CAPITAL ACCUMULATION UNDER NON-GAUSSIAN PROCESSES AND THE MARSHALLIAN LAW

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Abstract. We consider a risk-neutral, price-taking and value-maximizing firm under demand uncertainty. The firm chooses optimal investment strategies; the investment is irreversible. For a wide family of non-Gaussian processes, we derive an explicit formula for the boundary of the inaction region by using the Wiener-Hopf factorization method. As an application of the method, we suggest a Marshallian-like form for the investment rule. It is applicable when the price can move in both directions, and uses the infimum process of the price instead of the price process itself. We also write down an analytic formula for the expected level of the capital stock in terms of the infimum and supremum processes. Both results are new even for the Gaussian case.

1. Introduction

Consider a risk-neutral, competitive firm, maximizing its present value net of installation cost of capital, whose manager contemplates an increase of the capital stock. Assume that G, the production function of the firm, is differentiable, increasing, concave and satisfies the Inada conditions, and that investment is irreversible. For simplicity, we assume that all the uncertainty is on the demand side, i.e. the price of a unit of the firm's output, P, is stochastic, and the marginal cost of the capital, $C'(K)$, is constant and normalized to unity. A discount rate, r, is set to be constant as well.

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In the previous formulations, the price process $P = \{P_t\}$ was assumed to be Gaussian, the leading example being the log-normal process: $P_t = \exp X_t$, where X is a Gaussian process. However it is well documented (see, for example, Yang and Brorsen (1992) or Deaton and Laroque (1992)) that Gaussian models do not give very good fit to empirical data since the latter exhibit significant skewness and kurtosis, nothing to say about apparent fat tails of probability distribution functions. To capture these effects, several remedies have been suggested, the most popular ones being stochastic volatility models and jump-diffusion models (see Duffie (1996) and Dixit and Pindyck (1996)).

Stochastic volatility models introduce additional stochastic factors which are unobservable and hence the models are somewhat arbitrary. More severely, the addition of further stochastic factors increases the dimensionality of the problem and encumbers computation implementation. In order that jump volatility models can be fitted well to empirical data, many jumps must be incorporated, which makes the models not very efficient from both the analytical and computational points of view.

During the 90th, in Finance literature, there have appeared several types of non-Gaussian models which are

(i) capable of capturing non-Gaussian properties of real data,
(ii) almost as tractable as Gaussian models, and
(iii) can be applied to models in the theory of Investment under Uncertainty.

The first goal of this paper is to introduce this type of models, which belong to a class of Lévy processes (i.e. processes with stationary independent increments; for rigorous definition, see e.g. Bertoin (1996) and Sato (1999)), into the field of Real Options and show that they are tractable so that explicit analytical results can be obtained. The second goal is to demonstrate that in the non-Gaussian world, an adequate technique is the Wiener-Hopf factorization method.

Several useful results emerge when one uses the Wiener-Hopf method. The most operational one from the standpoint of economic applications is a modified version of the Marshallian law. About a century ago, Marshall suggested a rule that a firm should

> *invest as long as the expected marginal revenue is not less than the marginal cost of investment.*

However, as Dixit and Pindyck (1996) pointed it out, this rule does not take into consideration option-like characteristics of investment opportunities.

If the price process is non-decreasing almost surely, the irreversibility effect does not matter, and the Marshallian law is correct. Similarly, if the price process is non-increasing almost surely in t, the Marshallian law is evidently correct: the firm contemplating entering the market, installs the initial capital according to the standard Marshallian rule. Here, the Inada conditions at zero ensure that the entrance is always optimal, and since the current market conditions can only deteriorate, the firm will never increase the capital stock further. Thus in the sequel, we consider only processes, when the price can move in both directions with non-zero probability[1]. We analyze the optimal investment rule for a firm, which has too much capital for a current price level, since it suffered an adverse shock in the past. Should the price increase, it may become optimal to increase the capital. The methods we introduce in this paper allow us to restate the Marshallian law as follows. Starting with the original price process, define a new process, called the infimum process for the price of the firm's output: $N_t = \inf_{0\le s\le t} P_t$. Then $H(K)$, the current level of the price, which triggers the new investment for the firm, is determined from

$$E\left[\int_0^\infty e^{-rt} N_t G'(K)dt \mid N_0 = H(K)\right] = C'(K), \tag{1}$$

i.e., this is the same Marshallian law but with the infimum process started at the current level of the price instead of the initial price process. Thus, the correct investment rule is:

> *in the formula for the profit function, replace the price process with the infimum process started at the current level of the price and invest as long as the expected marginal revenue is not less than the marginal cost of investment.*

Notice that this investment rule reflects the "bad news principle", which was first spelled out by Bernanke (1983). The critical price which triggers new investment depends on downward moves, because the ability to avoid the consequences of "bad news" leads us to wait. The main advantage of our investment rule is that it obviates the need to introduce a correction factor, which solves the so-called "fundamental quadratic" equation, like in Dixit and Pindyck (1996). In other words, the firm's manager may remain in the Marshallian world, provided she has in mind infimum processes instead of real ones.

[1] This assumption was missing from the first draft of the paper, and I am grateful to J. M. Harrison for pointing out that the corrected Marshallian law does not hold for a deterministic process with a negative drift.

This implies, in particular, that the initial Marshallian law is the correct prescription either for a very gloomy world where there is no hope for future price increases and hence, for further increases of the capital, or for very cheerful one, where only price increases are anticipated.

The suggested form shows that in making the investment decisions under uncertainty, the anticipation about future negative shocks mostly matter. In addition, it allows us to separate two effects of the increasing uncertainty on the capital accumulation, working in the opposite directions: the increase of the downward uncertainty, measured by the infimum process, increases the investment threshold thereby decreasing the optimal level of the capital stock, but the increase of the upward uncertainty, measured by an increasing function of the supremum process, $M_t = \sup_{0\le s\le t} P_t$, increases the expected capital stock each instant the investment threshold is crossed. We obtain an analytical formula for the expected value $E[K_t]$ of the capital stock at time t, K_t, in which these contributions are factored out. We assume that the firm is newborn, the price process P starts at $P_0 = 1$, the marginal cost is constant and normalized to unity, and the production function is $G(K) = dK^\theta$, where $0 < \theta < 1$. Then

$$E[K_t] = (\theta d W_-)^{1/(1-\theta)} W_+(t), \tag{2}$$

where W_- and $W_+(t)$ are determined by the infimum process and supremum one, respectively:

$$W_- = E\left[\int_0^{+\infty} e^{-rt} N_t dt \mid N_0 = 1\right],$$

$$W_+(t) = E[M_t^{1/(1-\theta)} \mid M_0 = 1];$$

both processes are assumed to be non-trivial. For a similar factorization formula, in different terms, in a model of a firm which chooses both capital and labor under the Gaussian process, see Abel and Eberly (1999). As in Abel and Eberly (1999), we obtain explicit formulas for W_- and $W_+(t)$; the formula for $W_+(t)$ is fairly complicated but we show that for large t a good approximate formula is valid

$$W_+(t) = HE(\infty)e^{at} + O(e^{\epsilon t}),$$

where $HE(\infty)$ is given by a much simpler expression than $W_+(t)$ itself, $\epsilon > 0$ is arbitrary, and $a \equiv t^{-1} \ln E[P_t^{1/(1-\theta)} \mid P_0 = 1]$; a is assumed positive. If the last condition fails, then in the model of the completely reversible investment the expected level of the capital decreases with time. We study the dependence of $E[K_t]$, W_-, $W_+(t)$, a and $HE(\infty)$ on parameters of the process.

Following Abel and Eberly (1999), we compare (2) with the corresponding formula for the capital accumulation in the case of completely reversible investment

$$E[K_t^R] = (\theta d W_-^R)^{1/(1-\theta)} W_+^R(t),$$

where

$$W_-^R = E\left[\int_0^{+\infty} e^{-rt} P_t dt \mid P_0 = 1\right],$$

$$W_+^R(t) = E[P_t^{1/(1-\theta)} \mid P_0 = 1] = e^{at},$$

by considering the ratio of the expected levels of accumulated capital

$$\kappa(t) = \frac{E[K_t]}{E[K_t^R]} = UC \cdot HE(t),$$

where

$$UC = (W_-/W_-^R)^{1/(1-\theta)}$$

is the user-cost effect, and

$$HE(t) = \frac{E[M_t^{1/(1-\theta)} \mid M_0 = 1]}{E[P_t^{1/(1-\theta)} \mid P_0 = 1]}$$

– the hangover effect of the irreversibility, and study how both these effects and the ratio $\kappa(t)$ itself depend on parameters of the process. Clearly, $UC < 1$ (unless the process is non-decreasing), and $HE(t) > 1$ (unless the process is non-increasing), so the both effects work in the opposite directions, and the joint effect on the capital accumulation is ambiguous: $\kappa(t)$ can be larger or smaller than 1, depending on parameters of the process.

It is worth stressing again that the method suggested here not only provides a useful shortcut in the Gaussian case, which has already been worked out with the help of different methods, but also in the non-Gaussian case, where the investment rule has not been suggested so far, and the capital accumulation has not been studied. It can be shown, that similar arguments apply when one considers an investor assessing a new project or a firm entering the market; or a firm, planning the exit from the market; in all cases, the results can be formulated in terms of the supremum and infimum processes. To sum up, we may conclude that

> *the uncertainty relevant to investment decision-making is better represented by a pair: supremum–infimum processes than the process itself.*

This implies that attempts to find just one proxy for the uncertainty are hopeless and explains why the influence of the uncertainty is ambiguous.

Now we are going to explain in more detail the specifics of the new approach. The first thing that must be said about realistic models for shocks is that jumps of essentially any size have to be admitted, and hence the distribution of jumps should be described by an appropriate density. The concept of the density of jumps is well-defined for Lévy processes. It is denoted $F(dx)$ and called the Lévy density of a Lévy process X_t. The process can be uniquely defined by its generating triplet $(\sigma^2, b, F(dx))$, where σ^2 is the variance of the Gaussian diffusion component of the process, b is the drift, and the Lévy density $F(dx)$ must satisfy

$$\int_{-\infty}^{+\infty} \min\{1, x^2\} F(dx) < +\infty.$$

Notice that the process is Gaussian if and only if the Lévy density is zero. If the variance $\sigma^2 = 0$, then we have a pure non-Gaussian process.

Equivalently, a process can be defined by the characteristic exponent, ψ, which appears in the formula $E[e^{ikX_t}] = e^{-t\psi(k)}$. The Lévy-Khintchine formula

$$\psi(k) = \frac{\sigma^2}{2}k^2 - ibk + \int_{-\infty}^{+\infty} \left(1 - e^{iky} + iky\mathbf{1}_{|y|\le 1}(y)\right) F(dy) \tag{3}$$

relates the characteristic exponent and the generating triplet (see e.g. Bertoin (1996) and Sato (1999)).

The class of Lévy processes includes the Brownian motion and stable Lévy processes. We have pointed out already several disadvantages of the former, and the latter suggested as a model for stock returns by Mandelbrodt (1963) (see also Fama (1965)), are inadequate because their PDFs have too fat tails (polynomially decaying) and infinite second moments. Much more realistic models can be obtained with exponentially decaying Lévy densities, having a polynomial singularity at the origin, and several groups of researchers used different models of this sort in empirical studies of Financial Markets (see e.g. Barndorff-Nielsen (1998), Cont et al (1997), Eberlein et al (1998) and Madan et al (1999)). In Boyarchenko and Levendorskiĭ (1999, 2000b, 2000c), we described the main common properties these processes satisfy, and introduced a general class containing all of them as special cases[2]. We will

[2]We used the name Generalized Truncated Lévy Processes then but recently, Barndorff-Nielsen and Levendorskiĭ (2000) introduced a similar class of Feller processes, and suggested to call them regular Feller-Lévy processes of exponential type

show that this family of processes is analytically tractable for problems of investment under uncertainty.

We are going to solve the firm's problem here by using essentially the same approach as in Boyarchenko and Levendorskiǐ (2000a, 2000b), where the optimal exercise prices and rational prices of perpetual American options have been calculated; the formula (2) for the capital accumulation is an easy corollary. In the above paper, first, we establish some general properties of the price of the option, next we take any candidate for the exercise price and calculate the corresponding candidate for the price of the option by applying the Wiener-Hopf method, then we show that there is only one candidate, for which certain conditions can be satisfied, and finally, all the conditions are verified for this candidate.[3]

The aforementioned conditions have been known for the optimal stopping problem for perpetual American options for a long time. Here to solve the firm's problem we use sufficient conditions derived for the case of a general Markov process in Oksendal (2000), who applied them to diffusions in the one-dimensional case. We apply the same conditions to the case of regular Lévy processes of exponential type.

The rest of the paper is organized as follows. In Section 2, we provide the definition of the class of regular Lévy processes of exponential type together with some examples. In Section 3, the firm's problem for a general Lévy process is solved by using the Wiener-Hopf factorization theorem in a general stochastic form (Sato 1999), modulo some technical conditions, which are verified in the Appendix for regular Lévy processes of exponential type. We show that in the Gaussian case, our formula for the investment threshold coincides with the one in Dixit and Pindyck (1996).

The result makes sense for any Lévy process, and it is plausible that it can be proven in the full generality. On the other hand, the results obtained in this full generality are essentially formulated in terms of stochastic integrals of the supremum and infimum processes for the underlying process, and hence can in no case be regarded as final. Still, the stochastic version of the result allows one to state a modification of the Marshallian law.

In Section 4, we study the capital accumulation. In Section 5, we discuss numerical results, and possible applications and extensions of the

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and replace the name Generalized Truncated Lévy Processes with a new name *regular Lévy processes of exponential type.*

[3]Notice that in exactly the same way one can solve corresponding problems in Real Options theory.

model. Section 6 concludes. Explicit formulas (involving Riemann not stochastic integrals) for the investment threshold, the capital accumulation and the user-cost and hangover effects are derived in the Appendix by using analytical representation of the factors in the Wiener-Hopf factorization formula.

2. Regular Lévy processes of exponential type

Probably, the most natural way to explain why one should expect the appearance of non-Gaussian Lévy processes in economic reality, is a concept of the random business time. One may easily notice that relatively calm periods of business activity are randomly followed by the ones of hectic activity, so one may presume that the time process relevant for economics is not a usual one, and must be considered as a stochastic process, call it Z_t (for more discussion of this concept, see Geman, Madan and Yor[4] (1998)). So, if one believes that underlying process for the logarithm of the price of a unit of the output is a Brownian motion in the business time, one should use as a real process $X_t = Y_{Z_t}$, where Y_t is the Brownian motion. In order that this construction could work, Z_t must be a subordinator, i.e. an increasing process on $\mathbf{R}_+$, which trajectories do not reach infinity in the finite time almost surely (for the rigorous definition, see Bertoin (1996) and Sato (1999)). Let Ψ be its Laplace exponent:

$$E[e^{-uZ_t}] = e^{-t\Psi(u)}.$$

Let Y_t be a Lévy process with the characteristic exponent ψ_0, independent of Z_t. Then (see e.g. Theorem 30.1 in Sato (1999)) a process $X_t = Y_{Z_t}$ is a Lévy process with the characteristic exponent $\phi(k) = \Psi(\psi_0(k))$.

By applying this construction with a subordinator having the Laplace exponent $\Psi(u) = (d+u)^{\nu/2} - d^{\nu/2}$, where $d > 0$, $\nu \in (0,2)$ and Y_t – a Brownian motion with the characteristic exponent $\psi_0(\xi) = \frac{\sigma^2}{2}\xi^2 - ib\xi$ – we obtain a process with the characteristic exponent

$$\phi(k) = \left(d + \frac{\sigma^2}{2}k^2 - ibk\right)^{\nu/2} - d^{\nu/2}.$$

[4]As early as in 1973, Clark (1973) suggested that the distribution of the price change is subordinate to a normal distribution; he used the discrete time model. Geman, Madan and Yor used essentially the same concept of random business time in the continuous time model.

By making simple algebraic transformations and introducing new parameters, we can derive

$$\phi(k) = \delta[(\alpha^2 - (\beta + ik)^2)^{\nu/2} - (\alpha^2 - \beta^2)^{\nu/2}],$$

where $\delta > 0, \alpha > |\beta|$. Finally, by adding the drift, we arrive at a characteristic exponent

$$\psi(k) = -i\mu k + \delta[(\alpha^2 - (\beta + ik)^2)^{\nu/2} - (\alpha^2 - \beta^2)^{\nu/2}] \tag{4}$$

When $\nu = 1$, (4) becomes the characteristic exponent of the Normal Inverse Gaussian process constructed by Barndorff-Nielsen (1998).

The $\alpha - \beta$ describes the rate of exponential decay of the right tails of PDF, and $-\alpha - \beta$ describes the rate of exponential decay of the left tails of PDF $p_t(x)$, in the sense that

$$\ln p_t(x) \sim -(\alpha - \beta)x, \ x \to +\infty,$$

and

$$\ln p_t(x) \sim -(-\alpha - \beta)x, \ x \to -\infty.$$

Processes with exponentially decaying tails can be constructed without subordination as well, by using the Lévy-Khintchine formula directly. For instance, with a choice of the Lévy measure

$$F(dx) = \left(c_+ x_+^{-\nu-1} e^{-\lambda_+ x} + c_- x_-^{-\nu-1} e^{-\lambda_- x}\right) dx, \tag{5}$$

where $x_\pm = \max\{\pm x, 0\}$, $\nu \in (0, 2)$, $\nu \neq 1$, and $c_\pm, \lambda_+, -\lambda_-$ are positive parameters, we obtain characteristic exponents of the form

$$\begin{aligned} \psi(k) = & -i\mu k + c_+\Gamma(-\nu)[\lambda_+^\nu - (\lambda_+ + ik)^\nu] \\ & + c_-\Gamma(-\nu)[(-\lambda_-)^\nu - (-\lambda_- - ik)^\nu], \end{aligned} \tag{6}$$

where Γ is the Gamma-function. With $\lambda_+ = -\lambda_-$, these are characteristic exponents of processes of Koponen (1995) family; this restriction makes the model unrealistic for Economics and Finance since empirical PDF have usually fatter left tails than the right ones. A modification of Koponen's family with $\lambda_+ \neq -\lambda_-$ was introduced in Boyarchenko and Levendorskiĭ (1999, 2000a). Below, we will use (6) with $c_+ = c_-$ since the case $c_+ \neq c_-$ corresponds to processes with PDF asymmetric in the central part, which contradicts empirical data.

One can use (5) with $\nu = 0, 1$ as well but the characteristic exponents will be different from (6) then (see Boyarchenko and Levendorskiĭ (1999)); with $\nu = 0$, one obtains the characteristic exponents of the Variance Gamma Processes used by Madan and co-authors in a series of papers during 90th – see Madan et all (1998) and the bibliography therein.

In Boyarchenko and Levendorskiǐ (1999), it has been noticed that the characteristic exponents (4) and (6), and the ones of Hyperbolic Processes used by Eberlein et al (1998), exhibit the following common features (a definition, which differs a little from the original one, is used here in order to simplify the formulation):

a) the characteristic exponent, ψ, admits a representation

$$\psi(k) = -i\mu k + \phi(k), \tag{7}$$

where $\mu \in \mathbf{R}$, and ϕ satisfies the following two conditions:

b) there exist $\lambda_- \le -1 < 0 \le \lambda_+$ such that ϕ admits the analytic continuation into the strip $\Im k \in (\lambda_-, \lambda_+)$ and the continuous extension into the closed strip $\Im k \in [\lambda_-, \lambda_+]$;

c) there exist $c > 0, C > 0, \nu \in (0, 2]$ and $\nu_1 < \nu$ such that for all k in the strip $\Im k \in [\lambda_-, \lambda_+]$,

$$|\phi(k) - c|k|^\nu| \le C(1 + |k|)^{\nu_1}. \tag{8}$$

Parameter ν is called the order of the process. In Boyarchenko and Levendorskiǐ (1999), it was shown that if $\nu \ge 1$ or $\mu = 0$, ν coincides with the order of an operator in the Lévy-analog of the Black-Scholes equation, when stock returns follow the process X_t with the characteristic exponent ψ.

Following the suggestion made by Barndorff-Nielsen and Levendorskiǐ (2000), we will call a process with the characteristic exponent satisfying properties a)–c) a regular Lévy process of the order ν and the exponential type $[\lambda_-, \lambda_+]$. By using the Lévy-Khintchine formula (3) and the formula for the action of the infinitesimal generator, L, of the Lévy process X_t:

$$\begin{aligned} Lu(x) &= \frac{\sigma^2}{2}u''(x) + bu'(x) \\ &+ \int_{-\infty}^{+\infty} \left(u(x+y) - u(x) - yu'(x)\mathbf{1}_{|y|\le 1}(y)\right) F(dy), \end{aligned} \tag{9}$$

one easily computes the action of the infinitesimal generator on oscillating exponents:

$$Le^{ikx} = -\psi(k)e^{ikx}. \tag{10}$$

By using the Fourier transform

$$\hat{u}(k) = \int_{-\infty}^{+\infty} e^{-ikx}u(x)dx,$$

together with (9) and (10), we can define the action of L in a space of sufficiently good functions, e.g. C^∞ functions decaying at the infinity faster than any polynomial, together with all their derivatives, by

$$-Lu(x) = (2\pi)^{-1} \int_{-\infty}^{+\infty} e^{ixk}\psi(k)\hat{u}(k)dk. \tag{11}$$

Equation (11) means that $-L$ is a pseudo-differential operator (PDO) with the symbol $\psi(k)$, which is holomorphic in the strip $\Im k \in (\lambda_-, \lambda_+)$ and admits the continuous extension up to the boundary of the strip, and the very segment $[\lambda_-, \lambda_+]$ enters into the definition of a regular Lévy process of exponential type. If A is a PDO with the symbol a, one writes $A = a(x, D)$, and so we will write $L = -\psi(D)$; and use this notation from now on.[5]

3. The investment threshold

As we have already pointed out in the Introduction, we consider a process X with trajectories decreasing and increasing with non-zero probability, i.e. its probability density $\mu^t(dx)$ is non-zero on $(0, +\infty)$ and on $(-\infty, 0)$: $\mu^t(0, +\infty) > 0$, and $\mu^t(-\infty, 0) > 0$, for all $t > 0$. This condition is satisfied for a regular Lévy process unless it is a pure drift. One can see this by contradiction: if $\mu^t(0, +\infty) = 0$, then from the definition of the characteristic exponent $E[e^{ikX_t}] = e^{-t\psi(k)}$ it follows that ψ admits the analytic continuation into the lower half-plane. The characteristic exponent of a non-trivial regular Lévy process of order $\nu \in (0, 2)$ does not satisfy this property since $\phi(k) \sim c|k|^\nu$, as $k \to \infty$ in the strip $\Im k \in [\lambda_-, \lambda_+]$ – see (8). If the order of the process $\nu = 2$, the process contains non-trivial Gaussian component, but in this case, $\mu^t(0, +\infty) > 0$ as well. Similarly, $\mu^t(-\infty, 0) > 0$.

Suppose, $h = h(K)$ defines the boundary of the inaction region. Set

$$g(K, x) = E^x \left[\int_0^{+\infty} e^{-rt}(e^{X_t}G'(K) - r)\mathbf{1}_{(-\infty, h(K))}(X_t)dt \right].$$

A general characterization result for the boundary of the inaction region, obtained for the case of a general strong Markov process in Oksendal (2000) (see Definition 3.3 and sufficient conditions following it in the above paper), in our case can be formulated as follows:

Lemma 1. *Let the following conditions be satisfied:*
1) $g(K, x) \le 0$, *if* $x < h(K)$;
2) $g(K, x) = 0$, *if* $x = h(K)$;
3) if $e^x G'(K) - r < 0$, *then* $x < h(K)$;
4) $h(K_1) > h(K_2)$, *if* $K_1 > K_2$
Then $h = h(K)$ *is the boundary of the inaction region.*

[5] For the theory of pseudo-differential operators, see e.g. Eskin (1973).

To apply 1)–4), introduce

$$f(K,x) = (e^x G'(K) - r)\mathbf{1}_{(-\infty,h(K))}(x),$$

notice that $g(K,x) = (R_r f)(K,x)$ is the resolvent of f (K is assumed to be fixed here, and the resolvent operator acts with respect to the second variable), and use the following important result on the resolvent and generator (see Breiman (1968), p.342):

$$(r - L)g(K,x) = f(K,x), \quad x < h(K), \tag{12}$$

and

$$g(K,x) = 0, \quad x \geq h(K). \tag{13}$$

The result is valid provided g belongs to the domain of L.

Below we are going to consider the corresponding Wiener-Hopf equation on $(-\infty, h(K))$, i.e. (12) subject to (13), in appropriate Sobolev spaces of generalized functions (for details, see the Appendix). First we will find the unique solution in the sense of generalized functions for arbitrary $h(K)$, after that we will show that there is only one choice of a curve $x = h(K)$, which gives $g(K, \cdot)$ in the domain of L for all K, and these h and g satisfy 1)–4).

The standard tool of solving the Wiener-Hopf equation is the Wiener-Hopf factorization method. It uses the Wiener-Hopf factorization of $r + \psi(k)$, the symbol of $r - L$, the operator in (12). We formulate the factorization theorem from Sato (1999) for a general Lévy process first, in a form which does not give computationally effective formulas but is sufficient for the proof of our result; an explicit form will be given in the Appendix.

Sato (1999) uses a different definition of the characteristic exponent $E[e^{ikX_t}] = e^{t\psi(k)}$, hence $\psi = \psi_S$ in Sato (1999) and ψ here are related by $\psi_S(k) = -\psi(k)$. This explains the difference in formulation between Theorem 45.2 in Sato (1999) and its version below.

Theorem 2. *Let $r > 0$ be fixed and let $\mu^t(dx)$ be the probability density of X_t.*
There exists a unique pair of infinitely divisible distributions p_r^+ and p_r^- having drift 0 supported on $(-\infty, 0]$ and $[0, +\infty)$, respectively, such that their Fourier transforms $\phi_r^+(k)$ and $\phi_r^-(k)$ satisfy

$$r(r + \psi(k))^{-1} = \phi_r^+(k)\phi_r^-(k), \quad k \in \mathbf{R}. \tag{14}$$

The functions $\phi_r^\pm(k)$ have the following representations

$$\phi_r^+(k) = \exp\left[\int_0^{+\infty} t^{-1}e^{-rt}dt \int_0^{+\infty} (e^{ikx} - 1)\mu^t(dx)\right],$$

$$\phi_r^-(k) = \exp\left[\int_0^{+\infty} t^{-1}e^{-rt}dt \int_{-\infty}^0 (e^{ikx}-1)\mu^t(dx)\right].$$

If a solution to the Wiener-Hopf equation (12) exists and is unique, it is given by the formula

$$g = r^{-1}\phi_r^+(D)\mathbf{1}_{(-\infty,h(K)]}\phi_r^-(D)f \tag{15}$$

(see e.g. Eskin (1973)). For some operators and spaces of generalized functions, the solution of the Wiener-Hopf equation may not exist or be non-unique but this not the case here as it will be shown in the Appendix.

Since ϕ_r^- is the characteristic function of the probability distribution p_r^- supported at $[0,+\infty)$, we can conclude that, first,

$$\phi_r^-(D)f(x) = \int_{-\infty}^{+\infty} p_r^-(x-y)f(y)dy$$

(it follows from the definition of PDO and the fact that under the Fourier transform, the convolution $p_r^- * f$ of two functions p_r^- and f, i.e. an expression on the right hand side, becomes the multiplication of their Fourier images: $\widehat{p_r^- * f} = \widehat{p_r^-}\hat{f}$), and second, for $x \le h(K)$, the right hand side is equal to

$$\int_{-\infty}^{x} p_r^-(x-y)f(y)dy = \int_{-\infty}^{x} p_r^-(x-y)(e^yG'(K)-r)dy =$$

$$= \int_{-\infty}^{+\infty} p_r^-(x-y)(e^yG'(K)-r)dy = \phi_r^-(-i)G'(K)e^x - r.$$

By substituting into (15), we obtain

$$g(K,x) = r^{-1}\phi_r^+(D)\mathbf{1}_{(-\infty,h(K)]}w_K(x), \tag{16}$$

where $w_K(x) = \phi_r^-(-i)G'(K)e^x - r$, or equivalently,

$$g(K,x) = r^{-1}\int_{-\infty}^{h(K)} p_r^+(x-y)w_K(y)dy. \tag{17}$$

Determine $h(K)$ as a unique solution to the equation $w_K(x) = 0$, i.e.

$$h(K) = \ln[r/(\phi_r^-(-i)G'(K))]. \tag{18}$$

It will be shown in the Appendix, that with this choice of $h(K)$, $g(K,x)$ belongs to the domain of L. Further, w_K is continuous, negative on on $(-\infty,h(K))$ and vanishes at the end point. From (17), $g(K,x)$ is non-positive, the PDF p_r^+ being non-negative, and vanishes at the end point. Hence, conditions 1)–2) of Lemma 1 are satisfied; 4) follows since $G'(K)$ is decreasing, and to verify 3), we notice that

$$\phi_r^-(-i) = \int_{-\infty}^{+\infty} p_r^-(x)e^{-x}dx =$$

$$= \int_0^{+\infty} p_r^-(x)e^{-x}dx \le \int_0^{+\infty} p_r^-(x)dx = 1,$$

since p_r^- is the probability distribution, and therefore from $e^x G'(K) - r < 0$, it follows that $e^x G'(K)\phi_r^-(-i) - r < 0$. Since with $x = h(K)$ the last inequality turns into equality, we have $x < h(K)$.

We have proven (modulo verification of several technical points which are left for the Appendix), that the boundary $x = h(K)$ of the inaction region is determined by

$$\phi_r^-(-i)e^{h(K)}G'(K) = r.$$

Now we are going to explain the meaning of the abstract results presented above. Recall that r on the right hand side of the last equation is, in fact, r times $C'(K)$, the marginal cost of capital, which was assumed constant and normalized to 1, hence by dividing by r, we get

$$r^{-1}\phi_r^-(-i)e^{h(K)}G'(K) = C'(K). \tag{19}$$

Here comes the crucial step: define the infimum process $n_t = \inf_{0\le s\le t} X_s$. By using the formula (45.8) in Sato (1999):

$$r^{-1}\phi_r^-(k) = E\left[\int_0^{\infty} e^{-rt}e^{ikn_t}dt\right],$$

with $k = -i$, we conclude that

$$r^{-1}\phi_r^-(-i) = E\left[\int_0^{\infty} e^{-rt}e^{n_t}dt\right],$$

and rewrite (19) in the form

$$E\left[\int_0^{\infty} e^{-rt}e^{n_t}G'(K)dt \mid n_0 = h(K)\right] = C'(K).$$

This is exactly (1), the main result stated in the Introduction.

To conclude this Section, we are going to demonstrate that in the case of Gaussian processes our investment rule gives the prescription that is exactly the same as in Dixit and Pindyck (1996). Let X_t be the Brownian motion with the variance σ^2 and drift b. First, we compute the investment threshold by using the Marshallian law:

$$G'(K)H(K)E[\int_0^{\infty} e^{-rt}P_t dt \mid P_0 = 1] = C'(K),$$

or equivalently,

$$\frac{G'(K)H(K)}{r-b-\frac{\sigma^2}{2}} = C'(K),$$

$$H(K) = (r-b-\frac{\sigma^2}{2})\frac{C'(K)}{G'(K)}.$$

In Dixit and Pindyck (1996), the correction to the Marshallian law is derived by using the so-called fundamental quadratic

$$\frac{\sigma^2}{2}\lambda^2 + b\lambda - r = 0$$

which comes from

$$\frac{\sigma^2}{2}\frac{d^2 f}{dx^2} + b\frac{df}{dx} - rf = 0,$$

applied to $\exp(\lambda x)$. In our notation, the last equation is just

$$(L-r)f = 0.$$

Recall that

$$Le^{ixk} = -\psi(k)e^{ixk},$$

and apply it with $ik = \lambda$ which is $k = -i\lambda$:

$$Le^{\lambda x} = -\psi(-i\lambda)e^{\lambda x}.$$

We obtain that the fundamental quadratic can be rewritten as

$$-(r+\psi(-i\lambda)) = 0.$$

Let $\beta_+ > 0 > \beta_-$ be the roots of the fundamental quadratic. Then the characteristic polynomial is equal to

$$\frac{\sigma^2}{2}(\lambda-\beta_+)(\lambda-\beta_-).$$

If we substitute $\lambda = ik$ here, we obtain

$$r+\psi(k) = -\frac{\sigma^2}{2}(ik-\beta_+)(ik-\beta_-) =$$

$$= \frac{\sigma^2}{2}(\beta_+ - ik)(-\beta_- + ik).$$

Introduce the correction factor as in Dixit and Pindyck (1996):

$$H(K) = \frac{\beta_+}{\beta_+ - 1}(r-b-\frac{\sigma^2}{2})\frac{C'(K)}{G'(K)},$$

rewrite as

$$r^{-1}H(K) = \frac{\beta_+}{\beta_+ - 1}\cdot\frac{r-b-\frac{\sigma^2}{2}}{r}\cdot\frac{C'(K)}{G'(K)}, \tag{20}$$

and compare the result with the result of the present paper.

The factors in the Wiener-Hopf factorization (14) are uniquely defined by requirements:

ϕ_r^- (ϕ_r^+) is holomorphic in the lower (upper) half-plane and does not vanish there (due to the Liouville theorem, this fixes factors up to scalar multiples);

$\phi_r^\pm(0) = 1$, since they are characteristic functions of probability densities.

Now it is evident that

$$\phi_r^+(k) = \frac{\beta_+}{\beta_+ - ik}, \quad \phi_r^-(k) = \frac{-\beta_-}{-\beta_- + ik}. \tag{21}$$

Therefore the factor in Dixit-Pindyck formula (20) can be written as

$$\frac{\beta_+}{\beta_+ - 1} \cdot \frac{r + \psi(-i)}{r} = \phi_r^+(-i)(\phi_r^+(-i)\phi_r^-(-i))^{-1} = \frac{1}{\phi_r^-(-i)}.$$

By (19), the investment prescription in the present paper is given by

$$r^{-1}H(K) = \frac{1}{\phi_r^-(-i)} \cdot \frac{C'(K)}{G'(K)},$$

whence it is clear that the investment rules are identical.

We see that the factor $(\phi_r^-(-i))^{-1}$ "governs" the investment rule for the case of irreversible investment. Similarly, it can be shown that in the case of irreversible disinvestment, the factor $(\phi_r^+(-i))^{-1}$ governs. We also know that their product is just the factor

$$\frac{r + \psi(-i)}{r} = \frac{r - b - \frac{\sigma^2}{2}}{r}$$

in the Marshallian law, if we re-write the latter as

$$r^{-1}H(K) = \frac{r - b - \frac{\sigma^2}{2}}{r} \cdot \frac{C'(K)}{G'(K)}. \tag{22}$$

The last fact implies that the Marshallian law is correct when one can costlessly and continuously adjust ones capital in any direction (and hence, the investment and disinvestment are absolutely reversible). From the above exercise, we can notice that in the case of irreversible investment (respectively, disinvestment), the correction factor to the Marshallian law is $\phi_r^+(-i)$ (respectively, $\phi_r^-(-i)$), which is the inverse factor in the correct rule for irreversible disinvestment (respectively, investment).

4. Capital accumulation

Consider the capital accumulation of the firm born at $t = 0$ with the Cobb-Douglas production function $G(K) = dK^\theta$; the marginal cost is normalized to unity, both the infimum and supremum processes are non-trivial, and the initial level of the price is normalized to 1: $P_0 = 1$. Clearly, the capital stock increases when (and only when) the supremum process jumps, therefore from (18) we conclude that the capital stock evolves according to

$$M_t r^{-1}\phi_r^-(-i) = (d\theta K_t^{\theta-1})^{-1}.$$

By solving w.r.t. K_t:

$$K_t = (d\theta r^{-1}\phi_r^-(-i))^{1/(\theta-1)} M_t^{1/(\theta-1)}$$

and applying the expectation operator, we obtain (2). The first factor in (2), W_-, is expressed via the factor $\phi_r^-(-i)$ in the Wiener-Hopf factorization formula, and the second factor,

$$W_+(t) = E[e^{\omega m_t} \mid m_0 = 0],$$

where $m_t = \sup_{0\le s\le t} X_t$, and $\omega = 1/(1-\theta)$, can be found from the formula for its Laplace transform (see the equation (45.7) in Sato (1999)):

$$r^{-1}\phi_r^+(-i\omega) = \int_0^{+\infty} e^{-rt}W_+(t)dt. \tag{23}$$

In order that (23) be applicable, r must be so large that the integral in the RHS is finite, or equivalently, $\phi_r^+(k)$ must admit the analytic continuation into the half-plane $\Im k > -\omega$ and the continuous extension up to the boundary $\Im k = -\omega$, and be positive at the point $k = -i\omega$. Since $\phi_r^-(k)$ is holomorphic in the lower half-plane $\Im k < 0$, continuous in the closed half-plane $\Im k \le 0$, does not vanish there, and is positive on the imaginary negative half-axis, we can use (14) and obtain the holomorphic continuation of $\phi_r^+(k)$ into the strip where ψ is holomorphic:

$$r^{-1}\phi_r^+(k) = (r+\psi(k))^{-1}(\phi_r^-(k))^{-1}. \tag{24}$$

Suppose, $-i\omega$ is inside the strip (in real-world situations, it is usually the case since $\theta = 0.2 - 0.4$, hence $\omega \le 2$, and the width of the strip is much greater than 10 – see e.g. Eberlein et al. (1998)), therefore from (24), we conclude that (23) is applicable if

$$\psi(-i\omega) + r > 0. \tag{25}$$

This is a restriction on admissible values of parameters of the process. We also impose the restriction

$$\psi(-i\omega) < 0, \tag{26}$$

which is a necessary and sufficient condition for the expected level of the capital in the model of the completely reversible investment to grow with time since

$$E[P_t^\omega \mid P_0 = 1] = E[e^{\omega X_t} \mid X_0 = 0] = e^{at}, \tag{27}$$

where $a = -\psi(-i\omega)$. Apply the inverse Laplace transform to (23) and use (24); the result is

$$W_+(t) = (2\pi i)^{-1} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{ts}(s + \psi(-i\omega))^{-1}(\phi_s^-(-i\omega))^{-1} ds, \tag{28}$$

for any $\sigma \geq r$. From the equation (45.8) in Sato (1999), we conclude that

$$\phi_s^-(-i\omega) = s \int_0^{+\infty} e^{-st} E[e^{\omega n_t}] dt$$

is well-defined and holomorphic w.r.t. s in the half-plane $\Re s > 0$, and does not vanish there. Hence, the integrand in (28) is well-defined and holomorphic w.r.t. s in the half-plane $\Re s > 0$, with the only pole at $s = a$ – see (26). Clearly, the pole is simple, and therefore by shifting the line of integration to the left and using the residue theorem (its use can be justified for RLPE), we obtain

$$W_+(t) = e^{at} HE(\infty) + R(t), \tag{29}$$

where $HE(\infty) = 1/\phi_a^-(-i\omega)$, and

$$R(t) = (2\pi i)^{-1} \int_{\sigma_1 - i\infty}^{\sigma_1 + i\infty} e^{ts}(s + \psi(-i\omega))^{-1}(\phi_s^-(-i\omega))^{-1} ds, \tag{30}$$

for any $\sigma_1 \in (0, a)$. In the Appendix, we show that for RLPE, there exists $C = C(\sigma_1)$ such that

$$|R(t)| \leq C e^{\sigma_1 t} t^{-1}. \tag{31}$$

Hence, we can rewrite (29) as

$$W_+(t) = HE(\infty) e^{at} + O(e^{\sigma_1 t}), \quad t \to +\infty. \tag{32}$$

In the model of the completely reversible investment, the Marshallian law (22) is correct. Moreover, the capital stock changes each time the price changes (i.e., $H(K) = P$), therefore

$$W_-^R = (d\theta(r + \psi(-i))^{-1})^\omega,$$

and

$$W_+^R(t) = e^{at}.$$

Hence for the ratio of expected levels of the accumulated capital in the models of the irreversible investment and the reversible one, we obtain

$$\kappa(t) = \kappa(\infty) + O(e^{-t(a-\sigma_1)}), \tag{33}$$

for any $\sigma_1 > 0$, where $\kappa(\infty) = UC \cdot HE(\infty)$, and

$$UC = \left(\frac{r^{-1}\phi_r^-(-i)}{(r+\psi(-i))^{-1}}\right)^{\omega} = \phi_r^+(-i)^{-\omega}. \tag{34}$$

Numerical results will be given in Section 5, and we conclude the section by deriving explicit formulas for the case of the Gaussian process with the variance σ^2 and the drift b. For $s > 0$, let $\beta_\pm(s)$ be positive and negative roots of the "fundamental quadratic"

$$\frac{\sigma^2}{2}\lambda^2 + b\lambda - s = 0,$$

i.e.

$$\beta_\pm(s) = (-b \pm \sqrt{b^2 + 2s\sigma^2})/\sigma^2.$$

We have

$$a = -\psi(-i\omega) = \sigma^2\omega^2/2 + b\omega,$$

$$-\beta_-(a) = (b + \sqrt{b^2 + 2\sigma^2 b\omega + \sigma^4\omega^2})/\sigma^2 = (2b + \sigma^2\omega)/\sigma^2,$$

therefore by using (21), we obtain

$$UC = \left(\frac{\beta_+(r) - 1}{\beta_+}\right)^{\omega},$$

$$HE(\infty) = \left(\frac{-\beta_-(a)}{-\beta_-(a) + \omega}\right)^{-1} = \frac{2b + 2\sigma^2\omega}{2b + \sigma^2\omega}.$$

5. Possible applications and numerical examples

One of the straightforward applications of the model presented here is effective capital budgeting which is important to corporate survival. The real options literature to date has provided many insights into capital budgeting decision-making. The major result of the existing models of irreversible investment under uncertainty is that irreversibility increases the hurdle that projects must clear in order to be profitably undertaken.

Unfortunately, the current real option models are not widely used in corporate decision making. Among one of the primary reasons for that, Lander and Pinches (1999) point out that many of the required modeling assumptions are often violated in practical real options applications. In particular, this concerns the choice of the stochastic process for the underlying variable. As we already mentioned it in the Introduction, even though the normality of the process is rejected by empirical evidence, Gaussian processes are often used in the investment literature.

Due to the above, the first question which one may ask is how the investment threshold changes if one considers a regular Lévy process instead of a Gaussian process with the same first and second moments $\mu_{1,\Delta t}$ and $\mu_{2,\Delta t}$ of observed PDF $p_{\Delta t}(x)$ of the process X, for a chosen small time interval Δt; for small Δt, $\mu_{1,\Delta t}/\Delta t$ and $\mu_{2,\Delta t}/\Delta t$ are good proxies for the first two coefficients in the Taylor expansion of the characteristic exponent $\psi(k)$ at $k = 0$; they can be used to infer the parameters of a RLPE process (in the Gaussian modeling, they are the drift and variance).

To answer this question, we calculated the investment threshold for different regular Lévy processes of exponential type with the same limits (i.e. "drift" and "variance")

$$\lim_{\Delta t \downarrow 0} \mu_{1,\Delta t}/\Delta t, \quad \text{and} \quad \lim_{\Delta t \downarrow 0} \mu_{2,\Delta t}/\Delta t, \tag{35}$$

and we found that usually the threshold decreases (insignificantly) as we replace the Gaussian process with a regular Lévy processes of exponential type with the same "drift" and "variance"; only for "very non-Gaussian" processes of order close to zero we found sets of parameters for which the threshold increased. This result can be explained as follows. When one fits the Gaussian curve to real PDF, one is bound to disregard the extreme events, and so, in fact, the "drift" and "variance" for a Gaussian process inferred from the real data do not coincide with the ones in (35) which one obtains by using a larger portion of rare events than in the Gaussian modeling: (35) based on the larger data set, gives larger value of the variance and hence it is incorrect to compare Gaussian processes and non-Gaussian ones by using the first two moments – or even the first 3 or 4 moments, since all of them are suitable tools of describing the behavior of PDF near zero, being the coefficients of the Taylor series at zero.

Hence, much more natural question is: what can be an effect of taking into account more and more data on rare events. To answer this question, we consider a series of truncated Lévy processes with the characteristic exponent (6); we fix $r = 0.06, \mu = -0.05, c = c_- = c_+ = 0.15, \nu = 1.6$, and see how the moments $\lim_{\Delta t \downarrow 0} \mu_{j,\Delta t}/\Delta t$, $j = 2, 3, 4$ ("variance", "skewness" and "kurtosis") of the process and the investment threshold change with truncation parameters λ_+ and λ_-, which describe the fatness of the left and the right tail, respectively.

(Insert Fig.1 - Fig.4 here.)

The choice $\nu = 1.6$ means that near zero, the PDF does not differ too much from a Gaussian one, and the decreasing of λ_+ (respectively, $-\lambda_-$) can be interpreted as the taking into account the more and more of negative (respectively, positive) large jumps. We believe that the result of this computational experiment is instructive. We allow λ_- to vary from -30 to -22, and λ_+ from $-\lambda_-$ to $-\lambda_- - 8$, and we see that though the variance does not change much, the threshold changes more than four-fold; in particular, when $\lambda_- = -30$ remains fixed and λ_+ decreases from 30 to 22, the variance increases only by 7 percent, from 0.171 to 0.182, kurtosis by 55 percent, from 0.106×10^{-3} to 0.165×10^{-3}, and skewness changes from 0 to -0.0006, whereas the threshold increases by 423 percent. So, even this relatively small skewness and insignificant increase of variance (and a bit more significant increase of kurtosis), which can easily be disregarded in practice since they come mostly from the tails of PDF, can have a dramatic impact on the investment threshold.

The results of the numerical example highlight the importance of obtaining the correct estimates for the parameters of a regular Lévy process governing the underlying stochastic variable. In principle, it is possible to infer these parameters from the moments (up to the fourth order) of the corresponding probability distribution. However, this will provide a good fit for the central part of the distribution only, but not for the tails of the distribution. To obtain parameter estimates fitting the whole distribution, more sophisticated methods should be used (see, e.g. Barndorff-Nielsen and Shepard (2000) and the bibliography there).

Next problem examined in our model is the optimal amount of investment and long-run accumulation of capital. As it was stressed by Hubbard (1994), the benchmark models of investment under uncertainty introduced by Dixit and Pindyck (1996), do not suggest specific predictions about the level of investment. Since the investment rule itself is not observable, one has to use the data on investment and capital stock to evaluate investment models; this shows the importance of theoretical results on the long-run capital accumulation. Abel and Eberly (1999) examine the behavior of the capital stock in the long run and calculate explicitly the impacts of irreversibility and uncertainty on the expected long-run capital stock. They assume that an exogenous demand shock follows a geometric Brownian motion. Two types of effect are revealed: the user cost effect, which tends to reduce the capital stock, and the hangover effect, which arises because the irreversibility prevents a firm from selling capital when its marginal revenue product is low. Neither of these effects dominates globally, so the effect of increased uncertainty cannot be determined unambiguously.

Undoubtedly, the same problem has to be addressed in the case when the underlying stochastic variable evolves according to a regular Lévy process. Our formula (2), its realization (29) and corollaries (32)–(34) allows one to give quantitative answers. Consider the same firm and process as in the example for the investment threshold above, with $r = 0.06$, $\theta = 0.33$ and $d = r/\theta$ in the Cobb-Douglas function, $c = 0.15, \nu = 1.6$, and λ_+ and $-\lambda_-$ varying from 10 to 30. Following Abel and Eberly (1999), we determine the last parameter, μ, from the requirement that the expected rate of growth of the price remains constant:

$$\ln E[P_t \mid P_0 = 1]/t = \ln E[e^{X_t} \mid X_0 = 0]/t = -\psi(-i);$$

we set $-\psi(-i) = 0.05$, so that the requirement $r + \psi(-i) > 0$ is satisfied. In Fig. 5–13, we plot, respectively, variance, skewness, kurtosis, investment threshold, rate of the accumulation of the capital, user-cost effect, long-run hangover effect, long-run capital accumulation ratio, and expected capital accumulated during 10 years.

(Insert Fig.5 - Fig.13 here.)

The volatility is a convex decreasing function in $(\lambda_+, -\lambda_-)$, and the same holds for kurtosis, threshold, rate of capital accumulation, long-run hangover effect, and accumulated capital; on the contrary, user-cost effect and long-run capital accumulation ratio are concave increasing functions. It is seen that in this model, the user-cost effect dominates the long-run hangover effect, so that their product, the long-run capital accumulation ratio, satisfies the properties of the former but not the latter. Similarly, the effect of the decrease of the rate of accumulation dominates the effect of the irreversibility measured by the long-run capital accumulation ratio.

Now we consider errors which may arise, when one uses Gaussian models in non-Gaussian situations. In both the non-Gaussian and Gaussian models, we use the same expected rate of growth of the price, and for each set of parameters, which define a non-Gaussian process $X_t = \ln P_t$, we take the Gaussian process with the same variance as that of X_t. Numerical examples show that typically, when the skewness decreases, the threshold, the rate of growth of the capital and the capital accumulation in non-Gaussian case decrease as compared to the Gaussian case. In the example above these effects are not significant: of order 1 percent or less, but there the process does not deviate too much from the Gaussian: the order of the process, $\nu = 1.6$, is close to the Gaussian $\nu = 2$. Parameters $\lambda_+ \geq 10$ and $-\lambda_- \geq 10$, and so the tails of PDF decay as $\exp(-10|x|)$ or even faster.

In the next series of examples, we take $\nu = 1.1$, $c = 0.5$, and change λ_+ in a wider range from 2 to 30; λ_- changes in the former range from -30 to -10. So, here we allow for much fatter left tail. In Fig.14–19, we plot the ratios of the thresholds, the rates of growth of the capital, user-cost and hangover effects, and the capital accumulated in non-Gaussian and Gaussian models. We see that though the investment threshold in the non-Gaussian case decreases up to 15 percent, the rate of capital accumulation is depressed by the same 15 percent, and the capital accumulated during 10 years by 13 percent. This means that though the Gaussian model may be too pessimistic about the unobserved investment threshold, it certainly is too optimistic about observed quantities like: the rate of capital accumulation and the capital accumulated.

(Insert Fig.14 - Fig.19 here.)

The real-world processes may deviate from Gaussian ones even more (for instance, the left tail can be even fatter, and the figures clearly show that with the further decrease in λ_+, the capital accumulation will be depressed much more). This example illustrates that the capital accumulation is mainly depressed by the possibility of large downward jumps, which is not taken into account properly by Gaussian models.

Notice that there is a conceptual problem of proxy measures of uncertainty, which is not resolved so far even for investment models using Gaussian processes (see discussion in Carruth, Dickerson and Henley (2000)); our formulas (1)–(2) (and their analogs for disinvestment problems) suggest that the proper measure must be a couple of measures: for the supremum and infimum of the process. As the technical proxy in not too non-Gaussian situations, a pair volatility– the rate of the decay of the left tail can be used. Usually it is the left tail that is fat, and the rate of decay of the left tail gives us the probabilities of large downward jumps. This suggests one more possible application of our method.

The technique presented here can be applied to the problems of optimal capital structure, investment risk management and endogenous default (detailed discussions of existing treatments of these problems are given in Rogers (1999) and Zhou (1997)). Until recently, models of credit risk have relied almost exclusively on diffusion processes to model the evolution of firm value (see, for example, Leland (1994) or Leland and Toft (1996)). The diffusion approach predicts that the credit spreads on corporate bonds tend to zero, as the maturity tends to zero, which contradicts the empirical results. To capture the basic features of credit risk, it is necessary to allow for jumps; this can be achieved by modeling the evolution of the value of a firm as a Lévy process.

6. Conclusion

We suggested using a wide family of (non-Gaussian) regular Lévy processes of exponential type in models of Investment under Uncertainty. As an example, the problem of the gradual irreversible capacity expansion has been solved, for the case when the price can move in both directions with non-zero probability. It was shown that the result can be formulated as the Marshallian law but with the infimum process started at the current level of the price instead of the process for the price of the output itself. As an example, we considered the case of the Brownian Motion and recovered the optimal investment rule in Dixit and Pindyck (1996). We also obtain a formula for the capital accumulation, in the form of two factors, one depending on the infimum process, and the other one on the supremum process. Following Abel and Eberly (1999) we studied the effect of irreversibility by considering the ratios of these factors in the completely irreversible and completely reversible cases.

The main technical tool in the paper is the Wiener-Hopf factorization method. The method can also be applied to other models of the Investment under Uncertainty, for instance to entry and exit problems. In relatively simple cases the answers can be expressed in terms of the supremum and infimum processes; in more complicated cases of two thresholds, first approximations to the thresholds can be expressed in terms of these processes.

We produced a numerical example to show that if one fits a non-Gaussian process non-accurately to data, serious mistakes in the investment decisions may result, and examples to study the effects of the non-Gaussian uncertainty and irreversibility on the long-run accumulation of the capital. The examples clearly show that the non-Gaussianity makes the negative impact on the observed quantities like the rate of capital accumulation and the capital accumulated but not on the unobserved investment threshold.

More general regular Feller-Lévy processes of exponential type introduced recently by Barndorff-Nielsen and Levendorskiǐ (2000) can also be used. (These processes generalize regular Lévy processes of exponential type in the same spirit as diffusion processes with variable characteristics generalize the Brownian motion).

Appendix A. Explicit solutions

We start with the derivation of explicit analytic formulas for the factors in the Wiener-Hopf factorization formula for regular Lévy processes of the order $\nu \in (0,2]$ and the exponential type $[\lambda_-, \lambda_+]$. The formulas having been obtained in Boyarchenko and Levendorskiĭ (2000a, 2000b), here they will be given in cases $\nu \in (1,2]$ and $\nu \in (0,1], \mu = 0$; the case of $\nu \in (0,1]$ and $\mu \neq 0$ leads to more involved constructions. Notice that the case $\nu = 2$ includes Gaussian processes or allows for Gaussian components.

To simplify the construction, we assume that $\lambda_- \leq -1 < 0 < \lambda_+$, though the result obtains in the case $\lambda_- \leq 0 < 1 \leq \lambda_+$ as well. It can be shown (see Boyarchenko and Levendorskiĭ (1999), Lemma 2.6) that for any $\sigma_1 > 0$, there exist $\epsilon > 0$ and $\delta > 0$ such that for any s in the half-plane $\Re s \geq \sigma_1$, and any k in the strip $\Im k \in [-\epsilon, \epsilon]$,

$$\Re(s + \psi(k)) \geq \delta(1 + |k|)^\nu. \tag{36}$$

Set

$$B(s,k) \equiv c^{-1}(1 + k^2)^{-\nu}(s + \psi(k)), \tag{37}$$

where $c > 0$ is the constant in (8); then

$$\lim_{k \to \pm\infty} B(s,k) = 1. \tag{38}$$

Due to (36), $b(s,k) = \ln B(s,k)$ is well-defined for $\Im k \in [-\epsilon, \epsilon]$ and s in the half-plane $\Re s \geq \sigma_1$. For $\tau > -\epsilon$, $\tau_1 \in (-\epsilon, \tau)$ and real k, set

$$b_+(s, k + i\tau) = \frac{i}{2\pi} \int_{-\infty + i\tau_1}^{+\infty + i\tau_1} \frac{b(s,l)}{k + i\tau - l} dl. \tag{39}$$

Due to (8) and (38), the integral in (39) absolutely converges, and by the Cauchy theorem, $b_+(s, k + i\tau)$ is independent of a choice of τ_1. For $\tau < \epsilon$, $\tau_1 \in (\tau, \epsilon)$ and real k, set

$$b_-(s, k + i\tau) = -\frac{i}{2\pi} \int_{-\infty + i\tau_1}^{+\infty + i\tau_1} \frac{b(s,l)}{k + i\tau - l} dl. \tag{40}$$

Finally, for k in a half-plane $\pm \Im k \geq \mp\epsilon$ and the same s, set

$$a_\pm(s,k) = (1 \mp ik)^{\nu/2} \exp b_\pm(s,k). \tag{41}$$

In Boyarchenko and Levendorskiĭ (2000b), it has been shown that $a_\pm(s,k)$ and $\phi_s^\pm(k)$ in Theorem 2 are related by

$$\phi_s^\pm(k)^{-1} = a_\pm(s,k)/a_\pm(s,0). \tag{42}$$

(for $s > 0$; clearly, both sides are holomorphic in the half-plane $\Re s > 0$, and hence, (42) holds in this half-plane). By using (40) with $\tau_1 \in (0, \lambda_+)$, we obtain for any $\sigma > 0$

$$\begin{aligned}\phi_s^-(-i\sigma) = \frac{a_-(s,0)}{a_-(s,-i\sigma)} &= (1+\sigma)^{-\nu/2}\exp[b_-(s,0) - b_-(s,-i\sigma)]\\ &= (1+\sigma)^{-\nu/2}\exp\left[(2\pi i)^{-1}\int_{-\infty+i\tau_1}^{+\infty+i\tau_1}\left(\frac{b(s,l)}{-l} - \frac{b(s,l)}{-i\sigma - l}\right)dl\right]\\ &= (1+\sigma)^{-\nu/2}\exp\left[-\sigma(2\pi)^{-1}\int_{-\infty+i\tau_1}^{+\infty+i\tau_1}\frac{b(s,l)}{l(l+i\sigma)}dl\right],\end{aligned}$$

and finally,

$$\phi_s^-(-i\sigma)^{-1} =$$

$$= (1+\sigma)^{\nu/2}\exp\left[\sigma(2\pi)^{-1}\int_{-\infty+i\tau_1}^{+\infty+i\tau_1}\frac{\ln[c^{-1}(1+l^2)^{-\nu/2}(s+\psi(s,l))]}{l(l+i\sigma)}dl\right].$$

Now we can substitute into the formula (18) for the investment threshold in Section 3, and formulas $HE(\infty) = 1/\phi_a^-(-i\omega)$, (29)–(30) and (33)–(34), and obtain explicit results.

It remains to verify some technical points in the proof in Section 3. To simplify the consideration, we assume that there is no Gaussian component, i.e. $\nu < 2$ (it may seem strange, but the presence of a Gaussian component leads to some additional work).

Clearly, we may assume that for any $\epsilon > 0$, g, a solution to the problem (12)–(13), belongs to $L_2(\mathbf{R})$ with an exponential weight $e^{\epsilon x}$, i.e. $e^{x\epsilon/2}g \in L_2(\mathbf{R})$. By making the change $g(K,x) = e^{-x\epsilon/2}g_\epsilon(K,x)$, $f(x) = e^{-x\epsilon/2}f_\epsilon(K,x)$, multiplying (12) and (13) by $e^{x\epsilon/2}$ and noticing that $e^{x\epsilon/2}a(D)e^{-x\epsilon/2} = a(D+i\epsilon/2)$, we obtain the following problem, which is equivalent to (12)–(13):

$$(43)\qquad (r+\psi(D+i\epsilon/2))g_\epsilon(K,x) = f_\epsilon(K,x), \quad x < h(K),$$

$$(44)\qquad g_\epsilon(K,x) = 0, \quad x \ge h(K).$$

If $\epsilon > 0$ is small enough, $f_\epsilon \in L_2(\mathbf{R})$. Hence, f_ϵ belongs to the Sobolev space $H^{-\nu}(\mathbf{R})$. Since the factors $a_\pm(k)$ grow at the infinity as $|k|^{\nu/2}$, it follows from general theorems in Chapter 7 of Eskin (1973), that the solution to the problem (41)–(42) exists, and among the solutions, a bounded one is unique. It is given by

$$(45)\qquad g_\epsilon = r^{-1}\phi_r^+(D+i\epsilon/2)\mathbf{1}_{(-\infty,h(K)]}\phi_r^-(D+i\epsilon/2)f_\epsilon.$$

Certainly, $\epsilon > 0$ must be sufficiently small in order that (43) made sense. By making the inverse substitution $g_\epsilon(K,x) = e^{x\epsilon/2}g(K,x)$ and similarly for f_ϵ, we obtain (15).

With our choice of $h(K)$, the Fourier image of $v_K \equiv \mathbf{1}_{(-\infty,h(K)]}w_K$ decays at the infinity as $|k|^{-2}$ (and with any other choice, as $|k|^{-1}$):

$$\hat{v}_K(k) = \int_{-\infty}^{h(K)} e^{-iyk}(\phi_r^-(-i)G'(K)e^y - r)dy$$
$$= \frac{re^{-ikh(K)}}{(-ik)(1-ik)}.$$

Hence, from (16) and the fact that due to (39)–(42) $\phi_r^+(k)$ decays at the infinity as $|k|^{-\nu/2}$, we conclude that $\hat{g}$ decays at infinity as $|k|^{-2-\nu/2}$, along the strip where $\psi(k)$ is holomorphic. Since $r+\psi(k)$, the symbol of $r-L$, grows as $|k|^\nu$ at the infinity (due to (8) and our assumptions about ν and μ), and hence, the Fourier image of $(r-L)g$ decays at the infinity as $|k|^{-2+\nu/2}$. Since $\nu < 2$, this Fourier image is of the class $L_1(\mathbf{R})$, and hence, $(r-L)g \in C(\mathbf{R})$; by making some additional effort, it can be shown that $(r-L)g \in C(\mathbf{R})$ decays at infinity, i.e. $(r-L)g \in C_0(\mathbf{R})$. Hence, g is in the domain of L. By using similar but more detailed consideration, it can be shown that with any other choice of the boundary of the inaction region, $(r-L)g(x)$ is unbounded as x tends to the boundary of the inaction region; hence, one has no freedom with the choice of the boundary. This observation can be used as a substitute for the smooth pasting condition which fails for some regular Lévy processes of exponential type, as it was demonstrated in Boyarchenko and Levendorskiǐ (2000b) for perpetual American options (the smooth pasting condition fails in this model of irreversible investment if and only if the order of the process is less than 1 and the drift is negative).

It remains to prove the estimate (31) in Section 4. We integrate by part in (30) by using $e^{ts}ds = t^{-1}de^{ts}$. When we differentiate $f(s) := (s+\psi(-i\omega))^{-1}(\phi_s^-(-i\omega))^{-1}$, by using the Leibnitz formula, we obtain

$$f(s)(\alpha_1 f_1(s) + \alpha_2 f_2(s)),$$

where $\alpha_1, \alpha_2 \in \mathbb{C}$ are constants,

$$f_1(s) = (s+\psi(-i\omega))^{-1},$$

and

$$f_2(s) = \partial_s\left(\int_{-\infty}^{+\infty} \frac{\ln(s+\psi(l))}{-i\omega - l}dl\right)$$
$$= \int_{-\infty}^{+\infty} \frac{dl}{(s+\psi(l))(-i\omega - l)}.$$

Clearly, there exists C such that for s on the line $\Re s = \sigma_1 > 0$,

$$|f(s)| \le |\phi_{\sigma_1}^+(-i\omega)| \le C,$$

$$|f_1(s)| \le C|s|^{-1},$$

and

$$|f_2(s)| \le C\int_{-\infty}^{+\infty}(|s|+|l|^{\nu})^{-1}(1+|l|)^{-1}dl$$
$$\le C_{\epsilon}|s|^{-1+\epsilon},$$

for any $\epsilon > 0$. By gathering these estimates, we obtain that after the integration by part, the integrand in (30) admits an estimate via

$$Ct^{-1}e^{at}|s|^{-3/2},$$

and (31) follows.

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Fig.1. Volatility. Parameters: nu=1.6, mu=−0.05, c=0.15

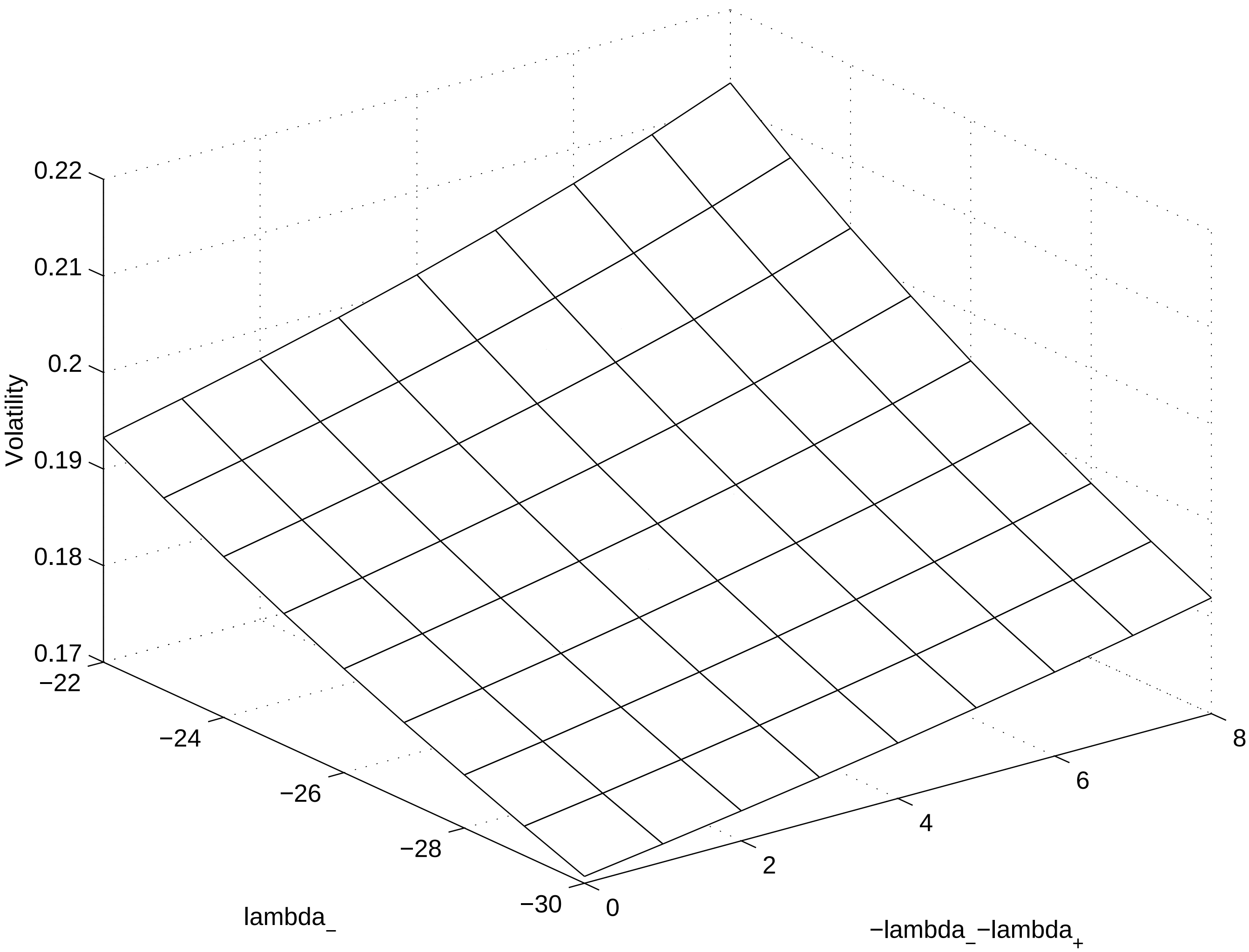

Fig.2. Skewness. Parameters: nu=1.6, mu=−0.05, c=0.15

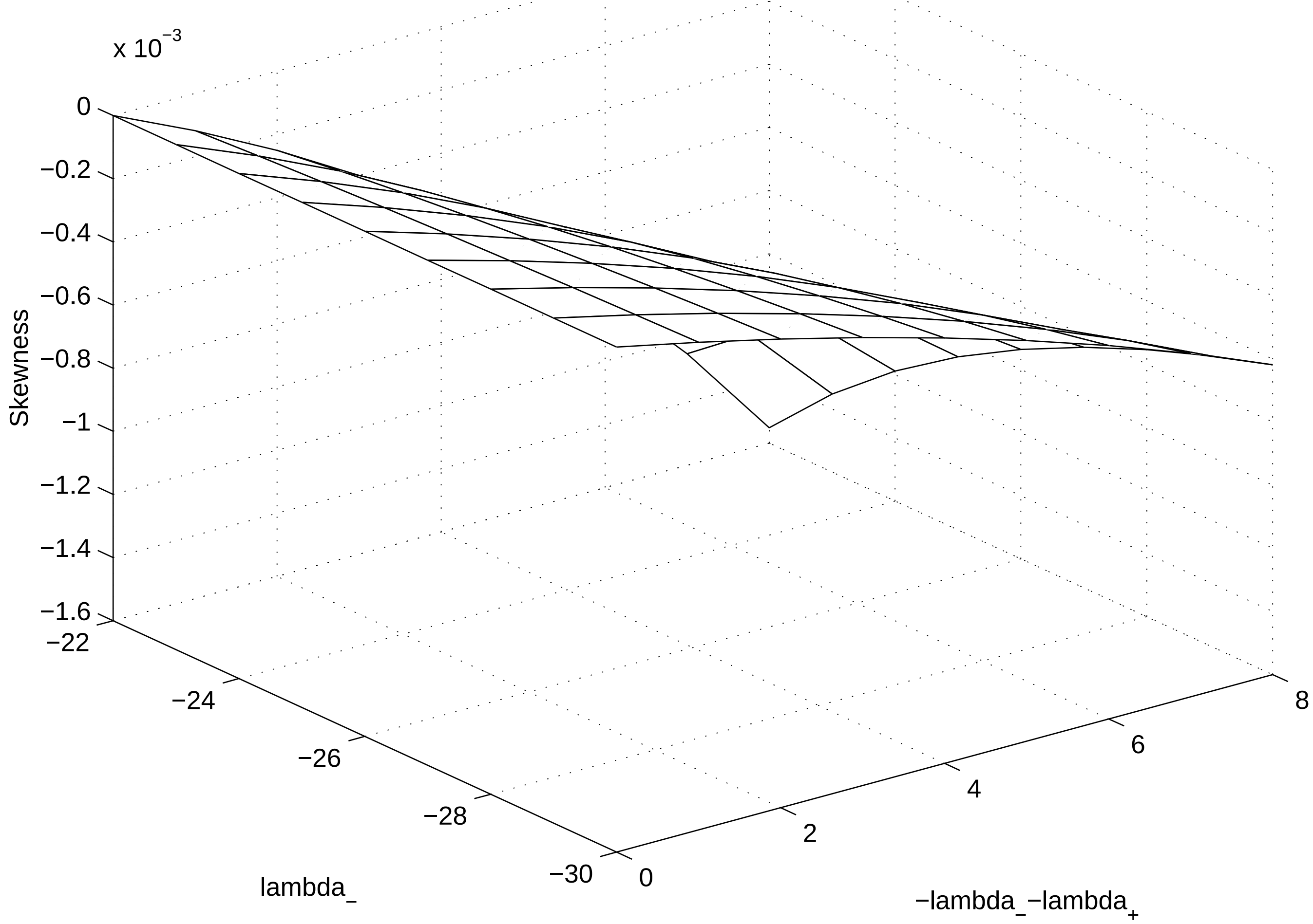

Fig.3. Kurtosis. Parameters: nu=1.6, mu=−0.05, c=0.15

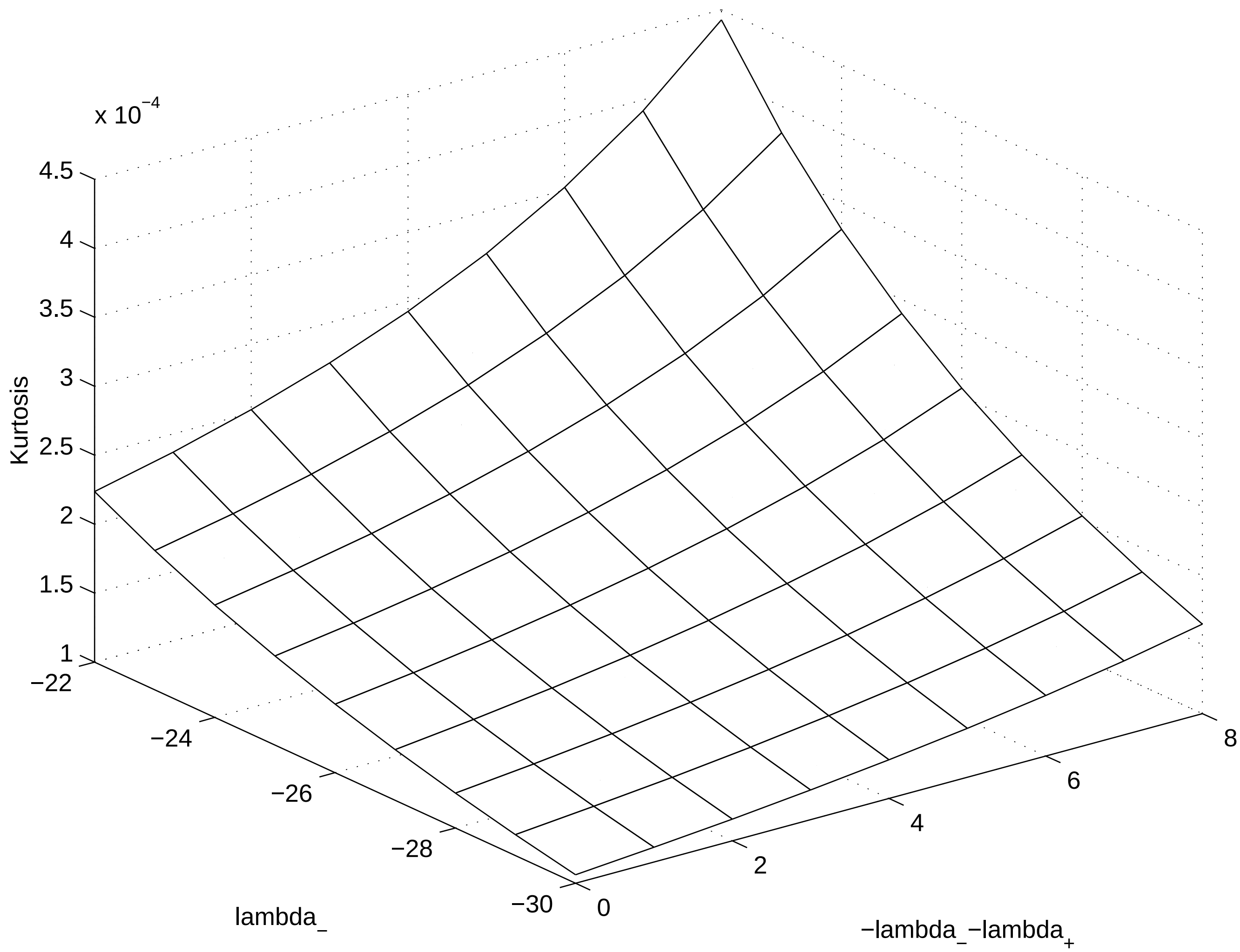

Fig.4. Investment Threshold. Parameters: r=0.06, nu=1.6, mu=−0.05, c=0.15

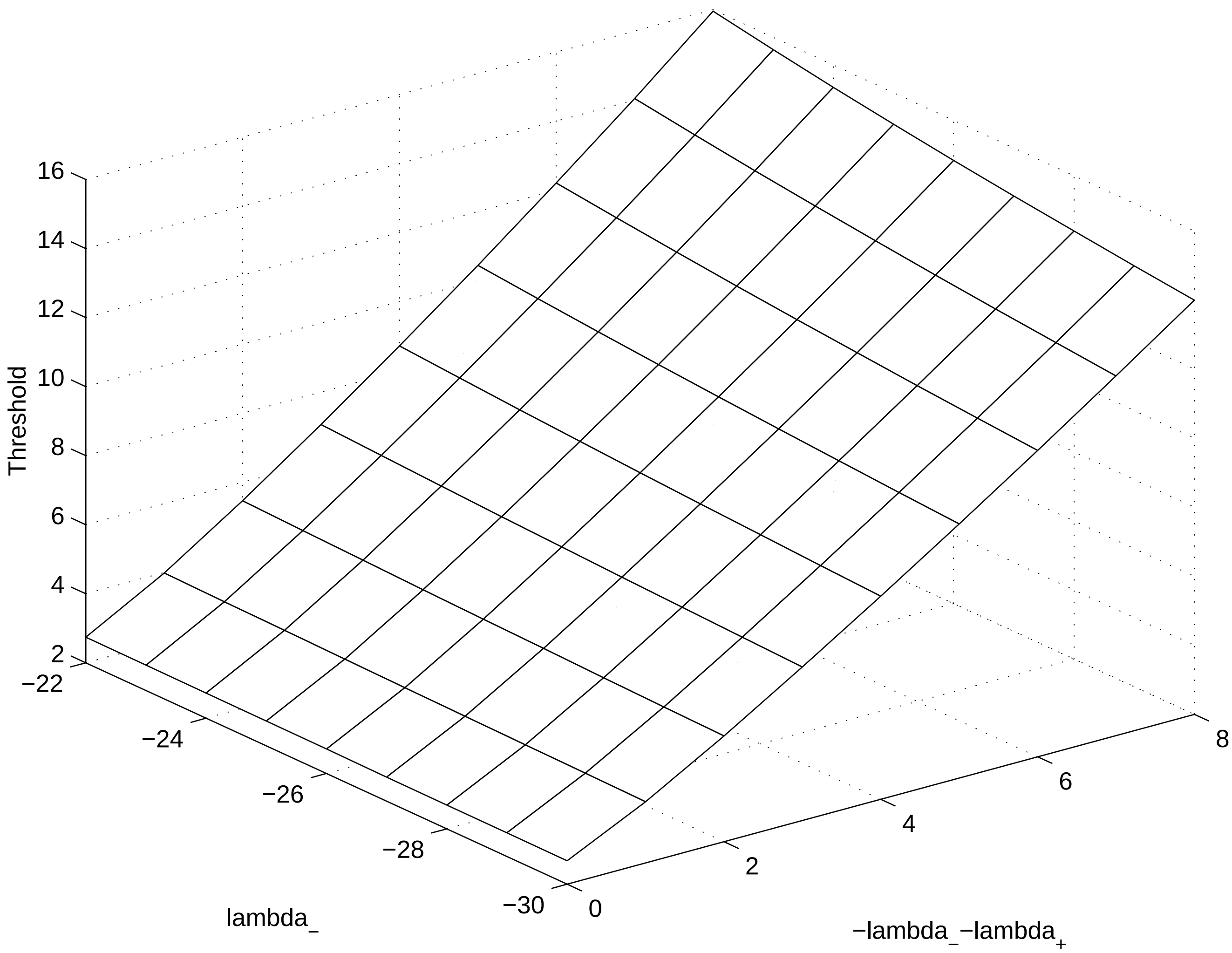

Fig.5. Volatility. Parameters: $\nu=1.6$, $-\psi(-i)=0.05$, $c=0.15$

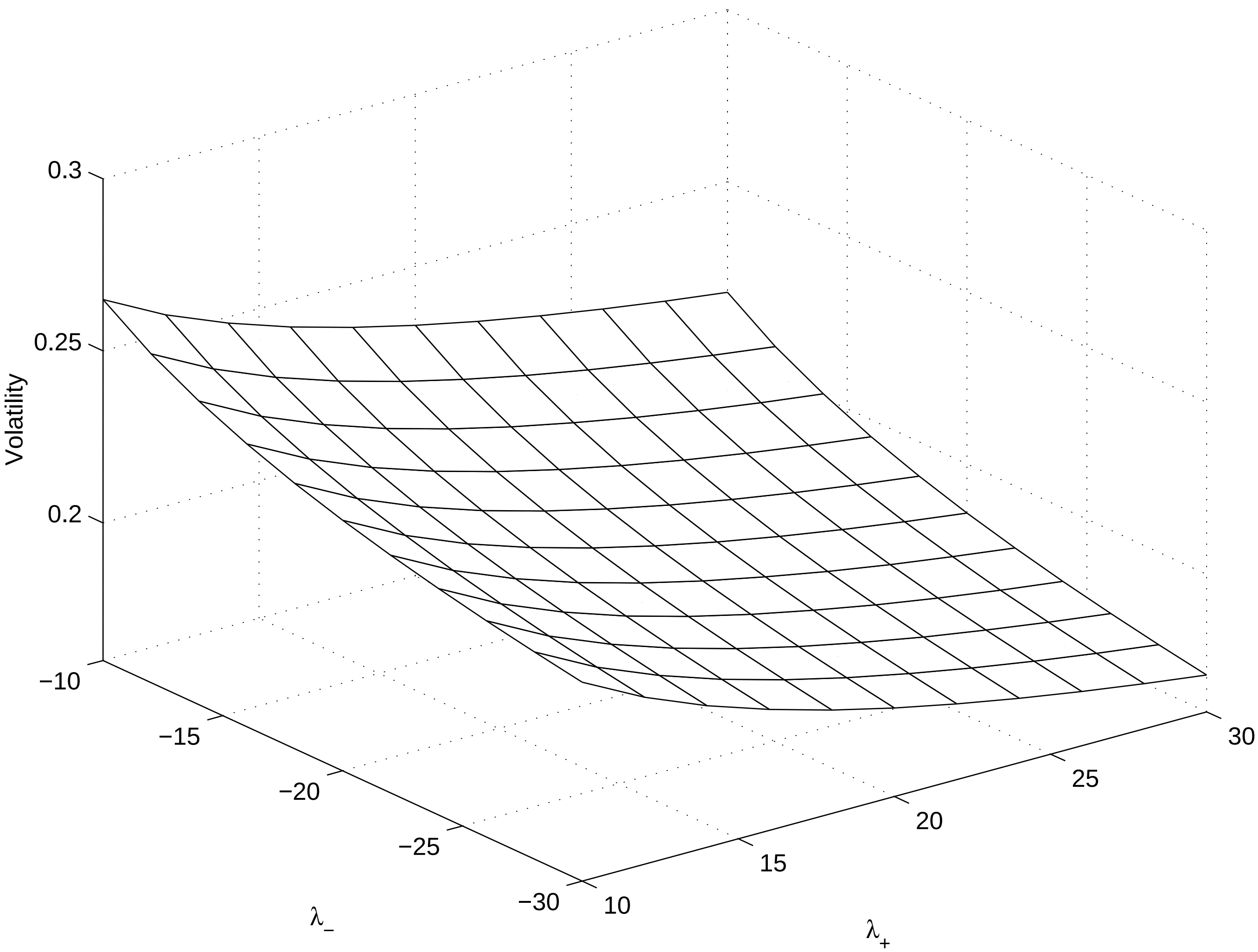

Fig.6. Skewness. Parameters: ν=1.6, $-\psi(-i)$=0.05, c=0.15

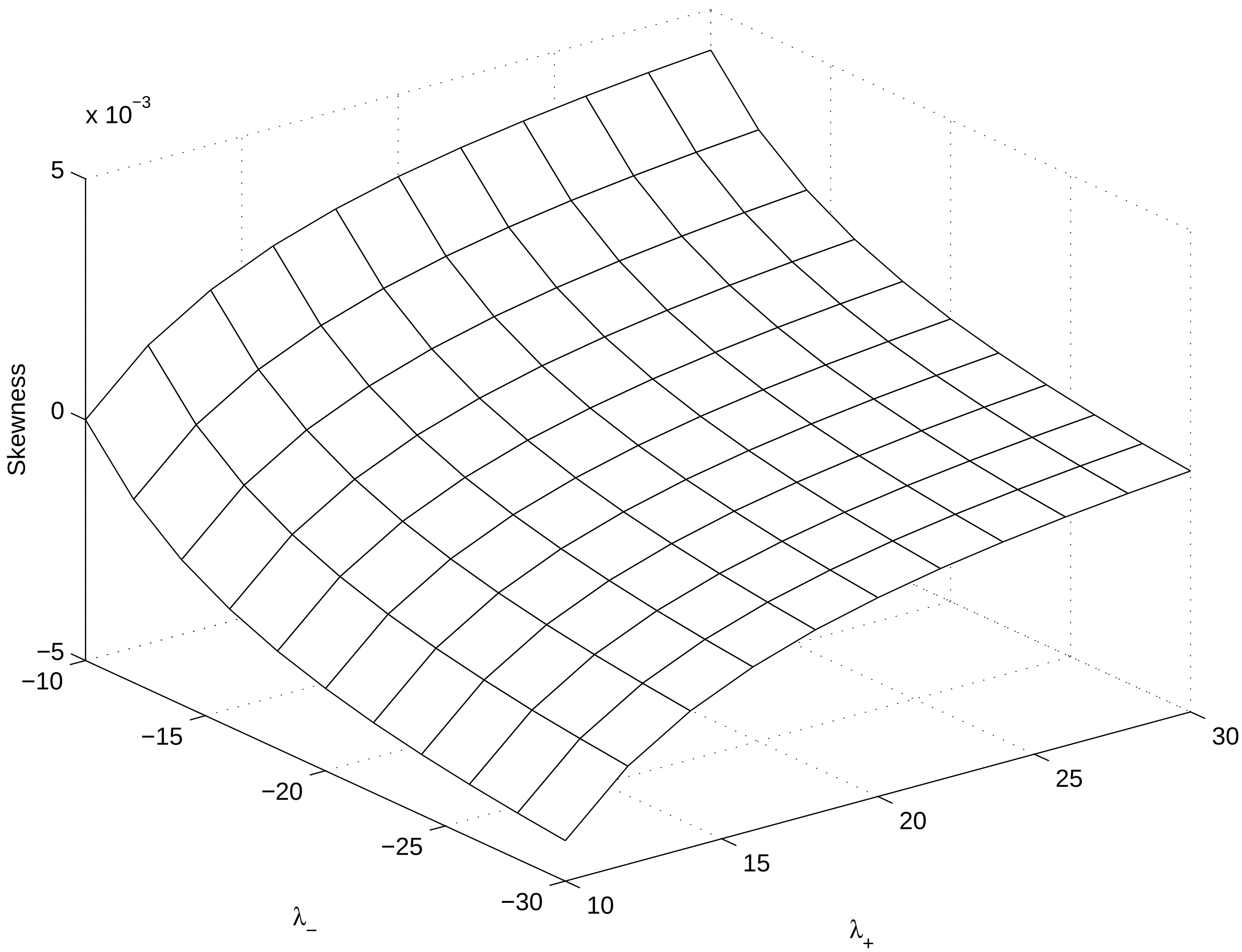

Fig.7. Kurtosis. Parameters: ν=1.6, $-\psi(-i)$=0.05, c=0.15

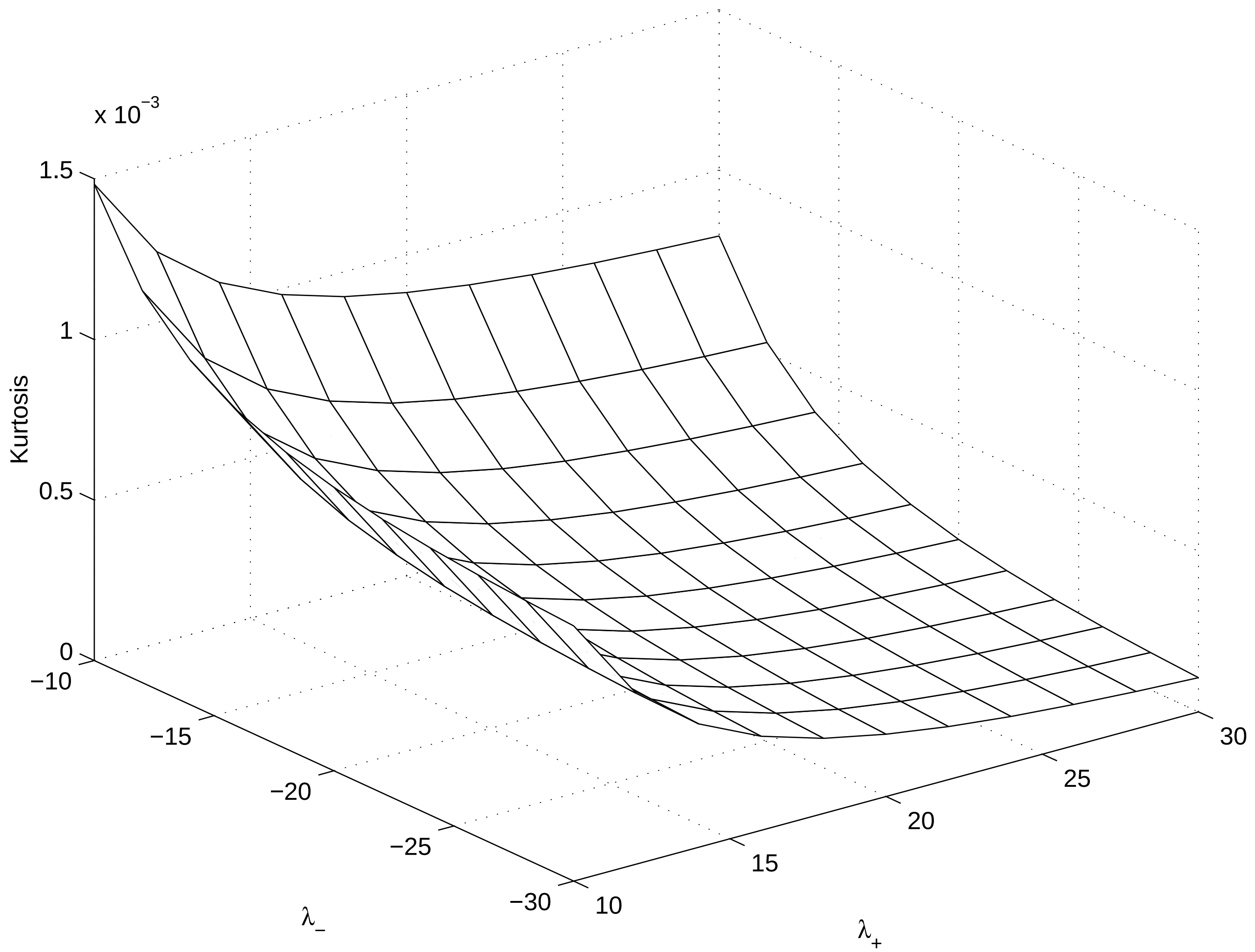

Fig.8. Investment Threshold. Parameters: r=0.06,ν=1.6, −ψ(−i)=0.05, c=0.15

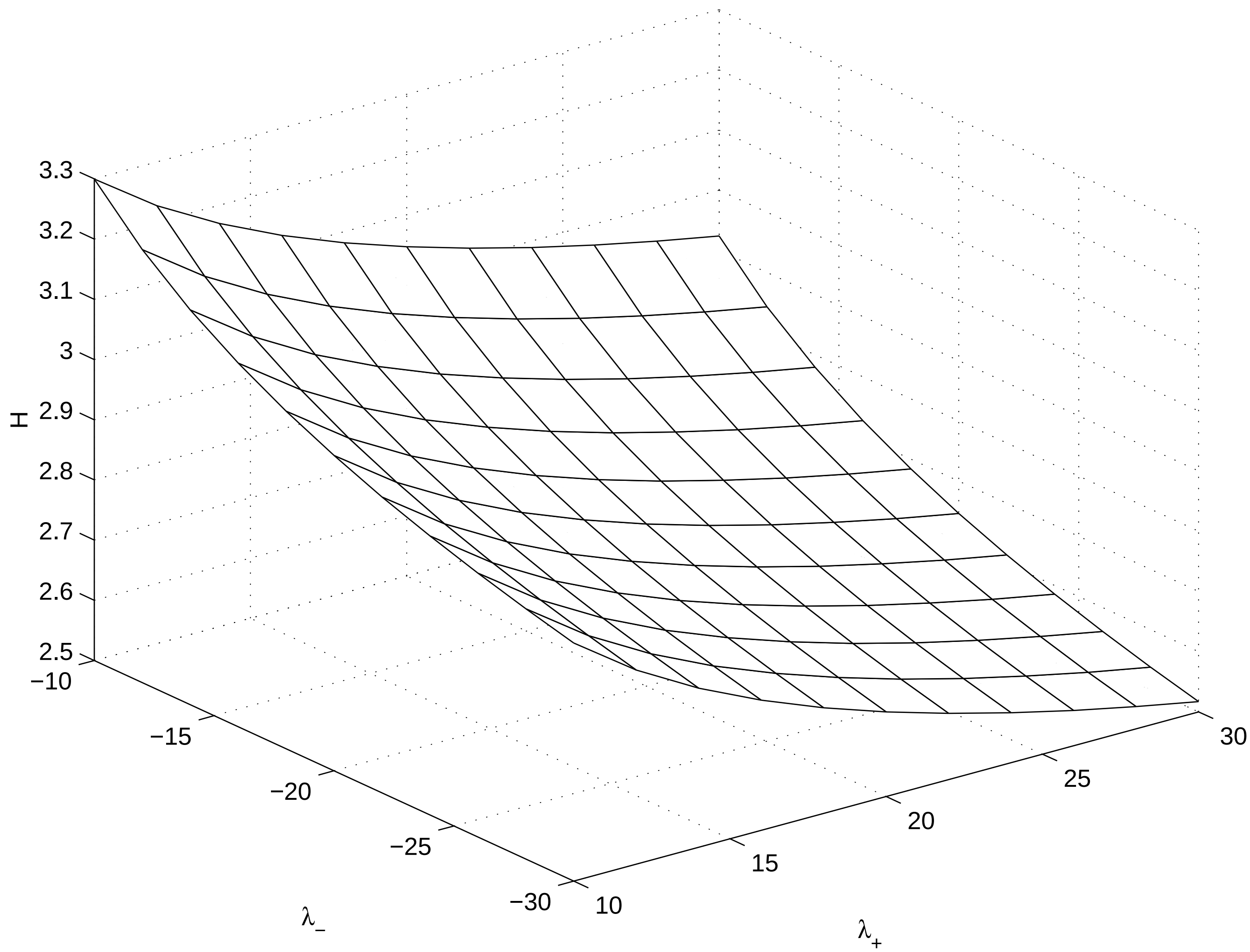

Fig.9. Rate of capital accumulation. Parameters: $r=0.06, \nu=1.6, -\psi(-i)=0.05, c=0.15$

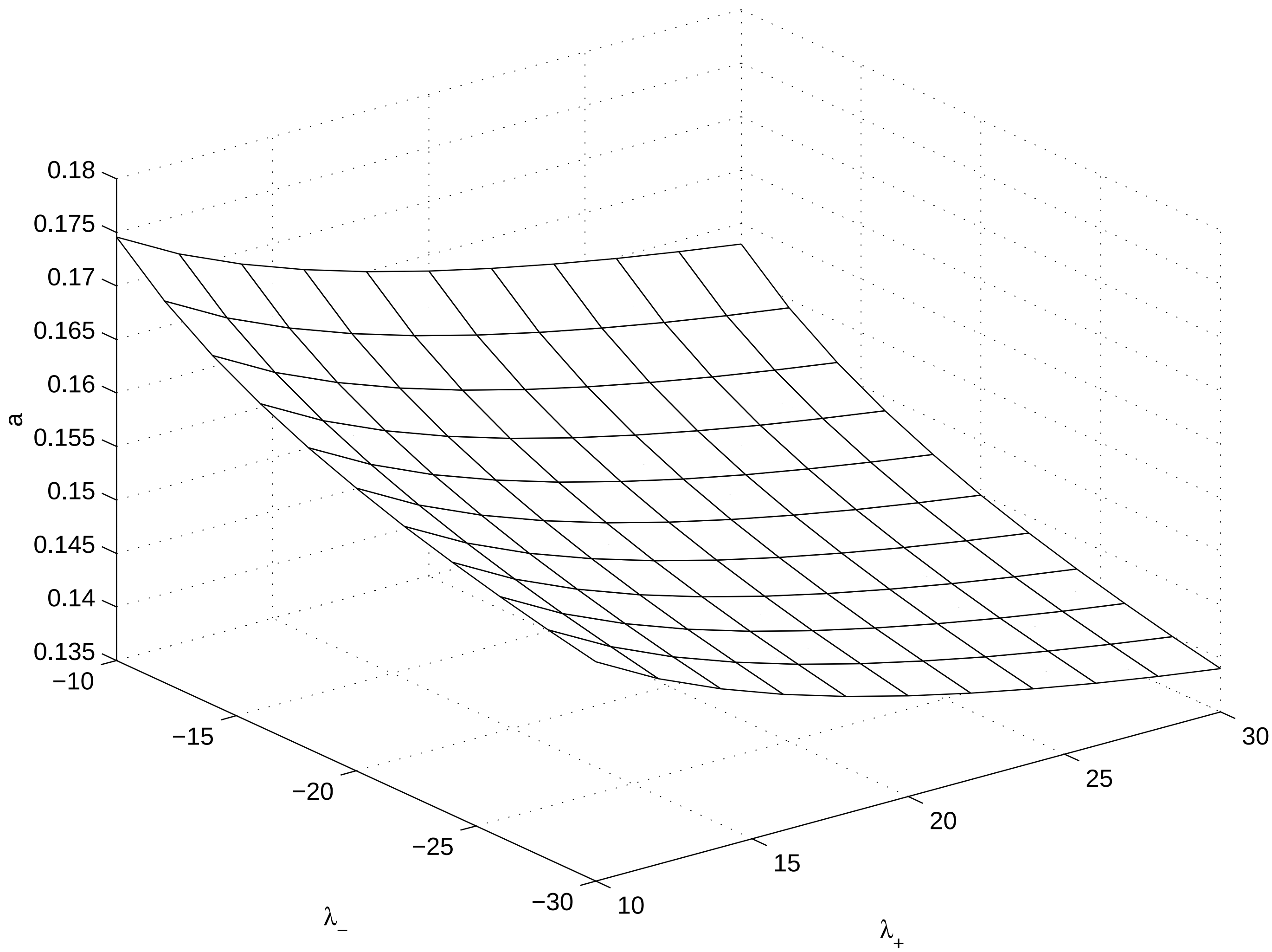

Fig.10. User−cost effect. Parameters: r=0.06,ν=1.6, −ψ(−i)=0.05, c=0.15, θ=0.33

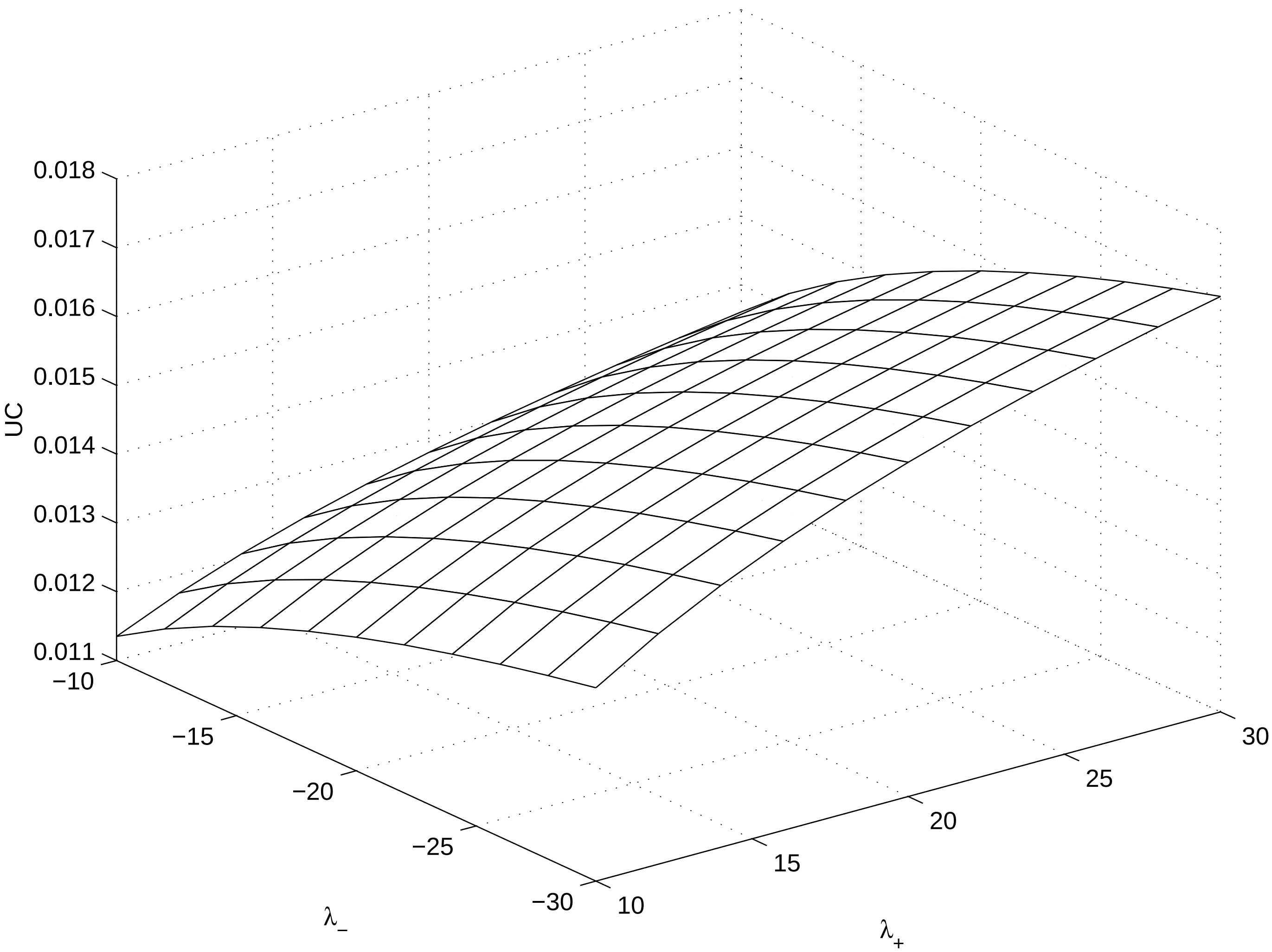

Fig.11. Long−run hangover effect. Parameters: $r=0.06, \nu=1.6, -\psi(-i)=0.05, c=0.15, \theta=0.33$

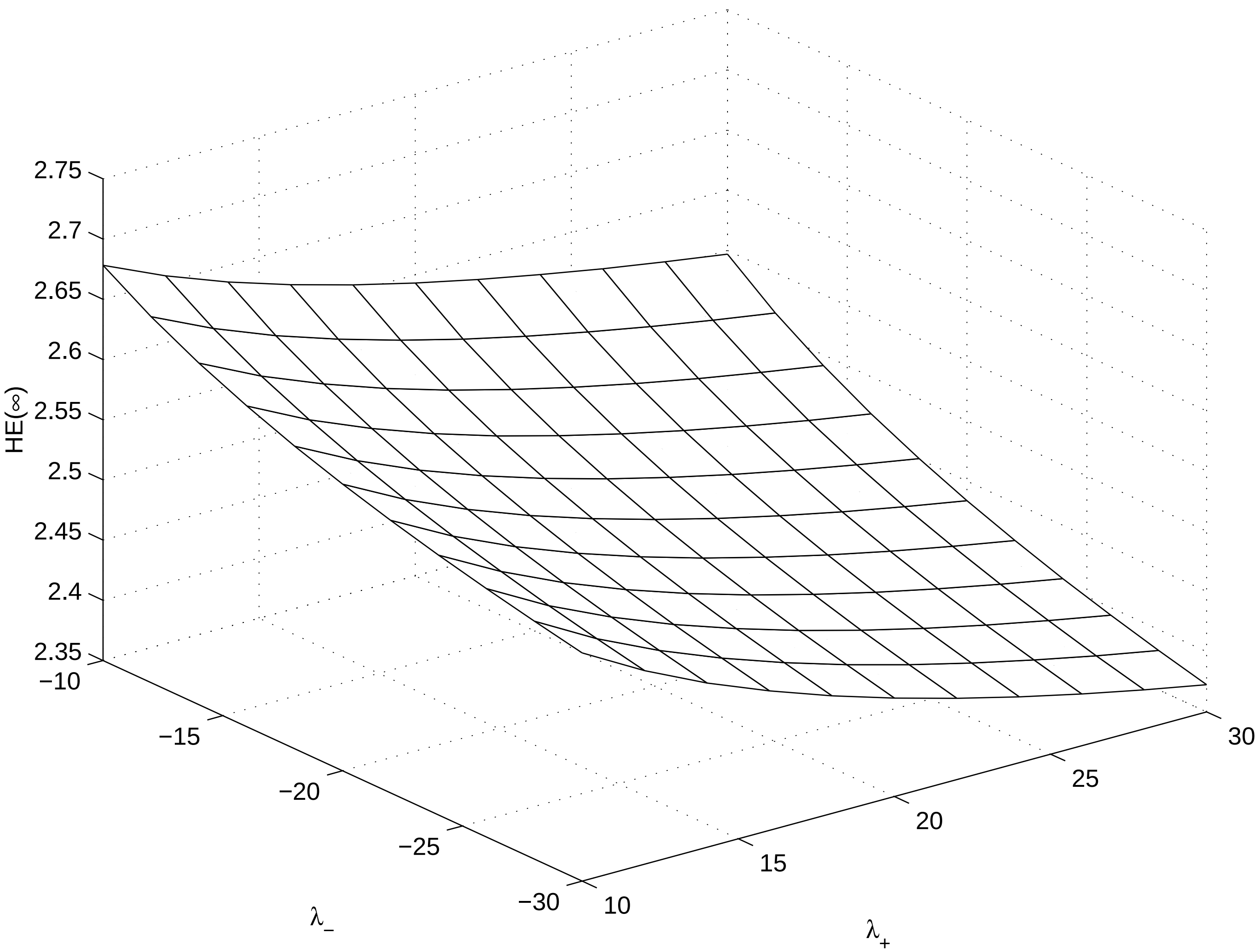

Fig.12. Long−run capital accumulation ratio; r=0.06, ν=1.6, −ψ(−i)=0.05, c=0.15, θ=0.33

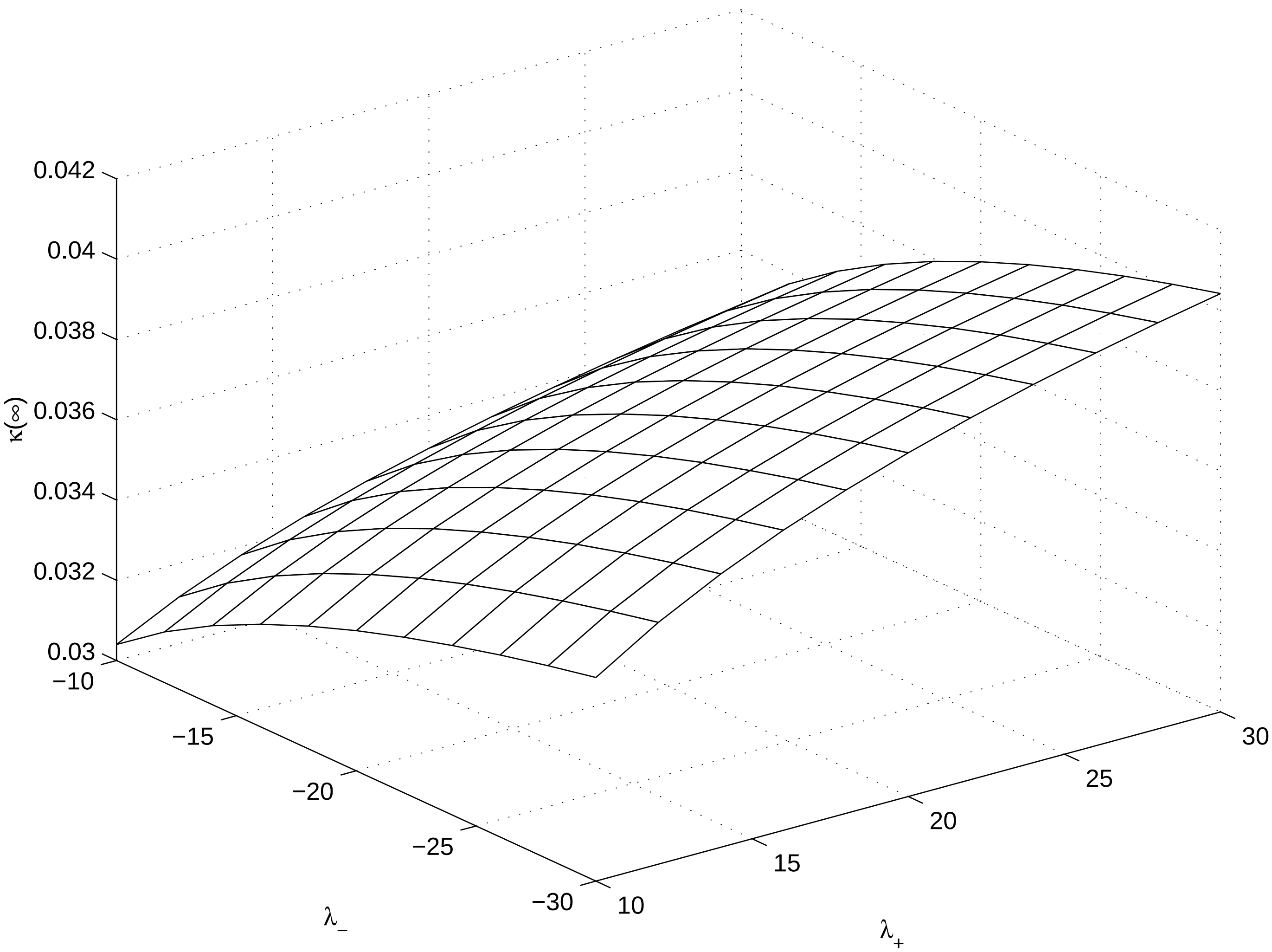

Fig.13. Accumulated capital; r=0.06, ν=1.6, −ψ(−i)=0.05, c=0.15, θ=0.33

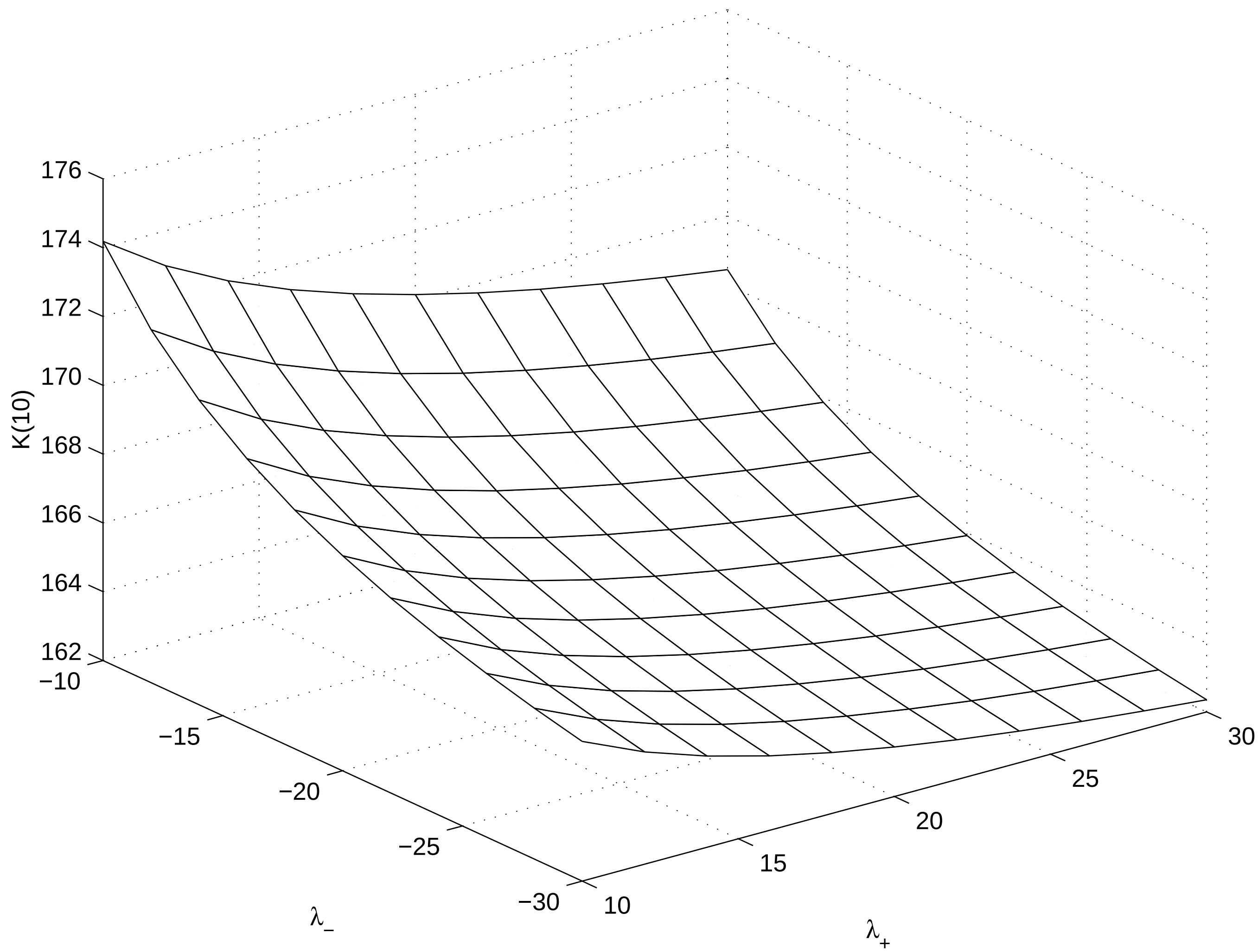

Fig.14. Threshold: Non−Gaussian vs Gaussian

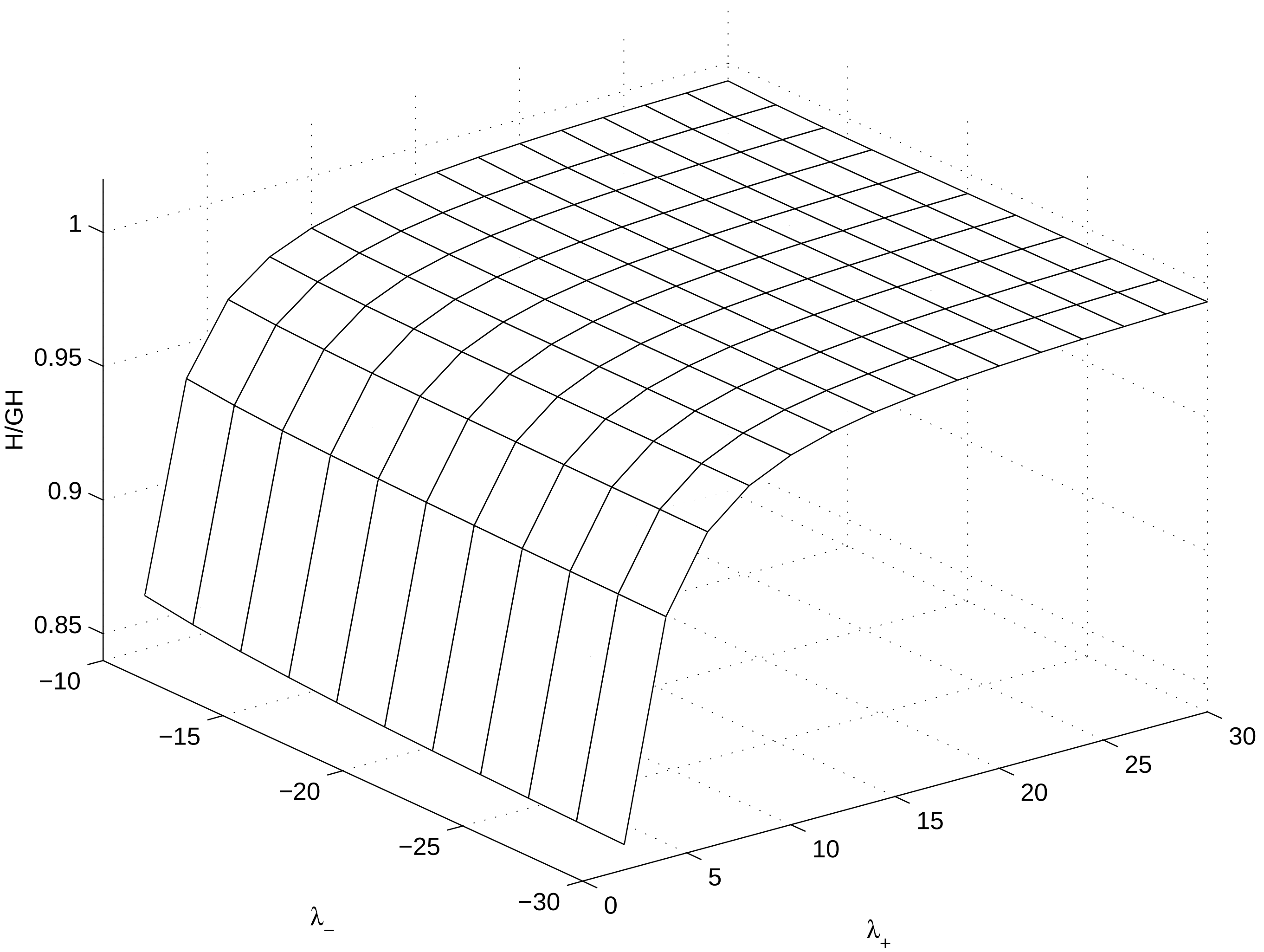

Fig.15. Rate of growth: Non−Gaussian vs Gaussian

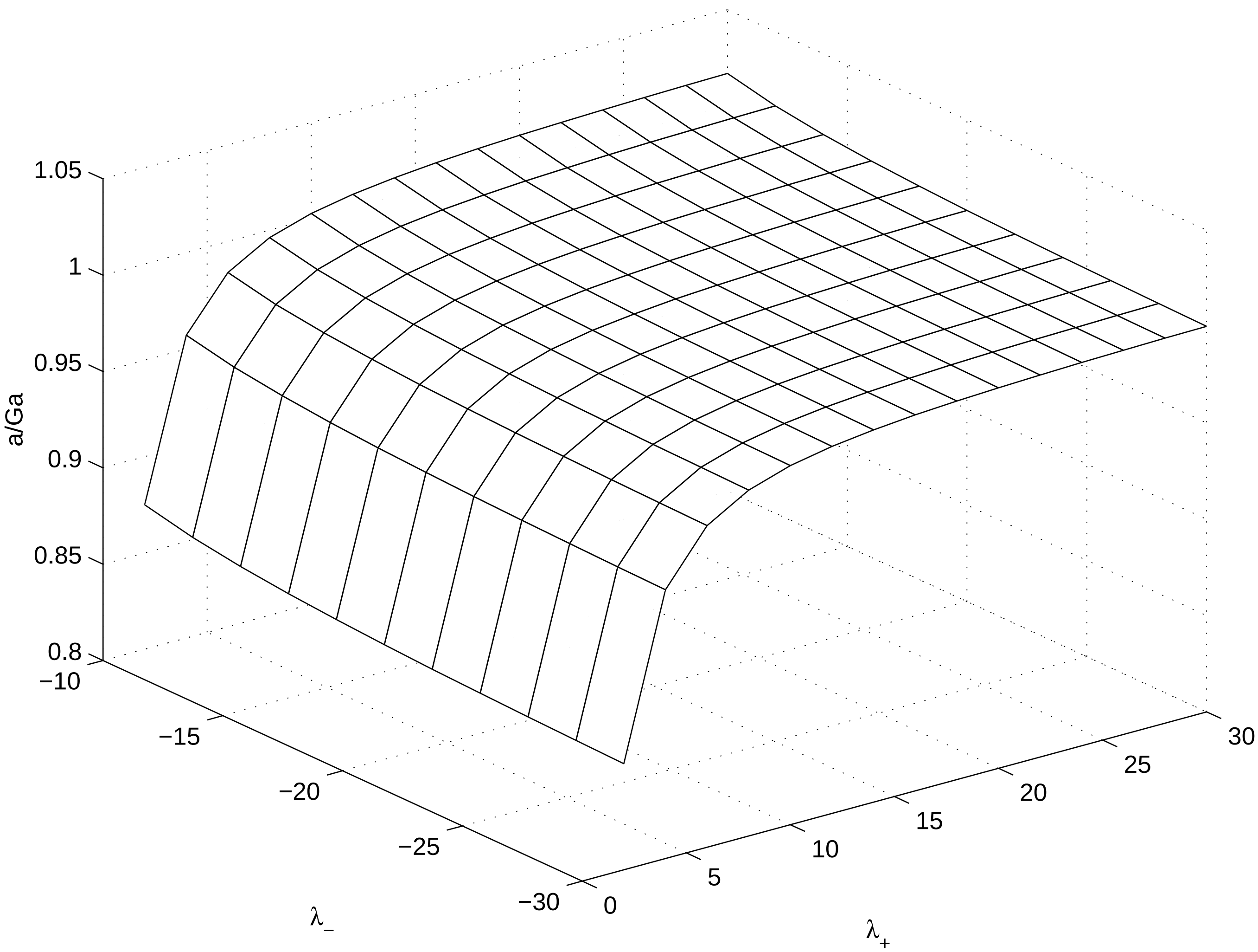

Fig.16. User−cost effect: Non−Gaussian vs Gaussian

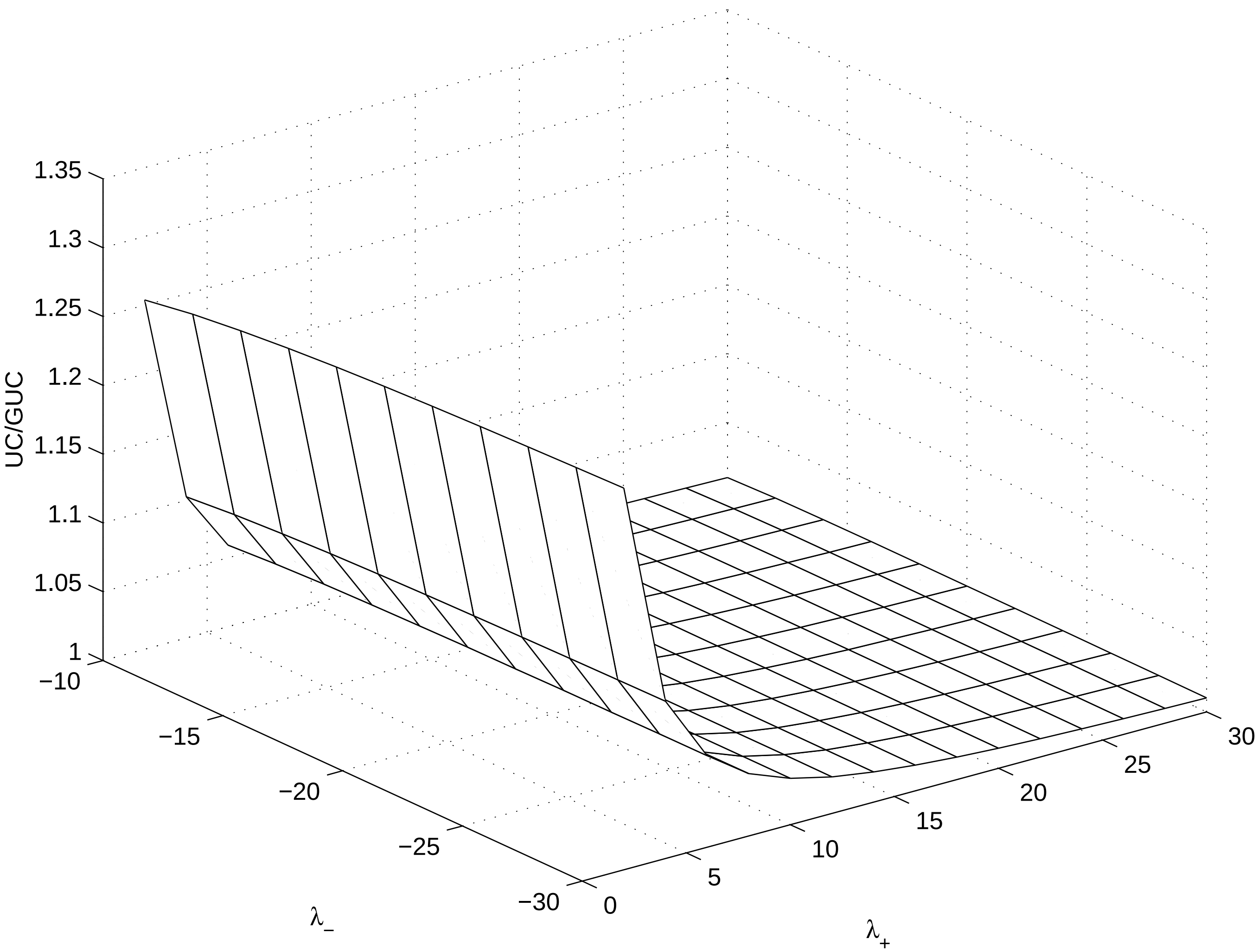

Fig.17. Long−run hangover effect: Non−Gaussian vs Gaussian

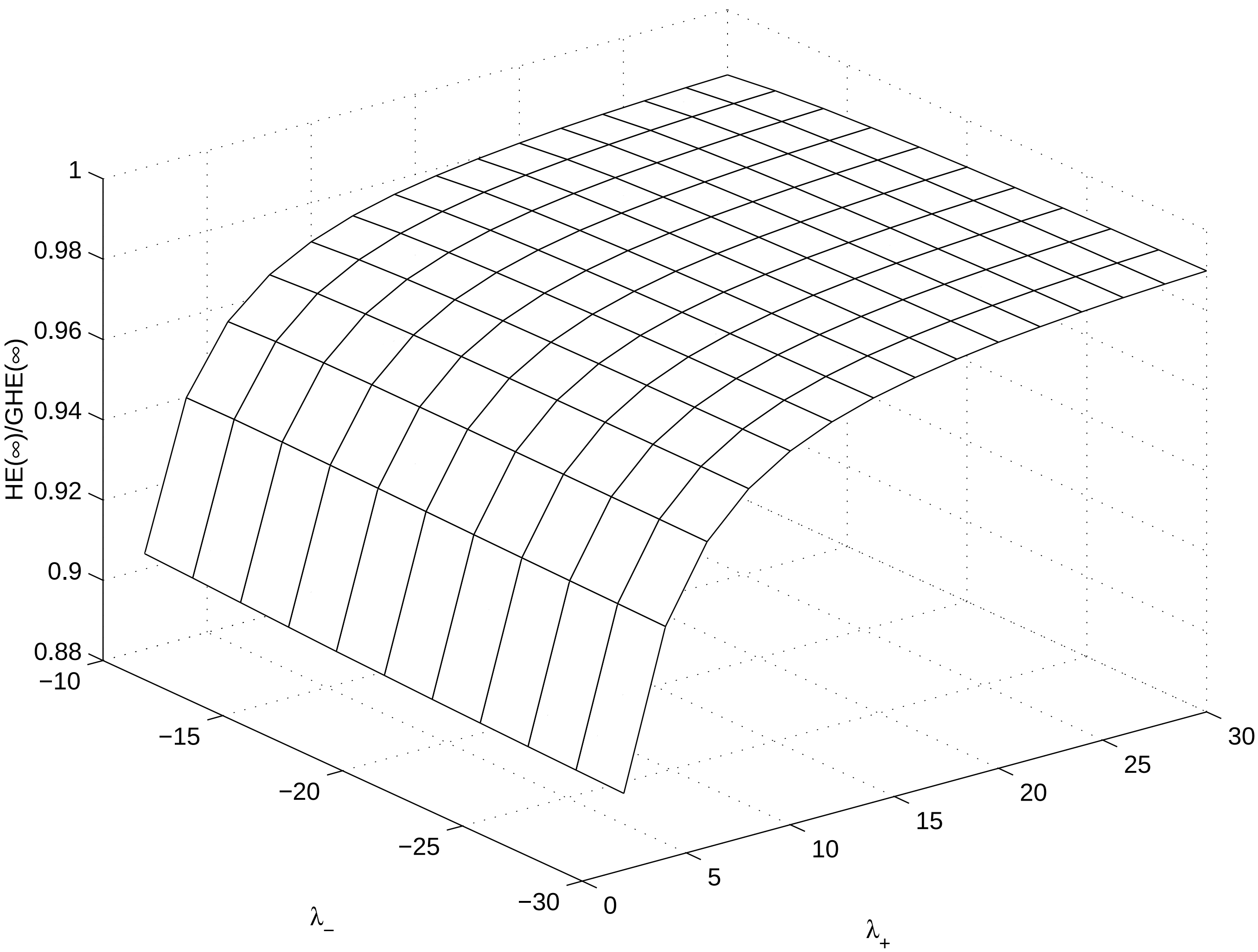

Fig.18. Long−run capital accumulation ratio: Non−Gaussian vs Gaussian

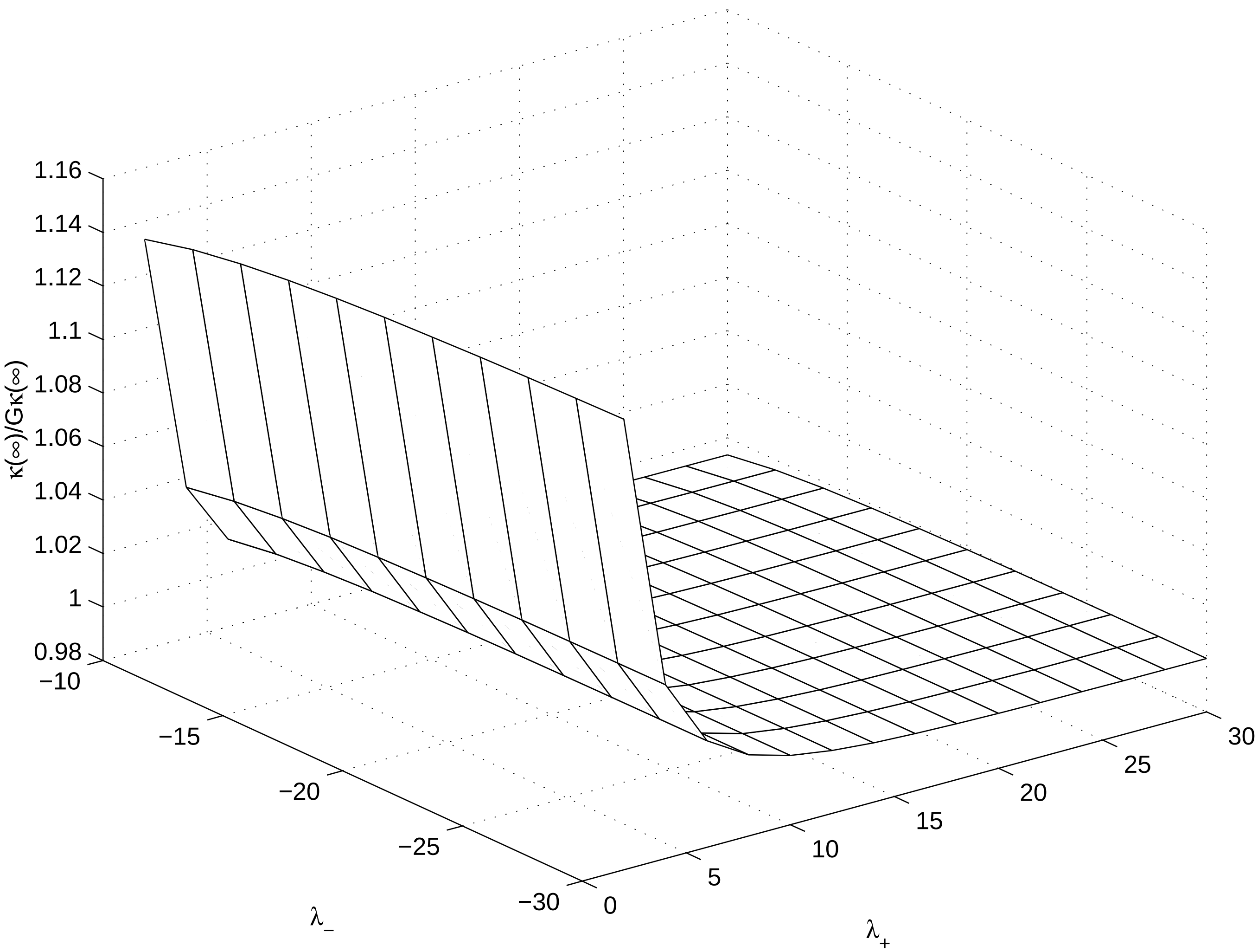

Fig.19. Capital accumulation, t=10 years: Non−Gaussian vs Gaussian