Informational Size, Incentive Compatibility and the Core of an Economy with Incomplete Information*

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Abstract

We examine the ex ante incentive compatible core, and show that generically, when agents are informationally small in the sense of McLean and Postlewaite (1999), the ex ante incentive compatible core is nonempty.

1. Introduction

While most of the game theoretic literature dealing with asymmetric information has focused primarily on noncooperative games, there is an expanding literature that studies the core in the presence of incomplete information, most of which is surveyed in Forges, Minelli and Vohra (2000). Several different definitions of the core in incomplete information environments are possible depending on whether incentive constraints are taken into account and on whether coalitional decisions are made ex ante (before agents learn their types) or at the interim stage (after agents learn their types). Our analysis deals with the ex ante incentive compatible core which, as the name suggests, treats the case in which decisions are made at the ex ante stage and incentive constraints are taken to matter.

While the core with complete information is nonempty under quite general circumstances, Vohra (1999) and Forges, Mertens and Vohra (2000) have recently

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shown that the ex ante core may be empty in well-behaved exchange economies. Our aim in this paper is to provide something akin to a continuity theorem: when the asymmetry of information among the agents in an exchange economy is small, the ex ante incentive compatible core is nonempty.

We analyze the core of an Arrow-Debreu pure exchange economy in which the agents are asymmetrically informed. Specifically, the agents' utility functions will depend on an underlying but unobserved state of nature and each agent will receive a private signal that is correlated with the state of nature. Roughly speaking, this corresponds to a "common value" model in which signals do not directly affect the underlying payoff functions but do affect expected utilities.

Vohra (1999) has shown that the ex ante incentive compatible core is nonempty under strong conditions limiting the amount of asymmetry of information among agents. Postlewaite and Schmeidler (1986) introduced the notion of non-exclusive information, under which no single agent's information was necessary to identify the correct state of the world. Vohra shows that, if information is non-exclusive, then the ex ante incentive compatible core is nonempty. When an information structure is non-exclusive then, when the information of all agents but one is known, that one agent's information cannot affect the conditional probability distribution over the states. For the common value model that we study in this paper, we use the concept of informational size that we developed in McLean and Postlewaite (2000). This notion of informational size extends the non-exclusive information concept by asking how much a given agent's information can affect the probability distribution over the states, given all other agents' information. Roughly speaking, an agent will be informationally small if, given other agents' information, it is very likely that the given agent's information will have a small effect on the probability distribution over the states.

Our theorem on the nonemptiness of the ex ante incentive compatible core depends on two other aspects of the information structure used in McLean and Postlewaite (2000): aggregate uncertainty and the variability of agents' beliefs. Aggregate uncertainty quantifies the degree to which the aggregate of agents' information resolves all uncertainty regarding the state of nature. Roughly speaking, the variability of an agent's beliefs quantifies the difference in the conditional distributions on the state space induced by the different types he might be. We show that generically, the ex ante incentive compatible core is nonempty if all agents are informationally small relative to the variability of their beliefs and aggregate uncertainty. We further show that in replica economies, agents' informational size goes to zero.

2. Basic Notation

Throughout the paper, let $J_q = \{1, .., q\}$ for each positive integer q and let $|| \cdot ||$ denote the 1-norm unless specified otherwise. Let $N = \{1, 2, ..., n\}$ denote the set of **economic agents**. Let $\Theta = \{\theta_1, .., \theta_m\}$ denote the (finite) **state space** and let $T_1, T_2, ..., T_n$ be finite sets where T_i represents the set of possible **signals** that agent i might receive. For each $S \subseteq N$, let $T_S \equiv \prod_{i \in S} T_i$. Elements of T_S will be written t_S. For notational simplicity, we will simply write T for T_N and t for t_N. If $t \in T$, then we will often write $t = (t_{N \backslash S}, t_S)$. If X is a finite set, define

$$\Delta_X := \{\rho \in \Re^{|X|} | \rho(x) \geq 0, \sum_{x \in X} \rho(x) = 1\}.$$

In our model, nature chooses an element $\theta \in \Theta$. The state of nature is unobservable but each agent i receives a "signal" t_i that is correlated with nature's choice of θ. More formally, let $(\widetilde{\theta}, \widetilde{t}_1, \widetilde{t}_2, ..., \widetilde{t}_n)$ be an (n+1)-dimensional random vector taking values in $\Theta \times T$ with associated distribution $P \in \Delta_{\Theta \times T}$ where

$$P(\theta, t_1, .., t_n) = \text{Prob}\{\widetilde{\theta} = \theta, \widetilde{t}_1 = t_1, ..., \widetilde{t}_n = t_n\}.$$

Without loss of generality, we will make the following assumption regarding the marginal distributions:

full support: $\text{supp}(\widetilde{\theta}) = \Theta$ i.e. for each $\theta \in \Theta$,

$$P(\theta) = \text{Prob}\{\widetilde{\theta} = \theta\} > 0$$

and for each $i \in N$, $\text{supp}(\widetilde{t}) = T$ i.e. for each $t = (t_1, .., t_n) \in T$,

$$P(t) = \text{Prob}\{\widetilde{t}_1 = t_1, .., \widetilde{t}_n = t_n\} > 0.$$

If $t \in T$, let $P_\Theta(\cdot|t) \in \Delta_\Theta$ denote the induced conditional probability measure on Θ. Let $\chi_\theta \in \Delta_\Theta$ denote the degenerate measure that puts probability one on state θ.

2.1. Economies

The **consumption set** of each agent is $\Re^\ell_+$ and for each $\theta \in \Theta, w_i \in \Re^\ell_{++}$ denotes the (state independent) of agent i in state θ. For each $\theta \in \Theta$, let $u_i(\cdot, \theta) : \Re^\ell_+ \to \Re$ be the utility function of agent i in state θ. The following assumptions are maintained throughout the paper:

(i) $u_i(\cdot, \theta)$ is continuous and concave

(ii) $u_i(0, \theta) = 0$

(iii) $u_i(\cdot, \theta)$ is monotonic: if $x, y \in \Re^{\ell}_{+}$, $x \geq y$ and $x \neq y$, then $u_i(x, \theta) > u_i(y, \theta)$.

Each $\theta \in \Theta$ gives rise to a pure exchange economy . Formally, let

$$e(\theta) = \{w_i, u_i(\cdot, \theta)\}_{i \in N}$$

denote the **Complete Information Economy** (CIE) corresponding to state θ. For each $\theta \in \Theta$, a **complete information economy (CIE) allocation** for $e(\theta)$ is a collection $\{x_i(\theta)\}_{i \in N}$ satisfying $x_i(\theta) \in \Re^{\ell}_{+}$ for each i and $\sum_{i \in N}(x_i(\theta) - w_i) \leq 0$.

The collection $(\{e(\theta)\}_{\theta \in \Theta}, \widetilde{\theta}, \tilde{t}, P)$ will be called a **private information economy** (PIE for short). A **private information economy allocation** $z = (z_1, z_2, ..., z_n)$ for the PIE is a collection of functions z_i: $T \rightarrow \Re^{\ell}_{+}$ satisfying $\sum_{i \in N}(z_i(t) - w_i) = 0$ for all $t \in T$. We will not distinguish between $w_i \in \Re^{\ell}_{++}$ and the constant allocation that assigns the bundle w_i to agent i for all $t \in T$.

For each $\pi \in \Delta_{\Theta}$ and each collection of PIE's $\{e(\theta)\}_{\theta \in \Theta}$, define the associated **auxiliary economy** to be the collection $\{\pi, \{e(\theta)\}_{\theta \in \Theta}\ \}$. A commodity vector for agent i in the auxiliary economy is a vector of state contingent bundles in $\Re^{\ell m}_{+}$ and is written as

$$(x_i(\theta_1), .., x_i(\theta_m)).$$

In the auxiliary economy, the initial endowment of agent i is the vector $\hat{w}_i = (w_i, .., w_i) \in R^{\ell m}_{++}$ and the utility of agent i is the function $v_i : \Re^{\ell m}_{+} \rightarrow \Re$ defined for each $(x_i(\theta_1), .., x_i(\theta_m)) \in \Re^{\ell m}_{+}$ as follows:

$$v_i(x_i(\theta_1), .., x_i(\theta_m)) := \sum_{k=1}^{m} u_i(x_i(\theta_k); \theta_k)\pi(\theta_k).$$

For each $S \subseteq N$, the set of S-feasible allocations in the auxiliary economy is the set

$$\Phi_S(w) = \{(x_i(\theta_1), .., x_i(\theta_m))_{i \in S} \in (\Re^{\ell m})^{|S|} | \sum_{i \in S} x_i(\theta) = \sum_{i \in S} w_i \text{ for each } \theta \in \Theta\}.$$

An allocation for the auxiliary economy is simply feasible if $(x_i(\theta_1), .., x_i(\theta_m))_{i \in N} \in \Phi_N$. Finally, the

An auxiliary economy gives rise to an NTU game in a natural way by defining the set of attainable payoffs as

$$V(S) = \{(y_i)_{i \in S} | \text{ for some } (x_i(\theta_1), .., x_i(\theta_m))_{i \in S} \in \Phi_S,\ y_i \leq v_i(x_i(\theta_1), .., x_i(\theta_m)) \text{ for all } i \in S\}.$$

Since each u_i is concave, a standard argument establishes that this NTU game is balanced and, therefore, has a nonempty core.

A feasible allocation $(\hat{x}_i(\theta_1),..,\hat{x}_i(\theta_m))_{i\in N} \in \Phi_N$ is a core allocation of the auxiliary economy if

$$(v_1(\hat{x}_1(\theta_1),..,\hat{x}_1(\theta_m)),..,v_n(\hat{x}_n(\theta_1),..,\hat{x}_n(\theta_m)))$$

is a payoff vector in the core of the NTU game V.

For an allocation $x \in \Phi_N$, denote the set of bundles weakly preferred by i to his part of the allocation by

$$\mathcal{P}_i(x) = \{y \in R_+^{\ell m} | v_i(y(\theta_1),..,y(\theta_m)) \geq v_i(x_i(\theta_1),..,x_i(\theta_m))\}.$$

For each $S \subseteq N$, let $\mathcal{P}_S(x) = \sum_{i\in S} \mathcal{P}_i(x)$, $w_S = \sum_{i\in S} w_i$ and $x_S(\theta) = \sum_{i\in S} x_i(\theta)$. Define $\hat{w}_S := (w_S,..,w_S)$ to be the point $(y(\theta_1),..,y(\theta_m)) \in R_+^{\ell m}$ with $y(\theta_k) = w_S$ for each k.

Definition 1: A core allocation $x = (x_i(\theta_1),..,x_i(\theta_m))_{i\in N}$ of the auxiliary economy $\{\pi, \{e(\theta)\}_{\theta\in\Theta}\}$ is a *strict* core allocation with respect to $w = (w_1,..,w_n)$ if

$$\hat{w}_S \notin \mathcal{P}_S(x)$$

whenever $S \neq N$.

Finally, an allocation $x = (x_i(\theta_1),..,x_i(\theta_m))_{i\in N} \in \Phi_N$ is a Walras equilibrium of the auxiliary economy $e = \{\pi, \{e(\theta)\}_{\theta\in\Theta}\}$ (with associated price vector $(p(\theta_1),..,p(\theta_m)) \in R_+^{\ell m}\backslash\{0\}$) if x is an equilibrium for the n-agent pure exchange economy in which agent i has utility function v_i and endowment vector $\hat{w}_i = (w_i,..,w_i) \in R_{++}^{\ell m}$.

3. Incentive Compatible Cores

3.1. Notions of Blocking

Let $(\{e(\theta)\}_{\theta\in\Theta}, \widetilde{\theta}, \tilde{t}, P)$ be a PIE. In order to define the core of an economy with incomplete information, it is necessary to propose a notion of "improve upon" or "blocking." For each $S \subseteq N$, let the set of *S-feasible allocations* for the PIE be defined as

$$A_S = \{(z_i)_{i\in S} | z_i : T_S \to \Re_+^\ell \text{ and } \sum_{i\in S}(z_i(t_S) - w_i) \leq 0 \text{ for all } t_S \in T_S\}.$$

An S-feasible allocation $(z_i)_{i\in S}$ is *incentive compatible* if

$$\sum_{\theta\in\Theta}\sum_{t_{S\setminus i}\in T_{S\setminus i}} u_i(z_i(t_{S\setminus i}, t'_i), \theta)P(\theta, t_{S\setminus i} \mid t_i) \geq \sum_{\theta\in\Theta}\sum_{t_{S\setminus i}\in T_{S\setminus i}} u_i(z_i(t_{S\setminus i}, t_i), \theta)P(\theta, t_{S\setminus i} \mid t_i)$$

for each $t_i, t'_i \in T_i$ and $i \in S$.

The set of incentive compatible, S-feasible allocations will be denoted A^*_S.

Let $(z_i)_{i\in N} \in A_N$ be an N-feasible allocation.

Definition 2.1: (*ex ante blocking*) A coalition $S \subseteq N$ can $X-$block $(z_i)_{i\in N}$ if there exists $(x_i)_{i\in S} \in A_S$ satisfying the following condition:

$$\sum_{t_S\in T_S}\sum_{\theta\in\Theta} u_i(x_i(t_S), \theta)P(\theta, t_S) > \sum_{t_S\in T_S}\sum_{t_{N\setminus S}\in T_{N\setminus S}}\sum_{\theta\in\Theta} u_i(z_i(t_{N\setminus S}, t_S), \theta)P(\theta, t_{N\setminus S}, t_S)$$

for all $i \in S$.

Definition 2.2: (*ex ante incentive compatible blocking*) A coalition $S \subseteq N$ can $ICX-$block $(z_i)_{i\in N}$ if there exists $(x_i)_{i\in S} \in A^*_S$ satisfying the following condition:

$$\sum_{t_S\in T_S}\sum_{\theta\in\Theta} u_i(x_i(t_S), \theta)P(\theta, t_S) > \sum_{t_S\in T_S}\sum_{t_{N\setminus S}\in T_{N\setminus S}}\sum_{\theta\in\Theta} u_i(z_i(t_{N\setminus S}, t_S), \theta)P(\theta, t_{N\setminus S}, t_S)$$

for all $i \in S$.

Definition 2.3: (*ex ante incentive compatible $\varepsilon-$blocking*) Suppose $\varepsilon > 0$. A coalition $S \subseteq N$ can $\varepsilon ICX-$block $(z_i)_{i\in N}$ if there exists $(x_i)_{i\in S} \in A^*_S$ satisfying the following condition:

$$\sum_{t_S\in T_S}\sum_{\theta\in\Theta} u_i(x_i(t_S), \theta)P(\theta, t_S) \geq \sum_{t_S\in T_S}\sum_{t_{N\setminus S}\in T_{N\setminus S}}\sum_{\theta\in\Theta} u_i(z_i(t_{N\setminus S}, t_S), \theta)P(\theta, t_{N\setminus S}, t_S) + \varepsilon$$

for all $i \in S$.

3.2. Incentive Compatible Cores

Definition 3.1: An N-feasible, incentive compatible allocation $(z_i)_{i\in N} \in A^*_N$ for the PIE $(\{e(\theta)\}_{\theta\in\Theta}, \widetilde{\theta}, \widetilde{t}, P)$ is an *Ex Ante Incentive Compatible Core Allocation* if $(z_i)_{i\in N}$ cannot be ICX-blocked by any $S \subseteq N$.

In general, ex ante IC core allocations do not exist.

Definition 3.2: An N-feasible, incentive compatible allocation $(z_i)_{i\in N} \in A_N^*$ for the PIE $(\{e(\theta)\}_{\theta\in\Theta}, \widetilde{\theta}, \tilde{t}, P)$ is an *Ex Ante Incentive Compatible* $\varepsilon-$*Core Allocation* if $(z_i)_{i\in N}$ cannot be ICX-blocked by N and $(z_i)_{i\in N}$ cannot be εICX-blocked by any $S \neq N$.

Obviously, every ex-ante IC core allocation is an ex ante IC ε- core allocation for each $\varepsilon > 0$.

4. Informational Size, Aggregate Uncertainty and Distributional Variability

We will show that there exist incentive compatible core allocations for economies with asymmetric information when agents' informational size is small relative to other properties of the information structure of the economy. In formulating the conditions under which the incentive compatible core is nonempty, we need the notions of informational size, aggregate uncertainty and distributional variability which we introduced in McLean and Postlewaite (2000). We will define these concepts below, but refer the reader to that paper for a full discussion of the concepts.

4.1. Informational Size

If $t \in T$, recall that $P_\Theta(\cdot|t) \in \Delta_\Theta$ denotes the induced conditional probability measure on Θ and $\chi_\theta \in \Delta_\Theta$ denotes the measure that puts probability one on θ. Any vector of agents' types $t = (t_{-i}, t_i) \in T$ induces a conditional distribution on Θ and, if agent i unilaterally changes his announced type from t_i to t_i', this conditional distribution will (in general) change. We consider agent i to be informationally small if, for each t_i, there is a "small" probability that he can induce a "large" change in the induced conditional distribution on Θ by changing his announced type from t_i to some other t_i'. We formalize this in the following definition.

Let

$$I_\varepsilon^i(t_i', t_i) = \{t_{-i} \in T_{-i}| \; ||P_\Theta(\cdot|t_{-i}, t_i) - P_\Theta(\cdot|t_{-i}, t_i')|| > \varepsilon\}.$$

The *informational size* of agent i is defined as

$$\nu_i^P = \max_{t_i\in T_i} \max_{t_i'\in T_i} \inf\{\varepsilon > 0| \text{ Prob}\{\tilde{t}_{-i} \in I_\varepsilon^i(t_i', t_i)|\tilde{t}_i = t_i\} \leq \varepsilon\}.$$

Loosely speaking, we will say that agent i is *informationally small* with respect to P if his informational size ν_i^P is "small." If agent i receives signal t_i but reports $t_i' \neq t_i$, then the effect of this misreport is a change in the conditional distribution on Θ from $P_\Theta(\cdot|t_{-i}, t_i)$ to $P_\Theta(\cdot|t_{-i}, t_i')$. If $t_{-i} \in I_\varepsilon(t_i', t_i)$, then this change is "large" in the sense that $||P_\Theta(\cdot|\hat{t}_{-i}, t_i) - P_\Theta(\cdot|\hat{t}_{-i}, t_i')|| > \varepsilon$. Therefore, $\text{Prob}\{\tilde{t}_{-i} \in I_\varepsilon(t_i', t_i)|\tilde{t}_i = t_i\}$ is the probability that i can have a "large" influence on the conditional distribution on Θ by reporting t_i' instead of t_i when his observed signal is t_i. An agent is informationally small if for each of his possible types t_i, he assigns small probability to the event that he can have a "large" influence on the distribution $P_\Theta(\cdot|t_{-i}, t_i)$, given his observed type.

4.2. Negligible Aggregate Uncertainty

We will next quantify aggregate uncertainty. Let

$$\mu_i^P = \max_{t_i \in T_i} \inf\{\varepsilon > 0 | \text{Prob}\{\tilde{t} \in T^* \text{ and } ||P_\Theta(\cdot|\tilde{t}) - \chi_\theta|| > \varepsilon \text{ for all } \theta \in \Theta | \tilde{t}_i = t_i\} \leq \varepsilon\}.$$

If μ_i^P is small for each i, then we will say that P exhibits *negligible aggregate uncertainty.* In this case, each agent knows that, conditional on his own signal, the aggregate information of all agents will, with high probability, provide a good prediction of the true state.

4.3. Distributional Variability

To define the measure of variability, we first define a metric d on Δ_Θ as follows: for each $\alpha, \beta \in \Delta_\Theta$, let

$$d(\alpha, \beta) = \left\| \frac{\alpha}{||\alpha||_2} - \frac{\beta}{||\beta||_2} \right\|_2$$

where $||\cdot||_2$ denotes the 2-norm. Hence, $d(\alpha, \beta)$ measures the Euclidean distance between the Euclidean normalizations of α and β. If $P \in \Delta_{\Theta \times T}$, let $P_\Theta(\cdot|t_i) \in \Delta_\Theta$ be the conditional distribution on Θ given that i receives signal t_i and define

$$\Lambda_i^P = \min_{t_i \in T_i} \min_{t_i' \in T_i \backslash t_i} d(P_\Theta(\cdot|t_i), P_\Theta(\cdot|t_i'))^2$$

This is the measure of the "variability" of the conditional distribution $P_\Theta(\cdot|t_i)$ as a function of t_i. Let

$$\Delta^*_{\Theta \times T} = \{P \in \Delta_{\Theta \times T} | \text{ for each i, } P_\Theta(\cdot|t_i) \neq P_\Theta(\cdot|t_i') \text{ whenever } t_i \neq t_i'\}.$$

The set $\Delta^*_{\Theta\times T}$ is the collection of distributions on $\Theta \times T$ for which the induced conditionals are different for different types. Hence, $\Lambda_i^P > 0$ for all i whenever $P \in \Delta^*_{\Theta\times T}$.

5. Existence Results

5.1. The Nonemptiness of the Incentive Compatible Core

We now present two results concerning the nonemptiness of the core in the presence of incomplete information.

Theorem 1: Let $\pi \in \Delta_\Theta$ and let

$$\Delta_{\Theta\times T}(\pi) = \{P \in \Delta_{\Theta\times T} | P_\Theta = \pi\}.$$

For every $\varepsilon > 0$, there exists a $\delta > 0$ such that, whenever $P \in \Delta_{\Theta\times T}(\pi)$ and satisfies

$$\max_i \mu_i^P \leq \delta \min_i \Lambda_i^P$$
$$\max_i \nu_i^P \leq \delta \min_i \Lambda_i^P$$

the ex ante incentive compatible $\varepsilon-$core of the PIE $(\{e(\theta)\}_{\theta\in\Theta}, \widetilde{\theta}, \tilde{t}, P)$ is nonempty.

The proof of Theorem 1 is found in the appendix, but we will sketch the proof here. Choose $\varepsilon > 0$ and let $(\zeta_i(\theta_1), .., \zeta_i(\theta_m))_{i\in N}$ be a core allocation of the auxiliary economy $\{\pi, \{e(\theta)\}_{\theta\in\Theta}\}$. There exists a $\delta > 0$ such that, if the conditions of the Theorem are satisfied, we can apply an approximation result in McLean and Postlewaite (2000) (see Lemma A in the appendix) that allows us to find an incentive efficient PIE allocation $(z_i)_{i\in N} \in A_N^*$ for the PIE $(\{e(\theta)\}_{\theta\in\Theta}, \widetilde{\theta}, \tilde{t}, P)$ satisfying

$$\sum_{t\in T}\sum_{\theta\in\Theta} u_i(z_i(t), \theta)P(\theta, t) + \varepsilon > v_i(\zeta_i(\theta_1), .., \zeta_i(\theta_m))$$

for each $i \in N$. Let $S \subseteq N, S \neq N$ and suppose that there exists $(x_i)_{i\in S} \in A_S$ satisfying the following condition:

$$\sum_{t_S\in T_S}\sum_{\theta\in\Theta} u_i(x_i(t_S), \theta)P(\theta, t_S) \geq \sum_{t\in T}\sum_{\theta\in\Theta} u_i(z_i(t), \theta)P(\theta, t) + \varepsilon$$

for all $i \in S$. Defining

$$\xi_i(\theta) = \sum_{t_S\in T_S} x_i(t_S)P(t_S|\theta)$$

for each $i \in S$, it follows that

$$v_i(\xi_i(\theta_1), .., \xi_i(\theta_m)) \geq \sum_k \sum_{t_S \in T_S} u_i(x_i(t_S), \theta_k) P(t_S, \theta_k)$$

since each $u_i(\cdot, \theta)$ is concave. Furthermore, $\sum_{i \in S} \xi_i(\theta) \leq \sum_{i \in S} w_i$. Combining these observations, we conclude that

$$v_i(\xi_i(\theta_1), .., \xi_i(\theta_m)) > v_i(\zeta_i(\theta_1), .., \zeta_i(\theta_m))$$

for each $i \in S$, contradicting the assumption that $(\zeta_i(\theta_1), .., \zeta_i(\theta_m))_{i \in N}$ is a core allocation of the auxiliary economy. Therefore, $(z_i)_{i \in N}$ is an incentive compatible $\varepsilon-$core allocation.

The concavity assumption regarding utility functions guarantees that the core of the auxiliary economy $(\pi, \{e(\theta)\}_{\theta \in \Theta})$ is nonempty and this is enough to show that the ex ante incentive compatible $\varepsilon-$ core is nonempty. However, we need the strict core of the auxiliary economy to be nonempty in order to prove the nonemptiness of the ex ante incentive compatible core.

Theorem 2: Let $\pi \in \Delta_\Theta$ and let

$$\Delta_{\Theta \times T}(\pi) = \{P \in \Delta_{\Theta \times T} | P_\Theta = \pi\}.$$

Furthermore, suppose that the strict core of the auxiliary economy $\{\pi, \{e(\theta)\}_{\theta \in \Theta}\}$ is nonempty. Then there exists a $\delta > 0$ such that, whenever $P \in \Delta_{\Theta \times T}(\pi)$ and satisfies

$$\max_i \mu_i^P \leq \delta \min_i \Lambda_i^P$$

$$\max_i \nu_i^P \leq \delta \min_i \Lambda_i^P$$

the ex ante incentive compatible core of the PIE $(\{e(\theta)\}_{\theta \in \Theta}, \widetilde{\theta}, \tilde{t}, P)$ is nonempty.

The proof of Theorem 2 is also found in the appendix but we will again sketch the idea of the proof. If the strict core of the auxiliary economy is nonempty, then there exists $\varepsilon > 0$ such that the following condition holds:

there is no coalition $S \neq N$ such that, for some $(y_i(\theta_1), .., y_i(\theta_m))_{i \in S} \in \Phi_S$,

$$v_i(y_i(\theta_1), .., y_i(\theta_m)) \geq v_i(x_i(\theta_1), .., x_i(\theta_m)) - \varepsilon$$

for all $i \in S$.

Applying the same approximation result in McLean and Postlewaite (2000) used in the proof of Theorem1, we can find an incentive efficient PIE allocation $z(\cdot)$ for the PIE $(\{e(\theta)\}_{\theta\in\Theta}, \widetilde{\theta}, \tilde{t}, P)$ satisfying

$$\sum_{t\in T}\sum_{\theta\in\Theta} u_i(z_i(t),\theta)P(\theta,t) + \varepsilon > v_i(\zeta_i(\theta_1),..,\zeta_i(\theta_m))$$

for each $i \in N$. Suppose that $S \subseteq N, S \neq N$ and suppose that there exists $(x_i)_{i\in S} \in A_S$ satisfying the following condition:

$$\sum_{t_S\in T_S}\sum_{\theta\in\Theta} u_i(x_i(t_S),\theta)P(\theta,t_S) > \sum_{t\in T}\sum_{\theta\in\Theta} u_i(z_i(t),\theta)P(\theta,t)$$

for all $i \in S$. Defining the allocation $(\xi_i(\theta_1),..,\xi_i(\theta_m))_{i\in N}$ as in the proof of Theorem 1, we can apply the same argument outlined above to conclude that

$$v_i(\xi_i(\theta_1),..,\xi_i(\theta_m)) + \varepsilon > v_i(\zeta_i(\theta_1),..,\zeta_i(\theta_m))$$

for each $i \in S$, a contradiction. Therefore, $(z_i)_{i\in N}$ is an incentive compatible core allocation.

5.2. Strict Cores of Auxiliary Economies

In order to show that the incentive compatible core is nonempty, we assume in Theorem 2 that the strict core of the auxiliary economy is nonempty. In the first result of this section, we provide conditions under which this assumption holds.

Proposition 1: Suppose that $x = (x_i(\theta_1),..,x_i(\theta_m))_{i\in N} \in \Phi_N$ is a Walras equilibrium of the auxiliary economy $e = \{\pi, \{e(\theta)\}_{\theta\in\Theta}\}$ with associated price vector $(p(\theta_1),..,p(\theta_m)) \in R_+^{\ell m}\backslash\{0\}$ satisfying the following assumption:

For each $i \in N$,

$$(y_i(\theta_1),..,y_i(\theta_m)) \in \mathcal{P}_i(x), y_i(\theta) \neq x_i(\theta) \text{ for some } \theta \Rightarrow \sum_\theta p(\theta)\cdot y_i(\theta) > \sum_\theta p(\theta)\cdot w_i$$

If $w_S \neq x_S(\theta)$ for some θ whenever $S \neq N$, then x is a strict core allocation with respect to w.

Proof: First, note that, for each S,

$$(y(\theta_1),..,y(\theta_m)) \in \mathcal{P}_S(x), y(\theta) \neq x_S(\theta) \text{ for some } \theta \Rightarrow \sum_\theta p(\theta)\cdot y(\theta) > \sum_\theta p(\theta)\cdot w_S.$$

To see this, choose $(y(\theta_1), .., y(\theta_m)) \in \mathcal{P}_S(x)$ with $y(\theta) \neq x_S(\theta)$ for some θ. From the definitions, it follows that there exist $z_i, i \in S$ such that $z_S(\theta) = y(\theta)$ for all θ and $z_i \in \mathcal{P}_i(x)$ for each $i \in S$. If $\sum_\theta p(\theta) \cdot z_i(\theta) < \sum_\theta p(\theta) \cdot w_i$, then the monotonicity assumption implies that z_i does not maximize i's utility on his budget set (recall that $(p(\theta_1), .., p(\theta_m)) \in R_+^{\ell m} \backslash \{0\}$). This in turn means that x is not a Walras equilibrium. Therefore, $\sum_\theta p(\theta) \cdot z_i(\theta) \geq \sum_\theta p(\theta) \cdot w_i$ for every $i \in S$. Since $y(\theta) \neq x_S(\theta)$ for some θ, it follows that $z_i(\theta) \neq x_i(\theta)$ for some θ and for some $i \in S$. Hence, the condition of the lemma implies that $\sum_\theta p(\theta) \cdot z_i(\theta) > \sum_\theta p(\theta) \cdot w_i$ for at least one i and we conclude that

$$\sum_\theta p(\theta) \cdot y(\theta) = \sum_\theta p(\theta) \cdot z_S(\theta) > \sum_\theta p(\theta) \cdot w_S.$$

Suppose that $w_S \neq x_S$. If $w_S \in \mathcal{P}_S(x)$, then $\sum_\theta p(\theta) \cdot w_S > \sum_\theta p(\theta) \cdot w_S$, an impossibility. Therefore, $w_S \notin \mathcal{P}_S(x)$.

The proof of Proposition 1 does not use the concavity assumption regarding utility functions. However, the Proposition does imply the following corollary: if $x = (x_i(\theta_1), .., x_i(\theta_m))_{i \in N} \in \Phi_N$ is a Walras equilibrium of the auxiliary economy $e = \{\pi, \{e(\theta)\}_{\theta \in \Theta}\}$, if the utilities $u_i(\cdot, \theta)$ are strictly concave and if $w_S \neq x_S(\theta)$ for some θ whenever $S \neq N$, then x is a strict core allocation with respect to w . We now turn to the question of the nonemptiness of the strict core of an auxiliary economy. We begin with two results that will lead to a "genericity" theorem.

Proposition 2: Suppose that the strict core of the auxiliary economy $\{\pi, \{u_i(\cdot, \theta), w_i\}_{\theta \in \Theta}\}$ is nonempty. Then there exists an $\varepsilon > 0$ such that the strict core of the auxiliary economy $\{\pi, \{u_i(\cdot, \theta), \omega_i\}_{\theta \in \Theta}\}$ is nonempty whenever $||w_i - \omega_i|| < \varepsilon$ for all i.

Proof: Suppose that x is a strict core allocation of the auxiliary economy $\{\pi, \{u_i(\cdot, \theta), w_i\}_{\theta \in \Theta}\}$. We claim that there exists an $\varepsilon > 0$ such that, for each $\omega = (\omega_1, .., \omega_n)$ satisfying $||w_i - \omega_i|| < \varepsilon$ for all i, there exists an $\xi \in \Phi(\omega)$ such that $\hat{\omega}_S \notin \mathcal{P}_S(\xi)$ whenever $S \neq N$. To see this, suppose instead that there exists a sequence $\omega^k \to w$ such that for each k, there exists an $S \neq N$ such that $\hat{\omega}_S^k \in \mathcal{P}_S(\xi)$ whenever $\xi \in \Phi(\omega^k)$. Since there are only finitely many coalitions, we may (choosing a subsequence if necessary) assume that S is independent of k. Let $y^k(\theta) = \omega_N^k - w_N$ for each θ and note that $y^k(\theta) \to 0$. Since $\sum_{i \in N} x_i(\theta) = w_N \in \Re_{++}^\ell$, it follows that, for all sufficiently large k, there exist $y_i^k(\theta)$ with $\sum_{i \in N} y_i^k(\theta) = y^k(\theta)$ such that $x_i(\theta) + y_i^k(\theta) \geq 0, y_i^k(\theta) \to 0$ and

$$\sum_{i \in N} \left[x_i(\theta) + y_i^k(\theta) \right] = \omega_N^k$$

for each θ. Defining $\xi_i^k(\theta) = x_i(\theta)+y_i^k(\theta)$, it follows that $\xi^k = (\xi_i^k(\theta_1),..,\xi_i^k(\theta_m))_{i\in N} \in \Phi_N(\omega^k)$. Therefore, $\hat{\omega}_S^k \in \mathcal{P}_S(\xi^k)$. This means that for all k sufficiently large, there exists $\zeta^k = (\zeta_i^k(\theta_1),..,\zeta_i^k(\theta_m))_{i\in N} \in \Phi_S(\omega^k)$ such that

$$v_i(\zeta_i^k(\theta_1),..,\zeta_i^k(\theta_m)) \geq v_i(\xi_i^k(\theta_1),..,\xi_i^k(\theta_m)).$$

Choosing a subsequence if necessary, we conclude that there exists $\zeta = (\zeta_i(\theta_1),..,\zeta_i(\theta_m))_{i\in N} \in \Phi_S(w)$ such that

$$v_i(\zeta_i(\theta_1),..,\zeta_i(\theta_m)) \geq v_i(x_i(\theta_1),..,x_i(\theta_m))$$

for each $i \in S$. This implies that $w_S \in \mathcal{P}_S(x)$, contradicting the assumption that x is a strict core allocation, and the proof of the claim is complete.

To complete the proof of the theorem, choose $\omega = (\omega_1,..,\omega_n)$ satisfying $||w_i - \omega_i|| < \varepsilon$ for all i and choose $\xi \in \Phi(\omega)$ such that $\hat{\omega}_S \notin \mathcal{P}_S(\xi)$ whenever $S \neq N$. Next, choose an efficient allocation $\hat{\xi}$ for the auxiliary economy satisfying $\hat{\xi} \in \mathcal{P}_i(\xi)$ for each $i \in N$. Since $\mathcal{P}_S(\hat{\xi}) \subseteq \mathcal{P}_S(\xi)$, it follows that $\hat{\omega}_S \notin \mathcal{P}_S(\hat{\xi})$ and we conclude that $\hat{\xi}$ is a strict core allocation.

Proposition 3: Suppose that $x = (x_i(\theta_1),..,x_i(\theta_m))_{i\in N} \in \Phi_N$ is a Walras equilibrium of the auxiliary economy $\{\pi, \{u_i(\cdot,\theta), w_i\}_{\theta\in\Theta}\}$ with associated price vector $(p(\theta_1),..,p(\theta_m)) \in R_+^{\ell m}\backslash\{0\}$ satisfying the following assumption:

For each $i \in N$,

$$(y_i(\theta_1),..,y_i(\theta_m)) \in \mathcal{P}_i(x), y_i(\theta) \neq x_i(\theta) \text{ for some } \theta \Rightarrow \sum_\theta p(\theta)\cdot y_i(\theta) > \sum_\theta p(\theta)\cdot w_i.$$

If there exists $S \neq N$ such that $w_S = x_S(\theta)$ for all θ, then for every $\delta > 0$ there exists a vector ω such that $\omega_i \in R_{++}^\ell$ for all i, $||w_i - \omega_i|| < \delta$ for all i, and x is a strict core allocation of the auxiliary economy $\{\pi, \{u_i(\cdot,\theta), \omega_i\}_{\theta\in\Theta}\}$.

Proof: Let

$$\mathcal{S} = \{S \subseteq N | S \neq N \text{ and } w_S = x_S(\theta) \text{ for all } \theta \in \Theta\}.$$

Let $y \neq 0$ be a net trade vector such that

$$\left[\sum_\theta p(\theta)\right] \cdot y = 0$$

where $(p(\theta_1),..,p(\theta_m)) \in R_+^{\ell m}\backslash\{0\}$ is the equilibrium price vector. Now consider the following perturbations to the endowment vector w:

$$\begin{aligned} w_i'(\varepsilon) &= w_i + \varepsilon y \text{ for } i \neq 1 \\ w_1'(\varepsilon) &= w_1 - (n-1)\varepsilon y \end{aligned}$$

The vector $w_i'(\varepsilon)$ will have positive components for sufficiently small ε. Also, note that

$$w_N'(\varepsilon) = w_N$$

while

$$w_S'(\varepsilon) \neq w_S$$

whenever $S \neq N$. If $S \in \mathcal{S}$, then $w_S'(\varepsilon) \neq x_S(\theta)$ for all θ and for all $\varepsilon > 0$. Since there are only finitely many coalitions and since $w_S'(\varepsilon) \rightarrow w_S$ as $\varepsilon \rightarrow 0$, it follows that $w_S'(\varepsilon) \neq x_S(\theta)$ for at least one θ whenever $S \notin \mathcal{S}$ and ε is small enough. Summarizing, there exists $\hat{\varepsilon} > 0$ such that, whenever $0 < \varepsilon < \hat{\varepsilon}$, there exists $\theta \in \Theta$ such that

$$w_S'(\varepsilon) \neq x_S$$

whenever $S \neq N$. To complete the proof, choose $\delta > 0$ and choose $\varepsilon^* < \hat{\varepsilon}$ so that $||w_i - w_i'(\varepsilon^*)|| < \delta$ for all i and define $\omega_i = w_i'(\varepsilon^*)$. Since

$$\left[\sum_\theta p(\theta)\right] \cdot \omega_i = \left[\sum_\theta p(\theta)\right] \cdot w_i'(\varepsilon^*) = \left[\sum_\theta p(\theta)\right] \cdot w_i$$

for each i, it follows that x is a Walras equilibrium of the auxiliary economy $e = (\pi, (u_i)_{i\in N}, (\omega_i)_{i\in N})$ with associated price vector $(p(\theta_1),..,p(\theta_m))$. Furthermore, x satisfies the following condition:

For each $i \in N$,

$$(y_i(\theta_1),..,y_i(\theta_m)) \in P_i(x), y_i(\theta) \neq x_i(\theta) \text{ for some } \theta \Rightarrow \sum_\theta p(\theta)\cdot y_i(\theta) > \sum_\theta p(\theta)\cdot \omega_i.$$

The proof is now completed by applying Proposition 1 to the auxiliary economy with endowment vector ω.

We can now combine Propositions 2 and 3 and provide a genericity result for strict cores of auxiliary economies.

Corollary: Let

$$X = \{(w_1,..,w_n) \in (\Re_{++}^\ell)^n | \text{ the strict core of } \{\pi, \{u_i(\cdot,\theta), w_i\}_{\theta\in\Theta}\} \text{ is nonempty.}$$

If each $u_i(\cdot,\theta)$ is strictly concave, then X is an open dense subset of $(\Re^{\ell}_{++})^n$.

Proof: Proposition 2 implies that X is open. If $w \notin X$, then the strict concavity assumption implies that there exists a Walras equilibrium of the auxiliary economy satisfying the hypothesis of Theorem 3. Applying Theorem 3, we conclude that X is dense in $(\Re^{\ell}_{++})^n$.

6. The Replica Problem

In the presence of a large number of agents, we might expect any single agent to be informationally small and replica economies are a natural framework in which to investigate this conjecture.

6.1. Notation and Definitions:

Let $\{e(\theta)\}_{\theta\in\Theta}$ be a collection of complete information economies and recall that $J_r = \{1, 2, ...r\}$. For each positive integer r and each θ, let $e^r(\theta) = \{w_{is}, u_{is}(\cdot,\theta)\}_{(i,s)\in N\times J_r}$ denote the r replicated Complete Information Economy (r-CIE) corresponding to state θ satisfying:

(1) $w_{is} = w_i$ for all $s \in J_r$

(2) $u_{is}(z,\theta) = u_i(z,\theta)$ for all $z \in \Re^{\ell}_{+}, i \in N$ and $s \in J_r$.

For any positive integer r, let $T^r = T\times\cdots\times T$ denote the r-fold Cartesian product and let $t^r = (t^r_{\cdot 1}, .., t^r_{\cdot r})$ denote a generic element of T^r where $t^r_{\cdot s} = (t^r_{1s}, .., t^r_{ns})$. If $P^r \in \Delta_{\Theta\times T^r}$, then $e^r = (\{e^r(\theta)\}_{\theta\in\Theta}, \widetilde{\theta}, \tilde{t}^r, P^r)$ is a PIE with nr agents. If $\mathcal{A} = \{\zeta(\theta)\}_{\theta\in\Theta}$ is a collection of CIE allocations for $\{e(\theta)\}_{\theta\in\Theta}$, let $\mathcal{A}^r = \{\zeta^r(\theta)\}_{\theta\in\Theta}$ be the associated "replicated" collection where $\zeta^r(\theta)$ is a CIE allocation for $e^r(\theta)$ satisfying

$$\zeta^r_{is}(\theta) = \zeta_i(\theta) \text{ for each } (i,s) \in N \times J_r$$

6.2. Replica Economies and the Replica Theorem

Definition 4: A sequence of replica economies $\{(\{e^r(\theta)\}_{\theta\in\Theta}, \widetilde{\theta}, \tilde{t}^r, P^r)\}_{r=1}^{\infty}$ is a **conditionally independent sequence** if there exists a $P \in \Delta^*_{\Theta\times T}$ such that

(a) For each r, each $s \in J_r$ and each $(\theta, t_1, .., t_n) \in \Theta \times T$,

$$\text{Prob}\{\widetilde{\theta} = \theta, \widetilde{t}^r_{1s} = t_1, \widetilde{t}^r_{2s} = t_2, ..., \widetilde{t}^r_{ns} = t_n\} = P(\theta, t_1, t_2, ..., t_n)$$

(b) For each r and each θ, the random vectors

$$(\widetilde{t}_{11}^r, \widetilde{t}_{21}^r, ..., \widetilde{t}_{n1}^r), .., (\widetilde{t}_{1r}^r, \widetilde{t}_{2r}^r, ..., \widetilde{t}_{nr}^r)$$

are independent given $\widetilde{\theta} = \theta$.

(c) For every $\theta, \hat{\theta}$ with $\theta \neq \hat{\theta}$, there exists a $t \in T$ such that $P(t|\theta) \neq P(t|\hat{\theta})$.

Thus a conditionally independent sequence is a sequence of PIE's with nr agents containing r "copies" of each agent $i \in N$. Each copy of an agent i is identical, i.e., has the same endowment and the same utility function. Furthermore, the realizations of type profiles across cohorts are independent given the true value of $\widetilde{\theta}$. As r increases each agent is becoming "small" in the economy in terms of endowment, and we can show that each agent is also becoming informationally small. Note that, for large r, an agent may have a small amount of private information regarding the preferences of everyone through his information about $\widetilde{\theta}$. We now state an analogue of Theorem 1 for replica economies.

Theorem 3: Let $\{(\{e^r(\theta)\}_{\theta\in\Theta}, \widetilde{\theta}, \tilde{t}^r, P^r)\}_{r=1}^{\infty}$ be a conditionally independent sequence. For every $\varepsilon > 0$, there exists an integer $\hat{r} > 0$ such that for all $r > \hat{r}$, the ex ante IC $\varepsilon-$core of the PIE $(\{e^r(\theta)\}_{\theta\in\Theta}, \widetilde{\theta}, \tilde{t}^r, P^r)$ is nonempty.

The proof of Theorem 3 (also found in the appendix) is essentially an application of Theorem 1 and again uses the ideas developed in McLean and Postlewaite (2000) (summarized in step 1 of the proof of Theorem 3 in the appendix.)

7. Discussion

1. In independent work, Krasa and Shafer (2000) investigate a question similar to that in this paper. They consider economies with asymmetric information in which agents receive noisy signals of the state of the world. They then study a sequence of economies with incomplete information that converges to an economy with complete information in the sense that the agents' signals are asymptotically perfect signals of the state of the world. Using our notion of strict core, Krasa and Shafer show that, for a sequence of asymmetric information economies that converge to a complete information economy for which the strict core is nonempty, the incentive compatible core will be nonempty for economies close to the limit (see their Theorem 2).

There is a close relationship between their notion of convergence to complete information and our concept of informational size. If the accuracy of all agents' information is increased uniformly, the agents' informational size will go to zero. It can be the case, however, that if the accuracy of one agent's information increases much faster than the accuracy of other agents, then that agent will not become informationally small. In McLean and Postlewaite (2000), it is discussed how our arguments can be extended to this case.

While uniformly increasing the accuracy of agents' signals of the state necessarily makes the agents informationally small, the converse is *not* true. This is demonstrated by our replica theorem. In this case, the agents' information is not becoming increasingly accurate, but they are nevertheless becoming informationally small. The relevant informational consideration for assuring a nonempty incentive compatible core is not that each agent necessarily have very accurate information about the environment. Rather, the relevant consideration is that no agent have a *monopoly* on information about the environment.

2. Forges, Mertens and Vohra (2000) construct an example of a three person asymmetric information economy in which the incentive compatible core is empty. It is straightforward to show that agents are informationally large, as they must be given our results. One can parametrize the information structure in their example so that the parameter controls the information size of agents. In the parametrized version, one can determine the limits on the informational size of the agents that would guarantee that the incentive compatible core is nonempty.

8. Proofs:

8.1. Preliminaries:

Lemma A: Let $\Theta = \{\theta_1, .., \theta_m\}$. Let $\{e(\theta)\}_{\theta\in\Theta}$ be a collection of CIE's and suppose that $\mathcal{A} = \{\zeta(\theta)\}_{\theta\in\Theta}$ is a collection of associated CIE allocations with $\zeta_i(\theta) \neq 0$ for each i and θ. For every $\eta > 0$, there exists a $\delta > 0$ such that, whenever $P \in \Delta_{\Theta\times T}$ and satisfies

$$\max_i \mu_i^P \leq \delta \min_i \Lambda_i^P$$

$$\max_i \nu_i^P \leq \delta \min_i \Lambda_i^P,$$

there exists an incentive compatible PIE allocation $z(\cdot)$ for the PIE $(\{e(\theta)\}_{\theta\in\Theta}, \widetilde{\theta}, \tilde{t}, P)$ and a collection $B_1, .., B_m$ of disjoint subsets of T such that $\text{Prob}\{\tilde{t} \in \cup_{k=1}^m B_k\} \geq 1 - \eta$ and, for all $\theta \in \Theta$ and all $t \in B_k$,

(i) $\text{Prob}\{\tilde{\theta} = \theta_k | \tilde{t} = t\} \geq 1 - \eta$

(ii) For all $i \in N$,

$$u_i(z_i(t);\theta_k) \geq u_i(\zeta_i(\theta_k);\theta_k) - \eta.$$

Proof: See McLean and Postlewaite (2000).

8.2. Proof of Theorem 1:

Choose $\varepsilon > 0$. Let $(\zeta_i(\theta_1), .., \zeta_i(\theta_m))_{i\in N}$ be a core allocation of the auxiliary economy $\{\pi, \{e(\theta)\}_{\theta\in\Theta}\ \}$. Let

$$M = \max_\theta \max_i u_i(\sum_{j=1}^n w_j;\theta),$$

and let

$$\eta = \frac{\varepsilon}{6M+1}.$$

Applying Lemma A, there exists a $\delta > 0$ such that, whenever $P \in \Delta_{\Theta\times T}(\pi)$ and satisfies

$$\max_i \mu_i^P \leq \delta \min_i \Lambda_i^P$$

$$\max_i \nu_i^P \leq \delta \min_i \Lambda_i^P,$$

there exists an incentive compatible PIE allocation $z(\cdot)$ for the PIE $(\{e(\theta)\}_{\theta\in\Theta}, \widetilde{\theta}, \tilde{t}, P)$ and a collection $B_1, .., B_m$ of disjoint subsets of T such that Prob$\{\tilde{t} \in \cup_{k=1}^m B_k\} \geq 1 - \eta$ and, for all $\theta \in \Theta$ and all $t \in B_k$,

(i) Prob$\{\tilde{\theta} = \theta_k | \tilde{t} = t\} \geq 1 - \eta$

(ii) For all $i \in N$,

$$u_i(z_i(t);\theta_k) \geq u_i(\zeta_i(\theta_k);\theta_k) - \eta.$$

Furthermore, $z(\cdot)$ may be chosen to be incentive efficient since the u_i are continuous and A_N^* is compact. Define $B_0 = T\backslash[\cup_{k=1}^m B_k]$.

Let $S \subseteq N, S \neq N$ and suppose that there exists $(x_i)_{i\in S} \in A_S$ satisfying the following condition:

$$\sum_{t_S\in T_S}\sum_{\theta\in\Theta} u_i(x_i(t_S),\theta)P(\theta,t_S) \geq \sum_{t\in T}\sum_{\theta\in\Theta} u_i(z_i(t),\theta)P(\theta,t) + \varepsilon$$

for all $i \in S$. We will show that this leads to a contradiction.

Step 1: For each $\theta \in \Theta$ and each $i \in S$, let

$$\xi_i(\theta) = \sum_{t_S \in T_S} x_i(t_S) P(t_S|\theta)$$

and note that

$$\begin{aligned} \sum_{i \in S} \xi_i(\theta) &= \sum_{t_S \in T_S} \left[\sum_{i \in S} x_i(t_S) \right] P(t_S|\theta) \\ &\leq \sum_{t_S \in T_S} \left[\sum_{i \in S} w_i \right] P(t_S|\theta) \\ &\leq \sum_{i \in S} w_i. \end{aligned}$$

Furthermore, for each $i \in S$,

$$\begin{aligned} v_i(\xi_i(\theta_1), .., \xi_i(\theta_k)) &= \sum_k u_i(\xi_i(\theta_k), \theta_k) \pi(\theta_k) \\ &= \sum_k u_i(\sum_{t_S \in T_S} x_i(t_S) P(t_S|\theta_k), \theta_k) P(\theta_k) \\ &\geq \sum_k \sum_{t_S \in T_S} P(t_S|\theta_k) u_i(x_i(t_S), \theta_k) P(\theta_k) \\ &= \sum_k \sum_{t_S \in T_S} u_i(x_i(t_S), \theta_k) P(t_S, \theta_k) \end{aligned}$$

Step 2: For each $i \in S$,

$$\begin{aligned} \sum_{t \in T} \sum_{\theta \in \Theta} u_i(z_i(t), \theta) P(\theta, t) &= \sum_k \sum_{t \in B_k} \sum_{\theta \in \Theta} u_i(z_i(t), \theta) P(\theta|t) P(t) + \sum_{t \in B_0} \sum_{\theta \in \Theta} u_i(z_i(t), \theta) P(\theta, t) \\ &\geq \sum_k \sum_{t \in B_k} \sum_j u_i(z_i(t), \theta_j) P(\theta_j|t) P(t) - M\eta \\ &\geq \sum_k \sum_{t \in B_k} u_i(\zeta_i(\theta_k), \theta_k) P(t) - (3M+1)\eta \\ &= \sum_k u_i(\zeta_i(\theta_k), \theta_k) \left[\sum_{t \in B_k} P(t) \right] - (3M+1)\eta \\ &\geq \sum_k u_i(\zeta_i(\theta_k), \theta_k) P(\theta_k) - 3M\eta - (3M+1)\eta \\ &= \sum_k u_i(\zeta_i(\theta_k), \theta_k) \pi(\theta) - (6M+1)\eta \\ &= v_i(\zeta_i(\theta_1), .., \zeta_i(\theta_m)) - (6M+1)\eta \end{aligned}$$

Step 3: Combining steps 1 and 2, we conclude that, for each $i \in S$,

$$\begin{aligned} v_i(\xi_i(\theta_1),..,\xi_i(\theta_m)) &\geq \sum_k \sum_{t_S \in T_S} u_i(x_i(t_S),\theta_k)P(t_S,\theta_k) \\ &\geq \sum_{t \in T} \sum_{\theta \in \Theta} u_i(z_i(t),\theta)P(\theta,t) + \varepsilon \\ &\geq v_i(\zeta_i(\theta_1),..,\zeta_i(\theta_m)) - (6M+1)\eta + \varepsilon \\ &> v_i(\zeta_i(\theta_1),..,\zeta_i(\theta_k)) \end{aligned}$$

contradicting the assumption that $\{(\zeta_i(\theta_1),..,\zeta_i(\theta_m))\}_{i \in N}$ is a core allocation of the auxiliary economy $e = \{\pi, \{e(\theta)\}_{\theta \in \Theta}\ \}$.

8.3. Proof of Theorem 2:

Let $\pi \in \Delta_\Theta$. Suppose that each u_i is concave and that ζ is a strict core allocation of $e = \{\pi, \{e(\theta)\}_{\theta \in \Theta}\ \}$.

Step 1: There exists $\varepsilon > 0$ such that the following condition holds:

there is no coalition $S \neq N$ such that, for some $(y_i(\theta_1),..,y_i(\theta_m))_{i \in S} \in \Phi_S$,

$$v_i(y_i(\theta_1),..,y_i(\theta_m)) \geq v_i(x_i(\theta_1),..,x_i(\theta_m)) - \varepsilon$$

for all $i \in S$.

To see this, suppose that, for every k, there is an $S \neq N$ and an allocation $(y_i^k(\theta_1),..,y_i^k(\theta_m))_{i \in S} \in \Phi_S$ such that

$$v_i(y_i^k(\theta_1),..,y_i^k(\theta_m)) \geq v_i(x_i(\theta_1),..,x_i(\theta_m)) - 1/k$$

for each $i \in S$. Since there are only finitely many coalitions, since each Φ_S is compact and each v_i is continuous, it follows that there exists an $S \neq N$ and $(y_i(\theta_1),..,y_i(\theta_m))_{i \in S} \in \Phi_S$ such that

$$v_i(y_i(\theta_1),..,y_i(\theta_m)) \geq v_i(x_i(\theta_1),..,x_i(\theta_m))$$

for each $i \in S$. In particular, $\hat{w}_S = (w_S,..,w_S) \in \mathcal{P}_S(\zeta)$ contradicting the assumption that ζ is a strict core allocation of $\{\pi, \{e(\theta)\}_{\theta \in \Theta}\ \}$.

Step 2: Let

$$M = \max_\theta \max_i u_i(\sum_{j=1}^n w_j;\theta),$$

and let

$$\eta = \frac{\varepsilon}{6M+1}.$$

Applying Lemma A, there exists a $\delta > 0$ such that, whenever $P \in \Delta_{\Theta\times T}(\pi)$ and satisfies

$$\max_i \mu_i^P \leq \delta \min_i \Lambda_i^P$$
$$\max_i \nu_i^P \leq \delta \min_i \Lambda_i^P,$$

there exists an incentive compatible PIE allocation $z(\cdot)$ for the PIE $(\{e(\theta)\}_{\theta\in\Theta}, \widetilde{\theta}, \tilde{t}, P)$ and a collection $A_1, .., A_m$ of disjoint subsets of T such that $\text{Prob}\{\tilde{t} \in \cup_{k=1}^m A_k\} \geq 1-\eta$ and, for all $\theta \in \Theta$ and all $t \in A_k$,

(i) $\text{Prob}\{\tilde{\theta} = \theta_k | \tilde{t} = t\} \geq 1 - \eta$

(ii) For all $i \in N$,

$$u_i(z_i(t); \theta_k) \geq u_i(\zeta_i(\theta_k); \theta_k) - \eta.$$

Furthermore, $z(\cdot)$ may be chosen to be incentive efficient since the u_i are continuous and A_N^* is compact.

Let $S \subseteq N, S \neq N$ and suppose that there exists $(x_i)_{i\in S} \in A_S$ satisfying the following condition:

$$\sum_{t_S\in T_S}\sum_{\theta\in\Theta} u_i(x_i(t_S), \theta)P(\theta, t_S) > \sum_{t\in T}\sum_{\theta\in\Theta} u_i(z_i(t), \theta)P(\theta, t)$$

for all $i \in S$. Defining $(\xi_i(\theta_1), .., \xi_i(\theta_m))_{i\in N}$ as in the proof of Theorem 1 and using the same steps found there, we conclude that, for each $i \in S$,

$$\begin{aligned} v_i(\xi_i(\theta_1), .., \xi_i(\theta_m)) &> \sum_k \sum_{t_S\in T_S} u_i(x_i(t_S), \theta_k)P(t_S, \theta_k) \\ &> \sum_{t\in T}\sum_{\theta\in\Theta} u_i(z_i(t), \theta)P(\theta, t) \\ &\geq v_i(\zeta_i(\theta_1), .., \zeta_i(\theta_m)) - (6M+1)\eta \\ &\geq v_i(\zeta_i(\theta_1), .., \zeta_i(\theta_m)) - \varepsilon. \end{aligned}$$

This contradicts the conclusion of Step 1 and it follows that the ex ante IC core is nonempty.

8.4. Proof of Theorem 3:

Let $\{(\{e^r(\theta)\}_{\theta\in\Theta}, \widetilde{\theta}, \tilde{t}^r, P^r)\}_{r=1}^{\infty}$ be a conditionally independent sequence. >From the definition, it follows that there exists $\pi \in \Delta_\Theta$ such that $P^r_\Theta = \pi$ for all r. Since each $u_i(\cdot;\theta)$ is concave and monotonic and each $w_i \in \Re^\ell_{++}$, it follows that the auxiliary economy has a Walras equilibrium $\zeta = (\zeta_i(\theta_1), .., \zeta_i(\theta_m))_{i\in N}$. We note that the r-replication of ζ^r of the waleas allocation ζ is a core allocation of the r- replicated auxiliary economy for every r. Let

$$M = \max_\theta \max_i u_i(\sum_{j=1}^n w_j; \theta).$$

Choose $\varepsilon > 0$ and let

$$\eta = \frac{\varepsilon}{6M+1}.$$

Step 1: There exists an $\hat{r} > 0$ such that, for all $r > \hat{r}$, there exists an incentive compatible PIE allocation $z^r(\cdot)$ for the PIE $(\{e^r(\theta)\}_{\theta\in\Theta}, \widetilde{\theta}, \tilde{t}^r, P^r)$ and a collection $B^r_1, .., B^r_m$ of disjoint subsets of T^r such that Prob$\{\tilde{t}^r \in \cup_{k=1}^m B^r_k\} \geq 1-\eta$ and, for all $k \in J_m$ and all $t^r \in B^r_k$,

(i) Prob$\{\tilde{\theta} = \theta_k | \tilde{t}^r = t^r\} \geq 1-\eta$

(ii) For all $i \in N$,

$$u_{is}(z^r_{is}(t^r); \theta_k) \geq u_i(\zeta_i(\theta_k); \theta_k) - \eta.$$

(iii) $z^r_{is}(t^r) = z^r_{is'}(t^r)$ whenever $s, s' \in J_r$.

(For a proof, see McLean and Postlewaite (2000).

Again, $z^r(\cdot)$ may be chosen to be incentive efficient since the u_i are continuous and A^{r*}_N is compact.

Define $N_r = N \times J_r$. Next, let $C \subseteq N_r, C \neq N_r$ and suppose that there exists $(x^r_{i,s})_{(i,s)\in C} \in A^r_C$ satisfying the following condition:

$$\sum_{t^r_C\in T^r_C} \sum_{\theta\in\Theta} u_i(x^r_{i,s}(t^r_C), \theta) P(\theta, t^r_C) \geq \sum_{t\in T^r} \sum_{\theta\in\Theta} u_i(z^r_{i,s}(t^r), \theta) P(\theta, t^r) + \varepsilon$$

for each $(i,s) \in C$. Let $C_i = \{s \in J_r | (i,s) \in C\}$ and let $I = \{i | C_i \neq \emptyset\}$.

In the remaining steps, we suppress the dependence of $T^r, T^r_C, t^r, t^r_C, t^r_{i,s}, A^r_k$ etc. on r.

Step 2: For each $\theta \in \Theta$ and each $i \in I$, let

$$\xi_i(\theta) = \frac{1}{|C_i|} \sum_{s \in C_i} \left[\sum_{t_C \in T_C} x_{i,s}(t_C) P(t_C|\theta) \right]$$

and note that

$$\begin{aligned} \sum_{i \in I} |C_i| \xi_i(\theta) &= \sum_{i \in I} \sum_{s \in C_i} \left[\sum_{t_C \in T_C} x_{i,s}(t_C) P(t_C|\theta) \right] \\ &\leq \sum_{t_C \in T_C} \left[\sum_{(i,s) \in C} w_i \right] P(t_C|\theta) \\ &= \sum_{i \in I} |C_i| w_i. \end{aligned}$$

Furthermore, for each $i \in I$,

$$\begin{aligned} v_i(\xi_i(\theta_1), .., \xi_i(\theta_k)) &= \sum_k u_i(\xi_i(\theta_k), \theta_k) \pi(\theta_k) \\ &= \sum_k u_i\left(\frac{1}{|C_i|} \sum_{s \in C_i} \left[\sum_{t_C \in T_C} x_{i,s}(t_C) P(t_C|\theta) \right], \theta_k\right) P(\theta_k) \\ &\geq \sum_k \frac{1}{|C_i|} \sum_{s \in C_i} \sum_{t_C \in T_C} P(t_C|\theta) u_i(x_{i,s}(t_C), \theta_k) P(\theta_k) \\ &= \frac{1}{|C_i|} \sum_{s \in C_i} \left[\sum_{\theta} \sum_{t_C \in T_C} u_i(x_{i,s}(t_C), \theta_k) P(t_C, \theta) \right] \end{aligned}$$

Step 3: For each $(i, s) \in C$,

$$\begin{aligned} \sum_{t \in T} \sum_{\theta \in \Theta} u_i(z_{i,s}(t), \theta) P(\theta, t) &= \sum_k \sum_{t \in A_k} \sum_{\theta \in \Theta} u_i(z_{i,s}(t), \theta) P(\theta|t) P(t) + \sum_{t \in A_0} \sum_{\theta \in \Theta} u_i(z_{i,s}(t), \theta) P(\theta, t) \\ &\geq \sum_k \sum_{t \in A_k} \sum_j u_i(z_{i,s}(t), \theta_j) P(\theta_j|t) P(t) - M\eta \\ &\geq \sum_k \sum_{t \in A_k} u_i(\zeta_i(\theta_k), \theta_k) P(t) - (3M + 1)\eta \\ &= \sum_k u_i(\zeta_i(\theta_k), \theta_k) \left[\sum_{t \in A_k} P(t) \right] - (3M + 1)\eta \end{aligned}$$

$$
\begin{aligned}
&\geq \sum_k u_i(\zeta_i(\theta_k), \theta_k) P(\theta_k) - 3M\eta - (3M+1)\eta \\
&= \sum_k u_i(\zeta_i(\theta_k), \theta_k) \pi(\theta) - (6M+1)\eta \\
&= v_i(\zeta_i(\theta_1), .., \zeta_i(\theta_m)) - (6M+1)\eta
\end{aligned}
$$

Step 4: Combining steps 2 and 3, together with the fact that $z^r_{is}(t^r) = z^r_{is'}(t^r)$ whenever $s, s' \in J_r$, we conclude that, for each $i \in I$,

$$
\begin{aligned}
v_i(\xi_i(\theta_1), .., \xi_i(\theta_m)) &\geq \frac{1}{|C_i|} \sum_{s \in C_i} \left[\sum_{\theta} \sum_{t_C \in T_C} u_i(x_{i,s}(t_C), \theta_k) P(t_C, \theta) \right] \\
&\geq \frac{1}{|C_i|} \sum_{s \in C_i} \left[\sum_{t \in T^r} \sum_{\theta \in \Theta} u_i(z_{i,s}(t), \theta) P(\theta, t) + \varepsilon \right] \\
&= \sum_{t \in T} \sum_{\theta \in \Theta} u_i(z_{i,s}(t), \theta) P(\theta, t) + \varepsilon \\
&\geq v_i(\zeta_i(\theta_1), .., \zeta_i(\theta_m)) - (6M+1)\eta + \varepsilon \\
&> v_i(\zeta_i(\theta_1), .., \zeta_i(\theta_m)).
\end{aligned}
$$

Therefore, the coalition C can improve upon the allocation ζ^r contradicting the assumption that ζ^r is a core allocation of the replicated auxiliary economy.

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