Penn Institute for Economic Research
Department of Economics
University of Pennsylvania
3718 Locust Walk
Philadelphia, PA 19104-6297
pier@econ.upenn.edu
http://economics.sas.upenn.edu/pier

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"Mechanism Design with Costly Verification and Limited Punishments "

BY

Yunan Li

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Mechanism Design with Costly Verification and Limited Punishments*

Yunan Li†

University of Pennsylvania

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Abstract

A principal has to allocate a good among a finite number of agents, each of whom values the good. Each agent has access to private information about the principal's payoff if he receives the good. There are no monetary transfers. The principal can inspect agents' reports at a cost and punish them, but punishments are limited because verification is imperfect or information arrives only after the good has been allocated for a set period of time. I characterize an optimal mechanism featuring two thresholds. Agents whose values are below the lower threshold and above the upper threshold are pooled, respectively. If the number of agents is small, then the pooling area at the top of the value distribution disappears. If the number of agents is large, then the two pooling areas meet and the optimal mechanism can be implemented via a shortlisting procedure. The fact that the optimal mechanism depends on the number of agents implies that small and large organizations should behave differently.

Keywords: Mechanism Design, Costly Verification, Limited Punishments

JEL Classification: D82

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†Author's addresses: Yunan Li, Economics Department, University of Pennsylvania, Philadelphia, PA, USA, *yunanli@sas.upenn.edu*.

1 Introduction

In many large organizations, scarce resources must be allocated internally without the benefit of prices. Examples include the head of personnel for an organization choosing one of several applicants for a job, venture capital firms choosing which startup to fund and funding agencies allocating a grant among researchers. In these settings, the principal must rely on the verification of agents' claims, which can be costly. For example, the head of personnel can confirm a job applicant's past work experience or monitor their performance once they are hired. A venture capital firm can investigate competing startups or audit the progress of a startup once it is funded. Furthermore, the principal can punish an agent if his claim is found to be false. For example, the head of personnel can reject an applicant, fire an employee or deny a promotion. Venture capitals and funding agencies can cut off funding.

Prior work has examined two extreme cases. In Ben-Porath et al. (2014), verification is costly but punishment is unlimited in the sense that an agent can be rejected and does not receive the resource. In Mylovanov and Zapechelnyuk (2014), verification is free but punishment is limited. In this paper, I consider a situation with both costly verification *and* limited punishment. I interpret verification as acquiring information (such as by requesting documentation, interviewing an agent or monitoring an agent at work), which could be costly. Moreover, punishment can be limited because verification is imperfect or information arrives only after an agent has been hired for some time.

I think it is important to consider this general setting with both costly verification and limited punishment for two reasons. Firstly, as will become clear, this general setting helps us to identify the role that the number of agents plays in shaping optimal mechanisms. In the concluding section, I provide a more detailed comparison of the results in this paper with those in previous papers regarding the role played by the number of agents. Secondly, in practice, it is possible that the principal can obtain more precise information by incurring a higher information acquisition cost, which leads in turn to a higher expected punishment. Although I take verification cost and punishment level as exogenous throughout this paper, its results readily extend if the principal can jointly optimize over verification cost and punishment level. Thus, the results in this paper help us to understand the interactions between verification cost and punishment level.

Specifically, in the model, there is one principal who has to allocate one indivisible object among a finite number of agents. She would like to give the object to the agent who has the highest value to her, but doing this encourages all agents to exaggerate their values. At her disposal, the principal has two devices to discourage agents from exaggeration: firstly, the principal can ration at the bottom or top of the distribution of values, but this reduces allocative efficiency; secondly, the principal can verify an agent's claim and punish him if his claim is found to be false, but verification is a costly procedure. The goal of this paper is to identify the optimal way to provide incentives via these two devices.

For most parts of this paper, I consider a symmetric environment and characterize an optimal symmetric mechanism in this setting. If the number of agents is sufficiently small, then a *one-threshold mechanism* as in Ben-Porath et al. (2014) is optimal. The allocation rule in this mechanism is efficient at the top of the value distribution and involves pooling only at the bottom. For intermediate and large numbers of agents, the allocation rule involves pooling at both the top and the bottom as in Mylovanov and Zapechelnyuk (2014). Specifically, the following *two-threshold mechanism* is optimal. If there are agents whose values are above the upper threshold, then one of them is chosen at random. If all agents' values are below the upper threshold but some are above the lower threshold, then the one with the highest value is chosen. If all agents' values are below the lower threshold, then one of them is chosen at random. It should be noted that a one-threshold mechanism can be viewed as a two-threshold mechanism whose upper threshold is equal to the upper-bound of the value support. For a sufficiently large number of agents, the two thresholds coincide, and the two-threshold mechanism can be implemented using a *shortlisting procedure*. In this shortlisting procedure, agents whose values are above a threshold are shortlisted with probability one, and agents whose values are below the threshold are shortlisted with some probability. The principal then chooses one agent from the shortlist at random. The fact that the optimal mechanism depends on the number of agents implies that small and large organizations should behave differently.

To understand the intuition behind these results, consider an agent with the lowest possible value to the principal. Intuitively, as the number of agents increases, this agent becomes worse off and has stronger incentives to exaggerate his value in a one-threshold mechanism as it is now more likely that another agent whose value is above the threshold exists. When punishments are limited, the principal can make exaggeration less attractive only by introducing distortions to the allocation rule at the top of the value distribution.

This distinction between small and intermediate numbers of agents is important because it allows us to establish a connection between Ben-Porath et al. (2014) and Mylovanov and Zapechelnyuk (2014). Note that this distinction is *absent* if either verification is free or punishment is unlimited. In Ben-Porath et al. (2014), an optimal mechanism never involves pooling at the top of the value distribution because punishment is unlimited. If punishment is limited, then pooling at the top is part of the optimal mechanism for a sufficiently large number of agents. In Mylovanov and Zapechelnyuk (2014), an optimal mechanism always involves pooling at the top because verification is free. If verification is costly, then pooling at the top disappears for a sufficiently small number of agents.

As an effort to understand the trade-off between verification (or information) cost and punishment level (or information quality), I provide some comparative statics results with respect to verification cost and punishment level in Section 4. An increase in verification cost has two opposite effects on the size of the pooling areas. Firstly, when verification becomes costlier, the optimal threshold mechanism involves more pooling at the bottom to save verification cost. Secondly, the enlarging pooling area at the bottom benefits agents with very low values and reduces their incentives to exaggerate their values, which leads to less or no pooling at the top. In this paper, I show that the second effect dominates and that one-threshold or two-threshold mechanisms consequently remain optimal for a larger number of agents as verification becomes costlier. The impact of a change in punishment level is ambiguous and more interesting. On the one hand, a reduction in punishment effectively makes verification costlier as the principal must inspect agents more frequently to maintain incentive compatibility. The above analysis implies that one-threshold or two-threshold mechanisms remain optimal for a larger number of agents as punishment becomes less severe. On the other hand, a reduction in punishment level makes it more difficult to prevent agents from exaggeration through punishments, which leads to larger pooling areas at both the bottom and the top to restore incentive compatibility. This in turn implies that one-threshold or two-threshold mechanisms remain optimal for a smaller number of agents as punishment becomes less severe. In general, the impact of a change in punishment level is not monotonic.

In Section 5.1, I study a general model with asymmetric agents. In this setting, threshold mechanisms remain optimal. The analysis, however, is much more complex. Although there is still a unique lower threshold for all agents, different agents may face different upper thresholds. Using this result, I revisit the symmetric environment and characterize the set of all optimal threshold mechanisms. I find that limiting the principal's ability to punish agents also limits her ability to treat agents differently. In particular, when a one-threshold mechanism is optimal, the set of all optimal threshold mechanisms shrinks as punishment becomes more limited. Eventually, the unique optimal threshold mechanism is symmetric. If punishment is sufficiently limited so that a two-threshold mechanism or a shortlisting procedure is optimal, then the principal can once again treat agents differently, although only to the extent that they share the same set of thresholds. The comparison is less clear in this case because the sets of optimal mechanisms are disjoint for different levels of punishments.

Technically, I follow Vohra (2012) and use tools from linear programming, which allows me to analyze Ben-Porath et al. (2014) and Mylovanov and Zapechelnyuk (2014) in a unified framework. It also allows me to obtain results on optimal mechanisms in the asymmetric environment with limited punishments, which are unavailable in Mylovanov and Zapechelnyuk (2014).

The rest of the paper is organized as follows. Section 1.1 discusses other related work. Section 2 presents the model. Section 3 characterizes an optimal symmetric mechanism when agents are ex ante identical. Section 4 discusses the properties of this optimal mechanism. Section 5.1 studies a general asymmetric environment. Section 5.2 considers other variations of verification and punishment technologies. Finally, Section 6 concludes the paper.

1.1 Other related literature

This paper is related to the literature on costly state verification. The first contribution in the series is Townsend (1979), who has studied a model of a principal and a single agent. In Townsend (1979), verification is deterministic. Border and Sobel (1987) and Mookherjee and Png (1989) have generalized it by allowing random inspection. Gale and Hellwig (1985) have considered the effects of costly verification in the context of credit markets. These models differ from what I consider here in that in their models there is only one agent and monetary transfers are allowed. Recently, Patel and Urgan (2017) have also studied the problem of a principal who must allocate a good among multiple agents when transfers are not allowed. As in Ben-Porath et al. (2014), in Patel and Urgan (2017), verification is costly and punishment is unlimited. But, in addition to costly verification, the principal can deploy another instrument: money burning. They have shown that both instruments are present in the optimal mechanism. Furthermore, the optimal mechanism admits monotonicity in the allocation probability with regards to an agent's value, and takes a threshold form where all the values below a certain threshold are not subject to verification or money burning.

Technically, this paper is related to the literature on reduced form implementation — see, e.g., Maskin and Riley (1984), Matthews (1984), Border (1991) and Mierendorff (2011). The most related paper is Pai and Vohra (2014), who also use reduced form implementation and linear programming to derive optimal mechanisms for financially constrained agents.

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2 Model

The set of agents is $\mathcal{I} := \{1, \ldots, n\}$. There is a single indivisible object to be allocated among them. The value of the principal of assigning the object to agent i is v_i, where v_i is agent i's private information. I assume $\{v_i\}$ to be independently distributed and that their density f_i is strictly positive on $V_i := [\underline{v}_i, \overline{v}_i] \subset \mathbb{R}_+$. The assumption that an agent's value to the principal is always non-negative simplifies some statements, but the results in this paper can easily extend to include negative values. I use F_i to denote the corresponding cumulative distribution function. Let $\mathcal{V} := \prod_i V_i$. Agent i gets a payoff of $b_i(v_i)$ if he receives the object, and 0 otherwise. The principal can verify agent i's report at a cost $k_i \geq 0$ if agent i receives the object, and at a cost $k_i^\beta \geq 0$ if agent i does not receive the object. I allow for verification costs to depend on whether an agent gets the object. This is natural in some environments. For example, if the object is a job slot and the private information is about an agent's ability, it is easier to inspect an agent who is hired.[1] Verification perfectly reveals an agent's type. The cost to an agent to have his report verified is zero. If agent i is inspected, then the principal can impose a penalty $c_i(v_i) \geq 0$ if agent i receives the object, and a penalty $c_i^\beta(v_i) \geq 0$ if agent i does not receive the object. In Ben-Porath et al. (2014), the principal can inspect an agent at the same cost regardless of whether he receives the object or not (i.e. $k_i = k_i^\beta$). However, the principal can only penalize an agent if he receives the object (i.e. $c_i^\beta = 0$). In Mylovanov and Zapechelnyuk (2014), the principal can only inspect and penalize an agent if he receives the object (i.e. $k_i^\beta = \infty$ and $c_i^\beta = 0$). For the rest of the paper, I follow Ben-Porath et al. (2014) and Mylovanov and Zapechelnyuk (2014) and assume that $c_i^\beta = 0$. The interpretation is that the principal can only penalize an agent by taking the object back, possibly after a number of periods (e.g., rejecting a job applicant or firing him after a certain length of employment). In Section 5.2, I discuss to what extent this assumption can be relaxed.

We say that punishment is *limited* if $c_i(v_i) < b_i(v_i)$ for all v_i. This is to say that the principal cannot reduce an agent's payoff to his outside option by punishing him. If we interpret verification as acquiring information, then punishment can be limited because information is imperfect. Throughout the paper, I take verification cost and punishment level to be exogenous. In practice, it is possible that the principal can obtain more precise information by incurring a higher information acquisition cost, which in turn leads to a severer expected punishment. In other words, by choosing a higher k_i, the principal can obtain a higher c_i. The results in this paper readily extend if the principal can jointly optimize over verification cost and punishment level.

[1] I will use the words "verify" and "inspect" interchangeably in this paper.

I invoke the Revelation Principle and focus on direct mechanisms in which truth-telling is a Bayes-Nash equilibrium. Clearly, if an agent is inspected, it is optimal to penalize him if and only if he is found to have lied. Using this result, a direct mechanism can be written as a triplet $(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{q}^\beta)$, where $\boldsymbol{p} := (p_1, \dots, p_n) : \mathcal{V} \to [0,1]^n$, $\boldsymbol{q} := (q_1, \dots, q_n) : \mathcal{V} \to [0,1]^n$ and $\boldsymbol{q}^\beta := (q_1^\beta, \dots, q_n^\beta) : \mathcal{V} \to [0,1]^n$. For each i and each profile of reported values, $\boldsymbol{v} \in \mathcal{V}$, $p_i(\boldsymbol{v})$ specifies the probability with which i is assigned the object, $q_i(\boldsymbol{v})$ specifies the probability of inspecting i conditional on the object being assigned to agent i, and $q_i^\beta(\boldsymbol{v})$ specifies the probability of inspecting i conditional on the object not being assigned to agent i. The utility of an agent whose true type is v_i and who reports v_i' is $p_i(v_i, v_{-i})b_i(v_i)$ if $v_i' = v_i$, and it is

$$p_i(v_i', v_{-i}) \left(b_i(v_i) - q_i(v_i', v_{-i})c_i(v_i)\right) - (1 - p_i(v_i', v_{-i}))q_i^\beta(v_i', v_{-i})c_i^\beta(v_i)$$

otherwise. A mechanism is *feasible* if $\sum_i p_i(\boldsymbol{v}) \leq 1$ for all $\boldsymbol{v} \in \mathcal{V}$. A mechanism satisfies the *incentive compatibility* (IC) constraints if, for each agent i,

$$\begin{aligned}&\mathbb{E}_{v_{-i}} \left[p_i(v_i, v_{-i})b_i(v_i)\right] \\ &\geq \mathbb{E}_{v_{-i}} \left[p_i(v_i', v_{-i}) \left(b_i(v_i) - q_i(v_i', v_{-i})c_i(v_i)\right) - (1 - p_i(v_i', v_{-i}))q_i^\beta(v_i', v_{-i})c_i^\beta(v_i)\right], \forall v_i, v_i'.\end{aligned}$$

The principal's objective is to maximize her expected gain from allocating the object minus the expected verification cost,

$$\mathbb{E}_{\boldsymbol{v}} \left[\sum_{i=1}^{n} p_i(\boldsymbol{v}) \left(v_i - q_i(\boldsymbol{v})k_i\right) - (1 - p_i(\boldsymbol{v}))q_i^\beta(\boldsymbol{v})k_i^\beta\right], \tag{1}$$

subject to the feasibility and IC constraints.

Because $c_i^\beta = 0$, it is clearly optimal to set $q_i^\beta = 0$. In what follows, I slightly abuse notation and denote a mechanism by a pair $(\boldsymbol{p}, \boldsymbol{q})$. The principal's objective function now becomes

$$\mathbb{E}_{\boldsymbol{v}} \left[\sum_{i=1}^{n} p_i(\boldsymbol{v}) \left(v_i - q_i(\boldsymbol{v})k_i\right)\right]. \tag{2}$$

The IC constraints become the following for each agent i:

$$\mathbb{E}_{v_{-i}} \left[p_i(v_i, v_{-i})b_i(v_i)\right] \geq \mathbb{E}_{v_{-i}} \left[p_i(v_i', v_{-i}) \left(b_i(v_i) - q_i(v_i', v_{-i})c_i(v_i)\right)\right], \forall v_i, v_i'. \tag{3}$$

Note that if $k_i = 0$, then the above principal's problem reduces to that considered in Mylovanov and Zapechelnyuk (2014); and if $c_i(v_i) = b_i(v_i)$ for all v_i, then it reduces to that considered in Ben-Porath et al. (2014).

For each agent i and each $v_i \in V_i$, let $P_i(v_i) := \mathbb{E}_{v_{-i}} \left[p_i(v_i, v_{-i})\right]$ denote the interim probability with which agent i is assigned the object. If $P_i(v_i) \neq 0$, then let $Q_i(v_i) := \mathbb{E}_{v_{-i}}\left[p_i(v_i, v_{-i})q_i(v_i, v_{-i})\right]/P_i(v_i)$; and otherwise let $Q_i(v_i) := 0$. Then $P_i(v_i)Q_i(v_i)$ is the interim probability with which agent i is inspected. Let $\boldsymbol{P} := (P_1, \ldots, P_n)$ and $\boldsymbol{Q} := (Q_1, \ldots, Q_n)$. The principal's problem can then be written in the following reduced form:

$$\max_{\boldsymbol{P},\boldsymbol{Q}} \sum_{i=1}^{n} \mathbb{E}_{v_i}\left[P_i(v_i)\left(v_i - Q_i(v_i)k_i\right)\right],$$

subject to

$$P_i(v_i)b_i(v_i) \geq P_i(v_i')\left(b_i(v_i) - Q_i(v_i')c_i(v_i)\right), \forall v_i, v_i', \tag{IC}$$

$$0 \leq Q_i(v_i) \leq 1, \forall v_i, \tag{F1}$$

$$\sum_i \int_{S_i} P_i(v_i)\mathrm{d}F_i(v_i) \leq 1 - \prod_i \left(1 - \int_{S_i} dF_i(v_i)\right), \forall S_i \subset V_i. \tag{AF2}$$

In particular, an allocate rule $\boldsymbol{p}$ is feasible if and only if the corresponding reduced form allocation rule $\boldsymbol{P}$ satisfies (AF2) by Theorem 2 in Mierendorff (2011), which generalizes the well-known Maskin-Riley-Matthews-Border conditions to asymmetric environments.

I begin solving the principal's problem by solving for the optimal $\boldsymbol{Q}$ for a given $\boldsymbol{P}$. In both Mylovanov and Zapechelnyuk (2014) and Ben-Porath et al. (2014), this exercise is simple. If $k_i = 0$, then $Q(v_i) = 1$ for all $v_i \in V_i$. If $c_i(v_i) = b_i(v_i)$ for all v_i, then (IC) becomes $P_i(v_i)b_i(v_i) \geq P_i(v_i')b_i(v_i)\left(1 - Q_i(v_i')\right)$ for all v_i and v_i'. Then, (IC) holds if and only if

$$\inf_{v_i} P_i(v_i) \geq P_i(v_i')\left(1 - Q_i(v_i')\right), \forall v_i'.$$

Because the principal's objective function is strictly decreasing in Q_i, it is optimal to set $Q_i(v_i) = 1 - \varphi_i/P_i(v_i)$ for all $v_i \in V_i$, where $\varphi_i := \inf_{v_i} P_i(v_i)$. In general, for $k_i > 0$ and $c_i(v_i) \neq b_i(v_i)$, solving for the optimal Q_i is difficult.

For tractability, I assume that $c_i(v_i) = c_i b_i(v_i)$ for some $0 < c_i \leq 1$. This assumption is natural in some applications. In the job slot example, this assumption is satisfied if an agent receives a private benefit for each period he is employed and the penalty is being fired after a pre-specified number of periods. In the example of venture capital firms or funding agencies, this assumption is satisfied if agents receive funds periodically and the penalty is cutting off funding after certain periods. Furthermore, this assumption allows us to obtain a clear analysis on the interaction between the verification cost (k) and the level of punishment (c). Lastly, this assumption can be relaxed, and the results in this paper can easily extend if $c_i(v_i)/b_i(v_i)$ is minimized at $\underline{v}_i$.[2]

Under the assumption that the penalty, $c_i(v_i)$, is proportional to the private benefit, $b_i(v_i)$, (IC) becomes $P_i(v_i) \geq P_i(v_i')\left(1 - c_i Q_i(v_i')\right)$ for all v_i and v_i'. Then (IC) holds if and only if

$$\varphi_i \geq P_i(v_i')\left(1 - c_i Q_i(v_i')\right), \forall v_i'. \tag{4}$$

Because $Q_i(v_i') \leq 1$, (4) holds only if

$$(1 - c_i)P_i(v_i') \leq \varphi_i, \forall v_i'. \tag{5}$$

Remark 1 *Note that if $c_i = 1$, as in Ben-Porath et al. (2014), then (5) is satisfied automatically. This explains why there is no pooling at the top of the value distribution in Ben-Porath et al. (2014). In contrast, if $0 < c_i < 1$, then (5) imposes an upper-bound on P_i and, as I demonstrate later, there can be pooling at the top for a sufficiently large number of agents.*

For the rest of the paper, I assume that $0 < c_i < 1$. If (5) holds, then it is optimal to set $Q_i(v_i) = \left(1 - \varphi_i/P_i(v_i)\right)/c_i$ for all $v_i \in V_i$. Substituting this into the principal's objective function results in

$$\sum_{i=1}^{n} \mathbb{E}_{v_i}\left[P_i(v_i)\left(v_i - \frac{k_i}{c_i}\right)\right] + \frac{\varphi_i k_i}{c_i}. \tag{6}$$

For the main part of the paper, I assume $\{v_i\}$ to be identically distributed and that their density f is strictly positive on $V = [\underline{v}, \overline{v}] \subset \mathbb{R}_+$. I use F to denote the corresponding cumulative distribution function. In addition, I assume $c_i = c$ and $k_i = k$ for all i. In this symmetric setting, there exists an optimal mechanism that is symmetric. Hence, I focus on symmetric mechanisms in sections 3 and 4. In what follows, I suppress the subscript i whenever the meaning is clear. The main results of the paper can be extended to environments in which the values (v_i) of different agents can follow different distributions (F_i), and both the punishments (c_i) and the verification costs (k_i) can be different for different agents. I discuss this general asymmetric setting in Section 5.1.

[2](IC) can be rewritten as: for each agent i,

$$Q_i(v_i') \geq \frac{b_i(v_i)}{c_i(v_i)}\left(1 - \frac{P_i(v_i)}{P_i(v_i')}\right), \forall v_i, v_i'.$$

Suppose that $c_i(v_i)/b_i(v_i)$ is minimized at $\underline{v}_i$ and $P_i(v_i)$ is non-decreasing. Then, for any given v_i', the left-hand side of the above inequality is maximized at $\underline{v}_i$. If redefining $c_i := c_i(\underline{v}_i)/b_i(\underline{v}_i)$, then (IC) hold if and only if (4) holds.

3 Optimal mechanisms

In this section, I demonstrate that a simple threshold mechanism is optimal. As an overview of the proof idea, I solve the principal's problem in two steps. In the first step, I characterize an optimal mechanism for any given lowest probability with which an agent receives the object (φ). In the second step, I solve for the optimal φ.

3.1 Optimal mechanisms for fixed φ

Fix $\varphi = \inf_v P(v) \leq 1/n$.[3] I first solve the following problem ($OPT - \varphi$):

$$\max_P \mathbb{E}_v \left[P(v) \left(v - \frac{k}{c} \right) \right] + \frac{\varphi k}{c},$$

subject to

$$\varphi \leq P(v) \leq \frac{\varphi}{1-c}, \forall v, \tag{IC'}$$

$$n \int_S P(v) \mathrm{d}F(v) \leq 1 - \left(1 - \int_S \mathrm{d}F(v) \right)^n, \forall S \subset V. \tag{F2}$$

In this symmetric setting, when mechanisms are symmetric, (AF2) can be simplifed to (F2). Recall that Q exists such that (F1) and (IC) hold if and only if P satisfies (IC′). To solve ($OPT - \varphi$), I approximate the continuum-type space with a finite partition, characterize an optimal mechanism in the finite model and take limits. Later, I show that the limiting mechanism is optimal in the original model.

3.1.1 Finite case

Fix an integer $m \geq 2$. For $t = 1, \ldots m$, let

$$v^t := \underline{v} + \frac{(2t-1)(\overline{v} - \underline{v})}{2m},$$
$$f^t := F\left(\underline{v} + \frac{t(\overline{v} - \underline{v})}{m} \right) - F\left(\underline{v} + \frac{(t-1)(\overline{v} - \underline{v})}{m} \right).$$

Consider the finite model in which v_i can take only m possible different values (i.e. $v_i \in \{v^1, \ldots, v^m\}$) and the probability mass function satisfies $f(v^t) = f^t$ for $t = 1, \ldots, m$. I slightly abuse notation and let $P := (P^1, \ldots, P^m)$, where P^t is the interim probability with which a type v^t agent is assigned the good. Then, the corresponding problem of ($OPT - \varphi$)

[3] Note that the problem ($OPT - \varphi$) is feasible only if $\varphi \leq 1/n$.

in the finite model, denoted by $(OPTm-\varphi)$, is given by

$$\max_{P}\sum_{t=1}^{m} f^t P^t\left(v^t-\frac{k}{c}\right)+\frac{\varphi k}{c},$$

subject to

$$\varphi \le P^t \le \frac{\varphi}{1-c}, \forall t, \tag{IC$'m$}$$

$$n\sum_{t\in S} f^t P^t \le 1-\left(\sum_{t\notin S} f^t\right)^n, \forall S\subset\{1,\dots,m\}. \tag{F2m}$$

To solve $(OPTm-\varphi)$, I first rewrite it as a polymatroid optimization problem. Define $G(S):=1-\left(\sum_{t\notin S} f^t\right)^n$ and $H(S):=G(S)-n\varphi\sum_{t\in S} f^t$ for all $S\subset\{1,\dots,m\}$. Define $z^t:=f^t(P^t-\varphi)$ for all $t=1,\dots,m$ and $z:=(z^1,\dots,z^m)$. Clearly, $P^t\ge\varphi$ if and only if $z^t\ge 0$ for all $t=1,\dots,m$. Using these notations, (F2m) can be rewritten as

$$n\sum_{t\in S} z^t \le H(S), \forall S\subset\{1,\dots,m\}.$$

It is easy to verify that $H(\emptyset)=0$ and H is submodular. However, H is not monotonic. Define $\overline{H}(S):=\min_{S'\supset S} H(S)$. Then $\overline{H}(\emptyset)=0$, and $\overline{H}$ is non-decreasing and submodular. Furthermore, by Lemma 2 in Appendix A,

$$\left\{z\,\middle|\, z\ge 0, n\sum_{t\in S} z^t\le H(S),\forall S\right\}=\left\{z\,\middle|\, z\ge 0, n\sum_{t\in S} z^t\le \overline{H}(S),\forall S\right\}.$$

Thus, $(OPTm-\varphi)$ can be rewritten as $(OPTm1-\varphi)$

$$\max_{z}\sum_{t=1}^{m} z^t\left(v^t-\frac{k}{c}\right)+\varphi\sum_{t=1}^{m} f^t v^t,$$

subject to

$$0\le z^t\le \frac{c\varphi f^t}{1-c}, \forall t, \tag{IC$'m1$}$$

$$n\sum_{t\in S} z^t \le \overline{H}(S), \forall S\subset\{1,\dots,m\}. \tag{F2$m1$}$$

Without the upper-bound on z^t in (IC$'m1$), this is a standard polymatroid optimization problem and can be solved using the greedy algorithm. With the upper-bound, this is a weighted polymatroid intersection problem and efficient algorithms exist that solve the optima if the weights $(v^t - k/c)$ are rational.[4] In this paper, I solve the problem using a "guess-and-verify" approach. Although we cannot directly apply the greedy algorithm to $(OPTm1 - \varphi)$, it is not difficult to conjecture the optimal solution. Intuitively, $z^t = 0$ if $v^t < k/c$. Consider $v^t \geq k/c$. Because $\overline{H}$ is non-decreasing and submodular, and the upper-bound on z^t is linear in f^t, the solution found by the greedy algorithm potentially violates the upper-bound for large t. Hence, it is natural to conjecture that a cutoff $\bar{t}$ exists such that the upper-bounds in (IC$'m1$) bind if and only if $t > \bar{t}$.

Formally, let $S^t := \{t, \ldots, m\}$ for all $t = 1, \ldots, m$, and $S^{m+1} := \emptyset$. If $\varphi \leq (1-c)/n$, let $\bar{t} := 0$; otherwise, I show in the proof of Lemma 1 that a unique $\bar{t} \in \{1, \ldots, m+1\}$ exists such that

$$\overline{H}(S^{\bar{t}}) \leq n \sum_{\tau=\bar{t}}^{m} \frac{c\varphi f^\tau}{1-c} \text{ and } \overline{H}(S^{\bar{t}+1}) > n \sum_{\tau=\bar{t}+1}^{m} \frac{c\varphi f^\tau}{1-c}.$$

Note that, by definition, if we assign the highest possible value allowed by (F2$m1$) to $\sum_{\tau=\bar{t}+1}^{m} z^\tau$, then (IC$'m1$) must be violated for some $t \geq \bar{t}+1$. However, it is possible to assign the highest possible value allowed by (F2$m1$) to $\sum_{\tau=\bar{t}}^{m} z^\tau$ while respecting (IC$'m1$) for all $t \geq \bar{t}$. Hence, it is natural to conjecture that $\bar{t}$ defined above is the cutoff above which the upper-bounds in (IC$'m1$) bind. I can now construct my candidate optimal solution of $(OPTm1 - \varphi)$ as follows

$$\hat{z}^t := \begin{cases} \bar{z}^t \text{ if } v^t \geq \frac{k}{c} \\ 0 \text{ if } v^t < \frac{k}{c} \end{cases}, \tag{7}$$

where

$$\bar{z}^t := \begin{cases} \frac{c\varphi f^t}{1-c} & \text{if } t > \bar{t} \\ \frac{1}{n}\overline{H}(S^{\bar{t}}) - \sum_{\tau=\bar{t}+1}^{m} \frac{c\varphi f^t}{1-c} & \text{if } t = \bar{t} \\ \frac{1}{n}\left[\overline{H}(S^t) - \overline{H}(S^{t+1})\right] & \text{if } t < \bar{t} \end{cases},$$

As previously discussed, if $t > \bar{t}$ and $v^t - k/c > 0$, then I conjecture that the upper-bound in (IC$'m1$) binds and let $\hat{z}^t = c\varphi f^t/(1-c)$. If $t \leq \bar{t}$ and $v^t - k/c > 0$, then, in the spirit of greedy algorithms, I start by assigning the highest possible value allowed by (F2$m1$) to $\hat{z}^{\bar{t}}$ and continue to assign values to $\hat{z}^{\bar{t}-1}, \hat{z}^{\bar{t}-2}, \ldots$ in the same fashion. Finally, it is clear that if $v^t - k/c < 0$, then it is optimal to set $\hat{z}^t = 0$. $\hat{z}$ is feasible following from the fact that $\overline{H}(\emptyset) = 0$, and $\overline{H}$ is non-decreasing and submodular. Furthermore, we can verify the optimality of $\hat{z}$ by the duality theorem:

Lemma 1 *$\hat{z}$ defined in (7) is an optimal solution to $(OPTm1 - \varphi)$.*

[4]See, for example, Cook et al. (2011) and Frank (2011).

For each $t = 1, \ldots, m$, let

$$P_m^t := \frac{\hat{z}^t}{f^t} + \varphi \tag{8}$$

The following corollary directly follows from Lemma 1:

Corollary 1 *P_m defined in (8) is an optimal solution to $(OPTm - \varphi)$.*

3.1.2 Continuum case

I characterize an optimal solution of $(OPT - \varphi)$ by taking m to infinity. Let v^l be such that $F(v^l)^{n-1} = n\varphi$ and

$$v^u := \inf\left\{ v \,\middle|\, 1 - F(v)^n - \frac{n\varphi}{1-c}\left[1 - F(v)\right] \geq 0 \right\}. \tag{9}$$

v^l is chosen so that if all agents whose values are below v^l are pooled together and ranked below any other agents with higher values, their interim probability of receiving the object $F(v^l)^{n-1}/n$ is equal to the lower-bound in (IC′), φ. The definition of v^u mirrors that of $\bar{t}$. Informally, v^u is chosen so that if all agents whose values are above v^u are pooled together and ranked above any other agents with lower values, then their interim probability of receiving the object $[1 - F(v)^n]/n[1 - F(v)]$ is equal to the upper-bound in (IC′), $\varphi/(1 - c)$. Note that if $\varphi \leq (1 - c)/n$, then $v^u = \underline{v}$. Let $\overline{P}_\varphi$ be defined as follows: If $v^l < v^u$, let

$$\overline{P}_\varphi(v) := \begin{cases} \frac{\varphi}{1-c} & \text{if } v \geq v^u \\ F(v)^{n-1} & \text{if } v^l < v < v^u \\ \varphi & \text{if } v \leq v^l \end{cases}.$$

If $v^l \geq v^u$, let

$$\hat{v} := \inf\left\{ v \,\middle|\, 1 - n\varphi F(v) - \frac{n\varphi}{1-c}\left[1 - F(v)\right] \geq 0 \right\} \in [v^u, v^l], \tag{10}$$

and

$$\overline{P}_\varphi(v) := \begin{cases} \frac{\varphi}{1-c} & \text{if } v \geq \hat{v} \\ \varphi & \text{if } v < \hat{v} \end{cases}.$$

Finally, let

$$P_\varphi^*(v) := \begin{cases} \overline{P}(v) & \text{if } v \geq \frac{k}{c} \\ \varphi & \text{if } v < \frac{k}{c} \end{cases}. \tag{11}$$

I show in Appendix A that P_φ^* is the "pointwise limit" of P_m as $m \to \infty$. Moreover, P_φ^* is an optimal solution to $(OPT - \varphi)$.

Theorem 1 *P^*_φ defined in (11) is an optimal solution to $(OPT-\varphi)$.*

3.2 Optimal φ

I complete the characterization of an optimal mechanism by solving for the optimal φ. Firstly, if verification is sufficiently costly or the principal's ability to punish an agent is sufficiently limited, then pure randomization is optimal.

Theorem 2 *If $\overline{v} - k/c \leq \mathbb{E}_v[v]$, then pure randomization is optimal: $P^* = 1/n$ and $Q^* = 0$.*

To make the problem more interesting, in what follows I assume that

Assumption 1 $\overline{v} - k/c > \mathbb{E}_v[v]$.

Recall that given φ, v^l is uniquely pinned down by $F(v^l)^{n-1} = n\varphi$ and v^u is uniquely pinned down by (9). Define v^* and v^{**} by the equations (12) and (13), respectively:

$$\mathbb{E}_v[v] - \mathbb{E}_v[\max\{v, v^*\}] + \frac{k}{c} = 0, \tag{12}$$

$$\mathbb{E}_v[v] - \mathbb{E}_v[\min\{v, v^{**}\}] + (1-c)\left[\mathbb{E}_v[v] - \mathbb{E}_v[\max\{v, v^{**}\}] + \frac{k}{c}\right] = 0. \tag{13}$$

They are well defined under Assumption 1. Furthermore, $v^{**} > v^* \geq k/c$. Finally, let

$$v^\natural := \sup\left\{v \,\middle|\, \frac{F(v)^{n-1}(1-F(v))}{1-c} - 1 + F(v)^n \leq 0\right\}. \tag{14}$$

An optimal mechanism is characterized by the following theorem:

Theorem 3 *Supposing that Assumption 1 holds, there are three cases.*

1. *If $F(v^*)^{n-1} \geq n(1-c)$, then the optimal $\varphi^* = F(v^*)^{n-1}/n$, the optimal inspection rule satisfies $Q^* = (1 - \varphi^*/P^*)/c$ and the following allocation rule is optimal:*

$$P^*(v) := \begin{cases} F(v)^{n-1} & \text{if } v \geq v^* \\ \varphi^* & \text{if } v < v^* \end{cases}.$$

2. *If $F(v^*)^{n-1} < n(1-c)$ and $v^{**} \leq v^\natural$, then the optimal $\varphi^* = (1-c)/n(1-cF(v^{**}))$, the optimal inspection rule satisfies $Q^* = (1 - \varphi^*/P^*)/c$ and the following allocation rule is optimal:*

$$P^*(v) := \begin{cases} \frac{\varphi^*}{1-c} & \text{if } v \geq v^{**} \\ \varphi^* & \text{if } v < v^{**} \end{cases}.$$

3. *If $F(v^*)^{n-1} < n(1-c)$ and $v^{**} > v^\natural$, then the optimal φ^* is defined by*

$$\mathbb{E}_v[v] - \mathbb{E}_v[\min\{v, v^u(\varphi^*)\}] + (1-c)\left[\mathbb{E}_v[v] - \mathbb{E}_v[\max\{v, v^l(\varphi^*)\}] + \frac{k}{c}\right] = 0, \qquad (15)$$

the optimal inspection rule satisfies $Q^* = (1 - \varphi^*/P^*)/c$ and the following allocation rule is optimal:

$$P^*(v) := \begin{cases} \frac{\varphi^*}{1-c} & \text{if } v \geq v^u(\varphi^*) \\ F(v)^{n-1} & \text{if } v^l(\varphi^*) < v < v^u(\varphi^*) \\ \varphi^* & \text{if } v \leq v^l(\varphi^*) \end{cases}.$$

To understand the result, consider the following implementation of the optimal mechanism in Theorem 3. There are two thresholds. I abuse notation here and denote them by v^l and v^u with $\underline{v} \leq v^l \leq v^u \leq \overline{v}$. If every agent reports a value below v^l, then an agent is selected uniformly at random and receives the good, and no one is inspected. If any agent reports a value above v^l but all reports are below v^u, then the agent with the highest reported value receives the good, is inspected with some probability (proportional to $1/c$) and is penalized if he is found to have lied. If any agent reports a value above v^u, then an agent is selected uniformly at random among all the agents whose reported values are above v^u, receives the good, is inspected with a probability of 1 and is penalized if he is found to have lied. I call a mechanism a one-threshold mechanism if $v^u = \overline{v}$, a two-threshold mechanism if $v^l < v^u < \overline{v}$, and a shortlisting mechanism if $v^l = v^u < \overline{v}$.

To understand conditions (12), (13) and (15), consider the impact of a reduction in v^l. Intuitively, this improves allocation efficiency at the bottom of the value distribution. After some algebra, one can verify that the increase in allocation efficiency is proportional to $\mathbb{E}_v[\max\{v, v^l\}] - \mathbb{E}_v[v]$. However, as v^l decreases, agents with low v's become worse off and have stronger incentives to exaggerate their types. To restore IC, the principal must now inspect agents more frequently, which raises the total verification cost by an amount proportional to k/c. Furthermore, because the principal's ability to penalize an agent is limited, more pooling at the top (i.e. a lower v^u) may also be required to restore IC. This reduces the allocation efficiency at the top by an amount proportional to $[\mathbb{E}_v[v] - \mathbb{E}_v[\min\{v, v^u\}]]/(1-c)$. In an optimal mechanism, the marginal gain from a reduction in v^l (proportional to the left-hand side of (16)) must equal the marginal cost (proportional to the right-hand side of (16)):

$$\mathbb{E}_v[\max\{v, v^l\}] - \mathbb{E}_v[v] = \frac{\mathbb{E}_v[v] - \mathbb{E}_v[\min\{v, v^u\}]}{1-c} + \frac{k}{c}. \qquad (16)$$

This is precisely the case captured by the third part of Theorem 3 (compare (16) with (15)).

If the limited punishment constraint does not bind (i.e. $v^u = \overline{v}$), there is no efficiency loss at the top and $[\mathbb{E}_v[v] - \mathbb{E}_v[\min\{v, v^u\}]]/(1-c) = 0$. In this case, (16) becomes (12) ($v^l = v^*$) and an optimal mechanism is characterized by the first part of Theorem 3. If the principal's ability to punish an agent is sufficiently limited so that $v^u = v^l (= v^{**})$, then (16) becomes (13) and an optimal mechanism is characterized by the second part of Theorem 3.

Remark 2 *If $k = 0$, then $v^* = \underline{v}$ and $F(v^*)^{n-1} = 0 < n(1-c)$ for any $0 < c < 1$. That is, when verification is free, there is always pooling at the top (Mylovanov and Zapechelnyuk (2014)).*

4 Properties of optimal mechanisms

Theorem 3 in the previous section shows that either one-threshold mechanisms, two-threshold mechanisms or shortlisting mechanisms are optimal. In this section, I show that which of the above three kinds of mechanisms are optimal crucially depends on the number of agents (n). Specifically, I show that there exist $n^*(\rho, c)$ and $n^{**}(\rho, c)$ with $n^*(\rho, c) < n^{**}(\rho, c)$ such that if $n \leq n^*(\rho, c)$, then one-threshold mechanisms are optimal; if $n^*(\rho, c) < n < n^{**}(\rho, c)$, then two-thresholds mechanisms are optimal; and if $n \geq n^{**}(\rho, c)$, then shortlisting mechanisms are optimal. Here $\rho := k/c \geq 0$ is referred as the *effective verification cost*. The effective verification cost, ρ, is strictly decreasing in c. This is because a smaller c implies a lower level of punishment, which essentially makes verification costlier as the principal must inspect agents more frequently to maintain IC.

Formally, let $n^*(\rho, c) < 1/(1-c)$ be defined by

$$F(v^*)^{n^*(\rho,c)-1} = n^*(\rho, c)(1-c), \tag{17}$$

where v^* is defined by

$$\mathbb{E}_v[v] - \mathbb{E}_v[\max\{v, v^*\}] + \rho = 0. \tag{12}$$

Because v^* is independent of n, by Theorem 3, one-threshold mechanisms are optimal if and only if $n \leq n^*(\rho, c)$. Intuitively, for fixed v^*, an agent whose type below v^* gets the object with probability

$$\varphi^* = \frac{1}{n} F(v^*)^{n-1},$$

which is strictly decreasing in n. In particular, an agent with the lowest type becomes worse off and has stronger incentives to exaggerate his type when the number of agents, n, increases. For an n sufficiently large, IC cannot be sustained without pooling at the top of the value distribution.

Because v^* is strictly increasing in ρ, the left-hand side of (17) is strictly decreasing in n and the right-hand side of (17) is strictly increasing in n, n^* is strictly increasing in ρ. Intuitively, as the effective verification cost (ρ) increases, the principal optimally reduces the use of verification and instead enlarges the pooling area at the bottom of the value distribution (v^* increases) to maintain IC. As a result, an agent with the lowest type becomes better off (φ increases), and IC can therefore be sustained without pooling at the top for a larger number of agents. For a fixed ρ, v^* is independent of c but the right-hand side of (17) is strictly decreasing in c. Hence, n^* is strictly increasing in c. Intuitively, the upper-bound on P in (IC′) becomes larger as c increases, and IC can therefore be sustained without pooling at the top for a larger number of agents.

Next, let $n^{**}(\rho, c) < 1/(1-c)$ be defined by

$$\frac{1 - F(v^{**})^{n^{**}(\rho,c)}}{1 - F(v^{**})} = \frac{F(v^{**})^{n^{**}(\rho,c)-1}}{1-c}, \tag{18}$$

where v^{**} is given by

$$\mathbb{E}_v[v] - \mathbb{E}_v[\min\{v, v^{**}\}] + (1-c)\left[\mathbb{E}_v[v] - \mathbb{E}_v[\max\{v, v^{**}\}] + \frac{k}{c}\right] = 0. \tag{13}$$

When comparing (18) with (14), it is easy to see that $v^{**} \leq v^\natural$ if and only if $n \geq n^{**}(\rho, c)$. Per Theorem 3, shortlisting mechanisms are optimal if and only if $n \geq n^{**}(\rho, c)$. As previously discussed, an agent with the lowest type becomes worse off and has stronger incentives to exaggerate his type when the number of agents, n, increases. As a result, pooling areas at both the bottom and the top of the value distribution must be enlarged to ensure that the mechanism is incentive compatible and to save verification cost. Formally, I show in Appendix 4 that $v^l(n, \rho, c)$ is strictly increasing in n and $v^u(n, \rho, c)$ is strictly decreasing in n. Eventually, for a sufficiently large number of agents, the two pooling areas meet and there is a unique threshold such that all agents whose values are above the threshold and all agents whose values are below the threshold are pooled, respectively.

Recall that $v^{**} > v^*$. Hence,

$$\frac{F(v^{**})^{n^*(\rho,c)-1}}{1-c} > \frac{F(v^*)^{n^*(\rho,c)-1}}{1-c} = n^*(\rho, c) \geq \frac{1 - F(v^{**})^{n^*(\rho,c)}}{1 - F(v^{**})}.$$

Because the left-hand side of (18) is strictly increasing in n, and the right-hand side of (18) is strictly decreasing in n, we have $n^{**}(\rho, c) > n^*(\rho, c)$. It is easy to see that $v^{**}(\rho, c)$ is strictly increasing in both ρ and c, and independent of n. Recall that $v^\natural$ is independent of ρ. I show in Lemma 4 in Appendix A that if $n(1-c) < 1$, then $v^\natural$ is strictly increasing in n

and strictly decreasing in c. Hence, $n^{**}(\rho, c)$ is strictly increasing in both ρ and c.

Fixed ρ, an increase in c has two opposite impacts on the size of the pooling areas. On the one hand, the upper-bound on P in (IC′) becomes larger as c increases, which reduces the pooling area at the top (v^u increases) needed to sustain IC. On the other hand, it follows from the analysis in Section 3 that the marginal cost from a reduction in v^l increases as c increases.[5] Hence, it is optimal for the principal to enlarge the pooling area at the bottom (v^l increases). Formally, I show in Appendix B that both $v^l(n, \rho, c)$ and $v^u(n, \rho, c)$ are strictly increasing in c. The analysis above on n^{**} shows that the first effect dominates, and two-thresholds mechanisms are optimal for a larger number of agents as c increases.

Fixed c, an increase in ρ also has two opposite impacts on the size of the pooling areas. On the one hand, as previously discussed, as the effective verification cost (ρ) increases, the principal optimally reduces the use of verification and instead enlarges the pooling area at the bottom of the value distribution to maintain IC. On the other hand, as the pooling area at the bottom increases, an agent with the lowest type becomes better off, and IC can be sustained with less pooling at the top (v^u increases). Formally, I show in Appendix B that both $v^l(n, \rho, c)$ and $v^u(n, \rho, c)$ are strictly increasing in ρ. The analysis above on n^{**} shows that the second effect dominates, and two-thresholds mechanisms are optimal for a larger number of agents as ρ increases.

These results are summarized by the following corollary:

Corollary 2 *Suppose that Assumption 1 holds. Given $k > 0$, $c \in (0, 1)$ and $\rho = k/c$, there exists $0 < n^*(\rho, c) < n^{**}(\rho, c) < 1/(1-c)$ such that the following statements are true:*

1. *If $n \leq n^*(\rho, c)$, then one-threshold mechanisms are optimal; if $n^*(\rho, c) < n < n^{**}(\rho, c)$, then two-thresholds mechanisms are optimal; if $n \geq n^{**}(\rho, c)$, then shortlisting mechanisms are optimal.*

2. *$n^*(\rho, c)$ and $n^{**}(\rho, c)$ are strictly increasing in ρ and c.*

3. *$v^*(n, \rho, c)$ is strictly increasing in ρ, and independent of n and c. v^{**} is strictly increasing in ρ and c, and independent of n. If $n^*(\rho, c) < n < n^{**}(\rho, c)$, then $v^l(n, \rho, c)$ is strictly increasing in n, ρ and c, and $v^u(n, \rho, c)$ is strictly decreasing in n, and strictly increasing in ρ and c.*

Corollary 2 gives comparative statics results in terms of (ρ, c). It is also interesting to see the comparative statics results with respect to the model primitives (k, c). The impact of k is straightforward. As k increases, verification becomes costlier. The optimal mechanism given

[5] Fixed $\rho = k/c$, the right-hand side of (16) is strictly increasing in c.

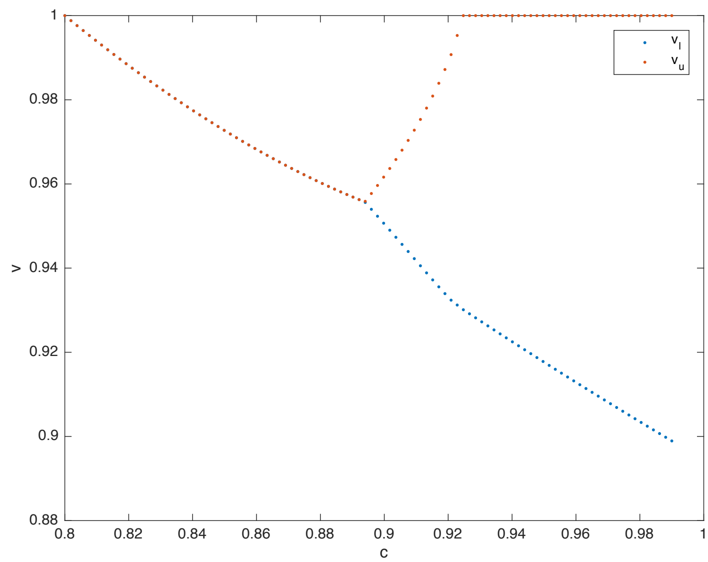

Figure 1: The impact of level of punishment (c)

in Theorem 3 sees more pooling at the bottom (measured by φ) in order to save verification cost. An increase in φ relaxes the upper-bound on P, which leads to less or no pooling at the top. The impact of c is ambiguous. On the one hand, given the amount of pooling at the bottom (measured by φ), a reduction in c lowers the upper-bound on P in (IC′), which implies more pooling at the top. On the other hand, a reduction in c makes verification costlier. Similar to the case of an increase in k, this change increases the amount of pooling at the bottom (φ increases) and relaxes the upper-bound on P. As a result, there may be less or no pooling at the top. The second channel is absent if verification is free ($k = 0$). The non-monotonicity of the pooling area at the top is further illustrated by the following numerical example.

Example 1 *Consider a numerical example in which $\{v_i\}$ are uniformly distributed on $[0, 1]$. There are $n = 8$ agents. The verification cost is $k = 0.4$. I slightly abuse notation and redefine $v^l = v^u = v^{**}$ if $v^l > v^u$. Figure 1 plots v^l and v^u as functions of c. Observe that the change of v^u is not monotonic. As c increases, the pooling area at the top first expands and then shrinks.*

Finally, a careful examination of (17) and (12) proves the following corollary:

Corollary 3 $\lim_{c\to 1} n^*(k/c, c) = \infty$ *and* $\lim_{k\to 0} n^*(k/c, c) = 0$.

Corollary 3 shows that as the principal's ability to punish an agent becomes unlimited, the model collapses to Ben-Porath et al. (2014); and as the verification cost diminishes, the model collapses to Mylovanov and Zapechelnyuk (2014).

5 Extensions

In this section, I consider two extensions. In Section 5.1, I consider the general asymmetric environment and show that a generalized threshold mechanism is optimal in this case. Using this result, I characterize the set of (possibly asymmetric) optimal mechanisms in the symmetric environment and show how limiting the principal's ability to punish agents also limits her ability to treat agents differently. The results in Section 5.1 also extend the analysis in Mylovanov and Zapechelnyuk (2014) to the asymmetric environments. In Section 5.2, I consider the case in which the principal can obtain information about and penalize an agent who does not receive the object, and show that threshold mechanism are still optimal in this environment.

5.1 Asymmetric environment

In this subsection, I consider the general model with asymmetric agents. Similar to that in Section 3, I first characterize an optimal mechanism given the lowest probabilities with which each agent receives the object ($\boldsymbol{\varphi} := (\varphi_1, \ldots, \varphi_n)$). Formally, fix $\varphi_i = \inf_{v_i} P_i(v_i)$ for all i and consider the following problem $(OPTA - \varphi)$:

$$\max_{\boldsymbol{P},\boldsymbol{Q}} \sum_{i=1}^{n} \mathbb{E}_{v_i}\left[P_i(v_i)\left(v_i - \frac{k_i}{c_i}\right)\right] + \frac{\varphi_i k_i}{c_i},$$

subject to

$$\varphi_i \leq P_i(v_i) \leq \frac{\varphi_i}{1-c_i}, \forall v_i, \tag{AIC$'$}$$

$$0 \leq Q_i(v_i) \leq 1, \forall v_i, \tag{F1}$$

$$\sum_i \int_{S_i} P_i(v_i)\mathrm{d}F_i(v_i) \leq 1 - \prod_i \left(1 - \int_{S_i} dF_i(v_i)\right), \forall S_i \subset V_i. \tag{AF2}$$

Clearly, $(OPTA - \varphi)$ is feasible only if $\sum_i \varphi_i \leq 1$. As in the symmetric case, I approximate the continuum type space with a finite partition, solve an optimal mechanism in the finite model and take limits. The following theorem gives an optimal solution to $(OPTA - \varphi)$:

Theorem 4 *There exist d^l and d_i^u for $i = 1, \ldots, n$ such that $\boldsymbol{P}^*$ defined by*

$$P_i^*(v_i) := \begin{cases} \overline{P}_i(v_i) & \text{if } v_i > \frac{k_i}{c_i} \\ \varphi_i & \text{if } v_i < \frac{k_i}{c_i}. \end{cases}, \tag{19}$$

where

$$\overline{P}_i(v_i) := \begin{cases} \frac{\varphi_i}{1-c_i} & \textit{if } v_i > d_i^u + \frac{k_i}{c_i} \\ \prod_{j \neq i, d_j^u \geq v_i - \frac{k_i}{c_i}} F_j\left(v_i - \frac{k_i}{c_i} + \frac{k_j}{c_j}\right) & \textit{if } d^l + \frac{k_i}{c_i} < v_i < d_i^u + \frac{k_i}{c_i} \\ \varphi_i & \textit{if } v_i < d^l + \frac{k_i}{c_i} \end{cases}. \tag{20}$$

is an optimal solution to $(OPTA - \varphi)$*.*

Unsurprisingly, agents are now ordered by their "net" values $v_i - k_i/c_i$, which is equal to their values to the principal minus the effective verification cost borne by the principal.[6]. As before, there is a unique lower threshold d^l such that all agents whose net values $v_i - k_i/c_i$ below the threshold are pooled. However, there can be up to n distinct upper thresholds d_i^u $(i = 1, \dots, n)$.

To illustrate how an optimal mechanism in Theorem 4 can be implemented, assume that there are two distinct upper thresholds: $d_1^u = \cdots = d_j^u > d_{j+1}^u = \dots d_n^u$. Then the first j agents whose net values are above d_1^u are pooled together, while the other $n-j$ agents whose net values are above d_{j+1}^u are pooled together and ranked below any of the first j agents whose net value is above d_{j+1}^u. Specifically, the following procedure implements the truth-telling equilibrium in a threshold mechanism: If there exists some agent i $(1 \leq i \leq j)$ whose net value $v_i - k_i/c_i$ is above d_1^u, then one such agent is selected at random, receives the good and is inspected with probability one. If $v_i - k_i/c_i < d_1^u$ for all $1 \leq i \leq j$ but $v_i - k_i/c_i \geq d_{j+1}^u$ for some $1 \leq i \leq j$, then the agent with the highest reported net value among the first j agents receives the good and is inspected with some probability. If $v_i - k_i/c_i < d_{j+1}^u$ for all $1 \leq i \leq j$ and $v_i - k_i/c_i \geq d_{j+1}^u$ for some $j+1 \leq i \leq n$, then one agent is selected at random among the last $n-j$ agents whose reported net values are above d_{j+1}^u, receives the good and is inspected with some probability. If $v_i - k_i/c_i < d_{j+1}^u$ for all i but $v_i - k_i/c_i \geq d^l$ for some i, then the agent with the highest reported net value receives the good and is inspected with some probability. If $v_i - k_i/c_i < d^l$ for all i, then one agent is selected at random and receives the good and no one is inspected. Finally, an agent is punished if and only if he is found to have lied.

Because of the complication of the pooling areas at the top, it is much harder to find an optimal solution to $(OPTA - \varphi)$. Specifically, d_i^u's are solved recursively from the largest to the smallest. Furthermore, to characterize the set of optimal $\boldsymbol{\varphi}$'s, without prior knowledge of which set of agents share the same upper threshold, one must consider 2^n different cases.[7]

[6] This is consistent with the result in Section 3 because when $k_i = k$ and $c_i = c$ for all i, ordering agents by net values produces the same result as ordering them by values.

[7] Assume, without loss of generality, that $d_1^u \geq \cdots \geq d_n^u$. If there are ν distinct upper thresholds, then there are C_n^ν possibilities to consider. In total, there are $\sum_{\nu=1}^n C_n^\nu = 2^n$ possibilities to consider.

Thus, I leave the full characterization of optimal mechanisms to future research.

Although it is generally extremely difficult to characterize the set of optimal $\boldsymbol{\varphi}$, in Appendix C.3, I characterize the set of $\boldsymbol{\varphi}$ when the upper-bounds on P_i in (AIC′) do not bind (i.e. $d_i^u = \overline{v}_i - \frac{k_i}{c_i}$ for all i). If $c_i = 1$ for all i, then these are the unique set of optimal mechanisms found in Ben-Porath et al. (2014).

5.1.1 Symmetric environment revisited

I conclude this subsection by revisiting the symmetric environment. Firstly, I argue that, in the symmetric environment, an optimal mechanism must satisfy: $d_1^u = \cdots = d_n^u$. To understand the intuition behind this result, note first that in the symmetric environment $d_i^u \geq d_j^u$ only if $\varphi_i \geq \varphi_j$. Assume, without loss of generality, that $d_1^u \geq \cdots \geq d_n^u$. Consider, for simplicity, a mechanism in which $\max_j\{\overline{v}_j - k_j/c_j\} > d_1^u > d_2^u > d_3^u$, which implies that $\varphi_1 > \varphi_2$. A new mechanism can then be constructed in which $\varphi_1^* = \varphi_2^* = \sum_{i=1}^{2} \varphi_i/2$ and $\varphi_i = \varphi_i^*$ for all $i \geq 3$. In this new mechanism, agents 1 and 2 share the same upper threshold $d^{u*} \in (d_1^u, d_2^u)$ and the upper thresholds of the other agents remain the same. If agents 1 and 2 are ex ante identical, then this new mechanism improves the principal's payoff by allocating the good between agents 1 and 2 more efficiently when their net values, $v_i - k_i/c_i$, lie between (d_1^u, d_2^u).

This property of optimal mechanisms facilitates our analysis of optimal $\boldsymbol{\varphi}$. Theorem 5 below characterizes the set of all optimal $\boldsymbol{\varphi}$. Let v^*, v^{**} and $v^\natural$ be defined by (12), (13) and (14), respectively.

Theorem 5 *Suppose that Assumption 1 holds. There are three cases.*

1. *If $F(v^*)^{n-1} \geq n(1-c)$, then the set of optimal $\boldsymbol{\varphi}$ is the convex hull of*

$$\left\{\boldsymbol{\varphi} \,\middle|\, \varphi_{i^*} = F(v^*)^{n-1} - (n-1)(1-c), \varphi_j = 1-c \;\forall j \neq i^*, i^* \in \mathcal{I}\right\}.$$

For each optimal $\boldsymbol{\varphi}^*$ and each agent i, the optimal inspection rule satisfies $Q_i^* = \left(1 - \varphi_i^*/P_i^*\right)/c_i$ and the following allocation rule is optimal:

$$P_i^*(v_i) := \begin{cases} F\left(v_i\right)^{n-1} & \text{if } v_i \geq v^* \\ \varphi_i^* & \text{if } v_i < v^* \end{cases}.$$

2. *If $F(v^*)^{n-1} < n(1-c)$ and $v^{**} \leq v^\natural$, then the set of optimal $\boldsymbol{\varphi}$ is the convex hull of*

$$\left\{\boldsymbol{\varphi} \,\middle|\, \begin{array}{l} \varphi_{i_j} = (1-c)F(v^{**})^{j-1} \text{ if } j \leq h-1, \varphi_{i_h} = \frac{1-c}{1-cF(v^{**})} - \sum_{j=1}^{h-1}(1-c)F(v^{**})^{j-1}, \\ \varphi_{i_j} = 0 \text{ if } j \geq h+1 \text{ and } (i_1, \ldots, i_n) \text{ is a permutation of } (1, \ldots, n) \end{array}\right\},$$

where $1 \leq h \leq n$ is such that

$$\frac{1-F(v^{**})^{h-1}}{1-F(v^{**})} \leq \frac{1}{1-cF(v^{**})} < \frac{1-F(v^{**})^{h}}{1-F(v^{**})}.$$

For each optimal $\boldsymbol{\varphi}^*$ and each agent i, the optimal inspection rule satisfies $Q_i^* = (1-\varphi_i^*/P_i^*)/c_i$ and the following allocation rule is optimal:

$$P_i^*(v_i) := \begin{cases} \frac{\varphi_i^*}{1-c} & \text{if } v_i \geq v^{**} \\ \varphi_i^* & \text{if } v_i < v^{**} \end{cases}.$$

3. *If $F(v^*)^{n-1} < n(1-c)$ and $v^{**} > v^{\natural}$, then the the set of optimal $\boldsymbol{\varphi}$ is the convex hull of*

$$\left\{\boldsymbol{\varphi} \,\middle|\, \varphi_{i_j} = (1-c)F(v^u(\varphi^*))^{j-1}\ \forall j \text{ and } (i_1,\dots,i_n) \text{ is a permutation of } (1,\dots,n)\right\},$$

where φ^* is defined by (15) and, for each φ, v^l is such that $F(v^l)^{n-1} = \varphi$ and v^u is defined by (9). For each optimal $\boldsymbol{\varphi}^*$ and each agent i, the optimal inspection rule satisfies $Q_i^* = (1-\varphi_i^*/P_i^*)/c_i$ and the following allocation rule is optimal:

$$P_i^*(v_i) := \begin{cases} \frac{\varphi_i^*}{1-c} & \text{if } v_i \geq v^u(\varphi^*) \\ F(v_i)^{n-1} & \text{if } v^l(\varphi^*) < v_i < v^u(\varphi^*) \\ \varphi_i^* & \text{if } v_i \leq v^l(\varphi^*) \end{cases}.$$

Theorem 5 illustrates how limiting the principal's ability to punish agents restricts her ability to treat agents differently. Supposing that $F(v^*)^{n-1} \geq n(1-c)$, the upper-bounds on P_i do not bind in an optimal mechanism. This inequality is trivially satisfied if $c = 1$ as in Ben-Porath et al. (2014). In their study, there is a class of optimal mechanisms called *favored-agent mechanisms*. In a favored-agent mechanism, there exists a favored-agent i^* whose $\varphi_{i^*} = F(v^*)^{n-1}$ while $\varphi_i = 0$ for any other agent $i \neq i^*$. However, if $c < 1$, then in an optimal mechanism it must be that $\varphi_i \geq 1-c$ for all i because otherwise some upper-bounds on P_i would be violated. Intuitively, the worse an agent is treated when he reports a low type, the stronger incentive he has to exaggerate his type. As a result, as the level of punishment declines, the extent to which the principal can favor one agent at the cost of others without violating IC also declines. Fix the ratio of $\rho = k/c$ so that v^* remains the same. The optimal set of $\boldsymbol{\varphi}$ shrinks as c becomes smaller. When c is such that $F(v^*)^{n-1} = n(1-c)$, the unique optimal $\boldsymbol{\varphi}^*$ is such that $\varphi_1^* = \cdots = \varphi_n^*$. These results are summarized in Corollary 4.

Corollary 4 *Suppose that Assumption 1 holds. Let $\Phi(\rho, c)$ denote the set of optimal $\boldsymbol{\varphi}^*$. If*

$c \geq 1 - F(v^*)^{n-1}/n$, *then* $c < c'$ *implies that* $\Phi(\rho, c) \subsetneq \Phi(\rho, c')$ *and*

$$\lim_{c \searrow 1 - F(v^*)^{n-1}/n} \Phi(\rho, c) = \left\{ \left(\frac{F(v^*)^{n-1}}{n}, \ldots, \frac{F(v^*)^{n-1}}{n} \right) \right\},$$

where v^* *is given by* (12).

If c is small enough so that $F(v^*)^{n-1} < n(1-c)$, then the comparison is less clear because the sets of optimal mechanisms are disjoint for different levels of punishment. In this case, the principal can again treat agents differently but only to the extent that they share the same upper threshold. Assume, without loss of generality, that an agent with a smaller index is more favored by the principal in terms of a larger φ_i. Then, in an optimal mechanism, the first h agents cannot be favored too much in the sense that $\sum_{i=1}^h \varphi_i \leq (1-c) \sum_{i=1}^h F(v^u)^{i-1}$ for all $h = 1, \ldots, n$.

5.2 Other verification and punishment technologies

In this subsection, I consider a variation of the model in which I allow for $k_i^\beta < \infty$ and $c_i^\beta > 0$. This means that the principal can obtain information about and penalize an agent who does not receive the object. I show that threshold mechanisms remain optimal in this environment.

For tractability, I assume in what follows that $c_i(v_i) = c_i b_i(v_i)$ and $c_i^\beta(v_i) = c_i^\beta b_i(v_i)$ for all v_i. For simplicity, I also assume that $k_i^\beta = k_i$ and $c_i^\beta = c_i$ for all i. The results in this subsection can readily extend to more general cases when, for example, it is costlier for the principal to acquire information about an agent who does not receive the object ($k_i^\beta \geq k_i$) and the punishment is also less severe for an agent who does not receive the object ($c_i^\beta \leq c_i$). Given $(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{q}^\beta)$, let $P_i(v_i) := \mathbb{E}_{v_{-i}}\left[p_i(v_i, v_{-i})\right]$ be the interim probability with which an agent receives the object and $\hat{P}_i(v_i) := \mathbb{E}_{v_{-i}}\left[p_i(v_i, v_{-i}) q_i(v_i, v_{-i}) + (1 - p_i(v_i, v_{-i})) q_i^\beta(v_i, v_{-i})\right]$ be the interim probability with which an agent is inspected.[8] The principal's problem can now be written in the following reduced form:

$$\max_{\boldsymbol{P}, \hat{\boldsymbol{P}}} \sum_{i=1}^n \mathbb{E}_{v_i}\left[P_i(v_i) v_i - \hat{P}(v_i) k_i\right],$$

[8]Because $k_i^\beta = k_i$ and $c_i^\beta = c_i$ for all i, the principal is indifferent between inspecting an agent when he receives the object and when he does not receive the object. In particularly, if we define Q_i and Q_i^β as in Section 2, then it is optimal to set $Q_i = Q_i^\beta = \hat{P}_i$.

subject to

$$P_i(v_i) \geq P_i(v_i') - \hat{P}_i(v_i')c_i, \forall v_i, v_i', \tag{IC-OT}$$

$$0 \leq \hat{P}_i(v_i) \leq 1, \forall v_i, \tag{F1-OT}$$

$$\sum_i \int_{S_i} P_i(v_i)\mathrm{d}F_i(v_i) \leq 1 - \prod_i \left(1 - \int_{S_i} dF_i(v_i)\right), \forall S_i \subset V_i. \tag{AF2}$$

Note that (IC-OT) holds if and only if

$$\varphi_i \geq P_i(v_i') - \hat{P}(v_i')c_i, \forall v_i'. \tag{21}$$

Because $\hat{P}_i(v_i') \leq 1$, (21) holds only if

$$P_i(v_i') \leq \varphi_i + c_i, \forall v_i'. \tag{22}$$

Supposing that (22) holds, it is optimal to set $\hat{P}_i(v_i) = (P_i(v_i) - \varphi_i)/c_i$ for all $v_i \in V_i$. Substituting this into the principal's objective function yields:

$$\sum_{i=1}^{n} \mathbb{E}_{v_i}\left[P_i(v_i)\left(v_i - \frac{k_i}{c_i}\right)\right] + \frac{\varphi_i k_i}{c_i}. \tag{6}$$

Note that, given $\{\varphi_i\}$, the principal's objective function is the same as that in the case of $c_i^\beta = 0$. The only difference between the principal's two problems is the upper-bound on P_i.[9]

There are two interesting observations. Firstly, the upper-bound on P_i does not bind in the original problem ($\varphi_i/(1-c_i) \geq 1$) if and only if it does not bind in the new problem ($\varphi_i + c_i \geq 1$). This implies that part 1 of Theorem 3 still applies here. Secondly, If the upper-bound binds in the original problem (i.e. $\varphi_i/(1-c_i) < 1$), then the new upper-bound is larger:

$$\varphi_i + c_i - \frac{\varphi_i}{1-c_i} = c_i\left(1 - \frac{\varphi_i}{1-c_i}\right) > 0.$$

This is intuitive because allowing for the principal to penalize an agent who does not receive the object clearly relaxes the principal's problem. Hence, any feasible solution to the new problem is also feasible in the original problem.

In the interest of the length of the paper, I only characterize an optimal mechanism in the symmetric environment. In what follows, I assume that $\{v_i\}$ are identically distributed and $c_i = c$ and $k_i = k$ for all i. Without loss of generality, I can focus on symmetric mechanisms. In what follows, I suppress the subscript i whenever the meaning is clear.

[9] Compare (22) with (5).

Firstly, as in the case of $k_i^\beta = 0$, if verification is sufficiently costly or the principal's ability to punish an agent is sufficiently limited, then pure randomization is optimal. In particular, Theorem 2 still applies here. To make the problem more interesting, in what follows, I assume that Assumption 1 holds. Let v^* be defined by (12) and redefine v^{**} by

$$\mathbb{E}_v[v] - \mathbb{E}_v[\min\{v, v^{**}\}] + \mathbb{E}_v[v] - \mathbb{E}_v[\max\{v, v^{**}\}] + \frac{k}{c} = 0. \tag{23}$$

v^* and v^{**} are well defined under Assumption 1. Furthermore, $v^{**} > v^* \geq k/c$. Finally, redefine $v^\natural$ as follows

$$v^\natural := \sup\left\{v \,\middle|\, \left(F(v)^{n-1} + nc\right)(1 - F(v)) - 1 + F(v)^n \leq 0\right\}. \tag{24}$$

Theorem 6 below characterizes an optimal symmetric mechanism. The proof is similar to that in Section 3 and neglected here.

Theorem 6 *Supposing that Assumption 1 holds, there are three cases.*

1. *If $F(v^*)^{n-1} \geq n(1-c)$, then the optimal $\varphi^* = F(v^*)^{n-1}/n$, the optimal $\hat{P}^* = (P^* - \varphi^*)/c$ and the following allocation rule is optimal:*

$$P^*(v) := \begin{cases} F(v)^{n-1} & \text{if } v \geq v^* \\ \varphi^* & \text{if } v < v^* \end{cases}.$$

2. *If $F(v^*)^{n-1} < n(1-c)$ and $v^{**} \leq v^\natural$, then the optimal $\varphi^* = (1-nc)/n(1-nc+ncF(v^{**}))$, the optimal $\hat{P}^* = (P^* - \varphi^*)/c$ and the following allocation rule is optimal:*

$$P^*(v) := \begin{cases} \varphi^* + c & \text{if } v \geq v^{**} \\ \varphi^* & \text{if } v < v^{**} \end{cases}.$$

3. *If $F(v^*)^{n-1} < n(1-c)$ and $v^{**} > v^\natural$, then the optimal φ^* is defined by*

$$\mathbb{E}_v[v] - \mathbb{E}_v[\min\{v, v^u(\varphi^*)\}] + \mathbb{E}_v[v] - \mathbb{E}_v[\max\{v, v^l(\varphi^*)\}] + \frac{k}{c} = 0, \tag{25}$$

the optimal $\hat{P}^* = (P^* - \varphi^*)/c$ and the following allocation rule is optimal:

$$P^*(v) := \begin{cases} \varphi^* + c & \text{if } v \geq v^u(\varphi^*) \\ F(v)^{n-1} & \text{if } v^l(\varphi^*) < v < v^u(\varphi^*) \\ \varphi^* & \text{if } v \leq v^l(\varphi^*) \end{cases}.$$

Note that the optimal mechanism obtained here is very similar to that obtained when the principal can only penalize an agent who receives the object. However, it has different thresholds when the limited punishment constraint is binding.

6 Concluding remarks

In this paper, I study the problem of a principal who has a single indivisible object to allocate among a number of agents. Each agent has private information about the principal's payoff of allocating the object to him. There are no monetary transfers. The principal can inspect agents' reports at a cost and punish them, but the punishments are limited. I show that some simple threshold mechanisms are optimal in this setting. This paper includes Ben-Porath et al. (2014) and Mylovanov and Zapechelnyuk (2014) as special cases and bridges their gaps. Specifically, if the number of agents is small, then the optimal mechanism only involves a pooling area at the bottom of value distribution as in Ben-Porath et al. (2014). As the number of agents increases, pooling at the top is required to guarantee incentive compatibility as in Mylovanov and Zapechelnyuk (2014). These results highlight the role played by the number of agents in shaping optimal mechanisms, which is absent or overlooked in previous work on mechanism design.

Firstly, earlier mechanism design papers studying an allocation problem have often focused on mechanisms with monetary transfers and ignored the possibility of the principal verifying agents' information. In these papers, a robust feature of optimal mechanisms is that they are independent of the number of agents. For example, in the seminal work of Myerson (1981), under some regularity conditions, the revenue-maximizing mechanism can be implemented by a first-price or second-price auction with a reserve price. In particular, this optimal reserve price is independent of the number of agents. This difference is mainly because the kinds of binding IC constraints are different in the two settings. In Myerson (1981), the binding IC constraints are between adjacent types, and the difference between two adjacent types' allocation rules is insensitive to a change in the number of agents. However, in this paper, the binding IC constraints correspond to those of the lowest possible type misreports as higher types. Note that, as the number of agents increases, the lowest possible type's probability of receiving the object declines much faster than that of a much higher type.

Secondly, in Ben-Porath et al. (2014), the optimal mechanisms are also independent of the number of agent.[10] What makes the difference? The analysis in this paper implies that,

[10]Recall that when punishment is unlimited, a one-threshold mechanism is optimal and the threshold is independent of the number of agents by the third part of Corollary 2.

given the rationing area at the bottom of the value distribution, the principal prefers to guarantee IC by verifying an agent's information and punishing him rather than by introducing rationing area to the top. If the level of punishment is limited as in this paper, then rationing at the top becomes indispensable as the number of agents increases. Furthermore, the size of the rationing area at the top required to sustain IC increases as the number of agents increases. This is why the optimal mechanisms depend on the number of agents. In contrast, if the level of punishment is high enough so that the principal can always guarantee IC by verification as in Ben-Porath et al. (2014), the optimal mechanisms are independent of the number of agents. In this sense, introducing limited punishment is important for us to understand the role played by the number of agents in shaping the optimal mechanisms.

A Omitted proofs in Sections 3

A polymatroid is a polytope of the following type

$$P(g) := \left\{ x \in \mathbb{R}^E \,\middle|\, x \geq 0, \sum_{e \in A} x_e \leq g(A) \text{ for all } A \subset E \right\}, \tag{26}$$

where E is a finite set and $g : 2^E \to \mathbb{R}_+$ is a submodular function.

Lemma 2 *There exists a monotone and submodular function $\overline{g} : 2^E \to \mathbb{R}_+$ with $\overline{g}(\emptyset) = 0$ and $P(g) = P(\overline{g})$.*

Proof. Let $\overline{g}(\emptyset) := 0$ and $\overline{g}(X) := \min_{A \supset X} g(A)$ for $X \neq \emptyset$. Let $X \subset Y \subset E$. If $X = \emptyset$, then $\overline{g}(X) = 0 \leq \overline{g}(Y)$. If $X \neq \emptyset$, then $A \supset Y$ implies that $A \supset X$, and therefore we have

$$\overline{g}(X) = \min_{A \supset X} g(A) \leq \min_{A \supset Y} g(A) = \overline{g}(Y).$$

Hence, $\overline{g}$ is monotone. Let $e \in E \backslash Y$. To show that $\overline{g}$ is submodular, it suffices to show that

$$\overline{g}(Y \cup \{e\}) - \overline{g}(Y) \leq \overline{g}(X \cup \{e\}) - \overline{g}(X).$$

Because $\overline{g}(\emptyset) = 0 \leq \min_A g(A)$, it suffices to show that

$$\min_{C \supset Y \cup \{e\}} g(C) + \min_{D \supset X} g(D) \leq \min_{A \supset X \cup \{e\}} g(A) + \min_{B \supset Y} g(B).$$

Let $A^* \in \arg\min_{A \supset X \cup \{e\}} g(A)$ and $B^* \in \arg\min_{B \supset Y} g(B)$. Then $A^* \cup B^* \supset Y \cup \{e\}$ and

$A^* \cap B^* \supset X$. Hence,

$$\begin{aligned}\min_{A \supset X \cup \{e\}} g(A) + \min_{B \supset Y} g(B) &= g(A^*) + g(B^*) \\ &\geq g(A^* \cup B^*) + g(A^* \cap B^*) \\ &\geq \min_{C \supset Y \cup \{e\}} g(C) + \min_{D \supset X} g(D),\end{aligned}$$

where the first inequality holds because g is submodular. Hence, $\overline{g}$ is submodular. Finally, I want to show that $P(g) = P(\overline{g})$. Because $g(A) \geq \overline{g}(A)$ for all $A \subset E$, we have $P(\overline{g}) \subset P(g)$. Suppose that there exists $x \in \mathbb{R}^E$ such that $x \in P(g)$ and $x \notin P(\overline{g})$. Then there exists $A \neq \emptyset$ such that $\sum_{e \in A} x_e > \overline{g}(A)$. By construction, there exists $B \supset A$ such that $\overline{g}(A) = g(B)$. However, then we have $\sum_{e \in B} x_e \geq \sum_{e \in A} x_e > \overline{g}(A) = g(B)$, which is a contradiction to that $x \in P(g)$. Hence, $P(g) = P(\overline{g})$. ■

Proof of Lemma 1. First, because $\overline{H}(\emptyset) = 0$, and $\overline{H}$ is non-decreasing and submodular, $\hat{z}^t$ is feasible. Next, I show that $\hat{z}^t$ is optimal.

I begin the analysis by characterizing $\overline{H}$. Clearly, there exists a unique $\underline{t} \in \{1, \ldots, m\}$ such that

$$\frac{1}{n}\left(\sum_{\tau=1}^{\underline{t}-1} f^\tau\right)^{n-1} < \varphi \leq \frac{1}{n}\left(\sum_{\tau=1}^{\underline{t}} f^\tau\right)^{n-1}.$$

Here, $\underline{t}$ is the minimum t such that if all agents whose values are weakly less than v^t are pooled together and ranked below any other agents with higher values, then they receive the object with probability of at least φ. It is easy to verify that[11]

$$\overline{H}(S^t) = \begin{cases} 1 - \left(\sum_{\tau=1}^{t-1} f^\tau\right)^n - n\varphi \sum_{\tau=t}^m f^\tau & \text{if } t > \underline{t} \\ 1 - n\varphi & \text{if } t \leq \underline{t} \end{cases}. \tag{27}$$

Let $\Delta(t) := \overline{H}(S^t) - n\sum_{\tau=t}^m \frac{c\varphi f^\tau}{1-c}$ for $t = 1, \ldots, m+1$. Then $\Delta(m+1) = 0$ and $\Delta(t) = \tilde{\Delta}\left(\sum_{\tau=1}^{t-1} f^\tau\right)$, where $\tilde{\Delta}(x) := 1 - \frac{n\varphi}{1-c} - x^n - \frac{n\varphi x}{1-c}$ is concave in x. If $\Delta(1) = 1 - n\varphi/(1-c) \geq 0$, then let $\overline{t} := 0$; otherwise, there exists a unique $\overline{t} \in \{1, \ldots, m+1\}$ such that

$$\overline{H}(S^{\overline{t}}) \leq n\sum_{\tau=\overline{t}}^m \frac{c\varphi f^\tau}{1-c} \text{ and } \overline{H}(S^{\overline{t}+1}) > n\sum_{\tau=\overline{t}+1}^m \frac{c\varphi f^\tau}{1-c}.$$

Let $\lambda := (\lambda^1, \ldots, \lambda^m)$ and $\mu := (\mu^1, \ldots, \mu^m)$ denote the dual variables corresponding to the upper-bounds and lower-bounds in (IC$'m$1), and $\beta := (\beta(S))_S$ denote the dual variables

[11] This result can be seen as a corollary of Lemmas 8 and 9 in Appendix C.

corresponding to (F$2m1$) in problem $(OPTm1-\varphi)$. Consider the dual to problem $(OPTm1-\varphi)$, denoted by $(DOPTm1-\varphi)$,

$$\min_{\lambda,\beta,\mu} \sum_{t=1}^{m} \frac{c\varphi f^t \lambda^t}{1-c} + \sum_{S} \beta(S)\overline{H}(S) + \varphi \sum_{t=1}^{m} f^t v^t,$$

subject to

$$v^t - \frac{k}{c} - \lambda^t + \mu^t - n\sum_{S \ni t} \beta(S) \geq 0, \forall t,$$
$$\lambda \geq 0, \beta \geq 0, \mu \geq 0.$$

Let $\hat{z}$ be define in (7), and $(\hat{\lambda}, \hat{\beta}, \hat{\mu})$ be the corresponding dual variables. Let t^0 be such that $v^t \geq k/c$ if and only if $t \geq t^0$.

Case 1: $v^{\bar{t}} < \frac{k}{c}$ or $\bar{t} < t^0$. In this case, we have

$$\hat{z}^t := \begin{cases} \frac{c\varphi f^t}{1-c} & \text{if } t > \bar{t} \\ 0 & \text{if } t \leq \bar{t} \end{cases}.$$

Let $\hat{\beta}(S) = 0$ for all S. If $v^t < k/c$, then let $\hat{\lambda}^t = 0$ and $\hat{\mu}^t = k/c - v^t > 0$; if $v^t \geq k/c$, then let $\hat{\mu}^t = 0$ and $\hat{\lambda}^t = v^t - k/c \geq 0$. It is easy to verify that this is a feasible solution to $(DOPTm1-\varphi)$, and the complementary slackness conditions are satisfied. Finally, the dual objective is equal to the primal objective:

$$\sum_{t=t^0}^{m} \frac{c\varphi f^t}{1-c}\left(v^t - \frac{k}{c}\right) + \varphi \sum_{t=1}^{m} f^t v^t.$$

By the duality theorem, $\hat{z}$ is an optimal solution to $(OPTm1-\varphi)$.

Case 2: $v^{\bar{t}} \geq \frac{k}{c}$ or $\bar{t} \geq t^0$. In this case, we have

$$\hat{z}^t := \begin{cases} \frac{c\varphi f^t}{1-c} & \text{if } t > \bar{t} \\ \frac{1}{n}\overline{H}(S^{\bar{t}}) - \sum_{\tau=\bar{t}+1}^{m} \frac{c\varphi f^t}{1-c} & \text{if } t = \bar{t} \\ \frac{1}{n}\left[\overline{H}(S^t) - \overline{H}(S^{t+1})\right] & \text{if } t^0 \leq t < \bar{t} \\ 0 \text{ if } t < t^0 \end{cases},$$

Let $\hat{\beta}(S) > 0$ if $S = S^t$ for $t^0 \leq t \leq \bar{t}$; and $\hat{\beta}(S) = 0$ otherwise. If $t < t^0$, then let $\hat{\lambda}^t = 0$ and $\hat{\mu}^t = k/c - v^t \geq 0$. If $t^0 \leq t \leq \bar{t}$, then let $\hat{\lambda}^t = \hat{\mu}^t = 0$, $\hat{\beta}(S^t) = (v^t - v^{t-1})/n$ for $t > t^0$ and $\hat{\beta}(S^{t^0}) = (v^{t^0} - k/c)/n$. If $t > \bar{t}$, then let $\hat{\lambda}^t = v^t - v^{\bar{t}}$ and $\hat{\mu}^t = 0$. It is easy to verify that this is a feasible solution to $(DOPTm1 - \varphi)$, and the complementary slackness conditions are satisfied. Finally, the dual objective is equal to the primal objective:

$$\frac{1}{n}\overline{H}(S^{t^0})\left(v^{t^0} - \frac{k}{c}\right) + \sum_{t=t^0+1}^{\bar{t}} \frac{1}{n}\overline{H}(S^t)\left(v^t - v^{t-1}\right) + \sum_{t=\bar{t}+1}^{m} \frac{c\varphi f^t}{1-c}\left(v^t - \frac{k}{c}\right) + \varphi \sum_{t=1}^{m} f^t v^t.$$

By the duality theorem, $\hat{z}$ is an optimal solution to $(OPTm1 - \varphi)$. ■

Lemma 3 *An optimal solution to $(OPT - \varphi)$ exists.*

Proof. Let $\mathscr{D}$ denote the set of feasible solutions, i.e., solutions satisfying (IC′) and (F2). Consider $\mathscr{D}$ as a subset of L_2, the set of square integrable functions with respect to the probability measure corresponding to F. Topologize L_2 with its weak*, or $\sigma(L_2, L_2)$, topology. It is straightforward to verify that $\mathscr{D}$ is $\sigma(L_2, L_2)$ compact. See, for example, Border (1991).

Let $V(\varphi) := \sup_{P \in \mathscr{D}} \mathbb{E}_v \left[P(v)\left(v - \frac{k}{c}\right)\right] + \frac{\varphi k}{c}$. Let $\{P_\nu\}$ be a sequence of feasible solutions to $(OPT - \varphi)$ such that

$$\int P_\nu(v)\left(v - \frac{k}{c}\right) \mathrm{d}F(v) + \frac{\varphi k}{c} \to V(\varphi).$$

By Helly's selection theorem, after taking subsequences, I can assume that there exists P such that $\{P_\nu\}$ converges pointwise to P. Because $\mathscr{D}$ is $\sigma(L_2, L_2)$ compact, after taking subsequences again, I can assume that there exists $P \in \mathscr{D}$ such that $\{P_\nu\}$ converges to P in $\sigma(L_2, L_2)$ topology. Because $v - k/c \in L_2$, the weak convergence of $\{P_\nu\}$ implies that

$$\int P(v)\left(v - \frac{k}{c}\right) \mathrm{d}F(v) + \frac{\varphi k}{c} = V(\varphi).$$

■

Proof of Theorem 1. Let $\{P_m\}$ be the sequence of optimal solutions to $(OPTm - \varphi)$ defined in Corollary 1. Let $\overline{P}_m^t := \overline{z}^t / f^t + \varphi$ for all t. Then

$$P_m^t := \begin{cases} \overline{P}_m^t & \text{if } v^t > \frac{k}{c} \\ \varphi & \text{if } v^t < \frac{k}{c} \end{cases}.$$

Recall that $\overline{H}$ is given by (27). Thus, there are three cases. If $\bar{t} > \underline{t}$, then

$$\overline{P}_m^t = \begin{cases} \frac{\varphi}{1-c} & \text{if } t > \bar{t} \\ \frac{\frac{1}{n} - \frac{1}{n}\left(\sum_{\tau=1}^{\bar{t}} f^\tau\right)^n - \sum_{\tau=\bar{t}+1}^{m} \frac{\varphi f^t}{1-c}}{f^{\bar{t}}} & \text{if } t = \bar{t} \\ \frac{\frac{1}{n}\left(\sum_{\tau=1}^{t} f^\tau\right)^n - \frac{1}{n}\left(\sum_{\tau=1}^{t-1} f^\tau\right)^n}{f^t} & \text{if } \underline{t} < t < \bar{t} \\ \frac{\frac{1}{n}\left(\sum_{\tau=1}^{t} f^\tau\right)^n - \varphi \sum_{\tau=1}^{t-1} f^\tau}{f^t} & \text{if } t = \underline{t} \\ \varphi & \text{if } t < \underline{t} \end{cases}.$$

If $\bar{t} = \underline{t}$, then

$$\overline{P}_m^t = \begin{cases} \frac{\varphi}{1-c} & \text{if } t > \bar{t} \\ \frac{\frac{1}{n} - \varphi \sum_{t=1}^{t-1} f^t - \sum_{\tau=t+1}^{m} \frac{\varphi f^t}{1-c}}{f^{\bar{t}}} & \text{if } t = \bar{t} \\ \varphi & \text{if } t < \bar{t} \end{cases}.$$

If $\bar{t} < \underline{t}$, then

$$\overline{P}_m^t = \begin{cases} \frac{\varphi}{1-c} & \text{if } t > \bar{t} \\ \varphi & \text{if } t < \bar{t} \end{cases}.$$

I can extend P_m to V by setting

$$P_m(v) := P_m^t \text{ for } v \in \left[\underline{v} + \frac{(t-1)(\overline{v} - \underline{v})}{m}, \underline{v} + \frac{t(\overline{v} - \underline{v})}{m}\right], t = 1, \dots, m.$$

Extend $\overline{P}_m$ to V in a similar fashion. Compare $\overline{P}_m$ and $\overline{P}_\varphi$. It is easy to see that $\{\overline{P}_m\}$ converges pointwise to $\overline{P}_\varphi$. Hence, $\{P_m\}$ converges pointwise to P_φ^*, which is a feasible solution to $(OPT - \varphi)$.

To show the optimality of P_φ^*, let $\hat{P}$ be an optimal solution to $(OPT - \varphi)$, which exists by Lemma 3 in the appendix. Define $\hat{P}_m$ be such that

$$\hat{P}_m^t := \frac{1}{f^t} \int_{\underline{v} + \frac{(t-1)(\overline{v} - \underline{v})}{m}}^{\underline{v} + \frac{t(\overline{v} - \underline{v})}{m}} \hat{P}(v) \mathrm{d}F(v) \text{ for } t = 1, \dots, m,$$

and it can be extended to V by setting

$$\hat{P}_m(v) := \hat{P}_m^t \text{ for } v \in \left[\underline{v} + \frac{(t-1)(\overline{v} - \underline{v})}{m}, \underline{v} + \frac{t(\overline{v} - \underline{v})}{m}\right], t = 1, \dots, m.$$

By the Lebesgue differentiation theorem, $\{\hat{P}_m\}$ converges pointwise to $\hat{P}$. It is easy to verify that $\hat{P}_m$ defined on $\{v^1, \ldots, v^m\}$ is a feasible solution to $(OPT - \varphi)$. Hence

$$\sum_{t=1}^{m} f^t \hat{P}_m^t \left(v^t - \frac{k}{c}\right) + \frac{\varphi k}{c} \leq \sum_{t=1}^{m} f^t P_m^t \left(v^t - \frac{k}{c}\right) + \frac{\varphi k}{c}$$

By the dominated convergence theorem,

$$\sum_{t=1}^{m} f^t \hat{P}_m^t \left(v^t - \frac{k}{c}\right) = \int_V \hat{P}_m(v) \left(v - \frac{k}{c}\right) \mathrm{d}F(v) \to \int_V \hat{P}(v) \left(v - \frac{k}{c}\right) \mathrm{d}F(v),$$

and

$$\sum_{t=1}^{m} f^t P_m^t \left(v^t - \frac{k}{c}\right) = \int_V P_m(v) \left(v - \frac{k}{c}\right) \mathrm{d}F(v) \to \int_V P_\varphi^*(v) \left(v - \frac{k}{c}\right) \mathrm{d}F(v).$$

Hence,

$$\int_V P_\varphi^*(v) \left(v - \frac{k}{c}\right) \mathrm{d}F(v) = \int_V \hat{P}(v) \left(v - \frac{k}{c}\right) \mathrm{d}F(v),$$

which implies that P_φ^* is optimal. ■

Lemma 4 *Suppose that* $(1-c)/n \leq \varphi \leq \min\{1/n, 1-c\}$. *Then* $v^l \geq v^u$ *if and only if* $v^l \leq v^\natural$, *where* $v^\natural$ *is defined by* (14). *Furthermore, if* $n(1-c) < 1$, *then* $v^\natural$ *is strictly increasing in* n *and strictly decreasing in* c.

Proof. Because $(1-c)/n \leq \varphi \leq \min\{1/n, 1-c\}$, v^l and v^u satisfies:

$$\frac{1 - F(v^u)^n}{1 - F(v^u)} = \frac{F(v^l)^{n-1}}{1-c}. \tag{28}$$

Define

$$\Delta(v) := \frac{F(v)^{n-1}(1 - F(v))}{1-c} - 1 + F(v)^n.$$

Then $\Delta(\underline{v}) = -1 < 0$ and $\Delta(\overline{v}) = 0$. Then

$$\Delta'(v) = \frac{F(v)^{n-2} f(v)}{1-c} \left[-cnF(v) + n - 1\right].$$

Clearly, the term in the brackets is strictly decreasing in v. Moreover, $\Delta'(\underline{v}) = n - 1 > 0$ and $\Delta'(\overline{v}) = n(1-c) - 1$.

If $n(1-c) \geq 1$, then $\Delta'(v) \geq 0$ for all v. Hence, $\Delta(v)$ is non-decreasing, and therefore

$\Delta(v) \leq 0$ for all v. Hence,

$$\frac{1-F(v^u)^n}{1-F(v^u)} = \frac{F(v^l)^{n-1}}{1-c} \leq \frac{1-F(v^l)^n}{1-F(v^l)},$$

which implies $v^l \geq v^u$.

If $n(1-c) < 1$, then there exists $v^\flat$ such that $\Delta'(v) > 0$ for $v \in [\underline{v}, v^\flat]$ and $\Delta'(v) < 0$ for $v \in [v^\flat, \overline{v}]$. Hence, $\Delta(v)$ in strictly increasing in $[\underline{v}, v^\flat]$, and strictly decreasing in $[v^\flat, \overline{v}]$. Hence, there exists a unique $v^\natural \in (\underline{v}, \overline{v})$ such that $\Delta(v) \leq 0$ if and only if $v \leq v^\natural$. By (28), this implies that $v^l \geq v^u$ if and only if $v^l \leq v^\natural$. Finally, for any v, $\Delta(v)$ is strictly decreasing in n and strictly increasing in c. Hence, $v^\natural$ is strictly increasing in n and strictly decreasing in c. ■

Proof of Theorem 3. First, if $\varphi \leq (1-c)/n$, then $v^u = \hat{v} = \underline{v}$, and

$$P^*_\varphi(v) := \begin{cases} \frac{\varphi}{1-c} & \text{if } v \geq \frac{k}{c} \\ \varphi & \text{if } v < \frac{k}{c} \end{cases}.$$

The principal's objective becomes

$$\frac{c\varphi}{1-c}\int_{\frac{k}{c}}^{\overline{v}}\left(v-\frac{k}{c}\right)\mathrm{d}F(v) + \varphi\int_{\underline{v}}^{\overline{v}} v\mathrm{d}F(v),$$

which is strictly increasing in φ. Hence, in optimum, $\varphi \geq (1-c)/n$.

Given φ, let $Z(\varphi)$ denote the principal's optimal payoff. Suppose that $\varphi \geq 1-c$ or equivalently $F(v^l)^{n-1} \geq n(1-c)$. Then $v^u = \overline{v}$ and the principal's payoff is $Z(\varphi) = Z_1(v^l(\varphi))$, where

$$\begin{aligned} Z_1(v^l) := &\int_{\max\left\{v^l, \frac{k}{c}\right\}}^{\overline{v}}\left(v-\frac{k}{c}\right)F(v)^{n-1}\mathrm{d}F(v) \\ &+ \frac{1}{n}F(v^l)^{n-1}\int_{\underline{v}}^{\max\left\{v^l, \frac{k}{c}\right\}}\left(v-\frac{k}{c}\right)\mathrm{d}F(v) + \frac{1}{n}F(v^l)^{n-1}\frac{k}{c}. \end{aligned}$$

If $v^l < k/c$, then $Z_1(v^l)$ is strictly increasing in v^l. If $v^l \geq k/c$, then

$$Z_1'(v^l) = \frac{n-1}{n}F(v^l)^{n-2}f(v^l)\left\{\mathbb{E}_v[v] - \mathbb{E}_v[\max\{v, v^l\}] + \frac{k}{c}\right\}.$$

Clearly, the term inside the braces is strictly decreasing in v^l. Recall that $v^* \geq k/c$ is defined by (12). Hence, $Z_1'(v^l) \geq 0$ if and only if $v^l \leq v^*$, and Z_1 achieves its maximum at $v^l = v^*$. I show in Lemma 5 below that, for any φ and the corresponding v^l, we have $Z(\varphi) \leq Z_1(v^l(\varphi))$.

Hence,

$$Z(\varphi) \leq Z_1(v^l(\varphi)) \leq Z_1(v^*).$$

Thus, if $F(v^*)^{n-1} \geq n(1-c)$, then it is optimal to set $\varphi^* = F(v^*)^{n-1}/n$ and $v^l = v^*$. This proves the first part of Theorem 3.

Suppose that $F(v^*)^{n-1} < n(1-c)$. Then in optimum $\varphi \leq 1-c$. Because $(1-c)/n \leq \varphi \leq 1/n$, there is a one-to-one correspondence between $\hat{v}$ and φ. Given φ, $\hat{v}(\varphi)$ is uniquely pinned down by

$$1 - n\varphi F(\hat{v}) - \frac{n\varphi}{1-c}\left[1 - F(\hat{v})\right] = 0.$$

If φ is such that $v^l \geq v^u$, then $Z(\varphi) = Z_2(\hat{v}(\varphi))$, where

$$\begin{aligned} Z_2(\hat{v}) :=& \frac{1-c}{n(1-cF(\hat{v}))} \int_{\underline{v}}^{\max\left\{\hat{v}, \frac{k}{c}\right\}} \left(v - \frac{k}{c}\right) \mathrm{d}F(v) \\ &+ \frac{1}{n(1-cF(\hat{v}))} \int_{\max\left\{\hat{v}, \frac{k}{c}\right\}}^{\overline{v}} \left(v - \frac{k}{c}\right) \mathrm{d}F(v) + \frac{1-c}{n(1-cF(\hat{v}))} \frac{k}{c}. \end{aligned}$$

If $\hat{v} < k/c$, then $Z_2(\hat{v})$ is strictly increasing in $\hat{v}$. If $\hat{v} \geq k/c$, then

$$Z_2'(\hat{v}) = \frac{cf(\hat{v})}{n(1-cF(\hat{v}))^2} \left\{ \mathbb{E}_v[v] - \mathbb{E}_v[\min\{v, \hat{v}\}] + (1-c)\left[\mathbb{E}_v[v] - \mathbb{E}_v[\max\{v, \hat{v}\}] + \frac{k}{c}\right] \right\}.$$

Clearly, the term inside the braces is strictly decreasing in $\hat{v}$. Recall that $v^{**} > v^* \geq k/c$ is defined by (13). Hence, $Z_2'(\hat{v}) \geq 0$ if and only if $\hat{v} \leq v^{**}$, and Z_2 achieves its maximum at $\hat{v} = v^{**}$. I show in Lemma 6 below that, for any $\varphi \leq 1-c$ and the corresponding $\hat{v}$, we have $Z(\varphi) \leq Z_2(\hat{v}(\varphi))$. Hence,

$$Z(\varphi) \leq Z_2(\hat{v}(\varphi)) \leq Z_2(v^{**}).$$

Finally, by Lemma 4, $v^l \geq v^u$ if and only if $v^l \leq v^\natural$. Thus, if $v^{**} \leq v^\natural$, then it is optimal to set $\varphi^* = F(v^{**})^{n-1}/n$ and $v^l = v^{**}$. This proves the second part of Theorem 3.

Suppose that $F(v^*)^{n-1} < n(1-c)$ and $v^{**} > v^\natural$. Then

$$\begin{aligned} Z(\varphi) =& \varphi \int_{\underline{v}}^{\max\left\{v^l, \frac{k}{c}\right\}} \left(v - \frac{k}{c}\right) \mathrm{d}F(v) + \int_{\max\left\{v^l, \frac{k}{c}\right\}}^{\max\left\{v^u, \frac{k}{c}\right\}} \left(v - \frac{k}{c}\right) F(v)^{n-1} \mathrm{d}F(v) \\ &+ \frac{\varphi}{1-c} \int_{\max\left\{v^u, \frac{k}{c}\right\}}^{\overline{v}} \left(v - \frac{k}{c}\right) \mathrm{d}F(v) + \frac{\varphi k}{c}. \end{aligned}$$

If φ is such that $v^l < k/c$, then $Z(\varphi)$ is strictly increasing in φ. If $v^l \geq k/c$, then

$$Z'(\varphi) = \frac{1}{1-c}\left[\mathbb{E}_v[v] - \mathbb{E}_v[\min\{v, v^u\}]\right] + \mathbb{E}_v[v] - \mathbb{E}_v[\max\{v, v^l\}] + \frac{k}{c}.$$

Because both v^l and v^u are strictly increasing in φ, $Z'(\varphi)$ is strictly decreasing in φ. Let φ^* be such that

$$\mathbb{E}_v[v] - \mathbb{E}_v[\min\{v, v^u(\varphi^*)\}] + (1-c)\left[\mathbb{E}_v[v] - \mathbb{E}_v[\max\{v, v^l(\varphi^*)\}] + \frac{k}{c}\right] = 0. \tag{15}$$

Compare (15) with (13) and (12). It is easy to see that $v^u(\varphi^*) > v^{**} > v^l(\varphi^*) > v^*$. Hence, $Z'(\varphi) \geq 0$ if and only if $\varphi \leq \varphi^*$, and Z achieves its maximum at $\varphi = \varphi^*$. This proves the third part of Theorem 3. ■

Lemma 5 *Let Z and Z_1 be defined as in the proof of Theorem 3. Then $Z(\varphi) \leq Z_1(v^l(\varphi))$.*

Proof. Fix φ and the corresponding v^l. Note that $Z_1(v^l)$ is attained by the following allocation rule

$$P_1(v) := \begin{cases} F(v)^{n-1} & \text{if } v \geq \max\left\{v^l, \frac{k}{c}\right\} \\ \varphi & \text{if } v < \max\left\{v^l, \frac{k}{c}\right\} \end{cases}.$$

It is easy to see that $P_1 - P^*_\varphi$ is non-decreasing and

$$\int_{\underline{v}}^{\overline{v}} P_1(v)\mathrm{d}F(v) = \int_{\underline{v}}^{\overline{v}} P^*_\varphi(v)\mathrm{d}F(v) = \frac{1}{n}.$$

Moreover, $v - k/c$ is non-decreasing in v. Hence, by Lemma 1 in Persico (2000), $Z(\varphi) \leq Z_1(v^l(\varphi))$. ■

Lemma 6 *Let Z and Z_2 be defined as in the proof of Theorem 3. If $\varphi \leq 1-c$, then $Z(\varphi) \leq Z_2(\hat{v}(\varphi))$.*

Proof. Fix φ and the corresponding $\hat{v}$. Note that $Z_2(\hat{v})$ is attained by the following allocation rule

$$P_2(v) := \begin{cases} \frac{\varphi}{1-c} & \text{if } v \geq \max\left\{\hat{v}, \frac{k}{c}\right\} \\ \varphi & \text{if } v < \max\left\{\hat{v}, \frac{k}{c}\right\} \end{cases}.$$

It is easy to see that $P_2 - P^*_\varphi$ is non-decreasing and

$$\int_{\underline{v}}^{\overline{v}} P_2(v)\mathrm{d}F(v) = \int_{\underline{v}}^{\overline{v}} P^*_\varphi(v)\mathrm{d}F(v) = \frac{1}{n},$$

Moreover, $v - k/c$ is non-decreasing in v. Hence, by Lemma 1 in Persico (2000), $Z(\varphi) \leq Z_1(v_2(\hat{\varphi}))$. ■

Proof of Theorem 2. Let Z and Z_1 be defined as in the proof of Theorem 3. If $\overline{v} - k/c \leq \mathbb{E}_v[v]$, then $Z_1(v)$ is strictly increasing in v^l and achieves its maximum when $v^l = \overline{v}$. By Lemma 5, $Z(\varphi) \leq Z_1(v^l(\varphi)) \leq Z_1(\overline{v})$. Note that $Z_1(\overline{v})$ can be achieved via pure randomization. This completes the proof. ■

B Omitted proofs in Section 4

Proof of Corollary 2. The analysis in Section 4 has proved most results of Corollary 2. What is left to prove is that if $n^*(\rho, c) < n < n^{**}(\rho, c)$, then $v^l(n, \rho, c)$ is strictly increasing in n, ρ and c and $v^u(n, \rho, c)$ is strictly decreasing in n and strictly increasing in ρ and c. If $n^*(\rho, c) < n < n^{**}(\rho, c)$, then v^l and v^u satisfy (28). By (28), v^u is strictly increasing in v^l and vice versa.

To prove the properties of v^l, let

$$\Delta_l(v^l, n, \rho, c) := \mathbb{E}_v[v] - \mathbb{E}_v[\min\{v, v^u\}] + (1-c)\left[\mathbb{E}_v[v] - \mathbb{E}_v[\max\{v, v^l\}] + \rho\right],$$

where v^u is a function of v^l, n and c defined by (28). Then $\Delta_l(v^l, n, \rho, c) \equiv 0$ by (15). Furthermore, we have

$$\begin{aligned}
\frac{\partial \Delta_l}{\partial v^l} &= -[1 - F(v^u)]\frac{\partial v^u}{\partial v^l} - (1-c)F(v^l) < 0, \\
\frac{\partial \Delta_l}{\partial n} &= -[1 - F(v^u)]\frac{\partial v^u}{\partial n} > 0, \\
\frac{\partial \Delta_l}{\partial \rho} &= 1 - c > 0, \\
\frac{\partial \Delta_l}{\partial c} &= -\left[\mathbb{E}_v[v] - \mathbb{E}_v[\max\{v, v^l\}] + \rho\right] > 0.
\end{aligned} \tag{29}$$

Hence, by the implicit function theorem, we have $\partial v^l/\partial n > 0$, $\partial v^l/\partial \rho > 0$ and $\partial v^l/\partial c > 0$. To see that $\partial v^u/\partial n < 0$ in the second line in (29), let

$$\Delta(v^u, v^l, n) := \frac{F(v^l)^{n-1}(1 - F(v^u))}{1-c} - 1 + F(v^u)^n.$$

Then $\Delta(v^u, v^l, n) \equiv 0$ by (28). Furthermore, we have

$$\frac{\partial \Delta}{\partial v^u} = \left[-\frac{F(v^l)^{n-1}}{1-c} + nF(v^u)^{n-1}\right] f(v^u) = \left[-\frac{1-F(v^u)^n}{1-F(v^u)} + nF(v^u)^{n-1}\right] f(v^u) < 0,$$
$$\frac{\partial \Delta}{\partial n} = \frac{F(v^l)^{n-1}[1-F(v^u)]\log F(v^l)}{1-c} + F(v^u)^n \log F(v^u) < 0.$$

Hence, by the implicit function theorem, $\partial v^u/\partial n = -(\partial\Delta/\partial n)/(\partial\Delta/\partial v^u) < 0$.

To prove the properties of v^u, let

$$\Delta_u(v^u, n, \rho, c) := \mathbb{E}_v[v] - \mathbb{E}_v[\min\{v, v^u\}] + (1-c)\left[\mathbb{E}_v[v] - \mathbb{E}_v[\max\{v, v^l\}] + \rho\right],$$

where v^l is a function of v^u, n and c defined by (28). Then $\Delta_u(v^u, n, \rho, c) \equiv 0$ by (28). Furthermore, we have

$$\begin{aligned}
\frac{\partial \Delta_u}{\partial v^u} &= -[1-F(v^u)] - (1-c)F(v^l)\frac{\partial v^l}{\partial v^u} < 0,\\
\frac{\partial \Delta_u}{\partial n} &= -(1-c)F(v^l)\frac{\partial v^l}{\partial n} < 0,\\
\frac{\partial \Delta_u}{\partial \rho} &= 1-c > 0,\\
\frac{\partial \Delta_u}{\partial c} &= -\left[\mathbb{E}_v[v] - \mathbb{E}_v[\max\{v, v^l\}] + \rho\right] > 0.
\end{aligned} \tag{30}$$

Hence, by the implicit function theorem, we have $\partial v^u/\partial n < 0$, $\partial v^u/\partial \rho > 0$ and $\partial v^u/\partial c > 0$. To see that $\partial v^l/\partial n > 0$ in the second line in (30), note that

$$\frac{\partial \Delta}{\partial v^l} = \frac{(n-1)F(v^l)^{n-2} f(v^l)[1-F(v^u)]}{1-c} > 0.$$

Hence, by the implicit function theorem, $\partial v^l/\partial n = -(\partial\Delta/\partial n)/(\partial\Delta/\partial v^l) > 0$. ■

C Asymmetric environment

C.1 Finite case

Let $\mathcal{D} := \cup_i [\underline{v}_i - k_i/c_i, \overline{v}_i - k_i/c_i]$. Let $\underline{d} := \inf \mathcal{D}$ and $\overline{d} := \sup \mathcal{D}$. Fix an integer $m \geq 2$. For $t = 1, \ldots m$, let

$$\begin{aligned} d^t :=& \underline{d} + \frac{(2t-1)(\overline{d} - \underline{d})}{2m}, \\ f_i^t :=& F_i\left(\underline{d} + \frac{t(\overline{d} - \underline{d})}{m} + \frac{k_i}{c_i}\right) - F_i\left(\underline{d} + \frac{(t-1)(\overline{d} - \underline{d})}{m} + \frac{k_i}{c_i}\right), i = 1, \ldots, n. \end{aligned}$$

Consider the finite model in which, for each agent i, $v_i - k_i/c_i$ can take only m possible different values (i.e. $v_i - k_i/c_i \in \{d^1, \ldots, d^m\}$) and the probability mass function satisfies $f_i(d^t) = f_i^t$ for $t = 1, \ldots, m$. It is possible that $f_i^t = 0$ for some t. The corresponding problem of $(OPTA - \varphi)$ in the finite model, denoted by $(OPTAm - \varphi)$, is given by:

$$\max_P \sum_{i=1}^n \left[\sum_{t=1}^m f_i^t P_i^t d^t + \frac{\varphi_i k_i}{c_i}\right],$$

subject to

$$\varphi_i \leq P_i^t \leq \frac{\varphi_i}{1 - c_i}, \forall t, \tag{AIC$'m$}$$

$$\sum_{i=1}^n \sum_{t \in S_i} f_i^t P_i^t \leq 1 - \prod_{i=1}^n \sum_{t \notin S_i} f_i^t, \forall S_i \subset \{1, \ldots, m\}. \tag{AF2m}$$

Define $H(\mathcal{S}) := 1 - \prod_{i=1}^n \sum_{t \notin S_i} f_i^t - \sum_{i=1}^n \sum_{t \in S_i} \varphi_i f_i^t$ for all $\mathcal{S} := (S_1, \ldots, S_n)$ and $S_i \subset \{1, \ldots, m\}$ for all i. Define $\overline{H}(\mathcal{S}) := \min_{\mathcal{S}' \supset \mathcal{S}} H(\mathcal{S}')$ for all $\mathcal{S}$. Let $z_i^t := f_i^t (P_i^t - \varphi_i)$ for all i and t. By Lemma 2, $(OPTAm - \varphi)$ can be rewritten as $(OPTAm1 - \varphi)$

$$\max_z \sum_{i=1}^n \sum_{t=1}^m z_i^t d^t + \sum_{i=1}^n \varphi_i \left(\sum_{t=1}^m f_i^t d^t + \frac{k_i}{c_i}\right),$$

subject to

$$0 \leq z_i^t \leq \frac{c_i \varphi_i f_i^t}{1 - c_i}, \forall i, \forall t, \tag{AIC$'m1$}$$

$$\sum_{i=1}^n \sum_{t \in S_i} z_i^t \leq \overline{H}(\mathcal{S}), \forall \mathcal{S} \subset \{1, \ldots, m\}^n. \tag{AF2$m1$}$$

Note that if $f_i^t = 0$, then $z_i^t = 0$ by definition and therefore satisfies (AIC$'m1$) automatically.

Algorithm 1 below describes an algorithm that finds a feasible solution to $(OPTAm-\varphi)$. I start by giving a verbal overview of the algorithm. It is in the spirit of greedy algorithms. It begins by assigning values to $\{z_i^m\}_i$ who have the largest weight d^m in the objective function. Let the set $\mathcal{I}_0^m$ collect all the agents whose highest net values are below d^m. If $i \in \mathcal{I}_0^m$, then $f_i^m = 0$ by definition and $z_i^m = 0$ by (AIC$'m1$). Next check whether there exists some agent $i \notin \mathcal{I}_0^m$ such that if z_i^m is assigned the highest value allowed by (AF2$m1$), the upper-bound on z_i^m in (AIC$'m1$) is respected. If so, assign z_i^m this highest value. Continue until no such agent can be found. Then, among all the agents whose z_i^m have not been assigned values yet, check whether there exists a pair of agents, a triple of agents and etc. until there does not exist a group of agents $\mathcal{I}'$ such that we can assign $\sum_{i\in\mathcal{I}'} z_i^m$ the highest value allowed by (AF2$m1$) while respecting the upper-bounds in (AIC$'m1$). If now there still exists an agent i whose z_i^m has not been assigned a value yet, then let $z_i^m = c_i\varphi_i f_i^t/(1-c_i)$ (the upper-bound on z_i^m in (AIC$'m1$)). Let the set $\mathcal{I}_1^m$ collect all the agents not in $\mathcal{I}_0^m$ and for whom the upper-bounds on z_i^m in (AIC$'m1$) do not bind. Continue to assign values to $\{z_i^{m-1}\}_i, \{z_i^{m-2}\}_i, \ldots \{z_i^1\}_i$ in the same fashion.

In order to define the algorithm formally, I introduce some notations. Let $S_i^t := \{t,\ldots,m\}$ and $S_i^{m+1} := \emptyset$ for all i and t, $\mathcal{S}^t := \{t,\ldots,m\}^n$ for all t and $\mathcal{S}^{m+1} := \emptyset$. Define $\mathcal{S} + (t,i) := (S_1,\ldots,S_{i-1},S_i\cup\{t\},S_{i+1}\ldots,S_n)$ and $\mathcal{S} - (t,i) := (S_1,\ldots,S_{i-1},S_i\backslash\{t\},S_{i+1}\ldots,S_n)$.

Algorithm 1 *Let* $\mathcal{I}_0^m := \{i\,|\,f_i^m = 0\}$ *and* $\overline{z}_i^m := 0$ *for all* $i \in \mathcal{I}_0^m$. *Define* $\mathcal{I}_1^m \subset \mathcal{I}\backslash\mathcal{I}_0^m$, n^m, $\{\pi^{m,1},\ldots,\pi^{m,n^m}\}$, $\{\mathcal{S}^{m,1},\ldots,\mathcal{S}^{m,n^m}\}$ *and* $\overline{z}_i^m$ *for all* $i \notin \mathcal{I}_0^m$ *recursively as follows.*

1. *Let* $\mathcal{I}_1^m = \emptyset$ *and* $\nu = 1$.

2. *If* $\mathcal{I}_1^m = \mathcal{I}\backslash\mathcal{I}_0^m$, *then go to step 5. Otherwise, let* $\iota = 1$ *and go to step 2.*

3. *If there exists* $\mathcal{I}' \neq \emptyset$ *such that* $|\mathcal{I}'| = \iota$, $\mathcal{I}' \cap (\mathcal{I}_0^m \cup \mathcal{I}_1^m) = \emptyset$ *and*

$$\overline{H}\left(\mathcal{S} + \sum_{i\in\mathcal{I}'}(m,i)\right) - \overline{H}(\mathcal{S}) \leq \sum_{i\in\mathcal{I}'} \frac{c_i\varphi_i f_i^m}{1-c_i},$$

where $S_j = S_j^m$ *if* $j \in \mathcal{I}_1^m$ *and* $S_j = S_j^{m+1}$ *otherwise, then let* $\overline{z}_i^m \leq c_i\varphi_i f_i^m/(1-c_i)$ *for* $i \in \mathcal{I}'$ *be such that*

$$\sum_{i\in\mathcal{I}'} \overline{z}_i^m = \overline{H}\left(\mathcal{S} + \sum_{i\in\mathcal{I}'}(m,i)\right) - \overline{H}(\mathcal{S}).$$

Let $\pi^{m,\nu} := \mathcal{I}'$ *and* $\mathcal{S}^{m,\nu} := \mathcal{S}$. *Redefine* ν *as* $\nu+1$ *and* $\mathcal{I}_1^m$ *as* $\mathcal{I}' \cup \mathcal{I}_1^m$, *and go to step 2. If there does not exist such an* $\mathcal{I}'$, *go to step 4.*

4. *If* $\iota < n - |\mathcal{I}_0^m \cup \mathcal{I}_1^m|$*, then redefine* ι *as* $\iota + 1$ *and go to step 3. If* $\iota = n - |\mathcal{I}_0^m \cup \mathcal{I}_1^m|$*, then go to step 5.*

5. *Let* $n^m := \nu - 1$ *and* $\overline{z}_i^m := c_i \varphi_i f_i^m / (1 - c_i)$ *for all* $i \in \mathcal{I} \backslash (\mathcal{I}_0^m \cup \mathcal{I}_1^m)$*.*

Let $1 \leq t \leq m-1$*. Suppose that we have defined* $\mathcal{I}_0^\tau$*,* $\mathcal{I}_1^\tau$*,* n^τ*,* $\{\pi^{\tau,1}, \ldots, \pi^{\tau,n^\tau}\}$*,* $\{\mathcal{S}^{\tau,1}, \ldots, \mathcal{S}^{\tau,n^\tau}\}$ *and* $\{\overline{z}_i^\tau\}_i$ *for all* $\tau \geq t+1$*. Let* $\mathcal{I}_0^t := \{i | f_i^t = 0\}$ *and* $\overline{z}_i^t := 0$ *for all* $i \in \mathcal{I}_0^t$*. Define* $\mathcal{I}_1^t \subset \mathcal{I} \backslash \mathcal{I}_0^t$*,* $\{\pi^{t,1}, \ldots, \pi^{t,n^t}\}$*,* $\{\mathcal{S}^{t,1}, \ldots, \mathcal{S}^{t,n^t}\}$ *and* $\overline{z}_i^t$ *for all* $i \notin \mathcal{I}_0^t$ *recursively as follows.*

1. *Let* $\mathcal{I}_1^t := \emptyset$ *and* $\nu = 1$*.*

2. *If* $\mathcal{I}_1^t = \mathcal{I} \backslash \mathcal{I}_0^t$*, then go to step 5. Otherwise, let* $\iota = 1$ *and go to step 2.*

3. *If there exists* $\mathcal{I}' \neq \emptyset$ *such that* $|\mathcal{I}'| = \iota$*,* $\mathcal{I}' \cap (\mathcal{I}_0^t \cup \mathcal{I}_1^t) = \emptyset$ *and*

$$\min_{\mathcal{S}} \overline{H} \left(\mathcal{S} + \sum_{i \in \mathcal{I}'} (t, i) \right) - \sum_{j=1}^{n} \sum_{\tau \in S_j} \overline{z}_j^\tau \leq \sum_{i \in \mathcal{I}'} \frac{c_i \varphi_i f_i^t}{1 - c_i},$$

where $\mathcal{S} = (S_1^{t_1}, \ldots, S_n^{t_n})$ *with* $t_j \geq t$ *if* $j \in \pi^{t,1} \cup \cdots \cup \pi^{t,\nu-1}$*,* $t_j = t+1$ *if* $j \in \mathcal{I}'$ *and* $t_j \geq t+1$ *otherwise, then let* $\overline{z}_i^t \leq c_i \varphi_i f_i^t / (1 - c_i)$ *for* $i \in \mathcal{I}'$ *be such that*

$$\sum_{i \in \mathcal{I}'} \overline{z}_i^t = \min_{\mathcal{S}} \overline{H} \left(\mathcal{S} + \sum_{i \in \mathcal{I}'} (m, i) \right) - \sum_{i=1}^{n} \sum_{\tau \in S_i} \overline{z}_i^\tau.$$

Let $\pi^{t,\nu} := \mathcal{I}'$ *and* $\mathcal{S}^{t,\nu}$ *as a minimizer of the right-hand side of the above equation such that there is no* $\mathcal{S} \supsetneq \mathcal{S}^{t,\nu}$ *which is also a minimizer. Redefine* ν *as* $\nu + 1$ *and* $\mathcal{I}_1^t$ *as* $\mathcal{I}' \cup \mathcal{I}_1^t$*, and go to step 2. If there does not exist such an* $\mathcal{I}'$*, then go to step 4.*

4. *If* $\iota < n - |\mathcal{I}_0^t \cup \mathcal{I}_1^t|$*, then redefine* ι *as* $\iota + 1$ *and go to step 3. If* $\iota = n - |\mathcal{I}_0^t \cup \mathcal{I}_1^t|$*, then go to step 5.*

5. *Let* $n^t := \nu - 1$ *and* $\overline{z}_i^t := c_i \varphi_i f_i^t / (1 - c_i)$ *for all* $i \in \mathcal{I} \backslash (\mathcal{I}_0^t \cup \mathcal{I}_1^t)$*.*

Note that $\{S^{t,\nu} + \sum_{i \in \pi^{t,\nu}} (t, i)\}$ is the collection of sets for which (AF2m1) bind.

Let $\overline{z}$ be a solution found by Algorithm 1. I first prove that $\overline{z}$ is a feasible solution to ($OPTAm1{-}\varphi$). For each i and t, let $\overline{P}_i^t := \overline{z}_i^t / f_i^t + \varphi_i$ if $f_i^t > 0$ and $\overline{P}_i^t := 0$ otherwise. Then $\overline{z}$ is a feasible solution to ($OPTAm1{-}\varphi$) if and only if $\overline{P}$ is a feasible solution to ($OPTAm{-}\varphi$). Lemma 11 below proves that $\overline{P}$ is non-decreasing. By Theorem 2 in Mierendorff (2011), $\overline{P}$

is a feasible solution to $(OPTAm-\varphi)$ if and only if for all $t_1,\ldots,t_n\in\{1,\ldots,m\}$

$$\sum_{i=1}^{n}\sum_{t\in S_i}\overline{P}_i^t\leq 1-\prod_{i=1}^{n}\sum_{t\notin S_i^{t_i}}f_i^t.$$

By construction, this is true if and only if for all $t_1,\ldots,t_n\in\{1,\ldots,m\}$,

$$\sum_{i=1}^{n}\sum_{t\in S_i}\overline{z}_i^t\leq\overline{H}(\mathcal{S}),\tag{31}$$

where $\mathcal{S}=(S_1^{t_1},\ldots,S_n^{t_n})$. Lemma 10 below proves that $\overline{z}$ satisfies (31).

Hence, $\overline{P}$ is a feasible solution to $(OPTAm-\varphi)$, or equivalently, $\overline{z}$ is a feasible solution to $(OPTAm1-\varphi)$. For each i and t, let

$$\hat{z}_i^t:=\begin{cases}\overline{z}_i^t & \text{if } d^t\geq 0\\ 0 & \text{if } d^t<0\end{cases}.\tag{32}$$

Clearly, $\hat{z}$ is also a feasible solution to $(OPTAm1-\varphi)$. Furthermore, one can verify that $\hat{z}$ is an optimal solution to $(OPTAm1-\varphi)$ by the duality theorem:

Lemma 7 *$\hat{z}$ define in (32) is an optimal solution to $(OPTAm1-\varphi)$.*

The formal proof of Lemma 7 can be found in Appendix C.1.3. Finally, let $\boldsymbol{P}^m:=(P_i^{m,t})_{i,t}$, where

$$P_i^{m,t}:=\begin{cases}\overline{P}_i^{m,t} & \text{if } d^t\geq 0\\ \varphi_i & \text{if } d^t<0\end{cases}.\tag{33}$$

The following corollary directly follows from Lemma 7:

Corollary 5 *$\boldsymbol{P}^m$ defined in (33) is an optimal solution to $(OPTAm-\varphi)$.*

The rest of this subsection is organized as follows. In Appendix C.1.1, I prove two technical lemmas on H and $\overline{H}$, which are useful for later proofs. In Appendix C.1.2, I prove that $\overline{z}$ is a feasible solution to $(OPTAm-\varphi)$ by proving Lemmas 10 and 11. In Appendix C.1.3, I prove that $\hat{z}$ is an optimal solution to $(OPTAm-\varphi)$. In Appendix C.1.4, I prove two technical lemmas that are useful in characterizing the limit of $\{P^m\}$.

C.1.1 Properties of H and $\overline{H}$

Here, I introduce two technical lemmas on H and $\overline{H}$. Lemma 8 proves a useful property of H. Lemma 9 characterizes $\overline{H}$.

Lemma 8 *If $H(\mathcal{S}) < 1 - \sum_{i=1}^n \varphi_i$ and $\mathcal{S}' \subset \mathcal{S}$, then $H(\mathcal{S}') \le H(\mathcal{S})$.*

Proof. Consider $\mathcal{S} = (S_1, \dots, S_n)$. We have

$$H(\mathcal{S}) - 1 + \sum_{i=1}^n \varphi_i = \sum_{i=1}^n \varphi_i \sum_{\tau \notin S_i} f_i^\tau - \prod_{i=1}^n \sum_{\tau \notin S_i} f_i^\tau.$$

Let $\mathcal{S}_i^{supp} := \{t | f_i^t > 0\}$. If $S_i = S_i^{supp}$ for some i, then $\sum_{\tau \notin S_i} f_i^\tau = 0$ and therefore $H(\mathcal{S}) \ge 1 - \sum_{i=1}^n \varphi_i$. Hence, $H(\mathcal{S}) < 1 - \sum_{i=1}^n \varphi_i$ implies that $S_i \ne S_i^{supp}$ or $\sum_{\tau \notin S_i} f_i^\tau > 0$ for all i. Thus, $\varphi_i \le \prod_{j \ne i} \sum_{\tau \notin S_j} f_j^\tau$ for all i. Let $\mathcal{S}' := (S_1, \dots, S_{i-1}, S_i \backslash \{t\}, S_{i+1} \dots, S_n)$. Then

$$H(\mathcal{S}) - H(\mathcal{S}') = f_i^t \left(\prod_{j \ne i} \sum_{\tau \notin S_j} f_j^\tau - \varphi_i \right) \ge 0.$$

Hence, $H(\mathcal{S}') \le H(\mathcal{S})$. By induction, $H(\mathcal{S}') \le H(\mathcal{S})$ for all $\mathcal{S}' \subset \mathcal{S}$. ■

Lemma 9 $\overline{H}(\mathcal{S}) = \min \{H(\mathcal{S}), 1 - \sum_{i=1}^n \varphi_i\}$.

Proof. Recall that $\overline{H}(\mathcal{S}) = \min_{\mathcal{S}'' \supset \mathcal{S}} H(S)$. Recall that $\mathcal{S}^1 := \{1, \dots, m\}^n$. Because $\mathcal{S}^1 \supset \mathcal{S}$ and $H(\mathcal{S}^1) = 1 - \sum_{i=1}^n \varphi_i$, we have $\overline{H}(\mathcal{S}) \le 1 - \sum_{i=1}^n \varphi_i$.

Suppose that $H(\mathcal{S}) \le 1 - \sum_{i=1}^n \varphi_i$. Let $\mathcal{S}'' \supset \mathcal{S}$. If $H(\mathcal{S}'') \ge 1 - \sum_{i=1}^n \varphi_i$, then $H(\mathcal{S}) \le 1 - \sum_{i=1}^n \varphi_i \le H(\mathcal{S}'')$. If $H(\mathcal{S}'') < 1 - \sum_{i=1}^n \varphi_i$, then $H(\mathcal{S}) \le H(\mathcal{S}'')$ by Lemma 8. Hence, $\overline{H}(\mathcal{S}) = H(\mathcal{S})$.

Suppose that $H(\mathcal{S}) > 1 - \sum_{i=1}^n \varphi_i$. I claim that $\overline{H}(\mathcal{S}) = 1 - \sum_{i=1}^n \varphi_i$. Suppose not, then there exists $\mathcal{S}'' \supset \mathcal{S}$ such that $H(\mathcal{S}'') < 1 - \sum_{i=1}^n \varphi_i$. Then, by Lemma 8, $H(\mathcal{S}) \le H(\mathcal{S}'') < 1 - \sum_{i=1}^n \varphi_i$, which is a contradiction to the fact that $H(\mathcal{S}) > 1 - \sum_{i=1}^n \varphi_i$. Hence, $\overline{H}(\mathcal{S}) = 1 - \sum_{i=1}^n \varphi_i$. ■

C.1.2 Proofs of feasibility

Lemma 10 *For all $t_1, \dots, t_n \in \{1, \dots, m\}$,*

$$\sum_{i=1}^n \sum_{t \in S_i} \overline{z}_i^t \le \overline{H}(\mathcal{S}), \tag{31}$$

where $\mathcal{S} = (S_1^{t_1}, \dots, S_n^{t_n})$.

Proof. For each t, let $\pi^{t,0} := \emptyset$ and $\pi^{t,n^t+1} := \mathcal{I} \backslash (\mathcal{I}_0^t \cup \mathcal{I}_1^t)$. Suppose that $\mathcal{S} \subset \mathcal{S}^m$, i.e.,

$t_i \geq m$ for all i. By Algorithm 1, we have

$$\sum_{i\in\pi^{m,1}} \overline{z}_i^m = \overline{H}\left(\emptyset + \sum_{i\in\pi^{m,1}} (m,i)\right),$$
$$\sum_{i\in\mathcal{I}''} \overline{z}_i^m \leq \sum_{i\in\mathcal{I}''} \frac{c_i\varphi_i f_i^m}{1-c_i} \leq \overline{H}\left(\emptyset + \sum_{i\in\mathcal{I}''}(m,i)\right), \forall\emptyset \neq \mathcal{I}'' \subsetneq \pi^{m,1},$$

where the second inequality in the second line holds because otherwise $|\pi^{m,1}| \leq |\mathcal{I}''|$ by Algorithm 1, which is a contradiction to $\mathcal{I}'' \subsetneq \pi^{m,1}$. Thus, (31) holds if $t_i = m+1$ for all $i \notin \pi^{m,1}$. Suppose that we have shown that (31) holds if $t_i = m+1$ for all $i \notin \pi^{m,1}\cup\cdots\cup\pi^{m,\nu-1}$ and $\nu \geq 2$. Suppose that $t_i = m+1$ for all $i \notin \pi^{m,1}\cup\cdots\cup\pi^{m,\nu}$. Let $\mathcal{S}' := \emptyset+\sum_{i\in\pi^{m,1}\cup\cdots\cup\pi^{m,\nu-1}}(m,i)$ and $\mathcal{I}'' := \{i \in \pi^{m,\nu}|t_i = m\} \subset \pi^{m,\nu}$. By Algorithm 1, we have

$$\sum_{i\in\mathcal{I}''} \overline{z}_i^m = \overline{H}\left(\mathcal{S}' + \sum_{i\in\mathcal{I}'}(m,i)\right) - \overline{H}(\mathcal{S}') \text{ if } \mathcal{I}'' = \pi^{m,\nu} \text{ and } \nu \leq n^m,$$
$$\sum_{i\in\mathcal{I}''} \overline{z}_i^m \leq \sum_{i\in\mathcal{I}''} \frac{c_i\varphi_i f_i^m}{1-c_i} \leq \overline{H}\left(\mathcal{S}' + \sum_{i\in\mathcal{I}''}(m,i)\right) - \overline{H}(\mathcal{S}') \text{ if } \mathcal{I}'' \subsetneq \pi^{m,\nu} \text{ or } \nu = n^m + 1.$$

Because $\mathcal{S} - \sum_{i\in\pi^{m,\nu}}(m,i) \subset \mathcal{S}'$, we have

$$\begin{aligned}\sum_{i=1}^{n} \sum_{t\in S_i'\setminus S_i} \overline{z}_i^t &= \sum_{i=1}^{n} \sum_{t\in S_i'} \overline{z}_i^t - \sum_{i\notin\pi^{m,\nu}} \sum_{t\in S_i} \overline{z}_i^t \\ &\geq \overline{H}(\mathcal{S}') - \overline{H}\left(\mathcal{S} - \sum_{i\in\pi^{m,\nu}}(m,i)\right) \\ &\geq \overline{H}\left(S' + \sum_{i\in\mathcal{I}'}(m,i)\right) - \overline{H}(\mathcal{S}),\end{aligned}$$

where the last inequality holds because $\overline{H}$ is submodular. Hence,

$$\begin{aligned}\sum_{i=1}^{n} \sum_{t\in S_i} \overline{z}_i^t &\leq \sum_{i=1}^{n} \sum_{t\in S_i'} \overline{z}_i^t - \sum_{i=1}^{n} \sum_{t\in S_i'\setminus S_i} \overline{z}_i^t + \overline{H}\left(\mathcal{S}' + \sum_{i\in\mathcal{I}'}(m,i)\right) - \overline{H}(\mathcal{S}') \\ &\leq \overline{H}(\mathcal{S}') - \overline{H}\left(\mathcal{S}' + \sum_{i\in\mathcal{I}'}(m,i)\right) + \overline{H}(\mathcal{S}) + \overline{H}\left(\mathcal{S}' + \sum_{i\in\mathcal{I}'}(m,i)\right) - \overline{H}(\mathcal{S}') \\ &= \overline{H}(\mathcal{S}).\end{aligned}$$

By induction, (31) holds for all $\mathcal{S} \subset \mathcal{S}^m$.

Suppose that $\mathcal{S} \subset \mathcal{S}^{t+1} + \sum_{i \in \pi^{t,1} \cup \cdots \cup \pi^{t,\nu}} (t,i)$ for $t \leq m-1$ and $1 \leq \nu \leq n^t + 1$. Let $\mathcal{I}' := \{i \in \pi^{t,\nu} | t_i = t\}$ and $\mathcal{S}' := \mathcal{S} - \sum_{i \in \mathcal{I}'} (t,i)$. Suppose, w.l.o.g., that $\mathcal{I}' \neq \emptyset$. If $\mathcal{I}' = \pi^{t,\nu}$, then, by Algorithm 1, we have

$$\sum_{i \in \mathcal{I}'} \overline{z}_i^\tau \leq \overline{H}\left(\mathcal{S}' + \sum_{i \in \mathcal{I}'} (t,i)\right) - \sum_{i=1}^n \sum_{\tau \in S_i'} \overline{z}_i^\tau = \overline{H}(\mathcal{S}) - \sum_{i=1}^n \sum_{\tau \in S_i'} \overline{z}_i^\tau.$$

If $\mathcal{I}' \subsetneq \pi^{t,\nu}$, then, by Algorithm 1, we have

$$\sum_{i \in \mathcal{I}'} \overline{z}_i^\tau \leq \sum_{i \in \mathcal{I}'} \frac{c_i \varphi_i f_i^t}{1-c_i} \leq \overline{H}\left(\mathcal{S}' + \sum_{i \in \mathcal{I}'} (t,i)\right) - \sum_{i=1}^n \sum_{\tau \in S_i'} \overline{z}_i^\tau = \overline{H}(\mathcal{S}) - \sum_{i=1}^n \sum_{\tau \in S_i'} \overline{z}_i^\tau,$$

where the second inequality holds because otherwise $|\pi^{t,\nu}| \leq |\mathcal{I}'|$ by Algorithm 1, which is a contradiction to $\mathcal{I}' \subsetneq \pi^{t,\nu}$. Hence, (31) holds for $\mathcal{S}$. ■

Lemma 11 *$\overline{P}_i^t$ is non-decreasing in t on $\{t \,|\, f_i^t > 0\}$.*

To prove Lemma 11, I first prove the following lemma which says that if the upper-bound in (AIC′m1) does not bind for z_i^{t+1}, then it does not bind for z_i^t.

Lemma 12 *Suppose that $f_i^t, f_i^{t+1} > 0$. Then $\overline{z}_i^{t+1} \in \mathcal{I}_1^{t+1}$ implies that $\overline{z}_i^t \in \mathcal{I}_1^t$.*

Proof. Suppose that $f_i^t, f_i^{t+1} > 0$ and $\overline{z}_i^{t+1} \in \mathcal{I}_1^{t+1}$. Then, by Algorithm 1, there exists $\mathcal{S}$ with $S_j = S_j^{t_j} \subset S_j^{t+1}$ for all $j \neq i$ and $S_i = S_i^{t+1}$ such that

$$\sum_{j=1}^n \sum_{\tau \in S_j} \overline{z}_j^\tau = \overline{H}(\mathcal{S}).$$

Suppose that $H(\mathcal{S}) < 1 - \sum_{j=1}^n \varphi_j$. Because, by Lemma 10,

$$\sum_{j \neq i} \sum_{\tau \in S_j} \overline{z}_j^\tau + \sum_{\tau \in S_i \setminus \{t+1\}} \overline{z}_i^\tau \leq \overline{H}(\mathcal{S} - (t+1,i)),$$

we have

$$\frac{c_i \varphi_i f_i^{t+1}}{1-c_i} \geq \overline{z}_i^{t+1} \geq \overline{H}(\mathcal{S}) - \overline{H}(\mathcal{S} - (t+1,i)) = f_i^{t+1}\left(\prod_{j \neq i} \sum_{\tau \in S_j} f_j^\tau - \varphi_i\right),$$

where the last equality holds by Lemmas 8 and 9. This implies that $\prod_{j \neq i} \sum_{\tau \in S_j} f_j^\tau \leq \frac{\varphi_i}{1-c_i}$.

Hence,

$$\begin{aligned}\overline{z}_i^t \leq & \overline{H}(\mathcal{S}+(t,i)) - \sum_{j=1}^{n}\sum_{\tau\in S_j}\overline{z}_j^\tau \\ \leq & H(\mathcal{S}+(t,i)) - \overline{H}(\mathcal{S}) \\ = & f_i^t\left(\prod_{j\neq i}\sum_{\tau\in S_j} f_j^\tau - \varphi_i\right) \leq \frac{c_i\varphi_i f_i^t}{1-c_i},\end{aligned}$$

where the equality holds by Lemmas 8 and 9.

Suppose that $H(\mathcal{S}) \geq 1-\sum_{j=1}^n \varphi_j$, then by Lemmas 8 and 9, $\overline{H}(\mathcal{S}) = \overline{H}(\mathcal{S}+(t,i)) = 1-\sum_{j=1}^n \varphi_j$. Hence,

$$\begin{aligned}\overline{z}_i^t \leq & \overline{H}(\mathcal{S}+(t,i)) - \sum_{j=1}^{n}\sum_{\tau\in S_j}\overline{z}_j^\tau \\ \leq & \overline{H}(\mathcal{S}+(t,i)) - \overline{H}(\mathcal{S}) \\ = & 0 \leq \frac{c_i\varphi_i f_i^t}{1-c_i}.\end{aligned}$$

Hence, $z_i^t \in \mathcal{I}_1^t$. ■

Proof of Lemma 11. Suppose that $f_i^t, f_i^{t+1} > 0$. Recall that $\overline{P}_i^t = \overline{z}_i^t/f_i^t + \varphi_i$ if $f_i^t > 0$. Suppose that $\overline{z}_i^{t+1} \notin \mathcal{I}_1^{t+1}$, then $\overline{P}_i^t \leq \frac{\varphi_i}{1-c_i} = \overline{P}_i^{t+1}$. Suppose that $\overline{z}_i^{t+1} \in \mathcal{I}_1^{t+1}$. Then there exists $\mathcal{S}$ with $S_j = S_j^{t_j} \subset S_j^{t+1}$ for all $j \neq i$ and $S_i = S_i^{t+1}$ such that

$$\sum_{j=1}^{n}\sum_{\tau\in S_j}\overline{z}_j^\tau = \overline{H}(\mathcal{S}).$$

Suppose that $H(\mathcal{S}) < 1-\sum_{j=1}^n \varphi_j$. In the proof of Lemma 12, we have shown that $\overline{z}_i^{t+1} \geq f_i^{t+1}\left(\prod_{j\neq i}\sum_{\tau\in S_j} f_j^\tau - \varphi_i\right)$ and $\overline{z}_i^t \leq f_i^t\left(\prod_{j\neq i}\sum_{\tau\in S_j} f_j^\tau - \varphi_i\right)$. Hence,

$$\overline{P}_i^t \leq \prod_{j\neq i}\sum_{\tau\in S_j} f_j^\tau \leq \overline{P}_i^{t+1}.$$

Suppose that $H(\mathcal{S}) \geq 1-\sum_{j=1}^n \varphi_j$. By the proof of Lemma 12, we have $\overline{P}_i^t = \varphi_i \leq \overline{P}_i^{t+1}$.

■

C.1.3 Proofs of optimality

Before proving Lemma 7, I first prove some useful properties of $\mathcal{S}^{t,\nu}$ and $\overline{z}$. Recall that $\{S^{t,\nu} + \sum_{i\in\pi^{t,\nu}}(t,i)\}$ is the collection of sets for which (AF2m1) bind. The result in Lemma 13 implies that this collection is a *nested sequence of sets*. In fact, Lemma 13 proves a stronger statement.

Lemma 13 *$\mathcal{S}^{t,1} \supset \mathcal{S}^{t+1,n^{t+1}} + \sum_{i\in\pi^{t+1,n^{t+1}}}(t+1,i)$ for $1 \le t \le m-1$; and $\mathcal{S}^{t,\nu+1} \supset \mathcal{S}^{t,\nu} + \sum_{i\in\pi^{t,\nu}}(t,i)$ for $1 \le t \le m$.*

Proof. By Algorithm 1, $\mathcal{S}^{m,\nu+1} \supset \mathcal{S}^{m,\nu} + \sum_{i\in\pi^{m,\nu}}(m,i)$. Let $t \le m-1$ and $\mathcal{I}' = \pi^{t,1}$. Let $\mathcal{S} := \mathcal{S}^{t+1,n^{t+1}} + \sum_{i\in\pi^{t+1,n^{t+1}}}(t+1,i)$. Then $\sum_{j=1}^{n}\sum_{\tau\in S_j}\overline{z}_j^\tau = \overline{H}(\mathcal{S})$. Suppose $\mathcal{S}^{t,1} \not\supset \mathcal{S}$. Let $\mathcal{S}' := \mathcal{S} \cup \mathcal{S}^{t,1}$. Then $S'_j = S_j^{t_j}$ for some $t_j \ge t+1$ for all j. By Lemma 10, we have

$$\begin{aligned}
&\sum_{j=1}^{n}\sum_{\tau\in S'_j\setminus S_j^{t,1}}\overline{z}_j^\tau\\
=&\sum_{j=1}^{n}\sum_{\tau\in S_j}\overline{z}_j^\tau - \sum_{j=1}^{n}\sum_{\tau\in S_j^{t,1}\cap S_j}\overline{z}_j^\tau\\
\ge&\overline{H}(\mathcal{S}) - \overline{H}(\mathcal{S}\cap\mathcal{S}^{t,1}).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\overline{H}\left(\mathcal{S}' + \sum_{i\in\mathcal{I}'}(t,i)\right) - \sum_{j=1}^{n}\sum_{\tau\in S'_j}\overline{z}_j^\tau - \overline{H}\left(\mathcal{S}^{t,1} + \sum_{i\in\mathcal{I}'}(t,i)\right) + \sum_{j=1}^{n}\sum_{\tau\in S_j^{t,1}}\overline{z}_j^\tau\\
=&\overline{H}\left(\mathcal{S}' + \sum_{i\in\mathcal{I}'}(t,i)\right) - \overline{H}\left(\mathcal{S}^{t,1} + \sum_{i\in\mathcal{I}'}(t,i)\right) - \sum_{j=1}^{n}\sum_{\tau\in S'_j\setminus S_j^{t,1}}\overline{z}_j^\tau\\
\le&\left[\overline{H}\left(\mathcal{S}' + \sum_{i\in\mathcal{I}'}(t,i)\right) - \overline{H}(\mathcal{S})\right] - \left[\overline{H}\left(\mathcal{S}^{t,1} + \sum_{i\in\mathcal{I}'}(t,i)\right) - \overline{H}(\mathcal{S}\cap\mathcal{S}^{t,1})\right]\\
\le&0,
\end{aligned}$$

where the last inequality holds because $\overline{H}$ is submodular, which is a contradiction to the definition of $\mathcal{S}^{t,\nu}$. Hence, $\mathcal{S}^{t,1} \supset \mathcal{S}^{t+1,n^{t+1}} + \sum_{i\in\pi^{t+1,n^{t+1}}}(t+1,i)$. By a similar argument, one can show that $\mathcal{S}^{t,\nu+1} \supset \mathcal{S}^{t,\nu} + \sum_{i\in\pi^{t,\nu}}(t,i)$ for all $t \le m-1$. ■

By Lemmas 8, 9 and 13, there exists $\underline{t}$ and $\overline{\nu}$ such that

$$\overline{H}\left(\mathcal{S}^{t,\nu}+\sum_{i\in\pi^{t,\nu}}(t,i)\right)=\begin{cases}1-\sum_i\varphi_i & \text{if } t<\underline{t} \text{ or } \nu\geq\overline{\nu}, t=\underline{t}\\ H\left(\mathcal{S}^{t,\nu}+\sum_{i\in\pi^{t,\nu}}(t,i)\right)<1-\sum_i\varphi_i & \text{otherwise}\end{cases}. \tag{34}$$

The definition of $\underline{t}$ is analogous to that in the symmetric case. By a similar argument to that in Lemma 12, we have

Lemma 14 *If $t<\underline{t}$, or $t=\underline{t}$ and $i\notin\pi^{t,1}\cup\cdots\cup\pi^{t,\overline{\nu}}$, then $\overline{z}_i^t=0$.*

Proof of Lemma 7. Consider the dual to problem $(OPTAm1-\varphi)$, which is denoted by $(DOPTAm1-\varphi)$,

$$\min_{\lambda,\beta,\mu}\sum_{i=1}^{n}\sum_{t=1}^{m}\frac{\lambda_i^t c_i\varphi_i f_i^t}{1-c_i}+\sum_{\mathcal{S}}\beta(\mathcal{S})\overline{H}(\mathcal{S})+\sum_{i=1}^{n}\varphi_i\left(\sum_{t=1}^{m}f_i^t d^t+\frac{k_i}{c_i}\right),$$

subject to

$$d^t-\lambda_i^t+\mu_i^t-\sum_{S_i\ni t}\beta(\mathcal{S})\geq 0 \text{ if } f_i^t>0, \forall i, \forall t,$$

$$\lambda\geq 0, \mu\geq 0, \beta\geq 0.$$

Let $\hat{z}$ be defined by (32) and $(\hat{\beta},\hat{\lambda},\hat{\mu})$ be the corresponding dual variables. Let t^0 be such that $d^{t^0}\geq 0$ if and only if $t\geq t^0$.

Let $\hat{\beta}\left(\mathcal{S}^{t,n^t}+\sum_{i\in\pi^{t,n^t}}(t,i)\right)\geq 0$ for $t\geq\max\{t^0,\underline{t}\}$ and $\hat{\beta}(\mathcal{S})=0$ otherwise. (i) If $t<\max\{t^0,\underline{t}\}$, then let $\hat{\lambda}_i^t=0$ and $\hat{\mu}_i^t=-d^t\geq 0$. (ii) If $t=\max\{t^0,\underline{t}\}$, then let $\hat{\beta}\left(\mathcal{S}^{t,n^t}+\sum_{j\in\pi^{t,n^t}}(t,j)\right)=d^t\geq 0$ and $\hat{\mu}_i^t=0$. If $i\in\mathcal{I}_1^t$, then let $\hat{\lambda}_i^t=0$. If $i\notin\mathcal{I}_0^t\cup\mathcal{I}_1^t$, let $\hat{\lambda}_i^t=d^t\geq 0$. (iii) If $t>\max\{t^0,\underline{t}\}$, let $\hat{\beta}\left(\mathcal{S}^{t,n^t}+\sum_{j\in\pi^{t,n^t}}(t,j)\right)=d^t-d^{t-1}\geq 0$ and $\hat{\mu}_i^t=0$. If $i\in\mathcal{I}_1^t$, then let $\hat{\lambda}_i^t=0$. If $i\notin\mathcal{I}_0^t\cup\mathcal{I}_1^t$ and $i\in\mathcal{I}_1^{\max\{t^0,\underline{t}\}}$, then let $\hat{\lambda}_i^t=d^t-d^{t^*}\geq 0$ where $t^*=\min\{t'\geq\max\{t^0,\underline{t}\}|\mathcal{I}_1^{t'}\ni i\}$. If $i\notin\mathcal{I}_1^{\max\{t^0,\underline{t}\}}$, then let $\hat{\lambda}_i^t=0$. Hence, $(\hat{\lambda},\hat{\mu},\hat{\beta})$ is a feasible solution to $(DOPTAm1-\varphi)$ and the complementary slackness conditions are satisfied. Finally, it is easy to verify that the dual objective is equal to the primal objective. By the duality theorem, $\hat{z}$ is an optimal solution to $(OPTAm1-\varphi)$. ■

C.1.4 Properties of $S^{t,\nu}$

Before moving on to the continuum case, I prove the following two lemmas which are useful in characterizing the limit of $\{P^m\}$.

Lemma 15 *Suppose* $\mathcal{S}^{t,\nu} = (S_1^{t_1^*}, \ldots, S_n^{t_n^*})$*. Then* $t_i^* = t$ *if* $i \in \pi^{t,1} \cup \cdots \cup \pi^{t,\nu-1}$*,* $t_i^* = t+1$ *if* $i \in \mathcal{I}_1^{t+1} \cup \pi^{t,\nu} \backslash (\pi^{t,1} \cup \cdots \cup \pi^{t,\nu-1})$*, and* $t_i^* \in \{t+1, m+1\}$ *otherwise. Furthermore, for* $h \notin \mathcal{I}_1^{t+1} \cup \pi^{t,1} \cup \cdots \cup \pi^{t,\nu}$*, we have*

1. *If* $\frac{\varphi_h}{1-c_h} - \prod_{i \neq h} \sum_{\tau=1}^{t_i^*-1} f_i^\tau \geq 0$*, then* $t_h^* = t+1$*.*
2. *If* $\frac{\varphi_h}{1-c_h} - \prod_{i \neq h} \sum_{\tau=1}^{t_i^*-1} f_i^\tau < 0$ *and* $\overline{H}\left(\mathcal{S}^{t,\nu} + \sum_{i \in \pi^{t,\nu}}(t,i)\right) < 1 - \sum_{i=1}^n \varphi_i$*, then* $t_h^* = m+1$*.*

Proof. By Algorithm 1, $t_i^* = t+1$ if $i \in \pi^{t,\nu}$. By Lemma 13, $t_i^* = t$ if $i \in \pi^{t,1} \cup \cdots \cup \pi^{t,\nu-1}$ and $t_i^* = t+1$ if $i \in \mathcal{I}_1^{t+1} \backslash (\pi^{t,1} \cup \cdots \cup \pi^{t,\nu-1})$. If $t = m$, then, by Algorithm 1, $t_i^* = m+1$ for $i \notin \pi^{t,1} \cup \cdots \cup \pi^{t,\nu-1}$.

Let $t \leq m-1$. For the ease of notation, let $\mathcal{I}' = \pi^{t,\nu}$ and $\mathcal{S} = (S_1^{t_1}, \ldots, S_n^{t_n})$ be such that $t_i = t$ if $i \in \pi^{t,1} \cup \cdots \cup \pi^{t,\nu-1}$, $t_i = t+1$ if $i \in \mathcal{I}_1^{t+1} \cup \pi^{t,\nu} \backslash (\pi^{t,1} \cup \cdots \cup \pi^{t,\nu-1})$ and $t_i \geq t+1$ otherwise. Fix $h \notin \mathcal{I}_1^{t+1} \cup \pi^{t,1} \cup \cdots \cup \pi^{t,\nu}$ and t_i for all $i \neq h$. Define

$$\Delta(t_h) := \overline{H}\left(\mathcal{S} + \sum_{i \in \mathcal{I}'}(t,i)\right) - \sum_{i=1}^n \sum_{\tau \in S_i} \overline{z}_i^\tau.$$

By Lemma 14 and the fact that $h \notin \mathcal{I}_1^{t+1}$, there exists $t \leq t^* \leq m+1$ such that if $t+1 \leq t_h \leq t^*$, then $\Delta(t_h) = 1 - \sum_{i=1}^n \varphi_i - \sum_{i=1}^n \sum_{\tau \in S_i} \overline{z}_i^\tau$; and if $t^* < t_h \leq m+1$, then

$$\Delta(t_h) = 1 - \left(\prod_{i \notin \mathcal{I}'} \sum_{\tau=1}^{t_i-1} f_i^\tau\right)\left(\prod_{i \in \mathcal{I}'} \sum_{\tau=1}^{t-1} f_i^\tau\right) - \sum_{i \notin \mathcal{I}'} \sum_{\tau=t_i}^{m} f_i^\tau \varphi_i - \sum_{i \in \mathcal{I}'} \sum_{\tau=t}^{m} f_i^\tau \varphi_i - \sum_{i=1}^n \sum_{\tau=t_i}^{m} \overline{z}_i^\tau.$$

Because $h \notin \mathcal{I}_1^{t+1}$, we have $\overline{z}_h^{t_h} = c_h \varphi_h f_h^{t_h}/(1-c_h)$ for all $t_h \geq t+1$. If $t_h < t^*$, then we have $\Delta(t_h+1) - \Delta(t_h) = c_h \varphi_h f_h^{t_h}/(1-c_h) \geq 0$. Hence, $\Delta(t+1) \leq \Delta(t_h)$ for all $t_h \leq t^*$. If $t_h > t^*$, we have

$$\begin{aligned}&\Delta(t_h+1) - \Delta(t_h)\\ =&f_h^{t_h}\left(\frac{\varphi_h}{1-c_h} - \left(\prod_{i \notin \mathcal{I}', i \neq h} \sum_{\tau=1}^{t_i-1} f_i^\tau\right)\left(\prod_{i \in \mathcal{I}'} \sum_{\tau=1}^{t-1} f_i^\tau\right)\right).\end{aligned}$$

If $t_h = t^*$, we have

$$\begin{aligned}&\Delta(t_h+1) - \Delta(t_h)\\ \geq&f_h^{t_h}\left(\frac{\varphi_h}{1-c_h} - \left(\prod_{i \notin \mathcal{I}', i \neq h} \sum_{\tau=1}^{t_i-1} f_i^\tau\right)\left(\prod_{i \in \mathcal{I}'} \sum_{\tau=1}^{t-1} f_i^\tau\right)\right).\end{aligned}$$

Hence, if $\frac{\varphi_h}{1-c_h} - \left(\prod_{i \notin \mathcal{I}', i \neq h} \sum_{\tau=1}^{t_i-1} f_i^\tau\right)\left(\prod_{i \in \mathcal{I}'} \sum_{\tau=1}^{t-1} f_i^\tau\right) \geq 0$, then $\Delta(t_h+1) \geq \Delta(t_h)$ for all $t_h \geq t^*$. Furthermore, because $\Delta(t+1) \leq \Delta(t_h)$ for all $t_h \leq t^*$, we have $\Delta(t+1) \leq \Delta(t_h)$ for all $t_h \geq t+1$, hence $t_h^* = t+1$.

If $\frac{\varphi_h}{1-c_h} - \left(\prod_{i \notin \mathcal{I}', i \neq h} \sum_{\tau=1}^{t_i-1} f_i^\tau\right)\left(\prod_{i \in \mathcal{I}'} \sum_{\tau=1}^{t-1} f_i^\tau\right) < 0$, then $\Delta(t_h+1) \leq \Delta(t_h)$ for all $t_h > t^*$. Hence, $\Delta(m+1) \leq \Delta(t_h)$ for all $t_h > t^*$. Recall that $\Delta(t+1) \leq \Delta(t_h)$ for all $t_h \leq t^*$. Hence, $t_h^* \in \arg\min\{\Delta(t+1), \Delta(m+1)\}$. If $\overline{H}\left(\mathcal{S}^{t,\nu} + \sum_{i \in \pi^{t,\nu}}(t,i)\right) < 1 - \sum_{i=1}^n \varphi_i$, then $t^* = t$ by definition, which implies that $t_h^* = m+1$. ■

Lemma 16 *Suppose* $\mathcal{S}^{t,\nu} = (S_1^{t_1^*}, \ldots, S_n^{t_n^*})$ *and* $h \notin \mathcal{I}_1^{t+1} \cup \pi^{t,1} \cup \cdots \cup \pi^{t,\nu}$*, then* $t_h^* = t+1$ *implies that* $h \in \mathcal{I}_1^t$*.*

Proof. Suppose $\overline{H}\left(\mathcal{S}^{t,\nu} + \sum_{i \in \pi^{t,\nu}}(t,i)\right) = 1 - \sum_{i=1}^n \varphi_i$, then by Lemma 14, $h \in \mathcal{I}_1^t$. Suppose $\overline{H}\left(\mathcal{S}^{t,\nu} + \sum_{i \in \pi^{t,\nu}}(t,i)\right) < 1 - \sum_{i=1}^n \varphi_i$. By Lemma 15, $\frac{\varphi_h}{1-c_h} \geq \prod_{i \neq h} \sum_{\tau=1}^{t_i^*-1} f_i^\tau$. Hence,

$$\begin{aligned}
&\overline{H}\left(\mathcal{S}^{t,\nu} + \sum_{i \in \pi^{t,\nu}}(t,i) + (t,h)\right) - \overline{H}\left(\mathcal{S}^{t,\nu} + \sum_{i \in \pi^{t,\nu}}(t,i)\right) \\
\leq& f_h^t\left(\prod_{i \neq h} \sum_{\tau=1}^{t_i^*-1} f_i^\tau - \varphi_h\right) \\
\leq& f_h^t\left(\frac{\varphi_h}{1-c_h} - \varphi_h\right) = \frac{c_h \varphi_h f_h^t}{1-c_h}.
\end{aligned}$$

By Algorithm 1, $h \in \mathcal{I}_1^t$. ■

C.2 Continuum case

I characterize an optimal solution in the continuum case by taking m to infinity. Let $\mathcal{I}_1^{m,t}$ denote $\mathcal{I}_1^t$, and $\underline{t}^m$ be defined by (34) when $\mathcal{D}$ is discretized by m grid points. Clearly, if $i \in \mathcal{I}_1^{m,t}$ then $i \in \mathcal{I}_1^{2m,2t-1}$. For each m and i, let $\bar{t}_i^m := \max\left\{t | i \in \mathcal{I}_1^{m,t}\right\}$ and $\overline{d}_i^m := \underline{d} + \frac{(\bar{t}_i^m - 1)(\overline{d} - \underline{d})}{m}$. Then the sequence of $\left\{\overline{d}_i^{2^\kappa}\right\}_\kappa$ is non-decreasing and bounded from above by $\overline{d}$. Hence, the sequence converges and let $d_i^u := \lim_{\kappa \to \infty} \overline{d}_i^{2^\kappa}$ denote its limit. For each κ, let $\underline{d}^{2^\kappa} := \underline{d} + \frac{(\underline{t}^{2^\kappa} - 1)(\overline{d} - \underline{d})}{2^\kappa}$, which is bounded. After taking subsequences, we can assume $\left\{\underline{d}^{2^\kappa}\right\}_\kappa$ converges and let $d^l := \lim_{\kappa \to \infty} \underline{d}^{2^\kappa}$ denote its limit. Let

$$\overline{P}_i(v_i) := \begin{cases} \frac{\varphi_i}{1-c_i} & \text{if } v_i > d_i^u + \frac{k_i}{c_i} \\ \prod_{j \neq i, d_j^u \geq v_i - \frac{k_i}{c_i}} F_j\left(v_i - \frac{k_i}{c_i} + \frac{k_j}{c_j}\right) & \text{if } d^l + \frac{k_i}{c_i} < v_i < d_i^u + \frac{k_i}{c_i} \\ \varphi_i & \text{if } v_i < d^l + \frac{k_i}{c_i} \end{cases}.$$

Finally, let $\boldsymbol{P}^* := (P_i^*)_i$ where

$$P_i^*(v_i) := \begin{cases} \overline{P}_i(v_i) & \text{if } v_i > \frac{k_i}{c_i} \\ \varphi_i & \text{if } v_i < \frac{k_i}{c_i} \end{cases} . \tag{19}$$

We are now ready to prove Theorem 4.

Proof of Theorem 4. We can extend $\overline{P}_i^m$ (P_i^m) to $[\underline{v}_i, \overline{v}_i]$ by setting, for each $t = 1, \ldots, m$,

$$\overline{P}_i^m(v_i) := \overline{P}_i^{m,t}(P_i^m(v_i) := P_i^{m,t}) \text{ for } v_i \in \left[\underline{d} + \frac{(t-1)(\overline{d}-\underline{d})}{m} + \frac{k_i}{c_i}, \underline{d} + \frac{t(\overline{d}-\underline{d})}{m} + \frac{k_i}{c_i}\right].$$

I show that, after taking subsequences, $\overline{P}_i^m$ converges to $\overline{P}_i$ pointwise.

First, by construction and Lemma 14, $\overline{P}_i^{2^\kappa}(v_i) = \varphi_i$ for all $v_i < \underline{d}^{2^\kappa} + \frac{k_i}{c_i}$, we have $\lim_{\kappa\to\infty} \overline{P}_i^{2^\kappa}(v_i) = \overline{P}_i(v_i)$ for all $v_i < d^l + \frac{k_i}{c_i}$. Similarly, by construction, $\overline{P}_i^{2^\kappa}(v_i) = \frac{\varphi_i}{1-c_i}$ for all $v_i > \overline{d}_i^{2^\kappa} + \frac{k_i}{c_i}$, we have $\lim_{\kappa\to\infty} \overline{P}_i^{2^\kappa}(v_i) = \overline{P}_i(v_i)$ for all $v_i > d_i^u + \frac{k_i}{c_i}$.

Suppose $d^l < v_i - \frac{k_i}{c_i} < d_i^u$. Assume without loss of generality that $d_1^u \geq \cdots \geq d_n^u \geq d^l$. If $d_i^u = d^l$, then we are done. Assume for the rest of the proof that $d_i^u > d^l$. Let $d_{n+1}^u := d^l$. Consider v_i such that $d_i^u \geq d_j^u > v_i - \frac{k_i}{c_i} > d_{j+1}^u$ for some $j \geq i$. For m sufficiently large, there exists t such that

$$d_{j+1}^u < \underline{d} + \frac{(t-1)(\overline{d}-\underline{d})}{m} < \underline{d} + \frac{t(\overline{d}-\underline{d})}{m} < v_i - \frac{k_i}{c_i} < \underline{d} + \frac{(t+1)(\overline{d}-\underline{d})}{m} < d_j^u \leq d_i^u.$$

Hence, by construction, we have $\mathcal{I}_1^{m,t} = \mathcal{I}_1^{m,t+1} = \{1, \ldots, j\}$. By Lemmas 15 and 16, there exists $\mathcal{S} = (S_1^{t_1}, \ldots, S_n^{t_n})$ such that $t_i = t+1$, $t_h \in \{t, t+1\}$ if $h \leq j$ and $h \neq i$, $t_h = m+1$ if $h > j$, and

$$f_i^t\left(\overline{P}_i^{m,t} - \varphi_i\right) = \overline{z}_i^{m,t} = \overline{H}(\mathcal{S} + (t,i)) - \overline{H}(\mathcal{S}).$$

Because $\overline{H}$ is submodular, we have

$$\begin{aligned} f_i^t\left(\overline{P}_i^{m,t} - \varphi_i\right) &\leq \overline{H}(\mathcal{S}' + (t,i)) - \overline{H}(\mathcal{S}') \\ &= f_i^t\left(\prod_{h\leq j, h\neq i} \sum_{\tau=1}^{t} f_h^\tau - \varphi_i\right), \end{aligned}$$

where $\mathcal{S}' = (S_1^{t+1}, \ldots, S_j^{t+1}, S_{j+1}^{m+1}, \ldots, S_n^{m+1})$; and

$$f_i^t\left(\overline{P}_i^{m,t} - \varphi_i\right) \geq \overline{H}(\mathcal{S}'' - (t,i)) - \overline{H}(\mathcal{S}')$$
$$= f_i^t\left(\prod_{h\leq j, h\neq i}\sum_{\tau=1}^{t-1} f_h^\tau - \varphi_i\right),$$

where $\mathcal{S}'' = (S_1^t, \ldots, S_j^t, S_{j+1}^{m+1}, \ldots, S_n^{m+1})$. Hence,

$$\prod_{h\leq j, h\neq i}\sum_{\tau=1}^{t-1} f_h^\tau \leq \overline{P}_i^{m,t} \leq \prod_{h\leq j, h\neq i}\sum_{\tau=1}^{t} f_h^\tau.$$

Take $m = 2^\kappa$ to infinity and we have $\lim_{\kappa\to\infty}\overline{P}_i^{2^\kappa}(v_i) = \overline{P}_i(v_i)$.

It follows that, after taking subsequences, P_i^m converges to P_i^* pointwise. $\boldsymbol{P}^*$ is feasible by a similar argument to that in the proof of Lemma 3, and optimal by a similar argument to that in the proof of Theorem 1. ■

C.3 Optimal one-threshold mechanism

By a similar argument to that in the proof of Theorem 2, we can show that pure randomization is optimal if verification is sufficiently costly or the principal's ability to punish an agent is sufficiently limited, i.e., $\overline{v}_i - k_i/c_i \leq \mathbb{E}_{v_i}[v_i]$ for all i. To make the problem interesting, in what follows, in what follows I assume that

Assumption 2 $\overline{v}_i - k_i/c_i > \mathbb{E}_{v_i}[v_i]$ *for some* i.

If $d_i^u = \overline{v}_i - \frac{k_i}{c_i}$ for all i, then $d^l \geq \max_j\left\{\underline{v}_j - k_j/c_j\right\}$ satisfies that

$$\sum_{i=1}^n \varphi_i F_i\left(d^l + \frac{k_i}{c_i}\right) = \prod_{i=1}^n F_i\left(d^l + \frac{k_i}{c_i}\right).$$

Lemma 17 below shows that there exists a unique d^l satisfying the above equation. Note that unless $\varphi_i = 0$ for all i, we have $d^l > \max_j\left\{\underline{v}_j - k_j/c_j\right\}$. Clearly, in optimum, $\varphi_i > 0$ for some i. Hence, $d^l > \max_j\left\{\underline{v}_j - k_j/c_j\right\}$. Let d_i^* $(i = 1, \ldots, n)$ be defined by

$$\mathbb{E}_{v_i}[v_i] - \mathbb{E}_{v_i}\left[\max\left\{v_i, d_i^* + \frac{k_i}{c_i}\right\}\right] + \frac{k_i}{c_i} = 0, \tag{35}$$

and $d^{l*} := \max_i d_i^*$. Now we are ready to state the main result in this subsection which characterizes the set of optimal $\boldsymbol{\varphi}$:

Theorem 7 *Suppose that Assumption 2 holds. If*

$$\sum_{i=1}^{n}(1-c_i)F_i\left(d^{l*}+\frac{k_i}{c_i}\right)\prod_{j\neq i}F_j\left(\overline{v}_j-\frac{k_i}{c_i}+\frac{k_j}{c_j}\right)\leq\prod_{i=1}^{n}F_i\left(d^{l*}+\frac{k_i}{c_i}\right),$$

then the set of optimal $\boldsymbol{\varphi}$ is the convex hull of

$$\left\{\boldsymbol{\varphi}\;\middle|\;\begin{array}{l} i^*\in\arg\max_i d_i^*, \varphi_i=(1-c_i)\prod_{j\neq i}F_j\left(\overline{v}_j-\frac{k_i}{c_i}+\frac{k_j}{c_j}\right)\ \forall i\neq i^*, \\ \varphi_{i^*}=\frac{\prod_{i=1}^{n}F_i\left(d^{l*}+\frac{k_i}{c_i}\right)-\sum_{i\neq i^*}(1-c_i)\prod_{j\neq i}F_j\left(\overline{v}_j-\frac{k_i}{c_i}+\frac{k_j}{c_j}\right)F_i\left(d^*+\frac{k_i}{c_i}\right)}{F_{i^*}\left(d^{l*}+\frac{k_{i^*}}{c_{i^*}}\right)}\end{array}\right\}.$$

For each optimal $\boldsymbol{\varphi}^*$, the following allocation rule is optimal:

$$P_i^{**}(v_i):=\begin{cases}\prod_{j\neq i}F_j\left(v_i-\frac{k_i}{c_i}+\frac{k_j}{c_j}\right) & \text{if } v_i\geq d^l+\frac{k_i}{c_i}\\ \varphi_i^* & \text{if } v_i<d^l+\frac{k_i}{c_i}\end{cases}.$$

Proof. Let $\Phi(d^l,d_1^u,\ldots,d_n^u)\subset\{\boldsymbol{\varphi}|\sum\varphi_i\leq 1\}$ denote the feasible set of $\boldsymbol{\varphi}$ given d^l and $d_1^u,\ldots,d_n^u$. I often abuse notation and use Φ to denote the feasible set when its meaning is clear. Fix $d^l>\max_j\left\{\underline{v}_j-k_j/c_j\right\}$ and $d_i^u=\overline{v}_i-\frac{k_i}{c_i}$ for all i. Then $\boldsymbol{\varphi}$ is feasible if and only if

$$\sum_{i=1}^{n}\varphi_iF_i\left(d^l+\frac{k_i}{c_i}\right)=\prod_{i=1}^{n}F_i\left(d^l+\frac{k_i}{c_i}\right),$$
$$\prod_{j\neq i}F_j\left(\overline{v}_i-\frac{k_i}{c_i}+\frac{k_j}{c_j}\right)\leq\frac{\varphi_i}{1-c_i},\forall i.$$

Hence, Φ is non-empty if and only if

$$\sum_{i=1}^{n}(1-c_i)F_i\left(d^l+\frac{k_i}{c_i}\right)\prod_{j\neq i}F_j\left(\overline{v}_i-\frac{k_i}{c_i}+\frac{k_j}{c_j}\right)\leq\prod_{i=1}^{n}F_i\left(d^l+\frac{k_i}{c_i}\right).$$

Suppose that Φ is non-empty. It is not hard to see that Φ is convex. Because the objective function is linear in $\boldsymbol{\varphi}$ and the feasible set is convex, there is an optimal $\boldsymbol{\varphi}$ which is an extreme point.

Clearly, $\boldsymbol{\varphi}$ is an extreme point of Φ if and only if there exists i^* such that

$$\sum_{i=1}^{n} \varphi_i F_i\left(d^l + \frac{k_i}{c_i}\right) = \prod_{i=1}^{n} F_i\left(d^l + \frac{k_i}{c_i}\right),$$
$$\varphi_j = (1-c_j)\prod_{i\neq j} F_i\left(\overline{v}_j - \frac{k_j}{c_j} + \frac{k_i}{c_i}\right), \forall j \neq i^*.$$

In this case, denote the principal's payoff by $Z_{1,i^*}(d^l)$. For ease of notation, let $i^* = 1$. Let $\bar{\varphi}_j := (1-c_j)\prod_{i\neq j} F_i\left(\overline{v}_j - \frac{k_j}{c_j} + \frac{k_i}{c_i}\right)$ for all j. Then the principal's payoff is given as follows:

$$\begin{aligned}
Z_{1,1}(d^l) :=& \sum_{i=1}^{n}\int_{\max\left\{d^l+\frac{k_i}{c_i},\frac{k_i}{c_i}\right\}}^{\overline{v}_i}\left(v_i - \frac{k_i}{c_i}\right)\prod_{j\neq i} F_j\left(v_i - \frac{k_i}{c_i} + \frac{k_j}{c_j}\right)\mathrm{d}F_i(v_i)\\
&+\sum_{i\neq 1}\int_{\underline{v}_i}^{\max\left\{d^l+\frac{k_i}{c_i},\frac{k_i}{c_i}\right\}}\left(v_i - \frac{k_i}{c_i}\right)\overline{\varphi}_i \mathrm{d}F_i(v_i)\\
&+\int_{\underline{v}_1}^{\max\left\{d^l+\frac{k_1}{c_1},\frac{k_1}{c_1}\right\}}\left(v_1 - \frac{k_1}{c_1}\right)\frac{\prod_{i=1}^{n} F_i\left(d^l + \frac{k_i}{c_i}\right) - \sum_{i\neq 1} F_i\left(d^l+\frac{k_i}{c_i}\right)\overline{\varphi}_i}{F_1\left(d^l + \frac{k_1}{c_1}\right)}\mathrm{d}F_1(v_1)\\
&+\sum_{i\neq 1}\frac{\overline{\varphi}_i k_i}{c_i} + \frac{k_1}{c_1}\frac{\prod_{i=1}^{n} F_i\left(d^l + \frac{k_i}{c_i}\right) - \sum_{i\neq 1} F_i\left(d^l+\frac{k_i}{c_i}\right)\overline{\varphi}_i}{F_1\left(d^l + \frac{k_1}{c_1}\right)}.
\end{aligned}$$

If $d^l < 0$, then it is not hard to show that $Z_{1,1}$ is strictly increasing in d^l. If $d^l \geq 0$, then, after some algebra, we have

$$\begin{aligned}
&Z'_{1,1}(d^l)\\
=&\left\{\sum_{i\neq 1}\left[f_i\left(d^l + \frac{k_i}{c_i}\right)\prod_{j\neq i,1} F_j\left(d^l + \frac{k_j}{c_j}\right) - \overline{\varphi}_i\frac{f_i\left(d^l+\frac{k_i}{c_i}\right)F_1\left(d^l+\frac{k_1}{c_1}\right) - F_i\left(d^l+\frac{k_i}{c_i}\right)f_1\left(d^l+\frac{k_1}{c_1}\right)}{F_1^2\left(d^l+\frac{k_1}{c_1}\right)}\right]\right\}\\
&\cdot\left[\int_{\underline{v}_1}^{d^l+\frac{k_1}{c_1}}\left(v_1 - d^l - \frac{k_1}{c_1}\right)\mathrm{d}F_1(v_1) + \frac{k_1}{c_1}\right].
\end{aligned}$$

Because $\overline{\varphi}_i \leq \prod_{j\neq i} F_j\left(d^l + \frac{k_j}{c_j}\right)$, the first-term in the above equation is strictly positive. The second-term is strictly decreasing in d^l. Let d_1^* be such that

$$\int_{\underline{v}_1}^{d_1^*+\frac{k_1}{c_1}}\left(v_1 - d_1^* - \frac{k_1}{c_1}\right)\mathrm{d}F_1(v_1) + \frac{k_1}{c_1} = 0. \tag{36}$$

Then $Z'_{1,1}(d^l) > 0$ if $d^l < d_1^*$ and $Z'_{1,1}(d^l) < 0$ if $d^l > d_1^*$. Hence, $Z_{1,1}(d^l)$ achieves its maximum at $d^l = d_1^*$.

Define d_i^* for all $i \geq 2$ as in (36). Suppose that $d_1^* \geq d_2^*$. By a similar argument to that in Lemma 17, $\Phi\left(d_2^*, \overline{v}_1 - \frac{k_1}{c_1}, \ldots, \overline{v}_n - \frac{k_n}{c_n}\right) \neq \emptyset$ implies that $\Phi\left(d_1^*, \overline{v}_1 - \frac{k_1}{c_1}, \ldots, \overline{v}_n - \frac{k_n}{c_n}\right) \neq \emptyset$. Suppose that both $\Phi\left(d_2^*, \overline{v}_1 - \frac{k_1}{c_1}, \ldots, \overline{v}_n - \frac{k_n}{c_n}\right)$ and $\Phi\left(d_1^*, \overline{v}_1 - \frac{k_1}{c_1}, \ldots, \overline{v}_n - \frac{k_n}{c_n}\right)$ are non-empty. Then

$$
\begin{aligned}
& Z_{1,1}(d^l) - Z_{1,2}(d^l) \\
= & \left[\prod_{i=1}^n F_i\left(d^l + \frac{k_i}{c_i}\right) - \sum_{i=1} F_i\left(d^l + \frac{k_i}{c_i}\right)\overline{\varphi}_i\right] \\
& \cdot \left\{\frac{1}{F_1\left(d^l + \frac{k_1}{c_1}\right)}\left[\int_{\underline{v}_1}^{d^l + \frac{k_1}{c_1}}\left(v_1 - \frac{k_1}{c_1}\right)\mathrm{d}F_1(v_1) + \frac{k_1}{c_1}\right]\right. \\
& \left. \quad - \frac{1}{F_2\left(d^l + \frac{k_2}{c_2}\right)}\left[\int_{\underline{v}_2}^{d^l + \frac{k_2}{c_2}}\left(v_1 - \frac{k_2}{c_2}\right)\mathrm{d}F_2(v_2) + \frac{k_2}{c_2}\right]\right\}
\end{aligned}
$$

If $d^l = d_2^*$, then by definition we have

$$
\begin{aligned}
& Z_{1,1}(d_2^*) - Z_{1,2}(d_2^*) \\
= & \left[\prod_{i=1}^n F_i\left(d_2^* + \frac{k_i}{c_i}\right) - \sum_{i=1} F_i\left(d_2^* + \frac{k_i}{c_i}\right)\overline{\varphi}_i\right] \\
& \cdot \left\{\frac{1}{F_1\left(d_2^* + \frac{k_1}{c_1}\right)}\left[\int_{\underline{v}_1}^{d_2^* + \frac{k_1}{c_1}}\left(v_1 - \frac{k_1}{c_1}\right)\mathrm{d}F_1(v_1) + \frac{k_1}{c_1}\right] - d_2^*\right\} \\
\geq & \left[\prod_{i=1}^n F_i\left(d_2^* + \frac{k_i}{c_i}\right) - \sum_{i=1} F_i\left(d_2^* + \frac{k_i}{c_i}\right)\overline{\varphi}_i\right](d_2^* - d_2^*) = 0,
\end{aligned}
$$

where the last inequality holds because $d_1^* \geq d_2^*$, and the inequality holds strictly if $d_1^* > d_2^*$. Hence, $Z_{1,1}(d_1^*) \geq Z_{1,2}(d_2^*)$ and the inequality holds strictly if $d_1^* > d_2^*$.

Let $d^{l*} := \max_i d_i^*$. If

$$
\sum_{i=1}^n (1 - c_i)\prod_{j \neq i} F_j\left(\overline{v}_j - \frac{k_i}{c_i} + \frac{k_j}{c_j}\right) F_i\left(d^* + \frac{k_i}{c_i}\right) \leq \prod_{i=1}^n F_i\left(d^{l*} + \frac{k_i}{c_i}\right),
$$

then $\Phi\left(d^{l*}, \overline{v}_1 - \frac{k_1}{c_1}, \ldots, \overline{v}_n - \frac{k_n}{c_n}\right)$ is feasible. This completes the proof. ■

Lemma 17 *There exists a unique* $d^l \geq \max_j \left\{\underline{v}_j - k_j/c_j\right\}$ *such that*

$$\sum_{i=1}^{n} \varphi_i F_i\left(d_l + \frac{k_i}{c_i}\right) = \prod_{i=1}^{n} F_i\left(d^l + \frac{k_i}{c_i}\right). \tag{37}$$

Proof. If $\varphi_i = 0$ for all i, then $d^l = \max_j \left\{\underline{v}_j - k_j/c_j\right\}$ is the unique solution to (37). Assume, for the rest of the proof, that $\varphi_i > 0$ for some i. Let

$$\Delta(d^l) := \sum_{i=1}^{n} \varphi_i F_i\left(d^l + \frac{k_i}{c_i}\right) - \prod_{i=1}^{n} F_i\left(d^l + \frac{k_i}{c_i}\right).$$

Then

$$\Delta'(d^l) = \sum_{i=1}^{n} f_i\left(d^l + \frac{k_i}{c_i}\right)\left[\varphi_i - \prod_{j\neq i} F_j\left(d^l + \frac{k_j}{c_j}\right)\right].$$

Because $\Delta_l\left(\max_j \left\{\underline{v}_j - k_j/c_j\right\}\right) > 0$, a solution to (37) must satisfy that $d^l > \max_j \left\{\underline{v}_j - k_j/c_j\right\}$. Assume, for the rest of the proof, that $d^l > \max_j \left\{\underline{v}_j - k_j/c_j\right\}$. Then $F_i\left(d^l + \frac{k_i}{c_i}\right) > 0$ for all i. If $\Delta(d^l) \leq 0$, then $\varphi_i \leq \prod_{j\neq i} F_j\left(d^l + \frac{k_j}{c_j}\right)$ for all i, and the strict inequality holds for some i, which implies that $\Delta'(d^l) < 0$. Hence, $\Delta(d^l)$ crosses zero at most once, in which case it does so from above. Because $\Delta_l\left(\max_j \left\{\underline{v}_j - k_j/c_j\right\}\right) > 0$ and $\Delta_l\left(\max_j \left\{\overline{v}_j - k_j/c_j\right\}\right) = \sum_i \varphi_i - 1 \leq 0$, there exists a unique d^l satisfying (37). ■

C.4 Symmetric environment revisited

Fix $\boldsymbol{\varphi}$ and let $\{d_i^u\}_i$ and d^l be the associated optimal thresholds. Assume, without loss of generality, that $d_1^u \geq \cdots \geq d_n^u \geq d^l$. Let $1 \leq \xi_1 < \cdots < \xi_L \leq n$ be such that $d_1^u = \cdots = d_{\xi_1}^u$, $d_{\xi_\iota}^u > d_{\xi_\iota+1}^u = \cdots = d_{\xi_{\iota+1}}^u$ for $\iota = 1, \ldots, L-1$ and $d_{\xi_L}^u > d_{\xi_L+1}^u = \cdots = d_n^u = d^l$. Note that in the symmetric environment $d_i^u \geq d_j^u$ only if $\varphi_i \geq \varphi_j$. The proof of Theorem 5 uses the following properties of d^l and d_i^u:

Lemma 18 *If* $\frac{\varphi_i}{1-c} \geq 1$ *for all* $i \leq \xi_1$, *then* $d_{\xi_1}^u = \overline{v} - \frac{k}{c}$; *otherwise* $\frac{\varphi_i}{1-c} < 1$ *for all* $i \leq \xi_1$ *and* $d_{\xi_1}^u$ *satisfies*

$$1 - F\left(d_{\xi_1}^u + \frac{k}{c}\right)^{\xi_1} = \sum_{i=1}^{\xi_1} \frac{\varphi_i}{1-c_i}\left[1 - F\left(d_{\xi_1}^u + \frac{k}{c}\right)\right],$$

$$1 - F(v)^{\xi_1} \lesseqgtr \sum_{i=1}^{\xi_1} \frac{\varphi_i}{1-c}\left[1 - F(v)\right] \text{ if } v \lesseqgtr d_{\xi_1}^u + \frac{k}{c},$$

$$\frac{\varphi_i}{1-c} \geq F\left(d_{\xi_1}^u + \frac{k}{c}\right)^{\xi_1 - 1}, \forall i = 1, \ldots, \xi_1.$$

For $\iota = 1, \ldots, L-1$, $\frac{\varphi_i}{1-c} < 1$ for $\xi_\iota + 1 \le i \le \xi_{\iota+1}$ and $d^u_{\xi_{\iota+1}}$ satisfies

$$F\left(d_{\xi_{\iota+1}} + \frac{k}{c}\right)^{\xi_\iota} - F\left(d_{\xi_{\iota+1}} + \frac{k}{c}\right)^{\xi_{\iota+1}} = \sum_{i=\xi_\iota+1}^{\xi_{\iota+1}} \frac{\varphi_i}{1-c_i}\left[1 - F\left(d_{\xi_{\iota+1}} + \frac{k}{c}\right)\right],$$

$$F(v)^{\xi_\iota} - F(v)^{\xi_{\iota+1}} \lesseqgtr \sum_{i=\xi_\iota+1}^{\xi_{\iota+1}} \frac{\varphi_i}{1-c}\left[1 - F(v)\right] \text{ if } v \lesseqgtr d_{\xi_{\iota+1}} + \frac{k}{c},$$

$$\frac{\varphi_i}{1-c} \ge F\left(d^u_{\xi_{\iota+1}} + \frac{k}{c}\right)^{\xi_{\iota+1}-1}, \forall i = \xi_\iota + 1, \ldots, \xi_{\iota+1}.$$

Finally, d^l satisfies

$$F\left(d^l + \frac{k}{c}\right)^{\xi_L} = \sum_{i=1}^{n} \varphi_i F\left(d^l + \frac{k}{c}\right) + \sum_{i=\xi_L+1}^{n} \frac{\varphi_i}{1-c}\left[1 - F\left(d^l + \frac{k}{c}\right)\right],$$

$$F(v)^{\xi_L} \lesseqgtr \sum_{i=1}^{n} \varphi_i F(v) + \sum_{i=\xi_L+1}^{n} \frac{\varphi_i}{1-c}\left[1 - F(v)\right] \text{ if } v \lesseqgtr d^l + \frac{k}{c}.$$

The arguments used to prove Lemma 18 are similar to that used to show that $\overline{P}_i^m$ converges to $\overline{P}_i$ if $d^l < v_i - \frac{k_i}{c_i} < d^u_i$, and are neglected here.

Proof of Theorem 5. The first part of the theorem directly follows from Theorem 7. Assume, for the rest of the proof, that $F(v^*)^{n-1} < n(1-c)$. Consider an optimal $\boldsymbol{\varphi}$. Let $\{d^u_i\}_i$ and d^l be the associated optimal thresholds. Assume, without loss of generality, that $d^u_1 \ge \cdots \ge d^u_n \ge d^l$. Let ξ_ι ($\iota = 1, \ldots, L$) be defined as in the beginning of this subsection.

First, I show that $L = 1$. Suppose, to the contrary, that $L \ge 2$. Suppose that $d^u_{\xi_2} < 0$, then the principal's objective function is strictly increasing in φ_i for $i > \xi_2$. Hence, in optimum, it must be that $d^u_{\xi_2} \ge 0$. Construct a new $\boldsymbol{\varphi}^*$ as follows: Let

$$\varphi^*_i = \frac{1}{\xi_2}\sum_{j=1}^{\xi_2} \varphi_j, \text{ for all } i = 1, \ldots, \xi_2,$$

and $\varphi^*_i = \varphi_i$ for all $i > \xi_2$. Let d^{u*}_i and d^{l*} be the optimal thresholds associated with $\boldsymbol{\varphi}^*$. Then $d^{u*}_1 = \ldots d^{u*}_{\xi_2}$ and $d^{u*}_i = d^u_i$ for all $i > \xi_2$. There are two cases: (1) $\varphi_i < 1-c$ for all $i \le \xi_1$ and (2) $\varphi_i \ge 1-c$ for all $i \le \xi_1$.

Case 1: $\varphi_i < 1-c$ for all $i \le \xi_1$. In this case, $\varphi^*_1 < 1-c$. Then $d^{u*}_{\xi_2}$ is defined by

$$\left[1 - F\left(d^{u*}_{\xi_2} + \frac{k}{c}\right)\right]\sum_{i=1}^{\xi_2} \frac{\varphi^*_i}{1-c} = 1 - F\left(d^{u^*}_{\xi_2} + \frac{k}{c}\right)^{\xi_2}. \tag{38}$$

Hence, $d^u_{\xi_2} < d^{u^*}_{\xi_2} < d^u_{\xi_1}$. Let $Z(\boldsymbol{\varphi})$ denote the principal's payoff given $\boldsymbol{\varphi}$. Then

$$\begin{aligned}
& Z(\boldsymbol{\varphi}^*) - Z(\boldsymbol{\varphi}) \\
=& \sum_{i=1}^{\xi_2}\left[\int_{d^{u*}_{\xi_2}+\frac{k}{c}}^{\overline{v}}\left(v_i-\frac{k}{c}\right)\frac{\varphi_i^*}{1-c}\mathrm{d}F(v_i)+\int_{d^u_{\xi_2}+\frac{k}{c}}^{d^{u^*}_{\xi_2}+\frac{k}{c}}\left(v_i-\frac{k}{c}\right)F(v_i)^{\xi_2-1}\mathrm{d}F(v_i)\right] \\
& - \sum_{i=\xi_1+1}^{\xi_2}\int_{d^u_{\xi_2}+\frac{k}{c}}^{\overline{v}}\left(v_i-\frac{k}{c}\right)\frac{\varphi_i}{1-c}\mathrm{d}F(v_i) \\
& - \sum_{i=1}^{\xi_1}\left[\int_{d^u_{\xi_2}+\frac{k}{c}}^{d^u_{\xi_1}+\frac{k}{c}}\left(v_i-\frac{k}{c}\right)F(v_i)^{\xi_1-1}\mathrm{d}F(v_i)+\int_{d^u_{\xi_1}+\frac{k}{c}}^{\overline{v}}\left(v_i-\frac{k}{c}\right)\frac{\varphi_i}{1-c}\mathrm{d}F(v_i)\right] \\
=& \int_{d^u_{\xi_2}+\frac{k}{c}}^{d^{u*}_{\xi_2}+\frac{k}{c}}\left(v-\frac{k}{c}\right)\xi_2F(v)^{\xi_2-1}\mathrm{d}F(v)+\int_{d^{u^*}_{\xi_2}+\frac{k}{c}}^{\overline{v}}\left(v-\frac{k}{c}\right)\sum_{i=1}^{\xi_2}\frac{\varphi_i^*}{1-c}\mathrm{d}F(v) \\
& - \int_{d^u_{\xi_2}+\frac{k}{c}}^{d^u_{\xi_1}+\frac{k}{c}}\left(v-\frac{k}{c}\right)\left(\sum_{i=\xi_1+1}^{\xi_2}\frac{\varphi_i}{1-c}+\xi_1F(v)^{\xi_1-1}\right)\mathrm{d}F(v)-\int_{d^u_{\xi_1}+\frac{k}{c}}^{\overline{u}}\left(v-\frac{k}{c}\right)\sum_{i=1}^{\xi_2}\frac{\varphi_i}{1-c}\mathrm{d}F(v) \\
=& \int_{d^u_{\xi_2}+\frac{k}{c}}^{d^{u*}_{\xi_2}+\frac{k}{c}}\left(v-\frac{k}{c}\right)\xi_2F(v)^{\xi_2-1}\mathrm{d}F(v)+\int_{d^{u^*}_{\xi_2}+\frac{k}{c}}^{d^u_{\xi_1}+\frac{k}{c}}\left(v-\frac{k}{c}\right)\sum_{i=1}^{\xi_2}\frac{\varphi_i}{1-c}\mathrm{d}F(v) \\
& - \int_{d^u_{\xi_2}+\frac{k}{c}}^{d^u_{\xi_1}+\frac{k}{c}}\left(v-\frac{k}{c}\right)\left(\sum_{i=\xi_1+1}^{\xi_2}\frac{\varphi_i}{1-c}+\xi_1F(v)^{\xi_1-1}\right)\mathrm{d}F(v) \\
=& d^u_{\xi_1}\left[F\left(d^u_{\xi_1}+\frac{k}{c}\right)\sum_{i=1}^{\xi_1}\frac{\varphi_i}{1-c}-F\left(d^u_{\xi_1}+\frac{k}{c}\right)^{\xi_1}\right]+d^{u*}_{\xi_2}\left[F\left(d^{u*}_{\xi_2}+\frac{k}{c}\right)^{\xi_2}-F\left(d^{u*}_{\xi_2}+\frac{k}{c}\right)\sum_{i=\xi_1+1}^{\xi_2}\frac{\varphi_i}{1-c}\right] \\
& - d^u_{\xi_2}\left[F\left(d^u_{\xi_2}+\frac{k}{c}\right)^{\xi_2}-F\left(d^u_{\xi_2}+\frac{k}{c}\right)\sum_{i=\xi_1+1}^{\xi_2}\frac{\varphi_i}{1-c}-F\left(d^u_{\xi_2}+\frac{k}{c}\right)^{\xi_1}\right] \\
& - \int_{d^u_{\xi_2}+\frac{k}{c}}^{d^{u*}_{\xi_2}+\frac{k}{c}}\left[F(v)^{\xi_2}-F(v)\sum_{i=\xi_1}^{\xi_2}\frac{\varphi_i}{1-c}-F(v)^{\xi_1}\right]\mathrm{d}v-\int_{d^{u*}_{\xi_2}+\frac{k}{c}}^{d^u_{\xi_1}+\frac{k}{c}}\left[F(v)\sum_{i=1}^{\xi_1+1}\frac{\varphi_i}{1-c}-F(v)^{\xi_1}\right]\mathrm{d}v,
\end{aligned}$$

where the third equality holds because $\sum_{i=1}^{\xi_2}\varphi_i = \sum_{i=1}^{\xi_2}\varphi_i^*$, and the last equality holds by integration by parts. Because $d^{u*}_{\xi_2}$ satisfies (38), $d^u_{\xi_1}$ satisfies that

$$1-F\left(d^u_{\xi_1}+\frac{k}{c}\right)^{\xi_1}=\left[1-F\left(d^u_{\xi_1}+\frac{k}{c}\right)\right]\sum_{i=1}^{\xi_1}\frac{\varphi_i}{1-c}$$

$$1-F\left(v\right)^{\xi_1}<\left[1-F\left(v\right)\right]\sum_{i=1}^{\xi_1}\frac{\varphi_i}{1-c},\forall v<d^u_{\xi_1}+\frac{k}{c},$$

and $d_{\xi_2}^{u}$ satisfies that

$$1-F\left(d_{\xi_2}^{u}+\frac{k}{c}\right)^{\xi_2}=\left[1-F\left(d_{\xi_2}^{u}+\frac{k}{c}\right)\right]\sum_{i=\xi_1+1}^{\xi_2}\frac{\varphi_i}{1-c}+1-F\left(d_{\xi_2}^{u}+\frac{k}{c}\right)^{\xi_1}$$

$$1-F(v)^{\xi_2}>[1-F(v)]\sum_{i=\xi_1+1}^{\xi_2}\frac{\varphi_i}{1-c}+1-F(v)^{\xi_1},\forall v>d_{\xi_2}^{u}+\frac{k}{c},$$

we have

$$\begin{aligned}&Z(\boldsymbol{\varphi}^*)-Z(\boldsymbol{\varphi})\\&>\left(d_{\xi_1}-d_{\xi_2}^{u^*}\right)\left(\sum_{i=1}^{\xi_1}\frac{\varphi_i}{1-c}-1\right)-\left(d_{\xi_2}^{u*}-d_{\xi_2}\right)\sum_{\xi_1+1}^{\xi_2}\frac{\varphi_i}{1-c}\\&+\int_{d_{\xi_2}^{u}+\frac{k}{c}}^{d_{\xi_2}^{u*}+\frac{k}{c}}\sum_{i=\xi_1+1}^{\xi_2}\frac{\varphi_i}{1-c}\mathrm{d}v-\int_{d_{\xi_2}^{u*}+\frac{k}{c}}^{d_{\xi_1}^{u}+\frac{k}{c}}\left(\sum_{i=1}^{\xi_1}\frac{\varphi_i}{1-c}-1\right)\mathrm{d}v=0,\end{aligned}$$

which is a contradiction to the optimality of $\boldsymbol{\varphi}$.

Case 2: $\varphi_i\geq 1-c$ for all $i\leq\xi_1$. If $\varphi_1^*\geq 1-c$, then $d_{\xi_2}^{u*}=\overline{d}$. In this case, we have

$$\begin{aligned}&Z(\boldsymbol{\varphi}^*)-Z(\boldsymbol{\varphi})\\=&\sum_{i=1}^{\xi_2}\int_{d_{\xi_2}^{u}+\frac{k}{c}}^{\overline{v}}\left(v_i-\frac{k}{c}\right)F(v_i)^{\xi_2-1}\mathrm{d}F(v_i)-\sum_{i=\xi_1+1}^{\xi_2}\int_{d_{\xi_2}^{u}+\frac{k}{c}}^{\overline{v}}\left(v_i-\frac{k}{c}\right)\frac{\varphi_i}{1-c}\mathrm{d}F(v_i)\\&-\sum_{i=1}^{\xi_1}\int_{d_{\xi_2}^{u}+\frac{k}{c}}^{\overline{v}}\left(v_i-\frac{k}{c}\right)F(v_i)^{\xi_1-1}\mathrm{d}F(v_i)\\=&\int_{d_{\xi_2}^{u}+\frac{k}{c}}^{\overline{v}}\left(v-\frac{k}{c}\right)\xi_2F(v)^{\xi_2-1}\mathrm{d}F(v)-\int_{d_{\xi_2}^{u}+\frac{k}{c}}^{\overline{v}}\left(v-\frac{k}{c}\right)\left(\sum_{i=\xi_1+1}^{\xi_2}\frac{\varphi_i}{1-c}+\xi_1F(v)^{\xi_1-1}\right)\mathrm{d}F(v)\\=&\left(v-\frac{k}{c}\right)\left[F(v)^{\xi_2}-F(v)\sum_{i=\xi_1+1}^{\xi_2}\frac{\varphi_i}{1-c}-F(v)^{\xi_1}\right]\Bigg|_{d_{\xi_2}^{u}+\frac{k}{c}}^{\overline{v}}\\&-\int_{d_{\xi_2}^{u}+\frac{k}{c}}^{\overline{v}}\left[F(v)^{\xi_2}-F(v)\sum_{i=\xi_1+1}^{\xi_2}\frac{\varphi_i}{1-c}-F(v)^{\xi_1}\right]\mathrm{d}v,\end{aligned}$$

where the last equality holds by integration by parts. Because $d^u_{\xi_2}$ satisfies that

$$\left[1-F\left(d^u_{\xi_2}+\frac{k}{c}\right)\right]\sum_{i=\xi_1+1}^{\xi_2}\frac{\varphi_i}{1-c}+1-F\left(d^u_{\xi_2}+\frac{k}{c}\right)^{\xi_1}=1-F\left(d^u_{\xi_2}+\frac{k}{c}\right)^{\xi_2},$$

$$[1-F(v)]\sum_{i=\xi_1+1}^{\xi_2}\frac{\varphi_i}{1-c}+1-F(v)^{\xi_1}<1-F(v)^{\xi_2},\forall v>d^u_{\xi_2}+\frac{k}{c},$$

we have

$$Z(\boldsymbol{\varphi}^*)-Z(\boldsymbol{\varphi})>-\left(\overline{v}-\frac{k}{c}-d^u_{\xi_2}\right)\sum_{i=1}^{\xi_2}\frac{\varphi_i}{1-c}+\int_{d^u_{\xi_2}+\frac{k}{c}}^{\overline{v}}\sum_{i=\xi_1+1}^{\xi_2}\frac{\varphi_i}{1-c}\mathrm{d}v=0,$$

which is a contradiction to the optimality of $\boldsymbol{\varphi}$. If $\varphi_1^*<1-c$, then let $d_1^{u*}=\cdots=d_{\xi_2}^{u*}$ be defined by (38). Note that if $\xi_1=1$ and $\varphi_1/(1-c)=1$, then the new mechanism using $\boldsymbol{\varphi}^*$ coincides with the old mechanism using $\boldsymbol{\varphi}$. In this case, we can redefine $d_1^u:=d^u_{\xi_2}$ without changing the mechanism. Except for this case, we can show, by a similar argument to that in Case 1, that $Z(\boldsymbol{\varphi}^*)-Z(\boldsymbol{\varphi})>0$, which is a contradiction to the optimality of $\boldsymbol{\varphi}$.

Hence, by induction, we have $L=1$. For ease of notation, let $j:=\xi_1$. Next, we show that $j=0$ or n. Suppose, to the contrary, that $0<j<n$. Suppose that $d^l<0$, then the principal's objective function is strictly increasing in φ_i for $i>j$. Hence, in optimum, it must be that $d^l\geq 0$. Construct a new $\boldsymbol{\varphi}^*$ as follows: Let

$$\varphi_i^*=\frac{1}{n}\sum_{j=1}^{n}\varphi_j,\text{ for all } i=1,\ldots,n.$$

Case 1: $\varphi_i<1-c$ for all $i\leq j$. In this case, $\varphi_1^*<1-c$. Let $d_1^{u*}=\cdots=d_n^{u*}$ be such that

$$1-F\left(d_j^{u*}+\frac{k}{c}\right)^n=\sum_{i=1}^{n}\frac{\varphi_i^*}{1-c}\left[1-F\left(d_j^{u*}+\frac{k}{c}\right)\right]. \tag{39}$$

Then $d_j^{u*}<d_j^u$. Let d^{l*} be such that

$$F\left(d^{l*}+\frac{k}{c}\right)^n=\sum_{i=1}^{n}\varphi_i^*F\left(d^{l*}+\frac{k}{c}\right). \tag{40}$$

Then $d^l<d^{l*}$. There are two subcases to consider: (i) $d^{l*}\leq d_j^{u*}$ and (ii) $d^{l*}>d_j^{u*}$.

(i) Suppose that $d^{l*} \le d_j^{u*}$. Then

$$
\begin{aligned}
& Z(\boldsymbol{\varphi}^*) - Z(\boldsymbol{\varphi}) \\
=& \sum_{i=1}^{n} \left[\int_{\underline{v}}^{d^{l^*}+\frac{k}{c}} \left(v_i - \frac{k}{c}\right) \varphi_i^* \mathrm{d}F(v_i) + \int_{d^{l*}+\frac{k}{c}}^{d_j^{u*}+\frac{k}{c}} \left(v_i - \frac{k}{c}\right) F(v_i)^{n-1} \mathrm{d}F(v_i) + \int_{d_j^{u*}+\frac{k}{c}}^{\overline{v}} \left(v_i - \frac{k}{c}\right) \frac{\varphi_i^*}{1-c} \mathrm{d}F(v_i) \right] \\
& - \sum_{i=1}^{n} \int_{\underline{v}}^{d^{l}+\frac{k}{c}} \left(v_i - \frac{k}{c}\right) \varphi_i \mathrm{d}F(v_i) - \sum_{i=j+1}^{n} \int_{d^{l}+\frac{k}{c}}^{\overline{v}} \left(v_i - \frac{k}{c}\right) \frac{\varphi_i}{1-c} \mathrm{d}F(v_i) \\
& - \sum_{i=1}^{j} \left[\int_{d^{l}+\frac{k}{c}}^{d_j^{u}+\frac{k}{c}} \left(v_i - \frac{k}{c}\right) F(v_i)^{j-1} \mathrm{d}F(v_i) + \int_{d_j^{u}+\frac{k}{c}}^{\overline{v}} \left(v_i - \frac{k}{c}\right) \frac{\varphi_i}{1-c} \mathrm{d}F(v_i) \right] \\
=& \int_{d^{l}+\frac{k}{c}}^{d^{l^*}+\frac{k}{c}} \left(v - \frac{k}{c}\right) \sum_{i=1}^{n} \varphi_i \mathrm{d}F(v) + \int_{d^{l*}+\frac{k}{c}}^{d_j^{u*}+\frac{k}{c}} \left(v - \frac{k}{c}\right) nF(v)^{n-1} \mathrm{d}F(v) + \int_{d_j^{u*}+\frac{k}{c}}^{d_j^{u}+\frac{k}{c}} \left(v - \frac{k}{c}\right) \sum_{i=1}^{n} \frac{\varphi_i}{1-c} \mathrm{d}F(v) \\
& - \int_{d^{l}+\frac{k}{c}}^{d_j^{u}+\frac{k}{c}} \left(v - \frac{k}{c}\right) \sum_{i=j+1}^{n} \frac{\varphi_i}{1-c} \mathrm{d}F(v) - \int_{d^{l}+\frac{k}{c}}^{d_j^{u}+\frac{k}{c}} \left(v - \frac{k}{c}\right) jF(v)^{j-1} \mathrm{d}F(v) \\
=& d_j^u \left[F\left(d_j^u + \frac{k}{c}\right) \sum_{i=1}^{j} \frac{\varphi_i}{1-c} - F\left(d_j^u + \frac{k}{c}\right)^j \right] + d_j^{u*} \left[F\left(d_j^{u*} + \frac{k}{c}\right)^n - F\left(d_j^{u*} + \frac{k}{c}\right) \sum_{i=1}^{n} \frac{\varphi_i}{1-c} \right] \\
& + d^{l*} \left[F\left(d^{l*} + \frac{k}{c}\right) \sum_{i=1}^{n} \varphi_i - F\left(d^{l*} + \frac{k}{c}\right)^n \right] \\
& + d^{l} \left[-F\left(d^{l} + \frac{k}{c}\right) \sum_{i=1}^{n} \varphi_i + F\left(d^{l} + \frac{k}{c}\right) \sum_{i=j+1}^{n} \frac{\varphi_i}{1-c} + F\left(d^{l} + \frac{k}{c}\right)^j \right] \\
& - \int_{d^{l}+\frac{k}{c}}^{d^{l*}+\frac{k}{c}} F(v) \sum_{i=1}^{n} \varphi_i \mathrm{d}v - \int_{d^{l*}+\frac{k}{c}}^{d_j^{u*}+\frac{k}{c}} F(v)^n \mathrm{d}v - \int_{d_j^{u}+\frac{k}{c}}^{d_j^{u*}+\frac{k}{c}} F(v) \sum_{i=1}^{n} \frac{\varphi_i}{1-c} \mathrm{d}v \\
& + \int_{d^{l}+\frac{k}{c}}^{d_j^{u}+\frac{k}{c}} \left[F(v) \sum_{i=j+1}^{n} \frac{\varphi_i}{1-c} + F(v)^j \right] \mathrm{d}v,
\end{aligned}
$$

where the second equality holds because $\sum_{i=1}^{n} \varphi_i^* = \sum_{i=1}^{n} \varphi_i$ and the last equality holds by integration by parts. Because d_j^{u*} satisfies (39), d^{l*} satisfies (40), d_j^u satisfies that

$$
1 - F\left(d_j^u + \frac{k}{c}\right)^j = \sum_{i=j+1}^{n} \frac{\varphi_i}{1-c} \left[1 - F\left(d_j^u + \frac{k}{c}\right)\right],
$$

$$
1 - F(v)^j < \sum_{i=j+1}^{n} \frac{\varphi_i}{1-c} \left[1 - F(v)\right], \forall v < d_j^u,
$$

and d^l satisfies that

$$1-F\left(d^l+\frac{k}{c}\right)^j+\sum_{i=j+1}^{n}\frac{\varphi_i}{1-c}\left[1-F\left(d^l+\frac{k}{c}\right)\right]+\sum_{i=1}^{n}\varphi_i F\left(d^l+\frac{k}{c}\right)=1,$$

$$1-F(v)^j+\sum_{i=j+1}^{n}\frac{\varphi_i}{1-c}\left[1-F(v)\right]+\sum_{i=1}^{n}\varphi_i F(v)<1,\forall v>d^l$$

$$F(v)^j-F(v)^n>\left[1-F(v)\right]\sum_{i=j+1}^{n}\frac{\varphi_i}{1-c},\forall v>d^l=d^u_{j+1},$$

we have

$$\begin{aligned}&Z(\boldsymbol{\varphi}^*)-Z(\boldsymbol{\varphi})\\ >&d^u_j\left(\sum_{i=1}^{j}\frac{\varphi_i}{1-c}-1\right)+d^{u*}_j\left(1-\sum_{i=1}^{n}\frac{\varphi_i}{1-c}\right)+d^l\sum_{i=j+1}^{n}\frac{\varphi_i}{1-c}\\ &-\int_{d^l+\frac{k}{c}}^{d^{l*}+\frac{k}{c}}\sum_{i=j+1}^{n}\frac{\varphi_i}{1-c}\mathrm{d}v-\int_{d^{l*}+\frac{k}{c}}^{d^{u*}_j+\frac{k}{c}}\sum_{i=j+1}^{n}\frac{\varphi_i}{1-c}\mathrm{d}v-\int_{d^u_j+\frac{k}{c}}^{d^{u*}_j+\frac{k}{c}}\left(\sum_{i=1}^{j}\frac{\varphi_i}{1-c}-1\right)\mathrm{d}v=0,\end{aligned}$$

which is a contradiction to the optimality of $\boldsymbol{\varphi}$.

(ii) Suppose that $d^{l*}>d^{u*}_j$. In this case, redefine $d^{l*}=d^{u*}_j$ such that

$$\sum_{i=1}^{n}\left[\varphi_i^* F\left(d^{l*}+\frac{k}{c}\right)+\frac{\varphi_i^*}{1-c}\left(1-F\left(d^{l*}+\frac{k}{c}\right)\right)\right]=1.$$

Then $d^l<d^{l*}=d^{u*}_j<d^u_j$. By a similar argument to that in case (ii), we can show that $Z(\boldsymbol{\varphi}^*)-Z(\boldsymbol{\varphi})>0$, which is a contradiction to the optimality of $\boldsymbol{\varphi}$.

Case 2: $\varphi_i\geq 1-c$ **for all** $i\leq j$**.** If $\varphi_1^*\geq 1-c$, then let $d^{u*}_1=\ldots d^{u*}_n=\overline{d}$ and d^{l*} be defined by (40). By a similar argument to that in Case 1, we can show that $Z(\boldsymbol{\varphi}^*)-Z(\boldsymbol{\varphi})>0$, which is a contradiction to the optimality of $\boldsymbol{\varphi}$.

If $\varphi_1^*<1-c$, then let $d^{u*}_1=\ldots d^{u*}_n=\overline{d}$ be defined by (39) and d^{l*} be defined by (40). Note that if $j=1$ and $\varphi_1/(1-c)=1$, then the new mechanism using $\boldsymbol{\varphi}^*$ coincides with the old mechanism using $\boldsymbol{\varphi}$. In this case, we can redefine $d^u_1:=d^l$ without changing the mechanism. Except for this case, we can show, by a similar argument to that in Case 1, that $Z(\boldsymbol{\varphi}^*)-Z(\boldsymbol{\varphi})>0$, which is a contradiction to the optimality of $\boldsymbol{\varphi}$.

Hence, $j=0$ or n.

Case 1: $j = 0$**.** In this case, for all i,

$$P_i^*(v_i) = \begin{cases} \frac{\varphi_i}{1-c} & \text{if } v_i \geq d^l + \frac{k}{c} \\ \varphi_i & \text{if } v_i < d^l + \frac{k}{c} \end{cases}.$$

is an optimal mechanism given $\boldsymbol{\varphi}$. Furthermore, $\boldsymbol{\varphi}$ and d^l must satisfy

$$\left[1 - F\left(d^l + \frac{k}{c}\right)\right] \sum_{j=1}^{i} \frac{\varphi_j}{1-c} \leq 1 - F\left(d^l + \frac{k}{c}\right)^i, \forall i \leq n, \tag{41}$$

$$F\left(d^l + \frac{k}{c}\right) \sum_{i=1}^{n} \varphi_i + \left[1 - F\left(d^l + \frac{k}{c}\right)\right] \sum_{i=1}^{n} \frac{\varphi_i}{1-c} = 1. \tag{42}$$

In particular, (41) holds for $i = n$, which implies

$$\sum_{i=1}^{n} \frac{\varphi_i}{1-c} \leq \frac{1 - F\left(d^l + \frac{k}{c}\right)^n}{1 - F\left(d^l + \frac{k}{c}\right)}.$$

Substituting this into (42) yields

$$F\left(d^l + \frac{k}{c}\right)^{n-1} \leq \sum_{i=1}^{n} \varphi_i \leq \frac{(1-c)\left[1 - F\left(d^l + \frac{k}{c}\right)^n\right]}{1 - F\left(d^l + \frac{k}{c}\right)}.$$

By the proof of the second part in Theorem 3, $j = 0$ is optimal if $v^{**} \leq v^\natural$, in which case the optimal $d^l = d_1^u = \cdots = d_n^u = v^{**} - \frac{k}{c}$. The set of optimal $\boldsymbol{\varphi}$ is given by $\Phi(d^l, d_1^u, \ldots, d_n^u)$. Clearly, $\boldsymbol{\varphi} \in \Phi$ if and only if $\boldsymbol{\varphi}$ satisfies conditions (41) and (42). Because $v^{**} \leq v^\natural$ implies that

$$1 \leq \frac{1}{1 - cF(v^{**})} \leq \frac{1 - F(v^{**})^n}{1 - F(v^{**})},$$

there exists $1 \leq h \leq n$ such that

$$\frac{1 - F(v^{**})^{h-1}}{1 - F(v^{**})} \leq \frac{1}{1 - cF(v^{**})} < \frac{1 - F(v^{**})^h}{1 - F(v^{**})}.$$

Hence, for all $i > h$, (41) holds if (42) holds. Given this, it is easy to see that the set of optimal $\boldsymbol{\varphi}$ is the convex hull of

$$\left\{ \boldsymbol{\varphi} \;\middle|\; \begin{array}{l} \varphi_{i_j} = (1-c)F(v^{**})^{j-1} \text{ if } j \leq h-1, \varphi_{i_h} = \frac{1-c}{1-cF(v^{**})} - \sum_{j=1}^{h-1}(1-c)F(v^{**})^{j-1}, \\ \varphi_{i_j} = 0 \text{ if } j \geq h+1 \text{ and } (i_1, \ldots, i_n) \text{ is a permutation of } (1, \ldots, n) \end{array} \right\}.$$

Case 2: $j = n$**.** In this case, let $d^u := d^u_1 = \cdots = d^u_n$, and

$$P^*_i(v_i) = \begin{cases} \frac{\varphi_i}{1-c} & \text{if } v_i \geq d^u + \frac{k}{c} \\ F(v)^{n-1} & \text{if } d^l + \frac{k}{c} < v_i < d^u + \frac{k}{c} \\ \varphi_i & \text{if } v_i \leq d^l + \frac{k}{c} \end{cases}.$$

Furthermore, $\boldsymbol{\varphi}$, d^l and d^u must satisfy that

$$\left[1 - F\left(d^u + \frac{k}{c}\right)\right] \sum_{j=1}^{i} \frac{\varphi_j}{1-c} \leq 1 - F\left(d^u + \frac{k}{c}\right)^i, \forall i \leq n-1, \tag{43}$$

$$\left[1 - F\left(d^u + \frac{k}{c}\right)\right] \sum_{i=1}^{n} \frac{\varphi_i}{1-c} = 1 - F\left(d^u + \frac{k}{c}\right)^n, \tag{44}$$

$$F\left(d^l + \frac{k}{c}\right) \sum_{i=1}^{n} \varphi_i = F\left(d^l + \frac{k}{c}\right)^n. \tag{45}$$

(44) and (45) imply that d^l and d^u satisfy that

$$\frac{1 - F\left(d^u + \frac{k}{c}\right)^n}{1 - F\left(d^u + \frac{k}{c}\right)} = \frac{F\left(d^l + \frac{k}{c}\right)^{n-1}}{1-c}.$$

By the proof of the third part in Theorem 3, $j = n$ is optimal if $v^{**} > v^{\natural}$, in which case the optimal $d^l = v^l(\varphi^*) - \frac{k}{c}$ and the optimal $d^u_1 = \cdots = d^u_n = v^u(\varphi^*) - \frac{k}{c}$. The set of optimal $\boldsymbol{\varphi}$ is given by $\Phi(d^l, d^u_1, \ldots, d^u_n)$. Clearly, $\boldsymbol{\varphi} \in \Phi$ if and only if $\boldsymbol{\varphi}$ satisfies conditions (43)-(45). It is easy to see that Φ is the convex hull of

$$\left\{\boldsymbol{\varphi} \,\middle|\, \varphi_{i_j} = (1-c)F(v^u(\varphi^*))^{j-1} \ \forall j \text{ and } (i_1, \ldots, i_n) \text{ is a permutation of } (1, \ldots, n)\right\}.$$

■

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