^\tau, x), \Phi_\tau(s^\tau, x) \right) \geq W(s_t)
$$
Consider a steady state of the planning problem in \((64)\) where \(\psi_t\) appropriately defined. Let \(\lambda_t\) be the lagrange multiplier associated with aggregate resource constraint \((65)\). Then it must be that in a steady state where \(\psi_t = \psi_{t+1}, \lambda_{t+1}/\lambda_t = q\) is constant. The ratio \(q\) is equal to \(\frac{1}{1+r^*}\) where \(r^*\) is the international interest rate considered in section 5.2. In particular, it can be simply shown that the state of each country can be represented by promised utility \(V_t\) as defined in section 3. Then, it can be shown that each individual country’s problem at each point in time is equivalent to the recursive formulation in \((P3)\).

Note that first order conditions of the above problem with respect to consumption imply that

\[
\left[ \hat{\beta}^t + \sum_{s^t \geq s^t} \hat{\beta}^{t-s^t} \mu_{s^t} (s^t, x) \right] \frac{1}{C_{t,0} (s^t, x)} = \lambda_t
\]

\[
\left[ \hat{\beta}^{t-1} + \sum_{s^t \geq s^t} \hat{\beta}^{t-1-s^t} \mu_{s^t} (s^t, x) \right] \frac{\beta}{C_{t-1,1} (s^t, x)} = \lambda_t
\]

Hence,

\[
\frac{C_{t+1,0} (s^{t+1}, x)}{C_{t,0} (s^t, x)} = \frac{\lambda_t \left( \hat{\beta}^{t+1} \right) + \sum_{s^t \geq s^{t+1}} \hat{\beta}^{t+1-s^t} \mu_{s^t} (s^t, x)}{\lambda_{t+1} \left( \hat{\beta}^t \right) + \sum_{s^t \geq s^t} \hat{\beta}^{t-s^t} \mu_{s^t} (s^t, x)} = \frac{\lambda_t}{\lambda_{t+1}} \left( \hat{\beta}^t + \sum_{s^t \geq s^t} \hat{\beta}^{t-s^t} \mu_{s^t} (s^t, x) \right)
\]

\[
\frac{C_{t,1} (s^{t+1}, x)}{C_{t-1,1} (s^t, x)} = \frac{\lambda_t \left( \hat{\beta}^{t+1} \right) + \sum_{s^t \geq s^{t+1}} \hat{\beta}^{t+1-s^t} \mu_{s^t} (s^t, x)}{\lambda_{t+1} \left( \hat{\beta}^t \right) + \sum_{s^t \geq s^t} \hat{\beta}^{t-s^t} \mu_{s^t} (s^t, x)} = \frac{\lambda_t}{\lambda_{t+1}} \left( \hat{\beta}^t + \sum_{s^t \geq s^t} \hat{\beta}^{t-s^t} \mu_{s^t} (s^t, x) \right)
\]

Note that the above equations imply that

\[
\frac{C_{t+1,0} (s^{t+1}, x)}{C_{t,0} (s^t, x)} \geq \frac{\hat{\beta}}{q}
\]

\[
\frac{C_{t,1} (s^{t+1}, x)}{C_{t-1,1} (s^t, x)} \geq \frac{\hat{\beta}}{q}
\]

with strict inequality if the sustainability constraint binds for \((s^{t+1}, x)\). This implies that we must have that in a steady state \(\frac{\hat{\beta}}{q} \leq 1\), otherwise aggregate consumption increases without bounds. Additionally, if the sustainability constraint binds with positive probability, we must have that \(\frac{\hat{\beta}}{q} < 1\). Hence, we have the following proposition:

**Proposition 21** Consider a steady state of the planning problem in \((64)\) where \(\lambda_{t+1}/\lambda_t = q\) is
the ratio of the multiplier on (65). Then it must be that \( \hat{\beta}/q \leq 1 \). Furthermore, if in steady state, the sustainability constraint (66) binds with positive probability, then \( \hat{\beta}/q < 1 \).

E  GHH Preferences

Let \( \gamma = 1 + \frac{1}{\epsilon} \). Then market utility is given by

\[
U^M_0 (C, Y; \Phi) = \max \sum \varphi^i \mu^i \log \left( c^i - \psi \left( l^i \right)^{\gamma} / \gamma \right)
\]

subject to

\[
\sum \mu^i c^i = C, \quad \sum \mu^i \theta^i l^i = Y
\]

Then FOCs are given by

\[
\frac{\varphi^i}{c^i - \psi \left( l^i \right)^{\gamma} / \gamma} = \lambda_C
\]

\[
\frac{\varphi^i}{c^i - \psi \left( l^i \right)^{\gamma} / \gamma} \psi \left( l^i \right)^{\gamma-1} = \lambda_Y \theta^i
\]

and we must have \( \psi \left( l^i \right)^{\gamma-1} = \frac{\theta^i \lambda_C}{\lambda_Y} \). Then

\[
l^i = \left( \frac{\theta^i \lambda_C}{\psi \lambda_Y} \right)^{\epsilon} \to \theta^i l^i = \left( \frac{\theta^i}{\theta^j} \right)^{1+\epsilon} \theta^j l^j \to \theta^i l^i = \frac{\left( \theta^i \right)^{1+\epsilon}}{\sum \mu^j (\theta^j)^{1+\epsilon} Y}
\]

We define average productivity as

\[
\hat{\theta}^{1+\epsilon} = \sum \mu^i (\theta^i)^{1+\epsilon}
\]

Then,

\[
l^i = \frac{(\theta^i)^{\epsilon}}{\hat{\theta}^{1+\epsilon} Y}
\]
Furthermore

\[
\frac{\phi_i}{\lambda C} = c^i - \psi \left( l^i \right)^{1+\epsilon} \gamma
\]

\[
\frac{1}{\lambda C} = \sum \mu_i c^i - \psi \sum \mu_i \left( l^i \right)^{1+\epsilon} \gamma
\]

\[
= C - \psi \frac{\sum \mu_i \left( \theta^i \right)^{1+\epsilon}}{\hat{\theta}^{(1+\epsilon)\gamma}} \gamma
\]

\[
= C - \frac{\psi \hat{\theta}^{(1+\epsilon)}}{\gamma \hat{\theta}^{(1+\epsilon)\gamma}} \gamma
\]

\[
= C - \frac{\psi \hat{\theta}^{(1+\epsilon)(1-\gamma)}}{\gamma \hat{\theta}^{(1+\epsilon)\gamma}} \gamma
\]

\[
= C - \frac{\psi \hat{\theta}^{-\gamma} \gamma Y}{\gamma}
\]

\[
= C - \nu \left( Y/\hat{\theta} \right)
\]

Then,

\[
c^i = \left( C - \nu \left( Y/\hat{\theta} \right) \right) \phi^i + \frac{\psi}{\gamma} \left( \frac{\left( \theta^i \right)^{\epsilon}}{\hat{\theta}^{1+\epsilon} \gamma} \right)
\]

\[
= \phi^i C - \phi^i \nu \left( Y/\hat{\theta} \right) + \frac{\psi}{\gamma} \left( \frac{\left( \theta^i \right)^{1+\epsilon}}{\hat{\theta}^{(1+\epsilon)\gamma}} \gamma \right)
\]

\[
= \phi^i C - \phi^i \nu \left( Y/\hat{\theta} \right) + \frac{\left( \theta^i \right)^{1+\epsilon}}{\hat{\theta}^{1+\epsilon} \gamma} \psi \hat{\theta}^{(1+\epsilon)(1-\gamma)} \gamma
\]

\[
= \phi^i C - \nu \left( Y/\hat{\theta} \right) \left( \phi^i - \left( \frac{\theta^i}{\hat{\theta}} \right)^{1+\epsilon} \right)
\]

then the market utility can be written as

\[
U \left( C, Y; \Phi \right) = \sum \mu_i \phi^i \log \left( \left( C - \nu \left( Y/\hat{\theta} \right) \right) \phi^i \right)
\]

\[
= \log \left( C - \nu \left( Y/\hat{\theta} \right) \right) + \sum \phi^i \mu_i \log \phi^i
\]

and so

\[
U_C = \frac{1}{C - \nu \left( Y/\hat{\theta} \right)}
\]

\[
U_Y = -\frac{1}{C - \nu \left( Y/\hat{\theta} \right)} \hat{\theta} \nu \left( Y/\hat{\theta} \right)
\]

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Hence, the budget constraint can be written as

\[
\frac{1}{C_0 - v(Y/\hat{\theta})} \left[ \varphi^i C_0 - v(Y/\hat{\theta}) \left( \varphi^i - \left( \frac{\theta^i}{\hat{\theta}} \right)^{1+\varepsilon} \right) \right] - \frac{1}{C_0 - v(Y/\hat{\theta})} \left( \frac{\theta^i}{\hat{\theta}} \right)^{1+\varepsilon} Y + \beta \varphi^i = \hat{T}
\]

The objective for the government is

\[
(1 + \beta) \sum \alpha_i \mu_i \log \varphi^i + \log \left( C_0 - v(Y/\hat{\theta}) \right) + \beta \log C_1
\]

So, we can write the recursive problem as

\[
B(V) = \max ZY - C_0 - \frac{1}{1 + r^s} C_1 + \frac{1}{1 + r^s} B(V')
\]

subject to

\[
(1 + \beta) \sum \alpha_i \mu_i \log \varphi^i + \log \left( C_0 - v(Y/\hat{\theta}) \right) + \beta \log C_1 + \hat{\beta} V' = V
\]

\[
\frac{\beta}{\hat{\beta}} \sum \alpha_i \mu_i \log \varphi^i + \log C_1 + V' \geq V
\]

\[
1 + \frac{1}{1 + \beta} \frac{v(1/\hat{\theta})}{C_0 Y - \gamma - v(1/\hat{\theta})} \left( \frac{\theta^i}{\hat{\theta}} \right)^{1+\varepsilon} = \varphi^i
\]

Some preliminary results:

Let \( H(Y, C_0) = \sum \alpha_i \mu_i \log \left( 1 + \frac{1}{1 + \beta} \frac{v(1/\hat{\theta})}{C_0 Y - \gamma - v(1/\hat{\theta})} \left( \frac{\theta^i}{\hat{\theta}} \right)^{1+\varepsilon} - \hat{\theta}^{1+\varepsilon} \right) \). Then we have:

**Lemma 22** The function \( \hat{H} \) is strictly increasing and concave. Furthermore, the function \( H(Y, C_0) \) is decreasing in \( Y \) and increasing in \( C_0 \) and if \( \gamma \geq 2 \), it is strictly concave in \( (Y, C_0) \).

**Proof.** Some algebra:
\[ \hat{H}' (x) = - \sum \frac{\alpha^i}{\varphi^i \mu^i} \frac{1}{1 + \beta} \frac{v (1/\hat{\theta})}{(x - v (1/\hat{\theta}))^2} \left( \theta^i \right)^{1+\epsilon} - \hat{\theta}^{1+\epsilon} > 0 \]

\[ \hat{H}'' (x) = 2 \sum \frac{\alpha^i}{\varphi^i \mu^i} \frac{1}{1 + \beta} \frac{v (1/\hat{\theta})}{(x - v (1/\hat{\theta}))^3} \left( \theta^i \right)^{1+\epsilon} - \hat{\theta}^{1+\epsilon} \]

\[ (\gamma + 1) \hat{H}' (x) + \gamma x \hat{H}'' (x) = - \frac{v (1/\hat{\theta})}{(x - v (1/\hat{\theta}))^2} (\gamma + 1) \sum \frac{\alpha^i}{\varphi^i \mu^i} \frac{1}{1 + \beta} \frac{(\theta^i)^{1+\epsilon} - \hat{\theta}^{1+\epsilon}}{\hat{\theta}^{1+\epsilon}} \]

\[ + 2\gamma x \frac{v (1/\hat{\theta})}{(x - v (1/\hat{\theta}))^3} \sum \frac{\alpha^i}{\varphi^i \mu^i} \frac{1}{1 + \beta} \frac{(\theta^i)^{1+\epsilon} - \hat{\theta}^{1+\epsilon}}{\hat{\theta}^{1+\epsilon}} \]

\[ - \gamma x \sum \frac{\alpha^i}{(\varphi^i)^2 \mu^i} \left( \frac{1}{1 + \beta} \frac{v (1/\hat{\theta})}{(x - v (1/\hat{\theta}))^2} \left( \theta^i \right)^{1+\epsilon} - \hat{\theta}^{1+\epsilon} \right)^2 \]

\[ = \frac{v (1/\hat{\theta})}{(x - v (1/\hat{\theta}))^3} \left[ (\gamma + 1) v (1/\hat{\theta}) + (\gamma - 1) x \right] \sum \frac{\alpha^i}{\varphi^i \mu^i} \frac{1}{1 + \beta} \frac{(\theta^i)^{1+\epsilon} - \hat{\theta}^{1+\epsilon}}{\hat{\theta}^{1+\epsilon}} \]

\[ - \gamma \hat{H}' (x) - \gamma x \hat{H}'' (x) = - \frac{v (1/\hat{\theta})}{(x - v (1/\hat{\theta}))^2} \gamma \sum \frac{\alpha^i}{\varphi^i \mu^i} \frac{1}{1 + \beta} \frac{(\theta^i)^{1+\epsilon} - \hat{\theta}^{1+\epsilon}}{\hat{\theta}^{1+\epsilon}} \]

\[ - 2\gamma x \frac{v (1/\hat{\theta})}{(x - v (1/\hat{\theta}))^3} \sum \frac{\alpha^i}{\varphi^i \mu^i} \frac{1}{1 + \beta} \frac{(\theta^i)^{1+\epsilon} - \hat{\theta}^{1+\epsilon}}{\hat{\theta}^{1+\epsilon}} \]

\[ + \gamma x \sum \frac{\alpha^i}{(\varphi^i)^2 \mu^i} \left( \frac{1}{1 + \beta} \frac{v (1/\hat{\theta})}{(x - v (1/\hat{\theta}))^2} \left( \theta^i \right)^{1+\epsilon} - \hat{\theta}^{1+\epsilon} \right)^2 \]

\[ = - \frac{v (1/\hat{\theta})}{(x - v (1/\hat{\theta}))^3} \left[ \gamma v (1/\hat{\theta}) + \gamma x \right] \sum \frac{\alpha^i}{\varphi^i \mu^i} \frac{1}{1 + \beta} \frac{(\theta^i)^{1+\epsilon} - \hat{\theta}^{1+\epsilon}}{\hat{\theta}^{1+\epsilon}} \]

\[ + \gamma x \sum \frac{\alpha^i}{(\varphi^i)^2 \mu^i} \left( \frac{1}{1 + \beta} \frac{v (1/\hat{\theta})}{(x - v (1/\hat{\theta}))^2} \left( \theta^i \right)^{1+\epsilon} - \hat{\theta}^{1+\epsilon} \right)^2 > 0 \]
Then

\[
\begin{align*}
H_C &= Y^{-\gamma} \hat{H}' (C_0 Y^{-\gamma}) > 0 \\
H_Y &= -\gamma C_0 Y^{-\gamma-1} \hat{H}' (C_0 Y^{-\gamma}) < 0 \\
H_C C &= Y^{-2\gamma} \hat{H}'' (C_0 Y^{-\gamma}) < 0 \\
H_Y &= \gamma (\gamma + 1) C_0 Y^{-\gamma-2} \hat{H}' (C_0 Y^{-\gamma}) \\
&\quad + \gamma^2 C_0^2 Y^{-2\gamma-2} \hat{H}'' (C_0 Y^{-\gamma}) \\
&= \gamma C_0 Y^{-\gamma-2} [(\gamma + 1) \hat{H}' (C_0 Y^{-\gamma}) + \gamma C_0 Y^{-\gamma} \hat{H}'' (C_0 Y^{-\gamma})] < 0 \\
H_Y H_C - (H_Y)^2 &= \gamma (\gamma + 1) C_0 Y^{-\gamma-2} \hat{H}' (C_0 Y^{-\gamma}) Y^{-2\gamma} \hat{H}'' (C_0 Y^{-\gamma}) \\
&\quad + \gamma^2 C_0^2 Y^{-2\gamma-2} \hat{H}'' (C_0 Y^{-\gamma}) Y^{-2\gamma} \hat{H}'' (C_0 Y^{-\gamma}) \\
&\quad - \gamma^2 Y^{-2\gamma-2} (\hat{H}' (C_0 Y^{-\gamma}) + C_0 Y^{-\gamma} \hat{H}'' (C_0 Y^{-\gamma}))^2 \\
&= \gamma (\gamma + 1) C_0 Y^{-3\gamma-2} \hat{H}' \hat{H}'' + \gamma^2 C_0^2 Y^{-4\gamma-2} (\hat{H}'')^2 \\
&\quad - \gamma^2 Y^{-2\gamma-2} (\hat{H}')^2 + 2 C_0 Y^{-\gamma} \hat{H}' \hat{H}'' + C_0^2 Y^{-2\gamma} (\hat{H}'')^2 \\
&= (\gamma - \gamma^2) C_0 Y^{-3\gamma-2} \hat{H}' \hat{H}'' - \gamma^2 Y^{-2\gamma-2} (\hat{H}')^2 \\
&= -\gamma Y^{-2\gamma-2} \hat{H}' ((\gamma - 1) C_0 Y^{-\gamma} \hat{H}'' + \gamma \hat{H}') \\
&= -\gamma Y^{-2\gamma-2} \hat{H}' ((\gamma - 1) x \hat{H}'' + \gamma \hat{H}')
\end{align*}
\]

The above expression is positive as long as \( \gamma \geq 2 \). This implies that the function \( H \) is concave, dec in \( Y \) and increasing in \( C \).

Q.E.D.
Let $\beta = \frac{\beta}{\beta}$.

Note that with the above definition, we can write the problem

$$B (V) = \max ZY - C_0 - \frac{1}{1 + r^s} C_1 + \frac{1}{1 + r^s} B (V')$$  \hspace{1cm} \text{(FE2)}$$

subject to

$$\log (C_0 - v (Y/\hat{\theta})) + (1 + \beta) H (C_0, Y) + \beta \log C_1 + \beta V' = V$$

$$\hat{\beta} \log C_1 + \beta H (C_0, Y) + V' \geq V$$

We analyze two cases:

**Case 1.** $V$ is high so that the sustainability constraint is not binding. Then FOCs are given by

$$-1 + \frac{\lambda}{C_0 - v (Y/\hat{\theta})} + (1 + \beta) \lambda H_C (C_0, Y) = 0$$

$$-\frac{1}{1 + r^s} + \frac{\beta \lambda}{C_1} = 0$$

$$Z - \frac{\lambda}{C_0 - v (Y/\hat{\theta})} v' (Y/\hat{\theta}) \frac{1}{\hat{\theta}} + (1 + \beta) \lambda H_Y (C_0, Y) = 0$$

$$\frac{1}{1 + r^s} B' (V') + \hat{\beta} \lambda = 0$$

$$\frac{1}{C_0 - v (Y/\hat{\theta})} + (1 + \beta) H_C (C_0, Y) = \frac{1}{\lambda}$$

$$\frac{1}{C_0 - v (Y/\hat{\theta})} v' (Y/\hat{\theta}) \frac{1}{\hat{\theta}} - (1 + \beta) H_Y (C_0, Y) = \frac{Z}{\lambda}$$

$$\frac{1}{C_0 - v (Y/\hat{\theta})} + (1 + \beta) Y^{-\gamma} \hat{H}' (C_0 Y^{-\gamma}) = \frac{1}{\lambda}$$

$$\frac{1}{C_0 - v (Y/\hat{\theta})} v' (Y/\hat{\theta}) \frac{1}{\hat{\theta}} + (1 + \beta) Y^{-\gamma - 1} C_0 \hat{H}' (C_0 Y^{-\gamma}) = \frac{Z}{\lambda}$$

Multiplying the first equation by $C_0$ and the second equation by $\frac{Y}{Y}$ and subtracting the second from the first, we have

$$1 = \frac{C_0 - ZY / \gamma}{\lambda} \rightarrow C_0 = \lambda + \frac{ZY}{\gamma}$$. 

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So, the equation that governs $Y$ can be written as

$$\frac{1}{\lambda + ZY/\gamma - \gamma v (1/\theta)} + (1 + \beta) Y^{-\gamma} \hat{H}' (\lambda + ZY/\gamma) Y^{-\gamma} = \frac{1}{\lambda}$$

We can rewrite the above equation as

$$\frac{1}{(\lambda + ZY/\gamma) Y^{-\gamma} - v (1/\theta)} + (1 + \beta) \gamma \hat{H}' (\lambda + ZY/\gamma) Y^{-\gamma} = \frac{\lambda}{\gamma}$$

As we have noted above, inequality is governed by the variable $x = C_0 Y^{-\gamma} = (\lambda + ZY/\gamma) Y^{-\gamma}$. As $x$ goes up inequality goes down. We want to characterize the behavior of various variables as a function of $V$. Note that since the value function is concave – the constraint set is convex and the objective is linear, $B' (V)$ is a decreasing function of $V$. Since by the envelope condition $B' (V) = -\lambda$ and hence as $V$ increases $\lambda$ increases as well. So the characterization comes down to the behavior of various variables as a function of $\lambda$. We start with inequality $x$. Note that

$$dx = Y^{-\gamma} d\lambda + \left[ -\gamma Y^{-\gamma-1} \lambda + \frac{1 - \gamma}{\gamma} ZY^{-\gamma} \right] dY$$

$$dY = \frac{dx - Y^{-\gamma} d\lambda}{t}$$

where $t = \gamma Y^{-1} - \gamma + \frac{\gamma - 1}{\gamma} ZY^{-\gamma} > 0$. Then we can write the above equation as

$$\lambda \left[ \frac{1}{x - \bar{v}} + (1 + \beta) \hat{H}' (x) \right] - \gamma Y = 0$$

Taking derivatives

$$d\lambda \left[ \frac{1}{x - \bar{v}} + (1 + \beta) \hat{H}' (x) \right] + \lambda \left[ -\frac{1}{x - \bar{v}} + (1 + \beta) \hat{H}'' (x) \right] dx - \gamma Y d\gamma = 0$$

or

$$d\lambda \left[ \frac{\gamma Y^{-1}}{t} + \frac{1}{x - \bar{v}} + (1 + \beta) \hat{H}' (x) \right] + \left[ -\frac{1}{x - \bar{v}} + (1 + \beta) \hat{H}'' (x) \right] \lambda - \frac{\gamma Y^{-1}}{t} \right] dx = 0$$

As a result

$$\frac{dx}{d\lambda} = -\frac{\gamma Y^{-1}}{t} + \frac{1}{x - \bar{v}} + (1 + \beta) \hat{H}' (x) \left( -\frac{1}{x - \bar{v}} + (1 + \beta) \hat{H}'' (x) \right) \lambda - \frac{\gamma Y^{-1}}{t} > 0$$
and this is positive since the numerator is negative and denominator is positive. This implies that $x$ increases with $V$ and as a result inequality decreases as $V$ increases.

**Case 2.** Suppose that the constraint is binding. Then we can write the program as

$$B(V) = \max ZY - C_0 - \frac{1}{1 + r^*} C_1 + \frac{1}{1 + r^*} B \left( V - \beta^{-1} \log C_1 - \beta^{-1} H(C_0, Y) \right)$$

subject to

$$\log (C_0 - v(Y/\hat{\theta})) + H(C_0, Y) + \hat{\beta} V = V$$

FOCs are

$$-1 - \frac{C_0 - v(Y/\hat{\theta})}{C_0 - v(Y/\hat{\theta})} - \frac{1}{\beta (1 + r^*)} B'(V') Y^{-\gamma} \hat{H}'(C_0 Y^{-\gamma}) + \lambda Y^{-\gamma} \hat{H}'(C_0 Y^{-\gamma}) = 0$$

$$- \frac{1}{1 + r^*} - \frac{1}{\beta (1 + r^*)} B'(V') \frac{1}{C_1} = 0$$

$$Z + \frac{1}{\beta (1 + r^*)} Y^{-\gamma - 1} \hat{H}'(C_0 Y^{-\gamma}) B'(V') - \frac{\lambda Y (Y/\hat{\theta}) / \hat{\theta}}{C_0 - v(Y/\hat{\theta})} - \lambda Y^{-\gamma} \hat{H}'(C_0 Y^{-\gamma}) = 0$$

Note that as before

$$C_0 = \lambda + \frac{ZY}{\gamma}$$

So similar to what we had before, we can write

$$dx = Y^{-\gamma} d\lambda + \left[ -\gamma Y^{-\gamma - 1} \lambda + \frac{1 - \gamma}{\gamma} ZY^{-\gamma} \right] dY$$

$$dY = \frac{dx - Y^{-\gamma} d\lambda}{t}$$

We can write the first FOC as

$$-Y^\gamma - \frac{\lambda}{C_0 Y^{-\gamma} - v(1/\hat{\theta})} - \frac{1}{\beta (1 + r^*)} B'(V') \hat{H}'(C_0 Y^{-\gamma}) + \lambda \hat{H}'(C_0 Y^{-\gamma}) = 0$$

or

$$-Y^\gamma - \frac{\lambda x - v}{x - \bar{v}} - \frac{1}{\beta (1 + r^*)} B'(V') \hat{H}'(x) + \lambda \hat{H}'(x) = 0$$

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Taking derivative we can write

\[-\gamma Y^{-1} dY + \lambda \left( -\frac{1}{(x-v)^2} + \hat{H}''(x) \right) dx + \left( \frac{1}{x-v} + \hat{H}'(x) \right) d\lambda \]

\[-\frac{1}{\beta (1+r^*)} B'(V') \hat{H}''(x) dx - \frac{1}{\beta (1+r^*)} \hat{H}'(x) B''(V') dV' = 0\]

From the FOCs, we have

\[
C_1 = -\frac{1}{\beta} B'(V') \rightarrow dC_1 = -\frac{1}{\beta} B''(V') dV' \\
V' = V - \hat{\beta}^{-1} \log C_1 - \hat{\beta}^{-1} \hat{H}(x) \rightarrow dV' = -\frac{1}{\beta C_1} dC_1 - \frac{1}{\beta} \hat{H}'(x) dx \\
dC_1 = -\frac{1}{\beta} B''(V') \left[ -\frac{1}{\beta C_1} dC_1 - \frac{1}{\beta} \hat{H}'(x) dx \right] \\
= -\frac{1}{\beta} B''(V') \left[ \frac{1}{B'(V')} dC_1 - \frac{1}{\beta} \hat{H}'(x) dx \right] \\
= -\frac{1}{\beta B'} dC_1 + \frac{1}{\beta B'} B'' \hat{H} dx \\
dC_1 = \frac{1}{\beta B'} \hat{H} dx \\
dV' = -\frac{1}{\beta B'} \hat{H} dC_1 \rightarrow dV' = -\frac{1}{1 + \frac{1}{\beta B'}} dV' \\
= -\frac{1}{1 + \frac{1}{\beta B'}} \hat{H} dx
\]

Note that \(\frac{B''}{B'} > 0\). So we can rewrite

\[-\gamma Y^{-1} \frac{dx}{t} - Y^{-\gamma} \frac{d\lambda}{t} + \lambda \left( -\frac{1}{(x-v)^2} + \hat{H}''(x) \right) dx + \left( \frac{1}{x-v} + \hat{H}'(x) \right) d\lambda \]

\[-\frac{1}{\beta (1+r^*)} B'(V') \hat{H}''(x) dx + \frac{1}{\beta (1+r^*)} \hat{H}'(x) B''(V') \frac{1}{1 + \frac{1}{\beta B'}} dx = 0\]
and we can write

$$\begin{align*}
\frac{dx}{d\lambda} &= \left[ -\frac{\gamma Y^{-1}}{t} + \lambda \left( -\frac{1}{(x - \bar{v})^2} + H''(x) \right) \right] - \frac{1}{\beta (1 + r^*)} B'(V') H''(x) \\
&\quad + \frac{1}{\beta (1 + r^*)} H'(x) B''(V') \left( \frac{1}{1 + \frac{1}{\beta} B''} \frac{1}{1 + \frac{1}{\beta} B''} \right) \\
&= \left[ -H' - \frac{1}{x - \bar{v}} - \frac{\gamma Y^{-1}}{t} \right] d\lambda \to \frac{dx}{d\lambda} > 0
\end{align*}$$

$$\frac{dV'}{d\lambda} = -\frac{1}{\beta} \frac{H'}{1 + \frac{1}{\beta} B''} \frac{dx}{d\lambda} < 0$$

This completes the proof that $V'$ is decreasing in $\lambda$ and as a result in $V$. Hence, we have the following proposition:

**Proposition 23** The continuation value for the policy function $V'(V)$ in (FE2) is U-shaped in $V$. 
