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"Testing for Indeterminacy:
An Application to U. S. Monetary Policy"

by

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An Application to U.S. Monetary Policy*

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Testing for Indeterminacy:

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Abstract

This paper considers a prototypical monetary business cycle model for the U.S. economy, in which the equilibrium is undetermined if monetary policy is 'inactive'. In previous multivariate studies it has been common practice to restrict parameter estimates to values for which the equilibrium is unique. We show how the likelihood-based estimation of dynamic stochastic general equilibrium models can be extended to allow for indeterminacies and sunspot fluctuations. We propose a posterior odds test for the hypothesis that the data are best explained by parameters that imply determinacy. Our empirical results show that the Volcker-Greenspan policy regime is consistent with determinacy, whereas the pre-Volcker regime is not. We find that before 1979 non-fundamental sunspot shocks may have contributed significantly to inflation and interest rate volatility, but essentially did not affect output fluctuations.

1 Introduction

Economists are increasingly making use of dynamic stochastic general equilibrium (DSGE) models for macroeconomic analysis. In order to solve these models and keep them tractable, linear rational expectations (LRE) models are typically used as local approximations. However, it is well known that LRE models can have multiple equilibria. This implies that for every sequence of exogenous shocks there exists more than one sequence of rational expectations errors for which the model dynamics are stable. Under equilibrium indeterminacy non-fundamental sunspot shocks can influence equilibrium allocations and induce business cycle movements that would not be present under determinacy.

Indeterminacy often arises in business cycle models where a monetary authority follows an interest rate rule. For instance, if the nominal interest rate is not raised aggressively enough in response to an increase in inflation, the local equilibrium dynamics are indeterminate. Hence, a central bank that wants to stabilize aggregate fluctuations should avoid policies that lead to this possibility. It has been suggested by Clarida, Galí, and Gertler [5] that the Federal Reserve's policy rule before 1979 was inconsistent with equilibrium determinacy. Beginning with Paul Volcker's tenure as Board Chairman and, later on, Alan Greenspan's the Federal Reserve implemented a much more aggressive rule that suppresses self-fulfilling expectations. Studying indeterminacy can therefore contribute to our understanding of the macroeconomic instability during the 1970s and further the design of beneficial policy rules.

The main objective of this paper is to develop an econometric framework for assessing the quantitative importance of indeterminacy. We use a Bayesian approach to evaluate the indeterminacy hypothesis by the posterior probability of the parameter region for which multiple stable equilibria exist. While in DSGE models with indeterminacies not all parameters are identifiable, we show that the structure of the models nevertheless generates testable implications for the endogenous dynamics. By introducing sunspot shocks we essentially characterize the set of multiple equilibria. Our approach enables us to estimate the stochastic properties of the sunspot shocks and to evaluate its contribution to the fluctuation of, for instance, output, inflation, and interest rates.

We apply our procedure to a New Keynesian business cycle model and revisit the question whether U.S. monetary policy was stabilizing pre- and post-Volcker. Our estimates confirm earlier univariate studies that U.S. monetary policy before 1979 has contributed to aggregate instability and that policy has become markedly more stabilizing during the Volcker-Greenspan period. Based on our variance decompositions, sunspot shocks increased the variability of inflation significantly prior to 1979, but essentially did not affect the volatility of output.

Previous empirical studies of indeterminacy fall broadly into two categories: First, calibration exercises, such as Farmer and Guo [7], Perli [24], Schmitt-Grohé [32, 33], or Thomas [37], that attempt to quantify the extent to which sunspot shocks are helpful in matching model properties to business cycle facts. They face the common difficulty of specifying the stochastic properties of the sunspot shock. The typical practice is to choose its variance to match the observed variance of output. While these authors demonstrate the qualitative importance of indeterminacy, their quantitative conclusions are more tenuous. As we show in this paper, equilibrium indeterminacy does not imply that aggregate fluctuations are in fact driven by sunspots. An alternative empirical approach is taken by Farmer and Guo [8], Salyer [29] and Salyer and Sheffrin [30]. They try to identify sunspot shocks from rational expectations residuals that are left unexplained by exogenous fundamentals. Although this approach imposes more structure than simple calibration, it cannot distinguish between omitted fundamentals and actual sunspots.

The closest theoretical and empirical precursors[1] to this paper are Kim [19], Ireland [16], and Rabanal and Rubio-Ramirez [27] who estimate monetary models similar to ours with likelihood-based techniques. Ireland [16] finds significant evidence

[1] In a much earlier contribution Jovanovic [17] provides a general characterization of the identification problems inherent in econometric analyses of models with multiple equilibria. Cooper [6] surveys different empirical approaches to indeterminacy from several fields.

of a change in monetary policy behavior after 1979. However, all of the above authors explicitly rule out indeterminate equilibria in their estimation strategy. Since the parameters of the monetary policy rule potentially imply indeterminacy, this approach may result in model misspecification and biased parameter estimates.

Our paper also builds on ideas in Clarida *et al.* [5], who estimate a univariate monetary policy rule by GMM over the two subsamples and argue that the pre-Volcker estimates imply indeterminacy. However, univariate estimation results tend to be fragile with respect to the choice of instruments. The potential presence of sunspot fluctuations may cause identification problems that are not readily transparent in a univariate analysis. We provide an example in which the policy parameter is only identifiable under indeterminacy. Indeterminacy is a property of a dynamic system and should therefore be studied through a multivariate analysis. Moreover, the boundary of the determinacy region is generally a function of both the policy parameters and the structural parameters of the model. An accurate assessment requires therefore the joint estimation of all parameters.

This paper is structured as follows. In the next section, we present a structural, New Keynesian monetary business cycle model where the possibility of indeterminacy arises due to the specification of the monetary policy rule. The log-linearized business cycle model belongs to the class of LRE models. In Section 3 we derive and compare the autocovariance properties of LRE models under both determinacy and indeterminacy. Section 4 develops a posterior odds test for indeterminacy. A simplified version of the monetary model is solved analytically in Section 5 in order to illustrate the test procedure. Empirical results for the full model are presented in Section 6. Section 7 concludes and the Appendix contains proofs and computational details.

2 A Model for the Analysis of Monetary Policy

We study a monetary business cycle model with optimizing households and monopolistically competitive firms that face nominal price adjustment costs. The central bank follows a monetary policy rule that adjusts the nominal interest rate in response to changes in the target variables. The model is a variant of what is often referred to as the New Keynesian IS-LM model. Related descriptions and derivations can be found, among others, in Galí and Gertler [11], King [20], King and Wolman [21], and Woodford [38].

2.1 The Household

The economy is populated by a representative household that derives utility from consumption c and real balances M/P, and disutility from working, where n denotes the labor supply:

$$U_t = E_t \left[\sum_{s=t}^{\infty} \beta^{s-t} \left(\frac{c_s^{1-\frac{1}{\tau}} - 1}{1 - \frac{1}{\tau}} + \chi \log \frac{M_s}{P_s} - n_s \right) \right]. \tag{1}$$

$0 < \beta < 1$ is the discount factor, $\tau > 0$ the intertemporal substitution elasticity, and $\chi > 0$ is a scale factor. P is the economy-wide nominal price level which the household takes as given. We define the (gross) inflation rate $\pi_t = P_t/P_{t-1}$.

The household supplies perfectly elastic labor services to the firms period by period for which it receives the real wage w. The household has access to a domestic capital market where nominal government bonds B are traded that pay (gross) interest R. Furthermore, it receives aggregate residual profits Π from the firms and has to pay lump-sum taxes T. Consequently, the household maximizes (1) subject to its budget constraint:

$$c_t + \frac{B_t}{P_t} + \frac{M_t}{P_t} + \frac{T_t}{P_t} = w_t n_t + \frac{M_{t-1}}{P_t} + R_{t-1}\frac{B_{t-1}}{P_t} + \Pi_t. \tag{2}$$

The usual transversality condition on asset accumulation applies which rules out Ponzi-schemes. Initial conditions are given by B_0. The household's behavior is described by the following first-order conditions:

$$c_t^{-1/\tau} = \mathbb{E}_t \left[\beta \frac{R_t}{\pi_{t+1}} c_{t+1}^{-1/\tau} \right], \tag{3}$$

$$\frac{M_t}{P_t} = \chi \frac{R_t}{R_t - 1} c_t^{1/\tau}, \tag{4}$$

$$w_t = c_t^{1/\tau}. \tag{5}$$

and the budget constraint (2).

Equation (3) is the usual consumption Euler equation. Expected consumption growth is thus an increasing function of the real interest rate, which is defined as the nominal interest rate adjusted for (expected) inflation. A monetary policy that controls the nominal interest rate thus affects the time path of consumption. It is through this channel that the possibility of indeterminacy arises. Equation (4) is a standard, interest-sensitive money-demand schedule. Once consumption and the nominal rate are determined, real balances adjust automatically in equilibrium. Specification of an interest rate rule is therefore enough to close the model, and the money demand equation plays no additional role in deriving a solution. Finally, (5) is the utility-weighted labor-supply equation.

2.2 The Firms

The production sector is described by a continuum of monopolistically competitive firms each of which faces a downward-sloping demand curve for its differentiated product:

$$p_t(j) = \left(\frac{y_t(j)}{y_t} \right)^{-1/\nu} P_t. \tag{6}$$

This demand function can be derived in the usual way from Dixit-Stiglitz preferences, whereby $p_t(j)$ is the profit-maximizing price consistent with production level $y_t(j)$. The parameter ν is the elasticity of substitution between two differentiated goods. As $\nu \rightarrow \infty$ the demand function becomes perfectly elastic and the differentiated goods become substitutes. The aggregate price level and aggregate demand y_t are beyond the control of the individual firm.

We introduce nominal rigidity by assuming that firms are subject to quadratic adjustment costs in nominal prices. When a firm wants to change its price beyond the general trend in prices, given by the economy-wide inflation rate π, it incurs 'menu costs' in the form of lost output. The parameter $\varphi \geq 0$ governs the degree of stickiness in the economy. Production is assumed to be linear in labor $n_t(j)$, which each firm hires from the household:

$$y_t(j) = A_t n_t(j). \tag{7}$$

Total factor productivity A_t is an exogenous process. Note that A_t affects all firms in the same way.

The firm j thus chooses factor input $n_t(j)$ to maximize:

$$I\!\!E_t\left[\sum_{s=t}^{\infty} q_s \Pi_s(j)\right] = I\!\!E_t\left[\sum_{s=t}^{\infty} q_s \left[\frac{p_s(j)}{P_s} y_s(j) - w_s n_s(j) - \frac{\varphi}{2}\left(\frac{p_s(j)}{p_{s-1}(j)} - \pi\right)^2 y_s\right]\right] \tag{8}$$

subject to (6) and (7). q is the (potentially) time-dependent discount factor that firms use to evaluate future profit streams. Under the assumption of perfect insurance markets $\frac{q_{t+1}}{q_t} = \beta\left(\frac{c_t}{c_{t+1}}\right)^{1/\tau}$. In other words, since the household is the recipient of the firms' residual payments it directs firms to make decisions based on the households intertemporal rate of substitution (see Kim [19]).

Although firms are heterogeneous ex ante, we impose the usual solution concept for this kind of production structure: We assume that firms behave in an identical way so that the individual firms can be aggregated into a single representative, monopolistically competitive firm. The firm's first-order conditions after imposing the homogeneity assumption are:

$$n_t = \frac{y_t}{w_t}\left(1 - \frac{1}{\vartheta_t}\right), \tag{9}$$

$$\frac{1}{\vartheta_t} = \frac{1}{\nu} I\!\!E_t\left[1 - \varphi\left(\pi_t - \pi\right)\pi_t + \beta\left(\frac{c_t}{c_{t+1}}\right)^{1/\tau} \varphi\left(\pi_{t+1} - \pi\right)\pi_{t+1}\frac{y_{t+1}}{y_t}\right]. \tag{10}$$

ϑ_t can be interpreted as the output demand elasticity augmented by the cost of price adjustment. Note that the mark-up of price over marginal cost is $\left(1 - \frac{1}{\vartheta_t}\right)^{-1}$. In steady state, the mark-up is $\frac{\nu}{\nu-1}$, so that ν indexes the degree of monopolistic distortion in the economy. In the perfectly competitive case, with $\nu \to \infty$, the mark-up is unity. Furthermore, when prices are perfectly flexible ($\varphi = 0$), or when the monetary authority perfectly stabilizes the price level ($\pi_t = \pi$ for all t) the mark-up is constant.

2.3 Monetary Policy Rules

In order to close the model we need to specify the behavior of the monetary authority. We assume that the central bank follows a nominal interest rate rule by adjusting its instrument in response to deviations of output and inflation from their respective target levels. The monetary authority is also concerned about a smooth interest rate path. We consider a monetary policy rule of the form

$$R_t = R_{t-1}^{\rho_R} R_{*,t}^{1-\rho_R} e^{\epsilon_{R,t}}, \tag{11}$$

where the target rate $R_{*,t}$ evolves according to

$$\frac{R_{*,t}}{R} = \left(\frac{\pi_t}{\pi}\right)^{\psi_1} \left(\frac{y_t}{y_{*,t}}\right)^{\psi_2}. \tag{12}$$

The shock $\epsilon_{R,t}$ can be interpreted as monetary policy implementation error or the unsystematic component of monetary policy. Potential output $y_{*,t}$ is defined as the level of output that would prevail under the absence of price stickiness ($\varphi = 0$).

Finally, we need to specify the behavior of the fiscal authority. It is assumed that the government levies a lump-sum tax (or subsidy) $\frac{T_t}{P_t}$ to finance any shortfall in government revenues (or to rebate any surplus):

$$\frac{T_t}{P_t} - \frac{M_t - M_{t-1}}{P_t} + \frac{B_t - R_{t-1}B_{t-1}}{P_t} = g_t^*. \tag{13}$$

The fiscal authority follows a 'passive' rule by endogenously adjusting the primary surplus to changes in the government's outstanding liabilities. The monetary authority, on the other hand, is free to pursue an active or a passive policy. g_t^* denotes aggregate government purchases. We assume for simplicity that the government consumes a fraction of each individual good: $g_t^*(j) = \zeta_t y_t(j)$.[2] Define $\gamma_t = 1/(1-\zeta_t)$. The aggregate demand can then be written in log-linear form as: $\ln y_t = \ln c_t + \ln \gamma_t$.

2.4 Equilibrium and Log-linear Approximation

In order to find a solution to the model, we log-linearize the equations describing the equilibrium around a deterministic steady state. The log-linearized equation

[2] See Galí [10] for a derivation and justification of this assumption.

system can be reduced to three equations in output $\widetilde{y}_t$, inflation $\widetilde{\pi}_t$ and the nominal interest rate $\widetilde{R}_t$, where $\widetilde{x}_t = \ln x_t - \ln x$:

$$
\begin{aligned}
\widetilde{y}_t &= I\!E_t[\widetilde{y}_{t+1}] - \tau(\widetilde{R}_t - I\!E_t[\widetilde{\pi}_{t+1}]) + \widetilde{\gamma}_t - I\!E_t[\widetilde{\gamma}_{t+1}], & (14)\\
\widetilde{\pi}_t &= \beta I\!E_t[\widetilde{\pi}_{t+1}] + \frac{\nu - 1}{\tau\varphi\pi^2}[\widetilde{y}_t - (\tau\widetilde{A}_t + \widetilde{\gamma}_t)], & (15)\\
\widetilde{R}_t &= \rho\widetilde{R}_{t-1} + (1-\rho)\Big(\psi_1\widetilde{\pi}_t + \psi_2[\widetilde{y}_t - (\tau\widetilde{A}_t + \widetilde{\gamma}_t)]\Big) + \epsilon_{R,t}, & (16)
\end{aligned}
$$

The potential output $\widetilde{y}_{*,t} = \tau\widetilde{A}_t + \widetilde{\gamma}_t$. The overall degree of distortion in the economy is measured by $\kappa = (\nu-1)/(\tau\varphi\pi^2)$. Eq. (14), often referred to as the New Keynesian IS-curve, is an intertemporal Euler-equation, while (15) is derived from firms' optimal price-setting problem and governs inflation dynamics around the steady state $\pi \geq 1$. This relation can be interpreted as an (expectational) Phillips-curve with slope κ. Eq. (16) is the log-linearized monetary policy rule where $0 \leq \rho < 1$ is the smoothing coefficient and $\psi_1, \psi_2 \geq 0$ are the policy parameters. We define the exogenous processes[3]

$$
\begin{aligned}
g_t &= \widetilde{\gamma}_t - I\!E_t[\widetilde{\gamma}_{t+1}] & (17)\\
z_t &= \tau\widetilde{A}_t + \widetilde{\gamma}_t & (18)
\end{aligned}
$$

and assume that both g_t and z_t evolve according to univariate AR(1) processes with autoregressive coefficients ρ_g and ρ_z, respectively. The innovations of the AR(1) processes are denoted by $\epsilon_{g,t}$ and $\epsilon_{z,t}$ and assumed to be correlated.

3 Autocovariance Properties of Linear Rational Expectations Models

The log-linearized DSGE model presented in the previous section can be written as a linear rational expectations (LRE) model of the form

$$\Gamma_0(\theta_{(1)})s_t = \Gamma_1(\theta_{(1)})s_{t-1} + \Psi(\theta_{(1)}) + \Pi(\theta_{(1)})\eta_t, \tag{19}$$

[3] Instead of imposing the cross-equation restrictions derived from the theoretical model we allow for a more general covariance structure. This follows common practice in the related literature (see, for instance, Galí [10]). We will return to this issue in Section 5.3.

where

$$s_t = [\widetilde{y}_t, \widetilde{\pi}_t, \widetilde{R}_t, I\!E_t[\widetilde{y}_{t+1}], I\!E_t[\widetilde{\pi}_{t+1}], g_t, z_t]'$$
$$\epsilon_t = [\epsilon_{R,t}, \epsilon_{g,t}, \epsilon_{z,t}]'$$
$$\eta_t = [(\widetilde{y}_t - I\!E_{t-1}[\widetilde{y}_t]), (\widetilde{\pi}_t - I\!E_{t-1}[\widetilde{\pi}_t])]'$$

and $\theta_{(1)} \in \Theta_{(1)}$ stacks the structural parameters of the DSGE model. In general, let s_t be a $n \times 1$ vector. Moreover, the dimension of the vector of fundamental shocks ϵ_t is $l \times 1$ and the dimension of the vector of expectation errors η_t is $k \times 1$. The LRE system has a stable solution if one can choose the expectation errors η_t to suppress the explosive components of s_t.

We assume that in addition to the fundamental shocks ϵ_t agents observe a $p \times 1$ vector of sunspot shocks ζ_t. These sunspot shocks can potentially influence the endogenous variables through the expectation formation embodied in η_t. Fundamental and sunspot shocks are regarded as exogenous. The expectation errors η_t are linear functions of ϵ_t and ζ_t

$$\eta_t = \eta_1(\epsilon_t) + \eta_2(\zeta_t) \tag{20}$$

with the property $\eta_i(0) = 0$ and hence considered endogenous. The solution of the LRE system is unique (*determinacy*) if there is only one mapping from the fundamental shocks to the expectation errors for which s_t is non-explosive and η_t does not depend on the sunspot shocks. *Indeterminacy* refers to the case in which there are many mappings from the fundamental shocks to the expectation errors and sunspot shocks potentially affect the expectation formation. Let $\nu_t = [\epsilon_t', \zeta_t']'$. Throughout this section we will assume that ν_t has mean zero and variance $\Sigma_{\nu\nu}$ conditional on the information set generated by past ν_t's.

The DSGE model is used to obtain a probabilistic representation for a $m \times 1$ vector of observable variables, related to s_t through the linear relationship

$$x_t = \Xi s_t. \tag{21}$$

Under the LRE model x_t is a linear function of current and past shocks ν_t. Since ν_t has a constant variance and we restrict ourselves to stable solutions of the LRE model the process x_t is covariance stationary and its stochastic properties can be characterized through its autocovariance sequence $\Gamma_{xx,h} = I\!\!E[x_t x'_{t-h}]$, $h = 0, 1, 2, \ldots$. The goal of this paper is to devise an empirical test of *determinacy* versus *indeterminacy*. Hence, we will begin by showing to what extent the autocovariance pattern of x_t is distinguishable under the two regimes.

In solving the LRE system we closely follow the approach by Sims [35], extended in Lubik and Schorfheide [23].[4] To keep the exposition simple we assume that the matrix Γ_0 in Eq. (19) is invertible.[5] The system can be rewritten as

$$s_t = \Gamma_1^* s_{t-1} + \Psi^* \epsilon_t + \Pi^* \eta_t. \tag{22}$$

Replace Γ_1^* by its Jordan decomposition $J\Lambda J^{-1}$ and define the vector of transformed model variables $w_t = J^{-1} s_t$. Let the i'th element of w_t be $w_{i,t}$ and denote the i'th row of $J^{-1}\Pi^*$ and $J^{-1}\Psi^*$ by $[J^{-1}\Pi^*]_{i.}$ and $[J^{-1}\Psi^*]_{i.}$, respectively. The model can be rewritten as a collection of AR(1) processes

$$w_{i,t} = \lambda_i w_{i,t-1} + [J^{-1}\Pi^*]_{i.}\epsilon_t + [J^{-1}\Psi^*]_{i.}\eta_t. \tag{23}$$

Define the set of stable AR(1) processes as

$$I_s(\theta_{(1)}) = \left\{ i \in \{1, \ldots n\} \,\middle|\, \left|\lambda_i(\theta_{(1)})\right| < 1 \right\} \tag{24}$$

and let $I_x(\theta_{(1)})$ be its complement. Let Ψ_x^J and Π_x^J be the matrices composed of the row vectors $[J^{-1}\Psi^*]_{i.}$ and $[J^{-1}\Pi^*]_{i.}$ that correspond to unstable eigenvalues, i.e., $i \in I_x(\theta_{(1)})$. To ensure stability of s_t the expectation errors η_t have to satisfy

$$\Psi_x^J \epsilon_t + \Pi_x^J \eta_t = 0 \tag{25}$$

[4]Sims [35] solution method generalizes the method proposed by Blanchard and Kahn [2]. In particular, it does not require the researcher to separate the list of endogenous variables into 'jump' and 'predetermined' variables. It recognizes that it is the structure of the coefficient matrices that implicitly pins down the solution. Instead of imposing ex ante which *individual* variables are 'predetermined', Sims' algorithm determines endogenously the *linear combination* of variables that has to be 'predetermined' for a solution to exist.

[5]If Γ_0 is singular, a generalized complex Schur decomposition (QZ) can be used to manipulate the system, see Sims [35].

for all t. Three cases can be distinguished: (i) Eq. (25) has no solution, (ii) one solution (*determinacy*), and (iii) multiple solutions (*indeterminacy*). We will focus on (ii) and (iii) and assume that the parameter space $\Theta_{(1)}$ is restricted to the set of $\theta_{(1)}$'s for which at least one solution to Equation (25) exists. We characterize the solution in the following proposition, which is proved in Lubik and Schorfheide [23].

Proposition 1 *Let UDV' be the singular value decomposition of Π_x^J and partition $U = [U_{.1}, U_{.2}]$, $D = diag(D_{11}, D_{22})$, $V = [V_{.1}, V_{.2}]$. The partitions $U_{.1}$ and $V_{.1}$ conform with the non-zero singular values D_{11}. If there exists a solution to Eq. (25) that expresses the forecast errors as function of the exogenous shocks ν_t, it is of the form*

$$\begin{aligned} \eta_t &= \eta_1(\epsilon_t) + \eta_2(\zeta_t) \\ &= (-V_{.1} D_{11}^{-1} U_{.1}' \Psi_x^J + V_{.2} M_1)\epsilon_t + V_{.2} M_2 \zeta_t, \end{aligned} \tag{26}$$

where r is the number of singular values that are equal to zero, M_1 is an $r \times l$ matrix, M_2 is an $r \times p$ matrix, and the dimension of $V_{.2}$ is $k \times r$. The solution is unique if $r = 0$ and $V_{.2}$ is empty.

The parameter space can be partitioned into two regions. For $\theta_{(1)} \in \Theta_{(1)}^D$ the LRE system has a unique stable solution, that is, $r = 0$. If $\theta_{(1)} \in \Theta_{(1)}^I$ then $r > 0$ and there exist multiple stable solutions. Eq. (26) shows that indeterminacy has two effects. First, without additional assumptions the impact of the structural shocks on the forecast errors is not determined, since $\theta_{(1)}$ does not pin down the matrix M_1. Second, the sunspot shocks ζ_t can influence the model dynamics through the expectation errors η_t ($M_2 \neq 0$).

We will introduce the parameter vector $\theta_{(2)} \in \Theta_{(2)}$ to index the matrices M_1 and M_2 that appear in Eq. (26). The vector $\theta_{(2)}$ essentially indexes the multiple equilibria that arise under indeterminacy. The structural parameter vector is extended to $\theta = [\theta_{(1)}', \theta_{(2)}']' \in \Theta$. We partition the domain of θ into Θ^D and Θ^I, which conform with the partitions of $\theta_{(1)}$. If $\theta_{(1)} \in \Theta_{(1)}^D$ then the value of $\theta_{(2)}$ does not matter, that is, the models $[\theta_{(1)}', \theta_{(2)}']'$ and $[\theta_{(1)}', \tilde{\theta}_{(2)}']'$ are observationally equivalent.

Under the assumption that $w_{i,0} = 0$ for $i \in I_x(\theta_{(1)})$ we obtain the following result

$$w_{i,t} = \begin{cases} \lambda_i(\theta_{(i)}) w_{i,t-1} + \mu_i(\theta_{(1)}, \theta_{(2)})\nu_t & i \in I_s(\theta_{(1)}) \\ 0 & i \in I_x(\theta_{(1)}), \end{cases} \tag{27}$$

where μ_i can be obtained by substituting (26) into Eq. (23). Thus, by construction of the solution some of the AR(1) processes are equal to zero. Since x_t is a linear combination of the $w_{i,t}$ processes, its autocovariances are given by

$$I\!E[x_{k,t} x_{l,t-h}] = \sum_{i \in I_s} \lambda_i^h \left[\sum_{j \in I_s} \frac{1}{1 - \lambda_i \lambda_j} \Xi_{k,i} \mu_i \Sigma_{\nu\nu} \mu_j' \Xi_{j,l} \right]. \tag{28}$$

More generally, the autocovariance matrices of x_t have a representation of the form

$$\Gamma_{xx,h}(\theta) = \sum_{i \in I_s(\theta_{(1)})} \lambda_i^h(\theta_{(1)}) D_i(\theta_{(1)}, \theta_{(2)}). \tag{29}$$

The autocovariance sequence $\{\Gamma_{xx,h}(\theta^*)\}_{h=0}^{\infty}$ provides enough information to classify the parameter $\theta^* \in \Theta^I$ as belonging to the indeterminacy region if there does not exist a $\theta \in \Theta^D$ such that $\{\Gamma_{xx,h}(\theta)\}_{h=0}^{\infty} = \{\Gamma_{xx,h}(\theta^*)\}_{h=0}^{\infty}$. Let $\Lambda(\theta)$ be the set of non-repeated stable eigenvalues $\lambda_i(\theta_{(1)})$, $i \in I_s(\theta_{(1)})$. The following proposition provides a sufficient condition for the identifiability of indeterminacy based on autocovariance sequences. The proposition is a direct consequence of Eq. (29).

Proposition 2 *Suppose $\theta^* \in \Theta^s$, where $s \in \{D, I\}$. Then θ^* can be correctly classified based on the sequence of autocovariances if $\Lambda(\theta) \neq \Lambda(\theta^*)$ for all $\theta \in \Theta \backslash \Theta^s$.*

Under indeterminacy the number of stable eigenvalues is generally larger than under determinacy. Thus, one can expect a richer autocovariance pattern that cannot be reproduced with parameters from the determinacy region. The intuition for our test procedure is that indeterminacy is potentially[6] identifiable based on the autocovariance sequence of x_t, which can be estimated from the data.

Since every $x_{i,t}$ is a linear combination of at most n AR(1) processes, the autocovariance structure of x_t can be reproduced with an ARMA(p,q) process, where

[6]Two notable exceptions are indeterminacy parameters that lead to repeated eigenvalues, and equilibria under indeterminacy for which $\mu_i(\theta_{(1)}, \theta_{(2)})\nu_t = 0$ for all t and some $i \in I_s(\theta_{(1)})$.

$p \leq n$ and $q \leq n-1$. We can represent

$$x_t = A_t + A_1 x_{t-1} + \ldots + A_n x_{t-n} + u_t + B_1 u_{t-1} + \ldots + B_{n-1} u_{t-n+1}, \qquad (30)$$

where the innovations u_t are functions of the fundamental shocks ϵ_t and the sunspot shocks ζ_t with conditional covariance Σ_{uu}. We express Σ_{uu} and the VARMA coefficient matrices as functions of a non-redundant parameter vector $\phi \in \Phi$. The DSGE model imposes nonlinear restrictions on the parameters of the VARMA representation.

4 Econometric Approach

This section develops a test of the hypothesis $\theta \in \Theta^D$ versus $\theta \in \Theta^I$ and outlines our strategy to obtain an estimate of θ based on a sequence of endogenous variables $X^T = \{x_1, \ldots, x_T\}$. For the remainder of the paper we assume that θ completely characterizes the probability distribution of the observables X^T so that we have a parametric likelihood function available. This can be achieved, for instance, by assuming that ν_t is normally distributed. We showed in the previous section that testing for determinacy is closely related to determining the number of latent (non-zero) autoregressive state variables $w_{i,t}$ that determine the evolution of x_t.

In the context of classical hypothesis testing, determining the number of latent state variables is a non-standard problem. Suppose under the null hypothesis, H_0, x_t is driven by two latent AR(1) processes, whereas under the alternative, H_a, x_t is a function of three latent AR(1) processes. The autoregressive coefficient of the third state process is not identifiable under the null hypothesis because the innovation to the third state process is zero under H_0. The econometric implications of parameters that are not identifiable under the null hypothesis have been studied from a frequentist perspective in, for instance, Andrews and Ploberger [1] and Hansen [14].

In the case of testing determinacy versus indeterminacy the number of latent states is only one piece of information that can be used to discriminate the two hypotheses. The second piece of information is embodied in the cross-equation restrictions that the DSGE model potentially imposes on the state space representation of

x_t. We will use a Bayesian approach that exploits both pieces of information simultaneously. From a joint probability distribution of the data X^T and the structural parameters θ a posterior distribution of θ given X^T will be derived. Our assessment of the empirical relevance of indeterminacy and sunspot fluctuations is based on the prior and posterior probability mass assigned to the indeterminacy region Θ^I.

The parameter $\phi \in \Phi$ of the reduced form VARMA representation (30) is used to index the probability distribution of the data. The likelihood function of ϕ is denoted by $L(\phi|X^T)$. The solution algorithm described in Section 3 provides a mapping $\mathcal{T}$ from the (extended) structural parameter space Θ into the reduced form parameter space, denoted by Φ. The mapping $\mathcal{T}$, however, is not one-to-one, which creates the identification problems in this model. For instance, if $\theta \in \Theta^D$, the parameters that are used to index multiple equilibria under indeterminacy, $\theta_{(2)}$, have no effect on the reduced form of x_t. The likelihood function for the structural parameters is given by $L(\mathcal{T}(\theta)|X^T)$.

We will use P and $P_{X,T}$ to denote the prior and posterior distribution of θ. Moreover, Q and $Q_{X,T}$ signify prior and posterior distribution of ϕ. According to Bayes Theorem, the posterior distribution of the structural parameters has the following density (with respect to its prior distribution)

$$\frac{dP_{X,T}}{dP} = \frac{L(\mathcal{T}(\theta)|X^T)}{\int L(\mathcal{T}(\theta)|X^T)dP}. \tag{31}$$

Since the mapping $\mathcal{T}$ is not one-to-one, the likelihood function is flat in some directions of the parameter space. More precisely, the likelihood function is constant on the sets $\{\theta : \mathcal{T}(\theta) = \phi\}$.

The denominator in Equation (31) can be rewritten as

$$\begin{aligned} \int L(\mathcal{T}(\theta)|X^T)dP &= \int [\int L(\mathcal{T}(\theta)|X^T)dP_\phi]dQ \\ &= \int L(\phi|X^T)dQ, \end{aligned} \tag{32}$$

where P_ϕ is the conditional prior distribution of θ given that $\mathcal{T}(\theta) = \phi$. The distribution of $\mathcal{T}(\theta)$ under the Bayesian posterior $I\!\!P_{X,T}$ is the posterior distribution of the reduced form parameters, denoted by $Q_{X,T}$. It has density

$$\frac{dQ_{X,T}}{dQ} = \frac{L(\phi|X^T)}{\int L(\phi|X^T)dQ} \tag{33}$$

with respect to Q.

The posterior expectation of a function $f(\theta)$ is computed as follows. First, integrate $f(\theta)$ on the sets $\{\theta : \mathcal{T}(\theta) = \phi\}$ with respect to the conditional prior distribution of θ given ϕ, P_ϕ. The likelihood function is constant on these sets and the data do not lead to an updating of the distribution P_ϕ.[7] Second, the reduced form parameter ϕ is integrated out with respect to its posterior, $Q_{X,T}$. Since the likelihood function is informative about ϕ its distribution is updated. Formally,

$$\begin{aligned}
\int f(\theta)dP_{X,T} &= \int f(\theta)\frac{L(\mathcal{T}(\theta)|X^T)}{\int L(\mathcal{T}(\theta)|X^T)dP}dP \qquad (34)\\
&= \int \left[\int f(\theta)\frac{L(\mathcal{T}(\theta)|X^T)}{\int L(\mathcal{T}(\theta)|X^T)dP}dP_\phi\right]dQ \\
&= \int \left[\int f(\theta)dP_\phi\right]dQ_{X,T}.
\end{aligned}$$

Despite a lack of identification of the structural parameters, the prior distribution of θ is updated in the directions in which the data are informative and the resulting posterior distribution can be used for inference.

The probability mass assigned to the indeterminacy region of the structural parameter space will be used to assess the evidence in favor of the indeterminacy hypothesis. The prior probability mass is given by

$$\pi_0(I) = \int \left[\int \{\theta \in \Theta^I\}dP_\phi\right]dQ, \tag{35}$$

whereas the posterior probability mass is

$$\pi_T(I) = \int \left[\int \{\theta \in \Theta^I\}dP_\phi\right]dQ_{X,T}. \tag{36}$$

Suppose that data are generated from the LRE model (19, 21) with parameters θ_0. Let $\phi_0 = \mathcal{T}(\theta_0)$. Under mild regularity conditions, e.g. Kim [18], the posterior

[7] This is a general feature of Bayesian inference for models with non-identifiable parameters, see Poirier [26].

distribution of the reduced form parameters begins to concentrate around ϕ_0 as more observations become available. The posterior probability of indeterminacy will asymptotically converge to the prior probability of indeterminacy conditional on the reduced form parameters taking the 'true' value ϕ_0. The result is summarized in the following proposition. '$\stackrel{p}{\longrightarrow}$' signifies convergence in distribution under the probability distribution of X^T.

Proposition 3 *Let a δ-neighborhood of ϕ_0, $\mathcal{N}_\delta(\phi_0)$, be a subset of Φ such that $|\phi - \phi_0| < \delta$. Suppose there exists a collection of δ-neighborhoods $\{\mathcal{N}_\delta(\phi_0)\}_{\delta>0}$ such that $g(\phi) = \int \{\theta \in \Theta^I\} dP_\phi$ is continuous at ϕ_0 within $\mathcal{N}_\delta(\phi_0)$ and the posterior distribution of ϕ concentrates in the δ-neighborhood in the sense $\int_{\mathcal{N}_\delta(\phi_0)} dQ_{X,T} \stackrel{p}{\longrightarrow} 1$, then*

$$\pi_T(I) \stackrel{p}{\longrightarrow} \int \{\theta \in \Theta^I\} dP_{\phi_0} \tag{37}$$

as $T \longrightarrow \infty$.

Suppose that data are generated from the indeterminacy region, $\theta_0 \in \Theta^I$, and the distribution of X^T under θ_0 is distinguishable, e.g., based on the number of latent AR(1) processes, from the distributions that can arise under determinacy, that is, $\mathcal{T}^{-1}(\phi_0) \in \Theta^I$.[8] The Proposition implies that in this case the posterior probability of indeterminacy will tend to one. Regardless of their prior distribution, econometricians will reach the same conclusion as $T \longrightarrow \infty$. If the reduced form representation is consistent with both regimes then the posterior probability of indeterminacy will converge to the prior probability conditional on ϕ_0. We will examine the concentration of the posterior and the consistency of the posterior-probability based assessment of the indeterminacy hypothesis in the specific context of the monetary DSGE model.

[8] The set $\mathcal{T}^{-1}(\phi)$ is defined as $\{\theta \in \Theta : \mathcal{T}(\theta) = \phi\}$.

5 A Simplified Version of the Monetary DSGE Model

Before applying our solution method and inference procedure to the full model described in Section 2, we will study a simplified version of the DSGE model analytically. We assume that the monetary authority does not attempt to smooth the nominal interest rate ($\rho_R = 0$), and it only targets current inflation ($\psi_2 = 0$). Furthermore, the exogenous processes g_t and z_t are serially uncorrelated ($\rho_g = \rho_z = 0$). Define the conditional expectations $\xi_t^y = I\!\!E_t[\widetilde{y}_{t+1}]$, $\xi_t^\pi = I\!\!E_t[\widetilde{\pi}_{t+1}]$, and the expectation errors $\eta_t^y = \widetilde{y}_t - \xi_{t-1}^y$, $\eta_t^\pi = \widetilde{\pi}_t - \xi_{t-1}^\pi$. Instead of applying the solution algorithm to the full DSGE model, we exploit its block-triangular structure. Notice that

$$\begin{bmatrix} \widetilde{y}_t \\ \widetilde{\pi}_t \\ \widetilde{R}_t \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & \psi_1 \end{bmatrix} \begin{bmatrix} \xi_t^y \\ \xi_t^\pi \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \epsilon_{R,t} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & \psi_1 \end{bmatrix} \begin{bmatrix} \eta_t^y \\ \eta_t^\pi \end{bmatrix}. \tag{38}$$

The conditional expectations $\xi_t = [\xi_t^y, \xi_t^\pi]'$ evolve according to

$$\xi_t = \underbrace{\begin{bmatrix} 1 + \frac{\kappa\tau}{\beta} & \tau(\psi_1 - \frac{1}{\beta}) \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix}}_{\Gamma_1^*} \xi_{t-1} + \underbrace{\begin{bmatrix} \tau & -1 & 0 \\ 0 & 0 & \kappa \end{bmatrix}}_{\Psi^*} \epsilon_t + \underbrace{\begin{bmatrix} 1 + \frac{\kappa\tau}{\beta} & \tau(\psi_1 - \frac{1}{\beta}) \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix}}_{\Pi^*} \eta_t. \tag{39}$$

The vector of expectation errors $\eta_t = [\eta_t^y, \eta_t^\pi]'$ can be determined by solving the two-dimensional subsystem (39). Replace Γ_1^* by its Jordan decomposition $J\Lambda J^{-1}$, define $w_t = J^{-1}\xi_t$, and rewrite the model as

$$w_t = \Lambda w_{t-1} + J^{-1}\Psi^*\epsilon_t + J^{-1}\Pi^*\eta_t. \tag{40}$$

Tedious but straightforward algebra yields the eigenvalues that appear in the diagonal matrix Λ in Eq. (40):

$$\lambda_1, \lambda_2 = \frac{1}{2}\left(1 + \frac{\kappa+1}{\beta}\right) \mp \frac{1}{2}\sqrt{\left(\frac{\kappa+1}{\beta} - 1\right)^2 + \frac{4\kappa}{\beta}(1 - \psi_1)}. \tag{41}$$

It can be shown[9] that the determinacy properties of this model solely hinge on the policy parameter ψ_1.

[9] See Bullard and Mitra [3] or Lubik and Marzo [22].

5.1 Determinacy

If $\psi_1 > 1$ then both λ_1 and λ_2 are greater than one in absolute value. Thus, $I_x(\theta_{(1)}) = \{1,2\}$ if $\theta_{(1)} \in \Theta^D_{(1)}$. The only stable solution of (39) is $\xi_t = 0$, which is obtained if $\xi_0 = 0$ and

$$\eta_t = -{\Pi^*}^{-1}\Psi^*\epsilon_t = -\frac{1}{1+\kappa\tau\psi_1}\begin{bmatrix} \tau & -1 & -\tau\kappa\psi_1 \\ \kappa\tau & -\kappa & \kappa \end{bmatrix}\epsilon_t. \tag{42}$$

Thus, the expectation errors η_t are uniquely determined as functions of the structural shocks and the evolution of output, inflation, and interest rates is given by

$$\begin{bmatrix} \widetilde{y}_t \\ \widetilde{\pi}_t \\ \widetilde{R}_t \end{bmatrix} = \frac{1}{1+\kappa\tau\psi_1}\begin{bmatrix} -\tau & 1 & \tau\kappa\psi_1 \\ -\kappa\tau & \kappa & -\kappa \\ 1 & \kappa\psi_1 & -\kappa\psi_1 \end{bmatrix}\begin{bmatrix} \epsilon_{R,t} \\ \epsilon_{g,t} \\ \epsilon_{z,t} \end{bmatrix}. \tag{43}$$

The model exhibits no dynamics because the solution suppresses the two roots of the autoregressive matrix Γ_1^*. An unanticipated monetary contraction leads to a one-period fall in output and inflation. An Euler-equation shock $\epsilon_{g,t}$ increases output, inflation, and interest rate. A Phillips-curve shock $\epsilon_{z,t}$ raises output, but lowers inflation and interest rates for one period.

5.2 Indeterminacy

If the inflation elasticity of the interest rate rule is less than one, then the smaller of the two eigenvalues, λ_1, is less than one and only the second element, $w_{2,t}$, of the 2×1 vector w_t is explosive. Thus, $I_x(\theta_{(1)}) = \{2\}$ and the stability condition (25) consists of one equation, that does not uniquely determine the 2×1 vector of expectation errors η_t. The full set of solutions for η_t is given by Proposition 1.

It is useful to transform the representation of η_t given in Proposition 1 as follows. Notice that in this model the set $I_x(\theta_{(1)}) = \{1,2\} = I_x^D$ is constant in the determinacy region $\theta_{(1)} \in \Theta^D_{(1)}$ and nests $I_x(\theta_{(1)}) = \{2\}$ for $\theta_{(1)} \in \Theta^I_{(1)}$. Thus, one solution to the stability condition can be obtained by using the equations

$$[J^{-1}\Pi^*]_{i.}\epsilon_t + [J^{-1}\Psi^*]_{i.}\eta_t \tag{44}$$

for $i \in I_x^D$ even though $\theta_{(1)} \in \Theta^I_{(1)}$. For the simple monetary model we obtain

$$\eta_t = \left(-\frac{1}{1+\kappa\tau\psi_1} \begin{bmatrix} \tau & -1 & -\tau\kappa\psi_1 \\ \kappa\tau & -\kappa & \kappa \end{bmatrix} + \begin{bmatrix} (\lambda_2 - 1 - \kappa\tau\psi_1) \\ \kappa\lambda_2 \end{bmatrix} \tilde{M}_1 \right) \epsilon_t + \begin{bmatrix} (\lambda_2 - 1 - \kappa\tau\psi_1) \\ \kappa\lambda_2 \end{bmatrix} \zeta_t^*, \tag{45}$$

where $\zeta_t^* = M_2\zeta_t$ is a normalized sunspot shock and $\tilde{M}_1$ is a 1×3 matrix that is not determined by the parameters $\theta_{(1)}$ of the DSGE model. It can be shown that both $\lambda_2 \geq 0$ and $\lambda_2 - 1 - \kappa\tau\psi \geq 0$. Thus a positive normalized sunspot shock raises the expectation errors in output and inflation. The effect of the structural shocks on the expectation errors is not uniquely determined.

Define the following linear combination of fundamental shock and sunspot shocks

$$\eta_t^* = \begin{bmatrix} (\lambda_2 - 1 - \kappa\tau\psi_1) \\ \kappa\lambda_2 \end{bmatrix} (\tilde{M}_1\epsilon_t + \zeta_t^*). \tag{46}$$

The law of motion for output, inflation, and interest rate becomes

$$\begin{bmatrix} \widetilde{y}_t \\ \widetilde{\pi}_t \\ \widetilde{R}_t \end{bmatrix} = \frac{1}{1+\kappa\tau\psi_1} \begin{bmatrix} -\tau & 1 & \tau\kappa\psi_1 & (\lambda_2 - 1 - \kappa\tau\psi_1) \\ -\kappa\tau & \kappa & -\kappa & \kappa\lambda_2 \\ 1 & \kappa\psi_1 & -\kappa\psi_1 & \psi_1\kappa\lambda_2 \end{bmatrix} \begin{bmatrix} \epsilon_{R,t} \\ \epsilon_{g,t} \\ \epsilon_{z,t} \\ \eta_t^* \end{bmatrix} + \begin{bmatrix} (\beta(\lambda_2 - 1) - \tau\kappa)/\kappa \\ 1 \\ \psi_1 \end{bmatrix} w_{1,t-1}, \tag{47}$$

where $w_{1,t}$ follows the AR(1) process

$$w_{1,t} = \lambda_1 w_{1,t-1} + \tilde{\mu}_1 \eta_t^*. \tag{48}$$

In the indeterminacy case only one root of Γ_1^* is suppressed and the endogenous variables are generally serially correlated through the process $w_{1,t}$. Notice that the fundamental shocks ϵ_t affect the latent process $w_{1,t}$ only indirectly through η_t^*. Thus, if $\tilde{M}_1 = 0$ the effects of fundamental shocks vanish after one period.

5.3 Orthogonal Shocks

The monetary DSGE model is driven by three fundamental shocks, namely a monetary policy shock $\epsilon_{R,t}$, a shock to the Euler equation, $\epsilon_{g,t}$, a shock to the Phillips curve, $\epsilon_{z,t}$, and potentially a sunspot shock, ζ_t^*. We assume that the sunspot shock ζ_t^* is unrelated to the fundamental shocks ϵ_t. Moreover, the monetary policy shock is uncorrelated with $\epsilon_{g,t}$ and $\epsilon_{z,t}$, as the reaction function of the central bank captures any systematic responses of the monetary policy to changes in the fundamentals. While we allow $\epsilon_{g,t}$ and $\epsilon_{z,t}$ to be generally correlated, we assume that they are functions of orthogonal 'demand' and 'supply' shocks. The 'demand' shock, such as a government spending shock, shifts both the Euler equation and the Phillips curve, whereas the 'supply' shock, such as a technology shock, only affects the price setting equation (see Section 2).

Let $\nu_t = [\epsilon_{R,t}, \epsilon_{g,t}, \epsilon_{z,t}, \eta_t^*]'$, where η_t^* is defined in Eq. (46). To complete the probabilistic specification of the model, let $\nu_t \sim iid\mathcal{N}(0, \Sigma_{\nu\nu})$, where

$$\Sigma_{\nu\nu} = \begin{bmatrix} \sigma_R^2 & 0 & 0 & \rho_{R\eta}\sigma_R\sigma_\eta \\ & \sigma_g^2 & \rho_{gz}\sigma_g\sigma_z & \rho_{g\eta}\sigma_g\sigma_\eta \\ & & \sigma_z^2 & \rho_{z\eta}\sigma_z\sigma_\eta \\ & & & \sigma_\eta^2 \end{bmatrix}. \tag{49}$$

The correlations are restricted to ensure that the covariance matrix is positive. The correlations $\rho_{R\eta}$, $\rho_{g\eta}$, and $\rho_{z\eta}$ arise because the effect of the fundamental shocks on the expectation errors, i.e., the matrix $\tilde{M}_1$ in Eq. (46), is not uniquely pinned down under indeterminacy. In the notation of Section 3, the correlations would enter the extension $\theta_{(2)}$ of the structural parameter vector.

The relationship between ν_t and the normalized and orthogonalized shocks is given by

$$\begin{bmatrix} \epsilon_{R,t} \\ \epsilon_{g,t} \\ \epsilon_{z,t} \\ \eta_t^* \end{bmatrix} = \begin{bmatrix} c_{11} & 0 & 0 & 0 \\ 0 & c_{22} & c_{23} & 0 \\ 0 & 0 & c_{33} & 0 \\ c_{41} & c_{42} & c_{43} & c_{44} \end{bmatrix} \begin{bmatrix} \tilde{\epsilon}_{R,t} \\ \tilde{\epsilon}_{g,t} \\ \tilde{\epsilon}_{z,t} \\ \tilde{\zeta}_t^* \end{bmatrix}. \tag{50}$$

The coefficients c_{ij} can be uniquely determined from the covariance matrix $\Sigma_{\nu\nu}$. If $\tilde{M}_1 = 0$ in Eq. (46) then $\rho_{R\eta} = \rho_{g\eta} = \rho_{z\eta} = 0$ and $c_{41} = c_{42} = c_{43} = 0$. In this case the orthogonalized fundamental shocks do not enter the model through η_t^* and the response of the endogenous variables to the fundamental shocks is continuous in the parameters at the boundary of the determinacy region $\psi_1 = 1$. If in addition the variance of the sunspot shock ζ_t^* is zero, then $\sigma_\eta = 0$ and Eq. (47) reduces to the law of motion obtained under determinacy in Eq. (43).

5.4 Identification of Indeterminacy Regimes in Large Samples

This section examines whether the posterior probability of indeterminacy is consistent in the sense that it converges to one if X^T was generated from the indeterminacy region and to zero otherwise. The (extended) vector of structural parameters of the simplified monetary DSGE model is

$$\theta = [\psi_1, \kappa, \tau, \rho_{gz}, \rho_{R\eta}, \rho_{g\eta}, \rho_{z\eta}, \sigma_R, \sigma_g, \sigma_z, \sigma_\eta]' \in \Theta \subset I\!R^{11}. \tag{51}$$

The results of Section 5.1 and 5.2 imply that the reduced form of the DSGE model is ARMA(1,1):

$$x_t = Ax_{t-1} + u_t + Bu_{t-1}, \quad u_t \sim iid\mathcal{N}(0, \Sigma_{uu}). \tag{52}$$

We express the ARMA coefficient matrices A, B, and Σ_{uu} as functions of the reduced parameter vector $\phi = [\phi_1', \phi_2']'$. We assume that the reduced form parameters are defined such that $\Sigma_{uu} = \Sigma_{uu}(\phi_1)$, $A = A(\phi_2)$, $B = B(\phi_2)$, $A(0) = 0$, and $B(0) = 0$. Thus, ϕ_1 indexes the covariance matrix Σ_{uu} of the reduced-form innovations u_t, whereas ϕ_2 determines the autoregressive and moving-average structure of the process. If $\phi_2 = 0$ then x_t has no serial correlation.

The prior distribution of θ is assumed to be continuous with the property that $0 < \pi_0(I) = \int \{\psi_1 > 1\} dP < 1$. According to Eq. (36) the first step of the analysis is to determine $\int \{\theta \in \Theta^I\} dP_\phi$:

$$\int \{\theta \in \Theta^I\} dP_\phi = \begin{cases} 1 & \text{if} \quad \phi_2 \neq 0 \\ 0 & \text{if} \quad \phi_2 = 0 \end{cases} . \tag{53}$$

If $\phi_2 \neq 0$ it can be deduced that $\mathcal{T}^{-1}(\phi) \in \Theta^I$ because only indeterminacy can generate autocorrelation. If $\phi = 0$, that is, x_t has no serial correlation, the information is inconclusive. The covariance matrix Σ_{uu} does not uniquely determine ψ_1 and the lack of serial correlation does not only arise in the determinacy case, but also under indeterminacy if the standard deviation of the composite shock η_t^* is equal to zero.

Given the other structural parameters, our prior assigns probability zero to $\{\sigma_\eta = 0\}$ and probability one to $\{\sigma_\eta > 0\}$. Since under determinacy $\phi_2 = 0$ for arbitrary values of σ_η we conclude that the prior probability of indeterminacy conditional on no serial correlation is equal to zero.[10]

The second step of the large sample analysis is to examine the continuity of $g(\phi) = \int \{\theta \in \Theta^I\} dP_\phi$ and the concentration of the posterior distribution $Q_{X,T}$. According to Eq. (53), $g(\phi)$ is continuous for $\phi_2 \neq 0$. If $\phi = [\phi_1', 0]'$, then $g(\phi)$ is in general discontinuous because an arbitrary small deviation from $\phi_2 = 0$ changes $g(\phi)$ from zero to one. However, continuity is preserved within neighborhoods that lie in the subspace of ϕ for which $\phi_2 = 0$. Thus, in order to apply Proposition 3 to $\phi_0 = [\phi_{1,0}', 0]$ it has to be shown that the posterior distribution $Q_{X,T}$ concentrates in the subspace of Φ for which $\phi_2 = 0$. Details are provided in the Appendix. The result is summarized in the following proposition.

Proposition 4 *Suppose X^T is generated according to the LRE model given in Eq. (38) and (39) with innovations $\nu_t \sim (0, \Sigma_{\nu\nu})$ and the prior distribution P of Θ is continuous then the posterior probability of the indeterminacy region converges to*

$$\pi_T(I) \xrightarrow{p} \begin{cases} 0 & \text{if} \quad \theta \in \Theta^D \\ 1 & \text{if} \quad \theta \in \Theta^I \end{cases} \tag{54}$$

except for a subset of the parameter space that has prior probability zero.

Proposition 4 states that the Bayesian test for indeterminacy is consistent.[11] It is instructive to compare our Bayesian test to an alternative procedure that has been

[10] Under a prior distribution that assigns non-zero mass to $\sigma_\eta = 0$ the prior probability of indeterminacy given $\phi_2 = 0$ could be non-zero.

[11] The proposition could be extended to cases in which X^T is generated from a more general

applied by Clarida *et al.* [5] in the context of the full model specified in Section 2. Since the inflation elasticity of the interest rate is a perfect regime classifier, it appears plausible to estimate ψ_1 based on

$$\widetilde{R}_t = \psi_1 \widetilde{\pi}_t + \epsilon_{R,t} \tag{55}$$

and compare the estimate to unity. According to the results in Section 5.1 and 5.2 the equilibrium inflation rate is a function of $\epsilon_{R,t}$. Hence simple OLS estimation of ψ_1 will lead to an inconsistent estimate.

Instrumental variable (IV) estimation is also problematic. As long as the sunspot shock $\tilde{\eta}_t$ is zero at all times, $\widetilde{\pi}_t$ is only a function of the current shocks. Since output is correlated both with the demand shock and the monetary policy shock it is not a valid instrument. Thus, there is no observable variable that could serve directly as an instrument.

The only remaining possibility is to use lagged values of inflation, output, or interest rates as instruments. However, this approach is only successful if the data were generated under the indeterminacy regime and $\sigma_{\tilde{\eta}} > 0$. Thus, although plausible at first glance, testing for indeterminacy based on a univariate estimate of the reaction function is not feasible in the model examined in this section.

In Section 6 we will apply the testing procedure to U.S. postwar data, without restricting the serial correlation coefficients ρ_R, ρ_g, and ρ_z to be zero. In the unrestricted model, output, inflation, and interest rates will be serially correlated also under determinacy. However, the autoregressive and moving-average order of the ARMA representation will be larger under indeterminacy due to one additional autoregressive factor in the state space representation. All the arguments presented in this section have a straightforward extension to the unrestricted model. However, analytical calculations are cumbersome.

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probability distribution $\mathcal{P}_T$. Under mild regularity conditions the pseudo-maximum-likelihood estimate of the reduced form parameter ϕ will converge to a ϕ_0^* that asymptotically minimizes the Kullback-Leibler distance between $\mathcal{P}^T$ and the parametric VARMA representation of X^T. The posterior distribution of ϕ will concentrate around this pseudo-true value ϕ_0^*. For details, see for instance Kim [18], Phillips [25], and Fernandez-Villaverde and Rubio-Ramirez [9].

6 Empirical Results

The log-linearized monetary DSGE model described in Section 2.4 is fitted to quarterly post-war U.S. data on output fluctuations $\widetilde{y}_t$, inflation $\widetilde{\pi}_t$, and nominal interest rate $\widetilde{R}_t$.[12] In line with the monetary policy literature[13], we consider the following sample periods: *pre-Volcker* sample from 1960:I to 1979:II, and *Volcker-Greenspan* sample from 1979:III to 1997:IV. All three series have been demeaned based on the *full* sample prior to estimation.

The first step in the empirical analysis is the specification of a prior distribution for the structural parameters. Prior means, 90% coverage intervals and the respective densities are reported in Table 1. The prior for the annualized real interest rate is centered at 2 percent which corresponds to a quarterly discount factor β of 0.995. The slope coefficient in the Phillips-curve is chosen to be consistent with the range of values typically found in the New-Keynesian Phillips-curve literature (see, for instance, Rotemberg and Woodford [28], Galí and Gertler [11], and Sbordone [31]). Its mean is set at 0.5, but we allow the slope to vary widely in the unit interval.[14] The prior confidence interval for the intertemporal substitution elasticity τ^{-1} ranges from 1.045 to 1.451, where 1 is log utility.

In choosing priors for the policy parameters we adopted an agnostic approach. The output gap coefficient ψ_2 is set to be roughly consistent with the values typically used in applications of the Taylor-rule, while the inflation coefficient ψ_1, centered at 1.1 with a wide confidence interval of [0.337, 1.852], is chosen to achieve 50 percent prior probability on the determinacy region. The prior confidence intervals for the autocorrelation parameters are set at a high degree of persistence. The priors for the correlations of the structural shocks are all centered at zero. A zero

[12] The time series are extracted from the DRI·WEFA database. Output is log real per capita GDP (GDPQ), HP detrended between 1955:I to 1998:IV, multiplied by 100 to convert into percent. Inflation is annualized percentage change of CPI-U (PUNEW), demeaned. Nominal interest rate is average Federal Funds Rate (FYFF) in percent, demeaned.

[13] See Clarida *et al.* [5] for further discussion.

[14] Note that in the estimation we put a prior on the coefficient $\kappa = \frac{\nu-1}{\tau\varphi\pi^2}$ instead of the individual structural parameters ν, τ, φ.

correlation between the sunspot shock and the fundamental shocks implies that the impulse response functions of the fundamental shocks are continuous at the border between the determinacy and indeterminacy region. Instead of using prior evidence on the variances of the fundamental innovations we chose them to get reasonable variance decompositions from the model's prior so that 'demand' and 'supply' shocks largely drive aggregate fluctuations. The unsystematic monetary policy shock only contributes 15%, while the sunspot shock is largely insignificant (see Table 4).

We now turn to a description of the multivariate estimation results of the DSGE model in Section 2.4. Details on the computational approach can be found in the Appendix. Table 2 contains the posterior estimates of the structural parameters. In the Volcker-Greenspan years, the posterior for ψ_1 indicates that monetary policy followed the Taylor-principle. The posterior probability of ψ_1 being greater than one is essentially one. 'Active' inflation-targeting is supported by a substantial degree of output-gap targeting as well as interest-rate smoothing which indicates a determinate equilibrium. Estimates of the other structural parameters are in line with other empirical studies. The posterior mean of the slope of the Phillips-curve is 0.78 with a confidence interval of [0.39, 1.15] which is on the high side, but not unreasonably so. Estimates of the sunspot parameters σ_η and the correlations $\rho_{.\eta}$ reveal the choice of the prior. As we argued in Section 4, from a classical perspective, these parameters are not 'identifiable' under determinacy. The posterior estimates are the same as the priors.

The posterior mean of ψ_1 based on the pre-Volcker sample is 0.86, and the 90 percent confidence interval lies below 1 which indicates an indeterminate equilibrium even in combination with output-gap targeting.[15] In general, however, the determinacy region does not only depend on the policy parameters, but also on the model's other structural parameters. Our framework allows us to compute posterior probabilities for the indeterminacy region that take the complicated shape of these regions into account. The results are reported in Table 3. Our choice of

[15]See Lubik and Marzo [22] for analytical results on the determinacy properties of this type of interest-rate rule.

priors put equal probability weight on the determinacy and indeterminacy region. The posterior probabilities reveal striking differences. The posterior probability of determinacy is 0.97 for the Volcker-Greenspan years which leads us to reject the hypothesis that sunpot shocks contributed to aggregate fluctuations. Pre-Volcker, however, the estimates show that that there is virtually no probability mass in the determinacy region. This is reflected in the estimates of the sunspot parameters. The posterior means of the correlation coefficients are negative between -0.21 to -0.13, but the parameters are fairly imprecisely estimated. The posterior estimate of the sunspot variance differs from the prior, revealing information in the data. Based on these estimates, we can not rule out the existence of a sunspot equilibrium without sunspots. That is, the use of a 'passive' monetary policy rule led to an indeterminate equilibrium.

In order to assess the importance of sunspots fluctuations we compute variance decompositions which are reported in Table 4. The estimates are based on the orthogonalization scheme described in Section 5.4. The orthogonal 'demand' shock affects both the Euler equation and and the Phillips curve, while the 'supply' shock only influences the latter. Under the determinate Volcker-Greenspan regime sunspot shocks do not influence aggregate business cycles.[16] Output is driven largely by supply shocks, while the demand shock mainly determines inflation and the nominal interest rate. There is a sizeable effect of unsystematic policy shocks on inflation which is in line with VAR studies of monetary policy. During the pre-Volcker years, sunspot shocks contribute significantly to inflation and interest rate variances. Interestingly, they do not affect output fluctuations which are solely determined by supply shocks. We can conclude that sunspots played a significant role during the pre-Volcker years in contributing to aggregate inflation volatility, but that this is not reflected in output.

Figures 1 and 2 report the estimated impulse response functions and 90% error

[16]Since the posterior probability of indeterminacy is slightly positive the posterior means of the sunspot shock contributions to the variances in Table 4 are non-zero, albeit very small. However, the 90 percent confidence intervals are exactly zero. The same is true for the sunspot shock responses in Figure 1.

bands to orthogonalized shocks for the pre-Volcker and Volcker-Greenspan regimes, respectively. Across both regimes, monetary policy shocks are contractionary, although more persistent under indeterminacy as is consistent with the structure of the model solution. The responses lack the characteristic hump-shaped pattern evident in VAR analyses which reflects the simplicity of the underlying theoretical model. Supply shocks, as well as aggregate demand shocks, are expansionary and have persistent effects. This is mainly due to the assumed structure of the unobserved shock processes since the theoretical model contains only very weak endogenous dynamics. Under indeterminacy sunspot shocks affect output and inflation in the same direction which we showed analytically in Section 5.2 for the simplified model. Consequently, the (orthogonalized) sunspot shock cannot be interpreted as a 'stagflation' shock during the 1970's in itself.

Indeterminacy potentially changes the effect of the fundamental shocks, since there are infinitely many ways to coordinate on expectations in response to fundamental innovations. In Figure 3 we compare posterior means of impulse response functions conditional on the determinacy and indeterminacy region of the parameter space. It turns out that the shapes of the mean responses are fairly similar across regions. This finding is consistent with the small estimates of the correlations between the composite shock η_t^* and the fundamental shocks reported in Table 2.

In summary, our empirical results show that the pre-Volcker years were characterized by a monetary policy that violated the Taylor-principle. This led to indeterminacy in aggregate business cycle dynamics in which sunspot shocks did play a substantial role. The specific sunspot equilibrium the U.S. found itself in can also be used to lend tentative support for the idea that sunpot shocks are behind this stagflationary episode. In the Volcker-Greenspan regime, on the other hand, monetary policy is sufficiently anti-inflationary to rule out any indeterminacy.

7 Conclusion

We develop and estimate a monetary business cycle model of the U.S. economy where monetary policy is characterized by an interest rate rule that attempts to stabilize output and inflation deviations around their target levels. It is well known that the application of such a rule may lead to (local) indeterminacy, thus opening the possibility of sunspot-driven aggregate fluctuations. Although previous research has acknowledged this problem and made some attempts to deal empirically with indeterminacy, our paper is, to the best of our knowledge, the first theoretically and empirically consistent attempt to estimate a DSGE model without restricting the parameters to the determinacy region.

We construct a multivariate test of *determinacy* versus *indeterminacy* that can be widely applied to assess the importance of sunspot fluctuations. In particular, our test takes into account the dependence of the determinacy and indeterminacy regions on all structural parameters and not just the policy parameters. This raises doubt about the applicability of a two-step approach to analyze models with indeterminate equilibria, which might pose subtle identification problems. Equilibrium indeterminacy is a property of a dynamic system and therefore has to be studied in a multivariate framework.

Empirical results confirm earlier studies that the behavior of the monetary authority has changed beginning with the tenure of Paul Volcker as Federal Reserve Chairman in 1979. During the Volcker-Greenspan years policy reacts very aggressively towards inflation which puts the U.S. economy into the determinacy region. On the other hand, monetary policy was much less active in the pre-Volcker period, and we cannot reject the possibility of a sunspot equilibrium. We conclude that while the U.S. was in a sunspot *equilibrium* before 1979, aggregate output fluctuations were not due to sunspot *shocks*, which did, however, contribute significantly to inflation and interest volatility.

The DSGE model in this paper, albeit widely employed in the recent monetary policy literature, is highly stylized. It may be premature to draw conclusions about the importance of indeterminacy and sunspot fluctuations based on the analysis in this paper alone. In particular, the model economy does not contain a strong endogenous propagation mechanism. It may be the case that the apparent sunspot dynamics can be explained by a richer economic environment inducing more persistence. It is worthwhile studying in future research whether our indeterminacy results obtain in a model with investment or habit persistence.

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A Proofs and Derivations

Proof of Proposition 3:

Let $I\!\!P$ signify the probability distribution of X^T. By assumption, the concentration of the posterior implies that

$$I\!\!P\left\{1-\int_{\mathcal{N}_\delta(\phi_0)} dQ_{X,T}<\epsilon\right\}\longrightarrow 1 \tag{56}$$

as $T\longrightarrow\infty$ for every $\epsilon>0$. Let $g(\phi)=\int\{\theta\in\Theta^I\}dP_\phi$ and write

$$\pi_T(I)=\int_{\Phi\backslash\mathcal{N}_\delta(\phi_0)} g(\phi)dQ_{X,T}+\int_{\mathcal{N}_\delta(\phi_0)} g(\phi)dQ_{X,T}. \tag{57}$$

Since $0\leq g(\phi)\leq 1$, the first term can be bounded above by $1-\int_{\mathcal{N}_\delta(\phi_0)} dQ_{X,T}$ which converges in probability to zero.

By continuity of $g(\phi)$ at ϕ_0 within $\{\mathcal{N}_\delta(\phi_0)\}_{\delta>0}$ the δ-neighborhood can be chosen such that $|g(\phi)-g(\phi_0)|<\nu$ for any $\nu>0$. Thus, we can bound

$$\left|\int_{\mathcal{N}_\delta(\phi_0)} g(\phi)dQ_{X,T}-g(\phi_0)\right|\leq\nu+1-\int_{\mathcal{N}_\delta(\phi_0)} dQ_{X,T}. \tag{58}$$

Since $1-\int_{\mathcal{N}_\delta(\phi_0)} dQ_{X,T}$ is less than ϵ with probability tending to one for any $\epsilon>0$, the left-hand side can be bounded by an arbitrary $\eta>0$ with probability tending to one. $\Box$

Proof of Proposition 4

We have to distinguish the following three cases:

(i) Suppose $\psi_1<1$, and $\sigma_\eta>0$. Thus, the "true" $\phi_{2,0}$ is not equal to zero. In this case $\int\{\theta\in\Theta^I\}dP_\phi$ is continuous at the "true" value ϕ_0. It is well known that the parameter vector ϕ of the VARMA representation can be consistently estimated, for instance by maximum likelihood. The mere existence of a consistent estimator implies the concentration of the posterior around ϕ_0 (Doob's Theorem, see Hartigan [15]). Alternative sufficient conditions for the concentration of the posterior are provided in Kim [18].

(ii) Suppose $\psi_1 \geq 1$ and the "true" $\phi_{2,0}$ equals zero. In this case we have to show that the posterior $Q_{X,T}$ concentrates in the subset of the parameter space for which $\phi_2 = 0$. Our prior for θ assigns positive probability to the determinacy region. Therefore, the implicitly specified prior for ϕ assigns positive prior probability to the lower-dimensional subspace of Φ in which $\phi_2 = 0$. The prior distribution can be represented as the mixture

$$Q = \pi_0(\phi_2 = 0)Q_{(\phi_2=0)} + \pi_0(\phi_2 \neq 0)Q_{(\phi_2 \neq 0)}, \tag{59}$$

where $\pi_0(\phi_2 = 0)$ is the prior probability assigned to $\phi_2 = 0$ and $Q_{(\phi_2=0)}$ is the prior distribution of ϕ conditional on $\phi_2 = 0$. The posterior odds ratio of $\{\phi_2 = 0\}$ versus $\{\phi_2 \neq 0\}$ is given by

$$\frac{\pi_T(\phi_2 = 0)}{\pi_T(\phi_2 \neq 0)} = \frac{\pi_0(\phi_2 = 0)}{\pi_0(\phi_2 \neq 0)} \frac{\int L(\phi|X^T)dQ_{(\phi_2=0)}}{\int L(\phi|X^T)dQ_{(\phi_2 \neq 0)}}. \tag{60}$$

This posterior odds ratio corresponds to the posterior odds of an VARMA(0,0) versus VARMA(1,1) representation. The marginal data densities $\int L(\phi|X^T)dQ_{(\phi_2=0)}$ and $\int L(\phi|X^T)dQ_{(\phi_2=0)}$ can be represented as maximized likelihood functions adjusted by a penalty term, see, for instance, Kim [18] and Phillips [25]. While

$$\left[\sup_{\phi \in \Phi, \phi_2 = 0} \ln L(\phi|X_T)\right] - \left[\sup_{\phi \in \Phi, \phi_2 \neq 0} \ln L(\phi|X_T)\right] \tag{61}$$

is stochastically bounded from below (if $\phi_{2,0} = 0$), the penalty term differential grows at rate $\ln T$ such that the posterior odds in favor of $\phi_2 = 0$ tend to infinity.

General proofs can be found, for instance, in Chao and Phillips [4] and Sin and White [36]. A rigorous argument has to pay special attention to the fact that as $\sigma_\eta \longrightarrow 0$ the autoregressive polynomial $I - A(\phi_2)z$ and the moving average polynomial $I + B(\phi_2)z$ cancel in the sense that $(I - A(\phi_2)z)^{-1}(I + B(\phi_2)z) \longrightarrow 0$ and the VARMA(1,1) representation reduces to a VARMA(0,0) representation in the limit. This is a common problem in order-selection for models with both autoregressive and moving average components. Techniques to formally account for the problem are provided, for instance, in Hannan [13].

(iii) Suppose $\psi_1 < 1$ and $\sigma_\eta = 0$. In this case $\phi_{2,0}$ and $\pi_T(I) \longrightarrow 0$, see (ii). Thus, the test will asymptotically lead to the wrong conclusion. However, our prior assigns probability zero to this event. $\square$

B Computational Issues

B.1 Model Solution

The construction of the model solution under indeterminacy in Section 5.2 is based on solving the set of equations

$$[J^{-1}\Pi^*]_{i.}\epsilon_t + [J^{-1}\Psi^*]_{i.}\eta_t \tag{62}$$

for $i \in I_x^D$, where I_x^D is the set of indices of the eigenvalues that are unstable in the determinacy region. Unfortunately, the numerical eigenvalue procedures available in GAUSS and Matlab do not maintain the same ordering of the $\lambda_i(\theta_{(1)})$ functions as $\Gamma_1^*(\theta_{(1)})$ changes. Thus, under indeterminacy it is not possible to determine which of the stable eigenvalues belong to the set I_x^D as the ordering produced by the numerical procedure is sensitive to $\theta_{(1)}$. To avoid this problem, we reduce the monetary model with $\rho_R \neq 0$, $\rho_g \neq 0$, and $\rho_z \neq 0$ to a three-dimensional system and use Cardan's formula to solve the cubic equation $|\Gamma_1^* - \lambda I| = 0$.

B.2 Bayesian Computations

Let $p(\theta)$ and $p(\theta|X^T)$ denote prior and posterior densities of θ, respectively. Since the likelihood function $L(\mathcal{T}(\theta)|X^T)$ is discontinuous at the boundary of the determinacy region for $\sigma_\eta > 0$, we conduct the computations for the two regions of the parameter space separately.

The overall prior distribution can be written as

$$\begin{aligned} p(\theta) &= \{\theta \in \Theta^D\}p(\theta) + \{\theta \in \Theta^I\}p(\theta) \\ &= \pi_0(D)p^D(\theta) + \pi_0(I)p^I(\theta), \end{aligned} \tag{63}$$

where $p^s(\theta) = p(\theta)\{\theta \in \Theta^s\}/\pi_0(s)$, is the prior density of θ conditional on region $s \in \{D, I\}$. The posterior has the decomposition

$$p(\theta|X^T) = \pi_T(D)p^D(\theta|X^T) + \pi_T(I)p^I(\theta|X^T), \tag{64}$$

where

$$\begin{aligned} p^s(\theta|X^T) &= \frac{L(\mathcal{T}(\theta)|X^T)p^s(\theta)}{\int L(\mathcal{T}(\theta)|X^T)p^s(\theta)d\theta} \\ \pi_T(s) &= \frac{\pi_0(s)\int L(\mathcal{T}(\theta)|X^T)p^s(\theta)d\theta}{\sum_{j\in\{D,I\}}\pi_0(j)\int L(\mathcal{T}(\theta)|X^T)p^j(\theta)d\theta}. \end{aligned}$$

Conditional on a parameter value θ, the likelihood function of the linearized DSGE model $L(\mathcal{T}(\theta)|X^T)$ can be evaluated with the Kalman filter. A numerical-optimization procedure is used to find the posterior mode in each region. The inverse Hessian is calculated at the posterior mode. For each region, 1,000,000 draws from $p^s(\theta|X^T)$ are generated with a random-walk Metropolis-Hastings Algorithm. The first 100,000 draws are discarded. The scaled inverse Hessian serves as a covariance matrix for the Gaussian proposal distribution used in the Metropolis-Hastings algorithm. If one of the two regions of the parameter space does not have a (local) mode, we use the inverse Hessian obtained from the other region. The marginal data densities $\int L(\mathcal{T}(\theta)|X^T)p^s(\theta)d\theta$ for the two regions are approximated with Geweke's [12] modified harmonic-mean estimator. The parameter draws θ are converted into impulse response functions and variance decompositions to generate the results reported in Section 6. Further details of these computations are discussed in Schorfheide [34].

Table 1: Prior Distributions for DSGE Model Parameters

Name	Range	Density	Mean	90% Interval
ψ_1	$I\!R^+$	Gamma	1.100	[0.337, 1.852]
ψ_2	$I\!R^+$	Gamma	0.250	[0.055, 0.436]
ρ_R	[0,1)	Beta	0.500	[0.182, 0.829]
r^*	$I\!R^+$	Gamma	2.000	[0.448, 3.477]
κ	$I\!R^+$	Gamma	0.500	[0.123, 0.868]
τ^{-1}	$I\!R^+$	Gamma	1.250	[1.045, 1.451]
ρ_g	[0.1)	Beta	0.700	[0.540, 0.860]
ρ_z	[0,1)	Beta	0.700	[0.540, 0.860]
ρ_{gz}	[-1,1]	Normal	0.000	[-0.656, 0.656]
$\rho_{R\eta}$	[-1,1]	Normal	0.000	[-0.328, 0.328]
$\rho_{g\eta}$	[-1,1]	Normal	0.000	[-0.328, 0.328]
$\rho_{z\eta}$	[-1,1]	Normal	0.000	[-0.328, 0.328]
σ_R	$I\!R^+$	Inv. Gamma	0.314	[0.133, 0.497]
σ_g	$I\!R^+$	Inv. Gamma	0.376	[0.159, 0.594]
σ_z	$I\!R^+$	Inv. Gamma	0.756	[0.335, 1.201]
σ_η	$I\!R^+$	Inv. Gamma	0.251	[0.108, 0.829]

Notes: The Inverse Gamma priors are of the form $p(\sigma|\nu,s) \propto \sigma^{-\nu-1}e^{-\nu s^2/2\sigma^2}$, where $\nu = 4$ and s equals 0.25, 0.3, 0.6, and 0.2, respectively. The prior is truncated to ensure that the covariance matrix of the shocks is positive-definite.

Table 2: Parameter Estimation Results

	Pre-Volcker		Volcker-Greenspan	
	Mean	Conf. Interval	Mean	Conf Interval
ψ_1	0.86	[0.56, 1.00]	1.58	[1.21, 1.98]
ψ_2	0.29	[0.03, 0.61]	0.26	[0.05, 0.45]
ρ_R	0.62	[0.50, 0.75]	0.69	[0.62, 0.77]
r^*	1.94	[0.42, 3.34]	1.99	[0.47, 3.44]
κ	0.59	[0.01, 0.97]	0.78	[0.39, 1.15]
τ^{-1}	1.33	[1.11, 1.53]	1.30	[1.08, 1.50]
ρ_g	0.80	[0.74, 0.85]	0.87	[0.81, 0.93]
ρ_z	0.71	[0.59, 0.84]	0.82	[0.75, 0.88]
ρ_{gz}	-0.79	[-1.00, -0.49]	-0.54	[-0.77, -0.31]
$\rho_{R\eta}$	-0.13	[-0.44, 0.19]	0.00	[-0.33, 0.33]
$\rho_{g\eta}$	-0.21	[-0.57, 0.12]	0.00	[-0.33, 0.33]
$\rho_{z\eta}$	-0.15	[-0.44, 0.13]	0.00	[-0.33, 0.33]
σ_R	0.25	[0.17, 0.32]	0.35	[0.29, 0.41]
σ_g	0.26	[0.15, 0.41]	0.25	[0.18, 0.32]
σ_z	1.09	[0.67, 1.38]	0.70	[0.60, 0.80]
σ_η	0.40	[0.15, 0.71]	0.25	[0.11, 0.83]

Notes: The table reports posterior means and 90 percent confidence intervals (in brackets). The posterior summary statistics are calculated from the output of the Metropolis algorithm.

Table 3: Determinacy and Model Fit

	Probability		Marginal Data Density	
	Determ.	Indeterm.	Determ.	Indeterm.
Prior	0.50	0.50	N/A	N/A
Pre-Volcker	1E-5	1.00	-377.92	-366.59
Volcker-Greenspan	0.97	0.03	-370.55	-374.23

Notes: The posterior probabilities are calculated based on the output of the Metropolis algorithm. Marginal data densities are approximated by Geweke's (1999) harmonic mean estimator.

Table 4: VARIANCE DECOMPOSITIONS

Shock	Prior	Pre-Volcker	Volcker-Greenspan
Output Deviations from Trend			
Policy	0.14 [0.00, 0.37]	0.03 [0.01, 0.05]	0.06 [0.03, 0.10]
Demand	0.37 [0.00, 0.81]	0.05 [0.00, 0.17]	0.08 [0.01, 0.15]
Supply	0.48 [0.00, 0.90]	0.88 [0.57, 0.99]	0.86 [0.75, 0.96]
Sunspot	0.01 [0.00, 0.03]	0.05 [0.00, 0.20]	0.00 [0.00, 0.00]
Inflation			
Policy	0.14 [0.00, 0.44]	0.10 [0.00, 0.18]	0.25 [0.15, 0.37]
Demand	0.48 [0.00, 0.91]	0.32 [0.02, 0.58]	0.66 [0.53, 0.82]
Supply	0.29 [0.00, 0.70]	0.14 [0.00, 0.28]	0.07 [0.00, 0.16]
Sunspot	0.09 [0.00, 0.33]	0.45 [0.09, 0.79]	0.01 [0.00, 0.00]
Interest Rates			
Policy	0.12 [0.00, 0.33]	0.10 [0.00, 0.23]	0.05 [0.02, 0.09]
Demand	0.49 [0.04, 0.95]	0.31 [0.03, 0.57]	0.83 [0.70, 0.96]
Supply	0.30 [0.00, 0.71]	0.14 [0.00, 0.29]	0.10 [0.00, 0.24]
Sunspot	0.09 [0.00, 0.35]	0.45 [0.10, 0.80]	0.01 [0.00, 0.00]

Notes: The table reports posterior means and 90 percent confidence intervals (in brackets). The posterior summary statistics are calculated from the output of the Metropolis algorithm.

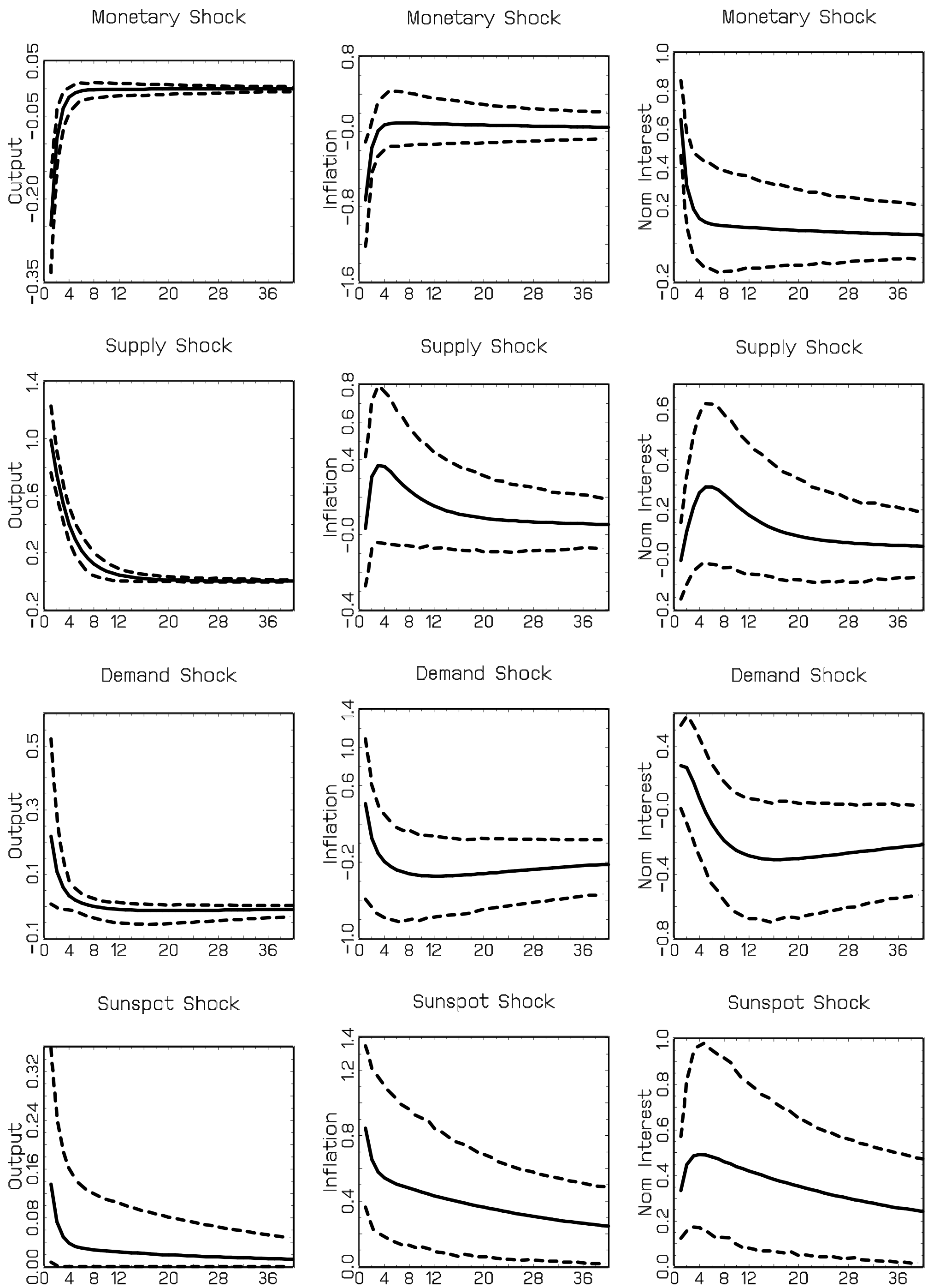

Figure 1: Impulse Responses: Pre-Volcker

Notes: Figure depicts posterior means (solid lines) and pointwise 90% posterior confidence intervals (dashed lines) for impulse responses of output, inflation, and the nominal interest rate to one-standard deviation orthogonalized shocks.

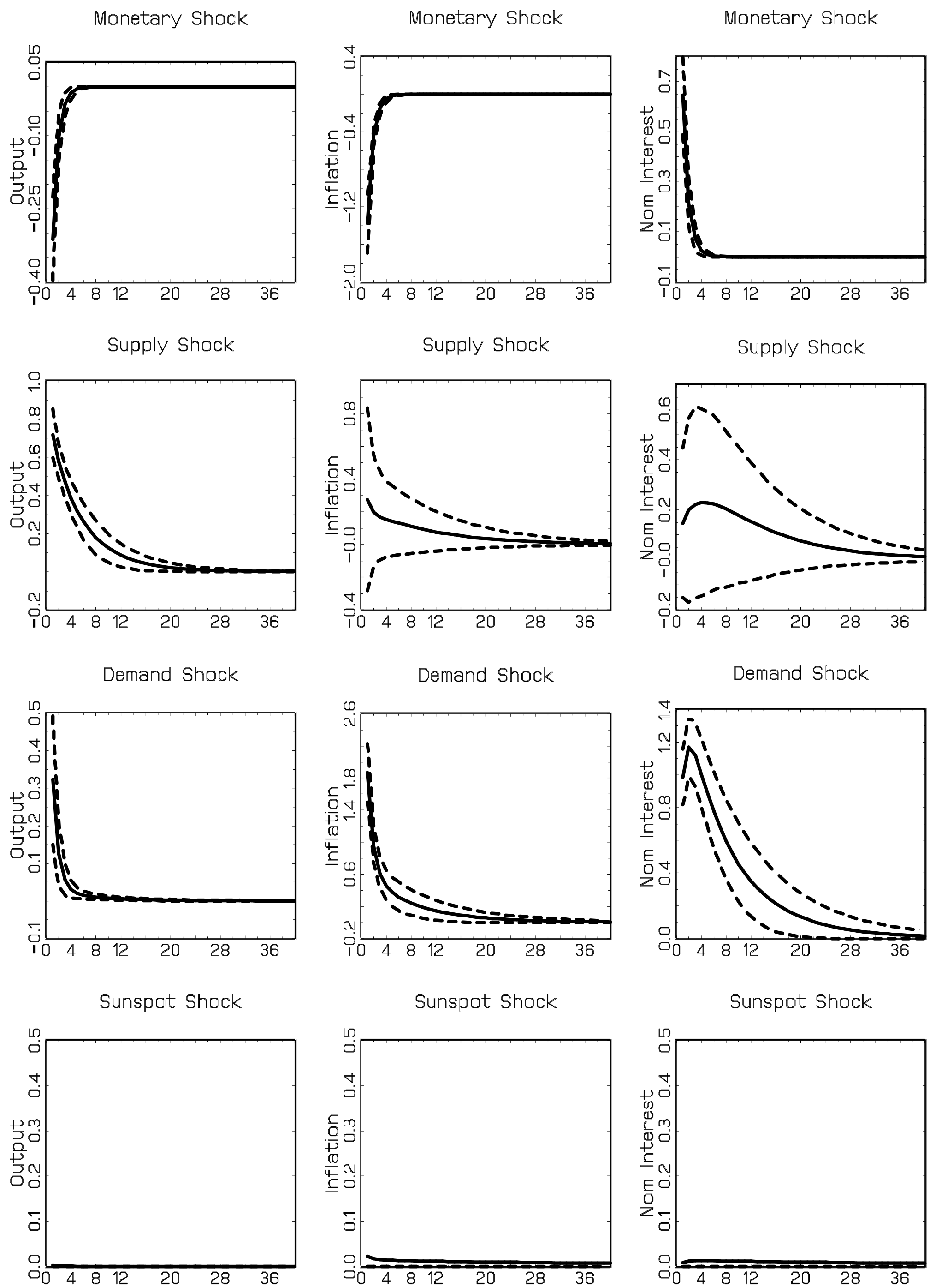

Figure 2: Impulse Responses: Volcker-Greenspan

Notes: Figure depicts posterior means (solid lines) and pointwise 90% posterior confidence intervals (dashed lines) for impulse responses of output, inflation, and the nominal interest rate to one-standard deviation orthogonalized shocks.

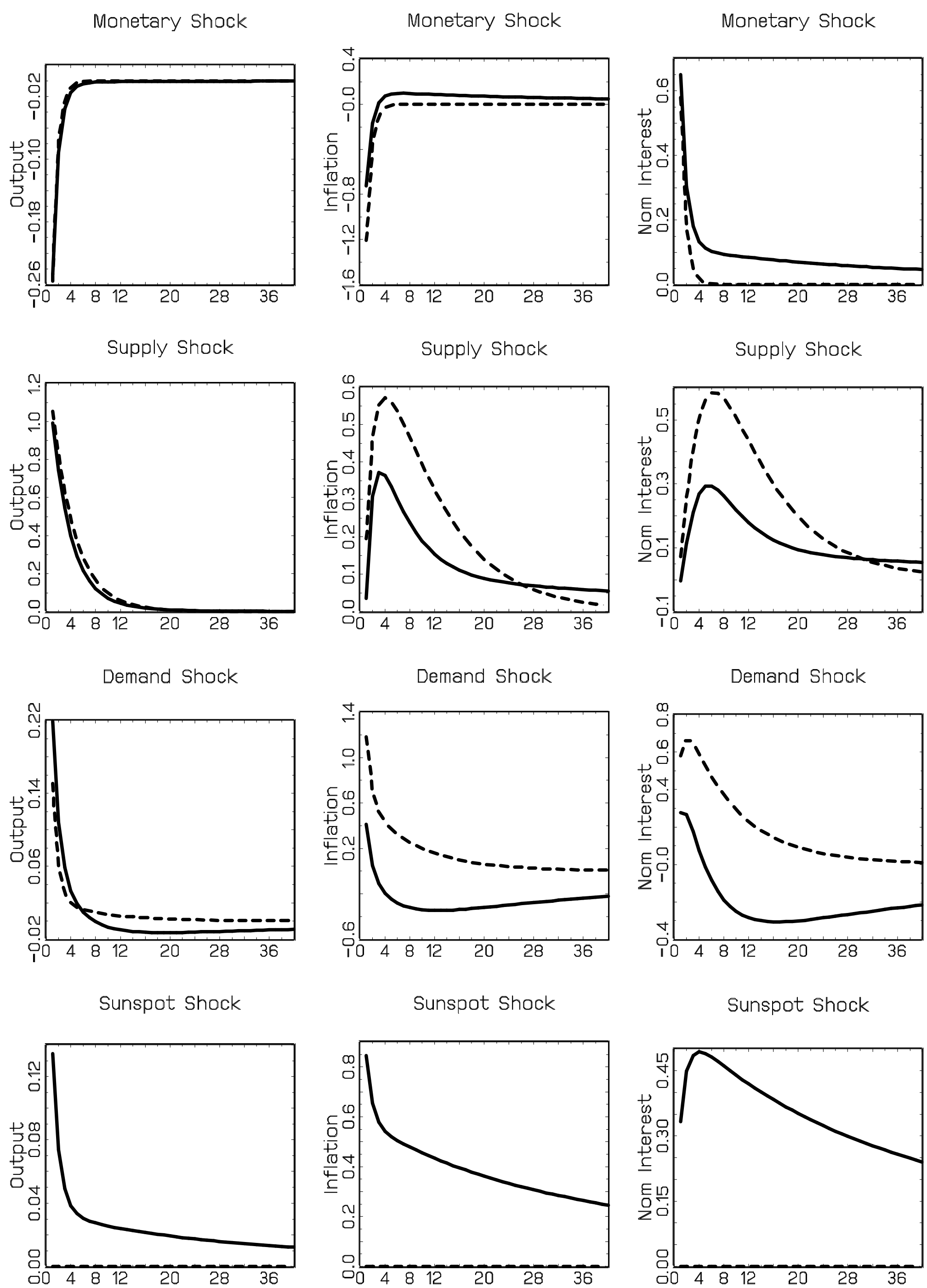

Figure 3: Determinacy vs Indeterminacy, Pre-Volcker

Notes: Figure depicts posterior mean of the indeterminacy region (solid lines) and posterior mean of the determinacy region (dashed lines) for impulse responses of output, inflation, and the nominal interest rate to one-standard deviation orthogonalized shocks.