Technical Supplement to "Uniform Inference in Panel Autoregression"

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Abstract

This Supplement comprises six appendices. Appendix SA states results on the asymptotic properties of the Anderson-Hsiao IV estimator and of the pooled ordinary least squares (POLS) estimator and uses these results to provide motivation for the design of the AIP estimator discussed in section 2 of the main paper. Appendix SB gives proofs of the results stated in Appendix SA and also supplies a proof of Lemma A1 in the main paper. Appendix SC analyzes the asymptotic properties of the Anderson-Rubin statistic in the context of the panel autoregression. Appendix SD provides proofs of the key supporting lemmas used in the proofs of the theorems stated in the main paper and in Appendix SA. Additional lemmas and their proofs as well as additional technical details are given in Appendix SE. Finally, Appendix SF provides additional Monte Carlo results comparing the finite sample performance of the AIP estimator with other point estimators of the autoregressive parameter ρ.

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Appendix SA: Asymptotic Properties of the IV and POLS Estimators

In this part of the supplement, we discuss the asymptotic properties of the Anderson-Hsiao IV estimator

$$\widehat{\rho}_{\mathrm{IVD}} = \left[\sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-2}\Delta y_{it-1}\right]^{-1}\left[\sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-2}\Delta y_{it}\right],$$

and the pooled ordinary least squares (POLS) estimator

$$\widehat{\rho}_{\mathrm{pols}} = \left[\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1}-\overline{y}_{-1,N,T}\right)^{2}\right]^{-1}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1}-\overline{y}_{-1,N,T}\right)y_{it},$$

where $\overline{y}_{-1,N,T} = N^{-1}(T-1)^{-1}\sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-1}$. More specifically, we provide an extensive catalogue of the asymptotic behavior of these two estimators under different stable, near unit root, and unit root parameter sequences. The account is fairly comprehensive and subsumes much earlier work, including recent results in Phillips (2014) and Phillips and Han (2015).

Our primary purpose in this discussion is to provide motivation for the AIP (average of IV and POLS) estimator introduced in section 2 of the main paper. As can be seen from the results given below, the Anderson-Hsiao IV estimator performs well when the underlying process is stable, whereas the POLS estimator is the superior estimator when the autoregressive parameter is unity or very nearly unity. As explained in section 2 of our main paper, the design of the AIP estimator seeks to exploit the differential strengths of the IV estimator vis-à-vis the POLS estimator, by creating an estimator that behaves like POLS in the unit root and near unit root regions of the parameter space but which behaves like IV in regions of the parameter space further away from unity.

Formal statements of the results for Anderson-Hsiao IV estimation and POLS estimation are given in Theorems SA-1 and SA-2 below, with Theorem SA-1 providing the asymptotic properties of the former and Theorem SA-2 giving comparable results for the latter. The proofs for these results on provided in the next appendix, Appendix SB.

Theorem SA-1:

Let $\mathcal{Z}_1$ and $\mathcal{Z}_2$ be independent $N(0,1)$ random variables. Under Assumptions 1-4, the following statements are true as $N, T \to \infty$ such that $N^{\kappa}/T \to \tau$, for $\kappa \in \left(\frac{1}{2}, \infty\right)$ and $\tau \in (0, \infty)$.

(a) Suppose that $\rho_T = 1$ for all T sufficiently large. Then,

$$\sqrt{T}\left(\widehat{\rho}_{\mathrm{IVD}} - \rho_T\right) \Rightarrow \frac{2\mathcal{Z}_1}{\mathcal{Z}_2}.$$

(b) Suppose that $\rho_T = \exp\{-1/q(T)\}$ such that $T^{1+\frac{1}{2\kappa}} \ll q(T)$. Then,

$$\sqrt{T}\left(\widehat{\rho}_{\mathrm{IVD}} - \rho_T\right) \Rightarrow \frac{2\mathcal{Z}_1}{\mathcal{Z}_2}.$$

(c) Suppose that $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \sim T^{1+\frac{1}{2\kappa}}$. Then, without further assumption on the convergence of $\upsilon(T) = T^{1+\frac{1}{2\kappa}}/q(T)$ as $T \to \infty$,

$$\sqrt{T}\left(\widehat{\rho}_{\mathrm{IVD}} - \rho_T\right) = O_p(1).$$

If, in addition, $T^{1+\frac{1}{2\kappa}}/q(T) \to \lambda \in (0,\infty)$, then

$$\sqrt{T}\left(\widehat{\rho}_{\mathrm{IVD}} - \rho_T\right) \Rightarrow \frac{2\mathcal{Z}_1}{\mathcal{Z}_2 - \lambda\tau^{\frac{1}{2\kappa}}/\sqrt{2}}.$$

(d) Suppose that $\rho_T = \exp\{-1/q(T)\}$ such that $T \ll q(T) \ll T^{1+\frac{1}{2\kappa}}$. Then,

$$\frac{\sqrt{N}T^{3/2}}{q(T)}\left(\widehat{\rho}_{\mathrm{IVD}} - \rho_T\right) \Rightarrow N(0,8).$$

(e) Suppose that $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \sim T$. Then,

$$\frac{V_{NT}}{\overline{\omega}_T}\frac{T^{3/2}\sqrt{N}}{q(T)}\left(\widehat{\rho}_{\mathrm{IVD}} - \rho_T\right) \Rightarrow N(0,1),$$

where

$$\begin{aligned}
V_{NT} &= \frac{1}{NT^2}\sum_{i=1}^{N} E\left[\left(\sum_{t=2}^{T}\frac{w_{it-1}\varepsilon_{it}}{\sigma}\right)^2\right] \\
&= \frac{\sigma^2}{4}\frac{q(T)^2}{T^2}\left[\exp\left\{-\frac{2T}{q(T)}\right\} + \frac{2T}{q(T)} - 1\right]\left[1 + O\left(\frac{1}{T}\right)\right], \\
\overline{\omega}_T &= \sigma^2\sqrt{1 + \frac{q(T)}{T}\left(\frac{1-\exp\{-2T/q(T)\}}{2}\right)}.
\end{aligned}$$

(f) Suppose that $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \to \infty$ but $q(T)/T \to 0$. Then,

$$\sqrt{NT}\left(\widehat{\rho}_{\mathrm{IVD}} - \rho_T\right) \Rightarrow N(0,4).$$

(g) Suppose that $\rho_T \in \mathcal{G}_{\mathrm{St}} = \left\{|\rho_T| = \exp\left\{-\frac{1}{q(T)}\right\} : q(T) \geq 0 \text{ and } q(T) = O(1) \text{ as } T \to \infty\right\}$. Then as $N, T \to \infty$

$$\sqrt{\frac{NT}{2(1+\rho_T)}}\left(\widehat{\rho}_{\mathrm{IVD}} - \rho_T\right) \Rightarrow N(0,1).$$

Theorem SA-2:

Under Assumptions 1-4, the following statements are true as $N, T \to \infty$ such that $N^{\kappa}/T \to \tau$, for $\kappa \in \left(\frac{1}{2}, \infty\right)$ and $\tau \in (0,\infty)$.

(a) Suppose that $\rho_T = 1$ for all T sufficiently large. Then,

$$\sqrt{N}T\left(\widehat{\rho}_{\mathrm{pols}} - \rho_T\right) \Rightarrow N(0,2).$$

(b) Suppose that $\rho_T = \exp\{-1/q(T)\}$ such that $T \ll q(T)$. Then,

$$\sqrt{N}T\left(\widehat{\rho}_{\text{pols}} - \rho_T\right) \Rightarrow N(0,2).$$

(c) Suppose that $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \sim T$. Then,

$$\sqrt{N}T\overline{V}_{NT}^{1/2}\left(\widehat{\rho}_{\text{pols}} - \rho_T\right) \Rightarrow N(0,2),$$

where $\overline{V}_{NT} = 2V_{NT}/\sigma^2$, with V_{NT} as defined in the statement of part (e) of Theorem SA-1 above.

(d) Suppose that $\rho_T = \exp\{-1/q(T)\}$ such that $T^{\frac{1+\kappa}{3\kappa}} \ll q(T) \ll T$. Then,

$$\sqrt{NTq(T)}\left(\widehat{\rho}_{\text{pols}} - \rho_T\right) \Rightarrow N(0,2).$$

(e) Suppose that $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \sim T^{\frac{1+\kappa}{3\kappa}} = N^{1/3}T^{1/3}$. Then,

$$\sqrt{NTq(T)}\left(\widehat{\rho}_{\text{pols}} - \rho_T - \frac{2\sigma_a^2}{q(T)^2\sigma^2}\right) \Rightarrow N(0,2).$$

(f) Suppose that $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \to \infty$ but $q(T)/T^{\frac{1+\kappa}{3\kappa}} \to 0$. Then,

$$q(T)^2\left(\widehat{\rho}_{\text{pols}} - \rho_T\right) = \frac{2\sigma_a^2}{\sigma^2} + o_p(1).$$

(g) Suppose that $\rho_T \in \mathcal{G}_{\text{St}} = \left\{|\rho_T| = \exp\left\{-\frac{1}{q(T)}\right\} : q(T) \geq 0 \text{ and } q(T) = O(1) \text{ as } T \to \infty\right\}$. Then,

$$\widehat{\rho}_{\text{pols}} - \rho_T = \frac{\left(1-\rho_T^2\right)(1-\rho_T)\sigma_a^2}{\left(1-\rho_T^2\right)\sigma_a^2 + \sigma^2} + o_p(1).$$

Theorem SA-2 above shows that the POLS estimator is a superior estimator when the underlying process either has a unit root or is a very nearly unit root process, i.e., cases where $T^{\frac{1+\kappa}{3\kappa}} \ll q(T)$ for $\kappa \in \left(\frac{1}{2}, \infty\right)$. In these cases, the POLS estimator is consistent and asymptotically normal with the fastest possible rate of convergence. POLS is, however, inconsistent when the underlying process is stable. On the other hand, Theorem SA-1 shows that the Anderson-Hsiao IV estimator performs well either when the underlying process is stable or when it can be modelled as a local-to-unity process that represents wider deviation from unity, more specifically, cases where $q(T) = O(T)$. The Anderson-Hsiao IV estimator is consistent, asymptotically normal, and also free of second order biases in these cases. Unfortunately, in local-to-unity cases where $T \ll q(T)$, the Anderson-Hsiao estimator suffers from a slower rate of convergence and could also have a nonstandard limiting distribution.

Given these results, it makes sense for us to construct an average of these two estimators to have a weight function that shifts from IV to POLS or vice versa in the (potentially thin) region of the parameter space that is characterized by the collection of localized parameter sequences $\mathcal{G}_{\cap} \equiv \left\{\rho_T = \exp\{-1/q(T)\} : T^{\frac{1+\kappa}{3\kappa}} \ll q(T) \ll T\right\}$, since this is a region where both estimators perform well. Careful examination of the proof of Theorem 2.1 shows that this desired feature is built into the AIP estimator of the form

$$\hat{\rho}_{\text{AIP}} = w_{IC}\hat{\rho}_{\text{IVD}} + (1 - w_{IC})\hat{\rho}_{\text{pols}}, \tag{1}$$

with weight function given by

$$w_{IC} = \frac{1}{1+\exp\left\{\frac{1}{2}\Delta_{IC}\right\}} \text{ and } \Delta_{IC} = \mathbb{T}_{NT} + \sqrt{N}L(T),$$

where $L(T)$ denotes some slowly varying function such that $L(T) \to \infty$ as $T \to \infty$ and where $\mathbb{T}_{NT}$ is the Studentized statistic for testing the unit root null hypothesis as defined in section 2 of the main paper. In particular, the proof of Theorem 2.1 shows that, asymptotically with probability approaching one, the weight function w_{IC} for this estimator lies strictly between zero and one only under parameter sequences belonging to the collection $\mathcal{G}_{\text{bdry}} \equiv \left\{\rho_T = \exp\{-1/q(T)\} : q(T) \sim T/(L(T))^2\right\}$ which is of course a subcollection of $\mathcal{G}_{\cap}$ for $\kappa \in \left(\frac{1}{2}, \infty\right)$. Moreover, for parameter sequences closer to unity relative to those belonging to $\mathcal{G}_{\text{bdry}}$, $w_{IC} \xrightarrow{p} 1$, so that, in these cases, the AIP estimator behaves like POLS in large samples. On the other hand, for parameter sequences further away from unity relative to those belonging to $\mathcal{G}_{\text{bdry}}$, $w_{IC} \xrightarrow{p} 0$, resulting in AIP behaving asymptotically like the Anderson-Hsiao IV estimator.

Appendix SB: Proof of Theorems on the AIP, IV and POLS Estimator

Proof of Theorem SA-1:

To show (a), first write

$$\sqrt{T}\left(\widehat{\rho}_{\text{IVD}} - \rho_T\right) = \left[\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-2}\Delta y_{it-1}\right]^{-1}\left[\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-2}\Delta\varepsilon_{it}\right]. \tag{2}$$

Applying part (a) of Lemmas SD-1 and SD-2 in Appendix SD below, we get

$$\begin{aligned}
\sqrt{T}\left(\widehat{\rho}_{\text{IVD}} - \rho_T\right) &= \left[\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=3}^{T} w_{it-2}\varepsilon_{it-1} + O_p\left(\frac{1}{\sqrt{T}}\right)\right]^{-1} \\
&\quad \times\left[-\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=4}^{T}\varepsilon_{it-2}\varepsilon_{it-1} + \frac{1}{\sqrt{NT}}\sum_{i=1}^{N} w_{iT-2}\varepsilon_{iT} + O_p\left(\frac{1}{\sqrt{T}}\right)\right] \\
&= \left[\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=3}^{T} w_{it-2}\varepsilon_{it-1}\right]^{-1}\left[-\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=4}^{T}\varepsilon_{it-2}\varepsilon_{it-1} + \frac{1}{\sqrt{NT}}\sum_{i=1}^{N} w_{iT-2}\varepsilon_{iT}\right] \\
&\quad + O_p\left(\frac{1}{\sqrt{T}}\right).
\end{aligned} \tag{3}$$

Moreover, it follows from Lemma SE-24 and part (b) of Lemma SE-20 given in Appendix SE below and from the asymptotic independence of the numerator and denominator terms of (3) that we have the joint weak convergence result

$$\begin{pmatrix} -\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=4}^{T}\varepsilon_{it-2}\varepsilon_{it-1} + \frac{1}{\sqrt{NT}}\sum_{i=1}^{N} w_{iT-2}\varepsilon_{iT} \\ \frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=3}^{T} w_{it-2}\varepsilon_{it-1} \end{pmatrix} \Rightarrow \begin{pmatrix} \sqrt{2}\sigma^2\mathcal{Z}_1 \\ \left(\sigma^2/\sqrt{2}\right)\mathcal{Z}_2 \end{pmatrix},$$

as $N, T \to \infty$. Hence, by the continuous mapping theorem, we deduce that

$$\sqrt{T}\left(\widehat{\rho}_{\mathrm{IVD}} - \rho_T\right) \Rightarrow 2\frac{\mathcal{Z}_1}{\mathcal{Z}_2},$$

as required.

To show part (b), we apply part (b) of Lemmas SD-1 and SD-2 to expression (2) above to obtain

$$\begin{aligned}
\sqrt{T}\left(\widehat{\rho}_{\mathrm{IVD}} - \rho_T\right) &= \left[\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=3}^{T} w_{it-2}\varepsilon_{it-1} + O_p\left(\max\left\{\frac{T^{1+\frac{1}{2\kappa}}}{q(T)}, \frac{1}{\sqrt{T}}\right\}\right)\right]^{-1} \\
&\quad \times \left[-\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=4}^{T}\varepsilon_{it-2}\varepsilon_{it-1} + \frac{1}{\sqrt{NT}}\sum_{i=1}^{N} w_{iT-2}\varepsilon_{iT} + O_p\left(\frac{1}{\sqrt{T}}\right)\right] \\
&= \left[\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=3}^{T} w_{it-2}\varepsilon_{it-1}\right]^{-1}\left[-\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=4}^{T}\varepsilon_{it-2}\varepsilon_{it-1} + \frac{1}{\sqrt{NT}}\sum_{i=1}^{N} w_{iT-2}\varepsilon_{iT}\right] \\
&\quad + O_p\left(\max\left\{\frac{T^{1+\frac{1}{2\kappa}}}{q(T)}, \frac{1}{\sqrt{T}}\right\}\right).
\end{aligned}$$

The rest of the argument then follows as in part (a) above.

Next, consider show (c). Applying part (c) of Lemmas SD-1 and part (b) of Lemma SD-2 to expression (2) above, we get in this case

$$\begin{aligned}
\sqrt{T}\left(\widehat{\rho}_{\mathrm{IVD}} - \rho_T\right) &= \left[\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=3}^{T} w_{it-2}\varepsilon_{it-1} - \frac{T\sqrt{N}}{q(T)}\left[\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-2}^2\right] + O_p\left(\frac{1}{\sqrt{T}}\right)\right]^{-1} \\
&\quad \times \left[-\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=4}^{T}\varepsilon_{it-2}\varepsilon_{it-1} + \frac{1}{\sqrt{NT}}\sum_{i=1}^{N} w_{iT-2}\varepsilon_{iT} + O_p\left(\frac{1}{\sqrt{T}}\right)\right] \\
&= \left[\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=3}^{T} w_{it-2}\varepsilon_{it-1} - \frac{T^{1+\frac{1}{2\kappa}}}{q(T)}\frac{\sqrt{N}}{T^{\frac{1}{2\kappa}}}\left(\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-2}^2\right)\right]^{-1} \\
&\quad \times \left[-\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=4}^{T}\varepsilon_{it-2}\varepsilon_{it-1} + \frac{1}{\sqrt{NT}}\sum_{i=1}^{N} w_{iT-2}\varepsilon_{iT}\right] + O_p\left(\frac{1}{\sqrt{T}}\right), \qquad (4)
\end{aligned}$$

as $N, T \to \infty$. Now, consider the case where $T^{1+\frac{1}{2\kappa}}/q(T) \to \lambda \in (0, \infty)$. In this case, making use of Lemma SE-15, part (a) of Lemma SE-20, and the Cramêr convergence theorem, we have that

$$\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=3}^{T} w_{it-2}\varepsilon_{it-1} - \frac{T^{1+\frac{1}{2\kappa}}}{q(T)}\frac{\sqrt{N}}{T^{\frac{1}{2\kappa}}}\left[\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-2}^2\right] \Rightarrow \frac{\sigma^2}{\sqrt{2}}\mathcal{Z}_2 - \frac{\lambda\tau^{\frac{1}{2\kappa}}}{2}\sigma^2,$$

as $N, T \to \infty$. Moreover, it follows from part (a) of Lemma SE-22 and the asymptotic independence of the numerator and denominator terms of (4) that we have the joint weak convergence result

$$\begin{pmatrix} -\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=4}^{T}\varepsilon_{it-2}\varepsilon_{it-1} + \frac{1}{\sqrt{NT}}\sum_{i=1}^{N} w_{iT-2}\varepsilon_{iT} \\ \frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=3}^{T} w_{it-2}\varepsilon_{it-1} - \tau^{\frac{1}{2\kappa}}\left[\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-2}^2\right] \end{pmatrix} \Rightarrow \begin{pmatrix} \sqrt{2}\sigma^2\mathcal{Z}_1 \\ \left(\sigma^2/\sqrt{2}\right)\mathcal{Z}_2 - \frac{\lambda\tau^{\frac{1}{2\kappa}}}{2}\sigma^2 \end{pmatrix},$$

as $N, T \to \infty$. Hence, by continuous mapping we deduce that

$$\sqrt{T}\left(\widehat{\rho}_{\mathrm{IVD}} - \rho_T\right) \Rightarrow \frac{\sqrt{2}\sigma^2 \mathcal{Z}_1}{\left(\sigma^2/\sqrt{2}\right)\mathcal{Z}_2 - \frac{\lambda\tau^{\frac{1}{2\kappa}}}{2}\sigma^2} \equiv \frac{2\mathcal{Z}_1}{\mathcal{Z}_2 - \lambda\tau^{\frac{1}{2\kappa}}/\sqrt{2}}.$$

On the other hand, if $q(T) \sim T^{1+\frac{1}{2\kappa}}$ but $\upsilon(T) = T^{1+\frac{1}{2\kappa}}/q(T)$ does not converge as $T \to \infty$; then, it is nevertheless the case that the denominator of (4)

$$\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=3}^{T} w_{it-2}\varepsilon_{it-1} - \frac{T^{1+\frac{1}{2\kappa}}}{q(T)}\frac{\sqrt{N}}{T^{\frac{1}{2\kappa}}}\left[\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-2}^2\right] \neq 0 \text{ wpa } 1,$$

given that $\mathcal{Z}_2$ is a continuous random variable. Since the numerator of (4) is $O_p(1)$, it follows that

$$\sqrt{T}\left(\widehat{\rho}_{\mathrm{IVD}} - \rho_T\right) = O_p(1,)$$

in this case as required.

To show (d), first write

$$\frac{\sqrt{N}T^{3/2}}{q(T)}\left(\widehat{\rho}_{\mathrm{IVD}} - \rho_T\right) = \left[\frac{q(T)}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-2}\Delta y_{it-1}\right]^{-1}\left[\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-2}\Delta\varepsilon_{it}\right].$$

Now, applying part (d) of Lemmas SD-1 and part (b) of Lemma SD-2, we get

$$\begin{aligned}
\frac{\sqrt{N}T^{3/2}}{q(T)}\left(\widehat{\rho}_{\mathrm{IVD}} - \rho_T\right) &= \left[-\left(\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-2}^2\right) + O_p\left(\frac{q(T)}{T^{1+\frac{1}{2\kappa}}}\right)\right]^{-1} \\
&\quad \times\left[-\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=4}^{T}\varepsilon_{it-2}\varepsilon_{it-1} + \frac{1}{\sqrt{NT}}\sum_{i=1}^{N} w_{iT-2}\varepsilon_{iT} + O_p\left(\frac{1}{\sqrt{T}}\right)\right] \\
&= -\left[\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-2}^2\right]^{-1}\left[-\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=4}^{T}\varepsilon_{it-2}\varepsilon_{it-1} + \frac{1}{\sqrt{NT}}\sum_{i=1}^{N} w_{iT-2}\varepsilon_{iT}\right] \\
&\quad + O_p\left(\max\left\{\frac{q(T)}{T^{1+\frac{1}{2\kappa}}}, \frac{1}{\sqrt{T}}\right\}\right).
\end{aligned}$$

Moreover, it follows from part (a) of Lemma SE-22 and Lemma SE-15 that

$$-\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=4}^{T}\varepsilon_{it-2}\varepsilon_{it-1} + \frac{1}{\sqrt{NT}}\sum_{i=1}^{N} w_{iT-2}\varepsilon_{iT} \Rightarrow \sqrt{2}\sigma^2\mathcal{Z}_1,$$

and

$$\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-2}^2 \xrightarrow{p} \frac{\sigma^2}{2},$$

as $N, T \to \infty$. Hence, by the Cramér convergence theorem, we deduce that

$$\frac{\sqrt{N}T^{3/2}}{q(T)}\left(\widehat{\rho}_{\mathrm{IVD}} - \rho_T\right) \Rightarrow -\frac{2\sqrt{2}}{\sigma^2}\sigma^2\mathcal{Z}_1 \equiv N(0,8).$$

as required.

To show part (e), write

$$\frac{V_{NT}}{\overline{\omega}_T}\frac{T^{3/2}\sqrt{N}}{q(T)}\left(\widehat{\rho}_{\mathrm{IVD}}-\rho_T\right)=\frac{V_{NT}}{\overline{\omega}_T}\left[\frac{q(T)}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T}y_{it-2}\Delta y_{it-1}\right]^{-1}\left[\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=2}^{T}y_{it-2}\Delta\varepsilon_{it}\right],$$

where

$$\begin{aligned}
V_{NT} &= \frac{1}{NT^2}\sum_{i=1}^{N}E\left[\left(\sum_{t=2}^{T}\frac{w_{it-1}\varepsilon_{it}}{\sigma}\right)^2\right] \\
&= \frac{\sigma^2}{4}\frac{q(T)^2}{T^2}\left[\exp\left\{-\frac{2T}{q(T)}\right\}+\frac{2T}{q(T)}-1\right]\left[1+O\left(\frac{1}{T}\right)\right], \\
\overline{\omega}_T &= \sigma^2\sqrt{1+\frac{q(T)}{T}\left(\frac{1-\exp\{-2T/q(T)\}}{2}\right)}.
\end{aligned}$$

Now, applying part (e) of Lemma SD-1 and part (c) of Lemma SD-2, we get

$$\begin{aligned}
&\frac{V_{NT}}{\overline{\omega}_T}\frac{T^{3/2}\sqrt{N}}{q(T)}\left(\widehat{\rho}_{\mathrm{IVD}}-\rho_T\right) \\
= \;& V_{NT}\left[-\left(\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T}w_{it-2}^2\right)+O_p\left(\frac{1}{\sqrt{N}}\right)\right]^{-1} \\
&\times\left[-\frac{1}{\overline{\omega}_T\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=4}^{T}\varepsilon_{it-2}\varepsilon_{it-1}+\frac{1}{\overline{\omega}_T\sqrt{NT}}\sum_{i=1}^{N}w_{iT-2}\varepsilon_{iT}+O_p\left(\frac{1}{\sqrt{T}}\right)\right] \\
= \;& \left[\frac{1}{V_{NT}}\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T}w_{it-2}^2\right]^{-1}\left[\frac{1}{\overline{\omega}_T\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=4}^{T}\varepsilon_{it-2}\varepsilon_{it-1}-\frac{1}{\overline{\omega}_T\sqrt{NT}}\sum_{i=1}^{N}w_{iT-2}\varepsilon_{iT}\right] \\
&+O_p\left(\frac{1}{\sqrt{T}}\right)+O_p\left(\frac{1}{\sqrt{N}}\right).
\end{aligned}$$

Moreover, it follows from part (b) of Lemma SE-22 and part (a) of Lemma SE-17 that

$$\frac{1}{\overline{\omega}_T\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=4}^{T}\varepsilon_{it-2}\varepsilon_{it-1}-\frac{1}{\overline{\omega}_T\sqrt{NT}}\sum_{i=1}^{N}w_{iT-2}\varepsilon_{iT}\Rightarrow\mathcal{Z}_1\equiv N(0,1),$$

and

$$\frac{1}{V_{NT}}\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T}w_{it-2}^2=1+\frac{1}{V_{NT}}\left[\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T}w_{it-2}^2-V_{NT}\right]=1+o_p(1)\xrightarrow{p}1,$$

as $N,T\to\infty$. Hence, by the Cramér convergence theorem, we deduce that

$$\frac{V_{NT}}{\overline{\omega}_T}\frac{T^{3/2}\sqrt{N}}{q(T)}\left(\widehat{\rho}_{\mathrm{IVD}}-\rho_T\right)\Rightarrow\mathcal{Z}_1\equiv N(0,1),$$

as required.

To show part (f),

$$\sqrt{NT}\left(\widehat{\rho}_{\mathrm{IVD}}-\rho_T\right)=\left[\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=2}^{T}y_{it-2}\Delta y_{it-1}\right]^{-1}\left[\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=2}^{T}y_{it-2}\Delta\varepsilon_{it}\right].$$

Now, applying part (f) of Lemma SD-1 and part (d) of Lemma SD-2, we get

$$\begin{aligned}\sqrt{NT}\left(\widehat{\rho}_{\mathrm{IVD}}-\rho_T\right) &= \left[-\frac{1}{NTq\left(T\right)}\sum_{i=1}^{N}\sum_{t=2}^{T}w_{it-2}^{2}+O_p\left(\max\left\{\sqrt{\frac{q\left(T\right)}{NT}},\frac{1}{q\left(T\right)}\right\}\right)\right]^{-1}\\ &\quad\times\left[-\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=4}^{T}\varepsilon_{it-2}\varepsilon_{it-1}+O_p\left(\max\left\{\sqrt{\frac{q\left(T\right)}{T}},\frac{1}{\sqrt{q\left(T\right)}}\right\}\right)\right]\\ &= \left[\frac{1}{NTq\left(T\right)}\sum_{i=1}^{N}\sum_{t=2}^{T}w_{it-2}^{2}\right]^{-1}\left[\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=4}^{T}\varepsilon_{it-2}\varepsilon_{it-1}\right]\\ &\quad+O_p\left(\max\left\{\sqrt{\frac{q\left(T\right)}{T}},\frac{1}{\sqrt{q\left(T\right)}}\right\}\right).\end{aligned}$$

Moreover, it follows from part (c) of Lemma SE-22 and part (b) of Lemma SE-17 that

$$-\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=4}^{T}\varepsilon_{it-2}\varepsilon_{it-1}\Rightarrow\sigma^2\mathcal{Z}_1,$$

and

$$-\frac{1}{NTq\left(T\right)}\sum_{i=1}^{N}\sum_{t=2}^{T}w_{it-2}^{2}\xrightarrow{p}\frac{\sigma^2}{-2},$$

as $N,T\to\infty$. Hence, by the Cramér convergence theorem, we deduce that

$$\sqrt{NT}\left(\widehat{\rho}_{\mathrm{IVD}}-\rho_T\right)\Rightarrow\left(\frac{-2}{\sigma^2}\right)\sigma^2\mathcal{Z}_1\equiv N\left(0,4\right).$$

as required.

Finally, to show part (g), note that by applying part (g) of Lemma SD-1, part (e) of Lemma SD-2, part (c) of Lemma SE-17, and Lemma SE-23; we obtain

$$\begin{aligned}&\sqrt{\frac{NT}{2\left(1+\rho_T\right)}}\left(\widehat{\rho}_{\mathrm{IVD}}-\rho_T\right)\\ =\;&\left[\frac{1+\rho_T}{NT}\sum_{i=1}^{N}\sum_{t=2}^{T}y_{it-2}\Delta y_{it-1}\right]^{-1}\left[\sqrt{\frac{1+\rho_T}{2}}\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=2}^{T}y_{it-2}\Delta\varepsilon_{it}\right]\\ =\;&\left[-\frac{\left(1-\rho_T^2\right)}{NT}\sum_{i=1}^{N}\sum_{t=2}^{T}w_{it-2}^{2}+O_p\left(\max\left\{\frac{1}{\sqrt{NT}},\frac{1}{T}\right\}\right)\right]^{-1}\\ &\times\left[\sigma^2\sqrt{\frac{1+\rho_T}{2\sigma^4}}\left(-\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=4}^{T}\varepsilon_{it-2}\varepsilon_{it-1}+\left(1-\rho_T\right)\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=4}^{T}w_{it-3}\varepsilon_{it-1}\right)+O_p\left(\frac{1}{\sqrt{T}}\right)\right]\\ \Rightarrow\;&-\frac{1}{\sigma^2}N\left(0,\sigma^4\right)\equiv N\left(0,1\right),\end{aligned}$$

as required. □

Proof of Theorem SA-2:

To proceed, first write

$$\begin{aligned}
\widehat{\rho}_{\text{pols}} &= \left[\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1}-\overline{y}_{-1,N,T}\right)^2\right]^{-1} \\
&\quad \times \sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1}-\overline{y}_{-1,N,T}\right)\left[a_i\left(1-\rho_T\right)+\rho_T y_{it-1}+\varepsilon_{it}\right] \\
&= \left[\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1}-\overline{y}_{-1,N,T}\right)^2\right]^{-1}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1}-\overline{y}_{-1,N,T}\right)a_i\left(1-\rho_T\right) \\
&\quad +\rho_T\left[\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1}-\overline{y}_{-1,N,T}\right)^2\right]^{-1}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1}-\overline{y}_{-1,N,T}\right)^2 \\
&\quad +\left[\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1}-\overline{y}_{-1,N,T}\right)^2\right]^{-1}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1}-\overline{y}_{-1,N,T}\right)\varepsilon_{it},
\end{aligned}$$

so that

$$\begin{aligned}
\widehat{\rho}_{\text{pols}}-\rho_T &= \left(1-\rho_T\right)\left[\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1}-\overline{y}_{-1,N,T}\right)^2\right]^{-1}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1}-\overline{y}_{-1,N,T}\right)a_i \\
&\quad +\left[\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1}-\overline{y}_{-1,N,T}\right)^2\right]^{-1}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1}-\overline{y}_{-1,N,T}\right)\varepsilon_{it}. \qquad (5)
\end{aligned}$$

Now, consider part (a), where we take $\rho_T = 1$ for all T sufficiently large. Hence, in this case, we can apply part (a) of Lemmas SD-3, SD-4, and SD-5 given in Appendix SD below, along with Lemma

SE-13 and part (b) of Lemma SE-20 given in Appendix SE to obtain

$$
\begin{aligned}
& T\sqrt{N}\left(\widehat{\rho}_{\text{pols}}-\rho_T\right) \\
= \; & (1-\rho_T)\left[\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1}-\overline{y}_{-1,N,T}\right)^2\right]^{-1}\frac{1}{T\sqrt{N}}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1}-\overline{y}_{-1,N,T}\right)a_i \\
& +\left[\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1}-\overline{y}_{-1,N,T}\right)^2\right]^{-1}\frac{1}{T\sqrt{N}}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1}-\overline{y}_{-1,N,T}\right)\varepsilon_{it} \\
= \; & \left[\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T}w_{it-1}^2+O_p\left(\frac{1}{T}\right)+O_p\left(\frac{1}{N}\right)\right]^{-1} \\
& \times\left[\frac{1}{T\sqrt{N}}\sum_{i=1}^{N}\sum_{t=2}^{T}w_{it-1}\varepsilon_{it}+O_p\left(\frac{1}{\sqrt{T}}\right)+O_p\left(\frac{1}{\sqrt{N}}\right)\right]+O_p\left((1-\rho_T)\max\left\{\sqrt{N},\sqrt{T}\right\}\right) \\
= \; & \left[\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T}w_{it-1}^2\right]^{-1}\left[\frac{1}{T\sqrt{N}}\sum_{i=1}^{N}\sum_{t=2}^{T}w_{it-1}\varepsilon_{it}\right]+O_p\left(\max\left\{\frac{1}{\sqrt{N}},\frac{1}{\sqrt{T}}\right\}\right) \\
\Rightarrow \; & \frac{2}{\sigma^2}N\left(0,\frac{\sigma^4}{2}\right)\equiv N\left(0,2\right).
\end{aligned}
$$

Next, consider part (b), where we assume that $\rho_T=\exp\left\{-1/q\left(T\right)\right\}$ such that $T\ll q\left(T\right)$. In this case, we apply part (b) of Lemmas SD-3, SD-4, and SD-5 along with Lemma SE-15 and part (a) of Lemma SE-20 to obtain

$$
\begin{aligned}
& T\sqrt{N}\left(\widehat{\rho}_{\text{pols}}-\rho_T\right) \\
= \; & (1-\rho_T)\left[\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1}-\overline{y}_{-1,N,T}\right)^2\right]^{-1}\frac{1}{T\sqrt{N}}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1}-\overline{y}_{-1,N,T}\right)a_i \\
& +\left[\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1}-\overline{y}_{-1,N,T}\right)^2\right]^{-1}\frac{1}{T\sqrt{N}}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1}-\overline{y}_{-1,N,T}\right)\varepsilon_{it} \\
= \; & \left[\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T}w_{it-1}^2+O_p\left(\frac{1}{T}\right)+O_p\left(\frac{1}{N}\right)\right]^{-1} \\
& \times\left[\frac{1}{T\sqrt{N}}\sum_{i=1}^{N}\sum_{t=2}^{T}w_{it-1}\varepsilon_{it}+O_p\left(\frac{1}{\sqrt{T}}\right)+O_p\left(\frac{1}{\sqrt{N}}\right)\right] \\
& +O\left(\frac{1}{q\left(T\right)}\right)O_p\left(1\right)O_p\left(\max\left\{\sqrt{N},\sqrt{T}\right\}\right) \\
= \; & \left[\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T}w_{it-1}^2\right]^{-1}\left[\frac{1}{T\sqrt{N}}\sum_{i=1}^{N}\sum_{t=2}^{T}w_{it-1}\varepsilon_{it}\right]+o_p\left(1\right) \\
\Rightarrow \; & \frac{2}{\sigma^2}N\left(0,\frac{\sigma^4}{2}\right)\equiv N\left(0,2\right).
\end{aligned}
$$

Now, we consider part (c), where we take $\rho_T=\exp\left\{-1/q\left(T\right)\right\}$ such that $q\left(T\right)\sim T$. To proceed,

note that

$$\overline{V}_{NT}=\frac{2}{\sigma^4}\frac{1}{NT^2}\overline{\omega}_{Z,N,T}^2,$$

where $\overline{\omega}_{Z,N,T}^2$ is defined in the statement of Lemma SE-27. It follows from the proof of part (a) of Lemma SE-27 that

$$\begin{aligned}\overline{V}_{NT} &= \frac{2}{\sigma^4}\frac{1}{NT^2}Nq\left(T\right)^2\frac{\sigma^4}{4}\left[\exp\left\{-\frac{2T}{q\left(T\right)}\right\}+\frac{2T}{q\left(T\right)}-1\right]\left[1+O\left(\frac{1}{T}\right)\right]\\ &= \frac{1}{2}\frac{q\left(T\right)^2}{T^2}\left[\exp\left\{-\frac{2T}{q\left(T\right)}\right\}+\frac{2T}{q\left(T\right)}-1\right]\left[1+O\left(\frac{1}{T}\right)\right].\end{aligned}$$

Applying part (c) of Lemmas SD-3, SD-4, and SD-5 along with part (a) of Lemma SE-17 and part (a) of Lemma SE-27; we can get

$$\begin{aligned}&\sqrt{N}T\overline{V}_{NT}^{1/2}\left(\widehat{\rho}_{\text{pols}}-\rho_T\right)\\ =\;&\left(1-\rho_T\right)\overline{V}_{NT}^{1/2}\left[\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1}-\overline{y}_{-1,N,T}\right)^2\right]^{-1}\frac{1}{T\sqrt{N}}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1}-\overline{y}_{-1,N,T}\right)a_i\\ &+\overline{V}_{NT}^{1/2}\left[\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1}-\overline{y}_{-1,N,T}\right)^2\right]^{-1}\frac{1}{T\sqrt{N}}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1}-\overline{y}_{-1,N,T}\right)\varepsilon_{it}\\ =\;&\overline{V}_{NT}^{1/2}\left[\frac{q\left(T\right)^2}{T^2}\frac{\sigma^2}{4}\left[\exp\left\{-\frac{2T}{q\left(T\right)}\right\}+\frac{2T}{q\left(T\right)}-1\right]+O_p\left(\max\left\{\frac{1}{\sqrt{N}},\frac{1}{\sqrt{T}}\right\}\right)\right]^{-1}\\ &\times\frac{1}{T\sqrt{N}}\sum_{i=1}^{N}\sum_{t=2}^{T}w_{it-1}\varepsilon_{it}\left[1+O_p\left(\frac{1}{\sqrt{T}}\right)+O_p\left(\frac{1}{\sqrt{N}}\right)\right]\\ &+O\left(\frac{1}{T}\right)O_p\left(1\right)O_p\left(1\right)O_p\left(\max\left\{\sqrt{N},\sqrt{T}\right\}\right)\\ =\;&\overline{V}_{NT}\left[\frac{q\left(T\right)^2}{T^2}\frac{\sigma^2}{4}\left[\exp\left\{-\frac{2T}{q\left(T\right)}\right\}+\frac{2T}{q\left(T\right)}-1\right]+O_p\left(\max\left\{\frac{1}{\sqrt{N}},\frac{1}{\sqrt{T}}\right\}\right)\right]^{-1}\\ &\times\frac{1}{\overline{V}_{NT}^{1/2}}\frac{1}{T\sqrt{N}}\sum_{i=1}^{N}\sum_{t=2}^{T}w_{it-1}\varepsilon_{it}\left[1+O_p\left(\frac{1}{\sqrt{T}}\right)+O_p\left(\frac{1}{\sqrt{N}}\right)\right]+O_p\left(\max\left\{\frac{\sqrt{N}}{T},\frac{1}{\sqrt{T}}\right\}\right)\\ =\;&\frac{1}{2}\frac{q\left(T\right)^2}{T^2}\left[\exp\left\{-\frac{2T}{q\left(T\right)}\right\}+\frac{2T}{q\left(T\right)}-1\right]\left[1+O\left(\frac{1}{T}\right)\right]\\ &\times\left[\frac{q\left(T\right)^2}{T^2}\frac{\sigma^2}{4}\left[\exp\left\{-\frac{2T}{q\left(T\right)}\right\}+\frac{2T}{q\left(T\right)}-1\right]+O_p\left(\max\left\{\frac{1}{\sqrt{N}},\frac{1}{\sqrt{T}}\right\}\right)\right]^{-1}\\ &\times\frac{\sigma^2}{\sqrt{2}\overline{\omega}_{Z,N,T}}\sum_{i=1}^{N}\sum_{t=2}^{T}w_{it-1}\varepsilon_{it}\left[1+O_p\left(\frac{1}{\sqrt{T}}\right)+O_p\left(\frac{1}{\sqrt{N}}\right)\right]+O_p\left(\max\left\{\frac{\sqrt{N}}{T},\frac{1}{\sqrt{T}}\right\}\right)\\ =\;&\frac{\sqrt{2}}{\overline{\omega}_{Z,N,T}}\sum_{i=1}^{N}\sum_{t=2}^{T}w_{it-1}\varepsilon_{it}\left[1+O_p\left(\max\left\{\frac{\sqrt{N}}{T},\frac{1}{\sqrt{N}},\frac{1}{\sqrt{T}}\right\}\right)\right]\\ \Rightarrow\;&N\left(0,2\right).\end{aligned}$$

We turn our attention to part (d) where we take $\rho_T = \exp\{-1/q(T)\}$ such that $T^{\frac{1+\kappa}{3\kappa}} \ll q(T) \ll T$. Here, using part (d) of Lemmas SD-3, SD-4, and SD-5 along with part (b) of Lemma SE-17 and part (b) of Lemma SE-27; we obtain

$$
\begin{aligned}
& \sqrt{NTq(T)}\left(\widehat{\rho}_{\text{pols}} - \rho_T\right) \\
= \; & (1-\rho_T)\left[\frac{1}{NTq(T)}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1} - \overline{y}_{-1,N,T}\right)^2\right]^{-1} \frac{1}{\sqrt{NTq(T)}}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1} - \overline{y}_{-1,N,T}\right)a_i \\
& + \left[\frac{1}{NTq(T)}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1} - \overline{y}_{-1,N,T}\right)^2\right]^{-1} \frac{1}{\sqrt{NTq(T)}}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1} - \overline{y}_{-1,N,T}\right)\varepsilon_{it} \\
= \; & \frac{2}{\sigma^2}\left[1 + O_p\left(\max\left\{\frac{q(T)}{T}, \frac{1}{q(T)}, \sqrt{\frac{q(T)}{NT}}, \frac{1}{\sqrt{T}}\right\}\right)\right] \\
& \times \left[\frac{1}{\sqrt{NTq(T)}}\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-1}\varepsilon_{it} + O_p\left(\max\left\{\frac{1}{\sqrt{q(T)}}, \sqrt{\frac{q(T)}{NT}}, \frac{\sqrt{q(T)}}{T}\right\}\right)\right] \\
& + O_p\left(\frac{1}{q(T)}\right) O_p(1)\, O_p\left(\max\left\{\sqrt{\frac{NT}{q(T)}}, \sqrt{q(T)}, \sqrt{\frac{q(T)N}{T}}\right\}\right) \\
= \; & \frac{2}{\sigma^2}\frac{1}{\sqrt{NTq(T)}}\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-1}\varepsilon_{it} + O_p\left(\max\left\{\frac{\sqrt{NT}}{q(T)^{3/2}}, \frac{1}{\sqrt{q(T)}}, \sqrt{\frac{q(T)}{NT}}\right\}\right) \\
\Rightarrow \; & \frac{2}{\sigma^2} N\left(0, \frac{\sigma^4}{2}\right) \equiv N(0,2).,
\end{aligned}
$$

Next, we consider part (e), where we take $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \sim T^{\frac{1+\kappa}{3\kappa}} = N^{1/3}T^{1/3} \ll T$ for $\kappa \in \left(\frac{1}{2}, \infty\right)$. In this case, we apply part (d) of Lemmas SD-3, SD-4, and SD-5

along with parts (a) and (b) of Lemma SE-11, and part (b) of Lemmas SE-17 and SE-27 to obtain

$$
\begin{aligned}
& \sqrt{NTq(T)}\left(\widehat{\rho}_{\text{pols}}-\rho_T\right) \\
= \; & (1-\rho_T)\left[\frac{1}{NTq(T)}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1}-\overline{y}_{-1,N,T}\right)^2\right]^{-1} \\
& \times\frac{1}{\sqrt{NTq(T)}}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1}-\overline{y}_{-1,N,T}\right)a_i \\
& +\left[\frac{1}{NTq(T)}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1}-\overline{y}_{-1,N,T}\right)^2\right]^{-1}\frac{1}{\sqrt{NTq(T)}}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1}-\overline{y}_{-1,N,T}\right)\varepsilon_{it} \\
= \; & \left(1-\left[1-\frac{1}{q(T)}+O\left(\frac{1}{q(T)^2}\right)\right]\right)\left[\frac{\sigma^2}{2}+O_p\left(\max\left\{\frac{q(T)}{T},\frac{1}{q(T)},\sqrt{\frac{q(T)}{NT}},\frac{1}{\sqrt{T}}\right\}\right)\right]^{-1} \\
& \times\left[\sqrt{\frac{NT}{q(T)}}\left[\frac{1}{N}\sum_{i=1}^{N}a_i^2-\overline{a}_N^2\right]+O_p\left(\max\left\{\sqrt{q(T)},\sqrt{\frac{q(T)N}{T}}\right\}\right)\right] \\
& +\left[\frac{\sigma^2}{2}+O_p\left(\max\left\{\frac{q(T)}{T},\frac{1}{q(T)},\sqrt{\frac{q(T)}{NT}},\frac{1}{\sqrt{T}}\right\}\right)\right]^{-1} \\
& \times\left[\frac{1}{\sqrt{NTq(T)}}\sum_{i=1}^{N}\sum_{t=2}^{T}w_{it-1}\varepsilon_{it}+O_p\left(\max\left\{\frac{1}{\sqrt{q(T)}},\sqrt{\frac{q(T)}{NT}},\frac{\sqrt{q(T)}}{T}\right\}\right)\right] \\
= \; & \frac{\sigma^2}{2}\frac{\sqrt{NT}}{q(T)^{3/2}}\left[\frac{1}{N}\sum_{i=1}^{N}a_i^2-\overline{a}_N^2\right]\left[1+O_p\left(\max\left\{\frac{q(T)}{T},\frac{1}{q(T)},\sqrt{\frac{q(T)}{NT}},\frac{1}{\sqrt{T}}\right\}\right)\right] \\
& \times\left[1+O_p\left(\max\left\{\frac{1}{\sqrt{q(T)}},\sqrt{\frac{N}{q(T)T}}\right\}\right)\right] \\
& +\frac{2}{\sigma^2}\frac{1}{\sqrt{NTq(T)}}\sum_{i=1}^{N}\sum_{t=2}^{T}w_{it-1}\varepsilon_{it}\left[1+O_p\left(\max\left\{\frac{q(T)}{T},\frac{1}{q(T)},\sqrt{\frac{q(T)}{NT}},\frac{1}{\sqrt{T}}\right\}\right)\right] \\
& \quad\times\left[1+O_p\left(\max\left\{\frac{1}{\sqrt{q(T)}},\sqrt{\frac{q(T)}{NT}},\frac{\sqrt{q(T)}}{T}\right\}\right)\right] \\
= \; & \left[\frac{2}{\sigma^2}\frac{\sqrt{NT}}{q(T)^{3/2}}\sigma_a^2+\frac{2}{\sigma^2}\frac{\sigma^2}{\sqrt{2}}\mathcal{Z}\right]\left[1+o_p(1)\right] \\
= \; & \left[\sqrt{NTq(T)}\frac{2\sigma_a^2}{\sigma^2q(T)^2}+\sqrt{2}\mathcal{Z}\right]\left[1+o_p(1)\right]
\end{aligned}
$$

where $\mathcal{Z}\equiv N(0,1)$. It follows that

$$
\sqrt{NTq(T)}\left(\widehat{\rho}_{\text{pols}}-\rho_T-\frac{2\sigma_a^2}{\sigma^2q(T)^2}\right)\Rightarrow N(0,2),
$$

as required for part (e).

Now, in part (f), we consider the case where $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \to \infty$ but $q(T)/T^{\frac{1+\kappa}{3\kappa}} \to 0$. In this case, note that applying part (d) of Lemmas SD-3 and SD-5 and part (e) of Lemma SD-4 along with parts (a) and (b) of Lemma SE-11, part (b) of Lemmas SE-17 and SE-27 yield

$$
\begin{aligned}
& q(T)^2 \left(\widehat{\rho}_{\text{pols}} - \rho_T\right) \\
= \; & (1-\rho_T)\, q(T) \left[\frac{1}{NTq(T)} \sum_{i=1}^{N}\sum_{t=2}^{T} \left(y_{it-1} - \overline{y}_{-1,N,T}\right)^2\right]^{-1} \frac{1}{NT} \sum_{i=1}^{N}\sum_{t=2}^{T} \left(y_{it-1} - \overline{y}_{-1,N,T}\right) a_i \\
& + \left[\frac{1}{NTq(T)} \sum_{i=1}^{N}\sum_{t=2}^{T} \left(y_{it-1} - \overline{y}_{-1,N,T}\right)^2\right]^{-1} \frac{q(T)}{NT} \sum_{i=1}^{N}\sum_{t=2}^{T} \left(y_{it-1} - \overline{y}_{-1,N,T}\right) \varepsilon_{it} \\
= \; & \left(1 - \left[1 - \frac{1}{q(T)} + O\left(\frac{1}{q(T)^2}\right)\right]\right) \left[\frac{\sigma^2}{2} + O_p\left(\max\left\{\frac{q(T)}{T}, \frac{1}{q(T)}, \sqrt{\frac{q(T)}{NT}}, \frac{1}{\sqrt{T}}\right\}\right)\right]^{-1} \\
& \times q(T) \left[\left[\frac{1}{N}\sum_{i=1}^{N} a_i^2 - \overline{a}_N^2\right] + O_p\left(\max\left\{\frac{q(T)}{\sqrt{NT}}, \frac{q(T)}{T}\right\}\right)\right] \\
& + \left[\frac{\sigma^2}{2} + O_p\left(\max\left\{\frac{q(T)}{T}, \frac{1}{q(T)}, \sqrt{\frac{q(T)}{NT}}, \frac{1}{\sqrt{T}}\right\}\right)\right]^{-1} \\
& \times \frac{q(T)^{3/2}}{\sqrt{NT}} \left[\frac{1}{\sqrt{NTq(T)}} \sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-1}\varepsilon_{it} + O_p\left(\max\left\{\frac{1}{\sqrt{q(T)}}, \sqrt{\frac{q(T)}{NT}}, \frac{\sqrt{q(T)}}{T}\right\}\right)\right],
\end{aligned}
$$

so that

$$
\begin{aligned}
& q(T)^2 \left(\widehat{\rho}_{\text{pols}} - \rho_T\right) \\
= \; & \frac{2}{\sigma^2}\left[\frac{1}{N}\sum_{i=1}^{N} a_i^2 - \overline{a}_N^2\right] \left[1 + O_p\left(\max\left\{\frac{q(T)}{T}, \frac{1}{q(T)}, \frac{q(T)}{\sqrt{NT}}, \frac{1}{\sqrt{T}}\right\}\right)\right] + O_p\left(\frac{q(T)^{3/2}}{\sqrt{NT}}\right) \\
= \; & \frac{2\sigma_a^2}{\sigma^2} + o_p(1),
\end{aligned}
$$

as required for part (f).

Finally, in part (g), we consider the case where $\rho_T \in \mathcal{G}_{\text{St}} = \left\{|\rho_T| = \exp\left\{-\frac{1}{q(T)}\right\} : q(T) \geq 0 \text{ and } q(T) = O(1) \text{ as } T \to \infty\right\}$. In this case, applying part

(e) of Lemmas SD-3 and SD-5 and part (f) of Lemma SD-4, we obtain

$$
\begin{aligned}
& \widehat{\rho}_{\text{pols}} - \rho_T \\
= \; & (1-\rho_T)\left[\frac{1-\rho_T^2}{NT}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1}-\overline{y}_{-1,N,T}\right)^2\right]^{-1}\frac{1-\rho_T^2}{NT}\sum_{i=1}^{,N}\sum_{t=2}^{T}\left(y_{it-1}-\overline{y}_{-1,N,T}\right)a_i \\
& +\left[\frac{1-\rho_T^2}{NT}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1}-\overline{y}_{-1,N,T}\right)^2\right]^{-1}\frac{1-\rho_T^2}{NT}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1}-\overline{y}_{-1,N,T}\right)\varepsilon_{it} \\
= \; & \frac{(1-\rho_T)}{\left(1-\rho_T^2\right)\sigma_a^2+\sigma^2}\left(1-\rho_T^2\right)\sigma_a^2\left[1+o_p(1)\right] \\
& +\frac{1}{\left(1-\rho_T^2\right)\sigma_a^2+\sigma^2}O_p\left(\frac{1}{\sqrt{NT}}\right)\left[1+o_p(1)\right] \\
= \; & \frac{\left(1-\rho_T^2\right)(1-\rho_T)\sigma_a^2}{\left(1-\rho_T^2\right)\sigma_a^2+\sigma^2}+o_p(1),
\end{aligned}
$$

as required. □

Proof of Lemma A1:

To proceed, note that, in the pathwise asymptotics considered here, N grows as a monotonically increasing function of T, so that the asymptotics can be taken to be single-indexed with $T \to \infty$. Hence, as in the statement of the lemma, we can set $N = N(T) = (\tau T)^{1/\kappa}$ and simplify notation by writing $\mathbb{C}_{\gamma,\alpha,N,T} = \mathbb{C}_{\gamma,\alpha,N(T),T} = \mathbb{C}_{\gamma,\alpha,T}$.

Consider first part (a), where we take $\rho_T \in \mathcal{G}_1^{\text{P}} = \{\rho_T : \rho_T = 1 \text{ for all } T \text{ sufficiently large}\}$. In this case, note that

$$
\begin{aligned}
& \Pr\left(\rho \notin \mathbb{C}_{\gamma,\alpha,T} | \rho = \rho_T \in \mathcal{G}_1^{\text{P}}\right) \\
= \; & \Pr\left(\rho \notin \mathbb{C}_{\alpha_1,T}^{\mathbb{M}}, \mathbb{T}_{1,T} \leq -z_{\gamma_1}, \mathbb{T}_{2,T} \leq -z_{\gamma_2} | \rho = \rho_T \in \mathcal{G}_1^{\text{P}}\right) \\
& + \Pr\left(\rho \notin \mathbb{C}_{\gamma_1,\alpha_2,T}^{\text{UR1}}, \mathbb{T}_{1,T} > -z_{\gamma_1} | \rho = \rho_T \in \mathcal{G}_1^{\text{P}}\right) \\
& + \Pr\left(\rho \notin \mathbb{C}_{\gamma_2,\alpha_2,T}^{\text{UR2}}, \mathbb{T}_{1,T} \leq -z_{\gamma_1}, \mathbb{T}_{2,T} > -z_{\gamma_2} | \rho = \rho_T \in \mathcal{G}_1^{\text{P}}\right) \\
\leq \; & \Pr\left(\rho \notin \mathbb{C}_{\alpha_1,T}^{\mathbb{M}} | \rho = \rho_T \in \mathcal{G}_1^{\text{P}}\right) + \Pr\left(\rho \notin \mathbb{C}_{\gamma_1,\alpha_2,T}^{\text{UR1}} | \rho = \rho_T \in \mathcal{G}_1^{\text{P}}\right) \\
& + \Pr\left(\rho \notin \mathbb{C}_{\gamma_2,\alpha_2,T}^{\text{UR2}} | \rho = \rho_T \in \mathcal{G}_1^{\text{P}}\right).
\end{aligned}
$$

Then, there exists a positive integer I_ρ such that, for all $T \geq I_\rho$,

$$
1-\sqrt{2}\frac{\left(z_{\gamma_1}+z_{\alpha_2}\right)}{T\sqrt{N}} = 1-\sqrt{2}\frac{\left(z_{\gamma_1}+z_{\alpha_2}\right)}{\tau^{\frac{1}{2\kappa}}T^{\frac{1}{2\kappa}+1}} \leq \rho_T = 1,
$$

and

$$
1-\frac{2\left(z_{\gamma_2}+z_{\alpha_2}\right)}{\sqrt{NT}} = 1-\sqrt{2}\frac{\left(z_{\gamma_2}+z_{\alpha_2}\right)}{\tau^{\frac{1}{2\kappa}}T^{\frac{1}{2}\left(1+\frac{1}{\kappa}\right)}} \leq \rho_T = 1.
$$

Thus, $\rho_T \in \mathbb{C}_{\gamma_1,\alpha_2,T}^{\text{UR1}}$ and $\rho_T \in \mathbb{C}_{\gamma_2,\alpha_2,T}^{\text{UR2}}$, for all T sufficiently large. Hence,

$\limsup_{T\to\infty}\Pr\left(\rho\notin\mathbb{C}^{\mathrm{UR1}}_{\gamma_1,\alpha_2,T}|\rho=\rho_T\in\mathcal{G}^{\mathrm{P}}_1\right)=0$ and $\limsup_{T\to\infty}\Pr\left(\rho\notin\mathbb{C}^{\mathrm{UR2}}_{\gamma_2,\alpha_2,T}|\rho=\rho_T\in\mathcal{G}^{\mathrm{P}}_1\right)=0$. Moreover, by part (a) of Theorem 3.1, $\lim_{T\to\infty}\Pr\left(\rho\notin\mathbb{C}^{\mathbb{M}}_{\alpha_1,T}|\rho=\rho_T\in\mathcal{G}^{\mathrm{P}}_1\right)=\alpha_1$. It follows that

$$\begin{aligned}
&\limsup_{T\to\infty}\Pr\left(\rho\notin\mathbb{C}_{\gamma,\alpha,T}|\rho=\rho_T\in\mathcal{G}^{\mathrm{P}}_1\right)\\
\leq\ &\limsup_{T\to\infty}\Pr\left(\rho\notin\mathbb{C}^{\mathbb{M}}_{\alpha_1,T}|\rho=\rho_T\in\mathcal{G}^{\mathrm{P}}_1\right)+\limsup_{T\to\infty}\Pr\left(\rho\notin\mathbb{C}^{\mathrm{UR1}}_{\gamma_1,\alpha_2,T}|\rho=\rho_T\in\mathcal{G}^{\mathrm{P}}_1\right)\\
&+\limsup_{T\to\infty}\Pr\left(\rho\notin\mathbb{C}^{\mathrm{UR2}}_{\gamma_2,\alpha_2,T}|\rho=\rho_T\in\mathcal{G}^{\mathrm{P}}_1\right)\\
=\ &\alpha_1+0+0=\alpha_1.
\end{aligned}$$

Next, consider part (b), where we take $\rho_T\in\mathcal{G}^{\mathrm{P}}_2=\left\{\rho_T=\exp\left\{-\frac{1}{q(T)}\right\}:\sqrt{N}T\sim T^{\frac{1}{2\kappa}+1}\ll q\left(T\right)\right\}$. Note that, in the present case, $T\sqrt{N}\sim T^{\frac{1}{2\kappa}+1}\ll q\left(T\right)$,

$$\begin{aligned}
1-\sqrt{2}\frac{\left(z_{\gamma_1}+z_{\alpha_2}\right)}{T\sqrt{N}}\ &=\ 1-\sqrt{2}\frac{\left(z_{\gamma_1}+z_{\alpha_2}\right)}{\tau^{\frac{1}{2\kappa}}T^{\frac{1}{2\kappa}+1}}\\
&\leq\ \rho_T=\exp\left\{-\frac{1}{q\left(T\right)}\right\}=1-\frac{1}{q\left(T\right)}+O\left(\frac{1}{q\left(T\right)^2}\right)\leq 1,
\end{aligned}$$

and

$$\begin{aligned}
1-\frac{2\left(z_{\gamma_2}+z_{\alpha_2}\right)}{\sqrt{NT}}\ &=\ 1-\frac{2\left(z_{\gamma_2}+z_{\alpha_2}\right)}{\tau^{\frac{1}{2\kappa}}T^{\frac{1}{2}\left(1+\frac{1}{\kappa}\right)}}\\
&\leq\ \rho_T=\exp\left\{-\frac{1}{q\left(T\right)}\right\}=1-\frac{1}{q\left(T\right)}+O\left(\frac{1}{q\left(T\right)^2}\right)\leq 1,
\end{aligned}$$

for all T sufficiently large, so that $\rho_T\in\mathbb{C}^{\mathrm{UR1}}_{\gamma_1,\alpha_2,T}$ and $\rho_T\in\mathbb{C}^{\mathrm{UR2}}_{\gamma_2,\alpha_2,T}$ eventually. Hence, $\limsup_{T\to\infty}\Pr\left(\rho\notin\mathbb{C}^{\mathrm{UR1}}_{\gamma_1,\alpha_2,T}|\rho=\rho_T\in\mathcal{G}^{\mathrm{P}}_2\right)=0$ and $\limsup_{T\to\infty}\Pr\left(\rho\notin\mathbb{C}^{\mathrm{UR2}}_{\gamma_2,\alpha_2,T}|\rho=\rho_T\in\mathcal{G}^{\mathrm{P}}_2\right)=0$. Moreover, by part (b) of Theorem 3.1, $\lim_{T\to\infty}\Pr\left(\rho\notin\mathbb{C}^{\mathbb{M}}_{\alpha_1,T}|\rho=\rho_T\in\mathcal{G}^{\mathrm{P}}_2\right)=\alpha_1$. It follows that

$$\begin{aligned}
&\limsup_{T\to\infty}\Pr\left(\rho\notin\mathbb{C}_{\gamma,\alpha,T}|\rho=\rho_T\in\mathcal{G}^{\mathrm{P}}_2\right)\\
\leq\ &\limsup_{T\to\infty}\Pr\left(\rho\notin\mathbb{C}^{\mathbb{M}}_{\alpha_1,T}|\rho=\rho_T\in\mathcal{G}^{\mathrm{P}}_2\right)+\limsup_{T\to\infty}\Pr\left(\rho\notin\mathbb{C}^{\mathrm{UR1}}_{\gamma_1,\alpha_2,T}|\rho=\rho_T\in\mathcal{G}^{\mathrm{P}}_2\right)\\
&+\limsup_{T\to\infty}\Pr\left(\rho\notin\mathbb{C}^{\mathrm{UR2}}_{\gamma_2,\alpha_2,T}|\rho=\rho_T\in\mathcal{G}^{\mathrm{P}}_2\right)\\
=\ &\alpha_1+0+0=\alpha_1<\alpha.
\end{aligned}$$

Consider part (c), where we assume

$$\rho_T\in\mathcal{G}^{\mathrm{P}}_3=\left\{\rho_T=\exp\left\{-\frac{1}{q\left(T\right)}\right\}:\left(\sqrt{N}T\sim T^{\frac{1}{2\kappa}+1}\sim q\left(T\right)\right)\cap\left(\rho_T\geq 1-\frac{\left(z_{\gamma_1}+z_{\alpha_2}\right)\sqrt{2}}{\sqrt{N}T}\text{ eventually}\right)\right\}.$$

In this case, $\rho_T\in\mathbb{C}^{\mathrm{UR1}}_{\gamma_1,\alpha_2,T}$ eventually by assumption. Since $\sqrt{NT}\ll\sqrt{N}T\sim q\left(T\right)$, we also have

$\rho_T \in \mathbb{C}^{\mathrm{UR2}}_{\gamma_2,\alpha_2,T}$ eventually. It follows by applying part (b) of Theorem 3.1 that

$$\begin{aligned}
& \limsup_{T\to\infty} \Pr\left(\rho \notin \mathbb{C}_{\gamma,\alpha,T} | \rho = \rho_T \in \mathcal{G}_3^{\mathrm{P}}\right) \\
\leq\ & \limsup_{T\to\infty} \Pr\left(\rho \notin \mathbb{C}^{\mathbb{M}}_{\alpha_1,T} | \rho = \rho_T \in \mathcal{G}_3^{\mathrm{P}}\right) + \limsup_{T\to\infty} \Pr\left(\rho \notin \mathbb{C}^{\mathrm{UR1}}_{\gamma_1,\alpha_2,T} | \rho = \rho_T \in \mathcal{G}_3^{\mathrm{P}}\right) \\
& + \limsup_{T\to\infty} \Pr\left(\rho \notin \mathbb{C}^{\mathrm{UR2}}_{\gamma_2,\alpha_2,T} | \rho = \rho_T \in \mathcal{G}_3^{\mathrm{P}}\right) \\
=\ & \alpha_1 + 0 + 0 = \alpha_1 < \alpha.
\end{aligned}$$

We turn our attention now to part (d) where we take

$$\rho_T \in \mathcal{G}_4^{\mathrm{P}} = \left\{ \rho_T = \exp\left\{-\frac{1}{q(T)}\right\} : \sqrt{N}T \sim q(T) \cap \rho_T < 1 - \frac{\left(z_{\gamma_1} + z_{\alpha_2}\right)\sqrt{2}}{\sqrt{N}T} \text{ eventually} \right\}.$$

To proceed, write $\mathbb{T}_{1,T} = \frac{M_{yy}^{1/2}\sqrt{N}T\left(\widehat{\rho}_{\mathrm{pols}} - \rho_T\right)}{\widehat{\sigma}} - \frac{M_{yy}^{1/2}\sqrt{N}T(1-\rho_T)}{\widehat{\sigma}} = \mathbb{T}^*_{1,T} - \theta^*_T$, where $M_{yy} = N^{-1}T^{-2}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1} - y_{-1,N,T}\right)^2$. In this case, by part (b) of Lemma SD-3, Lemma SD-12, and part (b) of Theorem SA-2; we have $\mathbb{T}^*_T \Rightarrow N(0,1)$ and $\theta^*_T = \widehat{\sigma}^{-1}M_{yy}^{1/2}\sqrt{N}T(1-\rho_T) = 2^{-1/2}\sqrt{N}T(1-\rho_T)[1+o_p(1)]$, so that $\mathbb{T}_{1,T} = \mathbb{T}^*_{1,T} - \theta^*_T = \mathbb{T}^*_{1,T} - \sqrt{N}T(1-\rho_T)/\sqrt{2} + \xi_T$, where $\xi_T = o_p(1)$. Now, let ϵ_T be a sequence of positive numbers such that, as $T \to \infty$, $\epsilon_T \to 0$ but $\xi_T/\epsilon_T = o_p(1)$. It follows that

$$\begin{aligned}
& \Pr\left(\mathbb{T}_{1,T} > -z_{\gamma_1} | \left\{q(T) \sim T^{\frac{1}{2\kappa}+1}\right\} \cap \left\{\rho_T < 1 - \frac{\left(z_{\gamma_1} + z_{\alpha_2}\right)\sqrt{2}}{\sqrt{N}T} \text{ eventually}\right\}\right) \\
=\ & \Pr\left(\left\{\mathbb{T}^*_{1,T} - \frac{\sqrt{N}T(1-\rho_T)}{\sqrt{2}} + \xi_T > -z_{\gamma_1}\right\} \cap \{|\xi_T| < \epsilon_T\} | \ \left\{q(T) \sim T^{\frac{1}{2\kappa}+1}\right\}\right. \\
& \left. \cap \left\{\rho_T < 1 - \frac{\left(z_{\gamma_1} + z_{\alpha_2}\right)\sqrt{2}}{\sqrt{N}T} \text{ eventually}\right\}\right) \\
& + \Pr\left(\left\{\mathbb{T}^*_{1,T} - \frac{\sqrt{N}T(1-\rho_T)}{\sqrt{2}} + \xi_T > -z_{\gamma_1}\right\} \cap \{|\xi_T| \geq \epsilon_T\} | \ \left\{q(T) \sim T^{\frac{1}{2\kappa}+1}\right\}\right. \\
& \left. \cap \left\{\rho_T < 1 - \frac{\left(z_{\gamma_1} + z_{\alpha_2}\right)\sqrt{2}}{\sqrt{N}T} \text{ eventually}\right\}\right) \\
\leq\ & \Pr\left(\mathbb{T}^*_{1,T} - \frac{\sqrt{N}T(1-\rho_T)}{\sqrt{2}} > -\left(z_{\gamma_1} + \epsilon_T\right) | q(T) \sim T^{\frac{1}{2\kappa}+1} \cap \rho_T < 1 - \frac{\left(z_{\gamma_1} + z_{\alpha_2}\right)\sqrt{2}}{\sqrt{N}T} \text{ eventually}\right) \\
& + \Pr\left(|\xi_T| \geq \epsilon_T | \ \left\{q(T) \sim T^{\frac{1}{2\kappa}+1}\right\} \cap \left\{\rho_T < 1 - \frac{\left(z_{\gamma_1} + z_{\alpha_2}\right)\sqrt{2}}{\sqrt{N}T} \text{ eventually}\right\}\right) \\
\leq\ & \Pr\left(\mathbb{T}^*_{1,T} > z_{\alpha_2} - \epsilon_T | \left\{q(T) \sim T^{\frac{1}{2\kappa}+1}\right\} \cap \left\{\rho_T < 1 - \frac{\left(z_{\gamma_1} + z_{\alpha_2}\right)\sqrt{2}}{\sqrt{N}T} \text{ eventually}\right\}\right) \\
& + \Pr\left(|\xi_T| \geq \epsilon_T | \left\{q(T) \sim T^{\frac{1}{2\kappa}+1}\right\} \cap \left\{\rho_T < 1 - \frac{\left(z_{\gamma_1} + z_{\alpha_2}\right)\sqrt{2}}{\sqrt{N}T} \text{ eventually}\right\}\right) \\
\to\ & 1 - \Phi\left(z_{\alpha_2}\right) = \alpha_2,
\end{aligned}$$

where the second inequality above results from the fact that in this case $\rho_T < 1-\sqrt{2}\left(z_{\gamma_1}+z_{\alpha_2}\right)N^{-1/2}T^{-1}$ $\Longleftrightarrow \sqrt{N}T\left(1-\rho_T\right)/\sqrt{2}-z_{\gamma_1} > z_{\alpha_2}$. Moreover, since $\sqrt{NT} \ll \sqrt{N}T \sim q\left(T\right)$ in this case, we again have $\rho_T \in \mathbb{C}^{\mathrm{UR2}}_{\gamma_2,\alpha_2,T}$ eventually. Hence, applying part (b) of Theorem 3.1, we obtain

$$\begin{aligned}
& \limsup_{T\to\infty} \Pr\left(\rho \notin \mathbb{C}_{\gamma,\alpha,T} | \rho = \rho_T \in \mathcal{G}_4^{\mathrm{P}}\right) \\
= \; & \Pr\left(\rho \notin \mathbb{C}^{\mathbb{M}}_{\alpha_1,T}, \mathbb{T}_{1,T} \leq -z_{\gamma_1}, \mathbb{T}_{2,T} \leq -z_{\gamma_2} | \rho = \rho_T \in \mathcal{G}_4^{\mathrm{P}}\right) \\
& + \Pr\left(\rho \notin \mathbb{C}^{\mathrm{UR1}}_{\gamma_1,\alpha_2,T}, \mathbb{T}_{1,T} > -z_{\gamma_1} | \rho = \rho_T \in \mathcal{G}_4^{\mathrm{P}}\right) \\
& + \Pr\left(\rho \notin \mathbb{C}^{\mathrm{UR2}}_{\gamma_2,\alpha_2,T}, \mathbb{T}_{1,T} \leq -z_{\gamma_1}, \mathbb{T}_{2,T} > -z_{\gamma_2} | \rho = \rho_T \in \mathcal{G}_4^{\mathrm{P}}\right) \\
\leq \; & \limsup_{T\to\infty} \Pr\left(\rho \notin C^{\mathbb{M}}_{\alpha_1} | \rho = \rho_T \in \mathcal{G}_4^{\mathrm{P}}\right) + \limsup_{T\to\infty} \Pr\left(\mathbb{T}_{1,T} > -z_{\gamma_1} | \rho = \rho_T \in \mathcal{G}_4^{\mathrm{P}}\right) \\
& + \limsup_{T\to\infty} \Pr\left(\rho \notin \mathbb{C}^{\mathrm{UR2}}_{\gamma_2,\alpha_2,T} | \rho = \rho_T \in \mathcal{G}_4^{\mathrm{P}}\right) \\
\leq \; & \alpha_1 + \alpha_2 + 0 = \alpha.
\end{aligned}$$

Consider part (e), where we take $\rho_T \in \mathcal{G}_5^{\mathrm{P}} = \left\{\rho_T = \exp\left\{-\frac{1}{q(T)}\right\} : \left(\sqrt{NT} \ll q\left(T\right)\right) \cap \left(T \ll q\left(T\right) \ll T^{\frac{1}{2\kappa}+1} \sim \sqrt{N}T\right)\right\}$. In this case, note that

$$\begin{aligned}
\theta_T^* &= \frac{M_{yy}^{1/2}\sqrt{N}T\left(1-\rho_T\right)}{\widehat{\sigma}} = \frac{1}{\sigma}\frac{\sigma}{\sqrt{2}}\sqrt{N}T\left[1-\exp\left\{-\frac{1}{q\left(T\right)}\right\}\right]\left[1+o_p\left(1\right)\right] \\
&= \frac{1}{\sqrt{2}}\sqrt{N}T\left[1-1+\frac{1}{q\left(T\right)}+O\left(\frac{1}{q\left(T\right)^2}\right)\right]\left[1+o_p\left(1\right)\right] \\
&= \frac{1}{\sqrt{2}}\frac{\sqrt{N}T}{q\left(T\right)}\left[1+O\left(\frac{1}{q\left(T\right)}\right)\right]\left[1+o_p\left(1\right)\right] \to \infty, \text{ wpa.1},
\end{aligned}$$

so that $\frac{q(T)}{\sqrt{N}T}\left(\mathbb{T}_{1,T}^* - \theta_T^* + z_{\gamma_1}\right) \xrightarrow{p} -\frac{1}{\sqrt{2}} < 0$. Thus,

$$\begin{aligned}
0 &\leq \Pr\left(\mathbb{T}_{1,T} > -z_{\gamma_1} | \rho = \rho_T \in \mathcal{G}_5^{\mathrm{P}}\right) \\
&= \Pr\left(\frac{q\left(T\right)}{\sqrt{N}T}\left(\mathbb{T}_{1,T}^* - \theta_T^* + z_{\gamma_1}\right) + \frac{1}{\sqrt{2}} > \frac{1}{\sqrt{2}} | \rho = \rho_T \in \mathcal{G}_5^{\mathrm{P}}\right) \\
&\leq \Pr\left(\left|\frac{q\left(T\right)}{\sqrt{N}T}\left(\mathbb{T}_{1,T}^* - \theta_T^* + z_{\gamma_1}\right) + \frac{1}{\sqrt{2}}\right| > \frac{1}{\sqrt{2}} | \rho = \rho_T \in \mathcal{G}_5^{\mathrm{P}}\right) \to 0.
\end{aligned}$$

Moreover, since $\sqrt{NT} \ll q\left(T\right)$, we have $\rho_T \in \mathbb{C}^{\mathrm{UR2}}_{\gamma_2,\alpha_2,T}$ eventually, as shown previously. Hence, applying part (b) of Theorem 3.1, we deduce that

$$\begin{aligned}
& \limsup_{T\to\infty} \Pr\left(\rho \notin \mathbb{C}_{\gamma,\alpha,T} | \rho = \rho_T \in \mathcal{G}_5^{\mathrm{P}}\right) \\
\leq \; & \limsup_{T\to\infty} \Pr\left(\rho \notin C^{\mathbb{M}}_{\alpha_1} | \rho = \rho_T \in \mathcal{G}_5^{\mathrm{P}}\right) + \limsup_{T\to\infty} \Pr\left(\mathbb{T}_{1,T} > -z_{\gamma_1} | \rho = \rho_T \in \mathcal{G}_5^{\mathrm{P}}\right) \\
& + \limsup_{T\to\infty} \Pr\left(\rho \notin \mathbb{C}^{\mathrm{UR2}}_{\gamma_2,\alpha_2,T} | \rho = \rho_T \in \mathcal{G}_5^{\mathrm{P}}\right) \\
\leq \; & \alpha_1 + 0 + 0 < \alpha.
\end{aligned}$$

Consider part (f), where we take

$\rho_T \in \mathcal{G}_6^{\mathrm{P}} = \left\{ \rho_T = \exp\left\{-\frac{1}{q(T)}\right\} : T \ll q(T) \sim T^{\frac{1}{2}\left(1+\frac{1}{\kappa}\right)} \sim \sqrt{NT} \cap \rho_T \geq 1 - \frac{2\left(z_{\gamma_2}+z_{\alpha_2}\right)}{\sqrt{NT}} \text{ eventually} \right\}.$

Following the same argument as part (e) above, we have $\theta_T^* = \frac{1}{\sqrt{2}}\frac{\sqrt{N}T}{q(T)}\left[1 + O\left(q(T)^{-1}\right)\right][1 + o_p(1)] \to \infty$ w.p.a.1, so that $\frac{q(T)}{\sqrt{N}T}\left(\mathbb{T}_{1,T}^* - \theta_T^* + z_{\gamma_1}\right) \xrightarrow{p} -1/\sqrt{2} < 0$. It follows that

$$\Pr\left(\mathbb{T}_{1,T} > -z_{\gamma_1} | \rho = \rho_T \in \mathcal{G}_6^{\mathrm{P}}\right) \leq \Pr\left(\left|\frac{q(T)}{\sqrt{N}T}\left(\mathbb{T}_{1,T}^* - \theta_T^* + z_{\gamma_1}\right) + \frac{1}{2}\right| > \frac{1}{2} | \rho = \rho_T \in \mathcal{G}_6^{\mathrm{P}}\right) \to 0.$$

Moreover, in this case, $\rho_T \in \mathbb{C}_{\gamma_2,\alpha_2,T}^{\mathrm{UR2}}$ eventually by assumption. Using these results and part (b) of Theorem 3.1, we get

$$\begin{aligned}
& \limsup_{T\to\infty} \Pr\left(\rho \notin \mathbb{C}_{\gamma,\alpha,T} | \rho = \rho_T \in \mathcal{G}_6^{\mathrm{P}}\right) \\
\leq\ & \limsup_{T\to\infty} \Pr\left(\rho \notin \mathbb{C}_{\alpha_1,T}^{\mathbb{M}} | \rho = \rho_T \in \mathcal{G}_6^{\mathrm{P}}\right) + \limsup_{T\to\infty} \Pr\left(\mathbb{T}_{1,T} > -z_{\gamma_1} | \rho = \rho_T \in \mathcal{G}_6^{\mathrm{P}}\right) \\
& + \limsup_{T\to\infty} \Pr\left(\rho \notin \mathbb{C}_{\gamma_2,\alpha_2,T}^{\mathrm{UR2}} | \rho = \rho_T \in \mathcal{G}_6^{\mathrm{P}}\right) \\
=\ & \alpha_1 + 0 + 0 < \alpha.
\end{aligned}$$

Consider part (g), where we take

$\rho_T \in \mathcal{G}_7^{\mathrm{P}} = \left\{ \rho_T = \exp\left\{-\frac{1}{q(T)}\right\} : T \ll q(T) \sim T^{\frac{1}{2}\left(1+\frac{1}{\kappa}\right)} \sim \sqrt{NT} \cap \rho_T < 1 - \frac{2\left(z_{\gamma_2}+z_{\alpha_2}\right)}{\sqrt{NT}} \text{ eventually} \right\}.$

Write $\mathbb{T}_{2,T} = \widehat{\omega}_{\mathrm{IVL}}\left(\widehat{\rho}_{\mathrm{IVL}} - 1\right) = \widehat{\omega}_{\mathrm{IVL}}\left(\widehat{\rho}_{\mathrm{IVL}} - \rho_T\right) - \widehat{\omega}_{\mathrm{IVL}}\left(1 - \rho_T\right) = \mathbb{T}_{2,T}^* - \vartheta_T^*$. In this case, by part (b) of Lemma SD-9,

$$\begin{aligned}
\mathbb{T}_{2,T}^* &= \widehat{\omega}_{\mathrm{IVL}}\left(\widehat{\rho}_{\mathrm{IVL}} - \rho_T\right) \\
&= \left(\frac{1}{\widehat{\sigma}^2}\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=3}^{T}\Delta y_{it-1}y_{it-1}\right)\left(\sum_{i=1}^{N}\sum_{t=3}^{T}\Delta y_{it-1}y_{it-1}\right)^{-1}\sum_{i=1}^{N}\sum_{t=3}^{T}\Delta y_{it-1}\left[a_i\left(1-\rho_T\right) + \varepsilon_{it}\right] \\
&= \frac{1}{\widehat{\sigma}^2}\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=3}^{T}\Delta y_{it-1}\left[a_i\left(1-\rho_T\right) + \varepsilon_{it}\right] \Rightarrow N(0,1).
\end{aligned}$$

In addition, by part (b) of Lemma SD-8, Lemma SD-12, and the Cramér convergence theorem, we have $\vartheta_T^* = \left(\widehat{\sigma}^{-2}N^{-1}T^{-1}\sum_{i=1}^{N}\sum_{t=3}^{T}\Delta y_{it-1}y_{it-1}\right)\sqrt{NT}\left(1-\rho_T\right) = \sqrt{NT}\left(1-\rho_T\right) + o_p(1)$, so that $\mathbb{T}_{2,T} = \mathbb{T}_{2,T}^* - \sqrt{NT}\left(1-\rho_T\right) + \zeta_T$, where $\zeta_T = o_p(1)$. Now, let ϵ_T be a sequence of positive numbers such that, as $T \to \infty$, $\epsilon_T \to 0$ but $\zeta_T/\epsilon_T = o_p(1)$. It follows by argument similar to that given in part (d) above

that

$$
\begin{aligned}
& \Pr\left(\mathbb{T}_{2,T} > -z_{\gamma_2} | \left\{T \ll q(T) \sim \sqrt{NT}\right\} \cap \left\{\rho_T < 1 - \frac{2\left(z_{\gamma_2} + z_{\alpha_2}\right)}{\sqrt{NT}} \text{ eventually}\right\}\right) \\
\leq\ & \Pr\left(\mathbb{T}^*_{2,T} > \sqrt{NT}(1-\rho_T) - \left(z_{\gamma_2} + \epsilon_T\right) | T \ll q(T) \sim \sqrt{NT} \cap \rho_T < 1 - \frac{2\left(z_{\gamma_2} + z_{\alpha_2}\right)}{\sqrt{NT}} \text{ eventually}\right) \\
& + \Pr\left(|\zeta_T| \geq \epsilon_T | \left\{T \ll q(T) \sim \sqrt{NT}\right\} \cap \left\{\rho_T < 1 - \frac{2\left(z_{\gamma_2} + z_{\alpha_2}\right)}{\sqrt{NT}} \text{ eventually}\right\}\right) \\
\leq\ & \Pr\left(\mathbb{T}^*_{2,T} > z_{\alpha_2} - \epsilon_T | \left\{T \ll q(T) \sim \sqrt{NT}\right\} \cap \left\{\rho_T < 1 - \frac{2\left(z_{\gamma_2} + z_{\alpha_2}\right)}{\sqrt{NT}} \text{ eventually}\right\}\right) \\
& + \Pr\left(|\zeta_T| \geq \epsilon_T | \left\{T \ll q(T) \sim \sqrt{NT}\right\} \cap \left\{\rho_T < 1 - \frac{2\left(z_{\gamma_2} + z_{\alpha_2}\right)}{\sqrt{NT}} \text{ eventually}\right\}\right) \\
\to\ & 1 - \Phi\left(z_{\alpha_2}\right) = \alpha_2.
\end{aligned}
$$

where the last inequality is due to the fact that $\rho_T < 1 - 2N^{-1/2}T^{-1/2}\left(z_{\gamma_2} + z_{\alpha_2}\right) \Longleftrightarrow \sqrt{NT}(1-\rho_T)/2 - z_{\gamma_2} > z_{\alpha_2}$, implying, in turn, that $\sqrt{NT}(1-\rho_T) - z_{\gamma_2} > z_{\alpha_2}$. Moreover, similar to the proof of part (e) above, we have in this case $\lim_{T\to\infty} \Pr\left(\mathbb{T}_{1,T} > -z_{\gamma_1} | \rho = \rho_T \in \mathcal{G}_7^{\mathrm{P}}\right) = 0$. Hence, using these results and applying part (b) of Theorem 3.1, we obtain

$$
\begin{aligned}
& \limsup_{T\to\infty} \Pr\left(\rho \notin \mathbb{C}_{\gamma,\alpha,T} | \rho = \rho_T \in \mathcal{G}_7^{\mathrm{P}}\right) \\
\leq\ & \limsup_{T\to\infty} \Pr\left(\rho \notin \mathbb{C}^{\mathbb{M}}_{\alpha_1,T} | \rho = \rho_T \in \mathcal{G}_7^{\mathrm{P}}\right) + \limsup_{T\to\infty} \Pr\left(\mathbb{T}_{1,T} > -z_{\gamma_1} | \rho = \rho_T \in \mathcal{G}_7^{\mathrm{P}}\right) \\
& + \limsup_{T\to\infty} \Pr\left(\mathbb{T}_{2,T} > -z_{\gamma_2} | \rho = \rho_T \in \mathcal{G}_7^{\mathrm{P}}\right) \\
\leq\ & \alpha_1 + 0 + \alpha_2 = \alpha.
\end{aligned}
$$

Consider part (h), where we take $\rho_T \in \mathcal{G}_8^{\mathrm{P}} = \left\{\rho_T = \exp\left\{-\frac{1}{q(T)}\right\} : \sqrt{NT} \ll q(T) \sim T\right\}$. Note that, in this case,

$$
\begin{aligned}
\theta^*_T &= \frac{1}{\sigma}\frac{\sigma}{2}\frac{q(T)}{T}\left[\exp\left\{-\frac{2T}{q(T)}\right\} + \frac{2T}{q(T)} - 1\right]^{1/2} \sqrt{N}T\left[1 - \exp\left\{-\frac{1}{q(T)}\right\}\right][1 + o_p(1)] \\
&= \frac{\sqrt{N}}{2}\left[\exp\left\{-\frac{2T}{q(T)}\right\} + \frac{2T}{q(T)} - 1\right]^{1/2}\left[1 + O\left(\frac{1}{q(T)}\right)\right][1 + o_p(1)] \to \infty, \text{ wpa.1},
\end{aligned}
$$

so that $N^{-1/2}\left[\exp\left\{-\frac{2T}{q(T)}\right\} + \frac{2T}{q(T)} - 1\right]^{-1/2}\left(\mathbb{T}^*_{1,T} - \theta^*_T + z_{\gamma_1}\right) \xrightarrow{p} -1/2 < 0$. Thus,

$$
\begin{aligned}
& \Pr\left(\mathbb{T}_{1,T} > -z_{\gamma_1} | \rho = \rho_T \in \mathcal{G}_8^{\mathrm{P}}\right) \\
\leq\ & \Pr\left(\left|\frac{1}{\sqrt{N}}\left[\exp\left\{-\frac{2T}{q(T)}\right\} + \frac{2T}{q(T)} - 1\right]^{-1/2}\left(\mathbb{T}^*_{1,T} - \theta^*_T + z_{\gamma_1}\right) + \frac{1}{2}\right| > \frac{1}{2} | \rho = \rho_T \in \mathcal{G}_8^{\mathrm{P}}\right) \to 0.
\end{aligned}
$$

Moreover, note that $\sqrt{NT} \ll q(T)$, so that $\rho_T \in \mathbb{C}^{\mathrm{UR2}}_{\gamma_2,\alpha_2,T}$ eventually in this case, from which we further deduce that $\lim_{T\to\infty} \Pr\left(\rho \notin \mathbb{C}^{\mathrm{UR2}}_{\gamma_2,\alpha_2,T} | \rho = \rho_T \in \mathcal{G}_8^{\mathrm{P}}\right) = 0$. Finally, applying part (c) of Theorem 3.1, we

obtain

$$\begin{aligned}
& \limsup_{T\to\infty} \Pr\left(\rho \notin \mathbb{C}_{\gamma,\alpha,T} | \rho = \rho_T \in \mathcal{G}_8^{\mathrm{P}}\right) \\
\leq\ & \limsup_{T\to\infty} \Pr\left(\rho \notin \mathbb{C}_{\alpha_1,T}^{\mathbb{M}} | \rho = \rho_T \in \mathcal{G}_8^{\mathrm{P}}\right) + \limsup_{T\to\infty} \Pr\left(\mathbb{T}_{1,T} > -z_{\gamma_1} | \rho = \rho_T \in \mathcal{G}_8^{\mathrm{P}}\right) \\
& + \limsup_{T\to\infty} \Pr\left(\rho \notin \mathbb{C}_{\gamma_2,\alpha_2,T}^{\mathrm{UR2}} | \rho = \rho_T \in \mathcal{G}_8^{\mathrm{P}}\right) \\
=\ & \alpha_1 + 0 + 0 < \alpha.
\end{aligned}$$

Consider part (i), where we take $\rho_T \in \mathcal{G}_9^{\mathrm{P}} = \left\{\rho_T = \exp\left\{-\frac{1}{q(T)}\right\} : \sqrt{NT} \sim T^{\frac{1}{2}\left(1+\frac{1}{\kappa}\right)} \ll q(T) \ll T\right\}$. Note that, for this case, $\theta_T^* = \overline{M}_{yy}^{1/2}\sqrt{NTq(T)}\,(1-\rho_T)/\widehat{\sigma} = \sqrt{\frac{NT}{2q(T)}}\left[1 + O\left(\frac{1}{q(T)}\right)\right][1 + o_p(1)] \to \infty$ wpa.1, where $\overline{M}_{yy} = N^{-1}T^{-1}q(T)^{-1}\sum_{i=1}^{N}\sum_{t=2}^{T}(y_{it-1} - y_{-1,NT})^2$. It follows that $\sqrt{\frac{q(T)}{NT}}\left(\mathbb{T}_{1,T}^* - \theta_T^* + z_{\gamma_1}\right) \xrightarrow{p} -1/\sqrt{2} < 0$. Thus,

$$\Pr\left(\mathbb{T}_{1,T} > -z_{\gamma_1} | \rho = \rho_T \in \mathcal{G}_9^{\mathrm{P}}\right) \leq \Pr\left(\left|\sqrt{\frac{q(T)}{NT}}\left(\mathbb{T}_{1,T}^* - \theta_T^* + z_{\gamma_1}\right) + \frac{1}{\sqrt{2}}\right| > \frac{1}{\sqrt{2}} | \rho = \rho_T \in \mathcal{G}_9^{\mathrm{P}}\right) \to 0.$$

Moreover, similar to previous parts, $\rho_T \in \mathbb{C}_{\gamma_2,\alpha_2,T}^{\mathrm{UR2}}$ eventually since $\sqrt{NT} \ll q(T)$, so that $\lim_{T\to\infty} \Pr\left(\rho \notin \mathbb{C}_{\gamma_2,\alpha_2,T}^{\mathrm{UR2}} | \rho = \rho_T \in \mathcal{G}_9^{\mathrm{P}}\right) = 0$. Hence, applying part (d) of Theorem 3.1, we obtain

$$\begin{aligned}
& \limsup_{T\to\infty} \Pr\left(\rho \notin \mathbb{C}_{\gamma,\alpha,T} | \rho = \rho_T \in \mathcal{G}_9^{\mathrm{P}}\right) \\
\leq\ & \limsup_{T\to\infty} \Pr\left(\rho \notin C_{\alpha_1}^{\mathbb{M}} | \rho = \rho_T \in \mathcal{G}_9^{\mathrm{P}}\right) + \limsup_{T\to\infty} \Pr\left(\mathbb{T}_{1,T} > -z_{\gamma_1} | \rho = \rho_T \in \mathcal{G}_9^{\mathrm{P}}\right) \\
& + \limsup_{T\to\infty} \Pr\left(\rho \notin \mathbb{C}_{\gamma_2,\alpha_2,T}^{\mathrm{UR2}} | \rho = \rho_T \in \mathcal{G}_9^{\mathrm{P}}\right) \\
\leq\ & \alpha_1 + 0 + 0 < \alpha.
\end{aligned}$$

Consider part (j), where we take
$\rho_T \in \mathcal{G}_{10}^{\mathrm{P}} = \left\{\rho_T = \exp\left\{-\frac{1}{q(T)}\right\} : q(T) \sim T^{\frac{1}{2}\left(1+\frac{1}{\kappa}\right)} \sim \sqrt{NT} \sim T \cap \rho_T \geq 1 - \frac{2\left(z_{\gamma_2}+z_{\alpha_2}\right)}{\sqrt{NT}} \text{ eventually}\right\}$.
Here, following the argument given in part (h) above, we have that $\Pr\left(\mathbb{T}_{1,T} > -z_{\gamma_1} | \rho = \rho_T \in \mathcal{G}_{10}^{\mathrm{P}}\right) \to 0$. Moreover, note that $\rho_T \in \mathbb{C}_{\gamma_2,\alpha_2,T}^{\mathrm{UR2}}$ eventually by assumption in this case, so that $\limsup_{T\to\infty} \Pr\left(\rho \notin \mathbb{C}_{\gamma_2,\alpha_2,T}^{\mathrm{UR2}} | \rho = \rho_T \in \mathcal{G}_{10}^{\mathrm{P}}\right) = 0$. It follows, by applying part (c) of Theorem 3.1, that

$$\begin{aligned}
& \limsup_{T\to\infty} \Pr\left(\rho \notin \mathbb{C}_{\gamma,\alpha,T} | \rho = \rho_T \in \mathcal{G}_{10}^{\mathrm{P}}\right) \\
\leq\ & \limsup_{T\to\infty} \Pr\left(\rho \notin \mathbb{C}_{\alpha_1,T}^{\mathbb{M}} | \rho = \rho_T \in \mathcal{G}_{10}^{\mathrm{P}}\right) + \limsup_{T\to\infty} \Pr\left(\mathbb{T}_{1,T} > -z_{\gamma_1} | \rho = \rho_T \in \mathcal{G}_{10}^{\mathrm{P}}\right) \\
& + \limsup_{T\to\infty} \Pr\left(\rho \notin \mathbb{C}_{\gamma_2,\alpha_2,T}^{\mathrm{UR2}} | \rho = \rho_T \in \mathcal{G}_{10}^{\mathrm{P}}\right) \\
=\ & \alpha_1 + 0 + 0 < \alpha.
\end{aligned}$$

Consider part (k), where we take

$\rho_T \in \mathcal{G}_{11}^{\mathrm{P}} = \left\{ \rho_T = \exp\left\{ -\frac{1}{q(T)} \right\} : \left(q(T) \sim \sqrt{NT} \sim T \right) \cap \left(\rho_T < 1 - \frac{2\left(z_{\gamma_2} + z_{\alpha_2}\right)}{\sqrt{NT}} \text{ eventually} \right) \right\}$. For this case, write $\mathbb{T}_{2,T} = \widehat{\omega}_{\mathrm{IVL}} \left(\widehat{\rho}_{\mathrm{IVL}} - 1 \right) = \widehat{\omega}_{\mathrm{IVL}} \left(\widehat{\rho}_{\mathrm{IVL}} - \rho_T \right) - \widehat{\omega}_{\mathrm{IVL}} \left(1 - \rho_T \right) = \mathbb{T}_{2,T}^* - \vartheta_T^*$. Applying part (c) of Lemma SD-9, Lemma SD-12, and the Cramér Convergence Theorem; we obtain

$$\begin{aligned}
\mathbb{T}_{2,T}^* &= \widehat{\omega}_{\mathrm{IVL}} \left(\widehat{\rho}_{\mathrm{IVL}} - \rho_T \right) \\
&= \left(\frac{1}{\widehat{\sigma}^2} \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=3}^{T} \Delta y_{it-1} y_{it-1} \right) \sqrt{NT} \left(\widehat{\rho}_{\mathrm{IVL}} - \rho_T \right) \\
&= \frac{1}{\widehat{\sigma}^2} \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=3}^{T} \Delta y_{it-1} \left[a_i \left(1 - \rho_T \right) + \varepsilon_{it} \right] \Rightarrow N(0,1),
\end{aligned}$$

while, applying part (c) of Lemma SD-8 and Lemma SD-12, we have

$$\begin{aligned}
\vartheta_T^* &= \widehat{\omega}_{\mathrm{IVL}} \left(1 - \rho_T \right) \\
&= \left(\frac{1}{\widehat{\sigma}^2} \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=3}^{T} \Delta y_{it-1} y_{it-1} \right) \sqrt{NT} \left(1 - \rho_T \right) \\
&= \left(1 - \frac{1}{4} \frac{q(T)}{T} \left[\exp\left\{ -\frac{2T}{q(T)} \right\} + \frac{2T}{q(T)} - 1 \right] \right) \sqrt{NT} \left(1 - \rho_T \right) + o_p(1).
\end{aligned}$$

These results imply that $\mathbb{T}_{2,T} = \mathbb{T}_{2,T}^* - \varphi\left(T/q(T) \right) \sqrt{NT} \left(1 - \rho_T \right) + \zeta_T$, where $\zeta_T = o_p(1)$ and where $\varphi\left(T/q(T) \right) = 1 - \frac{1}{4} \frac{q(T)}{T} \left[\exp\left\{ -\frac{2T}{q(T)} \right\} + \frac{2T}{q(T)} - 1 \right]$. Now, let ϵ_T be a sequence of positive numbers such that, as $T \to \infty$, $\epsilon_T \to 0$ but $\zeta_T / \epsilon_T = o_p(1)$. It follows by arguments similar to part (d) above that

$$\begin{aligned}
& \Pr\left(\mathbb{T}_{2,T} > -z_{\gamma_2} \middle| \left\{ q(T) \sim \sqrt{NT} \sim T \right\} \cap \left\{ \rho_T < 1 - \frac{2\left(z_{\gamma_2} + z_{\alpha_2}\right)}{\sqrt{NT}} \text{ eventually} \right\} \right) \\
\leq\ & \Pr\left(\mathbb{T}_{2,T}^* - \varphi\left(\frac{T}{q(T)} \right) \sqrt{NT} \left(1 - \rho_T \right) > -\left(z_{\gamma_2} + \epsilon_T \right) \middle| \left\{ q(T) \sim \sqrt{NT} \sim T \right\} \cap \right. \\
& \left. \left\{ \rho_T < 1 - \frac{2\left(z_{\gamma_2} + z_{\alpha_2}\right)}{\sqrt{NT}} \text{ eventually} \right\} \right) \\
& + \Pr\left(\left| \zeta_T \right| \geq \epsilon_T \middle| \left\{ q(T) \sim \sqrt{NT} \sim T \right\} \cap \left\{ \rho_T < 1 - \frac{2\left(z_{\gamma_2} + z_{\alpha_2}\right)}{\sqrt{NT}} \text{ eventually} \right\} \right) \\
\leq\ & \Pr\left(\mathbb{T}_{2,T}^* > \frac{\sqrt{NT}\left(1 - \rho_T\right)}{2} - \left(z_{\gamma_2} + \epsilon_T \right) \middle| q(T) \sim \sqrt{NT} \sim T \cap \rho_T < 1 - \frac{2\left(z_{\gamma_2} + z_{\alpha_2}\right)}{\sqrt{NT}} \text{ eventually} \right) \\
& + \Pr\left(\left| \zeta_T \right| \geq \epsilon_T \middle| \left\{ q(T) \sim \sqrt{NT} \sim T \right\} \cap \left\{ \rho_T < 1 - \frac{2\left(z_{\gamma_2} + z_{\alpha_2}\right)}{\sqrt{NT}} \text{ eventually} \right\} \right) \\
\leq\ & \Pr\left(\mathbb{T}_{2,T}^* > z_{\alpha_2} - \epsilon_T \middle| \left\{ q(T) \sim \sqrt{NT} \sim T \right\} \cap \left\{ \rho_T < 1 - \frac{2\left(z_{\gamma_2} + z_{\alpha_2}\right)}{\sqrt{NT}} \text{ eventually} \right\} \right) \\
& + \Pr\left(\left| \zeta_T \right| \geq \epsilon_T \middle| \left\{ q(T) \sim \sqrt{NT} \sim T \right\} \cap \left\{ \rho_T < 1 - \frac{2\left(z_{\gamma_2} + z_{\alpha_2}\right)}{\sqrt{NT}} \text{ eventually} \right\} \right) \\
\to\ & 1 - \Phi\left(z_{\alpha_2} \right) = \alpha_2,
\end{aligned}$$

where the second inequality above follows from the fact that, by Lemma SE-34, $\varphi\left(\frac{T}{q(T)}\right)\sqrt{NT}\left(1-\rho_T\right) \geq \frac{\sqrt{NT}(1-\rho_T)}{2}$ for $0 < \frac{T}{q(T)} < \infty$, while the last inequality above is due to the fact that $\rho_T < 1 - \frac{2\left(z_{\gamma_2}+z_{\alpha_2}\right)}{\sqrt{NT}} \iff \frac{\sqrt{NT}(1-\rho_T)}{2} - z_{\gamma_2} > z_{\alpha_2}$. Moreover, similar to the proof of part (h) above, we have $\lim_{T\to\infty}\Pr\left(\mathbb{T}_{1,T} > -z_{\gamma_1}|\rho=\rho_T\in\mathcal{G}_{11}^{\mathrm{P}}\right)=0$. Hence, applying part (c) of Theorem 3.1, we have

$$\begin{aligned}
& \limsup_{T\to\infty}\Pr\left(\rho\notin\mathbb{C}_{\gamma,\alpha,T}|\rho=\rho_T\in\mathcal{G}_{11}^{\mathrm{P}}\right) \\
\leq\ & \limsup_{T\to\infty}\Pr\left(\rho\notin\mathbb{C}_{\alpha_1,T}^{\mathbb{M}}|\rho=\rho_T\in\mathcal{G}_{11}^{\mathrm{P}}\right)+\limsup_{T\to\infty}\Pr\left(\mathbb{T}_{1,T} > -z_{\gamma_1}|\rho=\rho_T\in\mathcal{G}_{11}^{\mathrm{P}}\right) \\
& +\limsup_{T\to\infty}\Pr\left(\mathbb{T}_{2,T} > -z_{\gamma_2}|\rho=\rho_T\in\mathcal{G}_{11}^{\mathrm{P}}\right) \\
=\ & \alpha_1+0+\alpha_2=\alpha.
\end{aligned}$$

Consider part (l), where we take $\rho_T\in\mathcal{G}_{12}^{\mathrm{P}}=\left\{\rho_T=\exp\left\{-\frac{1}{q(T)}\right\}:\left(q\left(T\right)\sim\sqrt{NT}\ll T\right)\cap\left(\rho_T\geq 1-\frac{2\left(z_{\gamma_2}+z_{\alpha_3}\right)}{\sqrt{NT}}\text{ eventually}\right)\right\}$. For this case, note that, by arguments similar to those of part (i) above, we have $\Pr\left(\mathbb{T}_{1,T} > -z_{\gamma_1}|\rho=\rho_T\in\mathcal{G}_{12}^{\mathrm{P}}\right)\to 0$. $\rho_T\in\mathbb{C}_{\gamma_2,\alpha_2,T}^{\mathrm{UR2}}$ eventually by assumption in this case, so that $\lim_{T\to\infty}\Pr\left(\rho\notin\mathbb{C}_{\gamma_2,\alpha_2,T}^{\mathrm{UR2}}|\rho=\rho_T\in\mathcal{G}_{12}^{\mathrm{P}}\right)=0$. Using these results and part (d) of Theorem 3.1, we obtain

$$\begin{aligned}
& \limsup_{T\to\infty}\Pr\left(\rho\notin\mathbb{C}_{\gamma,\alpha,T}|\rho=\rho_T\in\mathcal{G}_{12}^{\mathrm{P}}\right) \\
\leq\ & \limsup_{T\to\infty}\Pr\left(\rho\notin\mathbb{C}_{\alpha_1,T}^{\mathbb{M}}|\rho=\rho_T\in\mathcal{G}_{12}^{\mathrm{P}}\right)+\limsup_{T\to\infty}\Pr\left(\mathbb{T}_{1,T} > -z_{\gamma_1}|\rho=\rho_T\in\mathcal{G}_{12}^{\mathrm{P}}\right) \\
& +\limsup_{T\to\infty}\Pr\left(\rho\notin\mathbb{C}_{\gamma_2,\alpha_2,T}^{\mathrm{UR2}}|\rho=\rho_T\in\mathcal{G}_{12}^{\mathrm{P}}\right) \\
=\ & \alpha_1+0+0<\alpha.
\end{aligned}$$

Consider part (m), where we take $\rho_T\in\mathcal{G}_{13}^{\mathrm{P}}=\left\{\rho_T=\exp\left\{-\frac{1}{q(T)}\right\}:q\left(T\right)\sim\sqrt{NT}\ll T\cap\rho_T<1-\frac{2\left(z_{\gamma_2}+z_{\alpha_2}\right)}{\sqrt{NT}}\text{ eventually}\right\}$. To proceed, again write $\mathbb{T}_{2,T}=\widehat{\omega}_{\mathrm{IVL}}\left(\widehat{\rho}_{\mathrm{IVL}}-\rho_T\right)-\widehat{\omega}_{\mathrm{IVL}}\left(1-\rho_T\right)=\mathbb{T}_{2,T}^{*}-\vartheta_T^{*}$. Here, note that $T^{\frac{1}{2}\left(\frac{1}{\kappa}-1\right)}\ll T^{\frac{1}{2}\left(\frac{1}{\kappa}+1\right)}\sim\sqrt{NT}\sim q\left(T\right)$ for $\kappa\in\left(\frac{1}{2},\infty\right)$, so that by part (d) of Lemma SD-9, Lemma SD-12, and the Cramér convergence theorem, we obtain $\mathbb{T}_{2,T}^{*}\Rightarrow N\left(0,1\right)$; and, by part (d) of Lemma SD-8, Lemma SD-12, and the Slutsky Theorem,

$$\vartheta_T^{*}=\widehat{\omega}_{\mathrm{IVL}}\left(1-\rho_T\right)=\left(\frac{1}{\widehat{\sigma}^2}\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=3}^{T}\Delta y_{it-1}y_{it-1}\right)\sqrt{NT}\left(1-\rho_T\right)=\frac{\sqrt{NT}\left(1-\rho_T\right)}{2}+o_p\left(1\right),$$

so that $\mathbb{T}_{2,T}=\mathbb{T}_{2,T}^{*}-\sqrt{NT}\left(1-\rho_T\right)/2+\zeta_{2,T}$, where $\zeta_{2,T}=o_p\left(1\right)$. Now, let ϵ_T be a sequence of positive numbers such that, as $T\to\infty$, $\epsilon_T\to 0$ but $\zeta_{2,T}/\epsilon_T=o_p\left(1\right)$. Similar to the proof of part (k) above, we have

$$\begin{aligned}
& \Pr\left(\mathbb{T}_{2,T} > -z_{\gamma_2}|\ \rho=\rho_T\in\mathcal{G}_{13}^{\mathrm{P}}\right) \\
\leq\ & \Pr\left(\mathbb{T}_{2,T}^{*} > z_{\alpha_2}-\epsilon_T|\rho=\rho_T\in\mathcal{G}_{13}^{\mathrm{P}}\right)+\Pr\left(|\zeta_T|\geq\epsilon_T|\rho=\rho_T\in\mathcal{G}_{13}^{\mathrm{P}}\right) \\
\to\ & 1-\Phi\left(z_{\alpha_2}\right)=\alpha_2.
\end{aligned}$$

Moreover, similar to the proof of part (i) above, we have $\lim_{T\to\infty}\Pr\left(\mathbb{T}_{1,T}>-z_{\gamma_1}|\rho=\rho_T\in\mathcal{G}_{13}^{\mathrm{P}}\right)=0$. Hence, applying part (d) of Theorem 3.1, we have

$$\begin{aligned}
&\limsup_{T\to\infty}\Pr\left(\rho\notin\mathbb{C}_{\gamma,\alpha,T}|\rho=\rho_T\in\mathcal{G}_{13}^{\mathrm{P}}\right)\\
\leq\ &\limsup_{T\to\infty}\Pr\left(\rho\notin\mathbb{C}_{\alpha_1,T}^{\mathbb{M}}|\rho=\rho_T\in\mathcal{G}_{13}^{\mathrm{P}}\right)+\limsup_{T\to\infty}\Pr\left(\mathbb{T}_{1,T}>-z_{\gamma_1}|\rho=\rho_T\in\mathcal{G}_{13}^{\mathrm{P}}\right)\\
&+\limsup_{T\to\infty}\Pr\left(\mathbb{T}_{2,T}\ >-z_{\gamma_2}|\rho=\rho_T\in\mathcal{G}_{13}^{\mathrm{P}}\right)\\
=\ &\alpha_1+0+\alpha_2=\alpha.
\end{aligned}$$

Consider part (n), where we take $\rho_T\in\mathcal{G}_{14}^{\mathrm{P}}=\left\{\rho_T=\exp\left\{-\frac{1}{q(T)}\right\}:T\ll q\left(T\right)\ll\sqrt{NT}\right\}$. By arguments similar to part (e) above, we have that $\Pr\left(\mathbb{T}_{1,T}>-z_{\gamma_1}|\rho=\rho_T\in\mathcal{G}_{14}^{\mathrm{P}}\right)\to 0$. Moreover, write $\mathbb{T}_{2,T}=\widehat{\omega}_{\mathrm{IVL}}\left(\widehat{\rho}_{\mathrm{IVL}}-1\right)=\widehat{\omega}_{\mathrm{IVL}}\left(\widehat{\rho}_{\mathrm{IVL}}-\rho_T\right)-\widehat{\omega}_{\mathrm{IVL}}\left(1-\rho_T\right)=\mathbb{T}_{2,T}^{*}-\vartheta_T^{*}$. In this case, by part (b) of Lemma SD-9, Lemma SD-12, and the Cramér convergence theorem, we obtain

$$\begin{aligned}
\mathbb{T}_{2,T}^{*}\ &=\ \widehat{\omega}_{\mathrm{IVL}}\left(\widehat{\rho}_{\mathrm{IVL}}-\rho_T\right)\\
&=\ \left(\frac{1}{\widehat{\sigma}^2}\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=3}^{T}\Delta y_{it-1}y_{it-1}\right)\sqrt{NT}\left(\widehat{\rho}_{\mathrm{IVL}}-\rho_T\right)\\
&=\ \frac{1}{\widehat{\sigma}^2}\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=3}^{T}\Delta y_{it-1}\left[a_i\left(1-\rho_T\right)+\varepsilon_{it}\right]\Rightarrow N\left(0,1\right),
\end{aligned}$$

In addition, by part (b) of Lemma SD-8 and Lemma SD-12,

$$\vartheta_T^{*}=\left(\frac{1}{\widehat{\sigma}^2}\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=3}^{T}\Delta y_{it-1}y_{it-1}\right)\sqrt{NT}\left(1-\rho_T\right)=\frac{\sqrt{NT}}{q\left(T\right)}\left[1+o_p\left(1\right)\right]\to\infty\ \text{ wpa.1.}$$

It follows that $\frac{q(T)}{\sqrt{NT}}\left(\mathbb{T}_{2,T}^{*}-\vartheta_T^{*}+z_{\gamma_2}\right)+1\xrightarrow{p}0$, from which we further deduce that

$$\begin{aligned}
&\Pr\left(\mathbb{T}_{2,T}>-z_{\gamma_2}|\rho=\rho_T\in\mathcal{G}_{14}^{\mathrm{P}}\right)\\
=\ &\Pr\left(\frac{q\left(T\right)}{\sqrt{NT}}\left(\mathbb{T}_{2,T}^{*}-\vartheta_T^{*}+z_{\gamma_2}\right)+1>1|\rho=\rho_T\in\mathcal{G}_{14}^{\mathrm{P}}\right)\\
\leq\ &\Pr\left(\left|\frac{q\left(T\right)}{\sqrt{NT}}\left(\mathbb{T}_{2,T}^{*}-\vartheta_T^{*}+z_{\gamma_2}\right)+1\right|>1|\rho=\rho_T\in\mathcal{G}_{14}^{\mathrm{P}}\right)\to 0
\end{aligned}$$

Using these results and part (b) of Theorem 3.1, we have

$$\begin{aligned}
&\limsup_{T\to\infty}\Pr\left(\rho\notin\mathbb{C}_{\gamma,\alpha,T}|\rho=\rho_T\in\mathcal{G}_{14}^{\mathrm{P}}\right)\\
\leq\ &\limsup_{T\to\infty}\Pr\left(\rho\notin\mathbb{C}_{\alpha_1,T}^{\mathbb{M}}|\rho=\rho_T\in\mathcal{G}_{14}^{\mathrm{P}}\right)+\limsup_{T\to\infty}\Pr\left(\mathbb{T}_{1,T}>-z_{\gamma_1}|\rho=\rho_T\in\mathcal{G}_{14}^{\mathrm{P}}\right)\\
&+\limsup_{T\to\infty}\Pr\left(\mathbb{T}_{2,T}\ >-z_{\gamma_2}|\rho=\rho_T\in\mathcal{G}_{14}^{\mathrm{P}}\right)\\
\leq\ &\alpha_1+0+0=\alpha_1.
\end{aligned}$$

Consider part (o), where we take $\rho_T\in\mathcal{G}_{15}^{\mathrm{P}}=\left\{\rho_T=\exp\left\{-\frac{1}{q(T)}\right\}:N^{1/3}T^{1/3}\ll q\left(T\right)\sim T\ll\sqrt{NT}\right\}$. By arguments similar to part (h) above, we have $\Pr\left(\mathbb{T}_{1,T}>-z_{\gamma_1}|\rho=\rho_T\in\mathcal{G}_{15}^{\mathrm{P}}\right)\to 0$. Moreover, write

$\mathbb{T}_{2,T} = \widehat{\omega}_{\mathrm{IVL}}\left(\widehat{\rho}_{\mathrm{IVL}} - 1\right) = \widehat{\omega}_{\mathrm{IVL}}\left(\widehat{\rho}_{\mathrm{IVL}} - \rho_T\right) - \widehat{\omega}_{\mathrm{IVL}}\left(1 - \rho_T\right) = \mathbb{T}^*_{2,T} - \vartheta^*_T$. In this case, $\mathbb{T}^*_{2,T} \Rightarrow N(0,1)$ by part (c) of Lemma SD-9, Lemma SD-12, and the Cramér convergence theorem. Moreover, by part (c) of Lemma SD-8 and Lemma SD-12,

$$\begin{aligned}
\vartheta^*_T &= \left(\frac{1}{\widehat{\sigma}^2}\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=3}^{T}\Delta y_{it-1}y_{it-1}\right)\sqrt{NT}\left(1-\rho_T\right) = \varphi\left(\frac{T}{q(T)}\right)\sqrt{NT}\left(1-\rho_T\right)\left[1+o_p(1)\right] \\
&= \varphi\left(\frac{T}{q(T)}\right)\frac{\sqrt{NT}}{q(T)}\left[1+o_p(1)\right] \to \infty \text{ wpa 1},
\end{aligned}$$

where $\varphi\left(T/q(T)\right) = 1 - \frac{1}{4}\frac{q(T)}{T}\left[\exp\left\{-\frac{2T}{q(T)}\right\} + \frac{2T}{q(T)} - 1\right] \geq \frac{1}{2}$ for all $T/q(T) > 0$. It follows that $\frac{q(T)}{\sqrt{NT}}\left(\mathbb{T}^*_{2,T} - \vartheta^*_T + z_{\gamma_2}\right) + \varphi\left(T/q(T)\right) \xrightarrow{p} 0$, from which we further deduce that

$$\begin{aligned}
& \Pr\left(\mathbb{T}_{2,T} > -z_{\gamma_2} | \rho = \rho_T \in \mathcal{G}^{\mathrm{P}}_{15}\right) \\
= \; & \Pr\left(\frac{q(T)}{\sqrt{NT}}\left(\mathbb{T}^*_{2,T} - \vartheta^*_T + z_{\gamma_2}\right) + \varphi\left(\frac{T}{q(T)}\right) > \varphi\left(\frac{T}{q(T)}\right) | \rho = \rho_T \in \mathcal{G}^{\mathrm{P}}_{15}\right) \\
\leq \; & \Pr\left(\left|\frac{q(T)}{\sqrt{NT}}\left(\mathbb{T}^*_{2,T} - \vartheta^*_T + z_{\gamma_2}\right) + \varphi\left(\frac{T}{q(T)}\right)\right| > \frac{1}{2} | \rho = \rho_T \in \mathcal{G}^{\mathrm{P}}_{15}\right) \to 0.
\end{aligned}$$

Using these results and part (c) of Theorem 3.1, we have

$$\begin{aligned}
& \limsup_{T\to\infty} \Pr\left(\rho \notin \mathbb{C}_{\gamma,\alpha,T} | \rho = \rho_T \in \mathcal{G}^{\mathrm{P}}_{15}\right) \\
\leq \; & \limsup_{T\to\infty} \Pr\left(\rho \notin \mathbb{C}^{\mathbb{M}}_{\alpha_1,T} | \rho = \rho_T \in \mathcal{G}^{\mathrm{P}}_{15}\right) + \limsup_{T\to\infty} \Pr\left(\mathbb{T}_{1,T} > -z_{\gamma_1} | \rho = \rho_T \in \mathcal{G}^{\mathrm{P}}_{15}\right) \\
& + \limsup_{T\to\infty} \Pr\left(\mathbb{T}_{2,T} > -z_{\gamma_2} | \rho = \rho_T \in \mathcal{G}^{\mathrm{P}}_{15}\right) \\
\leq \; & \alpha_1 + 0 + 0 = \alpha_1.
\end{aligned}$$

Consider part (p) where we take $\rho_T \in \mathcal{G}^{\mathrm{P}}_{16} = \left\{\rho_T = \exp\left\{-\frac{1}{q(T)}\right\} : \left(N^{1/3}T^{1/3} \sim T^{\frac{1+\kappa}{3\kappa}} \ll q(T) \ll \sqrt{NT}\right) \cap \left(q(T) \ll T\right)\right\}$. Here, again, by arguments similar to part (i), we have $\Pr\left(\mathbb{T}_{1,T} > -z_{\gamma_1} | \rho = \rho_T \in \mathcal{G}^{\mathrm{P}}_{16}\right) \to 0$. Moreover, note that, in this case, $T^{\frac{1}{2}\left(\frac{1}{\kappa}-1\right)} \ll T^{\frac{1+\kappa}{3\kappa}} \ll q(T)$ for $\kappa \in \left(\frac{1}{2}, \infty\right)$, so that by part (d) of Lemma SD-9, Lemma SD-12, and the Cramér convergence theorem, we obtain $\mathbb{T}^*_{2,T} \Rightarrow N(0,1)$. In addition, by part (d) of Lemma SD-8 and Lemma SD-12,

$$\begin{aligned}
\vartheta^*_T &= \left(\frac{1}{\widehat{\sigma}^2}\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=3}^{T}\Delta y_{it-1}y_{it-1}\right)\sqrt{NT}\left(1-\rho_T\right) \\
&= \frac{1}{\sigma^2}\frac{\sigma^2}{2}\frac{\sqrt{NT}}{q(T)}\left[1+o_p(1)\right] = \frac{1}{2}\frac{\sqrt{NT}}{q(T)}\left[1+o_p(1)\right] \to \infty \text{ wpa 1}.
\end{aligned}$$

It follows that $\frac{q(T)}{\sqrt{NT}}\left(\mathbb{T}^*_{2,T} - \vartheta^*_T + z_{\gamma_2}\right) \xrightarrow{p} -1/2 < 0$, from which we further deduce that

$$\Pr\left(\mathbb{T}_{2,T} > -z_{\gamma_2} | \rho = \rho_T \in \mathcal{G}^{\mathrm{P}}_{16}\right) \leq \Pr\left(\left|\frac{q(T)}{\sqrt{NT}}\left(\mathbb{T}^*_{2,T} - \vartheta^*_T + z_{\gamma_2}\right) + \frac{1}{2}\right| > \frac{1}{2} | \rho = \rho_T \in \mathcal{G}^{\mathrm{P}}_{16}\right) \to 0.$$

Hence, using these results and part (d) of Theorem 3.1, we have

$$\begin{aligned}
& \limsup_{T\to\infty} \Pr\left(\rho \notin \mathbb{C}_{\gamma,\alpha,T} | \rho = \rho_T \in \mathcal{G}_{16}^{\mathrm{P}}\right) \\
\leq\ & \limsup_{T\to\infty} \Pr\left(\rho \notin \mathbb{C}_{\alpha_1,T}^{\mathbb{M}} | \rho = \rho_T \in \mathcal{G}_{16}^{\mathrm{P}}\right) + \limsup_{T\to\infty} \Pr\left(\mathbb{T}_{1,T} > -z_{\gamma_1} | \rho = \rho_T \in \mathcal{G}_{16}^{\mathrm{P}}\right) \\
& + \limsup_{T\to\infty} \Pr\left(\mathbb{T}_{2,T} > -z_{\gamma_2} | \rho = \rho_T \in \mathcal{G}_{16}^{\mathrm{P}}\right) \\
=\ & \alpha_1 + 0 + 0 = \alpha_1 < \alpha.
\end{aligned}$$

Consider part (q) where we take $\rho_T \in \mathcal{G}_{17}^{\mathrm{P}} = \left\{\rho_T = \exp\left\{-\frac{1}{q(T)}\right\} : q(T) \sim T^{\frac{1+\kappa}{3\kappa}} \sim N^{1/3}T^{1/3}\right\}$. In this case, write $\mathbb{T}_{1,T} = \widehat{\mathbb{T}}_{1,T} - \widehat{\theta}_T$, where $\widehat{\mathbb{T}}_{1,T} = \widehat{\sigma}^{-1}\overline{M}_{yy}^{1/2}\sqrt{NTq(T)}\left(\widehat{\rho}_{\mathrm{pols}} - \rho_T - \frac{2\sigma_g^2}{q(T)^2\sigma^2}\right)$, $\widehat{\theta}_T = \widehat{\sigma}^{-1}\overline{M}_{yy}^{1/2}\sqrt{NTq(T)}\left(1 - \rho_T - \frac{2\sigma_g^2}{q(T)^2\sigma^2}\right)$, and $\overline{M}_{yy} = N^{-1}T^{-1}q(T)^{-1}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1} - \overline{y}_{-1,NT}\right)^2$. Applying part (e) of Theorem SA-2 along with part (d) of Lemma SD-3 and Lemma SD-12, we have $\widehat{\mathbb{T}}_{1,T} \Rightarrow N(0,1)$ and $\widehat{\theta}_T = \sqrt{\frac{NT}{2q(T)}}\left[1 + o_p(1)\right]$, so that $\sqrt{\frac{q(T)}{NT}}\left(\widehat{\mathbb{T}}_{1,T} - \widehat{\theta}_T + z_{\gamma_1}\right) \xrightarrow{p} -1/\sqrt{2} < 0$. Thus,

$$\Pr\left(\mathbb{T}_{1,T} > -z_{\gamma_1} | \rho = \rho_T \in \mathcal{G}_{17}^{\mathrm{P}}\right) \leq \Pr\left(\left|\sqrt{\frac{q(T)}{NT}}\left(\widehat{\mathbb{T}}_{1,T} - \widehat{\theta}_T + z_{\gamma_1}\right) + \frac{1}{\sqrt{2}}\right| > \frac{1}{\sqrt{2}} | \rho = \rho_T \in \mathcal{G}_{17}^{\mathrm{P}}\right) \to 0.$$

Moreover, note that, in this case, $T^{\frac{1}{2}\left(\frac{1}{\kappa}-1\right)} \ll T^{\frac{1+\kappa}{3\kappa}} \sim q(T)$ for $\kappa \in \left(\frac{1}{2}, \infty\right)$, so by arguments similar to part (o), we have $\Pr\left(\mathbb{T}_{2,T} > -z_{\gamma_2} | \rho = \rho_T \in \mathcal{G}_{17}^{\mathrm{P}}\right) \to 0$. Hence, using these results and part (d) of Theorem 3.1, we have

$$\begin{aligned}
& \limsup_{T\to\infty} \Pr\left(\rho \notin \mathbb{C}_{\gamma,\alpha,T} | \rho = \rho_T \in \mathcal{G}_{17}^{\mathrm{P}}\right) \\
\leq\ & \limsup_{T\to\infty} \Pr\left(\rho \notin \mathbb{C}_{\alpha_1,T}^{\mathbb{M}} | \rho = \rho_T \in \mathcal{G}_{17}^{\mathrm{P}}\right) + \limsup_{T\to\infty} \Pr\left(\mathbb{T}_{1,T} > -z_{\gamma_1} | \rho = \rho_T \in \mathcal{G}_{17}^{\mathrm{P}}\right) \\
& + \limsup_{T\to\infty} \Pr\left(\mathbb{T}_{2,T} > -z_{\gamma_2} | \rho = \rho_T \in \mathcal{G}_{17}^{\mathrm{P}}\right) \\
=\ & \alpha_1 + 0 + 0 = \alpha_1 < \alpha.
\end{aligned}$$

Consider part (r) where we take
$\rho_T \in \mathcal{G}_{18}^{\mathrm{P}} = \left\{\rho_T = \exp\left\{-\frac{1}{q(T)}\right\} : q(T) \to \infty \text{ such that } q(T)/T^{\frac{1+\kappa}{3\kappa}} \to 0\right\}$. In this case, again represent the unit root statistic $\mathbb{T}_{1,T}$ as $\mathbb{T}_{1,T} = \widehat{\mathbb{T}}_{1,T} - \widehat{\theta}_T$, where we take $\widehat{\mathbb{T}}_{1,T}$
$= \overline{M}_{yy}^{1/2}\sqrt{NTq(T)}\left(\widehat{\rho}_{\mathrm{pols}} - \rho_T - \frac{2\sigma_g^2}{q(T)^2\sigma^2}\right)/\widehat{\sigma} = \widehat{\sigma}^{-1}\overline{M}_{yy}^{1/2}\sqrt{\frac{NT}{q(T)^3}}q(T)^2\left(\widehat{\rho}_{\mathrm{pols}} - \rho_T - \frac{2\sigma_g^2}{q(T)^2\sigma^2}\right)$ and where $\widehat{\theta}$ and $\overline{M}_{yy}$ are as defined previously. Now, applying part (f) of Theorem SA-2 along with part (d) of Lemma SD-3 and Lemma SD-12, we have $\widehat{\mathbb{T}}_{1,T} = o_p\left(\frac{\sqrt{NT}}{q(T)^{3/2}}\right) = o_p(1)$ and $\widehat{\theta}_T = \sqrt{\frac{NT}{2q(T)}}\left[1 + o_p(1)\right]$, so that $\sqrt{\frac{q(T)}{NT}}\left(\widehat{\mathbb{T}}_{1,T} - \widehat{\theta}_T + z_{\gamma_1}\right) \xrightarrow{p} -1/\sqrt{2} < 0$. Thus,

$$\Pr\left(\mathbb{T}_{1,T} > -z_{\gamma_1} | \rho = \rho_T \in \mathcal{G}_{18}^{\mathrm{P}}\right) \leq \Pr\left(\left|\sqrt{\frac{q(T)}{NT}}\left(\widehat{\mathbb{T}}_{1,T} - \widehat{\theta}_T + z_{\gamma_1}\right) + \frac{1}{\sqrt{2}}\right| > \frac{1}{\sqrt{2}} | \rho = \rho_T \in \mathcal{G}_{18}^{\mathrm{P}}\right) \to 0.$$

Now, write $\mathbb{T}_{2,T} = \widehat{\omega}_{\mathrm{IVL}}\left(\widehat{\rho}_{\mathrm{IVL}} - 1\right) = \widehat{\omega}_{\mathrm{IVL}}\left(\widehat{\rho}_{\mathrm{IVL}} - \rho_T\right) - \widehat{\omega}_{\mathrm{IVL}}\left(1 - \rho_T\right) = \mathbb{T}_{2,T}^* - \vartheta_T^*$. Consider first the case where $T^{\frac{1}{2}\left(\frac{1}{\kappa}-1\right)} \ll q(T) \ll T^{\frac{1+\kappa}{3\kappa}}$. In this case, part (d) of Lemma SD-9, Lemma SD-12, and

the Cramér convergence theorem imply that

$$\frac{q(T)}{\sqrt{NT}}\mathbb{T}_{2,T}^{*}=\frac{q(T)}{\sqrt{NT}}\frac{1}{\widehat{\sigma}^{2}}\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=3}^{T}\Delta y_{it-1}\left[a_{i}\left(1-\rho_{T}\right)+\varepsilon_{it}\right]=O_{p}\left(\frac{q(T)}{\sqrt{NT}}\right).$$

Next, consider the case where $q(T)\rightarrow\infty$ such that $q(T)/\,T^{\frac{1}{2}\left(\frac{1}{\kappa}-1\right)}=O(1)$. In this case, we apply part (e) of Lemma SD-9 and Lemma SD-12 to obtain

$$\begin{aligned}\frac{q(T)}{\sqrt{NT}}\mathbb{T}_{2,T}^{*} &= \frac{q(T)}{\sqrt{NT}}\frac{1}{\widehat{\sigma}^{2}}\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=3}^{T}\Delta y_{it-1}\left[a_{i}\left(1-\rho_{T}\right)+\varepsilon_{it}\right]\\ &= O\left(\frac{q(T)}{\sqrt{NT}}\right)O_{p}\left(\max\left\{1,\frac{T^{\frac{1}{2}\left(\frac{1}{\kappa}-1\right)}}{q(T)}\right\}\right)=O_{p}\left(\max\left\{\frac{q(T)}{\sqrt{NT}},\frac{1}{T}\right\}\right).\end{aligned}$$

It follows that in either of the two cases above, we have that $\left(q(T)/\sqrt{NT}\right)\mathbb{T}_{2,T}^{*}=o_{p}(1)$. Moreover, note that applying part (d) of Lemma SD-8 and Lemma SD-12, we obtain

$$\begin{aligned}\frac{q(T)}{\sqrt{NT}}\vartheta_{T}^{*} &= \frac{q(T)}{\sqrt{NT}}\left(\frac{1}{\widehat{\sigma}^{2}}\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=3}^{T}\Delta y_{it-1}y_{it-1}\right)\sqrt{NT}\left(1-\rho_{T}\right)\\ &= \frac{1}{\sigma^{2}}\frac{\sigma^{2}}{2}\left[1+o_{p}(1)\right]=\frac{1}{2}\left[1+o_{p}(1)\right].\end{aligned}$$

Together, these results imply that $\left(q(T)/\sqrt{NT}\right)\left(\mathbb{T}_{2,T}^{*}-\vartheta_{T}^{*}+z_{\gamma_{2}}\right)\xrightarrow{p}-1/2$, from which we further deduce that

$$\Pr\left(\mathbb{T}_{2,T}>-z_{\gamma_{2}}|\rho=\rho_{T}\in\mathcal{G}_{18}^{\mathrm{P}}\right)\leq\Pr\left(\left|\frac{q(T)}{\sqrt{NT}}\left(\mathbb{T}_{2,T}^{*}-\vartheta_{T}^{*}+z_{\gamma_{2}}\right)+\frac{1}{2}\right|>\frac{1}{2}|\rho=\rho_{T}\in\mathcal{G}_{18}^{\mathrm{P}}\right)\rightarrow 0.$$

Hence, using these results and part (d) of Theorem 3.1, we have

$$\begin{aligned}&\limsup_{T\rightarrow\infty}\Pr\left(\rho\notin\mathbb{C}_{\gamma,\alpha,T}|\rho=\rho_{T}\in\mathcal{G}_{18}^{\mathrm{P}}\right)\\ \leq\;&\limsup_{T\rightarrow\infty}\Pr\left(\rho\notin\mathbb{C}_{\alpha_{1},T}^{\mathbb{M}}|\rho=\rho_{T}\in\mathcal{G}_{18}^{\mathrm{P}}\right)+\limsup_{T\rightarrow\infty}\Pr\left(\mathbb{T}_{1,T}>-z_{\gamma_{1}}|\rho=\rho_{T}\in\mathcal{G}_{18}^{\mathrm{P}}\right)\\ &+\limsup_{T\rightarrow\infty}\Pr\left(\mathbb{T}_{2,T}>-z_{\gamma_{2}}|\rho=\rho_{T}\in\mathcal{G}_{18}^{\mathrm{P}}\right)\\ =\;&\alpha_{1}+0+0=\alpha_{1}<\alpha.\end{aligned}$$

Finally, we consider part (s), where

$$\rho_{T}\in\mathcal{G}_{19}^{\mathrm{P}}=\left\{|\rho_{T}|=\exp\left\{-\frac{1}{q(T)}\right\}:q(T)\geq 0\text{ and }q(T)=O(1)\text{ as }T\rightarrow\infty\right\}.$$

In this case, we decompose $\mathbb{T}$ as $\mathbb{T}=\widetilde{\mathbb{T}}-\widetilde{\theta}$, where $\widetilde{M}_{yy}=\frac{1-\rho_{T}^{2}}{NT}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1}-\overline{y}_{-1,NT}\right)^{2}$, $\widetilde{\mathbb{T}}_{1,T}=\widehat{\sigma}^{-1}\widetilde{M}_{yy}^{1/2}\sqrt{\frac{NT}{1-\rho_{T}^{2}}}\left(\widehat{\rho}_{\mathrm{pols}}-\rho_{T}\right)$, and $\widetilde{\theta}_{T}=\widehat{\sigma}^{-1}\widetilde{M}_{yy}^{1/2}\sqrt{\frac{NT}{1-\rho_{T}^{2}}}\left(1-\rho_{T}\right)$. Using part (g) of Theorem SA-2

along with part (e) of Lemma SD-3 and Lemma SD-12, we have in this case

$$\begin{aligned}
\widetilde{M}_{yy} &= \left(1-\rho_T^2\right)\sigma_a^2+\sigma^2+o_p\left(1\right), \\
\frac{\widetilde{\mathbb{T}}_{1,T}}{\sqrt{NT}} &= \widehat{\sigma}^{-1}\widetilde{M}_{yy}^{1/2}\frac{\left(\widehat{\rho}_{\text{pols}}-\rho_T\right)}{\sqrt{1-\rho_T^2}}=\sigma^{-1}\sqrt{\frac{\left(1-\rho_T^2\right)\sigma_a^2+\sigma^2}{1-\rho_T^2}}\frac{\left(1-\rho_T\right)\sigma_a^2}{\sigma_a^2+\sigma^2/\left(1-\rho_T^2\right)}\left[1+o_p\left(1\right)\right] \\
&= \frac{\left(1-\rho_T\right)\sigma_a^2}{\sigma\sqrt{\sigma_a^2+\sigma^2/\left(1-\rho_T^2\right)}}\left[1+o_p\left(1\right)\right] \\
\frac{\widetilde{\theta}_T}{\sqrt{NT}} &= \widehat{\sigma}^{-1}\widetilde{M}_{yy}^{1/2}\frac{\left(1-\rho_T\right)}{\sqrt{1-\rho_T^2}}=\sigma^{-1}\sqrt{\sigma_a^2+\frac{\sigma^2}{1-\rho_T^2}}\left(1-\rho_T\right)\left[1+o_p\left(1\right)\right].
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{\mathbb{T}_{1,T}}{\sqrt{NT}} &= \frac{\widetilde{\mathbb{T}}_{1,T}}{\sqrt{NT}}-\frac{\widetilde{\theta}_T}{\sqrt{NT}} \\
&= \frac{1}{\sigma}\left[\sigma_a^2+\frac{\sigma^2}{1-\rho_T^2}\right]^{-1/2}\left[\left(1-\rho_T\right)\sigma_a^2-\left(\sigma_a^2+\frac{\sigma^2}{1-\rho_T^2}\right)\left(1-\rho_T\right)\right]\left[1+o_p\left(1\right)\right] \\
&= -\frac{1}{\sigma}\left[\sigma_a^2+\frac{\sigma^2}{1-\rho_T^2}\right]^{-1/2}\frac{\sigma^2\left(1-\rho_T\right)}{1-\rho_T^2}\left[1+o_p\left(1\right)\right] \\
&= -\frac{\sigma\sqrt{1-\rho_T^2}}{\left(1+\rho_T\right)\sqrt{\left(1-\rho_T^2\right)\sigma_a^2+\sigma^2}}\left[1+o_p\left(1\right)\right] \\
&= -\psi\left(\sigma^2,\sigma_a^2,\rho_T\right)\left[1+o_p\left(1\right)\right]<0 \quad \text{wpa 1.}
\end{aligned}$$

where $\psi\left(\sigma^2,\sigma_a^2,\rho_T\right)=\sigma\sqrt{1-\rho_T^2}\left[\left(1+\rho_T\right)\sqrt{\left(1-\rho_T^2\right)\sigma_a^2+\sigma^2}\right]^{-1}$, so that $\frac{1}{\sqrt{NT}}\left(\widetilde{\mathbb{T}}_{1,T}-\widetilde{\theta}_T+z_{\gamma_1}\right)+\psi\left(\sigma^2,\sigma_a^2,\rho_T\right)\xrightarrow{p}0$. Moreover, note that, by assumption, $q\left(T\right)=O\left(1\right)$, so that for some positive constant C_q, there exists some positive integer T^* such that for all $T\geq T^*$, $q\left(T\right)\leq C_q$, from which it follows that $0\leq\rho_T^2=\exp\left\{-\frac{2}{q(T)}\right\}\leq\exp\left\{-\frac{2}{C_q}\right\}<1$. It follows that

$$\psi\left(\sigma^2,\sigma_a^2,\rho_T\right)=\frac{\sigma\sqrt{1-\rho_T^2}}{\left(1+\rho_T\right)\sqrt{\left(1-\rho_T^2\right)\sigma_a^2+\sigma^2}}\geq\frac{\sigma\sqrt{1-\exp\left\{-\frac{2}{C_q}\right\}}}{2\sqrt{\sigma_a^2+\sigma^2}}=\underline{\psi}\left(\sigma^2,\sigma_a^2,C_q\right)>0.$$

Thus,

$$\begin{aligned}
& \Pr\left(\mathbb{T}_{1,T}>-z_{\gamma_1}|\rho=\rho_T\in\mathcal{G}_{19}^{\text{P}}\right) \\
= & \Pr\left(\frac{1}{\sqrt{NT}}\left(\widetilde{\mathbb{T}}_{1,T}-\widetilde{\theta}_T+z_{\gamma_1}\right)>0|\rho=\rho_T\in\mathcal{G}_{19}^{\text{P}}\right) \\
\leq & \Pr\left(\left|\frac{1}{\sqrt{NT}}\left(\widetilde{\mathbb{T}}_{1,T}-\widetilde{\theta}_T+z_{\gamma_1}\right)+\psi\left(\sigma^2,\sigma_a^2,\rho_T\right)\right|>\underline{\psi}\left(\sigma^2,\sigma_a^2,C_q\right)|\rho=\rho_T\in\mathcal{G}_{19}^{\text{P}}\right)\to 0.
\end{aligned}$$

Next, write $\mathbb{T}_{2,T} = \widehat{\omega}_{\mathrm{IVL}} \left(\widehat{\rho}_{\mathrm{IVL}} - 1\right) = \widehat{\omega}_{\mathrm{IVL}} \left(\widehat{\rho}_{\mathrm{IVL}} - \rho_T\right) - \widehat{\omega}_{\mathrm{IVL}} \left(1 - \rho_T\right) = \mathbb{T}^*_{2,T} - \vartheta^*_T$. Applying part (f) of Lemma SD-9 and Lemma SD-12, we have

$$\begin{aligned}
\frac{\mathbb{T}^*_{2,T}}{\sqrt{NT}} &= \frac{1}{\sqrt{NT}} \left(\frac{1}{\widehat{\sigma}^2} \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=3}^{T} \Delta y_{it-1} y_{it-1} \right) \sqrt{NT} \left(\widehat{\rho}_{\mathrm{IVL}} - \rho_T\right) \\
&= \frac{1}{\sqrt{NT}\sigma^2} \left(\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=3}^{T} \Delta y_{it-1} \left[a_i \left(1 - \rho_T\right) + \varepsilon_{it}\right] \right) \left[1 + o_p\left(1\right)\right] \\
&= O_p \left(\max \left\{ \frac{1}{\sqrt{NT}}, \frac{1}{T} \right\} \right) = o_p\left(1\right),
\end{aligned}$$

and, by part (e) of Lemma SD-8 and Lemma SD-12,

$$\begin{aligned}
\vartheta^*_T &= \widehat{\omega}_{\mathrm{IVL}} \left(1 - \rho_T\right) = \left(\frac{1}{\widehat{\sigma}^2} \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=3}^{T} \Delta y_{it-1} y_{it-1} \right) \sqrt{NT} \left(1 - \rho_T\right) \\
&= \frac{1}{\sigma^2} \frac{\sigma^2}{1 + \rho_T} \sqrt{NT} \left(1 - \rho_T\right) \left[1 + o_p\left(1\right)\right] = \sqrt{NT} \frac{1 - \rho_T}{1 + \rho_T} \left[1 + o_p\left(1\right)\right] \to \infty \text{ wpa 1.}
\end{aligned}$$

It follows that $N^{-1/2} T^{-1/2} \left(\mathbb{T}^*_{2,T} - \vartheta^*_T + z_{\gamma_2} \right) \xrightarrow{p} -\left(1 - \rho_T\right) / \left(1 + \rho_T\right) < 0$, from which we further deduce that

$$\Pr\left(\mathbb{T}_{2,T} > -z_{\gamma_2} | \rho = \rho_T \in \mathcal{G}^{\mathrm{P}}_{19}\right) \leq \Pr\left(\left| \frac{1}{\sqrt{NT}} \left(\mathbb{T}^*_{2,T} - \vartheta^*_T + z_{\gamma_2} \right) + \frac{1 - \rho_T}{1 + \rho_T} \right| > \frac{1 - \rho_T}{1 + \rho_T} | \rho = \rho_T \in \mathcal{G}^{\mathrm{P}}_{19} \right) \to 0.$$

Using this result and part (e) of Theorem 3.1, we obtain

$$\begin{aligned}
& \limsup_{T \to \infty} \Pr\left(\rho \notin \mathbb{C}_{\gamma,\alpha,T} | \rho = \rho_T \in \mathcal{G}^{\mathrm{P}}_{19}\right) \\
\leq\ & \limsup_{T \to \infty} \Pr\left(\rho \notin \mathbb{C}^{\mathbb{M}}_{\alpha_1,T} | \rho = \rho_T \in \mathcal{G}^{\mathrm{P}}_{19}\right) + \limsup_{T \to \infty} \Pr\left(\mathbb{T}_{1,T} > -z_{\gamma} | \rho = \rho_T \in \mathcal{G}^{\mathrm{P}}_{19}\right) \\
& + \limsup_{T \to \infty} \Pr\left(\mathbb{T}_{2,T} > -z_{\gamma_2} | \rho = \rho_T \in \mathcal{G}^{\mathrm{P}}_{19}\right) \\
=\ & \alpha_1 + 0 + 0 = \alpha_1 < \alpha. \ \square
\end{aligned}$$

Appendix SC: Asymptotic Properties of the Anderson-Rubin Statistic

An alternative way to formulate the estimation of ρ in a panel autoregression with individual effects is as an IV regression problem in the sense of Anderson and Hsiao (1981, 1982). The Anderson-Hsiao approach begins by first-differencing the panel $AR\left(1\right)$ model, given by equation (3) in the main paper, to obtain

$$\Delta y_{it} = \rho \Delta y_{it-1} + \Delta \varepsilon_{it}. \tag{6}$$

The autoregressive parameter ρ is then estimated from (6) using y_{it-2} as the instrument based on the implicit first-stage equation

$$\Delta y_{it-1} = a_i \left(1 - \rho_T\right) + \left(\rho_T - 1\right) y_{it-2} + \varepsilon_{it-1}. \tag{7}$$

It is well known that a weak instrument problem occurs for the Anderson-Hsiao IV estimator when ρ_T is near unity, as can be seen from equation (7) above. Hence, one may think that the robust confidence procedures which have been developed and used successfully in the weak instrument literature can also be applied in a straightforward manner here to yield asymptotically valid interval estimates in the panel data setting. That turns out not to be the case. In particular, a well-known confidence procedure which is shown to be robust to the effects of weak instruments in the IV regression setup is obtained by inverting the Anderson-Rubin test statistic as shown in Staiger and Stock (1997). This test is also known to have good properties in the just identified case. However, direct application of the Anderson-Rubin procedure based on the IV regression setup given by equations (6) and (7) above does not lead to a confidence interval for ρ that is asymptotically valid. More specifically, the Anderson-Rubin statistic in this case has the form

$$A_{NT}(\rho) = \frac{\left(\sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-2}\Delta\varepsilon_{it}\right)^2 \left(\sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-2}^2\right)^{-1}}{\frac{1}{NT-1}\left[\sum_{i=1}^{N}\sum_{t=2}^{T}(\Delta\varepsilon_{it})^2 - \left(\sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-2}\Delta\varepsilon_{it}\right)^2 \left(\sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-2}^2\right)^{-1}\right]}$$

which depends on ρ via the term $\Delta\varepsilon_{it} = \Delta y_{it} - \rho\Delta y_{it-1}$.

The following result gives the asymptotic behavior of $A_{NT}(\rho)$ under the null hypothesis $H_0 : \rho = \rho_T$ for alternative sequences $\{\rho_T\}$.

Theorem SC-1:

Suppose that Assumptions 1-4 hold. Then, the following statements are true as $N, T \to \infty$ such that $N^\kappa/T = \tau$, for $\kappa \in \left(\frac{1}{2}, \infty\right)$ and $\tau \in (0, \infty)$.

(a) Suppose that $\rho_T = 1$ for all T sufficiently large. Then,

$$TA_{NT}(\rho_T) \Rightarrow 2\chi_1^2$$

(b) Suppose that $\rho_T = \exp\{-1/q(T)\}$ such that $T/q(T) \to 0$. Then,

$$TA_{NT}(\rho_T) \Rightarrow 2\chi_1^2$$

(c) Suppose that $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \sim T$. Then,

$$TA_{NT}(\rho_T) = \Phi(T)\chi_1^2 + o_p(1),$$

or

$$\frac{T}{\Phi(T)}A_{NT}(\rho_T) \Rightarrow \chi_1^2,$$

where

$$\Phi(T) = \frac{T}{q(T)}\left[\frac{2T}{q(T)} + 1 - \exp\left\{-\frac{2T}{q(T)}\right\}\right]\left[\exp\left\{-\frac{2T}{q(T)}\right\} + \frac{2T}{q(T)} - 1\right]^{-1}.$$

(d) Suppose that $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \to \infty$ but $q(T)/T \to 0$,

$$q(T)A_{NT}(\rho_T) \Rightarrow \chi_1^2$$

(e) Suppose that $\rho_T \in \mathcal{G}_{\text{St}} = \left\{ |\rho_T| = \exp\left\{-\frac{1}{q(T)}\right\} : q(T) \geq 0 \text{ and } q(T) = O(1) \text{ as } T \to \infty \right\}$. Then,

$$\frac{1 + \left(1 - \rho_T^2\right)\left(\sigma_a^2 + \mu_a^2\right)/\sigma^2}{1 - \rho_T} A_{NT}(\rho_T) \Rightarrow \chi_1^2.$$

This result shows that $A_{NT}(\rho_T)$ is not uniformly convergent. In particular, note that $A_{NT}(\rho_T) = o_p(1)$ for all unit root and local-to-unity parameter sequences. On the other hand, for parameter sequences associated with a stable panel autoregressive process, $A_{NT}(\rho_T)$ does not converge in probability to zero and, in fact when appropriately rescaled, converges in distribution to a chi-squared distribution. Hence, the confidence interval obtained by inverting this statistic will not provide the correct asymptotic coverage uniformly over the parameter space $\rho \in (-1, 1]$.

Proof of Theorem SC-1:

To proceed, note first that

$$\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=2}^{T} E\left[(\Delta\varepsilon_{it})^2\right] = \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=2}^{T} E\left[\varepsilon_{it}^2 - 2\varepsilon_{it}\varepsilon_{it-1} + \varepsilon_{it-1}^2\right] = 2\sigma^2\left(\frac{T-1}{T}\right) = 2\sigma^2\left[1 + O\left(\frac{1}{T}\right)\right].$$

Moreover,

$$\begin{aligned}
& E\left(\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=2}^{T}\left[(\Delta\varepsilon_{it})^2 - 2\sigma^2\left(\frac{T-1}{T}\right)\right]\right)^2 \\
= & \frac{1}{N^2T^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=2}^{T}\sum_{s=2}^{T} E\left(\left[(\Delta\varepsilon_{it})^2 - 2\sigma^2\left(\frac{T-1}{T}\right)\right]\left[(\Delta\varepsilon_{js})^2 - 2\sigma^2\left(\frac{T-1}{T}\right)\right]\right) \\
= & \frac{1}{N^2T^2}\sum_{i=1}^{N}\sum_{t=2}^{T}\left\{E(\Delta\varepsilon_{it})^4 - 4\sigma^4\left(\frac{T-1}{T}\right)^2\right\} \\
= & \frac{1}{N^2T^2}\sum_{i=1}^{N}\sum_{t=2}^{T}\left\{E\left[\varepsilon_{it}^2 - 2\varepsilon_{it}\varepsilon_{it-1} + \varepsilon_{it-1}^2\right]^2 - 4\sigma^4\left(\frac{T-1}{T}\right)^2\right\} \\
= & \frac{1}{N^2T^2}\sum_{i=1}^{N}\sum_{t=2}^{T}\left\{E\left[\varepsilon_{it}^4 + 6\varepsilon_{it}^2\varepsilon_{it-1}^2 + \varepsilon_{it-1}^4 - 4\varepsilon_{it}^3\varepsilon_{it-1} - 4\varepsilon_{it}\varepsilon_{it-1}^3\right] - 4\sigma^4\left(\frac{T-1}{T}\right)^2\right\} \\
= & \frac{1}{N^2T^2}\sum_{i=1}^{N}\sum_{t=2}^{T}\left[2E\left[\varepsilon_{it}^4\right] + 6\sigma^4 - 4\sigma^4\left(\frac{T-1}{T}\right)^2\right] \\
= & \frac{1}{N^2T^2}\sum_{i=1}^{N}\sum_{t=2}^{T}\left[2E\left[\varepsilon_{it}^4\right] + 2\sigma^4 + \frac{8\sigma^4}{T} - \frac{4\sigma^4}{T^2}\right] \\
= & 2\frac{(T-1)}{NT^2}\left[\left(E\left[\varepsilon_{it}^4\right] + \sigma^4\right) + \frac{4\sigma^4}{T} - \frac{2\sigma^4}{T^2}\right] = O\left(\frac{1}{NT}\right)
\end{aligned}$$

It follows by Markov's inequality that

$$\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=2}^{T}(\Delta\varepsilon_{it})^2 = 2\sigma^2 + O\left(\frac{1}{T}\right) + O_p\left(\frac{1}{\sqrt{NT}}\right)$$

Consider first part (a), where we take $\rho_T = 1$ for all T sufficiently large. In this case, we have by part (a) of Lemma SD-2, Lemma SE-24, and the Cramér Covergence Theorem, that

$$\begin{aligned}
& \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2} \Delta \varepsilon_{it} \\
= & -\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} \varepsilon_{it-2} \varepsilon_{it-1} + \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} w_{iT-2} \varepsilon_{iT} + O_p\left(\frac{1}{\sqrt{T}}\right) \\
\Rightarrow & N\left(0, 2\sigma^4\right)
\end{aligned}$$

and, by part (a) of Lemma SE-11, and Lemma SE-14, and part (a) of Lemma SE-18 that

$$\begin{aligned}
\frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2}^2 &= \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} \left(a_i + w_{it-2}\right)^2 \\
&= \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-2}^2 + \frac{2}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} a_i w_{it-2} + \frac{T-1}{NT^2} \sum_{i=1}^{N} a_i^2 \\
&= \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-2}^2 + O_p\left(\frac{1}{\sqrt{NT}}\right) + O_p\left(\frac{1}{T}\right) \xrightarrow{p} \frac{\sigma^2}{2}
\end{aligned}$$

so that

$$\begin{aligned}
& TA_{NT}\left(\rho_T\right) \\
= & \frac{T \frac{1}{T} \left(\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2} \Delta \varepsilon_{it}\right)^2 \left(\frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2}^2\right)^{-1}}{\left[\left(\sum_{i=1}^{N} \sum_{t=2}^{T} \left(\Delta \varepsilon_{it}\right)^2\right) - \left(\sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2} \Delta \varepsilon_{it}\right)^2 \left(\frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2}^2\right)^{-1} \frac{1}{NT^2}\right] / \left(NT-1\right)} \\
= & \frac{\left(\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2} \Delta \varepsilon_{it}\right)^2 \left(\frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2}^2\right)^{-1}}{\left[\left(\frac{1}{NT-1} \sum_{i=1}^{N} \sum_{t=2}^{T} \left(\Delta \varepsilon_{it}\right)^2\right) - \left(\frac{1}{\sqrt{NT-1}} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2} \Delta \varepsilon_{it}\right)^2 \left(\frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2}^2\right)^{-1} \frac{1}{NT^2}\right]} \\
\Rightarrow & 2\sigma^4 \chi_1^2 \left(\frac{\sigma^2}{2}\right)^{-1} \frac{1}{2\sigma^2} \equiv 2\chi_1^2.
\end{aligned}$$

Next, consider part (b), where we take $\rho_T = \exp\left\{-1/q\left(T\right)\right\}$ such that $T/q\left(T\right) \to 0$. In this case, we have, by part (b) of Lemma SD-2, part (a) of Lemma SE-22, and the Cramér convergence theorem, that

$$\begin{aligned}
\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} y_{it-2} \Delta \varepsilon_{it} &= -\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} \varepsilon_{it-2} \varepsilon_{it-1} + \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} w_{iT-2} \varepsilon_{iT} + O_p\left(\frac{1}{\sqrt{T}}\right) \\
&\Rightarrow N\left(0, 2\sigma^4\right),
\end{aligned}$$

and, by part (a) of Lemma SE-11, Lemma SE-15, and part (b) of Lemma SE-18 that

$$\begin{aligned}
\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-2}^2 &= \frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(a_i + w_{it-2}\right)^2 \\
&= \frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-2}^2 + \frac{2}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T} a_i w_{it-2} + \frac{T-1}{NT^2}\sum_{i=1}^{N} a_i^2 \\
&= \frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-2}^2 + O_p\left(\frac{1}{\sqrt{NT}}\right) + O_p\left(\frac{1}{T}\right) \xrightarrow{p} \frac{\sigma^2}{2},
\end{aligned}$$

so that

$$\begin{aligned}
& TA_{NT}\left(\rho_T\right) \\
= & \frac{T\frac{1}{T}\left(\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-2}\Delta\varepsilon_{it}\right)^2\left(\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-2}^2\right)^{-1}}{\left[\left(\sum_{i=1}^{N}\sum_{t=2}^{T}\left(\Delta\varepsilon_{it}\right)^2\right) - \left(\sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-2}\Delta\varepsilon_{it}\right)^2\left(\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-2}^2\right)^{-1}\frac{1}{NT^2}\right]/\left(NT-1\right)} \\
= & \frac{\left(\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-2}\Delta\varepsilon_{it}\right)^2\left(\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-2}^2\right)^{-1}}{\left[\left(\frac{1}{NT-1}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(\Delta\varepsilon_{it}\right)^2\right) - \left(\frac{1}{\sqrt{NT-1}}\sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-2}\Delta\varepsilon_{it}\right)^2\left(\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-2}^2\right)^{-1}\frac{1}{NT^2}\right]} \\
\Rightarrow & \ 2\sigma^4\chi_1^2\left(\frac{\sigma^2}{2}\right)^{-1}\frac{1}{2\sigma^2} \equiv 2\chi_1^2.
\end{aligned}$$

Now, consider part (c), where we take $\rho_T = \exp\left\{-1/q\left(T\right)\right\}$ such that $q\left(T\right) \sim T$. In this case, we have, by part (c) of Lemma SD-2 and part (b) of Lemma SE-22, that

$$\begin{aligned}
& \frac{1}{\overline{\omega}_T\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-2}\Delta\varepsilon_{it} \\
= & -\frac{1}{\overline{\omega}_T\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=2}^{T}\varepsilon_{it-2}\varepsilon_{it-1} + \frac{1}{\overline{\omega}_T\sqrt{NT}}\sum_{i=1}^{N} w_{iT-2}\varepsilon_{iT} + O_p\left(\frac{1}{\sqrt{T}}\right) \\
\Rightarrow & \ N\left(0,1\right)
\end{aligned}$$

where $\omega_T = \sigma^2\sqrt{1 + \left(q\left(T\right)/2T\right)\left[1 - \exp\left\{-2T/q\left(T\right)\right\}\right]}$. Moreover, by part (a) of Lemma SE-11, part

(a) of Lemma SE-17, and part (c) of Lemma SE-18 that

$$\begin{aligned}
&\frac{1}{Nq(T)^2}\sum_{i=1}^{N}\sum_{t=2}^{T}y_{it-2}^2 \\
=\ &\frac{T^2}{q(T)^2}\left[\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T}w_{it-2}^2+\frac{2}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T}a_iw_{it-2}+\frac{T-1}{NT^2}\sum_{i=1}^{N}a_i^2\right] \\
=\ &\frac{T^2}{q(T)^2}\left[\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T}w_{it-2}^2+O_p\left(\frac{1}{\sqrt{NT}}\right)+O_p\left(\frac{1}{T}\right)\right] \\
=\ &\frac{\sigma^2}{4}\left[\exp\left\{-\frac{2T}{q(T)}\right\}+\frac{2T}{q(T)}-1\right]\left[1+O_p\left(\max\left\{\frac{1}{\sqrt{NT}},\frac{1}{T}\right\}\right)\right].
\end{aligned}$$

It follows that

$$\begin{aligned}
&TA_{NT}(\rho_T) \\
=\ &\frac{\frac{\omega_T^2T^2}{q(T)^2}\left(\frac{1}{\omega_T\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=2}^{T}y_{it-2}\Delta\varepsilon_{it}\right)^2\left(\frac{1}{Nq(T)^2}\sum_{i=1}^{N}\sum_{t=2}^{T}y_{it-2}^2\right)^{-1}}{\left[\left(\sum_{i=1}^{N}\sum_{t=2}^{T}(\Delta\varepsilon_{it})^2\right)-\left(\sum_{i=1}^{N}\sum_{t=2}^{T}y_{it-2}\Delta\varepsilon_{it}\right)^2\left(\frac{1}{Nq(T)^2}\sum_{i=1}^{N}\sum_{t=2}^{T}y_{it-2}^2\right)^{-1}\frac{1}{Nq(T)^2}\right]/(NT-1)} \\
=\ &\frac{\frac{\omega_T^2T^2}{q(T)^2}\left(\frac{1}{\omega_T\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=2}^{T}y_{it-2}\Delta\varepsilon_{it}\right)^2\left(\frac{1}{Nq(T)^2}\sum_{i=1}^{N}\sum_{t=2}^{T}y_{it-2}^2\right)^{-1}}{\left[\left(\frac{1}{NT-1}\sum_{i=1}^{N}\sum_{t=2}^{T}(\Delta\varepsilon_{it})^2\right)-\left(\frac{1}{\sqrt{NT-1}}\sum_{i=1}^{N}\sum_{t=2}^{T}y_{it-2}\Delta\varepsilon_{it}\right)^2\left(\frac{1}{Nq(T)^2}\sum_{i=1}^{N}\sum_{t=2}^{T}y_{it-2}^2\right)^{-1}\frac{1}{Nq(T)^2}\right]} \\
=\ &\frac{\omega_T^2T^2}{q(T)^2}\chi_1^2\left(\frac{\sigma^2}{4}\left[\exp\left\{-\frac{2T}{q(T)}\right\}+\frac{2T}{q(T)}-1\right]\right)^{-1}\frac{1}{2\sigma^2}+o_p(1) \\
=\ &\frac{T^2}{q(T)^2}\chi_1^2\frac{4\sigma^4}{2\sigma^4}\left(1+\frac{q(T)}{2T}\left[1-\exp\left\{-\frac{2T}{q(T)}\right\}\right]\right)\left[\exp\left\{-\frac{2T}{q(T)}\right\}+\frac{2T}{q(T)}-1\right]^{-1}+o_p(1) \\
=\ &\frac{2T^2}{q(T)^2}\chi_1^2\left(1+\frac{q(T)}{2T}\left[1-\exp\left\{-\frac{2T}{q(T)}\right\}\right]\right)\left[\exp\left\{-\frac{2T}{q(T)}\right\}-\frac{2T}{q(T)}-1\right]^{-1}+o_p(1) \\
=\ &\chi_1^2\frac{T}{q(T)}\left(\frac{2T}{q(T)}+1-\exp\left\{-\frac{2T}{q(T)}\right\}\right)\left[\exp\left\{-\frac{2T}{q(T)}\right\}+\frac{2T}{q(T)}-1\right]^{-1}+o_p(1) \\
=\ &\Phi(T)\chi_1^2+o_p(1)
\end{aligned}$$

where

$$\Phi(T)=\frac{T}{q(T)}\left[\frac{2T}{q(T)}+1-\exp\left\{-\frac{2T}{q(T)}\right\}\right]\left[\exp\left\{-\frac{2T}{q(T)}\right\}+\frac{2T}{q(T)}-1\right]^{-1}.$$

We turn our attention to part (d), where we take $\rho_T=\exp\{-1/q(T)\}$ such that $q(T)\to\infty$ but $q(T)/T\to 0$. In this case, we have, by part (d) of Lemma SD-2, part (c) of Lemma SE-22, and the

Cramér convergence theorem, that

$$\begin{aligned}
& \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-2}\Delta\varepsilon_{it} \\
= & -\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=4}^{T}\varepsilon_{it-2}\varepsilon_{it-1} + O_p\left(\max\left\{\sqrt{\frac{q(T)}{T}}, \frac{1}{\sqrt{q(T)}}\right\}\right) \\
\Rightarrow & \; N\left(0,\sigma^4\right),
\end{aligned}$$

and, by part (a) of Lemma SE-11, part (b) of Lemma SE-17, and part (d) of Lemma SE-18, that

$$\begin{aligned}
\frac{1}{NTq(T)}\sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-2}^2 &= \frac{1}{NTq(T)}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(a_i + w_{it-2}\right)^2 \\
&= \frac{1}{NTq(T)}\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-2}^2 + \frac{2}{NTq(T)}\sum_{i=1}^{N}\sum_{t=2}^{T} a_i w_{it-2} + \frac{T-1}{NTq(T)}\sum_{i=1}^{N} a_i^2 \\
&= \frac{1}{NTq(T)}\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-2}^2 + O_p\left(\frac{1}{\sqrt{NT}}\right) + O_p\left(\frac{1}{q(T)}\right) \xrightarrow{p} \frac{\sigma^2}{2}.
\end{aligned}$$

It follows that

$$\begin{aligned}
& q(T)\, A_{NT}(\rho_T) \\
= & \frac{q(T)\,\frac{\sigma^4}{q(T)}\left(\frac{1}{\sigma^2\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-2}\Delta\varepsilon_{it}\right)^2\left(\frac{1}{NTq(T)}\sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-2}^2\right)^{-1}}{\left[\left(\sum_{i=1}^{N}\sum_{t=2}^{T}\left(\Delta\varepsilon_{it}\right)^2\right) - \left(\sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-2}\Delta\varepsilon_{it}\right)^2\left(\frac{1}{NTq(T)}\sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-2}^2\right)^{-1}\frac{1}{NTq(T)}\right]/\left(NT-1\right)} \\
= & \frac{\sigma^4\left(\frac{1}{\sigma^2\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-2}\Delta\varepsilon_{it}\right)^2\left(\frac{1}{NTq(T)}\sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-2}^2\right)^{-1}}{\left[\left(\frac{1}{NT-1}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(\Delta\varepsilon_{it}\right)^2\right) - \left(\frac{1}{\sqrt{NT-1}}\sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-2}\Delta\varepsilon_{it}\right)^2\left(\frac{1}{NTq(T)}\sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-2}^2\right)^{-1}\frac{1}{NTq(T)}\right]} \\
\Rightarrow & \; \sigma^4\chi_1^2\left(\frac{\sigma^2}{2}\right)^{-1}\frac{1}{2\sigma^2} \equiv \chi_1^2.
\end{aligned}$$

Finally, consider part (e), where we take $\rho_T \in \mathcal{G}_{\mathrm{St}}$. In this case, we have, by part (e) of Lemma SD-2

and Lemma SE-23, that

$$\begin{aligned}
& \sqrt{\frac{1+\rho_T}{2\sigma^4}}\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-2}\Delta\varepsilon_{it} \\
= & -\sqrt{\frac{1+\rho_T}{2\sigma^4}}\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=4}^{T} \varepsilon_{it-2}\varepsilon_{it-1} \\
& +(1-\rho_T)\sqrt{\frac{1+\rho_T}{2\sigma^4}}\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=4}^{T} w_{it-3}\varepsilon_{it-1}+o_p(1) \\
\Rightarrow & \; N(0,1),
\end{aligned}$$

and, by part (a) of Lemma SE-11, part (c) of Lemma SE-17, and part (e) of Lemma SE-18, that

$$\begin{aligned}
& \frac{1-\rho_T^2}{NT}\sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-2}^2 \\
= & \frac{1-\rho_T^2}{NT}\sum_{i=1}^{N}\sum_{t=2}^{T} (a_i+w_{it-2})^2 \\
= & \frac{1-\rho_T^2}{NT}\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-2}^2+\left(1-\rho_T^2\right)\frac{T-1}{NT}\sum_{i=1}^{N} a_i^2+\frac{2\left(1-\rho_T^2\right)}{NT}\sum_{i=1}^{N}\sum_{t=2}^{T} a_i w_{it-2} \\
= & \frac{1-\rho_T^2}{NT}\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-2}^2+\left(1-\rho_T^2\right)\frac{T-1}{NT}\sum_{i=1}^{N} a_i^2+O_p\left(\frac{1}{\sqrt{NT}}\right) \\
= & \sigma^2+\left(1-\rho_T^2\right)\left(\sigma_a^2+\mu_a^2\right)+o_p(1).
\end{aligned}$$

It follows that

$$\begin{aligned}
& \frac{1+\left(1-\rho_T^2\right)\left(\sigma_a^2+\mu_a^2\right)/\sigma^2}{1-\rho_T}A_{NT}(\rho_T) \\
= & \frac{\frac{2\sigma^4\left[1+\left(1-\rho_T^2\right)\left(\sigma_a^2+\mu_a^2\right)/\sigma^2\right]\left(1-\rho_T^2\right)}{(1-\rho_T)(1+\rho_T)}\left(\sqrt{\frac{1+\rho_T}{2\sigma^4 NT}}\sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-2}\Delta\varepsilon_{it}\right)^2\left(\frac{1-\rho_T^2}{NT}\sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-2}^2\right)^{-1}}{\left[\left(\sum_{i=1}^{N}\sum_{t=2}^{T}(\Delta\varepsilon_{it})^2\right)-\left(\sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-2}\Delta\varepsilon_{it}\right)^2\left(\frac{1-\rho_T^2}{NT}\sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-2}^2\right)^{-1}\frac{1-\rho_T^2}{NT}\right]/(NT-1)} \\
= & \frac{2\sigma^2\left[\sigma^2+\left(1-\rho_T^2\right)\left(\sigma_a^2+\mu_a^2\right)\right]\left(\sqrt{\frac{1+\rho_T}{2\sigma^4 NT}}\sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-2}\Delta\varepsilon_{it}\right)^2\left(\frac{1-\rho_T^2}{NT}\sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-2}^2\right)^{-1}}{\left[\left(\frac{1}{NT-1}\sum_{i=1}^{N}\sum_{t=2}^{T}(\Delta\varepsilon_{it})^2\right)-\left(\frac{1}{\sqrt{NT-1}}\sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-2}\Delta\varepsilon_{it}\right)^2\left(\frac{1-\rho_T^2}{NT}\sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-2}^2\right)^{-1}\frac{1-\rho_T^2}{NT}\right]} \\
= & 2\sigma^2\frac{\left[\sigma^2+\left(1-\rho_T^2\right)\left(\sigma_a^2+\mu_a^2\right)\right]}{\sigma^2+\left(1-\rho_T^2\right)\left(\sigma_a^2+\mu_a^2\right)}\frac{1}{2\sigma^2}\chi_1^2+o_p(1) \\
\Rightarrow & \; \chi_1^2. \; \square
\end{aligned}$$

Appendix SD: Proof of Key Supporting Lemmas

Lemma SD-1:

Under Assumptions 1-4, the following statements are true as as $N, T \to \infty$ such that $N^{\kappa}/T \to \tau$, for constants $\kappa \in \left(\frac{1}{2}, \infty\right)$ and $\tau \in (0, \infty)$.

(a) If $\rho_T = 1$ for all T sufficiently large, then

$$\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-2}\Delta y_{it-1} = \frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-2}\varepsilon_{it-1} + O_p\left(\frac{1}{\sqrt{T}}\right).$$

(b) If $\rho_T = \exp\{-1/q(T)\}$ such that $T^{1+\frac{1}{2\kappa}} \ll q(T)$, then

$$\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-2}\Delta y_{it-1} = \frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-2}\varepsilon_{it-1} + O_p\left(\max\left\{\frac{T^{1+\frac{1}{2\kappa}}}{q(T)}, \frac{1}{\sqrt{T}}\right\}\right).$$

.

(c) If $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \sim T^{1+\frac{1}{2\kappa}}$, then

$$\begin{aligned}
&\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-2}\Delta y_{it-1} \\
= \;& \frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-2}\varepsilon_{it-1} - \frac{T\sqrt{N}}{q(T)}\left[\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-2}^2\right] + O_p\left(\frac{1}{\sqrt{T}}\right).
\end{aligned}$$

(d) If $\rho_T = \exp\{-1/q(T)\}$ such that $T \ll q(T) \ll T^{1+\frac{1}{2\kappa}}$, then

$$\frac{q(T)}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-2}\Delta y_{it-1} = -\left[\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-2}^2\right] + O_p\left(\frac{q(T)}{\sqrt{N}T}\right).$$

(e) If $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \sim T$, then

$$\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-2}\Delta y_{it-1} = -\frac{T}{q(T)}\left[\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-2}^2\right] + O_p\left(\frac{1}{\sqrt{N}}\right).$$

(f) If $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \to \infty$ but $q(T)/T \to 0$, then

$$\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-2}\Delta y_{it-1} = -\left[\frac{1}{NTq(T)}\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-2}^2\right] + O_p\left(\max\left\{\sqrt{\frac{q(T)}{NT}}, \frac{1}{q(T)}\right\}\right).$$

(g) If $\rho_T \in \mathcal{G}_{\mathrm{St}} = \left\{|\rho_T| = \exp\left\{-\frac{1}{q(T)}\right\} : q(T) \geq 0 \text{ and } q(T) = O(1) \text{ as } T \to \infty\right\}$, then

$$\frac{1+\rho_T}{NT}\sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-2}\Delta y_{it-1} = -\frac{\left(1-\rho_T^2\right)}{NT}\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-2}^2 + O_p\left(\max\left\{\frac{1}{\sqrt{NT}}, \frac{1}{T}\right\}\right).$$

Proof:

To proceed, we first make some preliminary calculations. Note that

$$
\begin{aligned}
& \sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-2}\Delta y_{it-1} \\
= & \sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-2}\varepsilon_{it-1} + (1-\rho_T)\sum_{i=1}^{N}\sum_{t=2}^{T} a_i y_{it-2} - (1-\rho_T)\sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-2}^2 \\
= & \sum_{i=1}^{N}\sum_{t=2}^{T} (a_i + w_{it-2})\,\varepsilon_{it-1} + (1-\rho_T)\sum_{i=1}^{N}\sum_{t=2}^{T} a_i\,(a_i + w_{it-2}) \\
& -(1-\rho_T)\sum_{i=1}^{N}\sum_{t=2}^{T}\left(a_i^2 + 2a_i w_{it-2} + w_{it-2}^2\right) \\
= & \sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-2}\varepsilon_{it-1} + \sum_{i=1}^{N}\sum_{t=2}^{T} a_i\varepsilon_{it-1} - (1-\rho_T)\sum_{i=1}^{N}\sum_{t=2}^{T} a_i w_{it-2} - (1-\rho_T)\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-2}^2.
\end{aligned}
$$

Now, consider part (a), where we take $\rho_T = 1$ for all T sufficiently large. In this case, we can apply part (c) of Lemma SE-11, Lemma SE-14, and part (a) of Lemma SE-18 to obtain

$$
\begin{aligned}
& \frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-2}\Delta y_{it-1} \\
= & \frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-2}\varepsilon_{it-1} + \frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=2}^{T} a_i\varepsilon_{it-1} - \frac{(1-\rho_T)}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=2}^{T} a_i w_{it-2} \\
& -\frac{(1-\rho_T)}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-2}^2 \\
= & \frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-2}\varepsilon_{it-1} + O_p\left(\frac{1}{\sqrt{T}}\right) + O_p\left((1-\rho_T)\max\left\{\sqrt{T},\sqrt{N}\right\}\right) \\
& +O_p\left((1-\rho_T)\sqrt{N}T\right) \\
= & \frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-2}\varepsilon_{it-1} + O_p\left(\frac{1}{\sqrt{T}}\right).
\end{aligned}
$$

To show parts (b)-(f), we further note that

$$
\begin{aligned}
& \sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-2}\Delta y_{it-1} \\
= & \sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-2}\varepsilon_{it-1} + \sum_{i=1}^{N}\sum_{t=2}^{T} a_i\varepsilon_{it-1} - (1-\rho_T)\sum_{i=1}^{N}\sum_{t=2}^{T} a_i w_{it-2} - (1-\rho_T)\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-2}^2 \\
= & \sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-2}\varepsilon_{it-1} + \sum_{i=1}^{N}\sum_{t=2}^{T} a_i\varepsilon_{it-1} - \left[1-\exp\left(-\frac{1}{q(T)}\right)\right]\sum_{i=1}^{N}\sum_{t=2}^{T} a_i w_{it-2} \\
& - \left[1-\exp\left(-\frac{1}{q(T)}\right)\right]\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-2}^2 \\
= & \sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-2}\varepsilon_{it-1} + \sum_{i=1}^{N}\sum_{t=2}^{T} a_i\varepsilon_{it-1} - \left[\frac{1}{q(T)} + O\left(\frac{1}{q(T)^2}\right)\right]\sum_{i=1}^{N}\sum_{t=2}^{T} a_i w_{it-2} \\
& - \left[\frac{1}{q(T)} + O\left(\frac{1}{q(T)^2}\right)\right]\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-2}^2 \\
= & \sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-2}\varepsilon_{it-1} + \sum_{i=1}^{N}\sum_{t=2}^{T} a_i\varepsilon_{it-1} - \frac{1}{q(T)}\sum_{i=1}^{N}\sum_{t=2}^{T} a_i w_{it-2}\left[1+O\left(\frac{1}{q(T)}\right)\right] \\
& - \frac{1}{q(T)}\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-2}^2\left[1+O\left(\frac{1}{q(T)}\right)\right]. \qquad (8)
\end{aligned}
$$

Next, consider part (b), where we take $\rho_T = \exp\{-1/q(T)\}$ such that $T^{1+\frac{1}{2\kappa}} \ll q(T)$. In this case, we apply parts (a) and (c) of Lemma SE-11, Lemma SE-15, and part (b) of Lemma SE-18 to expression (8) to obtain

$$
\begin{aligned}
& \frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-2}\Delta y_{it-1} \\
= & \frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-2}\varepsilon_{it-1} - \frac{T\sqrt{N}}{q(T)}\left[\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-2}^2\right]\left[1+O\left(\frac{1}{q(T)}\right)\right] \\
& + \frac{1}{\sqrt{T}}\left(\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=2}^{T} a_i\varepsilon_{it-1}\right) - \frac{1}{q(T)}\left[\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=2}^{T} a_i w_{it-2}\right]\left[1+O\left(\frac{1}{q(T)}\right)\right] \\
= & \frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-2}\varepsilon_{it-1} + O_p\left(\frac{T\sqrt{N}}{q(T)}\right) + O_p\left(\frac{1}{\sqrt{T}}\right) + O_p\left(\max\left\{\frac{\sqrt{T}}{q(T)}, \frac{\sqrt{N}}{q(T)}\right\}\right) \\
= & \frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-2}\varepsilon_{it-1} + O_p\left(\max\left\{\frac{T^{1+\frac{1}{2\kappa}}}{q(T)}, \frac{1}{\sqrt{T}}\right\}\right),
\end{aligned}
$$

as required for part (b).

Consider part (c), where we take $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \sim T^{1+\frac{1}{2\kappa}} \sim T\sqrt{N}$. In this case, applying parts (a) and (c) of Lemma SE-11, Lemma SE-15, and part (b) of Lemma SE-18 to

expression (8), we have

$$
\begin{aligned}
& \frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-2}\Delta y_{it-1} \\
= & \frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-2}\varepsilon_{it-1} - \frac{T\sqrt{N}}{q(T)}\left[\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-2}^2\right]\left[1+O\left(\frac{1}{q(T)}\right)\right] \\
& +\frac{1}{\sqrt{T}}\left(\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=2}^{T} a_i\varepsilon_{it-1}\right) - \frac{1}{q(T)}\left[\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=2}^{T} a_i w_{it-2}\right]\left[1+O\left(\frac{1}{q(T)}\right)\right] \\
= & \frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-2}\varepsilon_{it-1} - \frac{T\sqrt{N}}{q(T)}\left[\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-2}^2\right] + O_p\left(\frac{1}{\sqrt{T}}\right) \\
& +O_p\left(\max\left\{\frac{\sqrt{T}}{q(T)},\frac{\sqrt{N}}{q(T)}\right\}\right) \\
= & \frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-2}\varepsilon_{it-1} - \frac{T\sqrt{N}}{q(T)}\left[\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-2}^2\right] + O_p\left(\frac{1}{\sqrt{T}}\right).
\end{aligned}
$$

We turn our attention to part (d), where we take $\rho_T = \exp\{-1/q(T)\}$ such that $T \ll q(T) \ll T^{1+\frac{1}{2\kappa}}$. Note that, in this case, we can apply the results given in parts (a) and (c) of Lemma SE-11, Lemma SE-15, and part (b) of Lemma SE-18 to expression (8) to obtain

$$
\begin{aligned}
& \frac{q(T)}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-2}\Delta y_{it-1} \\
= & -\frac{q(T)}{q(T)}\left[\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-2}^2\right]\left[1+O\left(\frac{1}{q(T)}\right)\right] + \frac{q(T)}{T\sqrt{N}}\left[\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-2}\varepsilon_{it-1}\right] \\
& +\frac{q(T)}{\sqrt{N}T^{3/2}}\left(\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=2}^{T} a_i\varepsilon_{it-1}\right) - \frac{q(T)}{q(T)\sqrt{N}T}\left[\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=2}^{T} a_i w_{it-2}\right]\left[1+O\left(\frac{1}{q(T)}\right)\right] \\
= & \frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-2}^2\left[1+O\left(\frac{1}{q(T)}\right)\right] \\
& +O_p\left(\frac{q(T)}{T\sqrt{N}}\right) + O_p\left(\frac{q(T)}{\sqrt{N}T^{3/2}}\right) + O_p\left(\max\left\{\frac{1}{\sqrt{NT}},\frac{1}{T}\right\}\right) \\
= & -\left[\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-2}^2\right] + O_p\left(\frac{q(T)}{\sqrt{N}T}\right).
\end{aligned}
$$

To show part (e), where we take $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \sim T$. In this case, we make use of the results given in parts (a) and (c) of Lemma SE-11, part (a) of Lemma SE-17, and part (c) of

Lemma SE-18 to deduce

$$
\begin{aligned}
& \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-2}\Delta y_{it-1} \\
= & -\frac{T}{q(T)}\left[\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-2}^2\right]\left[1+O\left(\frac{1}{T}\right)\right]+\frac{1}{\sqrt{N}}\left[\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-2}\varepsilon_{it-1}\right] \\
& +\frac{1}{\sqrt{NT}}\left(\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=2}^{T} a_i\varepsilon_{it-1}\right)-\frac{1}{NTq(T)}\sum_{i=1}^{N}\sum_{t=2}^{T} a_i w_{it-2}\left[1+O\left(\frac{1}{T}\right)\right] \\
= & -\frac{T}{q(T)}\left[\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-2}^2\right]\left[1+O\left(\frac{1}{T}\right)\right]+O_p\left(\frac{1}{\sqrt{N}}\right)+O_p\left(\frac{1}{\sqrt{NT}}\right) \\
& +O_p\left(\max\left\{\frac{1}{\sqrt{NT}},\frac{1}{T}\right\}\right) \\
= & -\frac{T}{q(T)}\left[\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-2}^2\right]+O_p\left(\frac{1}{\sqrt{N}}\right).
\end{aligned}
$$

Consider part (f), where we take $\rho_T=\exp\{-1/q(T)\}$ such that $q(T)\to\infty$ but $q(T)/T\to 0$. In this case, we make use of the results given in parts (a) and (c) of Lemma SE-11, part (b) of Lemma SE-17, and part (d) of Lemma SE-18 to deduce

$$
\begin{aligned}
& \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-2}\Delta y_{it-1} \\
= & -\left[\frac{1}{NTq(T)}\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-2}^2\right]\left[1+O\left(\frac{1}{q(T)}\right)\right]+\sqrt{\frac{q(T)}{NT}}\left[\frac{1}{\sqrt{NTq(T)}}\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-2}\varepsilon_{it-1}\right] \\
& +\frac{1}{\sqrt{NT}}\left(\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=2}^{T} a_i\varepsilon_{it-1}\right)-\frac{1}{NTq(T)}\sum_{i=1}^{N}\sum_{t=2}^{T} a_i w_{it-2}\left[1+O\left(\frac{1}{q(T)}\right)\right] \\
= & -\left[\frac{1}{NTq(T)}\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-2}^2\right]\left[1+O\left(\frac{1}{q(T)}\right)\right]+O_p\left(\sqrt{\frac{q(T)}{NT}}\right)+O_p\left(\frac{1}{\sqrt{NT}}\right) \\
& +O_p\left(\max\left\{\frac{1}{\sqrt{NT}},\frac{1}{T}\right\}\right) \\
= & -\left[\frac{1}{NTq(T)}\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-2}^2\right]+O_p\left(\max\left\{\sqrt{\frac{q(T)}{NT}},\frac{1}{q(T)}\right\}\right),
\end{aligned}
$$

as required.

Finally, to show part (g), note that, in this case, we make use of the results given in parts (a) and

(c) of Lemma SE-11, part (c) of Lemma SE-17, and part (e) of Lemma SE-18 to deduce

$$\begin{aligned}
&\frac{1+\rho_T}{NT}\sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-2}\Delta y_{it-1} \\
= & -\frac{(1-\rho_T)(1+\rho_T)}{NT}\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-2}^2 - \frac{(1-\rho_T)(1+\rho_T)}{NT}\sum_{i=1}^{N}\sum_{t=2}^{T} a_i w_{it-2} \\
& +\frac{1+\rho_T}{NT}\sum_{i=1}^{N}\sum_{t=2}^{T} a_i\varepsilon_{it-1} + \frac{1+\rho_T}{NT}\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-2}\varepsilon_{it-1} \\
= & -\frac{\left(1-\rho_T^2\right)}{NT}\sum_{i=1}^{N}\sum_{t=3}^{T} w_{it-2}^2 + O_p\left(\max\left\{\frac{1}{\sqrt{NT}},\frac{1}{T}\right\}\right) + O_p\left(\frac{1}{\sqrt{NT}}\right) + O_p\left(\frac{1}{\sqrt{NT}}\right) \\
= & -\frac{\left(1-\rho_T^2\right)}{NT}\sum_{i=1}^{N}\sum_{t=3}^{T} w_{it-2}^2 + O_p\left(\max\left\{\frac{1}{\sqrt{NT}},\frac{1}{T}\right\}\right) \xrightarrow{p} -\sigma^2,
\end{aligned}$$

as required for part (f). □

Lemma SD-2:

Under Assumptions 1-4, the following statements are true as $N, T \to \infty$ such that $N^\kappa/T \to \tau$, for constants $\kappa \in \left(\frac{1}{2},\infty\right)$ and $\tau \in (0,\infty)$.

(a) If $\rho_T = 1$ for all T sufficiently large, then

$$\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-2}\Delta\varepsilon_{it} = -\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=4}^{T}\varepsilon_{it-2}\varepsilon_{it-1} + \frac{1}{\sqrt{NT}}\sum_{i=1}^{N} w_{iT-2}\varepsilon_{iT} + O_p\left(\frac{1}{\sqrt{T}}\right).$$

(b) If $\rho_T = \exp\{-1/q(T)\}$ such that $T/q(T) \to 0$, then

$$\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-2}\Delta\varepsilon_{it} = -\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=4}^{T}\varepsilon_{it-2}\varepsilon_{it-1} + \frac{1}{\sqrt{NT}}\sum_{i=1}^{N} w_{iT-2}\varepsilon_{iT} + O_p\left(\frac{1}{\sqrt{T}}\right).$$

(c) If $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \sim T$, then

$$\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-2}\Delta\varepsilon_{it} = -\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=4}^{T}\varepsilon_{it-2}\varepsilon_{it-1} + \frac{1}{\sqrt{NT}}\sum_{i=1}^{N} w_{iT-2}\varepsilon_{iT} + O_p\left(\frac{1}{\sqrt{T}}\right).$$

(d) If $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \to \infty$ but $q(T)/T \to 0$, then

$$\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-2}\Delta\varepsilon_{it} = -\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=4}^{T}\varepsilon_{it-2}\varepsilon_{it-1} + O_p\left(\max\left\{\sqrt{\frac{q(T)}{T}},\frac{1}{\sqrt{q(T)}}\right\}\right).$$

(e) If $\rho_T \in \mathcal{G}_{\mathrm{St}} = \left\{|\rho_T| = \exp\left\{-\frac{1}{q(T)}\right\} : q(T) \geq 0 \text{ and } q(T) = O(1) \text{ as } T \to \infty\right\}$, then

$$\sqrt{\frac{1+\rho_T}{2\sigma^4}}\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-2}\Delta\varepsilon_{it} = \sqrt{\frac{1+\rho_T}{2\sigma^4}}\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-2}\Delta\varepsilon_{it} + O_p\left(\frac{1}{\sqrt{T}}\right) \Rightarrow N(0,1).$$

Proof:

To proceed, write

$$\begin{aligned}\sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-2}\Delta\varepsilon_{it} &= \sum_{i=1}^{N}\sum_{t=2}^{T}\left[a_i + w_{it-2}\right]\Delta\varepsilon_{it} \\ &= \sum_{i=1}^{N}\sum_{t=2}^{T} a_i\Delta\varepsilon_{it} + \sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-2}\Delta\varepsilon_{it}. \end{aligned} \tag{9}$$

Applying partial summation we have that

$$\begin{aligned}&\sum_{i=1}^{N}\sum_{t=3}^{T} w_{it-2}\Delta\varepsilon_{it} \\ =\ & \sum_{i=1}^{N}\left\{\left[\sum_{t=4}^{T}\left(w_{it-3} - w_{it-2}\right)\varepsilon_{it-1}\right] + w_{iT-2}\varepsilon_{iT} - w_{i1}\varepsilon_{i2}\right\} \\ =\ & \sum_{i=1}^{N}\left\{\left[\sum_{t=4}^{T}\left(w_{it-3} - \left[\rho_T w_{it-3} + \varepsilon_{it-2}\right]\right)\varepsilon_{it-1}\right] + w_{iT-2}\varepsilon_{iT} - w_{i1}\varepsilon_{i2}\right\} \\ =\ & (1-\rho_T)\sum_{i=1}^{N}\sum_{t=4}^{T} w_{it-3}\varepsilon_{it-1} - \sum_{i=1}^{N}\sum_{t=4}^{T}\varepsilon_{it-2}\varepsilon_{it-1} + \sum_{i=1}^{N} w_{iT-2}\varepsilon_{iT} - \sum_{i=1}^{N}\varepsilon_{i1}\varepsilon_{i2}. \end{aligned} \tag{10}$$

Substituting (10) into (9), we have

$$\begin{aligned}&\sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-2}\Delta\varepsilon_{it} \\ =\ & -\sum_{i=1}^{N}\sum_{t=4}^{T}\varepsilon_{it-2}\varepsilon_{it-1} + \sum_{i=1}^{N} w_{iT-2}\varepsilon_{iT} + \sum_{i=1}^{N}\sum_{t=2}^{T} a_i\Delta\varepsilon_{it} - \sum_{i=1}^{N}\varepsilon_{i1}\varepsilon_{i2} + (1-\rho_T)\sum_{i=1}^{N}\sum_{t=4}^{T} w_{it-3}\varepsilon_{it-1}. \end{aligned}$$

Consider first part (a). Here, by assumption, $\rho_T = 1$ for all T sufficiently large. Hence, applying parts (g)-(i) of Lemma SE-11 and part (a) of Lemma SE-25 in Appendix SE below, we obtain

$$\begin{aligned}&\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-2}\Delta\varepsilon_{it} \\ =\ & -\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=4}^{T}\varepsilon_{it-2}\varepsilon_{it-1} + \frac{1}{\sqrt{NT}}\sum_{i=1}^{N} w_{iT-2}\varepsilon_{iT} + \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=2}^{T} a_i\Delta\varepsilon_{it} - \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\varepsilon_{i1}\varepsilon_{i2} \\ & + \frac{(1-\rho_T)}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=4}^{T} w_{it-3}\varepsilon_{it-1} \\ =\ & -\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=4}^{T}\varepsilon_{it-2}\varepsilon_{it-1} + \frac{1}{\sqrt{NT}}\sum_{i=1}^{N} w_{iT-2}\varepsilon_{iT} + O_p\left(\frac{1}{\sqrt{T}}\right) + O_p\left(\frac{1}{\sqrt{T}}\right) \\ & + O_p\left((1-\rho_T)\sqrt{T}\right) \\ =\ & -\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=4}^{T}\varepsilon_{it-2}\varepsilon_{it-1} + \frac{1}{\sqrt{NT}}\sum_{i=1}^{N} w_{iT-2}\varepsilon_{iT} + O_p\left(\frac{1}{\sqrt{T}}\right), \end{aligned}$$

as required.

Next, consider part (b). Here, we consider the case where $\rho_T = \exp\{-1/q(T)\}$ such that $T/q(T) \to 0$. In this case, using the results parts (g)-(i) of Lemma SE-11 and part (b) of Lemma SE-25, we deduce that

$$\begin{aligned}
&\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-2}\Delta\varepsilon_{it} \\
= & -\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=4}^{T}\varepsilon_{it-2}\varepsilon_{it-1} + \frac{1}{\sqrt{NT}}\sum_{i=1}^{N} w_{iT-2}\varepsilon_{iT} + O_p\left(\frac{1}{\sqrt{T}}\right) + O_p\left(\frac{\sqrt{T}}{q(T)}\right) \\
= & -\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=4}^{T}\varepsilon_{it-2}\varepsilon_{it-1} + \frac{1}{\sqrt{NT}}\sum_{i=1}^{N} w_{iT-2}\varepsilon_{iT} + O_p\left(\frac{1}{\sqrt{T}}\right)
\end{aligned}$$

which shows part (b).

Consider part (c), where we take $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \sim T$. In this case, using the results parts (g)-(i) of Lemma SE-11 and part (c) of Lemma SE-25, we deduce that

$$\begin{aligned}
&\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-2}\Delta\varepsilon_{it} \\
= & -\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=4}^{T}\varepsilon_{it-2}\varepsilon_{it-1} + \frac{1}{\sqrt{NT}}\sum_{i=1}^{N} w_{iT-2}\varepsilon_{iT} + O_p\left(\frac{1}{\sqrt{T}}\right) + O_p\left(\frac{\sqrt{T}}{q(T)}\right) \\
= & -\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=4}^{T}\varepsilon_{it-2}\varepsilon_{it-1} + \frac{1}{\sqrt{NT}}\sum_{i=1}^{N} w_{iT-2}\varepsilon_{iT} + O_p\left(\frac{1}{\sqrt{T}}\right),
\end{aligned}$$

which shows part (c).

Consider now part (d), where we take $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \to \infty$ but $q(T)/T \to 0$. In this case, we apply parts (g)-(i) of Lemma SE-11, part (d) of Lemma SE-21, and part (d) of Lemma SE-25 to obtain

$$\begin{aligned}
&\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=2}^{T} y_{it-2}\Delta\varepsilon_{it} \\
= & -\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=4}^{T}\varepsilon_{it-2}\varepsilon_{it-1} + \frac{1}{\sqrt{NT}}\sum_{i=1}^{N} w_{iT-2}\varepsilon_{iT} + O_p\left(\frac{1}{\sqrt{T}}\right) + O_p\left(\frac{1}{\sqrt{q(T)}}\right) \\
= & -\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=4}^{T}\varepsilon_{it-2}\varepsilon_{it-1} + O_p\left(\sqrt{\frac{q(T)}{T}}\right) + O_p\left(\frac{1}{\sqrt{T}}\right) + O_p\left(\frac{1}{\sqrt{q(T)}}\right) \\
= & -\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=4}^{T}\varepsilon_{it-2}\varepsilon_{it-1} + O_p\left(\max\left\{\sqrt{\frac{q(T)}{T}}, \frac{1}{\sqrt{q(T)}}\right\}\right),
\end{aligned}$$

as required.

Finally, to show part (e), note that, in this case,

$$\begin{aligned}\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=2}^{T}y_{it-2}\Delta\varepsilon_{it} &= \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=2}^{T}a_i\Delta\varepsilon_{it}+\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=2}^{T}w_{it-2}\Delta\varepsilon_{it}\\ &= \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=2}^{T}w_{it-2}\Delta\varepsilon_{it}+O_p\left(\frac{1}{\sqrt{T}}\right). \qquad (11)\end{aligned}$$

Applying partial summation the lead term on the right-hand side of the expression above, we get

$$\begin{aligned}&\sqrt{\frac{1+\rho_T}{2\sigma^4}}\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=3}^{T}w_{it-2}\Delta\varepsilon_{it}\\
=\;&\sqrt{\frac{1+\rho_T}{2\sigma^4}}\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\left\{\left[\sum_{t=4}^{T}\left(w_{it-3}-w_{it-2}\right)\varepsilon_{it-1}\right]+w_{iT-2}\varepsilon_{iT}-w_{i1}\varepsilon_{i2}\right\}\\
=\;&\sqrt{\frac{1+\rho_T}{2\sigma^4}}\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=4}^{T}\left[\sum_{j=1}^{t-3}\rho_T^{(t-3-j)}\varepsilon_{ij}-\sum_{j=1}^{t-2}\rho_T^{(t-2-j)}\varepsilon_{ij}\right]\varepsilon_{it-1}\\
&+\sqrt{\frac{1+\rho_T}{2\sigma^4}}\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{T-2}\rho_T^{(T-2-j)}\varepsilon_{ij}\varepsilon_{iT}-\sqrt{\frac{1+\rho_T}{2\sigma^4}}\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\varepsilon_{i1}\varepsilon_{i2}\\
=\;&\sqrt{\frac{1+\rho_T}{2\sigma^4}}\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=4}^{T}\left[\sum_{j=1}^{t-3}\rho_T^{(t-3-j)}\varepsilon_{ij}-\sum_{j=1}^{t-3}\rho_T^{(t-2-j)}\varepsilon_{ij}\right]\varepsilon_{it-1}\\
&-\sqrt{\frac{1+\rho_T}{2\sigma^4}}\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=4}^{T}\varepsilon_{it-2}\varepsilon_{it-1}+\sqrt{\frac{1+\rho_T}{2\sigma^4}}\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{T-2}\rho_T^{(T-2-j)}\varepsilon_{ij}\varepsilon_{iT}\\
&-\sqrt{\frac{1+\rho_T}{2\sigma^4}}\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\varepsilon_{i1}\varepsilon_{i2}\\
=\;&\sqrt{\frac{1+\rho_T}{2\sigma^4}}\frac{(1-\rho_T)}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=4}^{T}\sum_{j=1}^{t-3}\rho_T^{(t-3-j)}\varepsilon_{ij}\varepsilon_{it-1}-\sqrt{\frac{1+\rho_T}{2\sigma^4}}\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=4}^{T}\varepsilon_{it-2}\varepsilon_{it-1}\\
&+\sqrt{\frac{1+\rho_T}{2\sigma^4}}\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{T-2}\rho_T^{(T-2-j)}\varepsilon_{ij}\varepsilon_{iT}-\sqrt{\frac{1+\rho_T}{2\sigma^4}}\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\varepsilon_{i1}\varepsilon_{i2}.\end{aligned}$$

Applying parts (g)-(i) of Lemma SE-11 and part (e) of Lemma SE-21, we then have

$$\begin{aligned}
&\sqrt{\frac{1+\rho_T}{2\sigma^4}}\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=3}^{T}w_{it-2}\Delta\varepsilon_{it} \\
= & -\sqrt{\frac{1+\rho_T}{2\sigma^4}}\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=4}^{T}\varepsilon_{it-2}\varepsilon_{it-1}+\sqrt{\frac{1+\rho_T}{2\sigma^4}}\frac{(1-\rho_T)}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=4}^{T}\sum_{j=1}^{t-3}\rho_T^{(t-3-j)}\varepsilon_{ij}\varepsilon_{it-1} \\
& +\sqrt{\frac{1+\rho_T}{2\sigma^4}}\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}w_{iT-2}\varepsilon_{iT}-\sqrt{\frac{1+\rho_T}{2\sigma^4}}\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\varepsilon_{i1}\varepsilon_{i2} \\
= & \sqrt{\frac{1+\rho_T}{2\sigma^4}}\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(X_{i,T}+Y_{i,T}\right)+O_p\left(\frac{1}{\sqrt{T}}\right)
\end{aligned}$$

Combining this with expression (11) above, we have, by Lemma SE-23,

$$\begin{aligned}
\sqrt{\frac{1+\rho_T}{2\sigma^4}}\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=2}^{T}y_{it-2}\Delta\varepsilon_{it} &= \sqrt{\frac{1+\rho_T}{2\sigma^4}}\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=2}^{T}w_{it-2}\Delta\varepsilon_{it}+O_p\left(\frac{1}{\sqrt{T}}\right) \\
&= \sqrt{\frac{1+\rho_T}{2\sigma^4}}\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(X_{i,T}+Y_{i,T}\right)+O_p\left(\frac{1}{\sqrt{T}}\right) \\
&\Rightarrow N\left(0,1\right),
\end{aligned}$$

as required. □

Lemma SD-3:

Under Assumptions 1-4, the following statements are true as $N,T\to\infty$ such that $N^{\kappa}/T\to\tau$, for $\kappa\in\left(\frac{1}{2},\infty\right)$ and $\tau\in(0,\infty)$.

(a) If $\rho_T=1$ for all T sufficiently large, then

$$\begin{aligned}
&\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1}-\overline{y}_{-1,N,T}\right)^2 \\
= & \frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T}w_{it-1}^2+O_p\left(\frac{1}{T}\right)+O_p\left(\frac{1}{N}\right)\xrightarrow{p}\frac{\sigma^2}{2}.
\end{aligned}$$

(b) If $\rho_T=\exp\left\{-1/q\left(T\right)\right\}$ such that $T\ll q\left(T\right)$, then

$$\begin{aligned}
&\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1}-\overline{y}_{-1,N,T}\right)^2 \\
= & \frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T}w_{it-1}^2+O_p\left(\frac{1}{T}\right)+O_p\left(\frac{1}{N}\right)\xrightarrow{p}\frac{\sigma^2}{2}.
\end{aligned}$$

(c) If $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \sim T$, then

$$\begin{aligned}
&\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1}-\overline{y}_{-1,N,T}\right)^2 \\
= \;& \frac{q(T)^2}{T^2}\frac{\sigma^2}{4}\left[\exp\left\{-\frac{2T}{q(T)}\right\}+\frac{2T}{q(T)}-1\right]+O_p\left(\max\left\{\frac{1}{\sqrt{N}},\frac{1}{\sqrt{T}}\right\}\right).
\end{aligned}$$

(d) If $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \to \infty$ but $q(T)/T \to 0$, then

$$\frac{1}{NTq(T)}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1}-\overline{y}_{-1,N,T}\right)^2 = \frac{\sigma^2}{2}+O_p\left(\max\left\{\frac{1}{q(T)},\frac{q(T)}{T},\sqrt{\frac{q(T)}{NT}},\frac{1}{\sqrt{T}}\right\}\right).$$

(e) If $\rho_T \in \mathcal{G}_{\mathrm{St}} = \left\{|\rho_T| = \exp\left\{-\frac{1}{q(T)}\right\} : q(T) \geq 0 \text{ and } q(T) = O(1) \text{ as } T \to \infty\right\}$, then

$$\frac{1-\rho_T^2}{NT}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1}-\overline{y}_{-1,N,T}\right)^2 = \left(1-\rho_T^2\right)\sigma_a^2+\sigma^2+o_p(1),$$

as $N, T \to \infty$.

Proof:

To proceed, first write

$$\begin{aligned}
y_{it-1}-\overline{y}_{-1,N,T} &= a_i + w_{it-1} - \frac{1}{N(T-1)}\sum_{j=1}^{N}\sum_{s=2}^{T}(a_j+w_{is-1}) \\
&= a_i - \frac{1}{N}\sum_{j=1}^{N}a_j + w_{it-1} - \frac{1}{N(T-1)}\sum_{j=1}^{N}\sum_{s=2}^{T}w_{is-1} \\
&= a_i - \overline{a}_N + w_{it-1} - \overline{w}_{-1,N,T},
\end{aligned}$$

where

$$\overline{a}_N = \frac{1}{N}\sum_{j=1}^{N}a_j,\ \overline{w}_{-1,N,T} = \frac{1}{N(T-1)}\sum_{j=1}^{N}\sum_{s=2}^{T}w_{is-1},$$

so that

$$\begin{aligned}
&\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1}-\overline{y}_{-1,N,T}\right)^2 = \sum_{i=1}^{N}\sum_{t=2}^{T}(a_i-\overline{a}_N)^2 + \sum_{i=1}^{N}\sum_{t=2}^{T}\left(w_{it-1}-\overline{w}_{-1,N,T}\right)^2 \\
&+2\sum_{i=1}^{N}\sum_{t=2}^{T}(a_i-\overline{a}_N)\left(w_{it-1}-\overline{w}_{-1,N,T}\right) \\
= \;& (T-1)\sum_{i=1}^{N}a_i^2 - N(T-1)\overline{a}_N^2 + \sum_{i=1}^{N}\sum_{t=2}^{T}w_{it-1}^2 - N(T-1)\overline{w}_{-1,N,T}^2 \\
&+2\sum_{i=1}^{N}\sum_{t=2}^{T}a_i w_{it-1} - 2N(T-1)\overline{a}_N\overline{w}_{-1,N,T}.
\end{aligned}$$

Consider first part (a), where we take $\rho_T = 1$ for all T sufficiently large. In this case, we apply parts (a) and (b) of Lemma SE-11, part (a) of Lemma SE-18, and part (a) of Lemma SE-26 to obtain

$$\begin{aligned}
& \frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1}-\overline{y}_{-1,N,T}\right)^2 \\
= & \frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T}w_{it-1}^2+\frac{T-1}{NT^2}\sum_{i=1}^{N}a_i^2 \ -\frac{N(T-1)}{NT^2}\overline{a}_N^2 \\
& -\frac{N(T-1)}{NT^2}\overline{w}_{-1,N,T}^2-2\frac{N(T-1)}{NT^2}\overline{a}_N\overline{w}_{-1,N,T}+2\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T}a_iw_{it-1} \\
= & \frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T}w_{it-1}^2+O_p\left(\frac{1}{T}\right)+O_p\left(\frac{1}{N}\right)\xrightarrow{p}\frac{\sigma^2}{2},
\end{aligned}$$

as required.

Next, consider part (b), where we take $\rho_T = \exp\{-1/q(T)\}$ such that $T \ll q(T)$. Here, we make use of parts (a) and (b) of Lemma SE-11, part (b) of Lemma SE-18, and part (b) of Lemma SE-26 to deduce

$$\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1}-\overline{y}_{-1,N,T}\right)^2=\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T}w_{it-1}^2+O_p\left(\frac{1}{T}\right)+O_p\left(\frac{1}{N}\right)\xrightarrow{p}\frac{\sigma^2}{2},$$

which is the required result for part (b).

Consider now part (c), where we take $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \sim T$. To show the stated result for this case, note first that by part (a) of Lemma SE-17, we have

$$\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T}w_{it-1}^2=\frac{q(T)^2}{T^2}\frac{\sigma^2}{4}\left[\exp\left\{-\frac{2T}{q(T)}\right\}+\frac{2T}{q(T)}-1\right]+O_p\left(\frac{1}{\sqrt{N}}\right)+O_p\left(\frac{1}{\sqrt{T}}\right),$$

from which we deduce that $\sum_{i=1}^{N}\sum_{t=2}^{T}w_{it-1}^2=O_p\left(NT^2\right)$. Using this result and applying parts (a) and (b) of Lemma SE-11, part (c) of Lemma SE-18, and part (c) of Lemma SE-26, we get

$$\begin{aligned}
& \frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1}-\overline{y}_{-1,N,T}\right)^2 \\
= & \frac{T-1}{NT^2}\sum_{i=1}^{N}a_i^2 \ -\frac{N(T-1)}{NT^2}\overline{a}_N^2+\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T}w_{it-1}^2 \ -\frac{N(T-1)}{NT^2}\overline{w}_{-1,N,T}^2 \\
& +2\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T}a_iw_{it-1}-2\frac{N(T-1)}{NT^2}\overline{a}_N\overline{w}_{-1,N,T} \\
= & \frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T}w_{it-1}^2+O_p\left(\frac{1}{T}\right)+O_p\left(\frac{1}{N}\right) \\
= & \frac{q(T)^2}{T^2}\frac{\sigma^2}{4}\left[\exp\left\{-\frac{2T}{q(T)}\right\}+\frac{2T}{q(T)}-1\right]+O_p\left(\max\left\{\frac{1}{\sqrt{N}},\frac{1}{\sqrt{T}}\right\}\right).
\end{aligned}$$

Consider part (d), where we assume that $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \to \infty$ but $q(T)/T \to 0$. From part (b) of Lemma SE-17, we obtain

$$\frac{1}{NTq(T)}\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-1}^2 = \frac{\sigma^2}{2} + O_p\left(\max\left\{\frac{1}{q(T)}, \frac{q(T)}{T}, \sqrt{\frac{q(T)}{NT}}, \frac{1}{\sqrt{T}}\right\}\right).$$

Using this result along with parts (a) and (b) of Lemma SE-11, part (d) of Lemma SE-18, and part (d) of Lemma SE-26, we observe that

$$\begin{aligned}
& \frac{1}{NTq(T)}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1} - \overline{y}_{-1,N,T}\right)^2 \\
= & \frac{T-1}{NTq(T)}\sum_{i=1}^{N} a_i^2 - \frac{N(T-1)}{NTq(T)}\overline{a}_N^2 + \frac{1}{NTq(T)}\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-1}^2 - \frac{N(T-1)}{NTq(T)}\overline{w}_{-1,N,T}^2 \\
& +2\frac{1}{NTq(T)}\sum_{i=1}^{N}\sum_{t=2}^{T} a_i w_{it-1} - 2\frac{N(T-1)}{NTq(T)}\overline{a}_N\overline{w}_{-1,N,T} \\
= & \frac{1}{NTq(T)}\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-1}^2 + O_p\left(\frac{1}{q(T)}\right) + O_p\left(\frac{1}{q(T)}\right) + O_p\left(\max\left\{\frac{q(T)}{NT}, \frac{q(T)}{T^2}\right\}\right) \\
& +O_p\left(\max\left\{\frac{1}{\sqrt{NT}}, \frac{1}{T}\right\}\right) + O_p\left(\max\left\{\frac{1}{\sqrt{NT}}, \frac{1}{T}\right\}\right) \\
= & \frac{\sigma^2}{2} + O_p\left(\max\left\{\frac{1}{q(T)}, \frac{q(T)}{T}, \sqrt{\frac{q(T)}{NT}}, \frac{1}{\sqrt{T}}\right\}\right) \\
& +O_p\left(\max\left\{\frac{1}{q(T)}, \frac{q(T)}{NT}, \frac{1}{\sqrt{NT}}\right\}\right) \\
= & \frac{\sigma^2}{2} + O_p\left(\max\left\{\frac{1}{q(T)}, \frac{q(T)}{T}, \sqrt{\frac{q(T)}{NT}}, \frac{1}{\sqrt{T}}\right\}\right),
\end{aligned}$$

which shows part (d).

Finally, consider part (e), where we take $\rho_T \in \mathcal{G}_{\mathrm{St}} = \left\{|\rho_T| = \exp\left\{-\frac{1}{q(T)}\right\} : q(T) \geq 0 \text{ and } q(T) = O(1) \text{ as } T \to \infty\right\}$. In this case, we make use of parts (a) and (b) of Lemma SE-11, part (c) of Lemma SE-17, and part (e) of Lemmas SE-18 and SE-26 to deduce that

$$
\begin{aligned}
& \frac{1-\rho_T^2}{NT}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1}-\overline{y}_{-1,N,T}\right)^2 \\
= & \left(1-\rho_T^2\right)\frac{T-1}{NT}\sum_{i=1}^{N}a_i^2 \; -\frac{N\left(T-1\right)}{NT}\left(1-\rho_T^2\right)\overline{a}_N^2+\frac{\left(1-\rho_T^2\right)}{NT}\sum_{i=1}^{N}\sum_{t=2}^{T}w_{it-1}^2 \\
& -\left(1-\rho_T^2\right)\frac{N\left(T-1\right)}{NT}\overline{w}_{-1,N,T}^2+2\frac{\left(1-\rho_T^2\right)}{NT}\sum_{i=1}^{N}\sum_{t=2}^{T}a_iw_{it-1} \\
& -2\left(1-\rho_T^2\right)\frac{N\left(T-1\right)}{NT}\overline{a}_N\overline{w}_{-1,N,T} \\
= & \left(1-\rho_T^2\right)\left(\frac{1}{N}\sum_{i=1}^{N}a_i^2-\overline{a}_N^2\right)\left[1+O_p\left(\frac{1}{T}\right)\right]+\frac{\left(1-\rho_T^2\right)}{NT}\sum_{i=1}^{N}\sum_{t=2}^{T}w_{it-1}^2 \\
& +O_p\left(\max\left\{\frac{1}{NT},\frac{1}{T^2}\right\}\right)+O_p\left(\max\left\{\frac{1}{\sqrt{NT}},\frac{1}{T}\right\}\right)+O_p\left(\max\left\{\frac{1}{\sqrt{NT}},\frac{1}{T}\right\}\right) \\
= & \left(1-\rho_T^2\right)\sigma_a^2+\sigma^2+o_p\left(1\right)
\end{aligned}
$$

which completes the proof for part (e). □

Lemma SD-4:

Under Assumptions 1-4, the following statements are true as $N,T\to\infty$ such that $N^{\kappa}/T\to\tau$, for constants $\kappa\in\left(\frac{1}{2},\infty\right)$ and $\tau\in(0,\infty)$.

(a) If $\rho_T=1$ for all T sufficiently large, then

$$
\begin{aligned}
\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1}-\overline{y}_{-1,N,T}\right)a_i &= \sigma_a^2-\mu_a\overline{w}_{-1,N,T}+\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=2}^{T}a_iw_{it-1}+o_p\left(1\right) \\
&= O_p\left(\max\left\{1,\sqrt{\frac{T}{N}}\right\}\right).
\end{aligned}
$$

(b) If $\rho_T=\exp\left\{-1/q\left(T\right)\right\}$ such that $T\ll q\left(T\right)$, then

$$
\begin{aligned}
\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1}-\overline{y}_{-1,N,T}\right)a_i &= \sigma_a^2-\mu_a\overline{w}_{-1,N,T}+\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=2}^{T}a_iw_{it-1}+o_p\left(1\right) \\
&= O_p\left(\max\left\{1,\sqrt{\frac{T}{N}}\right\}\right).
\end{aligned}
$$

(c) If $\rho_T=\exp\left\{-1/q\left(T\right)\right\}$ such that $q\left(T\right)\sim T$, then

$$
\begin{aligned}
\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1}-\overline{y}_{-1,N,T}\right)a_i &= \sigma_a^2-\mu_a\overline{w}_{-1,N,T}+\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=2}^{T}a_iw_{it-1}+o_p\left(1\right) \\
&= O_p\left(\max\left\{1,\sqrt{\frac{T}{N}}\right\}\right).
\end{aligned}
$$

(d) If $\rho_T = \exp\{-1/q(T)\}$ such that $T^{\frac{1+\kappa}{3\kappa}}/q(T) = O(1)$ but $q(T)/T \to 0$, then

$$\begin{aligned}
& \frac{1}{q(T)^{3/2}\sqrt{NT}} \sum_{i=1}^{N}\sum_{t=2}^{T} \left(y_{it-1} - \overline{y}_{-1,N,T}\right) a_i \\
= \; & \frac{\sqrt{NT}}{q(T)^{3/2}} \left[\frac{1}{N}\sum_{i=1}^{N} a_i^2 - \overline{a}_N^2\right] + O_p\left(\max\left\{\frac{1}{\sqrt{q(T)}}, \sqrt{\frac{N}{q(T)\,T}}\right\}\right) \\
= \; & O_p\left(\max\left\{\frac{\sqrt{NT}}{q(T)^{3/2}}, \frac{1}{\sqrt{q(T)}}, \sqrt{\frac{N}{q(T)\,T}}\right\}\right).
\end{aligned}$$

(e) If $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \to \infty$ but $q(T)/T^{\frac{1+\kappa}{3\kappa}} \to 0$, then

$$\begin{aligned}
& \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=2}^{T} \left(y_{it-1} - \overline{y}_{-1,N,T}\right) a_i \\
= \; & \left[\frac{1}{N}\sum_{i=1}^{N} a_i^2 - \overline{a}_N^2\right] + O_p\left(\max\left\{\frac{q(T)}{\sqrt{NT}}, \frac{q(T)}{T}\right\}\right).
\end{aligned}$$

(f) If $\rho_T \in \mathcal{G}_{\mathrm{St}} = \left\{|\rho_T| = \exp\left\{-\frac{1}{q(T)}\right\} : q(T) \geq 0 \text{ and } q(T) = O(1) \text{ as } T \to \infty\right\}$, then

$$\frac{1-\rho_T^2}{NT}\sum_{i=1}^{N}\sum_{t=2}^{T} \left(y_{it-1} - \overline{y}_{-1,N,T}\right) a_i = \left(1-\rho_T^2\right)\sigma_a^2 + o_p(1),$$

as N, $T \to \infty$.

Proof:

To proceed, first write

$$\begin{aligned}
& \sum_{i=1}^{N}\sum_{t=2}^{T} \left(y_{it-1} - \overline{y}_{-1,N,T}\right) a_i = \sum_{i=1}^{N}\sum_{t=2}^{T} \left(a_i - \overline{a}_N + w_{it-1} - \overline{w}_{-1,N,T}\right) a_i \\
= \; & (T-1)\sum_{i=1}^{N} a_i^2 - N(T-1)\,\overline{a}_N^2 + \sum_{i=1}^{N}\sum_{t=2}^{T} a_i w_{it-1} \; - N(T-1)\,\overline{a}_N \overline{w}_{-1,N,T}.
\end{aligned}$$

Consider first part (a), where we take $\rho_T = 1$ for all T sufficiently large. Here, we apply parts (a)

and (b) of Lemma SE-11, part (a) of Lemma SE-18, and part (a) of Lemma SE-26 to obtain

$$\begin{aligned}
& \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1}-\overline{y}_{-1,N,T}\right)a_i \\
= & \frac{N(T-1)}{NT}\frac{1}{N}\sum_{i=1}^{N}a_i^2-\frac{N(T-1)}{NT}\overline{a}_N^2+\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=2}^{T}a_iw_{it-1}\ -\frac{N(T-1)}{NT}\overline{a}_N\overline{w}_{-1,N,T} \\
= & \left[\frac{1}{N}\sum_{i=1}^{N}a_i^2-\overline{a}_N^2-\overline{a}_N\overline{w}_{-1,N,T}\right]\left[1+O\left(\frac{1}{T}\right)\right]+\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=2}^{T}a_iw_{it-1} \\
= & \sigma_a^2-\mu_a\overline{w}_{-1,N,T}+\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=2}^{T}a_iw_{it-1}+o_p(1) \\
= & O_p(1)+O_p\left(\max\left\{1,\sqrt{\frac{T}{N}}\right\}\right)+O_p\left(\max\left\{1,\sqrt{\frac{T}{N}}\right\}\right)+o_p(1) \\
= & O_p\left(\max\left\{1,\sqrt{\frac{T}{N}}\right\}\right).
\end{aligned}$$

Next, consider part (b), where we take $\rho_T=\exp\{-1/q(T)\}$ such that $T\ll q(T)$. In this case, applying parts (a) and (b) of Lemma SE-11, part (b) of Lemma SE-18, and part (b) of Lemma SE-26; we obtain

$$\begin{aligned}
& \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1}-\overline{y}_{-1,N,T}\right)a_i \\
= & \sigma_a^2-\mu_a\overline{w}_{-1,N,T}+\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=2}^{T}a_iw_{it-1}+o_p(1) \\
= & O_p(1)+O_p\left(\max\left\{1,\sqrt{\frac{T}{N}}\right\}\right)+O_p\left(\max\left\{1,\sqrt{\frac{T}{N}}\right\}\right)+o_p(1) \\
= & O_p\left(\max\left\{1,\sqrt{\frac{T}{N}}\right\}\right),
\end{aligned}$$

as required.

Now, we turn our attention to part (c), where we take $\rho_T=\exp\{-1/q(T)\}$ such that $q(T)\sim T$. In this case, applying parts (a) and (b) of Lemma SE-11 and part (c) of Lemmas SE-18 and SE-26; we

have

$$
\begin{aligned}
& \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1}-\overline{y}_{-1,N,T}\right)a_i \\
= & \frac{N(T-1)}{NT}\frac{1}{N}\sum_{i=1}^{N}a_i^2-\frac{N(T-1)}{NT}\overline{a}_N^2+\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=2}^{T}a_iw_{it-1}\ -\frac{N(T-1)}{NT}\overline{a}_N\overline{w}_{-1,N,T} \\
= & \left[\frac{1}{N}\sum_{i=1}^{N}a_i^2-\overline{a}_N^2-\overline{a}_N\overline{w}_{-1,N,T}\right]\left[1+O\left(\frac{1}{T}\right)\right]+\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=2}^{T}a_iw_{it-1} \\
= & \sigma_a^2-\mu_a\overline{w}_{-1,N,T}+\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=2}^{T}a_iw_{it-1}+o_p(1) \\
= & O_p(1)+O_p\left(\max\left\{1,\sqrt{\frac{T}{N}}\right\}\right)+O_p\left(\max\left\{1,\sqrt{\frac{T}{N}}\right\}\right)+o_p(1) \\
= & O_p\left(\max\left\{1,\sqrt{\frac{T}{N}}\right\}\right).
\end{aligned}
$$

Consider part (d), where we assume that $\rho_T=\exp\{-1/q(T)\}$ such that $T^{\frac{1+\kappa}{3\kappa}}/q(T)=O(1)$ but $q(T)/T\to 0$. In this case, applying parts (a) and (b) of Lemma SE-11 and part (d) of Lemmas SE-18 and SE-26; we obtain

$$
\begin{aligned}
& \frac{1}{q(T)^{3/2}\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1}-\overline{y}_{-1,N,T}\right)a_i \\
= & \frac{N(T-1)}{q(T)^{3/2}\sqrt{NT}}\frac{1}{N}\sum_{i=1}^{N}a_i^2-\frac{N(T-1)}{q(T)^{3/2}\sqrt{NT}}\overline{a}_N^2 \\
& +\frac{1}{q(T)^{3/2}\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=2}^{T}a_iw_{it-1}\ -\frac{N(T-1)}{q(T)^{3/2}\sqrt{NT}}\overline{a}_N\overline{w}_{-1,N,T} \\
= & \frac{\sqrt{NT}}{q(T)^{3/2}}\left[\frac{1}{N}\sum_{i=1}^{N}a_i^2-\overline{a}_N^2\right]\left[1+O\left(\frac{1}{T}\right)\right]+O_p\left(\max\left\{\frac{1}{\sqrt{q(T)}},\sqrt{\frac{N}{q(T)T}}\right\}\right) \\
& +O_p\left(\max\left\{\frac{1}{\sqrt{q(T)}},\sqrt{\frac{N}{q(T)T}}\right\}\right) \\
= & \frac{\sqrt{NT}}{q(T)^{3/2}}\left[\frac{1}{N}\sum_{i=1}^{N}a_i^2-\overline{a}_N^2\right]+O_p\left(\max\left\{\frac{1}{\sqrt{q(T)}},\sqrt{\frac{N}{q(T)T}}\right\}\right) \\
= & O_p\left(\max\left\{\frac{\sqrt{NT}}{q(T)^{3/2}},\frac{1}{\sqrt{q(T)}},\sqrt{\frac{N}{q(T)T}}\right\}\right),
\end{aligned}
$$

as required for part (d).

Consider part (e), where we assume that $\rho_T=\exp\{-1/q(T)\}$ such that $q(T)\to\infty$ but $q(T)/T^{\frac{1+\kappa}{3\kappa}}\to 0$. In this case, again applying parts (a) and (b) of Lemma SE-11, and part (d) of Lemmas SE-18 and

SE-26; we obtain

$$\begin{aligned}
& \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1}-\overline{y}_{-1,N,T}\right)a_i \\
= & \frac{N(T-1)}{NT}\frac{1}{N}\sum_{i=1}^{N}a_i^2-\frac{N(T-1)}{NT}\overline{a}_N^2 \\
& +\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=2}^{T}a_i w_{it-1}\ -\frac{N(T-1)}{NT}\overline{a}_N\overline{w}_{-1,N,T} \\
= & \left[\frac{1}{N}\sum_{i=1}^{N}a_i^2-\overline{a}_N^2\right]\left[1+O\left(\frac{1}{T}\right)\right]+O_p\left(\max\left\{\frac{q(T)}{\sqrt{NT}},\frac{q(T)}{T}\right\}\right) \\
& +O_p\left(\max\left\{\frac{q(T)}{\sqrt{NT}},\frac{q(T)}{T}\right\}\right) \\
= & \left[\frac{1}{N}\sum_{i=1}^{N}a_i^2-\overline{a}_N^2\right]+O_p\left(\max\left\{\frac{q(T)}{\sqrt{NT}},\frac{q(T)}{T}\right\}\right)
\end{aligned}$$

which shows part (e).

Finally, consider part (f), where we take $\rho_T\in\mathcal{G}_{\mathrm{St}}=\left\{|\rho_T|=\exp\left\{-\frac{1}{q(T)}\right\}:q(T)\geq 0 \text{ and } q(T)=O(1) \text{ as } T\to\infty\right\}$. In this case, we make use of parts (a) and (b) of Lemma SE-11 and part (e) of Lemmas SE-18 and SE-26 to deduce

$$\begin{aligned}
& \frac{1-\rho_T^2}{NT}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1}-\overline{y}_{-1,N,T}\right)a_i=\left(1-\rho_T^2\right)\frac{N(T-1)}{NT}\frac{1}{N}\sum_{i=1}^{N}a_i^2-\left(1-\rho_T^2\right)\frac{N(T-1)}{NT}\overline{a}_N^2 \\
& +\frac{1-\rho_T^2}{NT}\sum_{i=1}^{N}\sum_{t=2}^{T}a_i w_{it-1}\ -\left(1-\rho_T^2\right)\frac{N(T-1)}{NT}\overline{a}_N\overline{w}_{-1,N,T} \\
= & \left(1-\rho_T^2\right)\left[\frac{1}{N}\sum_{i=1}^{N}a_i^2-\overline{a}_N^2\right]\left[1+O_p\left(\frac{1}{T}\right)\right]+O_p\left(\max\left\{\frac{1}{\sqrt{NT}},\frac{1}{T}\right\}\right) \\
& +O_p\left(\max\left\{\frac{1}{\sqrt{NT}},\frac{1}{T}\right\}\right) \\
= & \left(1-\rho_T^2\right)\sigma_a^2+o_p(1),
\end{aligned}$$

which completes the proof of part (f). □

Lemma SD-5:

Under Assumptions 1-4, the following statements are true as $N,T\to\infty$ such that $N^{\kappa}/T\to\tau$, for constants $\kappa\in\left(\frac{1}{2},\infty\right)$ and $\tau\in(0,\infty)$.

(a) If $\rho_T=1$ for all T sufficiently large, then

$$\frac{1}{T\sqrt{N}}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1}-\overline{y}_{-1,N,T}\right)\varepsilon_{it}=\frac{1}{T\sqrt{N}}\sum_{i=1}^{N}\sum_{t=2}^{T}w_{it-1}\varepsilon_{it}+O_p\left(\frac{1}{\sqrt{T}}\right)+O_p\left(\frac{1}{\sqrt{N}}\right).$$

(b) If $\rho_T = \exp\{-1/q(T)\}$ such that $T \ll q(T)$, then

$$\frac{1}{T\sqrt{N}}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1}-\overline{y}_{-1,N,T}\right)\varepsilon_{it} = \frac{1}{T\sqrt{N}}\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-1}\varepsilon_{it} + O_p\left(\frac{1}{\sqrt{T}}\right) + O_p\left(\frac{1}{\sqrt{N}}\right).$$

(c) If $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \sim T$, then

$$\frac{1}{T\sqrt{N}}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1}-\overline{y}_{-1,N,T}\right)\varepsilon_{it} = \frac{1}{T\sqrt{N}}\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-1}\varepsilon_{it} + O_p\left(\frac{1}{\sqrt{T}}\right) + O_p\left(\frac{1}{\sqrt{N}}\right).$$

(d) If $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \to \infty$ but $q(T)/T \to 0$, then

$$\begin{aligned}
&\frac{1}{\sqrt{NTq(T)}}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1}-\overline{y}_{-1,N,T}\right)\varepsilon_{it} \\
= \;& \frac{1}{\sqrt{NTq(T)}}\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-1}\varepsilon_{it} + O_p\left(\max\left\{\frac{1}{\sqrt{q(T)}}, \sqrt{\frac{q(T)}{NT}}, \frac{\sqrt{q(T)}}{T}\right\}\right).
\end{aligned}$$

(e) If $\rho_T \in \mathcal{G}_{\mathrm{St}} = \left\{|\rho_T| = \exp\left\{-\frac{1}{q(T)}\right\} : q(T) \geq 0 \text{ and } q(T) = O(1) \text{ as } T \to \infty\right\}$, then

$$\frac{1-\rho_T^2}{NT}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1}-\overline{y}_{-1,N,T}\right)\varepsilon_{it} = O_p\left(\frac{1}{\sqrt{NT}}\right),$$

as $N, T \to \infty$.

Proof:

To proceed, first write

$$\begin{aligned}
&\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1}-\overline{y}_{-1,N,T}\right)\varepsilon_{it} = \sum_{i=1}^{N}\sum_{t=2}^{T}\left(a_i - \overline{a}_N + w_{it-1} - \overline{w}_{-1,N,T}\right)\varepsilon_{it} \\
= \;& \sum_{i=1}^{N}\sum_{t=2}^{T} a_i\varepsilon_{it} - N(T-1)\,\overline{a}_N\overline{\varepsilon}_{N,T} + \sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-1}\varepsilon_{it} - N(T-1)\,\overline{w}_{-1,N,T}\overline{\varepsilon}_{N,T}.
\end{aligned}$$

Consider first part (a), where we take $\rho_T = 1$ for all T sufficiently large. Here, we make use of parts (b), (c), and (d) of Lemma SE-11, part (b) of Lemma SE-20, and part (a) of Lemma SE-26 to get

$$\begin{aligned}
&\frac{1}{T\sqrt{N}}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1}-\overline{y}_{-1,N,T}\right)\varepsilon_{it} \\
= \;& \frac{1}{T\sqrt{N}}\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-1}\varepsilon_{it} + \frac{1}{T\sqrt{N}}\sum_{i=1}^{N}\sum_{t=2}^{T} a_i\varepsilon_{it} - \frac{N(T-1)}{T\sqrt{N}}\overline{a}_N\overline{\varepsilon}_{N,T} - \frac{N(T-1)}{T\sqrt{N}}\overline{w}_{-1,N,T}\overline{\varepsilon}_{N,T} \\
= \;& \frac{1}{T\sqrt{N}}\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-1}\varepsilon_{it} + O_p\left(\frac{1}{\sqrt{T}}\right) + O_p\left(\frac{1}{\sqrt{T}}\right) + O_p\left(\max\left\{\frac{1}{\sqrt{N}}, \frac{1}{\sqrt{T}}\right\}\right) \\
= \;& \frac{1}{T\sqrt{N}}\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-1}\varepsilon_{it} + O_p\left(\max\left\{\frac{1}{\sqrt{N}}, \frac{1}{\sqrt{T}}\right\}\right) \Rightarrow N\left(0, \frac{\sigma^4}{2}\right),
\end{aligned}$$

which shows part (a).

Next, consider part (b), where we take $\rho_T = \exp\{-1/q(T)\}$ such that $T \ll q(T)$. In this case, applying parts (b), (c), and (d) of Lemma SE-11, part (a) of Lemma SE-20, and part (b) of Lemma SE-26; and, we obtain

$$\begin{aligned}
& \frac{1}{T\sqrt{N}}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1}-\overline{y}_{-1,N,T}\right)\varepsilon_{it} \\
= & \frac{1}{T\sqrt{N}}\sum_{i=1}^{N}\sum_{t=2}^{T}w_{it-1}\varepsilon_{it} + \frac{1}{T\sqrt{N}}\sum_{i=1}^{N}\sum_{t=2}^{T}a_i\varepsilon_{it} - \frac{N(T-1)}{T\sqrt{N}}\overline{a}_N\overline{\varepsilon}_{N,T} - \frac{N(T-1)}{T\sqrt{N}}\overline{w}_{-1,N,T}\overline{\varepsilon}_{N,T} \\
= & \frac{1}{T\sqrt{N}}\sum_{i=1}^{N}\sum_{t=2}^{T}w_{it-1}\varepsilon_{it} + O_p\left(\max\left\{\frac{1}{\sqrt{N}},\frac{1}{\sqrt{T}}\right\}\right) \Rightarrow N\left(0,\frac{\sigma^4}{2}\right),
\end{aligned}$$

as required.

We turn our attention to part (c), where we assume that $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \sim T$. In this case, applying parts (b), (c), and (d) of Lemma SE-11, part (c) of Lemma SE-25, and part (c) of Lemma SE-26; we have

$$\begin{aligned}
& \frac{1}{T\sqrt{N}}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1}-\overline{y}_{-1,N,T}\right)\varepsilon_{it} \\
= & \frac{1}{T\sqrt{N}}\sum_{i=1}^{N}\sum_{t=2}^{T}w_{it-1}\varepsilon_{it} + \frac{1}{T\sqrt{N}}\sum_{i=1}^{N}\sum_{t=2}^{T}a_i\varepsilon_{it} - \frac{N(T-1)}{T\sqrt{N}}\overline{a}_N\overline{\varepsilon}_{N,T} - \frac{N(T-1)}{T\sqrt{N}}\overline{w}_{-1,N,T}\overline{\varepsilon}_{N,T} \\
= & \frac{1}{T\sqrt{N}}\sum_{i=1}^{N}\sum_{t=2}^{T}w_{it-1}\varepsilon_{it} + O_p\left(\max\left\{\frac{1}{\sqrt{N}},\frac{1}{\sqrt{T}}\right\}\right)
\end{aligned}$$

which shows part (c).

Now, consider part (d), where we assume that $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \to \infty$ but $q(T)/T \to 0$. In this case, applying parts (b), (c), and (d) of Lemma SE-11 and part (d) of Lemmas SE-25 and SE-26; we obtain

$$\begin{aligned}
& \frac{1}{\sqrt{NTq(T)}}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1}-\overline{y}_{-1,N,T}\right)\varepsilon_{it} \\
= & \frac{1}{\sqrt{NTq(T)}}\sum_{i=1}^{N}\sum_{t=2}^{T}w_{it-1}\varepsilon_{it} + \frac{1}{\sqrt{NTq(T)}}\sum_{i=1}^{N}\sum_{t=2}^{T}a_i\varepsilon_{it} - \frac{N(T-1)}{\sqrt{NTq(T)}}\overline{a}_N\overline{\varepsilon}_{N,T} - \frac{N(T-1)}{\sqrt{NTq(T)}}\overline{w}_{-1,N,T}\overline{\varepsilon}_{N,T} \\
= & \frac{1}{\sqrt{NTq(T)}}\sum_{i=1}^{N}\sum_{t=2}^{T}w_{it-1}\varepsilon_{it} + O_p\left(\frac{1}{\sqrt{q(T)}}\right) + O_p\left(\frac{1}{\sqrt{q(T)}}\right) + O_p\left(\max\left\{\sqrt{\frac{q(T)}{NT}},\frac{\sqrt{q(T)}}{T}\right\}\right) \\
= & \frac{1}{\sqrt{NTq(T)}}\sum_{i=1}^{N}\sum_{t=2}^{T}w_{it-1}\varepsilon_{it} + O_p\left(\max\left\{\frac{1}{\sqrt{q(T)}},\sqrt{\frac{q(T)}{NT}},\frac{\sqrt{q(T)}}{T}\right\}\right),
\end{aligned}$$

as required for part (d).

Finally, consider part (e), where we take

$\rho_T \in \mathcal{G}_{\mathrm{St}} = \left\{ |\rho_T| = \exp\left\{-\frac{1}{q(T)}\right\} : q(T) \geq 0 \text{ and } q(T) = O(1) \text{ as } T \to \infty \right\}$. In this case, applying parts (b), (c), and (d) of Lemma SE-11 and part (e) of Lemmas SE-25 and SE-26; we have

$$\begin{aligned}
&\frac{1-\rho_T^2}{NT}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1}-\overline{y}_{-1,N,T}\right)\varepsilon_{it} = \frac{1-\rho_T^2}{NT}\sum_{i=1}^{N}\sum_{t=2}^{T} a_i\varepsilon_{it} - \frac{N(T-1)}{NT}\left(1-\rho_T^2\right)\overline{a}_N\overline{\varepsilon}_{N,T} \\
&+\frac{1-\rho_T^2}{NT}\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-1}\varepsilon_{it} - \frac{N(T-1)}{NT}\left(1-\rho_T^2\right)\overline{w}_{-1,N,T}\overline{\varepsilon}_{N,T} \\
= \;& O_p\left(\frac{1}{\sqrt{NT}}\right) + O_p\left(\frac{1}{\sqrt{NT}}\right) + O_p\left(\frac{1}{\sqrt{NT}}\right) + O_p\left(\max\left\{\frac{1}{NT}, \frac{1}{\sqrt{N}T^{3/2}}\right\}\right) \\
= \;& O_p\left(\frac{1}{\sqrt{NT}}\right),
\end{aligned}$$

which completes the proof of part (e). □

Lemma SD-6:

Under Assumptions 1-4, the following statements are true as $N, T \to \infty$ such that $N^\kappa/T \to \tau$, for $\kappa \in \left(\frac{1}{2}, \infty\right)$ and $\tau \in (0, \infty)$.

(a) If $\rho_T = 1$ for all T sufficiently large, then

$$\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=4}^{T}\left(y_{it-3}-y_{it-2}\right)^2 = \sigma^2 + O_p\left(\max\left\{\frac{1}{\sqrt{NT}}, \frac{1}{T}\right\}\right).$$

(b) If $\rho_T = \exp\{-1/q(T)\}$ such that $T/q(T) \to 0$,

$$\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=4}^{T}\left(y_{it-3}-y_{it-2}\right)^2 = \sigma^2 + O_p\left(\max\left\{\frac{1}{\sqrt{NT}}, \frac{1}{T}\right\}\right).$$

(c) If $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \sim T$,

$$\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=4}^{T}\left(y_{it-3}-y_{it-2}\right)^2 = \sigma^2 + O_p\left(\max\left\{\frac{1}{\sqrt{NT}}, \frac{1}{T}\right\}\right).$$

(d) If $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \to \infty$ but $q(T)/T \to 0$,

$$\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=4}^{T}\left(y_{it-3}-y_{it-2}\right)^2 = \sigma^2 + O_p\left(\max\left\{\frac{1}{\sqrt{NT}}, \frac{1}{q(T)}\right\}\right).$$

(e) If $\rho_T \in \mathcal{G}_{\mathrm{St}} = \left\{ |\rho_T| = \exp\left\{-\frac{1}{q(T)}\right\} : q(T) \geq 0 \text{ and } q(T) = O(1) \text{ as } T \to \infty \right\}$, then

$$\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=4}^{T}\left(y_{it-3}-y_{it-2}\right)^2 = \frac{2\sigma^2}{1+\rho_T} + o_p(1).$$

Proof of Lemma SD-6:

To show part (a), note that, by assumption here, there exists a positive integer I_ρ such that for all $T \geq I_\rho$, $\rho_T = 1$, so that

$$y_{it-3} - y_{it-2} = a_i + w_{it-3} - (a_i + w_{it-2}) = w_{it-3} - w_{it-2} = -\varepsilon_{it-2},$$

for all $T \geq I_\rho$. Hence, we can apply part (f) of Lemma SE-11 with $g = 2$ to obtain

$$\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=4}^{T}(y_{it-3} - y_{it-2})^2 = \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=4}^{T}\varepsilon_{it-2}^2 = \sigma^2 + O_p\left(\max\left\{\frac{1}{\sqrt{NT}}, \frac{1}{T}\right\}\right)$$

Now, to show parts (b)-(d), we first write

$$y_{it-3} - y_{it-2} = w_{it-3} - w_{it-2} = (1 - \rho_T)\, w_{it-3} - \varepsilon_{it-2},$$

from which we obtain

$$\begin{aligned}\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=4}^{T}(y_{it-3} - y_{it-2})^2 &= (1-\rho_T)^2 \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=4}^{T} w_{it-3}^2 - 2(1-\rho_T)\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=4}^{T} w_{it-3}\varepsilon_{it-2} \\ &\quad + \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=4}^{T}\varepsilon_{it-2}^2\end{aligned}$$

Next, consider part (b), where we take $\rho_T = \exp\{-1/q(T)\}$ such that $T/q(T) \to 0$. In this case, applying Lemma SE-15, we obtain

$$\begin{aligned}&(1-\rho_T)^2 \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=4}^{T} w_{it-3}^2 = T\left(1 - \exp\left\{-\frac{1}{q(T)}\right\}\right)^2 \frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=4}^{T} w_{it-3}^2 \\ &= T\left(1 - \left[1 - \frac{1}{q(T)} + O\left(\frac{1}{q(T)^2}\right)\right]\right)^2 \frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=4}^{T} w_{it-3}^2 \\ &= T\left(\frac{1}{q(T)} + O\left(\frac{1}{q(T)^2}\right)\right)^2 \frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=4}^{T} w_{it-3}^2 = \frac{T}{q(T)^2}\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=4}^{T} w_{it-3}^2\left[1 + O\left(\frac{1}{q(T)}\right)\right] \\ &= O_p\left(\frac{T}{q(T)^2}\right).\end{aligned}$$

and, by part (b) of Lemma SE-25

$$\begin{aligned}&2(1-\rho_T)\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=4}^{T} w_{it-3}\varepsilon_{it-2} = 2\left[\frac{1}{q(T)} + O\left(\frac{1}{q(T)^2}\right)\right]\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=4}^{T} w_{it-3}\varepsilon_{it-2} \\ &= \frac{2}{q(T)}\left[1 + O\left(\frac{1}{q(T)}\right)\right] O_p\left(\frac{1}{\sqrt{N}}\right) = O_p\left(\frac{1}{q(T)\sqrt{N}}\right).\end{aligned}$$

Using these results and part (f) of Lemma SE-11 with $g = 2$, we have

$$\begin{aligned}
&\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=4}^{T}\left(y_{it-3}-y_{it-2}\right)^2 \\
= \;& (1-\rho_T)^2\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=4}^{T}w_{it-3}^2 - 2\left(1-\rho_T\right)\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=4}^{T}w_{it-3}\varepsilon_{it-2} + \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=4}^{T}\varepsilon_{it-2}^2 \\
= \;& O_p\left(\frac{T}{q\left(T\right)^2}\right) + O_p\left(\frac{1}{q\left(T\right)\sqrt{N}}\right) + \sigma^2 + O_p\left(\max\left\{\frac{1}{\sqrt{NT}},\frac{1}{T}\right\}\right) \\
= \;& \sigma^2 + O_p\left(\max\left\{\frac{1}{\sqrt{NT}},\frac{1}{T}\right\}\right)
\end{aligned}$$

Consider part (c), where we take $\rho_T = \exp\left\{-1/q\left(T\right)\right\}$ such that $q\left(T\right)\sim T$. In this case, applying part (a) of Lemma SE-17, we obtain

$$\begin{aligned}
&(1-\rho_T)^2\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=4}^{T}w_{it-3}^2 = T\left(1-\exp\left\{-\frac{1}{q\left(T\right)}\right\}\right)^2\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=4}^{T}w_{it-3}^2 \\
= \;& \frac{T}{q\left(T\right)^2}\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=4}^{T}w_{it-3}^2\left[1+O\left(\frac{1}{q\left(T\right)}\right)\right] = O_p\left(\frac{1}{T}\right),
\end{aligned}$$

and, by part (c) of Lemma SE-25,

$$2\left(1-\rho_T\right)\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=4}^{T}w_{it-3}\varepsilon_{it-2} = 2\left[\frac{1}{q\left(T\right)}+O\left(\frac{1}{q\left(T\right)^2}\right)\right]\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=4}^{T}w_{it-3}\varepsilon_{it-2} = O_p\left(\frac{1}{T\sqrt{N}}\right).$$

Using these results along with part (f) of Lemma SE-11 with $g = 2$, we have

$$\begin{aligned}
&\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=4}^{T}\left(y_{it-3}-y_{it-2}\right)^2 = (1-\rho_T)^2\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=4}^{T}w_{it-3}^2 - 2\left(1-\rho_T\right)\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=4}^{T}w_{it-3}\varepsilon_{it-2} \\
&+\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=4}^{T}\varepsilon_{it-2}^2 \\
= \;& O_p\left(\frac{1}{T}\right) + O_p\left(\frac{1}{T\sqrt{N}}\right) + \sigma^2 + O_p\left(\max\left\{\frac{1}{\sqrt{NT}},\frac{1}{T}\right\}\right) \\
= \;& \sigma^2 + O_p\left(\max\left\{\frac{1}{\sqrt{NT}},\frac{1}{T}\right\}\right).
\end{aligned}$$

We turn our attention now to part (d), where we assume that $\rho_T = \exp\left\{-1/q\left(T\right)\right\}$ such that $q\left(T\right)\to\infty$ but $q\left(T\right)/T\to 0$. In this case, applying part (b) of Lemma SE-17, we obtain

$$\begin{aligned}
&(1-\rho_T)^2\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=4}^{T}w_{it-3}^2 = q\left(T\right)\left(1-\exp\left\{-\frac{1}{q\left(T\right)}\right\}\right)^2\frac{1}{NTq\left(T\right)}\sum_{i=1}^{N}\sum_{t=4}^{T}w_{it-3}^2 \\
= \;& \frac{1}{q\left(T\right)}\frac{1}{NTq\left(T\right)}\sum_{i=1}^{N}\sum_{t=4}^{T}w_{it-3}^2\left[1+O\left(\frac{1}{q\left(T\right)}\right)\right] = O_p\left(\frac{1}{q\left(T\right)}\right),
\end{aligned}$$

and, by part (d) of Lemma SE-25,

$$
\begin{aligned}
& 2\left(1-\rho_T\right)\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=4}^{T}w_{it-3}\varepsilon_{it-2}=2\left[\frac{1}{q\left(T\right)}+O\left(\frac{1}{q\left(T\right)^2}\right)\right]\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=4}^{T}w_{it-3}\varepsilon_{it-2} \\
= \; & \frac{2}{q\left(T\right)}\left[1+O\left(\frac{1}{q\left(T\right)}\right)\right]O_p\left(\sqrt{\frac{q\left(T\right)}{NT}}\right)=O_p\left(\frac{1}{\sqrt{NTq\left(T\right)}}\right).
\end{aligned}
$$

Using these results and part (f) of Lemma SE-11 with $g=2$, we have

$$
\begin{aligned}
& \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=4}^{T}\left(y_{it-3}-y_{it-2}\right)^2=\left(1-\rho_T\right)^2\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=4}^{T}w_{it-3}^2-2\left(1-\rho_T\right)\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=4}^{T}w_{it-3}\varepsilon_{it-2} \\
& +\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=4}^{T}\varepsilon_{it-2}^2 \\
= \; & O_p\left(\frac{1}{q\left(T\right)}\right)+O_p\left(\frac{1}{\sqrt{NTq\left(T\right)}}\right)+\sigma^2+O_p\left(\max\left\{\frac{1}{\sqrt{NT}},\frac{1}{T}\right\}\right) \\
= \; & \sigma^2+O_p\left(\max\left\{\frac{1}{\sqrt{NT}},\frac{1}{q\left(T\right)}\right\}\right).
\end{aligned}
$$

Finally, consider part (e), where we take $\rho_T\in\mathcal{G}_{\mathrm{St}}$. In this case, applying part (c) of Lemma SE-17, we have

$$
\begin{aligned}
\left(1-\rho_T\right)^2\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=4}^{T}w_{it-3}^2 &= \frac{\left(1-\rho_T\right)^2}{1-\rho_T^2}\frac{1-\rho_T^2}{NT}\sum_{i=1}^{N}\sum_{t=4}^{T}w_{it-3}^2=\frac{\left(1-\rho_T\right)^2}{1-\rho_T^2}\left[\sigma^2+o_p\left(1\right)\right] \\
&= \frac{1-\rho_T}{1+\rho_T}\sigma^2+o_p\left(1\right).
\end{aligned}
$$

Moreover, by part (e) of Lemma SE-25, we have

$$
2\left(1-\rho_T\right)\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=4}^{T}w_{it-3}\varepsilon_{it-2}=O_p\left(\frac{1}{\sqrt{NT}}\right).
$$

Hence, applying part (f) of Lemma SE-11, we deduce that

$$
\begin{aligned}
& \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=4}^{T}\left(y_{it-3}-y_{it-2}\right)^2 \\
= \; & \left[\left(1-\rho_T\right)^2\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=4}^{T}w_{it-3}^2-2\left(1-\rho_T\right)\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=4}^{T}w_{it-3}\varepsilon_{it-2}+\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=4}^{T}\varepsilon_{it-2}^2\right] \\
= \; & \frac{1-\rho_T}{1+\rho_T}\sigma^2+o_p\left(1\right)+O_p\left(\frac{1}{\sqrt{NT}}\right)+\sigma^2+O_p\left(\max\left\{\frac{1}{\sqrt{NT}},\frac{1}{T}\right\}\right) \\
= \; & \frac{2\sigma^2}{1+\rho_T}+o_p\left(1\right),
\end{aligned}
$$

as desired. □

Lemma SD-7:

Under Assumptions 1-4, the following statements are true as $N, T \to \infty$ such that $N^{\kappa}/T \to \tau$, for $\kappa \in \left(\frac{1}{2}, \infty\right)$ and $\tau \in (0, \infty)$.

(a) If $\rho_T = 1$ for all T sufficiently large, then

$$\frac{1}{NT}\sum_{i=1}^{N} y_{iT-2}^2 = \sigma^2 + O_p\left(\max\left\{\frac{1}{\sqrt{N}}, \frac{1}{\sqrt{T}}\right\}\right).$$

(b) If $\rho_T = \exp\{-1/q(T)\}$ such that $T/q(T) \to 0$, then

$$\frac{1}{NT}\sum_{i=1}^{N} y_{iT-2}^2 = \sigma^2 + O_p\left(\max\left\{\frac{1}{\sqrt{N}}, \frac{1}{\sqrt{T}}, \frac{T}{q(T)}\right\}\right).$$

(c) If $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \sim T$, then

$$\frac{1}{NT}\sum_{i=1}^{N} y_{iT-2}^2 = \frac{\sigma^2}{2}\frac{q(T)}{T}\left[1 - \exp\left\{-\frac{2T}{q(T)}\right\}\right] + O_p\left(\max\left\{\frac{1}{\sqrt{N}}, \frac{1}{\sqrt{T}}\right\}\right).$$

(d) If $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \to \infty$ but $q(T)/T \to 0$, then

$$\frac{1}{NT}\sum_{i=1}^{N} y_{iT-2}^2 = O_p\left(\frac{q(T)}{T}\right) + O_p\left(\sqrt{\frac{q(T)}{NT}}\right) = o_p(1).$$

(e) If $\rho_T \in \mathcal{G}_{\mathrm{St}} = \left\{|\rho_T| = \exp\left\{-\frac{1}{q(T)}\right\} : q(T) \geq 0 \text{ and } q(T) = O(1) \text{ as } T \to \infty\right\}$, then

$$\frac{1}{NT}\sum_{i=1}^{N} y_{iT-2}^2 = O_p\left(\frac{1}{T}\right).$$

Proof of Lemma SD-7:

To proceed, consider first part (a), where we take $\rho_T = 1$ for all T sufficiently large. In this case, we apply part (a) of Lemmas SE-11, SE-30, and SE-32 to obtain

$$\begin{aligned}
\frac{1}{NT}\sum_{i=1}^{N} y_{iT-2}^2 &= \frac{1}{NT}\sum_{i=1}^{N}(a_i + w_{iT-2})^2 = \frac{1}{NT}\sum_{i=1}^{N} w_{iT-2}^2 + 2\frac{1}{NT}\sum_{i=1}^{N} a_i w_{iT-2} + \frac{1}{NT}\sum_{i=1}^{N} a_i^2 \\
&= \sigma^2 + O_p\left(\max\left\{\frac{1}{\sqrt{N}}, \frac{1}{\sqrt{T}}\right\}\right) + O_p\left(\max\left\{\frac{1}{\sqrt{NT}}, \frac{1}{T}\right\}\right) + O_p\left(\frac{1}{T}\right) \\
&= \sigma^2 + O_p\left(\max\left\{\frac{1}{\sqrt{N}}, \frac{1}{\sqrt{T}}\right\}\right).
\end{aligned}$$

Next, consider part (b), where we take $\rho_T = \exp\{-1/q(T)\}$ such that $T/q(T) \to 0$. In this case, we apply part (a) of Lemmas SE-11 and part (b) of Lemmas SE-30 and SE-32 to obtain

$$\begin{aligned}
\frac{1}{NT}\sum_{i=1}^{N} y_{iT-2}^2 &= \frac{1}{NT}\sum_{i=1}^{N} w_{iT-2}^2 + 2\frac{1}{NT}\sum_{i=1}^{N} a_i w_{iT-2} + \frac{1}{NT}\sum_{i=1}^{N} a_i^2 \\
&= \sigma^2 + O_p\left(\max\left\{\frac{1}{\sqrt{N}}, \frac{1}{\sqrt{T}}, \frac{T}{q(T)}\right\}\right) + O_p\left(\max\left\{\frac{1}{\sqrt{NT}}, \frac{1}{T}\right\}\right) + O_p\left(\frac{1}{T}\right) \\
&= \sigma^2 + O_p\left(\max\left\{\frac{1}{\sqrt{N}}, \frac{1}{\sqrt{T}}, \frac{T}{q(T)}\right\}\right).
\end{aligned}$$

Consider part (c), where we take $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \sim T$. In this case, application of part (a) of Lemmas SE-11 and part (c) of Lemmas SE-30 and SE-32 yields

$$\begin{aligned}
&\frac{1}{NT}\sum_{i=1}^{N} y_{iT-2}^2 = \frac{1}{NT}\sum_{i=1}^{N} w_{iT-2}^2 + 2\frac{1}{NT}\sum_{i=1}^{N} a_i w_{iT-2} + \frac{1}{NT}\sum_{i=1}^{N} a_i^2 \\
= \;& \frac{\sigma^2}{2}\frac{q(T)}{T}\left[1-\exp\left\{-\frac{2T}{q(T)}\right\}\right] + O_p\left(\max\left\{\frac{1}{\sqrt{N}},\frac{1}{\sqrt{T}}\right\}\right) + O_p\left(\max\left\{\frac{1}{\sqrt{NT}},\frac{1}{T}\right\}\right) + O_p\left(\frac{1}{T}\right) \\
= \;& \frac{\sigma^2}{2}\frac{q(T)}{T}\left[1-\exp\left\{-\frac{2T}{q(T)}\right\}\right] + O_p\left(\max\left\{\frac{1}{\sqrt{N}},\frac{1}{\sqrt{T}}\right\}\right).
\end{aligned}$$

Now, we turn our attention to part (d), where we take $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \to \infty$ but $q(T)/T \to 0$. In this case, we apply part (a) of Lemmas SE-11 and part (d) of Lemmas SE-30 and SE-32 to obtain

$$\begin{aligned}
&\frac{1}{NT}\sum_{i=1}^{N} y_{iT-2}^2 = \frac{1}{NT}\sum_{i=1}^{N} w_{iT-2}^2 + 2\frac{1}{NT}\sum_{i=1}^{N} a_i w_{iT-2} + \frac{1}{NT}\sum_{i=1}^{N} a_i^2 \\
= \;& O_p\left(\frac{q(T)}{T}\right) + O_p\left(\frac{1}{T}\sqrt{\frac{q(T)}{N}}\right) + O_p\left(\frac{1}{T}\right) = O_p\left(\frac{q(T)}{T}\right).
\end{aligned}$$

Finally, consider part (e), where we take $\rho_T \in \mathcal{G}_{\mathrm{St}}$. In this case, we obtain part (a) of Lemmas SE-11 and part (e) of Lemmas SE-30 and SE-32 to obtain

$$\begin{aligned}
&\frac{1}{NT}\sum_{i=1}^{N} y_{iT-2}^2 = \frac{1}{NT}\sum_{i=1}^{N} w_{iT-2}^2 + 2\frac{1}{NT}\sum_{i=1}^{N} a_i w_{iT-2} + \frac{1}{NT}\sum_{i=1}^{N} a_i^2 \\
= \;& O_p\left(\frac{1}{T}\right) + O_p\left(\frac{1}{\sqrt{N}T}\right) + O_p\left(\frac{1}{T}\right) = O_p\left(\frac{1}{T}\right). \;\square
\end{aligned}$$

Lemma SD-8:

Suppose that Assumptions 1-4 hold. Then, the following statements are true as N, $T \to \infty$ such that $N^\kappa/T \to \tau$, for $\kappa \in \left(\frac{1}{2},\infty\right)$ and $\tau \in (0,\infty)$.

(a) If $\rho_T = 1$ for all T sufficiently large, then

$$\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=3}^{T}\Delta y_{it-1}y_{it-1} = \sigma^2 + O_p\left(\frac{1}{\sqrt{N}}\right).$$

(b) If $\rho_T = \exp\{-1/q(T)\}$ such that $T \ll q(T)$, then

$$\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=3}^{T}\Delta y_{it-1}y_{it-1} = \sigma^2 + O_p\left(\max\left\{\frac{T}{q(T)},\frac{1}{\sqrt{N}}\right\}\right).$$

(c) If $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \sim T$, then

$$\begin{aligned}
\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=3}^{T}\Delta y_{it-1}y_{it-1} \;=\;& \sigma^2\left(1-\frac{1}{4}\frac{q(T)}{T}\left[\exp\left\{-\frac{2T}{q(T)}\right\}+\frac{2T}{q(T)}-1\right]\right) \\
&+O_p\left(\max\left\{\frac{1}{\sqrt{N}},\frac{1}{\sqrt{T}}\right\}\right).
\end{aligned}$$

(d) If $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \to \infty$ but $q(T)/T \to 0$, then

$$\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=3}^{T}\Delta y_{it-1}y_{it-1} = \frac{\sigma^2}{2} + O_p\left(\max\left\{\frac{1}{q(T)}, \frac{q(T)}{T}, \sqrt{\frac{q(T)}{NT}}, \frac{1}{\sqrt{T}}\right\}\right).$$

(e) If $\rho_T \in \mathcal{G}_{\mathrm{St}} = \left\{|\rho_T| = \exp\left\{-\frac{1}{q(T)}\right\} : q(T) \geq 0 \text{ and } q(T) = O(1) \text{ as } T \to \infty\right\}$, then

$$\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=3}^{T}\Delta y_{it-1}y_{it-1} = \frac{\sigma^2}{1+\rho_T} + O_p\left(\frac{1}{\sqrt{T}}\right).$$

Proof of Lemma SD-8:

To proceed, first write

$$\begin{aligned}
&\sum_{i=1}^{N}\sum_{t=3}^{T}\Delta y_{it-1}y_{it-1} = \sum_{i=1}^{N}\sum_{t=3}^{T}\left[a_i(1-\rho_T) + (\rho_T - 1)y_{it-2} + \varepsilon_{it-1}\right]\left[a_i(1-\rho_T) + \rho_T y_{it-2} + \varepsilon_{it-1}\right] \\
= \;& (1-\rho_T)^2\sum_{i=1}^{N}\sum_{t=3}^{T}a_i^2 + \rho_T(\rho_T-1)\sum_{i=1}^{N}\sum_{t=3}^{T}y_{it-2}^2 + \sum_{i=1}^{N}\sum_{t=3}^{T}\varepsilon_{it-1}^2 + (1-\rho_T)[2\rho_T - 1]\sum_{i=1}^{N}\sum_{t=3}^{T}a_i y_{it-2} \\
&+2(1-\rho_T)\sum_{i=1}^{N}\sum_{t=3}^{T}a_i\varepsilon_{it-1} + (\rho_T-1)\sum_{i=1}^{N}\sum_{t=3}^{T}y_{it-2}\varepsilon_{it-1} + \rho_T\sum_{i=1}^{N}\sum_{t=3}^{T}y_{it-2}\varepsilon_{it-1} \\
= \;& (1-\rho_T)^2\sum_{i=1}^{N}\sum_{t=3}^{T}a_i^2 + \rho_T(\rho_T-1)\sum_{i=1}^{N}\sum_{t=3}^{T}a_i^2 + 2\rho_T(\rho_T-1)\sum_{i=1}^{N}\sum_{t=3}^{T}a_i w_{it-2} \\
&+\rho_T(\rho_T-1)\sum_{i=1}^{N}\sum_{t=3}^{T}w_{it-2}^2 + \sum_{i=1}^{N}\sum_{t=3}^{T}\varepsilon_{it-1}^2 + (1-\rho_T)[2\rho_T-1]\sum_{i=1}^{N}\sum_{t=3}^{T}a_i^2 \\
&+(1-\rho_T)[2\rho_T-1]\sum_{i=1}^{N}\sum_{t=3}^{T}a_i w_{it-2} + 2(1-\rho_T)\sum_{i=1}^{N}\sum_{t=3}^{T}a_i\varepsilon_{it-1} + (\rho_T-1)\sum_{i=1}^{N}\sum_{t=3}^{T}a_i\varepsilon_{it-1} \\
&+(\rho_T-1)\sum_{i=1}^{N}\sum_{t=3}^{T}w_{it-2}\varepsilon_{it-1} + \rho_T\sum_{i=1}^{N}\sum_{t=3}^{T}a_i\varepsilon_{it-1} + \rho_T\sum_{i=1}^{N}\sum_{t=3}^{T}w_{it-2}\varepsilon_{it-1} \\
= \;& (1-\rho_T)[1-\rho_T-\rho_T+2\rho_T-1]\sum_{i=1}^{N}\sum_{t=3}^{T}a_i^2 + (1-\rho_T)[2\rho_T-1-2\rho_T]\sum_{i=1}^{N}\sum_{t=3}^{T}a_i w_{it-2} \\
&+\rho_T(\rho_T-1)\sum_{i=1}^{N}\sum_{t=3}^{T}w_{it-2}^2 + \sum_{i=1}^{N}\sum_{t=3}^{T}\varepsilon_{it-1}^2 + [(1-\rho_T)(2-1)+\rho_T]\sum_{i=1}^{N}\sum_{t=3}^{T}a_i\varepsilon_{it-1} \\
&+(\rho_T-1+\rho_T)\sum_{i=1}^{N}\sum_{t=3}^{T}w_{it-2}\varepsilon_{it-1} \\
= \;& -(1-\rho_T)\sum_{i=1}^{N}\sum_{t=3}^{T}a_i w_{it-2} - \rho_T(1-\rho_T)\sum_{i=1}^{N}\sum_{t=3}^{T}w_{it-2}^2 + \sum_{i=1}^{N}\sum_{t=3}^{T}\varepsilon_{it-1}^2 + \sum_{i=1}^{N}\sum_{t=3}^{T}a_i\varepsilon_{it-1} \\
&+(2\rho_T-1)\sum_{i=1}^{N}\sum_{t=3}^{T}w_{it-2}\varepsilon_{it-1}
\end{aligned}$$

Consider part (a), where we take $\rho_T = 1$ for all T sufficiently large. In this case, we can apply parts (c) and (f) of Lemma SE-11, Lemma SE-14, part (a) of Lemma SE-18, and part (a) of Lemma SE-25 to obtain

$$\begin{aligned}
& \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=3}^{T}\Delta y_{it-1}y_{it-1} \\
= \; & \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=3}^{T}\varepsilon_{it-1}^2 + \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=3}^{T}a_i\varepsilon_{it-1} - \frac{(1-\rho_T)}{NT}\sum_{i=1}^{N}\sum_{t=3}^{T}a_i w_{it-2} \\
& -\frac{\rho_T(1-\rho_T)}{NT}\sum_{i=1}^{N}\sum_{t=3}^{T}w_{it-2}^2 + \frac{(2\rho_T-1)}{NT}\sum_{i=1}^{N}\sum_{t=3}^{T}w_{it-2}\varepsilon_{it-1} \\
= \; & \sigma^2 + O_p\left(\max\left\{\frac{1}{\sqrt{NT}},\frac{1}{T}\right\}\right) + O_p\left(\frac{1}{\sqrt{NT}}\right) + O_p\left((1-\rho_T)\max\left\{\sqrt{\frac{T}{N}},1\right\}\right) \\
& +O_p\left((1-\rho_T)T\right) + O_p\left(\frac{1}{\sqrt{N}}\right) \\
= \; & \sigma^2 + O_p\left(\frac{1}{\sqrt{N}}\right).
\end{aligned}$$

Now, consider part (b) where we take $\rho_T = \exp\{-1/q(T)\}$ such that $T \ll q(T)$. Here, applying parts (c) and (f) of Lemma SE-11, Lemma SE-15, part (b) of Lemma SE-18, and part (b) of Lemma SE-25; we obtain

$$\begin{aligned}
& \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=3}^{T}\Delta y_{it-1}y_{it-1} = \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=3}^{T}\varepsilon_{it-1}^2 + \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=3}^{T}a_i\varepsilon_{it-1} - \frac{(1-\rho_T)}{NT}\sum_{i=1}^{N}\sum_{t=3}^{T}a_i w_{it-2} \\
& -\frac{\rho_T(1-\rho_T)}{NT}\sum_{i=1}^{N}\sum_{t=3}^{T}w_{it-2}^2 + \frac{(2\rho_T-1)}{NT}\sum_{i=1}^{N}\sum_{t=3}^{T}w_{it-2}\varepsilon_{it-1} \\
= \; & \sigma^2 + O_p\left(\max\left\{\frac{1}{\sqrt{NT}},\frac{1}{T}\right\}\right) + O_p\left(\frac{1}{\sqrt{NT}}\right) + O_p\left(\frac{1}{q(T)}\max\left\{\sqrt{\frac{T}{N}},1\right\}\right) \\
& +O_p\left(\frac{T}{q(T)}\right) + O_p\left(\frac{1}{\sqrt{N}}\right) \\
= \; & \sigma^2 + O_p\left(\max\left\{\frac{T}{q(T)},\frac{1}{\sqrt{N}}\right\}\right).
\end{aligned}$$

Next, consider part (c) where we take $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \sim T$. Here, applying parts (c) and (f) of Lemma SE-11, part (a) of Lemma SE-17, part (c) of Lemma SE-18, and part (c) of

Lemma SE-25; we obtain

$$\begin{aligned}
&\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=3}^{T}\Delta y_{it-1}y_{it-1}=\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=3}^{T}\varepsilon_{it-1}^{2}-\frac{\rho_T\left(1-\rho_T\right)}{NT}\sum_{i=1}^{N}\sum_{t=3}^{T}w_{it-2}^{2}-\frac{\left(1-\rho_T\right)}{NT}\sum_{i=1}^{N}\sum_{t=3}^{T}a_iw_{it-2}\\
&+\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=3}^{T}a_i\varepsilon_{it-1}+\frac{\left(2\rho_T-1\right)}{NT}\sum_{i=1}^{N}\sum_{t=3}^{T}w_{it-2}\varepsilon_{it-1}\\
=\;&\sigma^2+O_p\left(\max\left\{\frac{1}{\sqrt{NT}},\frac{1}{T}\right\}\right)-\frac{1}{NTq\left(T\right)}\sum_{i=1}^{N}\sum_{t=3}^{T}w_{it-2}^{2}\left[1+O_p\left(\frac{1}{q\left(T\right)}\right)\right]\\
&+O_p\left(\max\left\{\frac{1}{\sqrt{NT}},\frac{1}{T}\right\}\right)+O_p\left(\frac{1}{\sqrt{NT}}\right)+O_p\left(\frac{1}{\sqrt{N}}\right)\\
=\;&\sigma^2\left(1-\frac{1}{4}\frac{q\left(T\right)}{T}\left[\exp\left\{-\frac{2T}{q\left(T\right)}\right\}+\frac{2T}{q\left(T\right)}-1\right]\right)+O_p\left(\max\left\{\frac{1}{\sqrt{N}},\frac{1}{\sqrt{T}}\right\}\right).
\end{aligned}$$

Consider part (d) where we take $\rho_T=\exp\left\{-1/q\left(T\right)\right\}$ such that $q\left(T\right)\to\infty$ but $q\left(T\right)/T\to 0$. Here, applying parts (c) and (f) of Lemma SE-11, part (b) of Lemma SE-17, part (d) of Lemma SE-18, and part (d) of Lemma SE-25; we obtain

$$\begin{aligned}
&\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=3}^{T}\Delta y_{it-1}y_{it-1}=\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=3}^{T}\varepsilon_{it-1}^{2}-\frac{\rho_T\left(1-\rho_T\right)}{NT}\sum_{i=1}^{N}\sum_{t=3}^{T}w_{it-2}^{2}-\frac{\left(1-\rho_T\right)}{NT}\sum_{i=1}^{N}\sum_{t=3}^{T}a_iw_{it-2}\\
&+\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=3}^{T}a_i\varepsilon_{it-1}+\frac{\left(2\rho_T-1\right)}{NT}\sum_{i=1}^{N}\sum_{t=3}^{T}w_{it-2}\varepsilon_{it-1}\\
=\;&\sigma^2+O_p\left(\max\left\{\frac{1}{\sqrt{NT}},\frac{1}{T}\right\}\right)-\frac{1}{NTq\left(T\right)}\sum_{i=1}^{N}\sum_{t=3}^{T}w_{it-2}^{2}\left[1+O_p\left(\frac{1}{q\left(T\right)}\right)\right]\\
&+O_p\left(\max\left\{\frac{1}{\sqrt{NT}},\frac{1}{T}\right\}\right)+O_p\left(\frac{1}{\sqrt{NT}}\right)+O_p\left(\sqrt{\frac{q\left(T\right)}{NT}}\right)\\
=\;&\sigma^2-\frac{\sigma^2}{2}\left[1+O_p\left(\max\left\{\frac{1}{q\left(T\right)},\frac{q\left(T\right)}{T},\sqrt{\frac{q\left(T\right)}{NT}},\frac{1}{\sqrt{T}}\right\}\right)\right]+O_p\left(\max\left\{\sqrt{\frac{q\left(T\right)}{NT}},\frac{1}{T}\right\}\right)\\
=\;&\frac{\sigma^2}{2}+O_p\left(\max\left\{\frac{1}{q\left(T\right)},\frac{q\left(T\right)}{T},\sqrt{\frac{q\left(T\right)}{NT}},\frac{1}{\sqrt{T}}\right\}\right).
\end{aligned}$$

Finally, consider part (e) where we take $\rho_T\in\mathcal{G}_{\mathrm{St}}=\left\{\left|\rho_T\right|=\exp\left\{-\frac{1}{q(T)}\right\}:q\left(T\right)\geq 0\text{ and }q\left(T\right)=O\left(1\right)\text{ as }T\to\infty\right\}$. Here, applying parts (c) and (f) of Lemma SE-11, part (c) of Lemma SE-17, part (e) of Lemma SE-18, and part (e) of Lemma

SE-25; we obtain

$$\begin{aligned}
&\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=3}^{T}\Delta y_{it-1}y_{it-1} = \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=3}^{T}\varepsilon_{it-1}^{2} - \frac{\rho_T(1-\rho_T)}{NT}\sum_{i=1}^{N}\sum_{t=3}^{T}w_{it-2}^{2} - \frac{(1-\rho_T)}{NT}\sum_{i=1}^{N}\sum_{t=3}^{T}a_i w_{it-2} \\
&+\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=3}^{T}a_i\varepsilon_{it-1} + \frac{(2\rho_T-1)}{NT}\sum_{i=1}^{N}\sum_{t=3}^{T}w_{it-2}\varepsilon_{it-1} \\
= \;& \sigma^2 + O_p\left(\max\left\{\frac{1}{\sqrt{NT}},\frac{1}{T}\right\}\right) - \frac{\rho_T}{1+\rho_T}\frac{1-\rho_T^2}{NT}\sum_{i=1}^{N}\sum_{t=3}^{T}w_{it-2}^{2} \\
&+O_p\left(\max\left\{\frac{1}{\sqrt{NT}},\frac{1}{T}\right\}\right) + O_p\left(\frac{1}{\sqrt{NT}}\right) + O_p\left(\frac{1}{\sqrt{NT}}\right) \\
= \;& \sigma^2 - \frac{\rho_T\sigma^2}{1+\rho_T}\left[1+O_p\left(\frac{1}{\sqrt{T}}\right)\right] + O_p\left(\max\left\{\frac{1}{\sqrt{NT}},\frac{1}{T}\right\}\right) \\
= \;& \frac{\sigma^2}{1+\rho_T} + O_p\left(\frac{1}{\sqrt{T}}\right).
\end{aligned}$$

Lemma SD-9:

Suppose that Assumptions 1-4 hold. Then, the following statements are true as N, $T \to \infty$ such that $N^{\kappa}/T \to \tau$ for some $\kappa \in \left(\frac{1}{2},\infty\right)$ and $\tau \in (0,\infty)$.

(a) If $\rho_T = 1$ for all T sufficiently large, then

$$\begin{aligned}
\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=3}^{T}\Delta y_{it-1}\left[a_i(1-\rho_T)+\varepsilon_{it}\right] &= \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=3}^{T}\varepsilon_{it-1}\varepsilon_{it} + O_p(1-\rho_T) \\
&\Rightarrow N\left(0,\sigma^4\right).
\end{aligned}$$

(b) If $\rho_T = \exp\{-1/q(T)\}$ such that $T \ll q(T)$, then

$$\begin{aligned}
\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=3}^{T}\Delta y_{it-1}\left[a_i(1-\rho_T)+\varepsilon_{it}\right] &= \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=3}^{T}\varepsilon_{it-1}\varepsilon_{it} + O_p\left(\frac{\sqrt{T}}{q(T)}\right) \\
&\Rightarrow N\left(0,\sigma^4\right).
\end{aligned}$$

(c) If $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \sim T$, then

$$\begin{aligned}
\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=3}^{T}\Delta y_{it-1}\left[a_i(1-\rho_T)+\varepsilon_{it}\right] &= \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=3}^{T}\varepsilon_{it-1}\varepsilon_{it} + O_p\left(\frac{1}{\sqrt{T}}\right) \\
&\Rightarrow N\left(0,\sigma^4\right).
\end{aligned}$$

(d) If $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \ll T$ but $T^{\frac{1}{2}\left(\frac{1}{\kappa}-1\right)}/q(T) \to 0$, then

$$\begin{aligned}
&\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=3}^{T}\Delta y_{it-1}\left[a_i(1-\rho_T)+\varepsilon_{it}\right] \\
= \;& \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=3}^{T}\varepsilon_{it-1}\varepsilon_{it} + O_p\left(\max\left\{\frac{1}{\sqrt{q(T)}},\frac{T^{\frac{1}{2}\left(\frac{1}{\kappa}-1\right)}}{q(T)}\right\}\right) \Rightarrow N\left(0,\sigma^4\right).
\end{aligned}$$

(e) If $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \to \infty$ but $q(T)/T^{\frac{1}{2}\left(\frac{1}{\kappa}-1\right)} = O(1)$, then

$$\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=3}^{T} \Delta y_{it-1} \left[a_i (1-\rho_T) + \varepsilon_{it}\right] = O_p\left(\max\left\{1, \frac{T^{\frac{1}{2}\left(\frac{1}{\kappa}-1\right)}}{q(T)}\right\}\right).$$

(f) If $\rho_T \in \mathcal{G}_{\mathrm{St}} = \left\{|\rho_T| = \exp\left\{-\frac{1}{q(T)}\right\} : q(T) \geq 0 \text{ and } q(T) = O(1) \text{ as } T \to \infty\right\}$, then

$$\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=3}^{T} \Delta y_{it-1} \left[a_i (1-\rho_T) + \varepsilon_{it}\right] = O_p\left(\max\left\{1, \sqrt{\frac{N}{T}}\right\}\right) = O_p\left(\max\left\{1, T^{\frac{1}{2}\left(\frac{1}{\kappa}-1\right)}\right\}\right).$$

Proof of Lemma SD-9:

To proceed, first write

$$\begin{aligned}
&\sum_{i=1}^{N} \sum_{t=3}^{T} \Delta y_{it-1} \left[a_i (1-\rho_T) + \varepsilon_{it}\right] = (1-\rho_T) \sum_{i=1}^{N} \sum_{t=3}^{T} \left[a_i (1-\rho_T) + (\rho_T - 1) y_{it-2} + \varepsilon_{it-1}\right] a_i \\
&+ \sum_{i=1}^{N} \sum_{t=3}^{T} \left[a_i (1-\rho_T) + (\rho_T - 1) y_{it-2} + \varepsilon_{it-1}\right] \varepsilon_{it} \\
= \;& (1-\rho_T)^2 \sum_{i=1}^{N} \sum_{t=3}^{T} a_i^2 - (1-\rho_T)^2 \sum_{i=1}^{N} \sum_{t=3}^{T} (a_i + w_{it-2}) a_i + (1-\rho_T) \sum_{i=1}^{N} \sum_{t=3}^{T} a_i \varepsilon_{it-1} \\
&+ (1-\rho_T) \sum_{i=1}^{N} \sum_{t=3}^{T} a_i \varepsilon_{it} - (1-\rho_T) \sum_{i=1}^{N} \sum_{t=3}^{T} (a_i + w_{it-2}) \varepsilon_{it} + \sum_{i=1}^{N} \sum_{t=3}^{T} \varepsilon_{it-1} \varepsilon_{it} \\
= \;& -(1-\rho_T)^2 \sum_{i=1}^{N} \sum_{t=3}^{T} a_i w_{it-2} + (1-\rho_T) \sum_{i=1}^{N} \sum_{t=3}^{T} a_i \varepsilon_{it-1} - (1-\rho_T) \sum_{i=1}^{N} \sum_{t=3}^{T} w_{it-2} \varepsilon_{it} + \sum_{i=1}^{N} \sum_{t=3}^{T} \varepsilon_{it-1} \varepsilon_{it}.
\end{aligned}$$

Consider part (a), where we take $\rho_T = 1$ for all T sufficiently large. In this case, we can apply parts (c) and (h) of Lemma SE-11, part (a) of Lemma SE-18, and part (a) of Lemma SE-25 to obtain

$$\begin{aligned}
&\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=3}^{T} \Delta y_{it-1} \left[a_i (1-\rho_T) + \varepsilon_{it}\right] = \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=3}^{T} \varepsilon_{it-1} \varepsilon_{it} - \frac{(1-\rho_T)}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=3}^{T} w_{it-2} \varepsilon_{it} \\
&- \frac{(1-\rho_T)^2}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=3}^{T} a_i w_{it-2} + \frac{(1-\rho_T)}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=3}^{T} a_i \varepsilon_{it-1} \\
= \;& \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=3}^{T} \varepsilon_{it-1} \varepsilon_{it} + O_p\left((1-\rho_T)\sqrt{T}\right) + O_p\left((1-\rho_T)^2 \max\left\{T, \sqrt{NT}\right\}\right) + O_p(1-\rho_T) \\
= \;& \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=3}^{T} \varepsilon_{it-1} \varepsilon_{it} + O_p\left((1-\rho_T)\sqrt{T}\right) \Rightarrow N\left(0, \sigma^4\right).
\end{aligned}$$

Now, consider part (b) where we take $\rho_T = \exp\{-1/q(T)\}$ such that $T \ll q(T)$. Here, applying

parts (c) and (h) Lemma SE-11, part (b) of Lemma SE-18, and part (b) of Lemma SE-25; we obtain

$$\begin{aligned}
& \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=3}^{T}\Delta y_{it-1}\left[a_i\left(1-\rho_T\right)+\varepsilon_{it}\right] \\
= & \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=3}^{T}\varepsilon_{it-1}\varepsilon_{it}-\frac{\left(1-\rho_T\right)}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=3}^{T}w_{it-2}\varepsilon_{it}-\frac{\left(1-\rho_T\right)^2}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=3}^{T}a_i w_{it-2} \\
& +\frac{\left(1-\rho_T\right)}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=3}^{T}a_i\varepsilon_{it-1} \\
= & \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=3}^{T}\varepsilon_{it-1}\varepsilon_{it}+O_p\left(\frac{\sqrt{T}}{q\left(T\right)}\right)+O_p\left(\max\left\{\frac{T}{q\left(T\right)^2},\frac{\sqrt{NT}}{q\left(T\right)^2}\right\}\right)+O_p\left(\frac{1}{q\left(T\right)}\right) \\
= & \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=3}^{T}\varepsilon_{it-1}\varepsilon_{it}+O_p\left(\frac{\sqrt{T}}{q\left(T\right)}\right)\Rightarrow N\left(0,\sigma^4\right).
\end{aligned}$$

Next, consider part (c) where we take $\rho_T=\exp\left\{-1/q\left(T\right)\right\}$ such that $q\left(T\right)\sim T$. Here, applying parts (c) and (h) Lemma SE-11, part (c) of Lemma SE-18, and part (c) of Lemma SE-25; we obtain

$$\begin{aligned}
& \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=3}^{T}\Delta y_{it-1}\left[a_i\left(1-\rho_T\right)+\varepsilon_{it}\right]=\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=3}^{T}\varepsilon_{it-1}\varepsilon_{it}-\frac{\left(1-\rho_T\right)}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=3}^{T}w_{it-2}\varepsilon_{it} \\
& -\frac{\left(1-\rho_T\right)^2}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=3}^{T}a_i w_{it-2}+\frac{\left(1-\rho_T\right)}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=3}^{T}a_i\varepsilon_{it-1} \\
= & \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=3}^{T}\varepsilon_{it-1}\varepsilon_{it}+O_p\left(\frac{1}{\sqrt{T}}\right)+O_p\left(\max\left\{\frac{1}{T},\frac{\sqrt{N}}{T^{3/2}}\right\}\right)+O_p\left(\frac{1}{T}\right) \\
= & \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=3}^{T}\varepsilon_{it-1}\varepsilon_{it}+O_p\left(\frac{1}{\sqrt{T}}\right)\Rightarrow N\left(0,\sigma^4\right).
\end{aligned}$$

Consider part (d) where we take $\rho_T=\exp\left\{-1/q\left(T\right)\right\}$ such that $q\left(T\right)\ll T$ but $T^{\frac{1}{2}\left(\frac{1}{\kappa}-1\right)}/q\left(T\right)\to 0$. Here, applying parts (c) and (h) Lemma SE-11, part (d) of Lemma SE-18, and part (d) of Lemma

SE-25; we obtain

$$
\begin{aligned}
& \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=3}^{T}\Delta y_{it-1}\left[a_i\left(1-\rho_T\right)+\varepsilon_{it}\right] \\
= & \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=3}^{T}\varepsilon_{it-1}\varepsilon_{it}-\frac{\left(1-\rho_T\right)}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=3}^{T}w_{it-2}\varepsilon_{it}-\frac{\left(1-\rho_T\right)^2}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=3}^{T}a_i w_{it-2} \\
& +\frac{\left(1-\rho_T\right)}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=3}^{T}a_i\varepsilon_{it-1} \\
= & \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=3}^{T}\varepsilon_{it-1}\varepsilon_{it}+O_p\left(\frac{1}{\sqrt{q\left(T\right)}}\right)+O_p\left(\max\left\{\frac{1}{q\left(T\right)},\frac{1}{q\left(T\right)}\sqrt{\frac{N}{T}}\right\}\right)+O_p\left(\frac{1}{q\left(T\right)}\right) \\
= & \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=3}^{T}\varepsilon_{it-1}\varepsilon_{it}+O_p\left(\max\left\{\frac{1}{\sqrt{q\left(T\right)}},\frac{T^{\frac{1}{2}\left(\frac{1}{\kappa}-1\right)}}{q\left(T\right)}\right\}\right)\quad\left(\text{for }\kappa\in\left(\frac{1}{2},\infty\right)\right) \\
\Rightarrow & N\left(0,\sigma^4\right).
\end{aligned}
$$

Consider part (e) where we take $\rho_T=\exp\left\{-1/q\left(T\right)\right\}$ such that $q\left(T\right)\to\infty$ but $q\left(T\right)/T^{\frac{1}{2}\left(\frac{1}{\kappa}-1\right)}=O\left(1\right)$. Here, applying parts (c) and (h) Lemma SE-11, part (d) of Lemma SE-18, and part (d) of Lemma SE-25; we obtain

$$
\begin{aligned}
& \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=3}^{T}\Delta y_{it-1}\left[a_i\left(1-\rho_T\right)+\varepsilon_{it}\right] \\
= & \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=3}^{T}\varepsilon_{it-1}\varepsilon_{it}-\frac{\left(1-\rho_T\right)^2}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=3}^{T}a_i w_{it-2}-\frac{\left(1-\rho_T\right)}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=3}^{T}w_{it-2}\varepsilon_{it} \\
& +\frac{\left(1-\rho_T\right)}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=3}^{T}a_i\varepsilon_{it-1} \\
= & O_p\left(1\right)+O_p\left(\max\left\{\frac{1}{q\left(T\right)},\frac{1}{q\left(T\right)}\sqrt{\frac{N}{T}}\right\}\right)+O_p\left(\frac{1}{\sqrt{q\left(T\right)}}\right)+O_p\left(\frac{1}{q\left(T\right)}\right) \\
= & O_p\left(\max\left\{1,\frac{T^{\frac{1}{2}\left(\frac{1}{\kappa}-1\right)}}{q\left(T\right)}\right\}\right),
\end{aligned}
$$

for $\kappa\in\left(\frac{1}{2},\infty\right)$.

Finally, consider part (f) where we take $\rho_T\in\mathcal{G}_{\mathrm{St}}=\left\{\left|\rho_T\right|=\exp\left\{-\frac{1}{q(T)}\right\}:q\left(T\right)\geq 0\text{ and }q\left(T\right)=O\left(1\right)\text{ as }T\to\infty\right\}$. Here, applying parts

(c) and (h) Lemma SE-11, part (e) of Lemma SE-18, and part (e) of Lemma SE-25; we obtain

$$\begin{aligned}
& \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=3}^{T}\Delta y_{it-1}\left[a_i\left(1-\rho_T\right)+\varepsilon_{it}\right] \\
= & \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=3}^{T}\varepsilon_{it-1}\varepsilon_{it}-\frac{\left(1-\rho_T\right)^2}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=3}^{T}a_i w_{it-2}-\frac{\left(1-\rho_T\right)}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=3}^{T}w_{it-2}\varepsilon_{it} \\
& +\frac{\left(1-\rho_T\right)}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=3}^{T}a_i\varepsilon_{it-1} \\
= & O_p\left(1\right)+O_p\left(\max\left\{1,\sqrt{\frac{N}{T}}\right\}\right)+O_p\left(1\right)+O_p\left(1\right) \\
= & O_p\left(\max\left\{1,\sqrt{\frac{N}{T}}\right\}\right). \quad \square
\end{aligned}$$

Lemma SD-10:

Let $\mathcal{G}_1, \mathcal{G}_2, ..., \mathcal{G}_J$ be J mutually exclusive collections of (parameter) sequences. Now, let $\left\{\rho_{j,T}\right\} \in \mathcal{G}_{s_j}$ (for $j = 1, ..., J$), i.e., $\left\{\rho_{j,T}\right\}$ is a sequence belonging to the collection $\mathcal{G}_{s_j}$, where $s_j \in \{1, ..., J\}$. Define $T_j = f_j\left(T\right)$ $(j = 1, ..., d)$, with $d \leq J$, where $f_j\left(\cdot\right) : \mathbb{N} \to \mathbb{N}$ is an increasing function in its argument, and let $\left\{\rho_{j,T_j}\right\}$ denote a subsequence of $\left\{\rho_{j,T}\right\}$. Furthermore, let $\mathcal{G}^*$ be a collection of parameter sequences, and for each $\{\rho_T\} \in \mathcal{G}^*$, suppose that

$$\{\rho_T\} = \bigcup_{j=1}^{d}\left\{\rho_{j,T_j}\right\},$$

where

$$\left\{\rho_{1,T_1}\right\} \in \mathcal{G}_{s_1}, ..., \left\{\rho_{d,T_d}\right\} \in \mathcal{G}_{s_d},$$

with $\mathcal{G}_{s_k} \neq \mathcal{G}_{s_\ell}$ for $k \neq \ell$ and where

$$\mathbb{N} = \bigcup_{k=1}^{d}\left\{T_k = f_k\left(T\right) : T \in \mathbb{N}\right\}.$$

Finally, let $\mathbb{S}_T\left(\rho_T\right)$ be a sequence of statistics, possibly depending on ρ_T, and let $\zeta\left(T\right)$ be a function of T such that $\zeta\left(T\right) \to 0$ as $T \to \infty$. Then, the following statements hold as $T \to \infty$.

(a) If for each $j \in \{1, .., d\}$ and each $\left\{\rho_{j,T}\right\} \in \mathcal{G}_{s_j}$

$$\mathbb{S}_T\left(\rho_{j,T}\right) = O_p\left(\zeta\left(T\right)\right),$$

then

$$\mathbb{S}_T\left(\rho_T\right) = O_p\left(\zeta\left(T\right)\right),$$

for all $\{\rho_T\} \in \mathcal{G}^*$.

(b) If for each $j \in \{1,..,d\}$ and each $\left\{\rho_{j,T}\right\} \in \mathcal{G}_{s_k}$

$$\mathbb{S}_T\left(\rho_{j,T}\right) \Rightarrow W, \quad \text{as } T \to \infty$$

for some random variable W, then

$$\mathbb{S}_T\left(\rho_T\right) \Rightarrow W, \quad \text{as } T \to \infty,$$

for all $\{\rho_T\} \in \mathcal{G}^*$.

Proof of Lemma SD-10:

To show (a), note that, by assumption, for any $\varepsilon > 0$ and for each $j \in \{1,..,d\}$, there exists a positive constant $K_{j,\varepsilon}$ and a positive integer L_j such that for all $T \geq L_j$

$$\Pr\left(\left|\frac{\mathbb{S}_T(\rho)}{\zeta(T)}\right| \geq K_{j,\varepsilon} | \rho = \rho_{j,T}\right) < \varepsilon.$$

Since, for $T \geq L_j$, $T_j = f_j(T) \geq T \geq L_j$ by Lemma SE-33 in Appendix SE below, we further deduce that

$$\Pr\left(\left|\frac{\mathbb{S}_{T_j}(\rho)}{\zeta(T_j)}\right| \geq K_{j,\varepsilon} | \rho = \rho_{j,T_j}\right) < \varepsilon.$$

for any subsequence $\left\{\rho_{j,T_j}\right\} \in \mathcal{G}_{s_j}$ and for all $j \in \{1,..,d\}$. Next, let $L^{\max} = \max\{f_1(L_1), ..., f_d(L_d)\}$ and $K_\varepsilon^{\max} = \max\{K_{1,\varepsilon}, ..., K_{d,\varepsilon}\}$. Consider any $T \geq L^{\max}$, and we must have

$$T = f_j(T^*),$$

for some $j = 1, ..., d$ and some $T^* \in \mathbb{N}$. By Lemma SE-33, we have that

$$T = f_j(T^*) \geq L^{\max} \geq f_j(L_j) \geq L_j,$$

from which we also deduce that $T^* \geq L_j$ since $f_j(\cdot)$ is a monotonically increasing function. Hence, for any $\{\rho_T\} \in \mathcal{G}^*$ and for all $T \geq L^{\max}$

$$\Pr\left(\left|\frac{\mathbb{S}_T(\rho)}{\zeta(T)}\right| \geq K_\varepsilon^{\max} | \rho = \rho_T\right) \leq \Pr\left(\left|\frac{\mathbb{S}_{f_j(T^*)}(\rho)}{\zeta(f_j(T^*))}\right| \geq K_{j,\varepsilon} | \rho = \rho_{f_j(T^*)}\right) < \varepsilon.$$

It follows that

$$\mathbb{S}_T(\rho_T) = O_p(\zeta(T))$$

for all $\{\rho_T\} \in \mathcal{G}^*$.

For part (b), note that, in this case, for any $\varepsilon > 0$ and for each $j \in \{1,..,d\}$, there exists a positive integer L_j such that for all $T \geq L_j$

$$\left|\Pr\left(\mathbb{S}_T(\rho) \leq x | \rho = \rho_{j,T}\right) - F_W(x)\right| < \varepsilon,$$

for each x which is a point of continuity of $F_W(\cdot)$, the cdf of W. Since, for $T \geq L_j$, $T_j = f_j(T) \geq T \geq L_j$ by Lemma SE-33 in Appendix SE below, we further deduce that

$$\left|\Pr\left(\mathbb{S}_{T_j}(\rho) \leq x | \rho = \rho_{j,T_j}\right) - F_W(x)\right| < \varepsilon,$$

for each x which is a point of continuity of $F_W(\cdot)$. Next, let $L^{\max} = \max\{f_1(L_1), ..., f_d(L_d)\}$. Consider any $T \geq L^{\max}$, and we must have

$$T = f_j(T^*),$$

for some $j = 1, ..., d$ and some $T^* \in \mathbb{N}$. Again, by Lemma SE-33, we have that

$$T = f_j(T^*) \geq L^{\max} \geq f_j(L_j) \geq L_j,$$

and, thus, $T \geq T^* \geq L_j$ from which it follows that, for any $\{\rho_T\} \in \mathcal{G}^*$ and for all $T \geq L^{\max}$,

$$\begin{aligned} & |\Pr(\mathbb{S}_T(\rho) \leq x | \rho = \rho_T) - F_W(x)| \\ = & \left|\Pr\left(\mathbb{S}_{f_j(T^*)}(\rho) \leq x | \rho = \rho_{f_j(T^*)}\right) - F_W(x)\right| \\ < & \varepsilon, \end{aligned}$$

for each x which is a point of continuity of $F_W(\cdot)$, as required. □

Lemma SD-11:

Let $\widehat{\rho}_{\text{pre}}$ be the preliminary estimator defined in footnote 4 of the main paper. Suppose that Assumptions 1-4 hold. Then, the following statements are true as $N, T \to \infty$ such that $N^\kappa / T = \tau$ for $\kappa \in \left(\frac{1}{2}, \infty\right)$ and $\tau \in (0, \infty)$.

(a) If $\rho_T = 1$ for all T sufficiently large, then

$$\widehat{\rho}_{\text{pre}} - \rho_T = O_p\left(\frac{1}{T\sqrt{N}}\right).$$

(b) If $\rho_T = \exp\{-1/q(T)\}$ such that $T/q(T) = O(1)$, then

$$\widehat{\rho}_{\text{pre}} - \rho_T = O_p\left(\frac{1}{T\sqrt{N}}\right).$$

(c) If $\rho_T = \exp\{-1/q(T)\}$ such that $T/(L(T))^2 \ll q(T) \ll T$, then

$$\widehat{\rho}_{\text{pre}} - \rho_T = O_p\left(\frac{1}{\sqrt{NTq(T)}}\right).$$

(d) If $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \to \infty$ but $q(T)(L(T))^2/T = O(1)$, then

$$\widehat{\rho}_{\text{pre}} - \rho_T = O_p\left(\frac{1}{\sqrt{NT}}\right).$$

(e) If $\rho_T \in \mathcal{G}_{\text{St}} = \left\{|\rho_T| = \exp\left\{-\frac{1}{q(T)}\right\} : q(T) \geq 0 \text{ and } q(T) = O(1) \text{ as } T \to \infty\right\}$, then

$$\widehat{\rho}_{\text{pre}} - \rho_T = O_p\left(\frac{1}{\sqrt{NT}}\right).$$

Proof of Lemma SD-11:

To proceed, we first consider part (a), where we assume that $\rho_T = 1$ for all T sufficiently large. In this case, by part (a) of Theorems SA-1 and SA-2, respectively, we have

$$\widehat{\rho}_{\text{IVD}} - 1 = O_p\left(\frac{1}{\sqrt{T}}\right) \text{ and } \widehat{\rho}_{\text{pols}} - 1 = O_p\left(\frac{1}{T\sqrt{N}}\right).$$

Moreover, in this case, there exists a positive integer I_ρ such that for all $T \geq I_\rho$

$$\begin{aligned}
\overline{\mathbb{T}} &= \sqrt{\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1} - \overline{y}_{-1,NT}\right)^2}\sqrt{N}T\left(\hat{\rho}_{\text{pols}} - 1\right) \\
&= \sqrt{\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1} - \overline{y}_{-1,NT}\right)^2}\sqrt{N}T\left(\hat{\rho}_{\text{pols}} - \rho_T\right).
\end{aligned}$$

Applying part (a) of Lemmas SD-3, Theorem SA-2, and the Cramér convergence theorem, we obtain

$$\left[\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1} - \overline{y}_{-1,NT}\right)^2\right]^{1/2}\sqrt{N}T\left(\hat{\rho}_{\text{pols}} - 1\right) \Rightarrow \sqrt{\frac{\sigma^2}{2}}\sqrt{2}\mathcal{Z} = \mathcal{Z} \equiv N\left(0, \sigma^2\right),$$

so that $\overline{\mathbb{T}} = O_p(1)$. It follows that in this case

$$\frac{\overline{\Delta}_{IC}}{\sqrt{N}L(T)} = \frac{\overline{\mathbb{T}} + \sqrt{N}L(T)}{\sqrt{N}L(T)} = 1 + \overline{\eta}_1(N,T),$$

where

$$\overline{\eta}_1(N,T) = \frac{\overline{\mathbb{T}}}{\sqrt{N}L(T)} = O_p\left(\frac{1}{\sqrt{N}L(T)}\right),$$

so that

$$\overline{w}_{IC} = \frac{1}{1 + \exp\left\{\frac{1}{2}\sqrt{N}L(T)\frac{\overline{\Delta}_{IC}}{\sqrt{N}L(T)}\right\}} = \exp\left\{-\frac{1}{2}\sqrt{N}L(T)\left[1 + \overline{\eta}_1(N,T)\right]\right\}\left[1 + o_p(1)\right].$$

Applying part (a) of Theorems SA-1 and SA-2, we obtain

$$\begin{aligned}
& \widehat{\rho}_{\text{pre}} - \rho_T \\
= \; & \overline{w}_{IC}\widehat{\rho}_{\text{IVD}} + \left(1 - \overline{w}_{IC}\right)\widehat{\rho}_{\text{pols}} - \rho_T \\
= \; & \overline{w}_{IC}\left(\widehat{\rho}_{\text{IVD}} - \rho_T\right) + \left(1 - \overline{w}_{IC}\right)\left(\widehat{\rho}_{\text{pols}} - \rho_T\right) \\
= \; & \exp\left\{-\frac{1}{2}\sqrt{N}L(T)\left[1 + \overline{\eta}_1(N,T)\right]\right\}\left(\widehat{\rho}_{\text{IVD}} - \rho_T\right)\left[1 + o_p(1)\right] + \left(\widehat{\rho}_{\text{pols}} - \rho_T\right) \\
& - \exp\left\{-\frac{1}{2}\sqrt{N}L(T)\left[1 + \overline{\eta}_1(N,T)\right]\right\}\left(\widehat{\rho}_{\text{pols}} - \rho_T\right)\left[1 + o_p(1)\right] \\
= \; & O_p\left(\exp\left\{-\frac{1}{2}\sqrt{N}L(T)\left[1 + \overline{\eta}_1(N,T) + \frac{\ln T}{\sqrt{N}L(T)}\right]\right\}\right) + O_p\left(\frac{1}{T\sqrt{N}}\right) \\
& + O_p\left(\exp\left\{-\frac{1}{2}\sqrt{N}L(T)\left[1 + \overline{\eta}_1(N,T) + \frac{2\ln T}{\sqrt{N}L(T)} + \frac{\ln N}{\sqrt{N}L(T)}\right]\right\}\right) \\
= \; & O_p\left(\frac{1}{T\sqrt{N}}\right).
\end{aligned}$$

Next, we consider part (b), where we take $\rho_T = \exp\{-1/q(T)\}$ such that $T/q(T) \to 0$. We break our analysis here into a number of different subcases:

Case b(i): $T\sqrt{N} \ll q(T)$.

In this case, by part (b) of Theorems SA-1 and SA-2, respectively, we have

$$\widehat{\rho}_{\text{IVD}} - \rho_T = O_p\left(\frac{1}{\sqrt{T}}\right) \text{ and } \widehat{\rho}_{\text{pols}} - \rho_T = O_p\left(\frac{1}{T\sqrt{N}}\right).$$

Moreover, write

$$\begin{aligned}
\overline{\mathbb{T}} &= \left[\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1} - \overline{y}_{-1,NT}\right)^2\right]^{1/2}\left(\hat{\rho}_{\text{pols}} - 1\right) \\
&= \sqrt{\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1} - \overline{y}_{-1,NT}\right)^2}\sqrt{N}T\left(\hat{\rho}_{\text{pols}} - \rho_T\right) \\
&\quad + \sqrt{\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1} - \overline{y}_{-1,NT}\right)^2}\sqrt{N}T\left(\rho_T - 1\right)
\end{aligned}$$

Applying part (b) of Lemmas SD-3, Theorem SA-2, and the Cramér convergence theorem, we obtain

$$\sqrt{\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1} - \overline{y}_{-1,NT}\right)^2}\sqrt{N}T\left(\hat{\rho}_{\text{pols}} - \rho_T\right) \Rightarrow \sqrt{\frac{\sigma^2}{2}}\sqrt{2}\mathcal{Z} = \mathcal{Z} \equiv N\left(0, \sigma^2\right)$$

From part (b) of Lemma SD-3, we also deduce that

$$\begin{aligned}
&\sqrt{\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1} - \overline{y}_{-1,NT}\right)^2}\sqrt{N}T\left(\rho_T - 1\right) \\
&= -\sqrt{\frac{\sigma^2}{2}}\frac{\sqrt{N}T}{q(T)}\left[1 + o_p(1)\right] = O_p\left(\frac{\sqrt{N}T}{q(T)}\right) = o_p(1).
\end{aligned}$$

Hence,

$$\overline{\mathbb{T}} = O_p(1) + o_p(1) = O_p(1).$$

The rest of the argument then follows in a manner similar to the proof of part (a) above. Hence, applying part (b) of Theorems SA-1 and SA-2 and following the derivation similar to that given for part (a), we also obtain in this case

$$\widehat{\rho}_{\text{pre}} - \rho_T = O_p\left(\frac{1}{T\sqrt{N}}\right).$$

Case b(ii): $q(T) \sim T\sqrt{N}$

For this case, by part (c) of Theorem SA-1, we have

$$\widehat{\rho}_{\text{IVD}} - \rho_T = O_p\left(\frac{1}{\sqrt{T}}\right),$$

and, by part (b) of Theorem SA-2, we again obtain

$$\widehat{\rho}_{\text{pols}} - \rho_T = O_p\left(\frac{1}{\sqrt{N}T}\right).$$

Applying part (b) of Lemma SD-3, we have

$$\begin{aligned} & \sqrt{\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1} - \overline{y}_{-1,NT}\right)^2}\sqrt{N}T\left(\rho_T - 1\right) \\ = & -\sqrt{\frac{\sigma^2}{2}}\frac{\sqrt{N}T}{q\left(T\right)}\left[1 + o_p\left(1\right)\right] = O_p\left(1\right), \end{aligned}$$

so that

$$\begin{aligned} \overline{\mathbb{T}} &= \sqrt{\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1} - \overline{y}_{-1,NT}\right)^2}\sqrt{N}T\left(\hat{\rho}_{\text{pols}} - \rho_T\right) \\ &\quad + \sqrt{\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1} - \overline{y}_{-1,NT}\right)^2}\sqrt{N}T\left(\rho_T - 1\right) \\ &= O_p\left(1\right) + O_p\left(1\right) = O_p\left(1\right). \end{aligned}$$

The rest of the argument is again similar to that of part (a), so that, by applying part (c) of Theorem SA-1 and part (b) of Theorem SA-2, we obtain

$$\widehat{\rho}_{\text{pre}} - \rho_T = O_p\left(\frac{1}{T\sqrt{N}}\right).$$

<u>Case b(iii):</u> $T \ll q\left(T\right) \ll T\sqrt{N}$

In this case, by part (d) of Theorem SA-1, we have

$$\widehat{\rho}_{\text{IVD}} - \rho_T = O_p\left(\frac{q\left(T\right)}{T^{3/2}\sqrt{N}}\right),$$

and, by part (b) of Theorem SA-2, we have.

$$\widehat{\rho}_{\text{pols}} - \rho_T = O_p\left(\frac{1}{\sqrt{N}T}\right).$$

Applying part (b) of Lemma SD-3 and part (b) of Theorem SA-2, we have

$$\frac{q\left(T\right)}{\sqrt{N}T}\overline{\mathbb{T}} = -\sqrt{\frac{\sigma^2}{2}} + o_p\left(1\right).$$

It follows that in this case

$$\frac{\overline{\Delta}_{IC}}{\sqrt{N}L\left(T\right)} = \frac{\overline{\mathbb{T}} + \sqrt{N}L\left(T\right)}{\sqrt{N}L\left(T\right)} = 1 + \overline{\eta}_1\left(N,T\right),$$

where

$$\overline{\eta}_1(N,T) = \frac{\overline{\mathbb{T}}}{\sqrt{N}L(T)} = \frac{T}{q(T)L(T)}\frac{q(T)}{\sqrt{N}T}\overline{\mathbb{T}} = O_p\left(\frac{T}{q(T)L(T)}\right) = o_p(1).$$

Now, write

$$\overline{w}_{IC} = \frac{1}{1+\exp\left\{\frac{1}{2}\sqrt{N}L(T)\frac{\overline{\Delta}_{IC}}{\sqrt{N}L(T)}\right\}} = \exp\left\{-\frac{1}{2}\sqrt{N}L(T)\left[1+\overline{\eta}_1(N,T)\right]\right\}\left[1+o_p(1)\right].$$

so that by applying part (d) of Theorems SA-1 and part (b) of Theorem SA-2, we obtain

$$\begin{aligned}
&\widehat{\rho}_{\text{pre}} - \rho_T \\
= \;& \overline{w}_{IC}\left(\widehat{\rho}_{\text{IVD}} - \rho_T\right) + \left(1-\overline{w}_{IC}\right)\left(\widehat{\rho}_{\text{pols}} - \rho_T\right) \\
= \;& \exp\left\{-\frac{1}{2}\sqrt{N}L(T)\left[1+\overline{\eta}_1(N,T)\right]\right\}\left(\widehat{\rho}_{\text{IVD}} - \rho_T\right)\left[1+o_p(1)\right] + \left(\widehat{\rho}_{\text{pols}} - \rho_T\right) \\
& - \exp\left\{-\frac{1}{2}\sqrt{N}L(T)\left[1+\overline{\eta}_1(N,T)\right]\right\}\left(\widehat{\rho}_{\text{pols}} - \rho_T\right)\left[1+o_p(1)\right] \\
= \;& O_p\left(\exp\left\{-\frac{1}{2}\sqrt{N}L(T)\left[1+\overline{\eta}_1(N,T)+\frac{3\ln T}{\sqrt{N}L(T)}+\frac{\ln N}{\sqrt{N}L(T)}-\frac{2\ln q(T)}{\sqrt{N}L(T)}\right]\right\}\right) \\
& + O_p\left(\frac{1}{T\sqrt{N}}\right) + O_p\left(\exp\left\{-\frac{1}{2}\sqrt{N}L(T)\left[1+\overline{\eta}_1(N,T)+\frac{2\ln T}{\sqrt{N}L(T)}+\frac{\ln N}{\sqrt{N}L(T)}\right]\right\}\right) \\
= \;& O_p\left(\frac{1}{T\sqrt{N}}\right).
\end{aligned}$$

<u>Case b(iv): $q(T) \sim T$</u>

Here, by part (e) of Theorem SA-1, we have

$$\widehat{\rho}_{\text{IVD}} - \rho_T = O_p\left(\frac{1}{\sqrt{NT}}\right),$$

and, by the proof of part (c) of Theorem SA-2, we have.

$$\widehat{\rho}_{\text{pols}} - \rho_T = O_p\left(\frac{1}{\sqrt{N}T}\right).$$

Moreover, applying part (c) Lemma SD-3 and Theorem SA-2, we obtain

$$\frac{q(T)}{\sqrt{N}T}\overline{\mathbb{T}} = -\frac{\sigma}{2}\frac{q(T)}{T}\sqrt{\exp\left\{-\frac{2T}{q(T)}\right\}+\frac{2T}{q(T)}-1}\left[1+o_p(1)\right] = O_p(1),$$

so that, similar to case b(iii) above,

$$\frac{\overline{\Delta}_{IC}}{\sqrt{N}L(T)} = \frac{\overline{\mathbb{T}}+\sqrt{N}L(T)}{\sqrt{N}L(T)} = 1+\overline{\eta}_1(N,T),$$

where

$$\overline{\eta}_1(N,T) = \frac{\overline{\mathbb{T}}}{\sqrt{N}L(T)} = \frac{T}{q(T)L(T)}\frac{q(T)}{\sqrt{N}T}\overline{\mathbb{T}} = O_p\left(\frac{1}{L(T)}\right) = o_p(1),$$

The rest of the argument follows similar to that given earlier, so that by applying part (e) of Theorem SA-1 along with part (c) of Theorem SA-2, we get

$$\begin{aligned}
& \widehat{\rho}_{\text{pre}} - \rho_T \\
= \; & \overline{w}_{IC} \left(\widehat{\rho}_{\text{IVD}} - \rho_T\right) + \left(1 - \overline{w}_{IC}\right) \left(\widehat{\rho}_{\text{pols}} - \rho_T\right) \\
= \; & \exp\left\{-\frac{1}{2}\sqrt{N} L(T) \left[1 + \overline{\eta}_1(N,T)\right]\right\} \left(\widehat{\rho}_{\text{IVD}} - \rho_T\right) \left[1 + o_p(1)\right] + \left(\widehat{\rho}_{\text{pols}} - \rho_T\right) \\
& - \exp\left\{-\frac{1}{2}\sqrt{N} L(T) \left[1 + \overline{\eta}_1(N,T)\right]\right\} \left(\widehat{\rho}_{\text{pols}} - \rho_T\right) \left[1 + o_p(1)\right] \\
= \; & O_p\left(\exp\left\{-\frac{1}{2}\sqrt{N} L(T) \left[1 + \overline{\eta}_1(N,T) + \frac{\ln T}{\sqrt{N} L(T)} + \frac{\ln N}{\sqrt{N} L(T)}\right]\right\}\right) + O_p\left(\frac{1}{T\sqrt{N}}\right) \\
& + O_p\left(\exp\left\{-\frac{1}{2}\sqrt{N} L(T) \left[1 + \overline{\eta}_1(N,T) + \frac{2\ln T}{\sqrt{N} L(T)} + \frac{\ln N}{\sqrt{N} L(T)}\right]\right\}\right) \\
= \; & O_p\left(\frac{1}{T\sqrt{N}}\right),
\end{aligned}$$

as required.

To complete the proof of part (b), note that for the pathwise asymptotics considered here N grows as a monotonically increasing function of T, so that the asymptotics employed here can be taken to be single-indexed with $T \to \infty$. Hence, we set $\mathbb{S}_T(\rho_T) = \widehat{\rho}_{\text{pre}} - \rho_T$ in Lemma SD-10 and apply part (a) of that lemma for the following collection of parameter sequences

$$\begin{aligned}
\mathcal{G}_1^{\rho} &= \left\{\rho_T = \exp\left\{-\frac{1}{q(T)}\right\} : \sqrt{N}T \sim T^{\frac{1}{2\kappa}+1} \ll q(T)\right\}, \\
\mathcal{G}_2^{\rho} &= \left\{\rho_T = \exp\left\{-\frac{1}{q(T)}\right\} : \sqrt{N}T \sim T^{\frac{1}{2\kappa}+1} \sim q(T)\right\}, \\
\mathcal{G}_3^{\rho} &= \left\{\rho_T = \exp\left\{-\frac{1}{q(T)}\right\} : T \ll q(T) \ll T^{\frac{1}{2\kappa}+1} \sim \sqrt{N}T\right\} \\
\mathcal{G}_4^{\rho} &= \left\{\rho_T = \exp\left\{-\frac{1}{q(T)}\right\} : q(T) \sim T\right\}.
\end{aligned}$$

Let

$$\mathcal{G}^* = \left\{\rho_T = \exp\left\{-\frac{1}{q(T)}\right\} : T/q(T) = O(1)\right\},$$

and note that, by the calculations given in subcases b(i)-b(iv) above and by part (a) of Lemma SD-10, we have that for every $\{\rho_T\} \in \mathcal{G}^*$,

$$\mathbb{S}_T(\rho_T) = \widehat{\rho}_{\text{pre}} - \rho_T = O_p\left(\frac{1}{T\sqrt{N}}\right),$$

which is the desired result.

Next, consider part (c), where we take $T/(L(T))^2 \ll q(T) \ll T$. From part (f) of Theorem SA-1, we see that in the case

$$\hat{\rho}_{\text{IVD}} - \rho_T = O_p\left(\frac{1}{\sqrt{NT}}\right).$$

Also, from the proof of part (d) of Theorem SA-2, we deduce that

$$\begin{aligned}\widehat{\rho}_{\text{pols}} &= \rho_T + O_p\left(\frac{1}{\sqrt{NTq(T)}}\right) \\ &= 1 - \frac{1}{q(T)} + O\left(\frac{1}{q(T)^2}\right) + O_p\left(\frac{1}{\sqrt{NTq(T)}}\right)\end{aligned}$$

or

$$q(T)\left(\hat{\rho}_{\text{pols}} - 1\right) = -1 + O\left(\frac{1}{q(T)}\right) + O_p\left(\sqrt{\frac{q(T)}{NT}}\right).$$

Using part (d) of Lemma SD-3 and part (d) of Theorem SA-2; it follows that in this case

$$\begin{aligned}\sqrt{\frac{q(T)}{NT}}\overline{\mathbb{T}} &= \sqrt{\frac{1}{NTq(T)}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1} - \overline{y}_{-1,NT}\right)^2 q(T)\left(\hat{\rho}_{\text{pols}} - 1\right)} \\ &= -\frac{\sigma}{\sqrt{2}} + o_p(1),\end{aligned}$$

so that

$$\frac{\overline{\Delta}_{IC}}{\sqrt{N}L(T)} = \frac{\overline{\mathbb{T}} + \sqrt{N}L(T)}{\sqrt{N}L(T)} = 1 + \overline{\eta}_1(N,T),$$

where

$$\overline{\eta}_1(N,T) = \frac{\overline{\mathbb{T}}}{\sqrt{N}L(T)} = \frac{1}{L(T)}\sqrt{\frac{T}{q(T)}}\sqrt{\frac{q(T)}{NT}}\overline{\mathbb{T}} = O_p\left(\frac{1}{L(T)}\sqrt{\frac{T}{q(T)}}\right) = o_p(1).$$

Now, write

$$\overline{w}_{IC} = \frac{1}{1 + \exp\left\{\frac{1}{2}\sqrt{N}L(T)\frac{\overline{\Delta}_{IC}}{\sqrt{N}L(T)}\right\}} = \exp\left\{-\frac{1}{2}\sqrt{N}L(T)\left[1 + \overline{\eta}_1(N,T)\right]\right\}\left[1 + o_p(1)\right].$$

Hence, by part (f) of Theorem SA-1 and part (d) of Theorem SA-2, we have

$$\begin{aligned}&\widehat{\rho}_{\text{pre}} - \rho_T \\ =\ & \overline{w}_{IC}\left(\widehat{\rho}_{\text{IVD}} - \rho_T\right) + \left(1 - \overline{w}_{IC}\right)\left(\widehat{\rho}_{\text{pols}} - \rho_T\right) \\ =\ & \exp\left\{-\frac{1}{2}\sqrt{N}L(T)\left[1 + \overline{\eta}_1(N,T)\right]\right\}\left(\widehat{\rho}_{\text{IVD}} - \rho_T\right)\left[1 + o_p(1)\right] + \left(\widehat{\rho}_{\text{pols}} - \rho_T\right) \\ & - \exp\left\{-\frac{1}{2}\sqrt{N}L(T)\left[1 + \overline{\eta}_1(N,T)\right]\right\}\left(\widehat{\rho}_{\text{pols}} - \rho_T\right)\left[1 + o_p(1)\right] \\ =\ & O_p\left(\exp\left\{-\frac{1}{2}\sqrt{N}L(T)\left[1 + \overline{\eta}_1(N,T) + \frac{\ln T}{\sqrt{N}L(T)} + \frac{\ln N}{\sqrt{N}L(T)}\right]\right\}\right) + O_p\left(\frac{1}{\sqrt{NTq(T)}}\right) \\ & + O_p\left(\exp\left\{-\frac{1}{2}\sqrt{N}L(T)\left[1 + \overline{\eta}_1(N,T) + \frac{\ln T}{\sqrt{N}L(T)} + \frac{\ln N}{\sqrt{N}L(T)} + \frac{\ln q(T)}{\sqrt{N}L(T)}\right]\right\}\right) \\ =\ & O_p\left(\frac{1}{\sqrt{NTq(T)}}\right).\end{aligned}$$

as required for part (c).

Now, for part (d), we consider the case where $q(T) \to \infty$ such that $q(T)(L(T))^2/T = O(1)$. It is helpful to break the analysis for this case into a number of subcases.

Case d(i): $q(T) \sim T/(L(T))^2$

Note that, in this case, by part (f) of Theorem SA-1, we have

$$\widehat{\rho}_{\text{IVD}} - \rho_T = O_p\left(\frac{1}{\sqrt{NT}}\right),$$

and, by the proof of part (d) of Theorem SA-2,

$$\begin{aligned}
\widehat{\rho}_{\text{pols}} &= \rho_T + O_p\left(\frac{1}{\sqrt{NTq(T)}}\right) \\
&= 1 - \frac{1}{q(T)} + O\left(\frac{1}{q(T)^2}\right) + O_p\left(\frac{1}{\sqrt{NTq(T)}}\right).
\end{aligned}$$

In addition, similar to part (c) above, we have

$$\sqrt{\frac{q(T)}{NT}}\overline{\mathbb{T}} = -\frac{\sigma}{\sqrt{2}} + o_p(1),$$

so that

$$\begin{aligned}
\frac{\overline{\Delta}_{IC}}{\sqrt{N}L(T)} &= \frac{\overline{\mathbb{T}} + \sqrt{N}L(T)}{\sqrt{N}L(T)} \\
&= 1 + \frac{1}{L(T)}\sqrt{\frac{T}{q(T)}}\sqrt{\frac{q(T)}{NT}}\overline{\mathbb{T}} \\
&= 1 - \frac{\sigma}{L(T)}\sqrt{\frac{T}{2q(T)}} + \overline{\eta}_2(N,T) \\
&= \xi(T) + \overline{\eta}_2(N,T),
\end{aligned}$$

where $\overline{\eta}_2(N,T) = o_p(1)$ and

$$\xi(T) = 1 - \frac{\sigma}{L(T)}\sqrt{\frac{T}{2q(T)}} = O(1).$$

Depending on the sequences $L(T)$ and $q(T)$, the sequence $\xi(T)$ could have positive or negative sign eventually or could be zero or have oscillating signs. Nevertheless, since $\overline{w}_{IC} \in [0,1]$ for all N, T; we have

$$\begin{aligned}
&\widehat{\rho}_{\text{pre}} - \rho_T \\
&= \overline{w}_{IC}(\widehat{\rho}_{\text{IVD}} - \rho_T) + (1 - \overline{w}_{IC})(\widehat{\rho}_{\text{pols}} - \rho_T) \\
&= O_p(1)\, O_p\left(\frac{1}{\sqrt{NT}}\right) + O_p(1)\, O_p\left(\frac{1}{\sqrt{NTq(T)}}\right) \\
&= O_p\left(\frac{1}{\sqrt{NT}}\right).
\end{aligned}$$

Case d(ii): $N^{1/3}T^{1/3} = T^{\frac{1+\kappa}{3\kappa}} \ll q(T)$ but $q(T)(L(T))^2/T \to 0$

To proceed, first write

$$\overline{\Delta}_{IC} = \overline{\mathbb{T}} + \sqrt{N}L(T) = \mathbb{T}^{\rho} + \overline{\theta} + \sqrt{N}L(T),$$

where

$$\begin{aligned}
\mathbb{T}^{\rho} &= \overline{M}_{yy}^{1/2}\sqrt{NTq(T)}\left(\widehat{\rho}_{\text{pols}} - \rho_T\right), \\
\overline{\theta} &= \overline{M}_{yy}^{1/2}\sqrt{NTq(T)}\left(\rho_T - 1\right), \\
\overline{M}_{yy} &= \frac{1}{NTq(T)}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1} - \overline{y}_{-1,NT}\right)^2.
\end{aligned}$$

In this case, applying part (d) of Theorem SA-2 and part (d) of Lemmas SD-3, we have

$$\begin{aligned}
\mathbb{T}^{\rho} &\Rightarrow N(0,1), \\
\overline{\theta} &= -\frac{\sigma}{\sqrt{2}}\sqrt{\frac{NT}{q(T)}}\left[1 + o_p(1)\right], \\
\sqrt{N}L(T) &= \sqrt{\frac{NT}{q(T)}}\sqrt{\frac{q(T)L(T)^2}{T}} = o\left(\sqrt{\frac{NT}{q(T)}}\right)
\end{aligned}$$

so that

$$\frac{\overline{\Delta}_{IC}}{\sqrt{NT/q(T)}} = \frac{\mathbb{T}^{\rho} + \overline{\theta} + \sqrt{N}L(T)}{\sqrt{NT/q(T)}} = -\frac{\sigma}{\sqrt{2}} + o_p(1).$$

Hence,

$$\begin{aligned}
\overline{w}_{IC} &= \frac{1}{1 + \exp\left\{\frac{1}{2}\sqrt{NT/q(T)}\frac{\overline{\Delta}_{IC}}{\sqrt{NT/q(T)}}\right\}} \\
&= \frac{1}{1 + \exp\left\{-\frac{\sigma}{2\sqrt{2}}\sqrt{NT/q(T)}\left[1 + o_p(1)\right]\right\}} \\
&= 1 + o_p(1).
\end{aligned}$$

It follows from applying the results of part (f) of Theorem SA-1 and part (d) of Theorem SA-2 that

$$\begin{aligned}
&\widehat{\rho}_{\text{pre}} - \rho_T \\
= \;& \overline{w}_{IC}\left(\widehat{\rho}_{\text{IVD}} - \rho_T\right) + \left(1 - \overline{w}_{IC}\right)\left(\widehat{\rho}_{\text{pols}} - \rho_T\right) \\
= \;& \left[1 + o_p(1)\right]O_p\left(\frac{1}{\sqrt{NT}}\right) + \left[1 - (1 + o_p(1))\right]O_p\left(\frac{1}{\sqrt{NTq(T)}}\right) \\
= \;& O_p\left(\frac{1}{\sqrt{NT}}\right).
\end{aligned}$$

Case d(iii): $q(T) \sim T^{\frac{1+\kappa}{3\kappa}} = N^{1/3}T^{1/3}$

In this case, by part (f) of Theorem SA-1, we have

$$\hat{\rho}_{\text{IVD}} - \rho_T = O_p\left(\frac{1}{\sqrt{NT}}\right).$$

Also, from the proof of part (e) of Theorem SA-2, we obtain

$$\widehat{\rho}_{\text{pols}} - \rho_T = O_p\left(\frac{1}{q(T)^2}\right) = O_p\left(\frac{1}{\sqrt{NTq(T)}}\right)$$

Next, write

$$\overline{\Delta}_{IC} = \overline{\mathbb{T}} + \sqrt{N}L(T) = \underline{\mathbb{T}} + \underline{\theta} + \sqrt{N}L(T),$$

where

$$\begin{aligned}
\underline{\mathbb{T}} &= \overline{M}_{yy}^{1/2}\sqrt{NTq(T)}\left(\widehat{\rho}_{\text{pols}} - \rho_T - \frac{2\sigma_a^2}{q(T)^2\sigma^2}\right), \\
\underline{\theta} &= \overline{M}_{yy}^{1/2}\sqrt{NTq(T)}\left(\rho_T + \frac{2\sigma_a^2}{q(T)^2\sigma^2} - 1\right), \\
\overline{M}_{yy} &= \frac{1}{NTq(T)}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1} - \overline{y}_{-1,NT}\right)^2.
\end{aligned}$$

In this case, applying part (e) of Theorem SA-2 along with part (d) of Lemmas SD-3, we have

$$\begin{aligned}
\underline{\mathbb{T}} &\Rightarrow N(0,1), \\
\underline{\theta} &= -\frac{\sigma}{\sqrt{2}}\sqrt{\frac{NT}{q(T)}}\left[1 + o_p(1)\right], \\
\sqrt{N}L(T) &= \sqrt{\frac{NT}{q(T)}}\sqrt{\frac{q(T)L(T)^2}{T}} = o\left(\sqrt{\frac{NT}{q(T)}}\right),
\end{aligned}$$

so that, similar to case d(ii) above,

$$\begin{aligned}
\frac{\overline{\Delta}_{IC}}{\sqrt{NT/q(T)}} &= \frac{\underline{\mathbb{T}}}{\sqrt{NT/q(T)}} + \frac{\underline{\theta}}{\sqrt{NT/q(T)}} + \frac{\sqrt{N}L(T)}{\sqrt{NT/q(T)}} \\
&= -\frac{\sigma}{\sqrt{2}} + o_p(1).
\end{aligned}$$

Hence,

$$\begin{aligned}
\overline{w}_{IC} &= \frac{1}{1 + \exp\left\{\frac{1}{2}\sqrt{NT/q(T)}\frac{\overline{\Delta}_{IC}}{\sqrt{NT/q(T)}}\right\}} \\
&= \frac{1}{1 + \exp\left\{-\frac{\sigma}{2\sqrt{2}}\sqrt{NT/q(T)}\left[1 + o_p(1)\right]\right\}} \\
&= 1 + o_p(1)
\end{aligned}$$

It follows from applying the results of part (f) of Theorem SA-1 and part (e) of Theorem SA-2 that

$$\begin{aligned}
& \widehat{\rho}_{\text{pre}} - \rho_T \\
= \; & \overline{w}_{IC}\left(\widehat{\rho}_{\text{IVD}} - \rho_T\right) + \left(1 - \overline{w}_{IC}\right)\left(\widehat{\rho}_{\text{pols}} - \rho_T\right) \\
= \; & \left[1 + o_p(1)\right] O_p\left(\frac{1}{\sqrt{NT}}\right) + \left[1 - (1 + o_p(1))\right] O_p\left(\frac{1}{q(T)^2}\right) \\
= \; & O_p\left(\frac{1}{\sqrt{NT}}\right) + o_p\left(\frac{1}{N^{2/3}T^{2/3}}\right) \\
= \; & O_p\left(\frac{1}{\sqrt{NT}}\right).
\end{aligned}$$

<u>Case d(iv): $q(T) \to \infty$ such that $q(T)/T^{\frac{1+\kappa}{3\kappa}} \to 0$</u>

To proceed, note that, by part (f) of Theorems SA-1 and SA-2, we have

$$\hat{\rho}_{\text{IVD}} - \rho_T = O_p\left(\frac{1}{\sqrt{NT}}\right) \text{ and } \widehat{\rho}_{\text{pols}} - \rho_T = O_p\left(\frac{1}{q(T)^2}\right).$$

Write

$$\overline{\Delta}_{IC} = \overline{\mathbb{T}} + \sqrt{N}L(T) = \underline{\mathbb{T}} + \underline{\theta} + \sqrt{N}L(T),$$

where

$$\begin{aligned}
\underline{\mathbb{T}} &= \overline{M}_{yy}^{1/2}\sqrt{NTq(T)}\left(\widehat{\rho}_{\text{pols}} - \rho_T - \frac{2\sigma_a^2}{q(T)^2\sigma^2}\right) \\
&= \overline{M}_{yy}^{1/2}\frac{\sqrt{NT}}{q(T)^{3/2}}q(T)^2\left(\widehat{\rho}_{\text{pols}} - \rho_T - \frac{2\sigma_a^2}{q(T)^2\sigma^2}\right) \\
\underline{\theta} &= \overline{M}_{yy}^{1/2}\sqrt{NTq(T)}\left(\rho_T + \frac{2\sigma_a^2}{q(T)^2\sigma^2} - 1\right), \\
\overline{M}_{yy} &= \frac{1}{NTq(T)}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1} - \overline{y}_{-1,NT}\right)^2.
\end{aligned}$$

In this case, applying part (f) of Theorem SA-2 along with part (d) of Lemma SD-3, we have

$$\begin{aligned}
\underline{\mathbb{T}} &= o_p\left(\frac{\sqrt{NT}}{q(T)^{3/2}}\right), \\
\underline{\theta} &= -\frac{\sigma}{\sqrt{2}}\sqrt{\frac{NT}{q(T)}}\left[1 + o_p(1)\right], \\
\sqrt{N}L(T) &= \sqrt{\frac{NT}{q(T)}}\sqrt{\frac{q(T)L(T)^2}{T}} = o\left(\sqrt{\frac{NT}{q(T)}}\right),
\end{aligned}$$

so that

$$\begin{aligned}
\frac{\overline{\Delta}_{IC}}{\sqrt{NT/q(T)}} &= \frac{\underline{\mathbb{T}}}{\sqrt{NT/q(T)}} + \frac{\underline{\theta}}{\sqrt{NT/q(T)}} + \frac{\sqrt{N}L(T)}{\sqrt{NT/q(T)}} \\
&= -\frac{\sigma}{\sqrt{2}}\left[1 + \overline{\eta}_3(N,T)\right],
\end{aligned}$$

where $\overline{\eta}_3(N,T) = o_p(1)$. Hence,

$$\begin{aligned} w_{IC} &= \frac{1}{1+\exp\left\{\frac{1}{2}\sqrt{NT/q(T)}\frac{\overline{\Delta}_{IC}}{\sqrt{NT/q(T)}}\right\}} \\ &= \frac{1}{1+\exp\left\{-\frac{\sigma}{2\sqrt{2}}\sqrt{NT/q(T)}\left[1+\overline{\eta}_3(N,T)\right]\right\}} \\ &= 1+O_p\left(\exp\left\{-\frac{\sigma}{2\sqrt{2}}\sqrt{NT/q(T)}\left[1+\overline{\eta}_3(N,T)\right]\right\}\right). \end{aligned}$$

It follows from applying the results of part (f) of Theorems SA-1 and SA-2 that

$$\begin{aligned} & \widehat{\rho}_{\text{pre}} - \rho_T \\ = \; & \overline{w}_{IC}\left(\widehat{\rho}_{\text{IVD}} - \rho_T\right) + \left(1-\overline{w}_{IC}\right)\left(\widehat{\rho}_{\text{pols}} - \rho_T\right) \\ = \; & \left[1+O_p\left(\exp\left\{-\frac{\sigma}{2\sqrt{2}}\sqrt{NT/q(T)}\left[1+\overline{\eta}_3(N,T)\right]\right\}\right)\right] O_p\left(\frac{1}{\sqrt{NT}}\right) \\ & + \left[1-\left(1+O_p\left(\exp\left\{-\frac{\sigma}{2\sqrt{2}}\sqrt{NT/q(T)}\left[1+\overline{\eta}_3(N,T)\right]\right\}\right)\right)\right] O_p\left(\frac{1}{q(T)^2}\right) \\ = \; & O_p\left(\frac{1}{\sqrt{NT}}\right) + O_p\left(\exp\left\{-\frac{\sigma}{2\sqrt{2}}\sqrt{NT/q(T)}\left[1+\overline{\eta}_3(N,T)+\frac{4\sqrt{2}}{\sigma}\sqrt{\frac{q(T)}{NT}}\ln q(T)\right]\right\}\right) \\ = \; & O_p\left(\frac{1}{\sqrt{NT}}\right). \end{aligned}$$

To complete the proof of part (d), again note that because N grows as monotonically increasing function of T, so that we can consider the asymptotics here as being single-indexed with $T \to \infty$. Hence, we set $\mathbb{S}_T(\rho_T) = \widehat{\rho}_{\text{pre}} - \rho_T$ in Lemma SD-10 and apply part (a) of that lemma for the following collection of parameter sequences

$$\begin{aligned} \mathcal{G}_5^\rho &= \left\{\rho_T = \exp\left\{-\frac{1}{q(T)}\right\} : \; q(T) \sim T/\left(L(T)\right)^2\right\}, \\ \mathcal{G}_6^\rho &= \left\{\rho_T = \exp\left\{-\frac{1}{q(T)}\right\} : N^{1/3}T^{1/3} = T^{\frac{1+\kappa}{3\kappa}} \ll q(T) \text{ but } q(T)\left(L(T)\right)^2/T \to 0\right\}, \\ \mathcal{G}_7^\rho &= \left\{\rho_T = \exp\left\{-\frac{1}{q(T)}\right\} : q(T) \sim T^{\frac{1+\kappa}{3\kappa}} = N^{1/3}T^{1/3}\right\} \\ \mathcal{G}_8^\rho &= \left\{\rho_T = \exp\left\{-\frac{1}{q(T)}\right\} : q(T) \to \infty \text{ such that } q(T)/T^{\frac{1+\kappa}{3\kappa}} \to 0\right\} \end{aligned}$$

Let

$$\mathcal{G}^* = \left\{\rho_T = \exp\left\{-\frac{1}{q(T)}\right\} : q(T) \to \infty \text{ such that } q(T)\left(L(T)\right)^2/T = O(1)\right\},$$

and note that, by the calculations given in subcases d(i)-d(iv) above and by part (a) of Lemma SD-10, we have that for every $\{\rho_T\} \in \mathcal{G}^*$,

$$\mathbb{S}_T(\rho_T) = \widehat{\rho}_{\text{pre}} - \rho_T = O_p\left(\frac{1}{\sqrt{NT}}\right),$$

which is the desired result.

Finally, we consider part (e), where we take $\rho_T \in \mathcal{G}_{\mathrm{St}}$. To proceed with this case, note first that from part (g) of Theorems SA-1 and SA-2, we have that

$$\hat{\rho}_{\mathrm{IVD}} - \rho_T = O_p\left(\frac{1}{\sqrt{NT}}\right) \text{ and } \hat{\rho}_{\mathrm{pols}} - \rho_T = \frac{\left(1-\rho_T^2\right)\left(1-\rho_T\right)\sigma_a^2}{\left(1-\rho_T^2\right)\sigma_a^2+\sigma^2}\left[1+o_p\left(1\right)\right].$$

Next, define

$$\begin{aligned}
\widetilde{M}_{yy} &= \frac{1-\rho_T^2}{NT}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1}-\overline{y}_{-1,NT}\right)^2, \\
\widetilde{\mathbb{T}} &= \widetilde{M}_{yy}^{1/2}\sqrt{\frac{NT}{1-\rho_T^2}}\left(\widehat{\rho}_{\mathrm{pols}}-\rho_T\right), \\
\widetilde{\theta} &= \widetilde{M}_{yy}^{1/2}\sqrt{\frac{NT}{1-\rho_T^2}}\left(\rho_T-1\right) = -\widetilde{M}_{yy}^{1/2}\sqrt{NT\frac{1-\rho_T}{1+\rho_T}}.
\end{aligned}$$

Using part (g) of Theorem SA-2 along with part (e) of Lemmas SD-3, we have in this case

$$\begin{aligned}
\widetilde{M}_{yy} &= \left(1-\rho_T^2\right)\sigma_a^2+\sigma^2+o_p\left(1\right), \\
\frac{\widetilde{\mathbb{T}}}{\sqrt{NT}} &= \widetilde{M}_{yy}^{1/2}\frac{\left(\widehat{\rho}_{\mathrm{pols}}-\rho_T\right)}{\sqrt{1-\rho_T^2}} \\
&= \sqrt{\frac{\left(1-\rho_T^2\right)\sigma_a^2+\sigma^2}{1-\rho_T^2}}\frac{\left(1-\rho_T^2\right)\left(1-\rho_T\right)\sigma_a^2}{\left(1-\rho_T^2\right)\sigma_a^2+\sigma^2}\left[1+o_p\left(1\right)\right] \\
&= \left(1-\rho_T\right)\sigma_a^2\sqrt{\frac{\left(1-\rho_T^2\right)}{\left(1-\rho_T^2\right)\sigma_a^2+\sigma^2}}\left[1+o_p\left(1\right)\right] \\
\frac{\widetilde{\theta}}{\sqrt{NT}} &= \widetilde{M}_{yy}^{1/2}\frac{\left(\rho_T-1\right)}{\sqrt{1-\rho_T^2}} = -\sqrt{\left(1-\rho_T^2\right)\sigma_a^2+\sigma^2}\sqrt{\frac{1-\rho_T}{1+\rho_T}}\left[1+o_p\left(1\right)\right].
\end{aligned}$$

Hence,

$$
\begin{aligned}
& \frac{\widetilde{\Delta}_{IC}}{\sqrt{NT}} \\
= & \frac{\widetilde{\mathbb{T}}}{\sqrt{NT}} + \frac{\widetilde{\theta}}{\sqrt{NT}} + \frac{\sqrt{N} L(T)}{\sqrt{NT}} \\
= & \left[(1-\rho_T)\sigma_a^2 \sqrt{\frac{(1-\rho_T^2)}{(1-\rho_T^2)\sigma_a^2+\sigma^2}} - \sqrt{(1-\rho_T^2)\sigma_a^2+\sigma^2}\sqrt{\frac{1-\rho_T}{1+\rho_T}} \right] [1+o_p(1)] \\
= & \frac{(1-\rho_T)\sigma_a^2\sqrt{(1+\rho_T)(1-\rho_T^2)} - \left[(1-\rho_T^2)\sigma_a^2+\sigma^2\right]\sqrt{1-\rho_T}}{\sqrt{(1+\rho_T)\left[(1-\rho_T^2)\sigma_a^2+\sigma^2\right]}} [1+o_p(1)] \\
= & -\frac{\sigma^2\sqrt{1-\rho_T}}{\sqrt{(1+\rho_T)\left[(1-\rho_T^2)\sigma_a^2+\sigma^2\right]}} [1+o_p(1)] \\
= & -\overline{\psi}\left(\rho_T, \sigma^2, \sigma_a^2\right)[1+o_p(1)] \qquad (12)
\end{aligned}
$$

where

$$
\overline{\psi}\left(\rho_T, \sigma^2, \sigma_a^2\right) = \frac{\sigma^2\sqrt{1-\rho_T}}{\sqrt{(1+\rho_T)\left[(1-\rho_T^2)\sigma_a^2+\sigma^2\right]}}.
$$

Next, note that in this case $|\rho_T| = \exp\left\{-\frac{1}{q(T)}\right\}$ with $q(T) = O(1)$, so that there exists a positive constant C_q and a positive integer T^* such that for all $T \geq T^*$

$$
|\rho_T| \leq \exp\left\{-\frac{1}{C_q}\right\} < 1.
$$

In consequence,

$$
\overline{\psi}\left(\rho_T, \sigma^2, \sigma_a^2\right) = \frac{\sigma^2\sqrt{1-\rho_T}}{\sqrt{(1+\rho_T)\left[(1-\rho_T^2)\sigma_a^2+\sigma^2\right]}} \geq \frac{\sigma^2\sqrt{1-\exp\{-1/C_q\}}}{\sqrt{2\left[\sigma_a^2+\sigma^2\right]}} > 0.
$$

Moreover, note that we can rewrite (12) as

$$
\frac{\overline{\Delta}_{IC}}{\sqrt{NT}} = -\overline{\psi}\left(\rho_T, \sigma^2, \sigma_a^2\right)\left[1+\overline{\eta}_4(N,T)\right],
$$

where $\overline{\eta}_4(N,T) = o_p(1)$, so that

$$
\begin{aligned}
\overline{w}_{IC} &= \frac{1}{1+\exp\left\{\frac{1}{2}\sqrt{NT}\frac{\overline{\Delta}_{IC}}{\sqrt{NT}}\right\}} \\
&= 1 + O_p\left(\exp\left\{-\frac{1}{2}\overline{\psi}\left(\rho_T, \sigma^2, \sigma_a^2\right)\sqrt{NT}\left[1+\overline{\eta}_4(N,T)\right]\right\}\right) \\
&= 1 + o_p(1).
\end{aligned}
$$

It follows from applying part (g) of Theorems SA-1 and SA-2 that

$$\begin{aligned}
& \widehat{\rho}_{\text{pre}} - \rho_T \\
= \; & \overline{w}_{IC}\left(\widehat{\rho}_{\text{IVD}} - \rho_T\right) + \left(1 - \overline{w}_{IC}\right)\left(\widehat{\rho}_{\text{pols}} - \rho_T\right) \\
= \; & \left[1 + O_p\left(\exp\left\{-\frac{1}{2}\overline{\psi}\left(\rho_T, \sigma^2, \sigma_a^2\right)\sqrt{NT}\left[1 + \overline{\eta}_4\left(N, T\right)\right]\right\}\right)\right] O_p\left(\frac{1}{\sqrt{NT}}\right) \\
& + \left[1 - 1 + O_p\left(\exp\left\{-\frac{1}{2}\overline{\psi}\left(\rho_T, \sigma^2, \sigma_a^2\right)\sqrt{NT}\left[1 + \overline{\eta}_4\left(N, T\right)\right]\right\}\right)\right] O_p\left(1\right) \\
= \; & O_p\left(\frac{1}{\sqrt{NT}}\right) + O_p\left(\exp\left\{-\frac{1}{2}\overline{\psi}\left(\rho_T, \sigma^2, \sigma_a^2\right)\sqrt{NT}\left[1 + \overline{\eta}_4\left(N, T\right)\right]\right\}\right) \\
= \; & O_p\left(\frac{1}{\sqrt{NT}}\right). \; \square
\end{aligned}$$

Lemma SD-12: Let

$$\begin{aligned}
\widehat{\sigma}^2 &= \frac{1}{N\left(T-1\right)}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it} - \overline{y}_i - \widehat{\rho}_{\text{pre}}\left[y_{it-1} - \overline{y}_{i,-1}\right]\right)^2, \\
\widetilde{\sigma}^2 &= \frac{1}{N\left(T-1\right)}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it} - \overline{y}_i - \widehat{\rho}_{\text{AIP}}\left[y_{it-1} - \overline{y}_{i,-1}\right]\right)^2
\end{aligned}$$

where $\widehat{\rho}_{\text{AIP}}$ are as defined by (1) and where $\widehat{\rho}_{\text{pre}}$ is as defined in footnote 4 of the main paper. Suppose that Assumptions 1-4 hold; then, as $N, T \to \infty$ such that $N^{\kappa}/T = \tau$, for $\kappa \in \left(\frac{1}{2}, \infty\right)$ and $\tau \in (0, \infty)$,

$$\begin{aligned}
\widehat{\sigma}^2 &= \sigma^2 + O_p\left(\max\left\{\frac{1}{N}, \frac{1}{T}\right\}\right) = \sigma^2 + O_p\left(\max\left\{\frac{1}{T^{1/\kappa}}, \frac{1}{T}\right\}\right), \\
\widetilde{\sigma}^2 &= \sigma^2 + O_p\left(\max\left\{\frac{1}{N}, \frac{1}{T}\right\}\right) = \sigma^2 + O_p\left(\max\left\{\frac{1}{T^{1/\kappa}}, \frac{1}{T}\right\}\right)
\end{aligned}$$

for every parameter sequences $\{\rho_T\} \in \mathcal{G}^* = \{\rho_T : -1 < \rho_T \leq 1 \text{ for all } T\}$.

Proof of Lemma SD-12:

We will only prove the result for $\widehat{\sigma}^2$ since the proof for $\widetilde{\sigma}^2$ follows in a similar manner. To proceed,

first write

$$
\begin{aligned}
\widehat{\sigma}^2 &= \frac{1}{N(T-1)}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it}-\overline{y}_i-\widehat{\rho}_{\text{pre}}\left[y_{it-1}-\overline{y}_{i,-1}\right]\right)^2 \\
&= \frac{1}{N(T-1)}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it}-\overline{y}_i-\rho_T\left[y_{it-1}-\overline{y}_{i,-1}\right]-\left[\widehat{\rho}_{\text{pre}}-\rho_T\right]\left[y_{it-1}-\overline{y}_{i,-1}\right]\right)^2 \\
&= \frac{1}{N(T-1)}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it}-\overline{y}_i-\rho_T\left[y_{it-1}-\overline{y}_{i,-1}\right]\right)^2 \\
&\quad -\frac{2\left[\widehat{\rho}_{\text{pre}}-\rho_T\right]}{N(T-1)}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it}-\overline{y}_i-\rho_T\left[y_{it-1}-\overline{y}_{i-}\right]\right)\left[y_{it-1}-\overline{y}_{i,-1}\right] \\
&\quad +\frac{\left[\widehat{\rho}_{\text{pre}}-\rho_T\right]^2}{N(T-1)}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it-1}-\overline{y}_{i,-1}\right)^2 \\
&= Q_{1,N,T}+2\left[\widehat{\rho}_{\text{pre}}-\rho_T\right]Q_{2,N,T}+\left[\widehat{\rho}_{\text{pre}}-\rho_T\right]^2 Q_{3,N,T},\ (say).
\end{aligned}
$$

Note that

$$
\begin{aligned}
& Q_{1,N,T} \\
&= \frac{1}{N(T-1)}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it}-\overline{y}_i-\rho_T\left[y_{it-1}-\overline{y}_{i,-1}\right]\right)^2 \\
&= \frac{1}{N(T-1)}\sum_{i=1}^{N}\sum_{t=2}^{T}\Big[a_i+w_{it}-\rho_T\left(a_i+w_{it-1}\right) \\
&\qquad\qquad -\frac{1}{T-1}\sum\nolimits_{s=2}^{T}\left(a_i+w_{is}\right)+\rho_T\frac{1}{T-1}\sum\nolimits_{s=2}^{T}\left(a_i+w_{is-1}\right)\Big]^2 \\
&= \frac{1}{N(T-1)}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(\varepsilon_{it}-\frac{1}{T-1}\sum_{s=2}^{T}\varepsilon_{is}\right)^2 \\
&= \frac{1}{N(T-1)}\sum_{i=1}^{N}\sum_{t=2}^{T}\varepsilon_{it}^2-\frac{1}{N}\sum_{i=1}^{N}\overline{\varepsilon}_i^2
\end{aligned}
$$

Next, note that

$$
\begin{aligned}
& Q_{2,N,T} \\
&= \frac{1}{N(T-1)}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(y_{it}-\overline{y}_i-\rho_T\left[y_{it-1}-\overline{y}_{i,-1}\right]\right)\left[y_{it-1}-\overline{y}_{i,-1}\right] \\
&= \frac{1}{N(T-1)}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(\varepsilon_{it}-\overline{\varepsilon}_i\right)\left(w_{it-1}-\overline{w}_{i,-1}\right) \\
&= \frac{1}{N(T-1)}\sum_{i=1}^{N}\sum_{t=2}^{T}w_{it-1}\varepsilon_{it}-\frac{1}{N}\sum_{i=1}^{N}\overline{\varepsilon}_i\overline{w}_{i,-1}.
\end{aligned}
$$

In addition,

$$
\begin{aligned}
& Q_{3,N,T} \\
= & \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} \left(y_{it-1} - \overline{y}_{i,-1}\right)^2 \\
= & \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} \left(a_i + w_{it-1} - \frac{1}{T-1} \sum_{s=2}^{T} (a_i + w_{is-1})\right)^2 \\
= & \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} \left(w_{it-1} - \frac{1}{T-1} \sum_{s=2}^{T} w_{is-1}\right)^2 \\
= & \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} \left[w_{it-1}^2 - 2w_{it-1} \frac{1}{T-1} \sum_{s=2}^{T} w_{is-1} + \left(\frac{1}{T-1} \sum_{s=2}^{T} w_{is-1}\right)^2\right] \\
= & \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1}^2 - \frac{1}{N} \sum_{i=1}^{N} \overline{w}_{i,-1}^2.
\end{aligned}
$$

Now, applying the results of parts (e) and (f) of Lemma SE-11, we get

$$
\begin{aligned}
& Q_{1,N,T} \\
= & \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} \left(y_{it} - \overline{y}_i - \rho_T \left[y_{it-1} - \overline{y}_{i,-1}\right]\right)^2 \\
= & \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} \varepsilon_{it}^2 - \frac{1}{N} \sum_{i=1}^{N} \overline{\varepsilon}_i^2 \\
= & \sigma^2 + O_p\left(\max\left\{\frac{1}{\sqrt{NT}}, \frac{1}{T}\right\}\right) + O_p\left(\frac{1}{T}\right) \\
= & \sigma^2 + O_p\left(\max\left\{\frac{1}{\sqrt{NT}}, \frac{1}{T}\right\}\right)
\end{aligned}
$$

We divide the rest of the proof into a number of cases.

<u>Case (i)</u>: $\rho_T = 1$ for all T sufficiently large.

In this case, we apply part (b) of Lemma SE-20 and part (a) of Lemma SE-29 to obtain

$$
\begin{aligned}
Q_{2,N,T} &= \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1}\varepsilon_{it} - \frac{1}{N} \sum_{i=1}^{N} \overline{\varepsilon}_i \overline{w}_{i,-1} \\
&= O_p\left(\frac{1}{\sqrt{N}}\right) + O_p(1) = O_p(1).
\end{aligned}
$$

In addition, by Lemma SE-14 and part (a) of Lemma SE-28, we get

$$
\begin{aligned}
Q_{3,N,T} &= \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1}^2 - \frac{1}{N} \sum_{i=1}^{N} \overline{w}_{i,-1}^2 \\
&= O_p(T) + O_p(T) = O_p(T).
\end{aligned}
$$

Furthermore, in this case,

$$\widehat{\rho}_{\text{pre}} - \rho_T = O_p\left(\frac{1}{\sqrt{N}T}\right),$$

by part (a) of Lemma SD-11. It follows that

$$\begin{aligned}
\widehat{\sigma}^2 &= Q_{1,N,T} + 2\left[\widehat{\rho}_{\text{pre}} - \rho_T\right] Q_{2,N,T} + \left[\widehat{\rho}_{\text{pre}} - \rho_T\right]^2 Q_{3,N,T} \\
&= \sigma^2 + O_p\left(\max\left\{\frac{1}{\sqrt{NT}}, \frac{1}{T}\right\}\right) + O_p\left(\frac{1}{\sqrt{N}T}\right) O_p(1) + O_p\left(\frac{1}{NT^2}\right) O_p(T) \\
&= \sigma^2 + O_p\left(\max\left\{\frac{1}{N}, \frac{1}{T}\right\}\right).
\end{aligned}$$

<u>Case (ii)</u>: $\rho_T = \exp\{-1/q(T)\}$ such that $T/q(T) \to 0$

Applying part (a) of Lemmas SE-20 and part (b) of Lemma SE-29 in this case, we obtain

$$\begin{aligned}
Q_{2,N,T} &= \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1}\varepsilon_{it} - \frac{1}{N} \sum_{i=1}^{N} \overline{\varepsilon}_i \overline{w}_{i,-1} \\
&= O_p\left(\frac{1}{\sqrt{N}}\right) + O_p(1) = O_p(1).
\end{aligned}$$

Moreover, applying Lemma SE-15 and part (b) of Lemma SE-28, we get

$$\begin{aligned}
Q_{3,N,T} &= \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1}^2 - \frac{1}{N} \sum_{i=1}^{N} \overline{w}_{i,-1}^2 \\
&= O_p(T) + O_p(T) = O_p(T).
\end{aligned}$$

In addition, from part (b) of Lemma SD-11, we have that in this case

$$\widehat{\rho}_{\text{pre}} - \rho_T = O_p\left(\frac{1}{\sqrt{N}T}\right).$$

It follows that

$$\begin{aligned}
\widehat{\sigma}^2 &= Q_{1,N,T} + 2\left[\widehat{\rho}_{\text{pre}} - \rho_T\right] Q_{2,N,T} + \left[\widehat{\rho}_{\text{pre}} - \rho_T\right]^2 Q_{3,N,T} \\
&= \sigma^2 + O_p\left(\max\left\{\frac{1}{\sqrt{NT}}, \frac{1}{T}\right\}\right) + O_p\left(\frac{1}{\sqrt{N}T}\right) O_p(1) + O_p\left(\frac{1}{NT^2}\right) O_p(T) \\
&= \sigma^2 + O_p\left(\max\left\{\frac{1}{N}, \frac{1}{T}\right\}\right).
\end{aligned}$$

<u>Case (iii)</u>: $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \sim T$

Here, the calculations are similar to case (ii) above, except that we apply part (a) of Lemma SE-27, part (c) of Lemma SE-29, part (a) of Lemma SE-17, part (c) of Lemma SE-28, and part (b) of Lemma SD-11 to obtain

$$\begin{aligned}
Q_{2,N,T} &= \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1}\varepsilon_{it} - \frac{1}{N} \sum_{i=1}^{N} \overline{\varepsilon}_i \overline{w}_{i,-1} = O_p(1), \\
Q_{3,N,T} &= \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1}^2 - \frac{1}{N} \sum_{i=1}^{N} \overline{w}_{i,-1}^2 = O_p(T), \\
\widehat{\rho}_{\text{pre}} - \rho_T &= O_p\left(\frac{1}{\sqrt{N}T}\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
\widehat{\sigma}^2 &= Q_{1,N,T} + 2\left[\widehat{\rho}_{\text{pre}} - \rho_T\right] Q_{2,N,T} + \left[\widehat{\rho}_{\text{pre}} - \rho_T\right]^2 Q_{3,N,T} \\
&= \sigma^2 + O_p\left(\max\left\{\frac{1}{\sqrt{NT}}, \frac{1}{T}\right\}\right) + O_p\left(\frac{1}{\sqrt{N}T}\right) O_p(1) + O_p\left(\frac{1}{NT^2}\right) O_p(T) \\
&= \sigma^2 + O_p\left(\max\left\{\frac{1}{N}, \frac{1}{T}\right\}\right).
\end{aligned}$$

<u>Case (iv)</u>: $\rho_T = \exp\{-1/q(T)\}$ such that $T/L(T)^2 \ll q(T) \ll T$

In this case, we apply part (b) of Lemma SE-27 and part (d) of Lemma SE-29 to obtain

$$\begin{aligned}
Q_{2,N,T} &= \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1}\varepsilon_{it} - \frac{1}{N} \sum_{i=1}^{N} \overline{\varepsilon}_i \overline{w}_{i,-1} \\
&= \sqrt{\frac{Tq(T)}{N(T-1)^2}} \frac{1}{\sqrt{NTq(T)}} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1}\varepsilon_{it} - \frac{1}{N} \sum_{i=1}^{N} \overline{\varepsilon}_i \overline{w}_{i,-1} \\
&= O_p\left(\sqrt{\frac{q(T)}{NT}}\right) + O_p\left(\frac{q(T)}{T}\right) = o_p\left(\sqrt{\frac{q(T)}{T}}\right).
\end{aligned}$$

Moreover, applying part (b) of Lemma SE-17 and part (d) of Lemma SE-28, we get

$$\begin{aligned}
Q_{3,N,T} &= \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1}^2 - \frac{1}{N} \sum_{i=1}^{N} \overline{w}_{i,-1}^2 \\
&= \frac{Tq(T)}{(T-1)} \frac{1}{NTq(T)} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1}^2 - \frac{1}{N} \sum_{i=1}^{N} \overline{w}_{i,-1}^2 \\
&= O_p(q(T)) + O_p\left(\frac{q(T)^2}{T}\right) = O_p(q(T)).
\end{aligned}$$

In addition, by part (c) of Lemma SD-11,

$$\widehat{\rho}_{\text{pre}} - \rho_T = O_p\left(\frac{1}{\sqrt{NTq(T)}}\right).$$

It follows that

$$\begin{aligned}
\widehat{\sigma}^2 &= Q_{1,N,T} + 2\left[\widehat{\rho}_{\text{pre}} - \rho_T\right] Q_{2,N,T} + \left[\widehat{\rho}_{\text{pre}} - \rho_T\right]^2 Q_{3,N,T} \\
&= \sigma^2 + O_p\left(\max\left\{\frac{1}{\sqrt{NT}}, \frac{1}{T}\right\}\right) + O_p\left(\frac{1}{\sqrt{NTq(T)}}\right) o_p\left(\sqrt{\frac{q(T)}{T}}\right) + O_p\left(\frac{1}{NTq(T)}\right) O_p(q(T)) \\
&= \sigma^2 + O_p\left(\max\left\{\frac{1}{N}, \frac{1}{T}\right\}\right).
\end{aligned}$$

<u>Case (v)</u>: $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \to \infty$ but $q(T)\,L(T)^2/T = O(1)$

Here, the calculations are similar to case (iv) above, except that, by part (d) of Lemma SD-11, we have

$$\widehat{\rho}_{\text{pre}} - \rho_T = O_p\left(\frac{1}{\sqrt{NT}}\right)$$

from which it follows that

$$\begin{aligned}
\widehat{\sigma}^2 &= Q_{1,N,T} + 2\left[\widehat{\rho}_{\text{pre}} - \rho_T\right] Q_{2,N,T} + \left[\widehat{\rho}_{\text{pre}} - \rho_T\right]^2 Q_{3,N,T} \\
&= \sigma^2 + O_p\left(\max\left\{\frac{1}{\sqrt{NT}}, \frac{1}{T}\right\}\right) + O_p\left(\frac{1}{\sqrt{NT}}\right) o_p\left(\sqrt{\frac{q(T)}{T}}\right) + O_p\left(\frac{1}{NT}\right) O_p\left(q(T)\right) \\
&= \sigma^2 + O_p\left(\max\left\{\frac{1}{N}, \frac{1}{T}\right\}\right).
\end{aligned}$$

<u>Case (vi)</u>: $\rho_T \in \mathcal{G}_{\text{St}} = \left\{|\rho_T| = \exp\left\{-\frac{1}{q(T)}\right\} : q(T) \geq 0 \text{ and } q(T) = O(1) \text{ as } T \to \infty\right\}$

In this case,

$$\begin{aligned}
& Q_{2,N,T} \\
&= \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1}\varepsilon_{it} - \frac{1}{N} \sum_{i=1}^{N} \overline{\varepsilon}_i \overline{w}_{i,-1} \\
&= \frac{\sqrt{NT}}{N(T-1)} \frac{1}{\sqrt{1-\rho_T^2}} \sqrt{\frac{1-\rho_T^2}{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1}\varepsilon_{it} - \frac{1}{N} \sum_{i=1}^{N} \overline{\varepsilon}_i \overline{w}_{i,-1} \\
&= \frac{1}{\sqrt{NT}} \frac{1}{\sqrt{1-\rho_T^2}} \sqrt{\frac{1-\rho_T^2}{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1}\varepsilon_{it} \left[1 + O\left(\frac{1}{T}\right)\right] - \frac{1}{N} \sum_{i=1}^{N} \overline{\varepsilon}_i \overline{w}_{i,-1}.
\end{aligned}$$

Here, we take $\rho_T^2 = \exp\left\{-\frac{2}{q(T)}\right\}$ with $q(T) = O(1)$, so that there exists a positive constant C_q and a positive integer T^* such that for all $T \geq T^*$

$$\rho_T^2 = \exp\left\{-\frac{2}{q(T)}\right\} \leq \exp\left\{-\frac{2}{C_q}\right\} < 1,$$

from which we deduce that for all $T \geq T^*$

$$\frac{1}{\sqrt{1-\rho_T^2}} \leq \frac{1}{\sqrt{1-\exp\{-2/C_q\}}} < \infty$$

Now, applying part (c) of Lemmas SE-27 and part (e) of Lemma SE-29, we obtain

$$\begin{aligned}
Q_{2,N,T} &= \frac{1}{\sqrt{NT}} \frac{1}{\sqrt{1-\rho_T^2}} \sqrt{\frac{1-\rho_T^2}{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1}\varepsilon_{it} \left[1 + O\left(\frac{1}{T}\right)\right] - \frac{1}{N} \sum_{i=1}^{N} \overline{\varepsilon}_i \overline{w}_{i,-1} \\
&= O_p\left(\frac{1}{\sqrt{NT}}\right) + O_p\left(\frac{1}{T}\right).
\end{aligned}$$

Moreover, write

$$\begin{aligned}
& Q_{3,N,T} \\
&= \frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1}^2 - \frac{1}{N} \sum_{i=1}^{N} \overline{w}_{i,-1}^2 \\
&= \frac{T}{T-1} \frac{1}{1-\rho_T^2} \frac{1-\rho_T^2}{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1}^2 - \frac{1}{N} \sum_{i=1}^{N} \overline{w}_{i,-1}^2,
\end{aligned}$$

and, using the upper bound above, we have that $T \geq T^*$

$$\frac{1}{1-\rho_T^2} \leq \frac{1}{1-\exp\{-2/C_q\}} < \infty.$$

Hence, applying part (c) of Lemma SE-17 and part (e) of Lemma SE-28, we get

$$\begin{aligned} & Q_{3,N,T} \\ = & \frac{T}{T-1}\frac{1}{1-\rho_T^2}\frac{1-\rho_T^2}{NT}\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-1}^2 - \frac{1}{N}\sum_{i=1}^{N}\overline{w}_{i,-1}^2 \\ = & O_p(1) + O_p\left(\frac{1}{T}\right) = O_p(1). \end{aligned}$$

In addition, in this case, we have by part (e) of Lemma SD-11,

$$\widehat{\rho}_{\text{pre}} - \rho_T = O_p\left(\frac{1}{\sqrt{NT}}\right).$$

It follows that

$$\begin{aligned} \widehat{\sigma}^2 &= Q_{1,N,T} + 2\left[\widehat{\rho}_{\text{pre}} - \rho_T\right] Q_{2,N,T} + \left[\widehat{\rho}_{\text{pre}} - \rho_T\right]^2 Q_{3,N,T} \\ &= \sigma^2 + O_p\left(\max\left\{\frac{1}{\sqrt{NT}}, \frac{1}{T}\right\}\right) + O_p\left(\frac{1}{\sqrt{NT}}\right) O_p\left(\max\left\{\frac{1}{\sqrt{NT}}, \frac{1}{T}\right\}\right) + O_p\left(\frac{1}{NT}\right) O_p(1) \\ &= \sigma^2 + O_p\left(\max\left\{\frac{1}{N}, \frac{1}{T}\right\}\right). \end{aligned}$$

To complete the proof of this lemma, note that, in the pathwise asymptotics considered here, N grows as a monotonically increasing function of T, so that the asymptotics here can be taken to be single-indexed with $T \to \infty$. Hence, we set $\mathbb{S}_T(\rho_T) = \widehat{\sigma}^2 - \sigma^2$ in Lemma SD-10 and apply part (a) of that lemma for the following collection of parameter sequences

$$\begin{aligned} \mathcal{G}_1^\sigma &= \{\rho_T : \ \rho_T = 1 \text{ for all } T \text{ sufficiently large}\}, \\ \mathcal{G}_2^\sigma &= \left\{\rho_T = \exp\left\{-\frac{1}{q(T)}\right\} : T/q(T) \to 0\right\}, \\ \mathcal{G}_3^\sigma &= \left\{\rho_T = \exp\left\{-\frac{1}{q(T)}\right\} : q(T) \sim T\right\}, \\ \mathcal{G}_4^\sigma &= \left\{\rho_T = \exp\left\{-\frac{1}{q(T)}\right\} : T/L(T)^2 \ll q(T) \ll T\right\}, \\ \mathcal{G}_5^\sigma &= \left\{\rho_T = \exp\left\{-\frac{1}{q(T)}\right\} : q(T) \to \infty \text{ but } q(T)\,L(T)^2/T = O(1)\right\} \\ \mathcal{G}_6^\sigma &= \mathcal{G}_{\text{St}} = \left\{|\rho_T| = \exp\left\{-\frac{1}{q(T)}\right\} : q(T) \geq 0 \text{ and } q(T) = O(1) \text{ as } T \to \infty\right\}. \end{aligned}$$

Note that the calculations given in cases (i)-(vi) above imply that

$$\widehat{\sigma}^2 = \sigma^2 + O_p\left(\max\left\{\frac{1}{N}, \frac{1}{T}\right\}\right) = \sigma^2 + O_p\left(\max\left\{\frac{1}{T^{1/\kappa}}, \frac{1}{T}\right\}\right),$$

under every sequence $\rho_T \in \mathcal{G}_\ell^\sigma$ and for every $\ell \in \{1, 2, .., 6\}$. Now, let $\left\{\rho_{j,T}\right\} \in \mathcal{G}_{s_j}^\sigma$ (for $j = 1, ..., 6$), i.e., $\left\{\rho_{j,T}\right\}$ is a sequence belonging to the collection $\mathcal{G}_{s_j}^\sigma$, where $s_j \in \{1, ..., 6\}$. In addition, define $T_j = f_j(T)$ $(j = 1, ..., d)$, with $d \leq 6$, where $f_j(\cdot) : \mathbb{N} \to \mathbb{N}$ is an increasing function in its argument, and let $\left\{\rho_{j,T_j}\right\}$ denote a subsequence of $\left\{\rho_{j,T}\right\}$. Furthermore, let

$$\mathcal{G}^* = \{\rho_T : -1 < \rho_T \leq 1 \text{ for all } T\},$$

and note that every parameter sequence $\rho_T \in \mathcal{G}^*$ can be represented as

$$\{\rho_T\} = \bigcup_{j=1}^{d} \left\{\rho_{j,T_j}\right\},$$

where

$$\left\{\rho_{1,T_1}\right\} \in \mathcal{G}_{s_1}^\sigma, ..., \left\{\rho_{d,T_d}\right\} \in \mathcal{G}_{s_d}^\sigma$$

with $\mathcal{G}_{s_k}^\sigma \neq \mathcal{G}_{s_\ell}^\sigma$ for $k \neq \ell$ and where

$$\mathbb{N} = \bigcup_{k=1}^{d} \{T_k = f_k(T) : T \in \mathbb{N}\}.$$

Hence, we can apply part (a) of Lemma SD-10 with $\zeta(T) = \max\left\{T^{-1/\kappa}, T^{-1}\right\}$ to conclude that for every sequence $\{\rho_T\} \in \mathcal{G}^*$,

$$\widehat{\sigma}^2 = \sigma^2 + O_p\left(\max\left\{\frac{1}{N}, \frac{1}{T}\right\}\right) = \sigma^2 + O_p\left(\max\left\{\frac{1}{T^{1/\kappa}}, \frac{1}{T}\right\}\right),$$

which is the desired result. □

Lemma SD-13:

Under Assumptions 1-4, the following statements are true as $N, T \to \infty$ such that $N^\kappa/T = \tau$ for $\kappa \in \left(\frac{1}{2}, \infty\right)$ and $\tau \in (0, \infty)$.

(a) If $\rho_T = 1$ for all T sufficiently large, then

$$\widehat{\omega}_{NT}^2 = 2\sigma^4 + O_p\left(\max\left\{\frac{1}{\sqrt{N}}, \frac{1}{\sqrt{T}}\right\}\right).$$

(b) If $\rho_T = \exp\{-1/q(T)\}$ such that $T/q(T) \to 0$, then

$$\widehat{\omega}_{NT}^2 = 2\sigma^4 + O_p\left(\max\left\{\frac{1}{\sqrt{N}}, \frac{1}{\sqrt{T}}, \frac{T}{q(T)}\right\}\right).$$

(c) If $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \sim T$, then

$$\widehat{\omega}_{NT}^2 = \sigma^4 + \frac{1}{2}\sigma^4 \frac{q(T)}{T}\left(1 - \exp\left\{-\frac{2T}{q(T)}\right\}\right) + O_p\left(\max\left\{\frac{1}{\sqrt{N}}, \frac{1}{\sqrt{T}}\right\}\right).$$

(d) If $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \to \infty$ but $q(T)/T \to 0$, then

$$\widehat{\omega}_{NT}^2 = \sigma^4 + O_p\left(\max\left\{\frac{1}{N}, \frac{1}{q(T)}, \frac{q(T)}{T}\right\}\right).$$

(e) If $\rho_T \in \mathcal{G}_{\mathrm{St}} = \left\{|\rho_T| = \exp\left\{-\frac{1}{q(T)}\right\} : q(T) \geq 0 \text{ and } q(T) = O(1) \text{ as } T \to \infty\right\}$, then

$$\widehat{\omega}_{NT}^2 = \frac{2\sigma^4}{1+\rho_T} + o_p(1).$$

Proof of Lemma SD-13:

To proceed, consider first part (a), where we take $\rho_T = 1$ for all T sufficiently large. In this case, we make use of part (a) of Lemmas SD-6 and SD-7 and Lemma SD-12 to obtain

$$\begin{aligned}
&\widehat{\omega}_{NT}^2 \\
= \;& \widetilde{\sigma}^2\left[\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=4}^{T}(y_{it-3}-y_{it-2})^2 + \frac{1}{NT}\sum_{i=1}^{N} y_{iT-2}^2\right] \\
= \;& \left[\sigma^2 + O_p\left(\max\left\{\frac{1}{N}, \frac{1}{T}\right\}\right)\right]\left[\sigma^2 + O_p\left(\max\left\{\frac{1}{\sqrt{NT}}, \frac{1}{T}\right\}\right) + \sigma^2 + O_p\left(\max\left\{\frac{1}{\sqrt{N}}, \frac{1}{\sqrt{T}}\right\}\right)\right] \\
= \;& 2\sigma^4 + O_p\left(\max\left\{\frac{1}{\sqrt{N}}, \frac{1}{\sqrt{T}}\right\}\right),
\end{aligned}$$

as required.

Next, consider part (b), where we take $\rho_T = \exp\{-1/q(T)\}$ such that $T/q(T) \to 0$. In this case, we apply part (b) of Lemmas SD-6 and SD-7 and Lemma SD-12 to obtain

$$\begin{aligned}
&\widehat{\omega}_{NT}^2 \\
= \;& \widetilde{\sigma}^2\left[\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=4}^{T}(y_{it-3}-y_{it-2})^2 + \frac{1}{NT}\sum_{i=1}^{N} y_{iT-2}^2\right] \\
= \;& \left[\sigma^2 + O_p\left(\max\left\{\frac{1}{N}, \frac{1}{T}\right\}\right)\right] \\
& \times\left[\sigma^2 + O_p\left(\max\left\{\frac{1}{\sqrt{NT}}, \frac{1}{T}\right\}\right) + \sigma^2 + O_p\left(\max\left\{\frac{1}{\sqrt{N}}, \frac{1}{\sqrt{T}}, \frac{T}{q(T)}\right\}\right)\right] \\
= \;& 2\sigma^4 + O_p\left(\max\left\{\frac{1}{\sqrt{N}}, \frac{1}{\sqrt{T}}, \frac{T}{q(T)}\right\}\right),
\end{aligned}$$

as required.

Now, consider part (c), where we take $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \sim T$. Here, applying

Lemmas part (c) of Lemmas SD-6 and SD-7 and Lemma SD-12, we have

$$\begin{aligned}
& \widehat{\omega}_{NT}^{2} \\
= \; & \widetilde{\sigma}^{2}\left[\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=4}^{T}\left(y_{it-3}-y_{it-2}\right)^{2}+\frac{1}{NT}\sum_{i=1}^{N}y_{iT-2}^{2}\right] \\
= \; & \left[\sigma^{2}+O_{p}\left(\max\left\{\frac{1}{N},\frac{1}{T}\right\}\right)\right]\left[\sigma^{2}+O_{p}\left(\max\left\{\frac{1}{\sqrt{NT}},\frac{1}{T}\right\}\right)\right] \\
& +\left[\sigma^{2}+O_{p}\left(\max\left\{\frac{1}{N},\frac{1}{T}\right\}\right)\right]\left[\frac{\sigma^{2}}{2}\frac{q\left(T\right)}{T}\left(1-\exp\left\{-\frac{2T}{q\left(T\right)}\right\}\right)+O_{p}\left(\max\left\{\frac{1}{\sqrt{N}},\frac{1}{\sqrt{T}}\right\}\right)\right] \\
= \; & \sigma^{4}+O_{p}\left(\max\left\{\frac{1}{N},\frac{1}{T}\right\}\right)+\frac{1}{2}\sigma^{4}\frac{q\left(T\right)}{T}\left(1-\exp\left\{-\frac{2T}{q\left(T\right)}\right\}\right)+O_{p}\left(\max\left\{\frac{1}{\sqrt{N}},\frac{1}{\sqrt{T}}\right\}\right) \\
= \; & \sigma^{4}+\frac{1}{2}\sigma^{4}\frac{q\left(T\right)}{T}\left(1-\exp\left\{-\frac{2T}{q\left(T\right)}\right\}\right)+O_{p}\left(\max\left\{\frac{1}{\sqrt{N}},\frac{1}{\sqrt{T}}\right\}\right),
\end{aligned}$$

as required.

We turn our attention to part (d), where we assume that $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \to \infty$ but $q(T)/T \to 0$. In this case, we apply part (d) of Lemmas SD-6 and SD-7 and Lemma SD-12 to obtain

$$\begin{aligned}
& \widehat{\omega}_{NT}^{2} \\
= \; & \widetilde{\sigma}^{2}\left[\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=4}^{T}\left(y_{it-3}-y_{it-2}\right)^{2}+\frac{1}{NT}\sum_{i=1}^{N}y_{iT-2}^{2}\right] \\
= \; & \left[\sigma^{2}+O_{p}\left(\max\left\{\frac{1}{N},\frac{1}{T}\right\}\right)\right] \\
& \times\left[\sigma^{2}+O_{p}\left(\max\left\{\frac{1}{\sqrt{NT}},\frac{1}{q\left(T\right)}\right\}\right)+O_{p}\left(\frac{q\left(T\right)}{T}\right)+O_{p}\left(\sqrt{\frac{q\left(T\right)}{NT}}\right)\right] \\
= \; & \sigma^{4}+O_{p}\left(\max\left\{\frac{1}{N},\frac{1}{q\left(T\right)},\frac{q\left(T\right)}{T}\right\}\right),
\end{aligned}$$

as required.

Finally, we consider part (e), where we assume that
$\rho_T \in \mathcal{G}_{\mathrm{St}} = \left\{|\rho_T| = \exp\left\{-\frac{1}{q(T)}\right\} : q(T) \geq 0 \text{ and } q(T) = O(1) \text{ as } T \to \infty\right\}$.
In this case, we apply part (e) of Lemmas SD-6 and SD-7 and Lemma SD-12 to obtain

$$\begin{aligned}
\widehat{\omega}_{NT}^{2} &= \widetilde{\sigma}^{2}\left[\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=4}^{T}\left(y_{it-3}-y_{it-2}\right)^{2}+\frac{1}{NT}\sum_{i=1}^{N}y_{iT-2}^{2}\right] \\
&= \left[\sigma^{2}+O_{p}\left(\max\left\{\frac{1}{N},\frac{1}{T}\right\}\right)\right]\left[\frac{2\sigma^{2}}{1+\rho_{T}}+o_{p}\left(1\right)+O_{p}\left(\frac{1}{T}\right)\right] \\
&= \frac{2\sigma^{4}}{1+\rho_{T}}+o_{p}\left(1\right). \quad \square
\end{aligned}$$

Appendix SE: Additional Lemmas and Technical Details

Lemma SE-1:

Let d and b both be some positive integer. The the following results hold as $T \to \infty$.

(a) If $T/q(T) \to 0$, then

$$\sum_{t=b}^{T}\sum_{j=1}^{t-b+1}\exp\left\{-d\frac{(t-b+1-j)}{q(T)}\right\} = \frac{T^2}{2}\left[1+O\left(\frac{1}{T}\right)+O\left(\frac{T}{q(T)}\right)\right].$$

(b) If $q(T) \sim T$, then

$$\sum_{t=b}^{T}\sum_{j=1}^{t-b+1}\exp\left\{-d\frac{(t-b+1-j)}{q(T)}\right\} = \frac{q(T)^2}{d^2}\left[\exp\left\{-\frac{dT}{q(T)}\right\}+\frac{dT}{q(T)}-1\right]\left[1+O\left(\frac{1}{T}\right)\right].$$

(c) If $q(T) \to \infty$ such that $q(T)/T \to 0$, then

$$\sum_{t=b}^{T}\sum_{j=1}^{t-b+1}\exp\left\{-d\frac{(t-b+1-j)}{q(T)}\right\} = \frac{Tq(T)}{d}\left[1+O\left(\frac{1}{q(T)}\right)+O\left(\frac{q(T)}{T}\right)\right].$$

Proof of Lemma SE-1:

To proceed, note first that for all $T \geq b$

$$\begin{aligned}
&\sum_{t=b}^{T}\sum_{j=1}^{t-b+1}\exp\left\{-d\frac{(t-b+1-j)}{q(T)}\right\} \\
=\ &\sum_{t=b}^{T}\frac{1-\exp\left\{-d\frac{t-b+1}{q(T)}\right\}}{1-\exp\left\{-\frac{d}{q(T)}\right\}} \\
=\ &\left[1-\exp\left\{-\frac{d}{q(T)}\right\}\right]^{-1}\left\{(T-b+1)-\sum_{t=b}^{T}\exp\left\{-d\left(\frac{t-b+1}{q(T)}\right)\right\}\right\} \\
=\ &\left[1-\exp\left\{-\frac{d}{q(T)}\right\}\right]^{-1}\left\{(T-b+1)-\exp\left\{-\frac{d}{q(T)}\right\}\sum_{t=b}^{T}\exp\left\{-d\left(\frac{t-b}{q(T)}\right)\right\}\right\} \\
=\ &\left[1-\exp\left\{-\frac{d}{q(T)}\right\}\right]^{-1} \\
&\times\left\{(T-b+1)-\exp\left\{-\frac{d}{q(T)}\right\}\left[1-\exp\left\{-\frac{d}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-d\left(\frac{T-b+1}{q(T)}\right)\right\}\right]\right\}.
\end{aligned}$$

For part (a), we consider the case $T/q(T) \to 0$. In this case, note that

$$
\begin{aligned}
& \sum_{t=b}^{T}\sum_{j=1}^{t-b+1}\exp\left\{-d\frac{(t-b+1-j)}{q(T)}\right\} \\
= & \left[1-\exp\left\{-\frac{d}{q(T)}\right\}\right]^{-1} \\
& \times\left\{(T-b+1)-\exp\left\{-\frac{d}{q(T)}\right\}\left[1-\exp\left\{-\frac{d}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-d\left(\frac{T-b+1}{q(T)}\right)\right\}\right]\right\} \\
= & \left[1-\exp\left\{-\frac{d}{q(T)}\right\}\right]^{-2}\left\{(T-b+1)\left[1-\exp\left\{-\frac{d}{q(T)}\right\}\right]\right. \\
& \left.-\exp\left\{-\frac{d}{q(T)}\right\}\left[1-\exp\left\{-d\frac{T}{q(T)}\right\}\exp\left\{\frac{d}{q(T)}(b-1)\right\}\right]\right\} \\
= & \left[1-1+\frac{d}{q(T)}+O\left(\frac{1}{q(T)^2}\right)\right]^{-2}(T-b+1)\left[\frac{d}{q(T)}-\frac{d^2}{2}\frac{1}{q(T)^2}+O\left(\frac{1}{q(T)^3}\right)\right] \\
& -\left[1-1+\frac{d}{q(T)}+O\left(\frac{1}{q(T)^2}\right)\right]^{-2}\left[1-\frac{d}{q(T)}+O\left(\frac{1}{q(T)^2}\right)\right] \\
& \quad\times\left(1-\left[1-\frac{dT}{q(T)}+\frac{d^2T^2}{2q(T)^2}+O\left(\frac{T^3}{q(T)^3}\right)\right]\left[1+\frac{d(b-1)}{q(T)}+O\left(\frac{1}{q(T)^2}\right)\right]\right) \\
= & \frac{q(T)^2}{d^2}\left[1+O\left(\frac{1}{q(T)}\right)\right]\left\{\frac{dT}{q(T)}-\frac{d(b-1)}{q(T)}-\frac{d^2}{2}\frac{T}{q(T)^2}+O\left(\frac{1}{q(T)^2}\right)\right. \\
& -\left[1-\frac{d}{q(T)}+O\left(\frac{1}{q(T)^2}\right)\right]\left[1-\left(1-\frac{dT}{q(T)}+\frac{d^2}{2}\frac{T^2}{q(T)^2}+\frac{d(b-1)}{q(T)}-\frac{d^2(b-1)T}{q(T)^2}\right.\right. \\
& \left.\left.\left.+O\left(\frac{T^3}{q(T)^3}\right)+O\left(\frac{1}{q(T)^2}\right)\right)\right]\right\} \\
= & \frac{q(T)^2}{d^2}\left[1+O\left(\frac{1}{q(T)}\right)\right]\left\{\frac{dT}{q(T)}-\frac{d(b-1)}{q(T)}-\frac{d^2}{2}\frac{T}{q(T)^2}+O\left(\frac{T}{q(T)^3}\right)-\right. \\
& \left[1-\frac{d}{q(T)}+O\left(\frac{1}{q(T)^2}\right)\right]\left(\frac{dT}{q(T)}-\frac{d^2T^2}{2q(T)^2}-\frac{d(b-1)}{q(T)}+\frac{d^2(b-1)T}{q(T)^2}\right. \\
& \left.\left.+O\left(\frac{T^3}{q(T)^3}\right)+O\left(\frac{1}{q(T)^2}\right)\right)\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{q(T)^2}{d^2}\left[1+O\left(\frac{1}{q(T)}\right)\right]\left\{\frac{dT}{q(T)}-\frac{d(b-1)}{q(T)}-\frac{d^2}{2}\frac{T}{q(T)^2}+O\left(\frac{T}{q(T)^3}\right)-\frac{dT}{q(T)}\right.\\
&\quad \left.+\frac{d^2}{2}\frac{T^2}{q(T)^2}+\frac{d(b-1)}{q(T)}-\frac{d^2(b-1)T}{q(T)^2}+\frac{d^2T}{q(T)^2}+O\left(\frac{T^3}{q(T)^3}\right)+O\left(\frac{1}{q(T)^2}\right)\right\}\\
&= \frac{q(T)^2}{d^2}\left[1+O\left(\frac{1}{q(T)}\right)\right]\left\{\frac{d^2}{2}\frac{T^2}{q(T)^2}-\left(\frac{2b-3}{2}\right)\frac{d^2T}{q(T)^2}+O\left(\frac{T^3}{q(T)^3}\right)+O\left(\frac{1}{q(T)^2}\right)\right\}\\
&= \left[\frac{T^2}{2}+O(T)+O\left(\frac{T^3}{q(T)}\right)+O(1)\right]\left[1+O\left(\frac{1}{q(T)}\right)\right]\\
&= \frac{T^2}{2}\left[1+O\left(\frac{1}{T}\right)+O\left(\frac{T}{q(T)}\right)+O\left(\frac{1}{T^2}\right)\right]\left[1+O\left(\frac{1}{q(T)}\right)\right]\\
&= \frac{T^2}{2}\left[1+O\left(\max\left\{\frac{1}{T},\frac{T}{q(T)}\right\}\right)\right]
\end{aligned}
$$

Next, consider part (b), where we take $q(T)\sim T$. In this case, we obtain by direct calculation

$$
\begin{aligned}
&\sum_{t=b}^{T}\sum_{j=1}^{t-b+1}\exp\left\{-d\frac{(t-b+1-j)}{q(T)}\right\}\\
&= \left[1-\exp\left\{-\frac{d}{q(T)}\right\}\right]^{-1}\\
&\quad \times\left\{(T-b+1)-\exp\left\{-\frac{d}{q(T)}\right\}\left[1-\exp\left\{-\frac{d}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-d\left(\frac{T-b+1}{q(T)}\right)\right\}\right]\right\}\\
&= \frac{q(T)}{d}\left[T-\left(\frac{q(T)}{d}\right)\left[1-\exp\left\{-\frac{dT}{q(T)}\right\}\exp\left\{\frac{d(b-1)}{q(T)}\right\}\right]\right]\left[1+O\left(\frac{1}{T}\right)\right]\\
&= \frac{q(T)}{d}\left[T+\frac{q(T)}{d}\left(\exp\left\{-\frac{dT}{q(T)}\right\}-1\right)+O(1)\right]\left[1+O\left(\frac{1}{T}\right)\right]\\
&= \frac{q(T)^2}{d^2}\left[\exp\left\{-\frac{dT}{q(T)}\right\}+\frac{dT}{q(T)}-1\right]\left[1+O\left(\frac{1}{T}\right)\right]\\
&= O\left(T^2\right),
\end{aligned}
$$

which shows the required result for part (b).

Finally, consider part (c). In this case, $q(T)\to\infty$ such that $q(T)/T\to 0$, and we have

$$
\begin{aligned}
& \sum_{t=b}^{T}\sum_{j=1}^{t-b+1}\exp\left\{-d\frac{(t-b+1-j)}{q(T)}\right\} \\
= & \left[1-\exp\left\{-\frac{d}{q(T)}\right\}\right]^{-1} \\
& \times\left\{(T-b+1)-\exp\left\{-\frac{d}{q(T)}\right\}\left[1-\exp\left\{-\frac{d}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-d\left(\frac{T-b+1}{q(T)}\right)\right\}\right]\right\} \\
= & \frac{q(T)}{d}\left[1+O\left(\frac{1}{q(T)}\right)\right] \\
& \times\left\{(T-b+1)-\exp\left\{-\frac{d}{q(T)}\right\}\left[1-\exp\left\{-\frac{d}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-d\left(\frac{T-b+1}{q(T)}\right)\right\}\right]\right\} \\
= & \frac{Tq(T)-q(T)(b-1)}{d}\left[1+O\left(\frac{1}{q(T)}\right)\right] \\
& -\frac{q(T)}{d}\frac{q(T)}{d}\left[1+O\left(\frac{1}{q(T)}\right)\right]\left[1-\exp\left\{-\frac{dT}{q(T)}\right\}\exp\left\{\frac{d(b-1)}{q(T)}\right\}\right] \\
= & \frac{Tq(T)-q(T)(b-1)}{d}\left[1+O\left(\frac{1}{q(T)}\right)\right]-\frac{q(T)^2}{d^2}\left(1-\exp\left\{-\frac{dT}{q(T)}\right\}\left[1+O\left(\frac{1}{q(T)}\right)\right]\right) \\
= & \frac{Tq(T)-q(T)(b-1)}{d}\left[1+O\left(\frac{1}{q(T)}\right)\right]-\frac{q(T)^2}{d^2}\left\{1+O\left(\exp\left\{-\frac{dT}{q(T)}\right\}\right)\right\} \\
= & \frac{Tq(T)}{d}+O(q(T))+O(T)+O\left(q(T)^2\right) \\
= & \frac{Tq(T)}{d}\left[1+O\left(\frac{1}{q(T)}\right)+O\left(\frac{q(T)}{T}\right)\right],
\end{aligned}
$$

as required. □

Lemma SE-2:

Let b be a positive integer and let g be a non-negative integer.

(a) If $T/q(T)\to 0$, then

$$
\sum_{t=g+1}^{T}\left[\sum_{j=1}^{t-g}\exp\left(-b\frac{(t-g-j)}{q(T)}\right)\right]^2=\frac{T^3}{3}\left[1+O\left(\frac{T}{q(T)}\right)+O\left(\frac{1}{T}\right)\right],
$$

as $T\to\infty$.

(b) If $q(T)\sim T$, then

$$
\begin{aligned}
& \sum_{t=g+1}^{T}\left[\sum_{j=1}^{t-g}\exp\left(-b\frac{(t-g-j)}{q(T)}\right)\right]^2 \\
= & \frac{q(T)^3}{2b^3}\left[\frac{2bT}{q(T)}+4\exp\left\{-\frac{bT}{q(T)}\right\}-\exp\left\{-\frac{2bT}{q(T)}\right\}-3\right]\left[1+O\left(\frac{1}{T}\right)\right],
\end{aligned}
$$

as $T \to \infty$.

(c) If $q(T) \to \infty$ such that $q(T)/T \to 0$, then

$$\sum_{t=g+1}^{T}\left[\sum_{j=1}^{t-g}\exp\left(-b\frac{(t-g-j)}{q(T)}\right)\right]^2 = \frac{Tq(T)^2}{b^2}\left[1+O\left(\frac{q(T)}{T}\right)\right],$$

as $T \to \infty$.

Proof of Lemma SE-2:

To proceed, note first that for all $T \geq g+1$, we have

$$\begin{aligned}
&\sum_{t=g+1}^{T}\left[\sum_{j=1}^{t-g}\exp\left\{-b\frac{(t-g-j)}{q(T)}\right\}\right]^2 \\
= &\left[1-\exp\left\{-\frac{b}{q(T)}\right\}\right]^{-2}\sum_{t=g+1}^{T}\left[1-\exp\left\{-b\frac{(t-g)}{q(T)}\right\}\right]^2 \\
= &\frac{q(T)^2}{b^2}\sum_{t=g+1}^{T}\left[1-2\exp\left\{-b\frac{(t-g)}{q(T)}\right\}+\exp\left\{-2b\frac{(t-g)}{q(T)}\right\}\right]\left[1+O\left(\frac{1}{q(T)}\right)\right] \\
= &\frac{q(T)^2}{b^2}\left[(T-g)-2\exp\left\{-\frac{b}{q(T)}\right\}\sum_{t=g+1}^{T}\exp\left\{-b\frac{(t-g-1)}{q(T)}\right\}\right. \\
&\left.+\exp\left\{-\frac{2b}{q(T)}\right\}\sum_{t=g+1}^{T}\exp\left\{-2b\frac{(t-g-1)}{q(T)}\right\}\right]\left[1+O\left(\frac{1}{q(T)}\right)\right] \\
= &\frac{q(T)^2}{b^2}\left[(T-g)-2\exp\left\{-\frac{b}{q(T)}\right\}\left(1-\exp\left\{-\frac{b}{q(T)}\right\}\right)^{-1}\left(1-\exp\left\{-b\frac{(T-g)}{q(T)}\right\}\right)\right. \\
&\left.+\exp\left\{-\frac{2b}{q(T)}\right\}\left(1-\exp\left\{-\frac{2b}{q(T)}\right\}\right)^{-1}\left(1-\exp\left\{-\frac{2b(T-g)}{q(T)}\right\}\right)\right]\left[1+O\left(\frac{1}{q(T)}\right)\right].
\end{aligned}$$

Now, consider part (a), where we take $T/q(T) \to 0$. Here, we have

$$
\begin{aligned}
& \sum_{t=g+1}^{T}\left[\sum_{j=1}^{t-g}\exp\left(-b\frac{(t-g-j)}{q(T)}\right)\right]^2 \\
= \; & \frac{q(T)^2}{b^2}\left[(T-g)-2\exp\left\{-\frac{b}{q(T)}\right\}\left(1-\exp\left\{-\frac{b}{q(T)}\right\}\right)^{-1}\left(1-\exp\left\{-\frac{bT}{q(T)}\right\}\exp\left\{\frac{gb}{q(T)}\right\}\right)\right. \\
& \left.+\exp\left\{-\frac{2b}{q(T)}\right\}\left(1-\exp\left\{-\frac{2b}{q(T)}\right\}\right)^{-1}\left(1-\exp\left\{-\frac{2bT}{q(T)}\right\}\exp\left\{\frac{2gb}{q(T)}\right\}\right)\right]\left[1+O\left(\frac{1}{q(T)}\right)\right] \\
= \; & \frac{q(T)^2}{b^2}\left\{(T-g)\right. \\
& -\frac{2q(T)}{b}\left(1-\left[1-\frac{bT}{q(T)}+\frac{b^2}{2}\frac{T^2}{q(T)^2}-\frac{1}{6}\frac{b^3T^3}{q(T)^3}+O\left(\frac{T^4}{q(T)^4}\right)\right]\left[1+\frac{gb}{q(T)}+O\left(\frac{1}{q(T)^2}\right)\right]\right) \\
& \times\left[1+O\left(\frac{1}{q(T)}\right)\right] \\
& +\frac{q(T)}{2b}\left(1-\left[1-\frac{2bT}{q(T)}+\frac{2b^2T^2}{q(T)^2}-\frac{8}{6}\frac{b^3T^3}{q(T)^3}+O\left(\frac{T^4}{q(T)^4}\right)\right]\left[1+\frac{2gb}{q(T)}+O\left(\frac{1}{q(T)^2}\right)\right]\right) \\
& \times\left[1+O\left(\frac{1}{q(T)}\right)\right] \\
= \; & \frac{q(T)^2}{b^2}\left\{(T-g)\right. \\
& -2\frac{q(T)}{b}\left[\frac{bT}{q(T)}-\frac{1}{2}\frac{b^2T^2}{q(T)^2}+\frac{1}{6}\frac{b^3T^3}{q(T)^3}-\frac{gb}{q(T)}+\frac{gb^2T}{q(T)^2}+O\left(\frac{T^4}{q(T)^4}\right)+O\left(\frac{T^2}{q(T)^3}\right)\right] \\
& \qquad\times\left[1+O\left(\frac{1}{q(T)}\right)\right] \\
& +\frac{q(T)}{2b}\left[\frac{2bT}{q(T)}-2\frac{b^2T^2}{q(T)^2}+\frac{4}{3}\frac{b^3T^3}{q(T)^3}-\frac{2gb}{q(T)}+\frac{4gb^2T}{q(T)^2}+O\left(\frac{T^4}{q(T)^4}\right)+O\left(\frac{T^2}{q(T)^3}\right)\right] \\
& \qquad\left.\times\left[1+O\left(\frac{1}{q(T)}\right)\right]\right\}
\end{aligned}
$$

$$
\begin{aligned}
= \; & \frac{q(T)^2}{b^2}\left\{(T-g)\right. \\
& -2\frac{q(T)}{b}\left[\frac{bT}{q(T)}-\frac{1}{2}\frac{b^2T^2}{q(T)^2}+\frac{1}{6}\frac{b^3T^3}{q(T)^3}-\frac{gb}{q(T)}+\frac{gb^2T}{q(T)^2}+O\left(\frac{T^4}{q(T)^4}\right)+O\left(\frac{T^2}{q(T)^3}\right)\right] \\
& \qquad\times\left[1+O\left(\frac{1}{q(T)}\right)\right] \\
& +\frac{q(T)}{2b}\left[\frac{2bT}{q(T)}-2\frac{b^2T^2}{q(T)^2}+\frac{4}{3}\frac{b^3T^3}{q(T)^3}-\frac{2gb}{q(T)}+\frac{4gb^2T}{q(T)^2}+O\left(\frac{T^4}{q(T)^4}\right)+O\left(\frac{T^2}{q(T)^3}\right)\right] \\
& \qquad\left.\times\left[1+O\left(\frac{1}{q(T)}\right)\right]\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{q(T)^2}{b^2}\left\{\left[(T-g)-2T+\frac{bT^2}{q(T)}-\frac{1}{3}\frac{b^2T^3}{q(T)^2}+2g-2\frac{gbT}{q(T)}+O\left(\frac{T^4}{q(T)^3}\right)+O\left(\frac{T^2}{q(T)^2}\right)\right]\right. \\
&\qquad \times\left[1+O\left(\frac{1}{q(T)}\right)\right] \\
&\quad \left.+\left[T-\frac{bT^2}{q(T)}+\frac{2}{3}\frac{b^2T^3}{q(T)^2}-g+2\frac{gbT}{q(T)}+O\left(\frac{T^4}{q(T)^3}\right)+O\left(\frac{T^2}{q(T)^2}\right)\right]\left[1+O\left(\frac{1}{q(T)}\right)\right]\right\} \\
&= \frac{q(T)^2}{b^2}\left[\frac{1}{3}\frac{b^2T^3}{q(T)^2}+O\left(\frac{T^4}{q(T)^3}\right)+O\left(\frac{T^2}{q(T)^2}\right)\right]\left[1+O\left(\frac{1}{q(T)}\right)\right] \\
&= \frac{T^3}{3}\left[1+O\left(\frac{T}{q(T)}\right)+O\left(\frac{1}{T}\right)\right],
\end{aligned}
$$

which completes the proof of part (a).

Next, consider part (b). In this case, $q(T)\sim T$, and we have

$$
\begin{aligned}
&\sum_{t=g+1}^{T}\left[\sum_{j=1}^{t-g}\exp\left(-b\frac{(t-g-j)}{q(T)}\right)\right]^2 \\
&= \frac{q(T)^2}{b^2}\left[(T-g)-2\exp\left\{-\frac{b}{q(T)}\right\}\left(1-\exp\left\{-\frac{b}{q(T)}\right\}\right)^{-1}\left(1-\exp\left\{-b\frac{(T-g)}{q(T)}\right\}\right)\right. \\
&\quad \left.+\exp\left\{-\frac{2b}{q(T)}\right\}\left(1-\exp\left\{-\frac{2b}{q(T)}\right\}\right)^{-1}\left(1-\exp\left\{-\frac{2b(T-g)}{q(T)}\right\}\right)\right]\left[1+O\left(\frac{1}{q(T)}\right)\right] \\
&= \frac{q(T)^2}{b^2}\left[(T-g)-2\exp\left\{-\frac{b}{q(T)}\right\}\left(1-\exp\left\{-\frac{b}{q(T)}\right\}\right)^{-1}\left(1-\exp\left\{-\frac{bT}{q(T)}\right\}\exp\left\{\frac{gb}{q(T)}\right\}\right)\right. \\
&\quad \left.+\exp\left\{-\frac{2b}{q(T)}\right\}\left(1-\exp\left\{-\frac{2b}{q(T)}\right\}\right)^{-1}\left(1-\exp\left\{-\frac{2bT}{q(T)}\right\}\exp\left\{\frac{2gb}{q(T)}\right\}\right)\right] \\
&\quad \times\left[1+O\left(\frac{1}{q(T)}\right)\right] \\
&= \frac{q(T)^2}{b^2}\left[(T-g)-2\frac{q(T)}{b}\left(1-\exp\left\{-\frac{bT}{q(T)}\right\}+O\left(\frac{1}{T}\right)\right)\left(1+O\left(\frac{1}{T}\right)\right)\right. \\
&\quad \left.+\frac{q(T)}{2b}\left(1-\exp\left\{-\frac{2bT}{q(T)}\right\}+O\left(\frac{1}{T}\right)\right)\left(1+O\left(\frac{1}{T}\right)\right)\right]\left[1+O\left(\frac{1}{T}\right)\right] \\
&= \frac{q(T)^3}{b^3}\left[\frac{bT}{q(T)}-2\left(1-\exp\left\{-\frac{bT}{q(T)}\right\}\right)+\frac{1}{2}\left(1-\exp\left\{-\frac{2bT}{q(T)}\right\}\right)\right]\left[1+O\left(\frac{1}{T}\right)\right] \\
&= \frac{q(T)^3}{b^3}\left[\frac{bT}{q(T)}+2\exp\left\{-\frac{bT}{q(T)}\right\}-\frac{1}{2}\exp\left\{-\frac{2bT}{q(T)}\right\}-\frac{3}{2}\right]\left[1+O\left(\frac{1}{T}\right)\right] \\
&= \frac{q(T)^3}{2b^3}\left[\frac{2bT}{q(T)}+4\exp\left\{-\frac{bT}{q(T)}\right\}-\exp\left\{-\frac{2bT}{q(T)}\right\}-3\right]\left[1+O\left(\frac{1}{T}\right)\right]
\end{aligned}
$$

as required.

Finally, consider part (c), where we take $q(T) \to \infty$ such that $q(T)/T \to 0$. In this case, we have

$$\begin{aligned}
& \sum_{t=g+1}^{T} \left[\sum_{j=1}^{t-g} \exp\left(-b \frac{(t-g-j)}{q(T)} \right) \right]^2 \\
= & \frac{q(T)^2}{b^2} \left\{ (T-g) - 2\frac{q(T)}{b} \left[1 + O\left(\frac{1}{q(T)} \right) \right] \left[1 + O\left(\exp\left\{ -b\frac{T}{q(T)} \right\} \right) \right] \right. \\
& \left. + \frac{q(T)}{2b} \left[1 + O\left(\frac{1}{q(T)} \right) \right] \left[1 + O\left(\exp\left\{ -2b\frac{T}{q(T)} \right\} \right) \right] \right\} \\
= & \frac{Tq(T)^2}{b^2} \left[1 + O\left(\frac{1}{T} \right) + O\left(\frac{q(T)}{T} \right) \right] \\
= & \frac{Tq(T)^2}{b^2} \left[1 + O\left(\frac{q(T)}{T} \right) \right],
\end{aligned}$$

as required. □

Lemma SE-3:

Let b and g be fixed positive integers; then, the following statements are true as $T \to \infty$.

(a) If $T/q(T) \to 0$, then

$$\begin{aligned}
& \sum_{j=1}^{T-g} \exp\left\{ -b \left(\frac{T-g-j}{q(T)} \right) \right\} \\
= & T \left[1 - \frac{g}{T} - \frac{b}{2}\frac{T}{q(T)} + O\left(\max\left\{ \frac{T^2}{q(T)^2}, \frac{1}{q(T)} \right\} \right) \right] \\
= & O(T)
\end{aligned}$$

(b) If $q(T) \sim T$, then

$$\sum_{j=1}^{T-g} \exp\left\{ -b \left(\frac{T-g-j}{q(T)} \right) \right\} = \frac{q(T)}{b} \left[1 - \exp\left\{ -\frac{bT}{q(T)} \right\} \right] \left[1 + O\left(\frac{1}{T} \right) \right].$$

(c) If $q(T) \to \infty$ such that $q(T)/T \to 0$, then

$$\sum_{j=1}^{T-g} \exp\left\{ -b \left(\frac{T-g-j}{q(T)} \right) \right\} = \frac{q(T)}{b} \left[1 + O\left(\frac{1}{q(T)} \right) \right] = O(q(T)).$$

Proof:

Focus first on part (a), where we consider the case in which $T/q(T) \to 0$. In this case,

$$
\begin{aligned}
& \sum_{j=1}^{T-g} \exp\left\{-b\left(\frac{T-g-j}{q(T)}\right)\right\} \\
= & \left[1-\exp\left\{-\frac{b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-b\left(\frac{T-g}{q(T)}\right)\right\}\right] \\
= & \left[1-\exp\left\{-\frac{b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{bT}{q(T)}\right\}\exp\left\{\frac{bg}{q(T)}\right\}\right] \\
= & \frac{q(T)}{b}\left[1+O\left(\frac{1}{q(T)}\right)\right] \\
& \times\left[1-\left(1-\frac{bT}{q(T)}+\frac{1}{2}\frac{b^2T^2}{q(T)^2}+O\left(\frac{T^3}{q(T)^3}\right)\right)\left(1+\frac{gb}{q(T)}+\frac{g^2b^2}{2!q(T)^2}+O\left(\frac{1}{q(T)^3}\right)\right)\right] \\
= & \frac{q(T)}{b}\left[1-\left(1-\frac{bT}{q(T)}+\frac{b^2T^2}{2q(T)^2}+\frac{gb}{q(T)}+O\left(\max\left\{\frac{T^3}{q(T)^3},\frac{T}{q(T)^2}\right\}\right)\right)\right] \\
= & \frac{q(T)}{b}\left[\frac{bT}{q(T)}-\frac{b^2T^2}{2q(T)^2}-\frac{gb}{q(T)}+O\left(\max\left\{\frac{T^3}{q(T)^3},\frac{T}{q(T)^2}\right\}\right)\right] \\
= & T-\frac{1}{2}\frac{bT^2}{q(T)}-g+O\left(\max\left\{\frac{T^3}{q(T)^2},\frac{T}{q(T)}\right\}\right) \\
= & T\left[1-\frac{g}{T}-\frac{b}{2}\frac{T}{q(T)}+O\left(\max\left\{\frac{T^2}{q(T)^2},\frac{1}{q(T)}\right\}\right)\right] \\
= & O(T),
\end{aligned}
$$

as required for part (a).

We now turn our attention to part (b), where we take $q(T) \sim T$. In this case,

$$
\begin{aligned}
= & \sum_{j=1}^{T-g} \exp\left\{-b\left(\frac{T-g-j}{q(T)}\right)\right\} \\
= & \left[1-\exp\left\{-\frac{b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{bT}{q(T)}\right\}\exp\left\{\frac{bg}{q(T)}\right\}\right] \\
= & \frac{q(T)}{b}\left[1+O\left(\frac{1}{T}\right)\right]\left[1-\exp\left\{-\frac{bT}{q(T)}\right\}\left(1+\frac{gb}{q(T)}+\frac{1}{2}\frac{g^2b^2}{q(T)^2}+O\left(\frac{1}{q(T)^3}\right)\right)\right] \\
= & \frac{q(T)}{b}\left[1-\exp\left\{-\frac{bT}{q(T)}\right\}-\frac{gb}{q(T)}\exp\left\{-\frac{bT}{q(T)}\right\}+O\left(\frac{1}{T^2}\right)\right]\left[1+O\left(\frac{1}{T}\right)\right] \\
= & \frac{q(T)}{b}\left[1-\exp\left\{-\frac{bT}{q(T)}\right\}\right]\left[1+O\left(\frac{1}{T}\right)\right] \\
= & O(T)
\end{aligned}
$$

Finally, we consider the case $q(T) \to \infty$ such that $q(T)/T \to 0$. In this case,

$$\begin{aligned}
& \sum_{j=1}^{T-2} \exp\left\{-b\left(\frac{T-g-j}{q(T)}\right)\right\} \\
= & \left[1-\exp\left\{-\frac{b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-b\left(\frac{T-g}{q(T)}\right)\right\}\right] \\
= & \frac{q(T)}{b}\left[1+O\left(\frac{1}{q(T)}\right)\right]\left[1+O\left(\exp\left\{-\frac{bT}{q(T)}\right\}\right)\right] \\
= & \frac{q(T)}{b}\left[1+O\left(\frac{1}{q(T)}\right)\right] \\
= & O(q(T)),
\end{aligned}$$

which gives the required result for part (c). □

Lemma SE-4:

Let d be a positive integer; then, the following statements are true as $T \to \infty$

(a) If $T/q(T) \to 0$, then

$$\begin{aligned}
& \sum_{r=2}^{T-2}\sum_{s=1}^{r-1} \exp\left\{-d\frac{T-2-r}{q(T)}\right\}\exp\left\{-d\frac{T-2-s}{q(T)}\right\} \\
= & \frac{T^2}{2}\left[1+O\left(\max\left\{\frac{T}{q(T)},\frac{1}{T}\right\}\right)\right] \\
= & O\left(T^2\right).
\end{aligned}$$

(b) If $q(T) \sim T$, then

$$\begin{aligned}
& \sum_{r=2}^{T-2}\sum_{s=1}^{r-1} \exp\left\{-d\frac{T-2-r}{q(T)}\right\}\exp\left\{-d\frac{T-2-s}{q(T)}\right\} \\
= & \frac{q(T)^2}{2d^2}\left[1-2\exp\left\{-\frac{dT}{q(T)}\right\}+\exp\left\{-\frac{2dT}{q(T)}\right\}\right]\left[1+O\left(\frac{1}{T}\right)\right] \\
= & O\left(T^2\right).
\end{aligned}$$

(c) If $q(T) \to \infty$ such that $q(T)/T \to 0$, then

$$\begin{aligned}
& \sum_{r=2}^{T-2}\sum_{s=1}^{r-1} \exp\left\{-d\frac{T-2-r}{q(T)}\right\}\exp\left\{-d\frac{T-2-s}{q(T)}\right\} \\
= & \frac{q(T)^2}{2d^2}\left[1+O\left(\frac{1}{q(T)}\right)+O\left(\exp\left\{-\frac{dT}{q(T)}\right\}\right)\right] \\
= & O\left(q(T)^2\right).
\end{aligned}$$

Proof of Lemma SE-4:

To proceed, note first that

$$
\begin{aligned}
& \sum_{r=2}^{T-2}\sum_{s=1}^{r-1}\exp\left\{-d\frac{T-2-r}{q(T)}\right\}\exp\left\{-d\frac{T-2-s}{q(T)}\right\} \\
= & \sum_{r=2}^{T-2}\exp\left\{-d\frac{T-2-r}{q(T)}\right\}\exp\left\{-d\frac{T-3}{q(T)}\right\}\sum_{s=1}^{r-1}\exp\left\{d\left(\frac{s-1}{q(T)}\right)\right\} \\
= & \left[\exp\left\{\frac{d}{q(T)}\right\}-1\right]^{-1}\sum_{r=2}^{T-2}\exp\left\{-d\frac{T-2-r}{q(T)}\right\}\exp\left\{-d\frac{T-3}{q(T)}\right\} \\
& \times\left[\exp\left\{d\left(\frac{r-1}{q(T)}\right)\right\}-1\right] \\
= & \left[\exp\left\{\frac{d}{q(T)}\right\}-1\right]^{-1}\sum_{r=2}^{T-2}\exp\left\{-2d\frac{T-2-r}{q(T)}\right\} \\
& -\left[\exp\left\{\frac{d}{q(T)}\right\}-1\right]^{-1}\exp\left\{-2d\frac{T}{q(T)}\right\}\exp\left\{\frac{7d}{q(T)}\right\}\sum_{r=2}^{T-2}\exp\left\{d\frac{r-2}{q(T)}\right\} \\
= & \left[\exp\left\{\frac{d}{q(T)}\right\}-1\right]^{-1}\sum_{r=2}^{T-2}\exp\left\{-2d\frac{T-2-r}{q(T)}\right\} \\
& -\left[\exp\left\{\frac{d}{q(T)}\right\}-1\right]^{-2}\exp\left\{-2d\frac{T}{q(T)}\right\}\exp\left\{\frac{7d}{q(T)}\right\}\left[\exp\left\{d\frac{T-3}{q(T)}\right\}-1\right] \\
= & \left[\exp\left\{\frac{d}{q(T)}\right\}-1\right]^{-1}\sum_{r=2}^{T-2}\exp\left\{-2d\frac{T-2-r}{q(T)}\right\} \\
& -\left[\exp\left\{\frac{d}{q(T)}\right\}-1\right]^{-2}\exp\left\{-d\frac{T}{q(T)}\right\}\exp\left\{\frac{4d}{q(T)}\right\} \\
& +\left[\exp\left\{\frac{d}{q(T)}\right\}-1\right]^{-2}\exp\left\{-2d\frac{T}{q(T)}\right\}\exp\left\{\frac{7d}{q(T)}\right\}
\end{aligned}
$$

Now, consider part (a) where we take $T/q(T)\to 0$. In this case, note that, applying part (a) of Lemma

SE-3 with $b = 2d$ and $g = 2$, we obtain

$$
\begin{aligned}
& \sum_{r=2}^{T-2}\sum_{s=1}^{r-1}\exp\left\{-d\frac{T-2-r}{q(T)}\right\}\exp\left\{-d\frac{T-2-s}{q(T)}\right\} \\
= & \left[\exp\left\{\frac{d}{q(T)}\right\}-1\right]^{-1}\sum_{r=1}^{T-2}\exp\left\{-2d\frac{T-2-r}{q(T)}\right\}-\left[\exp\left\{\frac{d}{q(T)}\right\}-1\right]^{-1}\exp\left\{-2d\frac{T-3}{q(T)}\right\} \\
& -\left[\exp\left\{\frac{d}{q(T)}\right\}-1\right]^{-2}\exp\left\{-d\frac{T}{q(T)}\right\}\exp\left\{\frac{4d}{q(T)}\right\} \\
& +\left[\exp\left\{\frac{d}{q(T)}\right\}-1\right]^{-2}\exp\left\{-2d\frac{T}{q(T)}\right\}\exp\left\{\frac{7d}{q(T)}\right\} \\
= & \frac{q(T)}{d}\left[1-\frac{d}{2q(T)}+O\left(\frac{1}{q(T)^2}\right)\right]T\left[1-\frac{2}{T}-d\frac{T}{q(T)}+O\left(\max\left\{\frac{T^2}{q(T)^2},\frac{1}{q(T)}\right\}\right)\right] \\
& -\frac{q(T)}{d}\left[1-\frac{d}{2q(T)}+O\left(\frac{1}{q(T)^2}\right)\right] \\
& \qquad\times\left[1-\frac{2dT}{q(T)}+\frac{6d}{q(T)}+\frac{2d^2T^2}{q(T)^2}+O\left(\max\left\{\frac{T^3}{q(T)^3},\frac{T}{q(T)^2}\right\}\right)\right] \\
& -\left[\frac{d}{q(T)}+\frac{d^2}{2}\frac{1}{q(T)^2}+\frac{1}{6}\frac{d^3}{q(T)^3}+O\left(\frac{1}{q(T)^4}\right)\right]^{-2} \\
& \times\left[1-d\frac{T}{q(T)}+\frac{d^2}{2}\frac{T^2}{q(T)^2}+O\left(\frac{T^3}{q(T)^3}\right)\right]\left[1+\frac{4d}{q(T)}+O\left(\frac{1}{q(T)^2}\right)\right] \\
& +\left[\frac{d}{q(T)}+\frac{d^2}{2}\frac{1}{q(T)^2}+\frac{1}{6}\frac{d^3}{q(T)^3}+O\left(\frac{1}{q(T)^4}\right)\right]^{-2} \\
& \times\left[1-\frac{2dT}{q(T)}+\frac{2d^2T^2}{q(T)^2}+O\left(\frac{T^3}{q(T)^3}\right)\right]\left[1+\frac{7d}{q(T)}+O\left(\frac{1}{q(T)^2}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{Tq(T)}{d} - 2\frac{q(T)}{d} - T^2 - \frac{q(T)}{d} + O\left(\max\left\{\frac{T^3}{q(T)}, T\right\}\right) \\
&\quad -\frac{q(T)^2}{d^2}\left[1 + \frac{d}{2q(T)} + \frac{d^2}{6q(T)^2} + O\left(\frac{1}{q(T)^3}\right)\right]^{-2} \\
&\quad \times\left[1 - d\frac{T}{q(T)} + \frac{4d}{q(T)} + \frac{d^2}{2}\frac{T^2}{q(T)^2} + O\left(\max\left\{\frac{T^3}{q(T)^3}, \frac{T}{q(T)^2}\right\}\right)\right] \\
&\quad +\frac{q(T)^2}{d^2}\left[1 + \frac{d}{2q(T)} + \frac{d^2}{6q(T)^2} + O\left(\frac{1}{q(T)^3}\right)\right]^{-2} \\
&\quad \times\left[1 - \frac{2dT}{q(T)} + \frac{7d}{q(T)} + 2\frac{d^2T^2}{q(T)^2} + O\left(\max\left\{\frac{T^3}{q(T)^3}, \frac{T}{q(T)^2}\right\}\right)\right] \\
&= \frac{Tq(T)}{d} - 2\frac{q(T)}{d} - T^2 - \frac{q(T)}{d} + O(T) \\
&\quad -\frac{q(T)^2}{d^2}\left[1 - \frac{d}{q(T)} + O\left(\frac{1}{q(T)^2}\right)\right] \\
&\qquad \times\left[1 - d\frac{T}{q(T)} + \frac{4d}{q(T)} + \frac{d^2}{2}\frac{T^2}{q(T)^2} + O\left(\max\left\{\frac{T^3}{q(T)^3}, \frac{T}{q(T)^2}\right\}\right)\right] \\
&\quad +\frac{q(T)^2}{d^2}\left[1 - \frac{d}{q(T)} + O\left(\frac{1}{q(T)^2}\right)\right] \\
&\qquad \times\left[1 - 2\frac{dT}{q(T)} + \frac{7d}{q(T)} + 2\frac{d^2T^2}{q(T)^2} + O\left(\max\left\{\frac{T^3}{q(T)^3}, \frac{T}{q(T)^2}\right\}\right)\right] \\
&= \frac{Tq(T)}{d} - 2\frac{q(T)}{d} - T^2 - \frac{q(T)}{d} - \frac{q(T)^2}{d^2} + \frac{q(T)}{d} - 4\frac{q(T)}{d} + \frac{Tq(T)}{d} - \frac{1}{2}T^2 \\
&\quad +\frac{q(T)^2}{d^2} - \frac{q(T)}{d} + 7\frac{q(T)}{d} - 2\frac{Tq(T)}{d} + 2T^2 + O\left(\max\left\{\frac{T^3}{q(T)}, T\right\}\right) \\
&= \left[-T^2 - \frac{1}{2}T^2 + 2T^2\right] + \left[\frac{Tq(T)}{d} + \frac{Tq(T)}{d} - 2\frac{Tq(T)}{d}\right] + \left[-\frac{q(T)^2}{d^2} + \frac{q(T)^2}{d^2}\right] \\
&\quad + \left[-2\frac{q(T)}{d} - \frac{q(T)}{d} + \frac{q(T)}{d} - 4\frac{q(T)}{d} - \frac{q(T)}{d} + 7\frac{q(T)}{d}\right] + O\left(\max\left\{\frac{T^3}{q(T)}, T\right\}\right) \\
&= \frac{T^2}{2} + O\left(\max\left\{\frac{T^3}{q(T)}, T\right\}\right) \\
&= \frac{T^2}{2}\left[1 + O\left(\max\left\{\frac{T}{q(T)}, \frac{1}{T}\right\}\right)\right]
\end{aligned}
$$

Next, consider part (b) where we take $q(T) \sim T$. In this case, we apply part (b) of Lemma SE-3

with $b = 2d$ and $g = 2$ to get

$$
\begin{aligned}
& \sum_{r=2}^{T-2}\sum_{s=1}^{r-1}\exp\left\{-d\frac{T-2-r}{T}\right\}\exp\left\{-d\frac{T-2-s}{T}\right\} \\
= & \left[\exp\left\{\frac{d}{q(T)}\right\}-1\right]^{-1}\sum_{r=1}^{T-2}\exp\left\{-2d\frac{T-2-r}{q(T)}\right\} \\
& -\left[\exp\left\{\frac{d}{q(T)}\right\}-1\right]^{-1}\exp\left\{-\frac{2dT}{q(T)}\right\}\exp\left\{\frac{6d}{q(T)}\right\} \\
& -\left[\exp\left\{\frac{d}{q(T)}\right\}-1\right]^{-2}\exp\left\{-\frac{dT}{q(T)}\right\}\exp\left\{\frac{4d}{q(T)}\right\} \\
& +\left[\exp\left\{\frac{d}{q(T)}\right\}-1\right]^{-2}\exp\left\{-\frac{2dT}{q(T)}\right\}\exp\left\{\frac{7d}{q(T)}\right\} \\
= & \frac{q(T)}{d}\left[1-\frac{d}{2q(T)}+O\left(\frac{1}{T^2}\right)\right]\frac{q(T)}{2d}\left[1-\exp\left\{-\frac{2dT}{q(T)}\right\}\right]\left[1+O\left(\frac{1}{T}\right)\right] \\
& -\frac{q(T)}{d}\left[1-\frac{d}{2q(T)}+O\left(\frac{1}{T^2}\right)\right]\exp\left\{-\frac{2dT}{q(T)}\right\}\left[1+\frac{6d}{q(T)}+O\left(\frac{1}{T^2}\right)\right] \\
& -\left[\frac{d}{q(T)}+\frac{d^2}{2q(T)^2}+O\left(\frac{1}{q(T)^3}\right)\right]^{-2}\exp\left\{-\frac{dT}{q(T)}\right\}\left[1+\frac{4d}{q(T)}+O\left(\frac{1}{T^2}\right)\right] \\
& +\left[\frac{d}{q(T)}+\frac{d^2}{2q(T)^2}+O\left(\frac{1}{q(T)^3}\right)\right]^{-2}\exp\left\{-\frac{2dT}{q(T)}\right\}\left[1+\frac{7d}{q(T)}+O\left(\frac{1}{T^2}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
= & \frac{q(T)^2}{2d^2}\left[1-\exp\left\{-\frac{2dT}{q(T)}\right\}\right]\left[1+O\left(\frac{1}{T}\right)\right] \\
& -\frac{q(T)^2}{d^2}\left[1+\frac{d}{2q(T)}+O\left(\frac{1}{T^2}\right)\right]^{-2}\exp\left\{-\frac{dT}{q(T)}\right\}\left[1+\frac{4d}{q(T)}+O\left(\frac{1}{T^2}\right)\right] \\
& +\frac{q(T)^2}{d^2}\left[1+\frac{d}{2q(T)}+O\left(\frac{1}{T^2}\right)\right]^{-2}\exp\left\{-\frac{2dT}{q(T)}\right\}\left[1+\frac{7d}{q(T)}+O\left(\frac{1}{T^2}\right)\right] \\
= & \frac{q(T)^2}{2d^2}\left[1-\exp\left\{-\frac{2dT}{q(T)}\right\}\right]\left[1+O\left(\frac{1}{T}\right)\right] \\
& -\frac{q(T)^2}{d^2}\left[1-\frac{d}{q(T)}+O\left(\frac{1}{T^2}\right)\right]\exp\left\{-\frac{dT}{q(T)}\right\}\left[1+\frac{4d}{q(T)}+O\left(\frac{1}{T^2}\right)\right] \\
& +\frac{q(T)^2}{d^2}\left[1-\frac{d}{q(T)}+O\left(\frac{1}{T^2}\right)\right]\exp\left\{-\frac{2dT}{q(T)}\right\}\left[1+\frac{7d}{q(T)}+O\left(\frac{1}{T^2}\right)\right] \\
= & \left(\frac{q(T)^2}{2d^2}\left[1-\exp\left\{-\frac{2dT}{q(T)}\right\}\right]+\frac{q(T)^2}{d^2}\left[\exp\left\{-\frac{2dT}{q(T)}\right\}-\exp\left\{-\frac{dT}{q(T)}\right\}\right]\right)\left[1+O\left(\frac{1}{T}\right)\right] \\
= & \frac{q(T)^2}{2d^2}\left[1-2\exp\left\{-\frac{dT}{q(T)}\right\}+\exp\left\{-\frac{2dT}{q(T)}\right\}\right]\left[1+O\left(\frac{1}{T}\right)\right]
\end{aligned}
$$

as required for part (b).

Finally, consider part (c) where we take $q(T) \to \infty$ such that $q(T)/T \to 0$. In this case, we apply part (c) of Lemma SE-3 with $b = 2d$ and $g = 2$ to obtain

$$\begin{aligned}
& \sum_{r=2}^{T-2}\sum_{s=1}^{r-1} \exp\left\{-d\frac{T-2-r}{q(T)}\right\}\exp\left\{-d\frac{T-2-s}{q(T)}\right\} \\
= & \left[\exp\left\{\frac{d}{q(T)}\right\}-1\right]^{-1}\sum_{r=1}^{T-2}\exp\left\{-2d\frac{T-2-r}{q(T)}\right\} \\
& -\left[\exp\left\{\frac{d}{q(T)}\right\}-1\right]^{-1}\exp\left\{-2d\frac{T-3}{q(T)}\right\} \\
& -\left[\exp\left\{\frac{d}{q(T)}\right\}-1\right]^{-2}\exp\left\{-d\frac{T}{q(T)}\right\}\exp\left\{\frac{4d}{q(T)}\right\} \\
& +\left[\exp\left\{\frac{d}{q(T)}\right\}-1\right]^{-2}\exp\left\{-2d\frac{T}{q(T)}\right\}\exp\left\{\frac{7d}{q(T)}\right\} \\
= & \frac{q(T)}{d}\left[1-\frac{d}{2q(T)}+O\left(\frac{1}{q(T)^2}\right)\right]\frac{q(T)}{2d}\left[1+O\left(\frac{1}{q(T)}\right)\right] \\
& +O\left(q(T)\exp\left\{-\frac{2dT}{q(T)}\right\}\right)+O\left(q(T)^2\exp\left\{-\frac{dT}{q(T)}\right\}\right)+O\left(q(T)^2\exp\left\{-\frac{2dT}{q(T)}\right\}\right) \\
= & \frac{q(T)^2}{2d^2}\left[1+O\left(\frac{1}{q(T)}\right)+O\left(\exp\left\{-\frac{dT}{q(T)}\right\}\right)\right] \\
= & O\left(q(T)^2\right),
\end{aligned}$$

as required, □

Lemma SE-5:

(a) For $r = 0, 1, .., T-2$

$$\begin{aligned}
& \sum_{t=r+2}^{T}(t-r-1)\exp\{x(t-r)\} \\
= & \frac{\exp\{2x\}-(T-r)\exp\{x(T-r+1)\}+(T-r-1)\exp\{x(T-r+2)\}}{(1-\exp\{x\})^2},
\end{aligned}$$

for $x \in \mathbb{R}$

(b) Let $r = 0, 1, ..., T-1$ and let α be a positive integer. In addition, suppose that $\rho \in [0, 1)$. Then,

$$\sum_{t=r}^{T}(t-r)\rho^{\alpha(t-r)} = \frac{\rho^{\alpha}-(T-r+1)\rho^{\alpha(T-r+1)}+(T-r)\rho^{\alpha(T-r+2)}}{(1-\rho^{\alpha})^2}$$

Proof of Lemma SE-5:

To show part (a), first set

$$\begin{aligned} f(x) &= \sum_{t=r+2}^{T} \exp\{x(t-r-1)\} = \sum_{t=r+1}^{T} \exp\{x(t-r-1)\} - 1 \\ &= \frac{1-\exp\{x(T-r)\}}{1-\exp\{x\}} - \frac{1-\exp\{x\}}{1-\exp\{x\}} = \frac{\exp\{x\}-\exp\{x(T-r)\}}{1-\exp\{x\}} \end{aligned}$$

Taking a derivative with respect to x, we get

$$\begin{aligned} & \sum_{t=r+2}^{T} (t-r-1)\exp\{x(t-r-1)\} = f'(x) \\ = & \frac{\exp\{x\}-(T-r)\exp\{x(T-r)\}}{1-\exp\{x\}} + \frac{(\exp\{x\}-\exp\{x(T-r)\})\exp\{x\}}{(1-\exp\{x\})^2} \\ = & (1-\exp\{x\})^{-2}[\exp\{x\}-\exp\{2x\}-(T-r)\exp\{x(T-r)\}+(T-r)\exp\{x(T-r+1)\} \\ & \qquad +\exp\{2x\}-\exp\{x(T-r+1)\}] \\ = & \frac{\exp\{x\}-(T-r)\exp\{x(T-r)\}+(T-r-1)\exp\{x(T-r+1)\}}{(1-\exp\{x\})^2}. \end{aligned}$$

It follows that

$$\begin{aligned} & \sum_{t=r+2}^{T} (t-r-1)\exp\{x(t-r)\} \\ = & \exp\{x\}\sum_{t=b+2}^{T} (t-r-1)\exp\{x(t-r-1)\} \\ = & \frac{\exp\{x\}[\exp\{x\}-(T-r)\exp\{x(T-r)\}+(T-r-1)\exp\{x(T-r+1)\}]}{(1-\exp\{x\})^2} \\ = & \frac{\exp\{2x\}-(T-r)\exp\{x(T-r+1)\}+(T-r-1)\exp\{x(T-r+2)\}}{(1-\exp\{x\})^2}. \quad \square \end{aligned}$$

Next, to show part (b), set

$$f(\rho) = \sum_{t=r}^{T} \rho^{\alpha(t-r)} = \frac{1-\rho^{\alpha(T-r+1)}}{1-\rho^{\alpha}}.$$

Taking a derivative with respect to ρ, we get

$$\begin{aligned}
& \sum_{t=r}^{n}(t-r)\,\rho^{\alpha(t-r)} \\
= \; & \frac{\rho}{\alpha}f'(\rho) \\
= \; & \frac{\rho}{\alpha}\left[\frac{\left(1-\rho^{\alpha(n-r+1)}\right)\alpha\rho^{\alpha-1}}{\left(1-\rho^{\alpha}\right)^{2}}-\frac{\alpha\,(n-r+1)\,\rho^{\alpha(n-r+1)-1}}{1-\rho^{\alpha}}\right] \\
= \; & \frac{\rho}{\alpha}\left[\frac{\alpha\rho^{\alpha-1}-\alpha\rho^{\alpha(n-r+2)-1}-\alpha\,(n-r+1)\,\rho^{\alpha(n-r+1)-1}\left(1-\rho^{\alpha}\right)}{\left(1-\rho^{\alpha}\right)^{2}}\right] \\
= \; & \frac{\rho}{\alpha}\left[\frac{\alpha\rho^{\alpha-1}-\alpha\rho^{\alpha(n-r+2)-1}-\alpha\,(n-r+1)\left(\rho^{\alpha(n-r+1)-1}-\rho^{\alpha(n-r+2)-1}\right)}{\left(1-\rho^{\alpha}\right)^{2}}\right] \\
= \; & \frac{\rho}{\alpha}\left[\frac{\alpha\rho^{\alpha-1}-\alpha\rho^{\alpha(n-r+2)-1}-\alpha\,(n-r+1)\,\rho^{\alpha(n-r+1)-1}+\alpha\,(n-r+1)\,\rho^{\alpha(n-r+2)-1}}{\left(1-\rho^{\alpha}\right)^{2}}\right] \\
= \; & \frac{\rho^{\alpha}-\rho^{\alpha(n-r+2)}-(n-r+1)\,\rho^{\alpha(n-r+1)}+(n-r+1)\,\rho^{\alpha(n-r+2)}}{\left(1-\rho^{\alpha}\right)^{2}} \\
= \; & \frac{\rho^{\alpha}-(n-r+1)\,\rho^{\alpha(n-r+1)}+(n-r)\,\rho^{\alpha(n-r+2)}}{\left(1-\rho^{\alpha}\right)^{2}},
\end{aligned}$$

as required. □

Lemma SE-6:

Let b be a positive integer, and $g \in \{1,2\}$.

(a) If $q(T) \to \infty$ such that $q(T)/T \to 0$, then

$$\begin{aligned}
& \sum_{t=g+2}^{T}\sum_{s=g+1}^{t-1}\sum_{j=1}^{s-g}\exp\left\{-b\frac{(t-g-j)}{q(T)}\right\}\exp\left\{-b\frac{(s-g-j)}{q(T)}\right\} \\
& \qquad\times\sum_{k=1}^{s-g}\exp\left\{-b\frac{(t-g-k)}{q(T)}\right\}\exp\left\{-b\frac{(s-g-k)}{q(T)}\right\} \\
= \; & \frac{1}{8}\frac{Tq(T)^{3}}{b^{3}}\left[1+O\left(\frac{1}{q(T)}\right)+O\left(\frac{q(T)}{T}\right)\right],
\end{aligned}$$

as $T \to \infty$.

(b) If $q(T) \sim T$, then

$$\begin{aligned}
&\sum_{t=g+2}^{T}\sum_{s=g+1}^{t-1}\sum_{j=1}^{s-g}\exp\left\{-b\frac{(t-g-j)}{q(T)}\right\}\exp\left\{-b\frac{(s-g-j)}{q(T)}\right\} \\
&\qquad\qquad \times\sum_{k=1}^{s-g}\exp\left\{-b\frac{(t-g-k)}{q(T)}\right\}\exp\left\{-b\frac{(s-g-k)}{q(T)}\right\} \\
=\;& \left\{\frac{q(T)^4}{32b^4}\left[\exp\left\{-\frac{4bT}{q(T)}\right\}+4(g-1)\exp\left\{-\frac{2bT}{q(T)}\right\}-5\right]+\frac{Tq(T)^3}{8b^3}\left[2\exp\left\{-\frac{2bT}{q(T)}\right\}+1\right]\right\} \\
&\times\left[1+O\left(\frac{1}{T}\right)\right],
\end{aligned}$$

as $T \to \infty$.

Proof of Lemma SE-6:

To proceed, note first that

$$\begin{aligned}
&\sum_{t=g+2}^{T}\sum_{s=g+1}^{t-1}\sum_{j=1}^{s-g}\exp\left\{-b\frac{(t-g-j)}{q(T)}\right\}\exp\left\{-b\frac{(s-g-j)}{q(T)}\right\} \\
&\qquad\qquad \times\sum_{k=1}^{s-g}\exp\left\{-b\frac{(t-g-k)}{q(T)}\right\}\exp\left\{-b\frac{(s-g-k)}{q(T)}\right\} \\
=\;& \sum_{t=g+2}^{T}\exp\left\{-\frac{2b(t-g)}{q(T)}\right\}\exp\left\{\frac{2b}{q(T)}\right\}\sum_{j=1}^{s-g}\exp\left\{\frac{2b}{q(T)}(j-1)\right\} \\
&\times\sum_{s=g+1}^{t-1}\exp\left\{-\frac{2b(s-g)}{q(T)}\right\}\exp\left\{\frac{2b}{q(T)}\right\}\sum_{k=1}^{s-g}\exp\left\{\frac{2b}{q(T)}(k-1)\right\} \\
=\;& \exp\left\{\frac{4b}{q(T)}\right\}\sum_{t=g+2}^{T}\exp\left\{-\frac{2b(t-g)}{q(T)}\right\}\sum_{j=1}^{s-g}\exp\left\{\frac{2b}{q(T)}(j-1)\right\} \\
&\times\sum_{s=g+1}^{t-1}\exp\left\{-\frac{2b(s-g)}{q(T)}\right\}\sum_{k=1}^{s-g}\exp\left\{\frac{2b}{q(T)}(k-1)\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \exp\left\{\frac{4b}{q(T)}\right\}\sum_{t=g+2}^{T}\sum_{s=g+1}^{t-1}\exp\left\{-\frac{2b(t-g)}{q(T)}\right\}\exp\left\{-\frac{2b(s-g)}{q(T)}\right\}\left[\sum_{j=1}^{s-g}\exp\left\{\frac{2b}{q(T)}(j-1)\right\}\right]^2 \\
&= \exp\left\{\frac{4b}{q(T)}\right\}\sum_{t=g+2}^{T}\sum_{s=g+1}^{t-1}\left[\exp\left\{-\frac{2b(t-g)}{q(T)}\right\}\exp\left\{-\frac{2b(s-g)}{q(T)}\right\}\right. \\
&\qquad\qquad \left.\times\left(\left[1-\exp\left\{\frac{2b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{\frac{2b}{q(T)}(s-g)\right\}\right]\right)^2\right] \\
&= \exp\left\{\frac{4b}{q(T)}\right\}\left[1-\exp\left\{\frac{2b}{q(T)}\right\}\right]^{-2} \\
&\quad \times\sum_{t=g+2}^{T}\exp\left\{-\frac{2b(t-g)}{q(T)}\right\} \\
&\quad \times\sum_{s=g+1}^{t-1}\exp\left\{-\frac{2b(s-g)}{q(T)}\right\}\left[1-2\exp\left\{\frac{2b}{q(T)}(s-g)\right\}+\exp\left\{\frac{4b}{q(T)}(s-g)\right\}\right] \\
&= \exp\left\{\frac{2b}{q(T)}\right\}\left[1-\exp\left\{\frac{2b}{q(T)}\right\}\right]^{-2}\sum_{t=g+2}^{T}\exp\left\{-2b\frac{(t-g)}{q(T)}\right\}\sum_{s=g+1}^{t-1}\exp\left\{-\frac{2b(s-g-1)}{q(T)}\right\} \\
&\quad -2\exp\left\{\frac{4b}{q(T)}\right\}\left[1-\exp\left\{\frac{2b}{q(T)}\right\}\right]^{-2}\sum_{t=g+2}^{T}\exp\left\{-2b\frac{(t-g)}{q(T)}\right\}(t-g-1) \\
&\quad +\exp\left\{\frac{6b}{q(T)}\right\}\left[1-\exp\left\{\frac{2b}{q(T)}\right\}\right]^{-2}\sum_{t=g+2}^{T}\exp\left\{-2b\frac{(t-g)}{q(T)}\right\}\sum_{s=g+1}^{t-1}\exp\left\{2\frac{b(s-g-1)}{q(T)}\right\} \\
&= \exp\left\{\frac{2b}{q(T)}\right\}\left[1-\exp\left\{\frac{2b}{q(T)}\right\}\right]^{-2}\left[1-\exp\left\{-\frac{2b}{q(T)}\right\}\right]^{-1} \\
&\quad \times\sum_{t=g+2}^{T}\exp\left\{-2b\frac{(t-g)}{q(T)}\right\}\left[1-\exp\left\{-2b\frac{(t-g-1)}{q(T)}\right\}\right] \\
&\quad -2\exp\left\{\frac{4b}{q(T)}\right\}\left[1-\exp\left\{\frac{2b}{q(T)}\right\}\right]^{-2}\left[1-\exp\left\{-\frac{2b}{q(T)}\right\}\right]^{-2}\exp\left\{-\frac{4b}{q(T)}\right\} \\
&\quad +2\exp\left\{\frac{4b}{q(T)}\right\}\left[1-\exp\left\{\frac{2b}{q(T)}\right\}\right]^{-2}\left[1-\exp\left\{-\frac{2b}{q(T)}\right\}\right]^{-2}(T-g)\exp\left\{-\frac{2b(T-g+1)}{q(T)}\right\} \\
&\quad -2\exp\left\{\frac{4b}{q(T)}\right\}\left[1-\exp\left\{\frac{2b}{q(T)}\right\}\right]^{-2}\left[1-\exp\left\{-\frac{2b}{q(T)}\right\}\right]^{-2} \\
&\quad \times(T-g-1)\exp\left\{-\frac{2b(T-g+2)}{q(T)}\right\} \\
&\quad +\exp\left\{\frac{6b}{q(T)}\right\}\left[1-\exp\left\{\frac{2b}{q(T)}\right\}\right]^{-3}\sum_{t=g+2}^{T}\exp\left\{-\frac{2b(t-g)}{q(T)}\right\}\left[1-\exp\left\{\frac{2b(t-g-1)}{q(T)}\right\}\right]
\end{aligned}
$$

where the last equality above follows from part (a) of Lemma SE-5. It follows by further calculation

that

$$
\begin{aligned}
&\sum_{t=g+2}^{T}\sum_{s=g+1}^{t-1}\sum_{j=1}^{s-g}\exp\left\{-b\frac{(t-g-j)}{q(T)}\right\}\exp\left\{-b\frac{(s-g-j)}{q(T)}\right\} \\
&\qquad\times\sum_{k=1}^{s-g}\exp\left\{-b\frac{(t-g-k)}{q(T)}\right\}\exp\left\{-b\frac{(s-g-k)}{q(T)}\right\} \\
=\;&\exp\left\{-\frac{2b}{q(T)}\right\}\left[1-\exp\left\{\frac{2b}{q(T)}\right\}\right]^{-2}\left[1-\exp\left\{-\frac{2b}{q(T)}\right\}\right]^{-2}\left[1-\exp\left\{-\frac{2b(T-g-1)}{q(T)}\right\}\right] \\
&-\exp\left\{-\frac{4b}{q(T)}\right\}\left[1-\exp\left\{\frac{2b}{q(T)}\right\}\right]^{-2}\left[1-\exp\left\{-\frac{2b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{4b}{q(T)}\right\}\right]^{-1} \\
&\times\left[1-\exp\left\{-\frac{4b(T-g-1)}{q(T)}\right\}\right] \\
&-2\exp\left\{\frac{4b}{q(T)}\right\}\left[1-\exp\left\{\frac{2b}{q(T)}\right\}\right]^{-2}\left[1-\exp\left\{-\frac{2b}{q(T)}\right\}\right]^{-2}\exp\left\{-\frac{4b}{q(T)}\right\} \\
&+2\exp\left\{\frac{4b}{q(T)}\right\}\left[1-\exp\left\{\frac{2b}{q(T)}\right\}\right]^{-2}\left[1-\exp\left\{-\frac{2b}{q(T)}\right\}\right]^{-2} \\
&\times(T-g)\exp\left\{-\frac{2b}{q(T)}(T-g+1)\right\} \\
&-2\exp\left\{\frac{4b}{q(T)}\right\}\left[1-\exp\left\{\frac{2b}{q(T)}\right\}\right]^{-2}\left[1-\exp\left\{-\frac{2b}{q(T)}\right\}\right]^{-2} \\
&\times(T-g-1)\exp\left\{-\frac{2b(T-g+2)}{q(T)}\right\} \\
&+\exp\left\{\frac{2b}{q(T)}\right\}\left[1-\exp\left\{\frac{2b}{q(T)}\right\}\right]^{-3}\left[1-\exp\left\{-\frac{2b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{2b(T-g-1)}{q(T)}\right\}\right] \\
&-\exp\left\{\frac{4b}{q(T)}\right\}\left[1-\exp\left\{\frac{2b}{q(T)}\right\}\right]^{-3}(T-g-1)
\end{aligned}
$$

Now, consider part (a), where we take $q(T)\to\infty$ such that $q(T)/T\to 0$. In this case, we have

from the above calculations that

$$
\begin{aligned}
&\sum_{t=g+2}^{T}\sum_{s=g+1}^{t-1}\sum_{j=1}^{s-g}\exp\left\{-\frac{b\left(t-g-j\right)}{q\left(T\right)}\right\}\exp\left\{-\frac{b\left(s-g-j\right)}{q\left(T\right)}\right\}\\
&\qquad\times\sum_{k=1}^{s-g}\exp\left\{-\frac{b\left(t-g-k\right)}{q\left(T\right)}\right\}\exp\left\{-\frac{b\left(s-g-k\right)}{q\left(T\right)}\right\}\\
=\;&\exp\left\{-\frac{2b}{q\left(T\right)}\right\}\left[1-\exp\left\{\frac{2b}{q\left(T\right)}\right\}\right]^{-2}\left[1-\exp\left\{-\frac{2b}{q\left(T\right)}\right\}\right]^{-2}\left[1-\exp\left\{-\frac{2b\left(T-g-1\right)}{q\left(T\right)}\right\}\right]\\
&-\exp\left\{-\frac{4b}{q\left(T\right)}\right\}\left[1-\exp\left\{\frac{2b}{q\left(T\right)}\right\}\right]^{-2}\left[1-\exp\left\{-\frac{2b}{q\left(T\right)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{4b}{q\left(T\right)}\right\}\right]^{-1}\\
&\times\left[1-\exp\left\{-\frac{4b\left(T-g-1\right)}{q\left(T\right)}\right\}\right]\\
&-2\left[1-\exp\left\{\frac{2b}{q\left(T\right)}\right\}\right]^{-2}\left[1-\exp\left\{-\frac{2b}{q\left(T\right)}\right\}\right]^{-2}\\
&-2\exp\left\{\frac{4b}{q\left(T\right)}\right\}\left[1-\exp\left\{\frac{2b}{q\left(T\right)}\right\}\right]^{-2}\left[1-\exp\left\{-\frac{2b}{q\left(T\right)}\right\}\right]^{-2}\\
&\qquad\times\left(T-g\right)\exp\left\{-\frac{2b\left(T-g+1\right)}{q\left(T\right)}\right\}\left[\exp\left\{-\frac{2b}{q\left(T\right)}\right\}-1\right]\\
&+2\exp\left\{\frac{4b}{q\left(T\right)}\right\}\left[1-\exp\left\{\frac{2b}{q\left(T\right)}\right\}\right]^{-2}\left[1-\exp\left\{-\frac{2b}{q\left(T\right)}\right\}\right]^{-2}\exp\left\{-\frac{2b\left(T-g+2\right)}{q\left(T\right)}\right\}\\
&+\exp\left\{\frac{2b}{q\left(T\right)}\right\}\left[1-\exp\left\{\frac{2b}{q\left(T\right)}\right\}\right]^{-3}\left[1-\exp\left\{-\frac{2b}{q\left(T\right)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{2b\left(T-g-1\right)}{q\left(T\right)}\right\}\right]\\
&-\exp\left\{\frac{4b}{q\left(T\right)}\right\}\left[1-\exp\left\{\frac{2b}{q\left(T\right)}\right\}\right]^{-3}\left(T-g-1\right)\\
=\;&\frac{1}{16}\frac{q\left(T\right)^{4}}{b^{4}}\left[1+O\left(\frac{1}{q\left(T\right)}\right)\right]\left[1+O\left(\exp\left\{-2b\frac{T}{q\left(T\right)}\right\}\right)\right]\\
&-\frac{1}{32}\frac{q\left(T\right)^{4}}{b^{4}}\left[1+O\left(\frac{1}{q\left(T\right)}\right)\right]\left[1+O\left(\exp\left\{-4b\frac{T}{q\left(T\right)}\right\}\right)\right]\\
&-\frac{1}{8}\frac{q\left(T\right)^{4}}{b^{4}}\left[1+O\left(\frac{1}{q\left(T\right)}\right)\right]+O\left(Tq\left(T\right)^{3}\exp\left\{-2b\frac{T}{q\left(T\right)}\right\}\right)+O\left(q\left(T\right)^{4}\exp\left\{-2b\frac{T}{q\left(T\right)}\right\}\right)\\
&-\frac{1}{16}\frac{q\left(T\right)^{4}}{b^{4}}\left[1+O\left(\frac{1}{q\left(T\right)}\right)\right]\left[1+O\left(\exp\left\{-2b\frac{T}{q\left(T\right)}\right\}\right)\right]\\
&+\frac{1}{8}\frac{Tq\left(T\right)^{3}}{b^{3}}\left[1+O\left(\frac{1}{q\left(T\right)}\right)\right]\left[1+O\left(\frac{1}{T}\right)\right]\\
=\;&\frac{1}{8}\frac{Tq\left(T\right)^{3}}{b^{3}}\left[1+O\left(\frac{1}{q\left(T\right)}\right)+O\left(\frac{q\left(T\right)}{T}\right)+O\left(\exp\left\{-2b\frac{T}{q\left(T\right)}\right\}\right)\right]\\
=\;&\frac{1}{8}\frac{Tq\left(T\right)^{3}}{b^{3}}\left[1+O\left(\frac{1}{q\left(T\right)}\right)+O\left(\frac{q\left(T\right)}{T}\right)\right],
\end{aligned}
$$

as required.

Next, we turn our attention to part (b), where we take $q(T) \sim T$. In this case, from previous calcuations we have

$$
\begin{aligned}
& \sum_{t=g+2}^{T} \sum_{s=g+1}^{t-1} \sum_{j=1}^{s-g} \exp\left\{-b\frac{(t-g-j)}{q(T)}\right\} \exp\left\{-b\frac{(s-g-j)}{q(T)}\right\} \\
& \times \sum_{k=1}^{s-g} \exp\left\{-b\frac{(t-g-k)}{q(T)}\right\} \exp\left\{-b\frac{(s-g-k)}{q(T)}\right\} \\
= \; & \exp\left\{-\frac{2b}{q(T)}\right\} \left[1-\exp\left\{\frac{2b}{q(T)}\right\}\right]^{-2} \left[1-\exp\left\{-\frac{2b}{q(T)}\right\}\right]^{-2} \left[1-\exp\left\{-\frac{2b(T-g-1)}{q(T)}\right\}\right] \\
& -\exp\left\{-\frac{4b}{q(T)}\right\} \left[1-\exp\left\{\frac{2b}{q(T)}\right\}\right]^{-2} \left[1-\exp\left\{-\frac{2b}{q(T)}\right\}\right]^{-1} \left[1-\exp\left\{-\frac{4b}{q(T)}\right\}\right]^{-1} \\
& \times \left[1-\exp\left\{-\frac{4b(T-g-1)}{q(T)}\right\}\right] \\
& -2\left[1-\exp\left\{\frac{2b}{q(T)}\right\}\right]^{-2} \left[1-\exp\left\{-\frac{2b}{q(T)}\right\}\right]^{-2} \\
& +2\exp\left\{\frac{4b}{q(T)}\right\} \left[1-\exp\left\{\frac{2b}{q(T)}\right\}\right]^{-2} \left[1-\exp\left\{-\frac{2b}{q(T)}\right\}\right]^{-2} (T-g)\exp\left\{-\frac{2b(T-g+1)}{q(T)}\right\} \\
& -2\exp\left\{\frac{4b}{q(T)}\right\} \left[1-\exp\left\{\frac{2b}{q(T)}\right\}\right]^{-2} \left[1-\exp\left\{-\frac{2b}{q(T)}\right\}\right]^{-2} (T-g-1)\exp\left\{-\frac{2b(T-g+2)}{q(T)}\right\} \\
& +\exp\left\{\frac{2b}{q(T)}\right\} \left[1-\exp\left\{\frac{2b}{q(T)}\right\}\right]^{-3} \left[1-\exp\left\{-\frac{2b}{q(T)}\right\}\right]^{-1} \left[1-\exp\left\{-\frac{2b(T-g-1)}{q(T)}\right\}\right] \\
& -\exp\left\{\frac{4b}{q(T)}\right\} \left[1-\exp\left\{\frac{2b}{q(T)}\right\}\right]^{-3} (T-g-1),
\end{aligned}
$$

so that

$$
\begin{aligned}
& \sum_{t=g+2}^{T} \sum_{s=g+1}^{t-1} \sum_{j=1}^{s-g} \exp\left\{-\frac{b\left(t-g-j\right)}{q\left(T\right)}\right\} \exp\left\{-\frac{b\left(s-g-j\right)}{q\left(T\right)}\right\} \\
& \qquad \times \sum_{k=1}^{s-g} \exp\left\{-\frac{b\left(t-g-k\right)}{q\left(T\right)}\right\} \exp\left\{-\frac{b\left(s-g-k\right)}{q\left(T\right)}\right\} \\
= \; & \exp\left\{-\frac{2b}{q\left(T\right)}\right\} \left[1-\exp\left\{\frac{2b}{q\left(T\right)}\right\}\right]^{-2} \left[1-\exp\left\{-\frac{2b}{q\left(T\right)}\right\}\right]^{-2} \\
& \times \left[1-\exp\left\{-\frac{2bT}{q\left(T\right)}\right\} \exp\left\{\frac{2b\left(g+1\right)}{q\left(T\right)}\right\}\right] \\
& -\exp\left\{-\frac{4b}{q\left(T\right)}\right\} \left[1-\exp\left\{\frac{2b}{q\left(T\right)}\right\}\right]^{-2} \left[1-\exp\left\{-\frac{2b}{q\left(T\right)}\right\}\right]^{-1} \left[1-\exp\left\{-\frac{4b}{q\left(T\right)}\right\}\right]^{-1} \\
& \times \left[1-\exp\left\{-\frac{4bT}{q\left(T\right)}\right\} \exp\left\{\frac{4b\left(g+1\right)}{q\left(T\right)}\right\}\right] \\
& -2\left[1-\exp\left\{\frac{2b}{q\left(T\right)}\right\}\right]^{-2} \left[1-\exp\left\{-\frac{2b}{q\left(T\right)}\right\}\right]^{-2} \\
& +2\exp\left\{\frac{4b}{q\left(T\right)}\right\} \left[1-\exp\left\{\frac{2b}{q\left(T\right)}\right\}\right]^{-2} \left[1-\exp\left\{-\frac{2b}{q\left(T\right)}\right\}\right]^{-2} \\
& \times \left(T-g\right) \exp\left\{-\frac{2bT}{q\left(T\right)}\right\} \exp\left\{\frac{2b\left(g-1\right)}{q\left(T\right)}\right\} \\
& -2\exp\left\{\frac{4b}{q\left(T\right)}\right\} \left[1-\exp\left\{\frac{2b}{q\left(T\right)}\right\}\right]^{-2} \left[1-\exp\left\{-\frac{2b}{q\left(T\right)}\right\}\right]^{-2} \left(T-g-1\right) \exp\left\{-\frac{2bT}{q\left(T\right)}\right\} \\
& \times \exp\left\{\frac{2b\left(g-2\right)}{q\left(T\right)}\right\} \\
& +\exp\left\{\frac{2b}{q\left(T\right)}\right\} \left[1-\exp\left\{\frac{2b}{q\left(T\right)}\right\}\right]^{-3} \left[1-\exp\left\{-\frac{2b}{q\left(T\right)}\right\}\right]^{-1} \\
& \qquad \times \left[1-\exp\left\{-\frac{2bT}{q\left(T\right)}\right\} \exp\left\{\frac{2b\left(g+1\right)}{q\left(T\right)}\right\}\right] \\
& -\exp\left\{\frac{4b}{q\left(T\right)}\right\} \left[1-\exp\left\{\frac{2b}{q\left(T\right)}\right\}\right]^{-3} \left(T-g-1\right)
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \exp\left\{-\frac{2b}{q(T)}\right\}\left[1-\exp\left\{\frac{2b}{q(T)}\right\}\right]^{-2}\left[1-\exp\left\{-\frac{2b}{q(T)}\right\}\right]^{-2} \\
& \times\left[1-\exp\left\{-\frac{2bT}{q(T)}\right\}\exp\left\{\frac{2b(g+1)}{q(T)}\right\}\right] \\
& -\exp\left\{-\frac{4b}{q(T)}\right\}\left[1-\exp\left\{\frac{2b}{q(T)}\right\}\right]^{-2}\left[1-\exp\left\{-\frac{2b}{q(T)}\right\}\right]^{-1} \\
& \times\left[1-\exp\left\{-\frac{4b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{4bT}{q(T)}\right\}\exp\left\{\frac{4b(g+1)}{q(T)}\right\}\right] \\
& -2\left[1-\exp\left\{\frac{2b}{q(T)}\right\}\right]^{-2}\left[1-\exp\left\{-\frac{2b}{q(T)}\right\}\right]^{-2} \\
& +2T\left[1+\frac{4b}{q(T)}+\frac{8b^2}{q(T)^2}+O\left(\frac{1}{T^3}\right)\right]\left[-\frac{2b}{q(T)}-2\frac{b^2}{q(T)^2}+O\left(\frac{1}{T^3}\right)\right]^{-2} \\
& \times\left[\frac{2b}{q(T)}-\frac{2b^2}{q(T)^2}+O\left(\frac{1}{T^3}\right)\right]^{-2}\exp\left\{-\frac{2bT}{q(T)}\right\} \\
& \times\left[1+\frac{2b(g-1)}{q(T)}+\frac{2b^2(g-1)^2}{q(T)^2}+O\left(\frac{1}{T^3}\right)\right] \\
& -2g\exp\left\{\frac{4b}{q(T)}\right\}\left[1-\exp\left\{\frac{2b}{q(T)}\right\}\right]^{-2}\left[1-\exp\left\{-\frac{2b}{q(T)}\right\}\right]^{-2}\exp\left\{-\frac{2bT}{q(T)}\right\} \\
& \qquad \times\exp\left\{\frac{2b(g-1)}{q(T)}\right\} \\
& -2T\left[1+4\frac{b}{q(T)}+8\frac{b^2}{q(T)^2}+O\left(\frac{1}{T^3}\right)\right]\left[-\frac{2b}{q(T)}-2\frac{b^2}{q(T)^2}+O\left(\frac{1}{T^3}\right)\right]^{-2} \\
& \times\left[\frac{2b}{q(T)}-\frac{2b^2}{q(T)^2}+O\left(\frac{1}{T^3}\right)\right]^{-2}\exp\left\{-\frac{2bT}{q(T)}\right\}\exp\left\{\frac{2b(g-2)}{q(T)}\right\} \\
& +2(g+1)\exp\left\{\frac{4b}{q(T)}\right\}\left[1-\exp\left\{\frac{2b}{q(T)}\right\}\right]^{-2}\left[1-\exp\left\{-\frac{2b}{q(T)}\right\}\right]^{-2} \\
& \times\exp\left\{-\frac{2bT}{q(T)}\right\}\exp\left\{\frac{2b(g-2)}{q(T)}\right\} \\
& +\exp\left\{\frac{2b}{q(T)}\right\}\left[1-\exp\left\{\frac{2b}{q(T)}\right\}\right]^{-3}\left[1-\exp\left\{-\frac{2b}{q(T)}\right\}\right]^{-1} \\
& \times\left[1-\exp\left\{-\frac{2bT}{q(T)}\right\}\exp\left\{\frac{2b(g+1)}{q(T)}\right\}\right] \\
& -\exp\left\{\frac{4b}{q(T)}\right\}\left[1-\exp\left\{\frac{2b}{q(T)}\right\}\right]^{-3}(T-g-1)
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \frac{1}{16}\frac{q(T)^4}{b^4}\left[1-\exp\left\{-\frac{2bT}{q(T)}\right\}\right]\left[1+O\left(\frac{1}{T}\right)\right] \\
& -\frac{1}{32}\frac{q(T)^4}{b^4}\left[1-\exp\left\{-\frac{4bT}{q(T)}\right\}\right]\left[1+O\left(\frac{1}{T}\right)\right]-\frac{1}{8}\frac{q(T)^4}{b^4}\left[1+O\left(\frac{1}{T}\right)\right] \\
& +2T\left[1+\frac{4b}{q(T)}+\frac{8b^2}{q(T)^2}+O\left(\frac{1}{T^3}\right)\right]\left[-\frac{2b}{q(T)}-\frac{2b^2}{q(T)^2}+O\left(\frac{1}{T^3}\right)\right]^{-2} \\
& \times\left[\frac{2b}{q(T)}-\frac{2b^2}{q(T)^2}+O\left(\frac{1}{T^3}\right)\right]^{-2}\exp\left\{-\frac{2bT}{q(T)}\right\} \\
& \times\left[1+\frac{2b(g-1)}{q(T)}+\frac{2b^2(g-1)^2}{q(T)^2}+O\left(\frac{1}{T^3}\right)\right] \\
& -\frac{g}{8}\frac{q(T)^4}{b^4}\exp\left\{-\frac{2bT}{q(T)}\right\}\left[1+O\left(\frac{1}{T}\right)\right] \\
& -2T\left[1+4\frac{b}{q(T)}+8\frac{b^2}{q(T)^2}+O\left(\frac{1}{T^3}\right)\right]\left[-\frac{2b}{q(T)}-2\frac{b^2}{q(T)^2}+O\left(\frac{1}{T^3}\right)\right]^{-2} \\
& \times\left[\frac{2b}{q(T)}-\frac{2b^2}{q(T)^2}+O\left(\frac{1}{T^3}\right)\right]^{-2}\exp\left\{-\frac{2bT}{q(T)}\right\} \\
& \times\left[1+\frac{2b(g-2)}{q(T)}+\frac{2b^2(g-2)^2}{q(T)^2}+O\left(\frac{1}{T^3}\right)\right] \\
& +\frac{(g+1)}{8}\frac{q(T)^4}{b^4}\exp\left\{-\frac{2bT}{q(T)}\right\}\left[1+O\left(\frac{1}{T}\right)\right] \\
& -\frac{1}{16}\frac{q(T)^4}{b^4}\left[1-\exp\left\{-\frac{2bT}{q(T)}\right\}\right]\left[1+O\left(\frac{1}{T}\right)\right] \\
& +\frac{1}{8}\frac{Tq(T)^3}{b^3}\left[1+O\left(\frac{1}{T}\right)\right]-\frac{(g+1)}{8}\frac{q(T)^3}{b^3}\left[1+O\left(\frac{1}{T}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \frac{1}{16}\frac{q(T)^4}{b^4}\left[1-\exp\left\{-\frac{2bT}{q(T)}\right\}\right]\left[1+O\left(\frac{1}{T}\right)\right] \\
& -\frac{1}{32}\frac{q(T)^4}{b^4}\left[1-\exp\left\{-\frac{4bT}{q(T)}\right\}\right]\left[1+O\left(\frac{1}{T}\right)\right]-\frac{1}{8}\frac{q(T)^4}{b^4}\left[1+O\left(\frac{1}{T}\right)\right] \\
& +2T\left[1+\frac{4b}{q(T)}+\frac{8b^2}{q(T)^2}+O\left(\frac{1}{T^3}\right)\right]\frac{q(T)^2}{4b^2}\left[1+\frac{b}{q(T)}+O\left(\frac{1}{T^2}\right)\right]^{-2} \\
& \times\frac{q(T)^2}{4b^2}\left[1-\frac{b}{q(T)}+O\left(\frac{1}{T^2}\right)\right]^{-2}\exp\left\{-\frac{2bT}{q(T)}\right\} \\
& \times\left[1+\frac{2b(g-1)}{q(T)}+\frac{2b^2(g-1)^2}{q(T)^2}+O\left(\frac{1}{T^3}\right)\right] \\
& -\frac{g}{8}\frac{q(T)^4}{b^4}\exp\left\{-\frac{2bT}{q(T)}\right\}\left[1+O\left(\frac{1}{T}\right)\right] \\
& -2T\left[1+\frac{4b}{q(T)}+\frac{8b^2}{q(T)^2}+O\left(\frac{1}{T^3}\right)\right]\frac{q(T)^2}{4b^2}\left[1+\frac{b}{q(T)}+O\left(\frac{1}{T^2}\right)\right]^{-2} \\
& \times\frac{q(T)^2}{4b^2}\left[1-\frac{b}{q(T)}+O\left(\frac{1}{T^2}\right)\right]^{-2}\exp\left\{-\frac{2bT}{q(T)}\right\}\left[1+\frac{2b(g-2)}{q(T)}+\frac{2b^2(g-2)^2}{q(T)^2}+O\left(\frac{1}{T^3}\right)\right] \\
& +\frac{(g+1)}{8}\frac{q(T)^4}{b^4}\exp\left\{-\frac{2bT}{q(T)}\right\}\left[1+O\left(\frac{1}{T}\right)\right]\ -\frac{1}{16}\frac{q(T)^4}{b^4}\left[1-\exp\left\{-\frac{2bT}{q(T)}\right\}\right]\left[1+O\left(\frac{1}{T}\right)\right] \\
& +\frac{1}{8}\frac{Tq(T)^3}{b^3}\left[1+O\left(\frac{1}{T}\right)\right]-\frac{(g+1)}{8}\frac{q(T)^3}{b^3}\left[1+O\left(\frac{1}{T}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \frac{1}{16}\frac{q(T)^4}{b^4}\left[1-\exp\left\{-\frac{2bT}{q(T)}\right\}\right]\left[1+O\left(\frac{1}{T}\right)\right] \\
& -\frac{1}{32}\frac{q(T)^4}{b^4}\left[1-\exp\left\{-\frac{4bT}{q(T)}\right\}\right]\left[1+O\left(\frac{1}{T}\right)\right]-\frac{1}{8}\frac{q(T)^4}{b^4}\left[1+O\left(\frac{1}{T}\right)\right] \\
& +\frac{1}{8}\frac{Tq(T)^4}{b^4}\left[1+\frac{4b}{q(T)}+\frac{8b^2}{q(T)^2}+O\left(\frac{1}{T^3}\right)\right]\left[1-\frac{2b}{q(T)}+O\left(\frac{1}{T^2}\right)\right] \\
& \times\left[1+\frac{2b}{q(T)}+O\left(\frac{1}{T^2}\right)\right]\left[1+\frac{2b(g-1)}{q(T)}+\frac{2b^2(g-1)^2}{q(T)^2}+O\left(\frac{1}{T^3}\right)\right]\exp\left\{-\frac{2bT}{q(T)}\right\} \\
& -\frac{g}{8}\frac{q(T)^4}{b^4}\exp\left\{-\frac{2bT}{q(T)}\right\}\left[1+O\left(\frac{1}{T}\right)\right] \\
& -\frac{1}{8}\frac{Tq(T)^4}{b^4}\left[1+\frac{4b}{q(T)}+\frac{8b^2}{q(T)^2}+O\left(\frac{1}{T^3}\right)\right]\left[1-\frac{2b}{q(T)}+O\left(\frac{1}{T^2}\right)\right] \\
& \times\left[1+\frac{2b}{q(T)}+O\left(\frac{1}{T^2}\right)\right]\left[1+\frac{2b(g-2)}{q(T)}+\frac{2b^2(g-2)^2}{q(T)^2}+O\left(\frac{1}{T^3}\right)\right]\exp\left\{-\frac{2bT}{q(T)}\right\} \\
& +\frac{(g+1)}{8}\frac{q(T)^4}{b^4}\exp\left\{-\frac{2bT}{q(T)}\right\}\left[1+O\left(\frac{1}{T}\right)\right] \\
& -\frac{1}{16}\frac{q(T)^4}{b^4}\left[1-\exp\left\{-\frac{2bT}{q(T)}\right\}\right]\left[1+O\left(\frac{1}{T}\right)\right]+\frac{1}{8}\frac{Tq(T)^3}{b^3}\left[1+O\left(\frac{1}{T}\right)\right] \\
& -\frac{(g+1)}{8}\frac{q(T)^3}{b^3}\left[1+O\left(\frac{1}{T}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \frac{1}{16}\frac{q(T)^4}{b^4}\left[1-\exp\left\{-\frac{2bT}{q(T)}\right\}\right]\left[1+O\left(\frac{1}{T}\right)\right] \\
& -\frac{1}{32}\frac{q(T)^4}{b^4}\left[1-\exp\left\{-\frac{4bT}{q(T)}\right\}\right]\left[1+O\left(\frac{1}{T}\right)\right]-\frac{1}{8}\frac{q(T)^4}{b^4}\left[1+O\left(\frac{1}{T}\right)\right] \\
& +\frac{1}{8}\frac{Tq(T)^4}{b^4}\left[1+\frac{2b}{q(T)}+O\left(\frac{1}{T^2}\right)\right]\left[1+\frac{2bg}{q(T)}+O\left(\frac{1}{T^2}\right)\right]\exp\left\{-\frac{2bT}{q(T)}\right\} \\
& -\frac{g}{8}\frac{q(T)^4}{b^4}\exp\left\{-\frac{2bT}{q(T)}\right\}\left[1+O\left(\frac{1}{T}\right)\right] \\
& -\frac{1}{8}\frac{Tq(T)^4}{b^4}\left[1+\frac{4b}{q(T)}+O\left(\frac{1}{T^2}\right)\right]\left[1+\frac{2b(g-1)}{q(T)}+O\left(\frac{1}{T^2}\right)\right]\exp\left\{-\frac{2bT}{q(T)}\right\} \\
& +\frac{(g+1)}{8}\frac{q(T)^4}{b^4}\exp\left\{-\frac{2bT}{q(T)}\right\}\left[1+O\left(\frac{1}{T}\right)\right] \\
& -\frac{1}{16}\frac{q(T)^4}{b^4}\left[1-\exp\left\{-\frac{2bT}{q(T)}\right\}\right]\left[1+O\left(\frac{1}{T}\right)\right] \\
& +\frac{1}{8}\frac{Tq(T)^3}{b^3}\left[1+O\left(\frac{1}{T}\right)\right]-\frac{(g+1)}{8}\frac{q(T)^3}{b^3}\left[1+O\left(\frac{1}{T}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
= \; & \frac{1}{16}\frac{q(T)^4}{b^4}\left[1-\exp\left\{-\frac{2bT}{q(T)}\right\}\right]\left[1+O\left(\frac{1}{T}\right)\right] \\
& -\frac{1}{32}\frac{q(T)^4}{b^4}\left[1-\exp\left\{-\frac{4bT}{q(T)}\right\}\right]\left[1+O\left(\frac{1}{T}\right)\right]-\frac{1}{8}\frac{q(T)^4}{b^4}\left[1+O\left(\frac{1}{T}\right)\right] \\
& +\frac{1}{8}\frac{Tq(T)^4}{b^4}\left[1+\frac{2b(g+1)}{q(T)}+O\left(\frac{1}{T^2}\right)\right]\exp\left\{-\frac{2bT}{q(T)}\right\} \\
& -\frac{g}{8}\frac{q(T)^4}{b^4}\exp\left\{-\frac{2bT}{q(T)}\right\}\left[1+O\left(\frac{1}{T}\right)\right] \\
& -\frac{1}{8}\frac{Tq(T)^4}{b^4}\left[1+\frac{2bg}{q(T)}+O\left(\frac{1}{T^2}\right)\right]\exp\left\{-\frac{2bT}{q(T)}\right\} \\
& +\frac{(g+1)}{8}\frac{q(T)^4}{b^4}\exp\left\{-\frac{2bT}{q(T)}\right\}\left[1+O\left(\frac{1}{T}\right)\right] \\
& -\frac{1}{16}\frac{q(T)^4}{b^4}\left[1-\exp\left\{-\frac{2bT}{q(T)}\right\}\right]\left[1+O\left(\frac{1}{T}\right)\right]+\frac{1}{8}\frac{Tq(T)^3}{b^3}\left[1+O\left(\frac{1}{T}\right)\right] \\
& -\frac{(g+1)}{8}\frac{q(T)^3}{b^3}\left[1+O\left(\frac{1}{T}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
= \; & \frac{1}{16}\frac{q(T)^4}{b^4}\left[1-\exp\left\{-\frac{2bT}{q(T)}\right\}\right]\left[1+O\left(\frac{1}{T}\right)\right] \\
& -\frac{1}{32}\frac{q(T)^4}{b^4}\left[1-\exp\left\{-\frac{4bT}{q(T)}\right\}\right]\left[1+O\left(\frac{1}{T}\right)\right]-\frac{1}{8}\frac{q(T)^4}{b^4}\left[1+O\left(\frac{1}{T}\right)\right] \\
& +\frac{(g+1)}{4}\frac{Tq(T)^3}{b^3}\exp\left\{-\frac{2bT}{q(T)}\right\}\left[1+O\left(\frac{1}{T}\right)\right]-\frac{1}{4}\frac{q(T)^4}{b^4}\exp\left\{-\frac{2bT}{q(T)}\right\}\left[1+O\left(\frac{1}{T}\right)\right] \\
& -\frac{g}{4}\frac{Tq(T)^3}{b^3}\exp\left\{-\frac{2bT}{q(T)}\right\}\left[1+O\left(\frac{1}{T}\right)\right]+\frac{g+1}{8}\frac{q(T)^4}{b^4}\exp\left\{-\frac{2bT}{q(T)}\right\}\left[1+O\left(\frac{1}{T}\right)\right] \\
& -\frac{1}{16}\frac{q(T)^4}{b^4}\left[1-\exp\left\{-\frac{2bT}{q(T)}\right\}\right]\left[1+O\left(\frac{1}{T}\right)\right]+\frac{1}{8}\frac{Tq(T)^3}{b^3}\left[1+O\left(\frac{1}{T}\right)\right] \\
& -\frac{g+1}{8}\frac{q(T)^3}{b^3}\left[1+O\left(\frac{1}{T}\right)\right] \\
= \; & \left\{\frac{q(T)^4}{32b^4}\left[\exp\left\{-\frac{4bT}{q(T)}\right\}+4(g-1)\exp\left\{-\frac{2bT}{q(T)}\right\}-5\right]+\frac{Tq(T)^3}{8b^3}\left[2\exp\left\{-\frac{2bT}{q(T)}\right\}+1\right]\right\} \\
& \times\left[1+O\left(\frac{1}{T}\right)\right],
\end{aligned}
$$

which shows (b). □

Lemma SE-7:

Let b and g be a positive integer. Then, the following statements are true as $T \to \infty$

(a) If $T/q(T) \to 0$, then

$$\begin{aligned}
&\sum_{t=g+2}^{T}\sum_{s=g+1}^{t-1}\sum_{j=1}^{s-g}\exp\left\{-b\frac{(t-g-j)}{q(T)}\right\}\exp\left\{-b\frac{(s-g-j)}{q(T)}\right\} \\
= \;& \frac{1}{6}T^3\left[1+O\left(\max\left\{\frac{T}{q(T)},\frac{1}{T}\right\}\right)\right].
\end{aligned}$$

(b) If $q(T) \sim T$, then

$$\begin{aligned}
&\sum_{t=g+2}^{T}\sum_{s=g+1}^{t-1}\sum_{j=1}^{s-g}\exp\left\{-b\frac{(t-g-j)}{q(T)}\right\}\exp\left\{-b\frac{(s-g-j)}{q(T)}\right\} \\
= \;& \frac{Tq(T)^2}{2b^2}\left[1-\frac{3}{2b}\frac{q(T)}{T}+\frac{2}{b}\frac{q(T)}{T}\exp\left\{-\frac{bT}{q(T)}\right\}-\frac{1}{2b}\frac{q(T)}{T}\exp\left\{-\frac{2bT}{q(T)}\right\}\right]\left[1+O\left(\frac{1}{T}\right)\right].
\end{aligned}$$

(c) If $q(T) \to \infty$ such that $q(T)/T \to 0$, then

$$\begin{aligned}
&\sum_{t=g+2}^{T}\sum_{s=g+1}^{t-1}\sum_{j=1}^{s-g}\exp\left\{-b\frac{(t-g-j)}{q(T)}\right\}\exp\left\{-b\frac{(s-g-j)}{q(T)}\right\} \\
= \;& \frac{Tq(T)^2}{2b^2}\left[1+O\left(\frac{q(T)}{T}\right)\right].
\end{aligned}$$

Proof of Lemma SE-7:

To proceed, note first that

$$\begin{aligned}
&\sum_{t=g+2}^{T}\sum_{s=g+1}^{t-1}\sum_{j=1}^{s-g}\exp\left\{-b\frac{(t-g-j)}{q(T)}\right\}\exp\left\{-b\frac{(s-g-j)}{q(T)}\right\} \\
= \;& \sum_{t=g+2}^{T}\sum_{s=g+1}^{t-1}\exp\left\{-\frac{b(t-g)}{q(T)}\right\}\exp\left\{-\frac{b(s-g)}{q(T)}\right\}\exp\left\{\frac{2b}{q(T)}\right\}\sum_{j=1}^{s-g}\exp\left\{2b\left(\frac{j-1}{q(T)}\right)\right\} \\
= \;& \left[1-\exp\left\{\frac{2b}{q(T)}\right\}\right]^{-1}\sum_{t=g+2}^{T}\sum_{s=g+1}^{t-1}\left[\exp\left\{-\frac{b(t-g)}{q(T)}\right\}\exp\left\{-\frac{b(s-g)}{q(T)}\right\}\right. \\
& \left.\times\exp\left\{\frac{2b}{q(T)}\right\}\left(1-\exp\left\{2b\left(\frac{s-g}{q(T)}\right)\right\}\right)\right] \\
= \;& \exp\left\{\frac{2b}{q(T)}\right\}\exp\left\{-\frac{b}{q(T)}\right\}\left[1-\exp\left\{\frac{2b}{q(T)}\right\}\right]^{-1}\sum_{t=g+2}^{T}\exp\left\{-\frac{b(t-g)}{q(T)}\right\} \\
& \times\sum_{s=g+1}^{t-1}\exp\left\{-\frac{b(s-g-1)}{q(T)}\right\} \\
& -\exp\left\{\frac{2b}{q(T)}\right\}\exp\left\{\frac{b}{q(T)}\right\}\left[1-\exp\left\{\frac{2b}{q(T)}\right\}\right]^{-1}\sum_{t=g+2}^{T}\exp\left\{-\frac{b(t-g)}{q(T)}\right\} \\
& \times\sum_{s=g+1}^{t-1}\exp\left\{b\left(\frac{s-g-1}{q(T)}\right)\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \exp\left\{\frac{b}{q(T)}\right\}\left[1-\exp\left\{\frac{2b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{b}{q(T)}\right\}\right]^{-1} \\
&\quad \times \sum_{t=g+2}^{T}\exp\left\{-\frac{b(t-g)}{q(T)}\right\}\left[1-\exp\left\{-b\left(\frac{t-g-1}{q(T)}\right)\right\}\right] \\
&\quad -\exp\left\{\frac{3b}{q(T)}\right\}\left[1-\exp\left\{\frac{2b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{\frac{b}{q(T)}\right\}\right]^{-1} \\
&\quad \times \sum_{t=g+2}^{T}\exp\left\{-b\left(\frac{t-g}{q(T)}\right)\right\}\left[1-\exp\left\{b\left(\frac{t-g-1}{q(T)}\right)\right\}\right] \\
&= \exp\left\{\frac{b}{q(T)}\right\}\left[1-\exp\left\{\frac{2b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{b}{q(T)}\right\}\right]^{-1}\exp\left\{-\frac{2b}{q(T)}\right\} \\
&\quad \times\left[\sum_{t=g+2}^{T}\exp\left\{-\frac{b(t-g-2)}{q(T)}\right\}-\exp\left\{-\frac{b}{q(T)}\right\}\sum_{t=g+2}^{T}\exp\left\{-2b\left(\frac{t-g-2}{q(T)}\right)\right\}\right] \\
&\quad -\exp\left\{\frac{3b}{q(T)}\right\}\left[1-\exp\left\{\frac{2b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{\frac{b}{q(T)}\right\}\right]^{-1} \\
&\quad \times\left[\exp\left\{-\frac{2b}{q(T)}\right\}\sum_{t=g+2}^{T}\exp\left\{-\frac{b(t-g-2)}{q(T)}\right\}-\exp\left\{-\frac{b}{q(T)}\right\}(T-g-1)\right]
\end{aligned}
$$

$$
\begin{aligned}
= &\left[1-\exp\left\{\frac{2b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{b}{q(T)}\right\}\right]^{-2}\exp\left\{-\frac{b}{q(T)}\right\} \\
&-\left[1-\exp\left\{\frac{2b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{b}{q(T)}\right\}\right]^{-2}\exp\left\{-\frac{b}{q(T)}\right\}\exp\left\{-\frac{b(T-g-1)}{q(T)}\right\} \\
&-\left[1-\exp\left\{\frac{2b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{2b}{q(T)}\right\}\right]^{-1}\exp\left\{-\frac{2b}{q(T)}\right\} \\
&+\left[1-\left\{\exp\frac{2b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{2b}{q(T)}\right\}\right]^{-1} \\
&\qquad \times\exp\left\{-\frac{2b}{q(T)}\right\}\exp\left\{-\frac{2b(T-g-1)}{q(T)}\right\} \\
&-\left[1-\exp\left\{\frac{2b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{\frac{b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{b}{q(T)}\right\}\right]^{-1}\exp\left\{\frac{b}{q(T)}\right\} \\
&+\left[1-\exp\left\{\frac{2b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{\frac{b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{b}{q(T)}\right\}\right]^{-1} \\
&\qquad \times\exp\left\{\frac{b}{q(T)}\right\}\exp\left\{-\frac{b(T-g-1)}{q(T)}\right\} \\
&+\exp\left\{\frac{2b}{q(T)}\right\}\left[1-\exp\left\{\frac{2b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{\frac{b}{q(T)}\right\}\right]^{-1}(T-g-1)
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \left[1-\exp\left\{\frac{2b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{b}{q(T)}\right\}\right]^{-2}\exp\left\{-\frac{b}{q(T)}\right\} \\
& -\left[1-\exp\left\{\frac{2b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{b}{q(T)}\right\}\right]^{-2}\exp\left\{-\frac{b(T-g)}{q(T)}\right\} \\
& -\left[1-\exp\left\{\frac{2b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{2b}{q(T)}\right\}\right]^{-1}\exp\left\{-\frac{2b}{q(T)}\right\} \\
& +\left[1-\left\{\exp\frac{2b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{2b}{q(T)}\right\}\right]^{-1} \\
& \qquad \times\exp\left\{-\frac{2b(T-g)}{q(T)}\right\} \\
& -\left[1-\exp\left\{\frac{2b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{\frac{b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{b}{q(T)}\right\}\right]^{-1}\exp\left\{\frac{b}{q(T)}\right\} \\
& +\left[1-\exp\left\{\frac{2b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{\frac{b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{b}{q(T)}\right\}\right]^{-1} \\
& \qquad \times\exp\left\{-\frac{b(T-g-2)}{q(T)}\right\} \\
& +\exp\left\{\frac{2b}{q(T)}\right\}\left[1-\exp\left\{\frac{2b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{\frac{b}{q(T)}\right\}\right]^{-1}(T-g-1)
\end{aligned}
$$

Next, note that

$$
\begin{aligned}
& \left[1-\exp\left(\frac{2b}{q(T)}\right)\right]^{-1}\left[1-\exp\left(-\frac{b}{q(T)}\right)\right]^{-2} \\
= & \left[-\frac{2b}{q(T)}-\frac{4b^2}{2q(T)^2}-\frac{8}{6}\frac{b^3}{q(T)^3}-\frac{16}{24}\frac{b^4}{q(T)^4}+O\left(\frac{1}{q(T)^5}\right)\right]^{-1} \\
& \times\left[-\frac{b}{q(T)}+\frac{b^2}{2q(T)^2}-\frac{1}{6}\frac{b^3}{q(T)^3}+\frac{1}{24}\frac{b^4}{q(T)^4}+O\left(\frac{1}{q(T)^5}\right)\right]^{-2} \\
= & -\frac{q(T)^3}{2b^3}\left[1+\frac{b}{q(T)}+\frac{2}{3}\frac{b^2}{q(T)^2}+\frac{1}{3}\frac{b^3}{q(T)^3}+O\left(\frac{1}{q(T)^4}\right)\right]^{-1} \\
& \times\left[1-\frac{b}{2q(T)}+\frac{b^2}{6q(T)^2}-\frac{1}{24}\frac{b^3}{q(T)^3}+O\left(\frac{1}{q(T)^4}\right)\right]^{-2} \\
= & -\frac{q(T)^3}{2b^3}\left[1-\frac{b}{q(T)}-\frac{2}{3}\frac{b^2}{q(T)^2}+\frac{b^2}{q(T)^2}-\frac{1}{3}\frac{b^3}{q(T)^3}+\frac{4}{3}\frac{b^3}{q(T)^3}-\frac{b^3}{q(T)^3}+O\left(\frac{1}{q(T)^4}\right)\right] \\
& \times\left[1+\frac{b}{q(T)}-\frac{1}{3}\frac{b^2}{q(T)^2}+\frac{3}{4}\frac{b^2}{q(T)^2}+\frac{1}{12}\frac{b^3}{q(T)^3}-\frac{1}{2}\frac{b^3}{q(T)^3}+\frac{1}{2}\frac{b^3}{q(T)^3}+O\left(\frac{1}{q(T)^4}\right)\right] \\
= & -\frac{q(T)^3}{2b^3}\left[1-\frac{b}{q(T)}+\frac{1}{3}\frac{b^2}{q(T)^2}+O\left(\frac{1}{q(T)^4}\right)\right] \\
& \times\left[1+\frac{b}{q(T)}+\frac{5}{12}\frac{b^2}{q(T)^2}+\frac{1}{12}\frac{b^3}{q(T)^3}+O\left(\frac{1}{q(T)^4}\right)\right] \\
= & \frac{q(T)^3}{2b^3c^3}\left[1+\frac{b}{q(T)}-\frac{b}{q(T)}+\frac{5}{12}\frac{b^2}{q(T)^2}-\frac{b^2}{q(T)^2}+\frac{1}{3}\frac{b^2}{q(T)^2}+\frac{1}{12}\frac{b^3}{q(T)^3}\right. \\
& \left.-\frac{5}{12}\frac{b^3}{q(T)^3}+\frac{1}{3}\frac{b^3}{q(T)^3}+O\left(\frac{1}{q(T)^4}\right)\right] \\
= & -\frac{q(T)^3}{2b^3}\left[1-\frac{1}{4}\frac{b^2}{q(T)^2}+O\left(\frac{1}{q(T)^4}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
& \left[1-\exp\left\{\frac{2b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{2b}{q(T)}\right\}\right]^{-1} \\
= & \left[-\frac{2b}{q(T)}-\frac{4b^2}{2q(T)^2}-\frac{8b^3}{6q(T)^3}-\frac{16}{24}\frac{b^4}{q(T)^4}+O\left(\frac{1}{q(T)^5}\right)\right]^{-1} \\
& \times-\left[-\frac{b}{q(T)}+\frac{b^2}{2q(T)^2}-\frac{b^3}{6q(T)^3}+\frac{1}{24}\frac{b^4}{q(T)^4}+O\left(\frac{1}{q(T)^5}\right)\right]^{-1} \\
& \times-\left[-\frac{2b}{q(T)}+\frac{4b^2}{2q(T)^2}-\frac{8b^3}{6q(T)^3}+\frac{16}{24}\frac{b^4}{q(T)^4}+O\left(\frac{1}{q(T)^5}\right)\right]^{-1} \\
= & -\frac{q(T)^3}{4b^3}\left[1+\frac{b}{q(T)}+\frac{2}{3}\frac{b^2}{q(T)^2}+\frac{1}{3}\frac{b^3}{q(T)^3}+O\left(\frac{1}{q(T)^4}\right)\right]^{-1} \\
& \times\left[1-\frac{b}{2q(T)}+\frac{b^2}{6q(T)^2}-\frac{1}{24}\frac{b^3}{q(T)^3}+O\left(\frac{1}{q(T)^4}\right)\right]^{-1} \\
& \times\left[1-\frac{b}{q(T)}+\frac{2}{3}\frac{b^2}{q(T)^2}-\frac{1}{3}\frac{b^3}{q(T)^3}+O\left(\frac{1}{q(T)^4}\right)\right]^{-1} \\
= & -\frac{q(T)^3}{4b^3}\left[1-\frac{b}{q(T)}-\frac{2}{3}\frac{b^2}{q(T)^2}+\frac{b^2}{q(T)^2}-\frac{1}{3}\frac{b^3}{q(T)^3}+\frac{4}{3}\frac{b^3}{q(T)^3}-\frac{b^3}{q(T)^3}+O\left(\frac{1}{q(T)^4}\right)\right] \\
& \times\left[1+\frac{b}{2q(T)}-\frac{1}{6}\frac{b^2}{q(T)^2}+\frac{1}{4}\frac{b^2}{q(T)^2}+\frac{1}{24}\frac{b^3}{q(T)^3}+\frac{1}{8}\frac{b^3}{q(T)^3}-\frac{1}{6}\frac{b^3}{q(T)^3}+O\left(\frac{1}{q(T)^4}\right)\right] \\
& \times\left[1+\frac{b}{q(T)}-\frac{2}{3}\frac{b^2}{q(T)^2}+\frac{b^2}{q(T)^2}+\frac{1}{3}\frac{b^3}{q(T)^3}-\frac{4}{3}\frac{b^3}{q(T)^3}+\frac{b^3}{q(T)^3}+O\left(\frac{1}{q(T)^4}\right)\right] \\
= & -\frac{q(T)^3}{4b^3}\left[1-\frac{b}{q(T)}+\frac{1}{3}\frac{b^2}{q(T)^2}+O\left(\frac{1}{q(T)^4}\right)\right]\left[1+\frac{b}{2q(T)}+\frac{1}{12}\frac{b^2}{q(T)^2}+O\left(\frac{1}{q(T)^4}\right)\right] \\
& \times\left[1+\frac{b}{q(T)}+\frac{1}{3}\frac{b^2}{q(T)^2}+O\left(\frac{1}{q(T)^4}\right)\right] \\
= & -\frac{q(T)^3}{4b^3}\left[1-\frac{b}{q(T)}+\frac{b}{2q(T)}+\frac{1}{12}\frac{b^2}{q(T)^2}+\frac{1}{3}\frac{b^2}{q(T)^2}-\frac{1}{2}\frac{b^2}{q(T)^2}-\frac{1}{12}\frac{b^3}{q(T)^3}\right. \\
& \left.+\frac{1}{6}\frac{b^3}{q(T)^3}+O\left(\frac{1}{q(T)^4}\right)\right]\left[1+\frac{b}{q(T)}+\frac{1}{3}\frac{b^2}{q(T)^2}+O\left(\frac{1}{q(T)^4}\right)\right] \\
= & -\frac{q(T)^3}{4b^3}\left[1-\frac{1}{2}\frac{b}{q(T)}-\frac{1}{12}\frac{b^2}{q(T)^2}+\frac{1}{12}\frac{b^3}{q(T)^3}+O\left(\frac{1}{q(T)^4}\right)\right] \\
& \left[1+\frac{b}{q(T)}+\frac{1}{3}\frac{b^2}{q(T)^2}+O\left(\frac{1}{q(T)^4}\right)\right] \\
= & -\frac{q(T)^3}{4b^3}\left[1+\frac{b}{q(T)}-\frac{1}{2}\frac{b}{q(T)}+\frac{1}{3}\frac{b^2}{q(T)^2}-\frac{1}{2}\frac{b^2}{q(T)^2}-\frac{1}{12}\frac{b^2}{q(T)^2}-\frac{1}{12}\frac{b^3}{q(T)^3}\right. \\
& \left.+\frac{1}{12}\frac{b^3}{q(T)^3}-\frac{1}{6}\frac{b^3}{q(T)^3}+O\left(\frac{1}{q(T)^4}\right)\right] \\
= & -\frac{q(T)^3}{4b^3}\left[1+\frac{1}{2}\frac{b}{q(T)}-\frac{1}{4}\frac{b^2}{q(T)^2}-\frac{1}{6}\frac{b^3}{q(T)^3}+O\left(\frac{1}{q(T)^4}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
& \left[1-\exp\left\{\frac{2b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{\frac{b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{b}{q(T)}\right\}\right]^{-1} \\
= & \left[-\frac{2b}{q(T)}-\frac{4b^2}{2q(T)^2}-\frac{8b^3}{6q(T)^3}-\frac{16}{24}\frac{b^4}{q(T)^4}+O\left(\frac{1}{q(T)^5}\right)\right]^{-1} \\
& \times\left[-\frac{b}{q(T)}-\frac{b^2}{2q(T)^2}-\frac{b^3}{6q(T)^3}-\frac{1}{24}\frac{b^4}{q(T)^4}+O\left(\frac{1}{q(T)^5}\right)\right]^{-1} \\
& \times-\left[-\frac{b}{q(T)}+\frac{b^2}{2q(T)^2}-\frac{b^3}{6q(T)^3}+\frac{1}{24}\frac{b^4}{q(T)^4}+O\left(\frac{1}{q(T)^5}\right)\right]^{-1} \\
= & \frac{q(T)^3}{2b^3}\left[1+\frac{b}{q(T)}+\frac{2}{3}\frac{b^2}{q(T)^2}+\frac{1}{3}\frac{b^3}{q(T)^3}+O\left(\frac{1}{q(T)^4}\right)\right]^{-1} \\
& \times\left[1+\frac{b}{2q(T)}+\frac{b^2}{6q(T)^2}+\frac{1}{24}\frac{b^3}{q(T)^3}+O\left(\frac{1}{q(T)^4}\right)\right]^{-1} \\
& \times\left[1-\frac{b}{2q(T)}+\frac{b^2}{6q(T)^2}-\frac{1}{24}\frac{b^3}{q(T)^3}+O\left(\frac{1}{q(T)^4}\right)\right]^{-1} \\
= & \frac{q(T)^3}{2b^3}\left[1-\frac{b}{q(T)}-\frac{2}{3}\frac{b^2}{q(T)^2}+\frac{b^2}{q(T)^2}-\frac{1}{3}\frac{b^3}{q(T)^3}+\frac{4}{3}\frac{b^3}{q(T)^3}-\frac{b^3}{q(T)^3}+O\left(\frac{1}{q(T)^4}\right)\right] \\
& \times\left[1-\frac{b}{2q(T)}-\frac{1}{6}\frac{b^2}{q(T)^2}+\frac{1}{4}\frac{b^2}{q(T)^2}-\frac{1}{24}\frac{b^3}{q(T)^3}-\frac{1}{8}\frac{b^3}{q(T)^3}+\frac{1}{6}\frac{b^3}{q(T)^3}+O\left(\frac{1}{q(T)^4}\right)\right] \\
& \times\left[1+\frac{b}{2q(T)}-\frac{1}{6}\frac{b^2}{q(T)^2}+\frac{1}{4}\frac{b^2}{q(T)^2}+\frac{1}{24}\frac{b^3}{q(T)^3}-\frac{1}{6}\frac{b^3}{q(T)^3}+\frac{1}{8}\frac{b^3}{q(T)^3}+O\left(\frac{1}{q(T)^4}\right)\right] \\
= & \frac{q(T)^3}{2b^3}\left[1-\frac{b}{q(T)}+\frac{1}{3}\frac{b^2}{q(T)^2}+O\left(\frac{1}{q(T)^4}\right)\right]\left[1-\frac{b}{2q(T)}+\frac{1}{12}\frac{b^2}{q(T)^2}+O\left(\frac{1}{q(T)^4}\right)\right] \\
& \times\left[1+\frac{b}{2q(T)}+\frac{1}{12}\frac{b^2}{q(T)^2}+O\left(\frac{1}{q(T)^4}\right)\right] \\
= & \frac{q(T)^3}{2b^3}\left[1-\frac{b}{q(T)}-\frac{b}{2q(T)}+\frac{1}{12}\frac{b^2}{q(T)^2}+\frac{1}{3}\frac{b^2}{q(T)^2}+\frac{1}{2}\frac{b^2}{q(T)^2}-\frac{1}{12}\frac{b^3}{q(T)^3}\right. \\
& \left.-\frac{1}{6}\frac{b^3}{q(T)^3}+O\left(\frac{1}{q(T)^4}\right)\right]\left[1+\frac{b}{2q(T)}+\frac{1}{12}\frac{b^2}{q(T)^2}+O\left(\frac{1}{q(T)^4}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{q(T)^3}{2b^3}\left[1-\frac{3}{2}\frac{b}{q(T)}+\frac{11}{12}\frac{b^2}{q(T)^2}-\frac{1}{4}\frac{b^3}{q(T)^3}+O\left(\frac{1}{q(T)^4}\right)\right] \\
&\quad \times\left[1+\frac{b}{2q(T)}+\frac{1}{12}\frac{b^2}{q(T)^2}+O\left(\frac{1}{q(T)^4}\right)\right] \\
&= \frac{q(T)^3}{2b^3}\left[1+\frac{1}{2}\frac{b}{q(T)}-\frac{3}{2}\frac{b}{q(T)}+\frac{11}{12}\frac{b^2}{q(T)^2}-\frac{3}{4}\frac{b^2}{q(T)^2}+\frac{1}{12}\frac{b^2}{q(T)^2}-\frac{1}{8}\frac{b^3}{q(T)^3}\right. \\
&\quad \left.+\frac{11}{24}\frac{b^3}{q(T)^3}-\frac{1}{4}\frac{b^3}{q(T)^3}+O\left(\frac{1}{q(T)^4}\right)\right] \\
&= \frac{q(T)^3}{2b^3}\left[1-\frac{b}{q(T)}+\frac{1}{4}\frac{b^2}{q(T)^2}+\frac{1}{12}\frac{b^3}{q(T)^3}+O\left(\frac{1}{q(T)^4}\right)\right],
\end{aligned}
$$

$$
\begin{aligned}
&\left[1-\exp\left\{\frac{2b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{\frac{b}{q(T)}\right\}\right]^{-1} \\
&= \left[-\frac{2b}{q(T)}-\frac{4b^2}{2q(T)^2}-\frac{8}{6}\frac{b^3}{q(T)^3}-\frac{16}{24}\frac{b^4}{q(T)^4}+O\left(\frac{1}{q(T)^5}\right)\right]^{-1} \\
&\quad \times\left[-\frac{b}{q(T)}-\frac{b^2}{2q(T)^2}-\frac{b^3}{6q(T)^3}-\frac{1}{24}\frac{b^4}{q(T)^4}+O\left(\frac{1}{q(T)^5}\right)\right]^{-1} \\
&= \frac{q(T)^2}{2b^2}\left[1+\frac{b}{q(T)}+\frac{2}{3}\frac{b^2}{q(T)^2}+\frac{1}{3}\frac{b^3}{q(T)^3}+O\left(\frac{1}{q(T)^4}\right)\right]^{-1} \\
&\quad \times\left[1+\frac{b}{2q(T)}+\frac{b^2}{6q(T)^2}+\frac{1}{24}\frac{b^3}{q(T)^3}+O\left(\frac{1}{q(T)^4}\right)\right]^{-1} \\
&= \frac{q(T)^2}{2b^2}\left[1-\frac{b}{q(T)}-\frac{2}{3}\frac{b^2}{q(T)^2}+\frac{b^2}{q(T)^2}-\frac{1}{3}\frac{b^3}{q(T)^3}+\frac{4}{3}\frac{b^3}{q(T)^3}-\frac{b^3}{q(T)^3}+O\left(\frac{1}{q(T)^4}\right)\right] \\
&\quad \times\left[1-\frac{b}{2q(T)}-\frac{1}{6}\frac{b^2}{q(T)^2}+\frac{1}{4}\frac{b^2}{q(T)^2}-\frac{1}{24}\frac{b^3}{q(T)^3}-\frac{1}{8}\frac{b^3}{q(T)^3}+\frac{1}{6}\frac{b^3}{q(T)^3}+O\left(\frac{1}{q(T)^4}\right)\right] \\
&= \frac{q(T)^2}{2b^2}\left[1-\frac{b}{q(T)}+\frac{1}{3}\frac{b^2}{q(T)^2}+O\left(\frac{1}{q(T)^4}\right)\right]\left[1-\frac{b}{2q(T)}+\frac{1}{12}\frac{b^2}{q(T)^2}+O\left(\frac{1}{q(T)^4}\right)\right] \\
&= \frac{q(T)^2}{2b^2}\left[1-\frac{b}{q(T)}-\frac{b}{2q(T)}+\frac{1}{12}\frac{b^2}{q(T)^2}+\frac{1}{3}\frac{b^2}{q(T)^2}+\frac{1}{2}\frac{b^2}{q(T)^2}-\frac{1}{12}\frac{b^3}{q(T)^3}\right. \\
&\quad \left.-\frac{1}{6}\frac{b^3}{q(T)^3}+O\left(\frac{1}{q(T)^4}\right)\right] \\
&= \frac{q(T)^2}{2b^2}\left[1-\frac{3}{2}\frac{b}{q(T)}+\frac{11}{12}\frac{b^2}{q(T)^2}-\frac{1}{4}\frac{b^3}{q(T)^3}+O\left(\frac{1}{q(T)^4}\right)\right],
\end{aligned}
$$

$$
\begin{aligned}
& \exp\left\{-\frac{b\left(T-g\right)}{q\left(T\right)}\right\} \\
= \; & \exp\left\{\frac{gb}{q\left(T\right)}\right\}\exp\left\{-b\frac{T}{q\left(T\right)}\right\} \\
= \; & \left[1+\frac{gb}{q\left(T\right)}+\frac{1}{2}\frac{g^2b^2}{q\left(T\right)^2}+\frac{1}{6}\frac{g^3b^3}{q\left(T\right)^3}+O\left(\frac{1}{q\left(T\right)^4}\right)\right] \\
& \times\left[1-\frac{bT}{q\left(T\right)}+\frac{b^2T^2}{2q\left(T\right)^2}-\frac{1}{6}\frac{b^3T^3}{q\left(T\right)^3}+O\left(\frac{T^4}{q\left(T\right)^4}\right)\right] \\
= \; & 1-\frac{bT}{q\left(T\right)}+\frac{gb}{q\left(T\right)}+\frac{b^2T^2}{2q\left(T\right)^2}-\frac{gb^2T}{q\left(T\right)^2}-\frac{1}{6}\frac{b^3T^3}{q\left(T\right)^3}+\frac{1}{2}\frac{g^2b^2}{q\left(T\right)^2}+\frac{g}{2}\frac{b^3T^2}{q\left(T\right)^3} \\
& -\frac{1}{2}\frac{g^2b^3T}{q\left(T\right)^3}+\frac{1}{6}\frac{g^3b^3}{q\left(T\right)^3}+O\left(\frac{T^4}{q\left(T\right)^4}\right)
\end{aligned}
$$

$$
\begin{aligned}
& \exp\left\{-\frac{2b\left(T-g\right)}{q\left(T\right)}\right\} \\
= \; & \exp\left\{\frac{2gb}{q\left(T\right)}\right\}\exp\left\{-\frac{2bT}{q\left(T\right)}\right\} \\
= \; & \left[1+\frac{2gb}{q\left(T\right)}+\frac{4g^2}{2}\frac{b^2}{q\left(T\right)^2}+\frac{4g^3}{3}\frac{b^3}{q\left(T\right)^3}+O\left(\frac{1}{q\left(T\right)^4}\right)\right] \\
& \times\left[1-\frac{2bT}{q\left(T\right)}+2\frac{b^2T^2}{q\left(T\right)^2}-\frac{4}{3}\frac{b^3T^3}{q\left(T\right)^3}+O\left(\frac{T^4}{q\left(T\right)^4}\right)\right] \\
= \; & 1-\frac{2bT}{q\left(T\right)}+\frac{2gb}{q\left(T\right)}+2\frac{b^2T^2}{q\left(T\right)^2}-4g\frac{b^2T}{q\left(T\right)^2}-\frac{4}{3}\frac{b^3T^3}{q\left(T\right)^3}+\frac{2g^2b^2}{q\left(T\right)^2}+4g\frac{b^3T^2}{q\left(T\right)^3} \\
& -4g^2\frac{b^3T}{q\left(T\right)^3}+\frac{4g^3}{3}\frac{b^3}{q\left(T\right)^3}+O\left(\frac{T^4}{q\left(T\right)^4}\right)
\end{aligned}
$$

$$
\begin{aligned}
& \exp\left\{-\frac{b\left(T-g-2\right)}{q\left(T\right)}\right\} \\
= \; & \exp\left\{\frac{\left(g+2\right)b}{q\left(T\right)}\right\}\exp\left\{-b\frac{T}{q\left(T\right)}\right\} \\
= \; & \left[1+\frac{\left(g+2\right)b}{q\left(T\right)}+\frac{\left(g+2\right)^2}{2}\frac{b^2}{q\left(T\right)^2}+\frac{\left(g+2\right)^3}{6}\frac{b^3}{q\left(T\right)^3}+O\left(\frac{1}{q\left(T\right)^4}\right)\right] \\
& \times\left[1-\frac{bT}{q\left(T\right)}+\frac{1}{2}\frac{b^2T^2}{q\left(T\right)^2}-\frac{1}{6}\frac{b^3T^3}{q\left(T\right)^3}+O\left(\frac{T^4}{q\left(T\right)^4}\right)\right] \\
= \; & 1-\frac{bT}{q\left(T\right)}+\frac{\left(g+2\right)b}{q\left(T\right)}+\frac{1}{2}\frac{b^2T^2}{q\left(T\right)^2}-\left(g+2\right)\frac{b^2T}{q\left(T\right)^2}-\frac{1}{6}\frac{b^3T^3}{q\left(T\right)^3}+\frac{\left(g+2\right)^2}{2}\frac{b^2}{q\left(T\right)^2} \\
& +\frac{g+2}{2}\frac{b^3T^2}{q\left(T\right)^3}-\frac{\left(g+2\right)^2}{2}\frac{b^3T}{q\left(T\right)^3}+\frac{\left(g+2\right)^3}{6}\frac{b^3}{q\left(T\right)^3}+O\left(\frac{T^4}{q\left(T\right)^4}\right).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\sum_{t=g+2}^{T}\sum_{s=g+1}^{t-1}\sum_{j=1}^{s-g}\exp\left\{-b\frac{(t-g-j)}{q(T)}\right\}\exp\left\{-b\frac{(s-g-j)}{q(T)}\right\}\\
=\;& -\frac{q(T)^3}{2b^3}\left[1-\frac{1}{4}\frac{b^2}{q(T)^2}+O\left(\frac{1}{q(T)^4}\right)\right]\left[1-\frac{b}{q(T)}+\frac{b^2}{2q(T)^2}-\frac{1}{6}\frac{b^3}{q(T)^3}+O\left(\frac{1}{q(T)^4}\right)\right]\\
&+\frac{q(T)^3}{2b^3}\left[1-\frac{1}{4}\frac{b^2}{q(T)^2}+O\left(\frac{1}{q(T)^4}\right)\right]\left[1-\frac{bT}{q(T)}+\frac{gb}{q(T)}+\frac{b^2T^2}{2q(T)^2}-\frac{gb^2T}{q(T)^2}-\frac{1}{6}\frac{b^3T^3}{q(T)^3}\right.\\
&\qquad\left.+\frac{1}{2}\frac{g^2b^2}{q(T)^2}+\frac{g}{2}\frac{b^3T^2}{q(T)^3}-\frac{1}{2}\frac{g^2b^3T}{q(T)^3}+\frac{1}{6}\frac{g^3b^3}{q(T)^3}+O\left(\frac{T^4}{q(T)^4}\right)\right]\\
&+\frac{q(T)^3}{4b^3}\left[1+\frac{1}{2}\frac{b}{q(T)}-\frac{1}{4}\frac{b^2}{q(T)^2}-\frac{1}{6}\frac{b^3}{q(T)^3}+O\left(\frac{1}{q(T)^4}\right)\right]\\
&\times\left[1-2\frac{b}{q(T)}+2\frac{b^2}{q(T)^2}-\frac{4}{3}\frac{b^3}{q(T)^3}+O\left(\frac{1}{q(T)^4}\right)\right]\\
&-\frac{q(T)^3}{4b^3}\left[1+\frac{1}{2}\frac{b}{q(T)}-\frac{1}{4}\frac{b^2}{q(T)^2}-\frac{1}{6}\frac{b^3}{q(T)^3}+O\left(\frac{1}{q(T)^4}\right)\right]\left[1-\frac{2bT}{q(T)}+\frac{2gb}{q(T)}+2\frac{b^2T^2}{q(T)^2}\right.\\
&\left.-4g\frac{b^2T}{q(T)^2}-\frac{4}{3}\frac{b^3T^3}{q(T)^3}+\frac{2g^2b^2}{q(T)^2}+4g\frac{b^3T^2}{q(T)^3}-4g^2\frac{b^3T}{q(T)^3}+\frac{4g^3}{3}\frac{b^3}{q(T)^3}+O\left(\frac{T^4}{q(T)^4}\right)\right]\\
&-\frac{q(T)^3}{2b^3}\left[1-\frac{b}{q(T)}+\frac{1}{4}\frac{b^2}{q(T)^2}+\frac{1}{12}\frac{b^3}{q(T)^3}+O\left(\frac{1}{q(T)^4}\right)\right]\\
&\times\left[1+\frac{b}{q(T)}+\frac{1}{2}\frac{b^2}{q(T)^2}+\frac{1}{6}\frac{b^3}{q(T)^3}+O\left(\frac{1}{q(T)^4}\right)\right]\\
&+\frac{q(T)^3}{2b^3}\left[1-\frac{b}{q(T)}+\frac{1}{4}\frac{b^2}{q(T)^2}+\frac{1}{12}\frac{b^3}{q(T)^3}+O\left(\frac{1}{q(T)^4}\right)\right]\\
&\times\left[1-\frac{bT}{q(T)}+\frac{(g+2)b}{q(T)}+\frac{1}{2}\frac{b^2T^2}{q(T)^2}-(g+2)\frac{b^2T}{q(T)^2}-\frac{1}{6}\frac{b^3T^3}{q(T)^3}+\frac{(g+2)^2}{2}\frac{b^2}{q(T)^2}\right.\\
&\left.+\frac{g+2}{2}\frac{b^3T^2}{q(T)^3}-\frac{(g+2)^2}{2}\frac{b^3T}{q(T)^3}+\frac{(g+2)^3}{6}\frac{b^3}{q(T)^3}+O\left(\frac{T^4}{q(T)^4}\right)\right]\\
&+\frac{q(T)^2}{2b^2}\left[1-\frac{3}{2}\frac{b}{q(T)}+\frac{11}{12}\frac{b^2}{q(T)^2}-\frac{1}{4}\frac{b^3}{q(T)^3}+O\left(\frac{1}{q(T)^4}\right)\right](T-g-1)\\
&\qquad\times\left[1+\frac{2b}{q(T)}+2\frac{b^2}{q(T)^2}+\frac{4}{3}\frac{b^3}{q(T)^3}+O\left(\frac{1}{q(T)^4}\right)\right],
\end{aligned}
$$

so that

$$
\begin{aligned}
&\sum_{t=g+2}^{T}\sum_{s=g+1}^{t-1}\sum_{j=1}^{s-g}\exp\left\{-b\frac{(t-g-j)}{q(T)}\right\}\exp\left\{-b\frac{(s-g-j)}{q(T)}\right\}\\
=\;& -\frac{q(T)^3}{2b^3}\left[1-\frac{b}{q(T)}+\frac{1}{2}\frac{b^2}{q(T)^2}-\frac{1}{4}\frac{b^2}{q(T)^2}-\frac{1}{6}\frac{b^3}{q(T)^3}+\frac{1}{4}\frac{b^3}{q(T)^3}+O\left(\frac{1}{q(T)^4}\right)\right]\\
&+\frac{q(T)^3}{2b^3}\left[1-\frac{bT}{q(T)}+\frac{gb}{q(T)}+\frac{b^2T^2}{2q(T)^2}-\frac{gb^2T}{q(T)^2}-\frac{1}{6}\frac{b^3T^3}{q(T)^3}+\frac{g^2}{2}\frac{b^2}{q(T)^2}-\frac{1}{4}\frac{b^2}{q(T)^2}\right.\\
&\qquad\left.+\frac{g}{2}\frac{b^3T^2}{q(T)^3}-\frac{g^2}{2}\frac{b^3T}{q(T)^3}+\frac{1}{4}\frac{b^3T}{q(T)^3}+\frac{g^3}{6}\frac{b^3}{q(T)^3}-\frac{g}{4}\frac{b^3}{q(T)^3}+O\left(\frac{T^4}{q(T)^4}\right)\right]\\
&+\frac{q(T)^3}{4b^3}\left[1-2\frac{b}{q(T)}+\frac{1}{2}\frac{b}{q(T)}+2\frac{b^2}{q(T)^2}-\frac{b^2}{q(T)^2}-\frac{1}{4}\frac{b^2}{q(T)^2}-\frac{4}{3}\frac{b^3}{q(T)^3}+\frac{b^3}{q(T)^3}+\frac{1}{2}\frac{b^3}{q(T)^3}\right.\\
&\qquad\left.-\frac{1}{6}\frac{b^3}{q(T)^3}+O\left(\frac{1}{q(T)^4}\right)\right]\\
&-\frac{q(T)^3}{4b^3}\left[1-\frac{2bT}{q(T)}+\frac{2gb}{q(T)}+\frac{1}{2}\frac{b}{q(T)}+2\frac{b^2T^2}{q(T)^2}-4g\frac{b^2T}{q(T)^2}-\frac{b^2T}{q(T)^2}-\frac{4}{3}\frac{b^3T^3}{q(T)^3}\right.\\
&\qquad+\frac{2g^2b^2}{q(T)^2}+\frac{gb^2}{q(T)^2}-\frac{1}{4}\frac{b^2}{q(T)^2}+\frac{4gb^3T^2}{q(T)^3}+\frac{b^3T^2}{q(T)^3}-\frac{4g^2b^3T}{q(T)^3}-\frac{2gb^3T}{q(T)^3}\\
&\qquad\left.+\frac{1}{2}\frac{b^3T}{q(T)^3}+\frac{4g^3}{3}\frac{b^3}{q(T)^3}+\frac{g^2b^3}{q(T)^3}-\frac{g}{2}\frac{b^3}{q(T)^3}-\frac{1}{6}\frac{b^3}{q(T)^3}+O\left(\frac{T^4}{q(T)^4}\right)\right]\\
&-\frac{q(T)^3}{2b^3}\left[1+\frac{b}{q(T)}-\frac{b}{q(T)}+\frac{1}{2}\frac{b^2}{q(T)^2}-\frac{b^2}{q(T)^2}+\frac{1}{4}\frac{b^2}{q(T)^2}-\frac{1}{2}\frac{b^3}{q(T)^3}+\frac{1}{6}\frac{b^3}{q(T)^3}\right.\\
&\qquad\left.+\frac{1}{4}\frac{b^3}{q(T)^3}+\frac{1}{12}\frac{b^3}{q(T)^3}+O\left(\frac{1}{q(T)^4}\right)\right]\\
&+\frac{q(T)^3}{2b^3}\left[1-\frac{bT}{q(T)}+\frac{(g+2)b}{q(T)}-\frac{b}{q(T)}+\frac{b^2T^2}{2q(T)^2}-\frac{(g+2)b^2T}{q(T)^2}+\frac{b^2T}{q(T)^2}-\frac{b^3T^3}{6q(T)^3}+\frac{(g+2)^2b^2}{2q(T)^2}\right.\\
&\qquad-\frac{(g+2)b^2}{q(T)^2}+\frac{1}{4}\frac{b^2}{q(T)^2}+\frac{g+2}{2}\frac{b^3T^2}{q(T)^3}-\frac{1}{2}\frac{b^3T^2}{q(T)^3}-\frac{(g+2)^2}{2}\frac{b^3T}{q(T)^3}+\frac{(g+2)b^3T}{q(T)^3}\\
&\qquad\left.-\frac{b^3T}{4q(T)^3}+\frac{(g+2)^3}{6}\frac{b^3}{q(T)^3}-\frac{(g+2)^2}{2}\frac{b^3}{q(T)^3}+\frac{(g+2)}{4}\frac{b^3}{q(T)^3}+\frac{b^3}{12q(T)^3}+O\left(\frac{T^4}{q(T)^4}\right)\right]\\
&+\frac{Tq(T)^2}{2b^2}\left[1+\frac{2b}{q(T)}-\frac{3}{2}\frac{b}{q(T)}+2\frac{b^2}{q(T)^2}-3\frac{b^2}{q(T)^2}+\frac{11}{12}\frac{b^2}{q(T)^2}+\frac{4}{3}\frac{b^3}{q(T)^3}-3\frac{b^3}{q(T)^3}\right.\\
&\qquad\left.+\frac{11}{6}\frac{b^3}{q(T)^3}-\frac{1}{4}\frac{b^3}{q(T)^3}+O\left(\frac{1}{q(T)^4}\right)\right]\\
&-\frac{g+1}{2}\frac{q(T)^2}{b^2}\left[1+\frac{2b}{q(T)}-\frac{3}{2}\frac{b}{q(T)}+2\frac{b^2}{q(T)^2}-3\frac{b^2}{q(T)^2}+\frac{11}{12}\frac{b^2}{q(T)^2}+\frac{4}{3}\frac{b^3}{q(T)^3}-3\frac{b^3}{q(T)^3}\right.\\
&\qquad\left.+\frac{11}{6}\frac{b^3}{q(T)^3}-\frac{1}{4}\frac{b^3}{q(T)^3}+O\left(\frac{1}{q(T)^4}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{t=g+2}^{T}\sum_{s=g+1}^{t-1}\sum_{j=1}^{s-g}\exp\left\{-b\frac{(t-g-j)}{q(T)}\right\}\exp\left\{-b\frac{(s-g-j)}{q(T)}\right\}\\
=\;& -\frac{q(T)^3}{2b^3}\left[1-\frac{b}{q(T)}+\frac{1}{4}\frac{b^2}{q(T)^2}+\frac{1}{12}\frac{b^3}{q(T)^3}+O\left(\frac{1}{q(T)^4}\right)\right]\\
&+\frac{q(T)^3}{2b^3}\left[1-\frac{bT}{q(T)}+\frac{gb}{q(T)}+\frac{b^2T^2}{2q(T)^2}-\frac{gb^2T}{q(T)^2}-\frac{b^3T^3}{6q(T)^3}+\frac{2g^2-1}{4}\frac{b^2}{q(T)^2}\right.\\
&\qquad\left.+\frac{g}{2}\frac{b^3T^2}{q(T)^3}-\left(\frac{2g^2-1}{4}\right)\frac{b^3T}{q(T)^3}+\frac{g\left(2g^2-3\right)}{12}\frac{b^3}{q(T)^3}+O\left(\frac{T^4}{q(T)^4}\right)\right]\\
&+\frac{q(T)^3}{4b^3}\left[1-\frac{3}{2}\frac{b}{q(T)}+\frac{3}{4}\frac{b^2}{q(T)^2}+O\left(\frac{1}{q(T)^4}\right)\right]\\
&-\frac{q(T)^3}{4b^3}\left[1-\frac{2bT}{q(T)}+\frac{4g+1}{2}\frac{b}{q(T)}+2\frac{b^2T^2}{q(T)^2}-\frac{(4g+1)b^2T}{q(T)^2}-\frac{4}{3}\frac{b^3T^3}{q(T)^3}\right.\\
&\qquad+\frac{8g^2+4g-1}{4}\frac{b^2}{q(T)^2}+\frac{(4g+1)b^3T^2}{q(T)^3}-\frac{8g^2+4g-1}{2}\frac{b^3T}{q(T)^3}\\
&\qquad\left.+\frac{8g^3+6g^2-3g-1}{6}\frac{b^3}{q(T)^3}+O\left(\frac{T^4}{q(T)^4}\right)\right]\\
&-\frac{q(T)^3}{2b^3}\left[1-\frac{1}{4}\frac{b^2}{q(T)^2}+O\left(\frac{1}{q(T)^4}\right)\right]\\
&+\frac{q(T)^3}{2b^3}\left[1-\frac{bT}{q(T)}+\frac{(g+1)b}{q(T)}+\frac{1}{2}\frac{b^2T^2}{q(T)^2}-\frac{(g+1)b^2T}{q(T)^2}-\frac{1}{6}\frac{b^3T^3}{q(T)^3}\right.\\
&\qquad+\frac{2(g+2)^2-4(g+2)+1}{4}\frac{b^2}{q(T)^2}+\frac{g+1}{2}\frac{b^3T^2}{q(T)^3}\\
&\qquad-\frac{2(g+2)^2-4(g+2)+1}{4}\frac{b^3T}{q(T)^3}\\
&\qquad\left.+\frac{2(g+2)^3-6(g+2)^2+3(g+2)+1}{12}\frac{b^3}{q(T)^3}+O\left(\frac{T^4}{q(T)^4}\right)\right]\\
&+\frac{Tq(T)^2}{2b^2}\left[1+\frac{1}{2}\frac{b}{q(T)}-\frac{1}{12}\frac{b^2}{q(T)^2}-\frac{1}{12}\frac{b^3}{q(T)^3}+O\left(\frac{1}{q(T)^4}\right)\right]\\
&-\frac{g+1}{2}\frac{q(T)^2}{b^2}\left[1+\frac{1}{2}\frac{b}{q(T)}-\frac{1}{12}\frac{b^2}{q(T)^2}-\frac{1}{12}\frac{b^3}{q(T)^3}+O\left(\frac{1}{q(T)^4}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{t=g+2}^{T}\sum_{s=g+1}^{t-1}\sum_{j=1}^{s-g}\exp\left\{-b\frac{(t-g-j)}{q(T)}\right\}\exp\left\{-b\frac{(s-g-j)}{q(T)}\right\} \\
= \;& \left[-\frac{q(T)^3}{2b^3}+\frac{1}{2}\frac{q(T)^2}{b^2}-\frac{1}{8}\frac{q(T)}{b}-\frac{1}{24}+O\left(\frac{1}{q(T)}\right)\right] \\
&+\left[\frac{q(T)^3}{2b^3}-\frac{1}{2}\frac{Tq(T)^2}{b^2}+\frac{g}{2}\frac{q(T)^2}{b^2}+\frac{T^2q(T)}{4b}-\frac{g}{2}\frac{Tq(T)}{b}-\frac{1}{12}T^3\right. \\
&\quad\left.+\left(\frac{2g^2-1}{8}\right)\frac{q(T)}{b}+\frac{g}{4}T^2-\left(\frac{2g^2-1}{8}\right)T+\frac{5}{12}+O\left(\frac{T^4}{q(T)}\right)\right] \\
&+\left[\frac{q(T)^3}{4b^3}-\frac{3}{8}\frac{q(T)^2}{b^2}+\frac{3}{16}\frac{q(T)}{b}+O\left(\frac{1}{q(T)}\right)\right] \\
&-\left[\frac{q(T)^3}{4b^3}-\frac{1}{2}\frac{Tq(T)^2}{b^2}+\frac{4g+1}{8}\frac{q(T)^2}{b^2}+\frac{1}{2}\frac{T^2q(T)}{b}-\frac{4g+1}{4}\frac{Tq(T)}{b}-\frac{1}{3}T^3\right. \\
&\quad+\frac{8g^2+4g-1}{16}\frac{q(T)}{b}+\frac{4g+1}{4}T^2-\frac{8g^2+4g-1}{8}T \\
&\quad\left.+\frac{8g^3+6g^2-3g-1}{24}+O\left(\frac{T^4}{q(T)}\right)\right] \\
&-\left[\frac{q(T)^3}{2b^3}-\frac{1}{8}\frac{q(T)}{b}+O\left(\frac{1}{q(T)}\right)\right] \\
&+\left[\frac{q(T)^3}{2b^3}-\frac{1}{2}\frac{Tq(T)^2}{b^2}+\frac{g+1}{2}\frac{q(T)^2}{b^2}+\frac{1}{4}\frac{T^2q(T)}{b}-\frac{g+1}{2}\frac{Tq(T)}{b}\right. \\
&-\frac{1}{12}T^3+\frac{2(g+2)^2-4(g+2)+1}{8}\frac{q(T)}{b}+\frac{g+1}{4}T^2-\frac{2(g+2)^2-4(g+2)+1}{8}T \\
&\left.+\frac{2(g+2)^3-6(g+2)^2+3(g+2)+1}{24}+O\left(\frac{T^4}{q(T)}\right)\right] \\
&+\left[\frac{Tq(T)^2}{2b^2}+\frac{1}{4}\frac{Tq(T)}{b}-\frac{1}{24}T-\frac{1}{24}\frac{bT}{q(T)}+O\left(\frac{T}{q(T)^2}\right)\right] \\
&-\left[\frac{g+1}{2}\frac{q(T)^2}{b^2}+\frac{g+1}{4}\frac{q(T)}{b}-\frac{g+1}{24}-\frac{g+1}{24}\frac{b}{q(T)}+O\left(\frac{1}{q(T)^2}\right)\right] \\
= \;& \frac{1}{6}T^3\left[1+O\left(\max\left\{\frac{T}{q(T)},\frac{1}{T}\right\}\right)\right]
\end{aligned}
$$

as required for part (a).

Next, consider part (b), where we take $q(T)\sim T$. Specializing the calculations given in the proof

of part (a), we have that

$$
\begin{aligned}
& \sum_{t=g+2}^{T}\sum_{s=g+1}^{t-1}\sum_{j=1}^{s-2}\exp\left\{-b\frac{(t-g-j)}{q(T)}\right\}\exp\left\{-b\frac{(s-g-j)}{q(T)}\right\} \\
= \; & \left[1-\exp\left\{\frac{2b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{b}{q(T)}\right\}\right]^{-2}\exp\left\{-\frac{b}{q(T)}\right\} \\
& -\left[1-\exp\left\{\frac{2b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{b}{q(T)}\right\}\right]^{-2}\exp\left\{\frac{bg}{q(T)}\right\}\exp\left\{-\frac{bT}{q(T)}\right\} \\
& -\left[1-\exp\left\{\frac{2b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{2b}{q(T)}\right\}\right]^{-1}\exp\left\{-\frac{2b}{q(T)}\right\} \\
& +\left[1-\left\{\exp\frac{2b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{2b}{q(T)}\right\}\right]^{-1} \\
& \qquad \times\exp\left\{\frac{2bg}{q(T)}\right\}\exp\left\{-\frac{2bT}{q(T)}\right\} \\
& -\left[1-\exp\left\{\frac{2b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{\frac{b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{b}{q(T)}\right\}\right]^{-1}\exp\left\{\frac{b}{q(T)}\right\} \\
& +\left[1-\exp\left\{\frac{2b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{\frac{b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{b}{q(T)}\right\}\right]^{-1} \\
& \qquad \times\exp\left\{\frac{b(g+2)}{q(T)}\right\}\exp\left\{-\frac{bT}{q(T)}\right\} \\
& +\exp\left\{\frac{2b}{q(T)}\right\}\left[1-\exp\left\{\frac{2b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{\frac{b}{q(T)}\right\}\right]^{-1}(T-g-1) \\
= \; & -\frac{q(T)^3}{2b^3}\left[1+O\left(\frac{1}{T}\right)\right]+\frac{q(T)^3}{2b^3}\exp\left\{-\frac{bT}{q(T)}\right\}\left[1+O\left(\frac{1}{T}\right)\right]+\frac{q(T)^3}{4b^3}\left[1+O\left(\frac{1}{T}\right)\right] \\
& -\frac{q(T)^3}{4b^3}\exp\left\{-\frac{2bT}{q(T)}\right\}\left[1+O\left(\frac{1}{T}\right)\right]-\frac{q(T)^3}{2b^3}\left[1+O\left(\frac{1}{T}\right)\right]+\frac{q(T)^3}{2b^3}\exp\left\{-\frac{bT}{q(T)}\right\}\left[1+O\left(\frac{1}{T}\right)\right] \\
& +\frac{Tq(T)^2}{2b^2}\left[1+O\left(\frac{1}{T}\right)\right] \\
= \; & \frac{Tq(T)^2}{2b^2}\left[1-\frac{3}{2b}\frac{q(T)}{T}+\frac{2}{b}\frac{q(T)}{T}\exp\left\{-\frac{bT}{q(T)}\right\}-\frac{1}{2b}\frac{q(T)}{T}\exp\left\{-\frac{2bT}{q(T)}\right\}\right]\left[1+O\left(\frac{1}{T}\right)\right],
\end{aligned}
$$

as required to show part (b).

Finally, consider part (c), where we take $q(T) \to \infty$ such that $q(T)/T \to 0$. In this case, we have

$$
\begin{aligned}
& \sum_{t=g+2}^{T} \sum_{s=g+1}^{t-1} \sum_{j=1}^{s-2} \exp\left\{-b\frac{(t-g-j)}{q(T)}\right\} \exp\left\{-b\frac{(s-g-j)}{q(T)}\right\} \\
= \; & \left[1-\exp\left\{\frac{2b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{b}{q(T)}\right\}\right]^{-2}\exp\left\{-\frac{b}{q(T)}\right\} \\
& -\left[1-\exp\left\{\frac{2b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{b}{q(T)}\right\}\right]^{-2}\exp\left\{\frac{bg}{q(T)}\right\}\exp\left\{-\frac{bT}{q(T)}\right\} \\
& -\left[1-\exp\left\{\frac{2b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{2b}{q(T)}\right\}\right]^{-1}\exp\left\{-\frac{2b}{q(T)}\right\} \\
& +\left[1-\left\{\exp\frac{2b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{2b}{q(T)}\right\}\right]^{-1} \\
& \quad \times \exp\left\{\frac{2bg}{q(T)}\right\}\exp\left\{-\frac{2bT}{q(T)}\right\} \\
& -\left[1-\exp\left\{\frac{2b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{\frac{b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{b}{q(T)}\right\}\right]^{-1}\exp\left\{\frac{b}{q(T)}\right\} \\
& +\left[1-\exp\left\{\frac{2b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{\frac{b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{b}{q(T)}\right\}\right]^{-1} \\
& \quad \times \exp\left\{\frac{b(g+2)}{q(T)}\right\}\exp\left\{-\frac{bT}{q(T)}\right\} \\
& +\exp\left\{\frac{2b}{q(T)}\right\}\left[1-\exp\left\{\frac{2b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{\frac{b}{q(T)}\right\}\right]^{-1}(T-g-1) \\
= \; & -\frac{q(T)^3}{2b^3}\left[1+O\left(\frac{1}{T}\right)\right]+\frac{q(T)^3}{2b^3}\exp\left\{-\frac{bT}{q(T)}\right\}\left[1+O\left(\frac{1}{T}\right)\right]+\frac{q(T)^3}{4b^3}\left[1+O\left(\frac{1}{T}\right)\right] \\
& -\frac{q(T)^3}{4b^3}\exp\left\{-\frac{2bT}{q(T)}\right\}\left[1+O\left(\frac{1}{T}\right)\right]-\frac{q(T)^3}{2b^3}\left[1+O\left(\frac{1}{T}\right)\right]+\frac{q(T)^3}{2b^3}\exp\left\{-\frac{bT}{q(T)}\right\}\left[1+O\left(\frac{1}{T}\right)\right] \\
& +\frac{Tq(T)^2}{2b^2}\left[1+O\left(\frac{1}{T}\right)\right] \\
= \; & \frac{Tq(T)^2}{2b^2}\left[1+O\left(\frac{q(T)}{T}\right)\right]\left[1+O\left(\frac{1}{T}\right)\right] \\
= \; & \frac{Tq(T)^2}{2b^2}\left[1+O\left(\frac{q(T)}{T}\right)\right],
\end{aligned}
$$

as required to show part (c). $\square$.

Lemma SE-8:

Let b be a positive integer.

(a) Suppose that $q(T) \sim T$. Then,

$$\begin{aligned}
&\sum_{t=3}^{T}\sum_{s=2}^{t-1}\sum_{j=1}^{s-1}\exp\left\{-b\frac{t-1-s}{q(T)}\right\}\exp\left\{-b\frac{s-1-j}{q(T)}\right\} \\
= &\left(\frac{Tq(T)^2}{b^2}\left[1+\exp\left\{-\frac{Tb}{q(T)}\right\}\right]-\frac{2q(T)^3}{b^3}\left[1-\exp\left\{-\frac{Tb}{q(T)}\right\}\right]\right)\left[1+O\left(\frac{1}{T}\right)\right],
\end{aligned}$$

as $T \to \infty$.

(b) Suppose that $q(T) \to \infty$ such that $q(T)/T \to 0$. Then,

$$\sum_{t=3}^{T}\sum_{s=2}^{t-1}\sum_{j=1}^{s-1}\exp\left\{-b\frac{t-1-s}{q(T)}\right\}\exp\left\{-b\frac{s-1-j}{q(T)}\right\} = \frac{Tq(T)^2}{b^2}\left[1+O\left(\frac{1}{q(T)}\right)+O\left(\frac{q(T)}{T}\right)\right],$$

as $T \to \infty$.

Proof of Lemma SE-8:

To proceed, note first that

$$\begin{aligned}
&\sum_{t=3}^{T}\sum_{s=2}^{t-1}\sum_{j=1}^{s-1}\exp\left\{-b\frac{t-1-s}{q(T)}\right\}\exp\left\{-b\frac{s-1-j}{q(T)}\right\} \\
= &\sum_{t=3}^{T}\sum_{s=2}^{t-1}\exp\left\{-b\frac{t-1-s}{q(T)}\right\}\exp\left\{-b\frac{s-1}{q(T)}\right\}\exp\left\{\frac{b}{q(T)}\right\}\sum_{j=1}^{s-1}\exp\left\{b\frac{j-1}{q(T)}\right\} \\
= &\sum_{t=3}^{T}\exp\left\{-b\frac{t-1}{q(T)}\right\}\sum_{s=2}^{t-1}\exp\left\{\frac{2b}{q(T)}\right\}\left[1-\exp\left\{\frac{b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{b\frac{s-1}{q(T)}\right\}\right] \\
= &\exp\left\{\frac{2b}{q(T)}\right\}\left[1-\exp\left\{\frac{b}{q(T)}\right\}\right]^{-1} \\
&\times\left[\sum_{t=3}^{T}\exp\left\{-b\frac{t-1}{q(T)}\right\}(t-2)-\exp\left\{\frac{b}{q(T)}\right\}\sum_{t=3}^{T}\exp\left\{-b\frac{t-1}{q(T)}\right\}\sum_{s=2}^{t-1}\exp\left\{b\frac{s-2}{q(T)}\right\}\right].
\end{aligned}$$

Next, we apply part (a) of Lemma SE-5; and, after further calculation, we obtain

$$
\begin{aligned}
& \sum_{t=3}^{T}\sum_{s=2}^{t-1}\sum_{j=1}^{s-1}\exp\left\{-b\frac{t-1-s}{q(T)}\right\}\exp\left\{-b\frac{s-1-j}{q(T)}\right\} \\
= \; & \exp\left\{\frac{2b}{q(T)}\right\}\left[1-\exp\left\{\frac{b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{b}{q(T)}\right\}\right]^{-2} \\
& \times\left[\exp\left\{-\frac{2b}{q(T)}\right\}-(T-1)\exp\left\{-b\frac{T}{q(T)}\right\}+(T-2)\exp\left\{-b\frac{T+1}{q(T)}\right\}\right] \\
& -\exp\left\{\frac{3b}{q(T)}\right\}\left[1-\exp\left\{\frac{b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{\frac{b}{q(T)}\right\}\right]^{-1} \\
& \times\sum_{t=3}^{T}\exp\left\{-b\frac{t-1}{q(T)}\right\}\left[1-\exp\left\{b\frac{t-2}{q(T)}\right\}\right] \\
= \; & \exp\left\{\frac{2b}{q(T)}\right\}\left[1-\exp\left\{\frac{b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{b}{q(T)}\right\}\right]^{-2} \\
& \times\left[\exp\left\{-\frac{2b}{q(T)}\right\}-(T-1)\exp\left\{-b\frac{T}{q(T)}\right\}+(T-2)\exp\left\{-b\frac{T+1}{q(T)}\right\}\right] \\
& -\exp\left\{\frac{3b}{q(T)}\right\}\left[1-\exp\left\{\frac{b}{q(T)}\right\}\right]^{-2}\sum_{t=3}^{T}\exp\left\{-b\frac{t-1}{q(T)}\right\}\left[1-\exp\left\{b\frac{t-2}{q(T)}\right\}\right] \\
= \; & \exp\left\{\frac{2b}{q(T)}\right\}\left[1-\exp\left\{\frac{b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{b}{q(T)}\right\}\right]^{-2} \\
& \times\left[\exp\left\{-\frac{2b}{q(T)}\right\}-(T-1)\exp\left\{-b\frac{T}{q(T)}\right\}+(T-2)\exp\left\{-b\frac{T+1}{q(T)}\right\}\right] \\
& -\exp\left\{\frac{b}{q(T)}\right\}\left[1-\exp\left\{\frac{b}{q(T)}\right\}\right]^{-2}\left[1-\exp\left\{-\frac{b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-b\frac{T-2}{q(T)}\right\}\right] \\
& -\exp\left\{\frac{2b}{q(T)}\right\}\left[1-\exp\left\{\frac{b}{q(T)}\right\}\right]^{-2}(T-2).
\end{aligned}
$$

Now, consider part (a), where we take $q(T)\sim T$. From the calculations we have performed above, we have

$$
\begin{aligned}
& \sum_{t=3}^{T}\sum_{s=2}^{t-1}\sum_{j=1}^{s-1}\exp\left\{-b\frac{t-1-s}{q(T)}\right\}\exp\left\{-b\frac{s-1-j}{q(T)}\right\} \\
= \; & \exp\left\{\frac{2b}{q(T)}\right\}\left[1-\exp\left\{\frac{b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{b}{q(T)}\right\}\right]^{-2} \\
& \times\left[\exp\left\{-\frac{2b}{q(T)}\right\}-(T-1)\exp\left\{-b\frac{T}{q(T)}\right\}+(T-2)\exp\left\{-b\frac{T}{q(T)}\right\}\exp\left\{-\frac{b}{q(T)}\right\}\right] \\
& -\exp\left\{\frac{b}{q(T)}\right\}\left[1-\exp\left\{\frac{b}{q(T)}\right\}\right]^{-2}\left[1-\exp\left\{-\frac{b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-b\frac{T-2}{q(T)}\right\}\right] \\
& +\exp\left\{\frac{2b}{q(T)}\right\}\left[1-\exp\left\{\frac{b}{q(T)}\right\}\right]^{-2}(T-2) \\
= \; & \exp\left\{\frac{2b}{q(T)}\right\}\left[1-\exp\left\{\frac{b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{b}{q(T)}\right\}\right]^{-2} \\
& \times\left[\exp\left\{-\frac{2b}{q(T)}\right\}-(T-1)\exp\left\{-b\frac{T}{q(T)}\right\}\right. \\
& \left.+(T-2)\exp\left\{-b\frac{T}{q(T)}\right\}\left(1-\frac{b}{q(T)}+O\left(\frac{1}{T^2}\right)\right)\right] \\
& -\exp\left\{\frac{b}{q(T)}\right\}\left[1-\exp\left\{\frac{b}{q(T)}\right\}\right]^{-2}\left[1-\exp\left\{-\frac{b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-b\frac{T-2}{q(T)}\right\}\right] \\
& +\exp\left\{\frac{2b}{q(T)}\right\}\left[1-\exp\left\{\frac{b}{q(T)}\right\}\right]^{-2}(T-2) \\
= \; & \exp\left\{\frac{2b}{q(T)}\right\}\left[1-\exp\left\{\frac{b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{b}{q(T)}\right\}\right]^{-2} \\
& \times\left[\exp\left\{-\frac{2b}{q(T)}\right\}-T\exp\left\{-\frac{bT}{q(T)}\right\}+\exp\left\{-b\frac{T}{q(T)}\right\}\right. \\
& \left.+T\exp\left\{-b\frac{T}{q(T)}\right\}-\frac{bT}{q(T)}\exp\left\{-\frac{bT}{q(T)}\right\}-2\exp\left\{-b\frac{T}{q(T)}\right\}+O\left(\frac{1}{T}\right)\right] \\
& -\exp\left\{\frac{b}{q(T)}\right\}\left[1-\exp\left\{\frac{b}{q(T)}\right\}\right]^{-2}\left[1-\exp\left\{-\frac{b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-b\frac{T-2}{q(T)}\right\}\right] \\
& +\exp\left\{\frac{2b}{q(T)}\right\}\left[1-\exp\left\{\frac{b}{q(T)}\right\}\right]^{-2}(T-2)
\end{aligned}
$$

$$
\begin{aligned}
&= \left(-\frac{q(T)}{b}\right)\left(\frac{q(T)}{b}\right)^2\left[1-\exp\left\{-\frac{Tb}{q(T)}\right\}\right]\left[1+O\left(\frac{1}{T}\right)\right] \\
&\quad -\left(-\frac{q(T)}{b}\right)\left(\frac{q(T)}{b}\right)^2 b\frac{T}{q(T)}\exp\left\{-\frac{Tb}{q(T)}\right\}\left[1+O\left(\frac{1}{T}\right)\right] \\
&\quad -\left(-\frac{q(T)}{b}\right)^2\left(\frac{q(T)}{b}\right)\left[1-\exp\left\{-\frac{Tb}{q(T)}\right\}\right]\left[1+O\left(\frac{1}{T}\right)\right] \\
&\quad +\left(-\frac{q(T)}{b}\right)^2 T\left[1+O\left(\frac{1}{T}\right)\right] \\
&= \left(\frac{Tq(T)^2}{b^2}\left[1+\exp\left\{-\frac{Tb}{q(T)}\right\}\right]-\frac{2q(T)^3}{b^3}\left[1-\exp\left\{-\frac{Tb}{q(T)}\right\}\right]\right)\left[1+O\left(\frac{1}{T}\right)\right]
\end{aligned}
$$

which is the required result.

Next, we turn our attention to part (b), where we take $q(T)\to\infty$ such that $q(T)/T\to 0$. Here, from previous calculations, we have

$$
\begin{aligned}
&\sum_{t=3}^{T}\sum_{s=2}^{t-1}\sum_{j=1}^{s-1}\exp\left\{-b\frac{t-1-s}{q(T)}\right\}\exp\left\{-b\frac{s-1-j}{q(T)}\right\} \\
&= \exp\left\{\frac{2b}{q(T)}\right\}\left[1-\exp\left\{\frac{b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{b}{q(T)}\right\}\right]^{-2} \\
&\quad \times\left[\exp\left\{-\frac{2b}{q(T)}\right\}-(T-1)\exp\left\{-b\frac{T}{q(T)}\right\}+(T-2)\exp\left\{-b\frac{T+1}{q(T)}\right\}\right] \\
&\quad -\exp\left\{\frac{b}{q(T)}\right\}\left[1-\exp\left\{\frac{b}{q(T)}\right\}\right]^{-2}\left[1-\exp\left\{-\frac{b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-b\frac{T-2}{q(T)}\right\}\right] \\
&\quad +\exp\left\{\frac{2b}{q(T)}\right\}\left[1-\exp\left\{\frac{b}{q(T)}\right\}\right]^{-2}(T-2) \\
&= \left(-\frac{q(T)}{b}\right)\left(\frac{q(T)}{b}\right)^2\left[1+O\left(\frac{1}{q(T)}\right)\right] \\
&\quad -\left(-\frac{q(T)}{b}\right)\left(\frac{q(T)}{b}\right)^2\frac{Tb}{q(T)}\exp\left\{-\frac{bT}{q(T)}\right\}\left[1+O\left(\frac{1}{q(T)}\right)\right] \\
&\quad -\left(-\frac{q(T)}{b}\right)\left(\frac{q(T)}{b}\right)^2\exp\left\{-\frac{bT}{q(T)}\right\}\left[1+O\left(\frac{1}{q(T)}\right)\right] \\
&\quad -\left(-\frac{q(T)}{b}\right)^2\left(\frac{q(T)}{b}\right)\left[1+O\left(\frac{1}{q(T)}\right)\right]\left[1+O\left(\exp\left\{-\frac{bT}{q(T)}\right\}\right)\right] \\
&\quad +T\left(-\frac{q(T)}{b}\right)^2\left[1+O\left(\frac{1}{q(T)}\right)\right] \\
&= \frac{Tq(T)^2}{b^2}\left[1+O\left(\frac{1}{q(T)}\right)+O\left(\frac{q(T)}{T}\right)\right],
\end{aligned}
$$

which is the required result. □

Lemma SE-9:

Let b and c be positive integers.

(a) If $q(T) \sim T$, then

$$\begin{aligned}
&\sum_{t=3}^{T}\sum_{s=2}^{t-1}\sum_{j=1}^{t-1}\sum_{k=1}^{s-1}\exp\left\{-b\frac{t-1-j}{q(T)}\right\}\exp\left\{-c\frac{s-1-k}{q(T)}\right\} \\
= &\left(\frac{T^2q(T)^2}{2bc}-\frac{q(T)^4}{b^3c}\left[1-\exp\left\{-b\frac{T}{q(T)}\right\}\right]+\frac{Tq(T)^3}{b^2c}\exp\left\{-\frac{bT}{q(T)}\right\}-\frac{Tq(T)^3}{bc^2}\right. \\
&+\frac{q(T)^4}{bc^3}\left[1-\exp\left\{-c\frac{T}{q(T)}\right\}\right]+\frac{q(T)^4}{b^2c^2}\left[1-\exp\left\{-b\frac{T}{q(T)}\right\}\right] \\
&\left.-\frac{q(T)^4}{bc^2(b+c)}\left[1-\exp\left\{-(b+c)\frac{T}{q(T)}\right\}\right]\right)\left[1+O\left(\frac{1}{T}\right)\right],
\end{aligned}$$

as $T \to \infty$.

(b) If $q(T) \to \infty$ such that $q(T)/T \to 0$, then

$$\begin{aligned}
&\sum_{t=3}^{T}\sum_{s=2}^{t-1}\sum_{j=1}^{t-1}\sum_{k=1}^{s-1}\exp\left\{-b\frac{t-1-j}{q(T)}\right\}\exp\left\{-c\frac{s-1-k}{q(T)}\right\} \\
= &\frac{T^2q(T)^2}{2bc}\left[1+O\left(\frac{1}{q(T)}\right)+O\left(\frac{q(T)}{T}\right)\right],
\end{aligned}$$

as $T \to \infty$.

Proof of Lemma SE-9:

To proceed, note first that

$$\begin{aligned}
&\sum_{t=3}^{T}\sum_{s=2}^{t-1}\sum_{j=1}^{t-1}\exp\left\{-b\frac{t-1-j}{q(T)}\right\}\sum_{k=1}^{s-1}\exp\left\{-c\frac{s-1-k}{q(T)}\right\} \\
= &\left[1-\exp\left\{-\frac{b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{c}{q(T)}\right\}\right]^{-1} \\
&\times\sum_{t=3}^{T}\sum_{s=2}^{t-1}\left[1-\exp\left\{-b\frac{t-1}{q(T)}\right\}\right]\left[1-\exp\left\{-c\frac{s-1}{q(T)}\right\}\right] \\
= &\left[1-\exp\left\{-\frac{b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{c}{q(T)}\right\}\right]^{-1} \\
&\times\sum_{t=3}^{T}\sum_{s=2}^{t-1}\left[1-\exp\left\{-b\frac{t-1}{q(T)}\right\}-\exp\left\{-c\frac{s-1}{q(T)}\right\}+\exp\left\{-b\frac{t-1}{q(T)}\right\}\exp\left\{-c\frac{s-1}{q(T)}\right\}\right] \\
= &\left[1-\exp\left\{-\frac{b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{c}{q(T)}\right\}\right]^{-1}\sum_{t=3}^{T}\left[(t-2)-(t-2)\exp\left\{-b\frac{t-1}{q(T)}\right\}\right. \\
&\left.-\exp\left\{-\frac{c}{q(T)}\right\}\sum_{s=2}^{t-1}\exp\left\{-c\frac{s-2}{q(T)}\right\}+\exp\left\{-b\frac{t-1}{q(T)}\right\}\exp\left\{-\frac{c}{q(T)}\right\}\sum_{s=2}^{t-1}\exp\left\{-c\frac{s-2}{q(T)}\right\}\right].
\end{aligned}$$

Applying part (a) of Lemma SE-5 and performing additional calculation, we get

$$
\begin{aligned}
& \sum_{t=3}^{T}\sum_{s=2}^{t-1}\sum_{j=1}^{t-1}\exp\left\{-b\frac{t-1-j}{q(T)}\right\}\sum_{k=1}^{s-1}\exp\left\{-c\frac{s-1-k}{q(T)}\right\} \\
= & \left[1-\exp\left\{-\frac{b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{c}{q(T)}\right\}\right]^{-1}\frac{(T-2)(T-1)}{2} \\
& -\left[1-\exp\left\{-\frac{b}{q(T)}\right\}\right]^{-3}\left[1-\exp\left\{-\frac{c}{q(T)}\right\}\right]^{-1}\left[\exp\left\{-\frac{2b}{q(T)}\right\}\right. \\
& \left.-(T-1)\exp\left\{-\frac{bT}{q(T)}\right\}+(T-2)\exp\left\{-b\frac{T+1}{q(T)}\right\}\right] \\
& -\left[1-\exp\left\{-\frac{b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{c}{q(T)}\right\}\right]^{-2}\exp\left\{-\frac{c}{q(T)}\right\} \\
& \times\sum_{t=3}^{T}\left[1-\exp\left\{-c\frac{t-2}{q(T)}\right\}\right] \\
& +\left[1-\exp\left\{-\frac{b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{c}{q(T)}\right\}\right]^{-2}\exp\left\{-\frac{c}{q(T)}\right\} \\
& \times\sum_{t=3}^{T}\exp\left\{-b\frac{t-1}{q(T)}\right\}\left[1-\exp\left\{-c\frac{t-2}{q(T)}\right\}\right] \\
= & \left[1-\exp\left\{-\frac{b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{c}{q(T)}\right\}\right]^{-1}\frac{(T-2)(T-1)}{2} \\
& -\left[1-\exp\left\{-\frac{b}{q(T)}\right\}\right]^{-3}\left[1-\exp\left\{-\frac{c}{q(T)}\right\}\right]^{-1}\left[\exp\left\{-\frac{2b}{q(T)}\right\}\right. \\
& \left.-(T-1)\exp\left\{-\frac{bT}{q(T)}\right\}+(T-2)\exp\left\{-b\frac{T+1}{q(T)}\right\}\right] \\
& -\left[1-\exp\left\{-\frac{b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{c}{q(T)}\right\}\right]^{-2}\exp\left\{-\frac{c}{q(T)}\right\} \\
& \times\left\{(T-2)-\exp\left\{-\frac{c}{q(T)}\right\}\left[1-\exp\left\{-\frac{c}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-c\frac{T-2}{q(T)}\right\}\right]\right\} \\
& +\left[1-\exp\left\{-\frac{b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{c}{q(T)}\right\}\right]^{-2}\exp\left\{-\frac{c}{q(T)}\right\} \\
& \times\exp\left\{-\frac{2b}{q(T)}\right\}\left[1-\exp\left\{-\frac{b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-b\frac{T-2}{q(T)}\right\}\right] \\
& -\left[1-\exp\left\{-\frac{b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{c}{q(T)}\right\}\right]^{-2}\exp\left\{-\frac{c}{q(T)}\right\} \\
& \times\exp\left\{-\frac{2b}{q(T)}\right\}\exp\left\{-\frac{c}{q(T)}\right\}\sum_{t=3}^{T}\exp\left\{-(b+c)\frac{t-3}{q(T)}\right\}
\end{aligned}
$$

$$
\begin{aligned}
= & \left[1-\exp\left\{-\frac{b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{c}{q(T)}\right\}\right]^{-1}\frac{(T-2)(T-1)}{2} \\
& -\left[1-\exp\left\{-\frac{b}{q(T)}\right\}\right]^{-3}\left[1-\exp\left\{-\frac{c}{q(T)}\right\}\right]^{-1}\left[\exp\left\{-\frac{2b}{q(T)}\right\}\right. \\
& \left.-(T-1)\exp\left\{-\frac{bT}{q(T)}\right\}+(T-2)\exp\left\{-b\frac{T+1}{q(T)}\right\}\right] \\
& -\left[1-\exp\left\{-\frac{b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{c}{q(T)}\right\}\right]^{-2}\exp\left\{-\frac{c}{q(T)}\right\} \\
& \times\left\{(T-2)-\exp\left\{-\frac{c}{q(T)}\right\}\left[1-\exp\left\{-\frac{c}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-c\frac{T-2}{q(T)}\right\}\right]\right\} \\
& +\left[1-\exp\left\{-\frac{b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{c}{q(T)}\right\}\right]^{-2}\exp\left\{-\frac{c}{q(T)}\right\} \\
& \times\left(\exp\left\{-\frac{2b}{q(T)}\right\}\left[1-\exp\left\{-\frac{b}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-b\frac{T-2}{q(T)}\right\}\right]\right. \\
& \left.-\exp\left\{-\frac{2b}{q(T)}\right\}\exp\left\{-\frac{c}{q(T)}\right\}\left[1-\exp\left\{-\frac{b+c}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-(b+c)\frac{T-2}{q(T)}\right\}\right]\right)
\end{aligned}
$$

For this case, we have Now, consider part (a), where we take $q(T)\sim T$. In this case,

$$
\begin{aligned}
& \sum_{t=3}^{T}\sum_{s=2}^{t-1}\sum_{j=1}^{t-1}\exp\left\{-b\frac{t-1-j}{q(T)}\right\}\sum_{k=1}^{s-1}\exp\left\{-c\frac{s-1-k}{q(T)}\right\} \\
= & \frac{T^2q(T)^2}{2bc}\left[1+O\left(\frac{1}{T}\right)\right]-\frac{q(T)^4}{b^3c}\left[1+O\left(\frac{1}{T}\right)\right] \\
& +\frac{Tq(T)^3}{b^2c}\exp\left\{-\frac{bT}{q(T)}\right\}\left[1+O\left(\frac{1}{T}\right)\right]+\frac{q(T)^4}{b^3c}\exp\left\{-\frac{bT}{q(T)}\right\}\left[1+O\left(\frac{1}{T}\right)\right] \\
& -\frac{Tq(T)^3}{bc^2}\left[1+O\left(\frac{1}{T}\right)\right]+\frac{q(T)^4}{bc^3}\left[1-\exp\left\{-c\frac{T}{q(T)}\right\}\right]\left[1+O\left(\frac{1}{T}\right)\right] \\
& +\frac{q(T)^4}{b^2c^2}\left[1-\exp\left\{-b\frac{T}{q(T)}\right\}\right]\left[1+O\left(\frac{1}{T}\right)\right] \\
& -\frac{q(T)^4}{bc^2(b+c)}\left[1-\exp\left\{-(b+c)\frac{T}{q(T)}\right\}\right]\left[1+O\left(\frac{1}{T}\right)\right] \\
= & \left(\frac{T^2q(T)^2}{2bc}-\frac{q(T)^4}{b^3c}\left[1-\exp\left\{-b\frac{T}{q(T)}\right\}\right]+\frac{Tq(T)^3}{b^2c}\exp\left\{-\frac{bT}{q(T)}\right\}-\frac{Tq(T)^3}{bc^2}\right. \\
& +\frac{q(T)^4}{bc^3}\left[1-\exp\left\{-c\frac{T}{q(T)}\right\}\right]+\frac{q(T)^4}{b^2c^2}\left[1-\exp\left\{-b\frac{T}{q(T)}\right\}\right] \\
& \left.-\frac{q(T)^4}{bc^2(b+c)}\left[1-\exp\left\{-(b+c)\frac{T}{q(T)}\right\}\right]\right)\left[1+O\left(\frac{1}{T}\right)\right].
\end{aligned}
$$

Next, we turn our attention to part (b), where we take $q(T)\to\infty$ such that $q(T)/T\to 0$. For this

case, we have

$$
\begin{aligned}
& \sum_{t=3}^{T}\sum_{s=2}^{t-1}\sum_{j=1}^{t-1}\exp\left\{-b\frac{t-1-j}{q(T)}\right\}\sum_{k=1}^{s-1}\exp\left\{-c\frac{s-1-k}{q(T)}\right\} \\
= \; & \frac{T^2 q(T)^2}{2bc}\left[1+O\left(\frac{1}{q(T)}\right)\right]-\frac{q(T)^4}{b^3c}\left[1+O\left(\frac{1}{q(T)}\right)\right] \\
& +\frac{Tq(T)^3}{b^2c}\exp\left\{-\frac{bT}{q(T)}\right\}\left[1+O\left(\frac{1}{q(T)}\right)\right]+\frac{q(T)^4}{b^3c}\exp\left\{-\frac{bT}{q(T)}\right\}\left[1+O\left(\frac{1}{q(T)}\right)\right] \\
& -\frac{Tq(T)^3}{bc^2}\left[1+O\left(\frac{1}{q(T)}\right)\right]+\frac{q(T)^4}{bc^3}\left[1+O\left(\frac{1}{q(T)}\right)\right]\left[1+O\left(\exp\left\{-c\frac{T}{q(T)}\right\}\right)\right] \\
& +\frac{q(T)^4}{b^2c^2}\left[1+O\left(\frac{1}{q(T)}\right)\right]\left[1+O\left(\exp\left\{-b\frac{T}{q(T)}\right\}\right)\right] \\
& -\frac{q(T)^4}{bc^2(b+c)}\left[1+O\left(\frac{1}{q(T)}\right)\right]\left[1+O\left(\exp\left\{-(b+c)\frac{T}{q(T)}\right\}\right)\right] \\
= \; & \frac{T^2 q(T)^2}{2bc}\left[1+O\left(\frac{1}{q(T)}\right)+O\left(\frac{q(T)}{T}\right)\right],
\end{aligned}
$$

which shows the desired result. □

Lemma SE-10 (Phillips and Moon, 1999, Theorem 2): Let $\{Y_{i,T}\}$ be independent random variables such that $E[Y_{i,T}]=0, \quad E\left[Y_{i,T}^2\right]=\omega_{i,T}^2$. Define $\xi_{i,N,T}=\frac{Y_{i,T}}{\omega_{N,T}}$, where $\omega_{N,T}^2=\sum_{i=1}^{N}\omega_{i,T}^2$. Suppose that for all $\epsilon>0$

$$\lim_{N,T\to\infty}\sum_{i=1}^{N}E\left[\xi_{i,N,T}^2\mathbb{I}\left\{\left|\xi_{i,N,T}\right|>\epsilon\right\}\right]=0 \tag{13}$$

Then, as $N,T\to\infty$ jointly $\sum_{i=1}^{N}\xi_{i,N,T}\Rightarrow N(0,1)$.

Lemma SE-11:

Given Assumptions 1-3, the following results hold.

(a) $N^{-1}(T-1)^{-1}\sum_{i=1}^{N}\sum_{t=2}^{T}a_i^2=\mu_a^2+\sigma_a^2+O_p\left(N^{-1/2}\right)$;

(b) $N^{-1}\sum_{i=1}^{N}a_i=\mu_a+O_p\left(N^{-1/2}\right)$;

(c) $\sum_{i=1}^{N}\sum_{t=2}^{T}a_i\varepsilon_{it-g}=O_p\left(\sqrt{NT}\right)$ for $g\in\{0,1\}$;

(d) $\overline{\varepsilon}_{NT}=N^{-1}(T-1)^{-1}\sum_{i=1}^{N}\sum_{t=2}^{T}\varepsilon_{it}=O_p\left(N^{-1/2}T^{-1/2}\right)$;

(e) $N^{-1}\sum_{i=1}^{N}\overline{\varepsilon}_i^2=\frac{\sigma^2}{(T-1)}+O_p\left(\frac{1}{T\sqrt{N}}\right)=O_p\left(\frac{1}{T}\right)$, where $\overline{\varepsilon}_i=(T-1)^{-1}\sum_{s=2}^{T}\varepsilon_{is}$;

(f) $N^{-1}(T-1)^{-1}\sum_{i=1}^{N}\sum_{t=g+2}^{T}\varepsilon_{it-g}^{2}=\sigma^{2}+O_{p}\left(\max\left\{N^{-1/2}T^{-1/2},T^{-1}\right\}\right)$ for (fixed) non-negative integer g;

(g) $\sum_{i=1}^{N}\sum_{t=2}^{T}a_{i}\Delta\varepsilon_{it}=O_{p}\left(\sqrt{N}\right)$;

(h) $N^{-1/2}T^{-1/2}\sum_{i=1}^{N}\sum_{t=4}^{T}\varepsilon_{it-g-1}\varepsilon_{it-g}\Rightarrow N\left(0,\sigma^{4}\right)$, for (fixed) non-negative integer g;

(i) $\sum_{i=1}^{N}\varepsilon_{i1}\varepsilon_{i2}=O_{p}\left(\sqrt{N}\right)$.

Proof of Lemma SE-11:

To show part (a), note that, by Assumption 2,

$$\begin{aligned}E\left[\frac{1}{N(T-1)}\sum_{i=1}^{N}\sum_{t=2}^{T}a_{i}^{2}\right] &= \frac{1}{N(T-1)}\sum_{i=1}^{N}\sum_{t=2}^{T}E\left[a_{i}^{2}\right]=\frac{1}{N(T-1)}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(\mu_{a}^{2}+\sigma_{a}^{2}\right)\\ &= \frac{N(T-1)}{N(T-1)}\left(\mu_{a}^{2}+\sigma_{a}^{2}\right)=\mu_{a}^{2}+\sigma_{a}^{2}.\end{aligned}$$

Moreover, by part (b) of Assumption 2,

$$\begin{aligned}&Var\left(\frac{1}{N(T-1)}\sum_{i=1}^{N}\sum_{t=2}^{T}a_{i}^{2}\right)=E\left(\frac{1}{N(T-1)}\sum_{i=1}^{N}\sum_{t=2}^{T}\left(a_{i}^{2}-\left[\mu_{a}^{2}+\sigma_{a}^{2}\right]\right)\right)^{2}\\ =&\ \frac{1}{N^{2}(T-1)^{2}}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=2}^{T}\sum_{s=2}^{T}E\left[\left(a_{i}^{2}-\left[\mu_{a}^{2}+\sigma_{a}^{2}\right]\right)\left(a_{j}^{2}-\left[\mu_{a}^{2}+\sigma_{a}^{2}\right]\right)\right]\\ =&\ \frac{1}{N^{2}}\sum_{i=1}^{N}E\left[\left(a_{i}^{2}-\left[\mu_{a}^{2}+\sigma_{a}^{2}\right]\right)^{2}\right]=\frac{E\left[a_{i}^{4}\right]-\left(\mu_{a}^{2}+\sigma_{a}^{2}\right)^{2}}{N}=O\left(\frac{1}{N}\right).\end{aligned}$$

Using Markov's inequality, we deduce that

$$\frac{1}{N(T-1)}\sum_{i=1}^{N}\sum_{t=2}^{T}a_{i}^{2}=\mu_{a}^{2}+\sigma_{a}^{2}+O_{p}\left(\frac{1}{\sqrt{N}}\right).$$

To show part (b), note that

$$E\left[\frac{1}{N}\sum_{i=1}^{N}a_{i}\right]=\frac{1}{N}\sum_{i=1}^{N}E\left[a_{i}\right]=\mu_{a},$$

and

$$\begin{aligned}Var\left(\frac{1}{N}\sum_{i=1}^{N}a_{i}\right) &= E\left(\frac{1}{N}\sum_{i=1}^{N}\left[a_{i}-\mu_{a}\right]\right)^{2}=\frac{1}{N^{2}}\sum_{i=1}^{N}\sum_{j=1}^{N}E\left[\left(a_{i}-\mu_{a}\right)\left(a_{j}-\mu_{a}\right)\right]\\ &= \frac{1}{N^{2}}\sum_{i=1}^{N}E\left[\left(a_{i}-\mu_{a}\right)^{2}\right]=\frac{\sigma_{a}^{2}}{N}.\end{aligned}$$

Using Markov's inequality, we deduce that

$$\frac{1}{N}\sum_{i=1}^{N} a_i = \mu_a + O_p\left(\frac{1}{\sqrt{N}}\right).$$

To show (c), note that

$$\begin{aligned} E\left(\sum_{i=1}^{N}\sum_{t=2}^{T} a_i\varepsilon_{it-g}\right)^2 &= \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=2}^{T}\sum_{s=2}^{T} E\left[a_i a_j\right] E\left[\varepsilon_{i,t-g}\varepsilon_{j,s-g}\right] = \sum_{i=1}^{N}\sum_{t=2}^{T} E\left[a_i^2\right] E\left[\varepsilon_{i,t-g}^2\right] \\ &= \sigma^2\left(\mu_a^2+\sigma_a^2\right) N\left(T-1\right) = O\left(NT\right), \end{aligned}$$

from which it follows by applying the Markov's inequality that

$$\sum_{i=1}^{N}\sum_{t=2}^{T} a_i\varepsilon_{it-g} = O_p\left(\sqrt{NT},\right)$$

as required.

To show part (d), note that

$$\begin{aligned} E\left(\frac{1}{N\left(T-1\right)}\sum_{i=1}^{N}\sum_{t=2}^{T}\varepsilon_{it}\right)^2 &= \frac{1}{N^2\left(T-1\right)^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=2}^{T}\sum_{s=2}^{T} E\left[\varepsilon_{it}\varepsilon_{js}\right] = \frac{1}{N^2\left(T-1\right)^2}\sum_{i=1}^{N}\sum_{t=2}^{T} E\left[\varepsilon_{it}^2\right] \\ &= \frac{\sigma^2}{N\left(T-1\right)} = O\left(\frac{1}{NT}\right). \end{aligned}$$

It follows from the Markov's inequality that

$$\frac{1}{N\left(T-1\right)}\sum_{i=1}^{N}\sum_{t=2}^{T}\varepsilon_{it} = O_p\left(\frac{1}{\sqrt{NT}}\right),$$

as required.

To show (e), note first that

$$\frac{1}{N}\sum_{i=1}^{N} E\left[\overline{\varepsilon}_i^2\right] = \frac{1}{N\left(T-1\right)^2}\sum_{i=1}^{N}\sum_{t=2}^{T}\sum_{s=2}^{T} E\left[\varepsilon_{it}\varepsilon_{is}\right] = \frac{\sigma^2}{\left(T-1\right)} = O\left(T^{-1}\right),$$

and

$$
\begin{aligned}
& E\left[\frac{1}{N}\sum_{i=1}^{N}\overline{\varepsilon}_i^2-\frac{\sigma^2}{(T-1)}\right]^2 \\
= \; & E\left[\frac{1}{N(T-1)^2}\sum_{i=1}^{N}\sum_{t=2}^{T}\sum_{s=2}^{T}\varepsilon_{it}\varepsilon_{is}\right]^2-2\frac{1}{N}\sum_{i=1}^{N}E\left[\overline{\varepsilon}_i^2\right]\frac{\sigma^2}{(T-1)}+\frac{\sigma^4}{(T-1)^2} \\
= \; & \frac{1}{N^2(T-1)^4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=2}^{T}\sum_{s=2}^{T}\sum_{g=2}^{T}\sum_{v=2}^{T}E\left[\varepsilon_{it}\varepsilon_{is}\varepsilon_{jg}\varepsilon_{jv}\right]-\frac{\sigma^4}{(T-1)^2} \\
= \; & \frac{1}{N^2(T-1)^4}\sum_{i=1}^{N}\sum_{t=2}^{T}E\left[\varepsilon_{it}^4\right]+\frac{1}{N^2(T-1)^4}\sum_{i\neq j}\sum_{t=2}^{T}\sum_{g=2}^{T}\sigma^4+\frac{3}{N^2(T-1)^4}\sum_{i=1}^{N}\sum_{t\neq s}\sigma^4 \\
& -\frac{\sigma^4}{(T-1)^2} \\
= \; & \frac{E\left[\varepsilon_{it}^4\right]}{N(T-1)^3}+\frac{\sigma^4}{N^2(T-1)^2}N(N-1)+\frac{3\sigma^4}{N(T-1)^4}(T-2)(T-1)-\frac{\sigma^4}{(T-1)^2} \\
= \; & -\frac{\sigma^4}{N(T-1)^2}+\frac{3\sigma^4(T-2)}{N(T-1)^3}+\frac{E\left[\varepsilon_{it}^4\right]}{N(T-1)^3}=O\left(\frac{1}{NT^2}\right).
\end{aligned}
$$

It follows by Markov's inequality that

$$
\frac{1}{N}\sum_{i=1}^{N}\overline{\varepsilon}_i^2=\frac{\sigma^2}{(T-1)}+O_p\left(\frac{1}{T\sqrt{N}}\right).
$$

To show part (f), note that, for all positive integer $T>g+2$

$$
E\left[\frac{1}{N(T-1)}\sum_{i=1}^{N}\sum_{t=g+2}^{T}\varepsilon_{it-g}^2\right]=\frac{1}{N(T-1)}\sum_{i=1}^{N}\sum_{t=g+2}^{T}E\left[\varepsilon_{it}^2\right]=\frac{T-g-1}{T-1}\sigma^2=\sigma^2+O\left(\frac{1}{T}\right).
$$

Moreover, for all $T > g+2$, we have

$$
\begin{aligned}
& E\left[\frac{1}{N(T-1)}\sum_{i=1}^{N}\sum_{t=g+2}^{T}\varepsilon_{it-g}^{2}-\frac{T-g-1}{T-1}\sigma^{2}\right]^{2} \\
= & \frac{1}{N^{2}(T-1)^{2}}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{s=g+2}^{T}\sum_{t=g+2}^{T}E\left[\varepsilon_{js-g}^{2}\varepsilon_{it-g}^{2}\right]-2\frac{\sigma^{2}}{N(T-1)}\sum_{i=1}^{N}\sum_{t=g+2}^{T}E\left[\varepsilon_{it-g}^{2}\right] \\
& +\left(\frac{T-g-1}{T-1}\right)^{2}\sigma^{4} \\
= & \frac{1}{N^{2}(T-1)^{2}}\sum_{i\neq j}\sum_{s\neq t}E\left[\varepsilon_{js-g}^{2}\right]E\left[\varepsilon_{it-g}^{2}\right]+\frac{1}{N^{2}(T-1)^{2}}\sum_{i\neq j}\sum_{t=g+2}^{T}E\left[\varepsilon_{jt-g}^{2}\right]E\left[\varepsilon_{it-g}^{2}\right] \\
& +\frac{1}{N^{2}(T-1)^{2}}\sum_{i=1}^{N}\sum_{s\neq t}E\left[\varepsilon_{is-g}^{2}\right]E\left[\varepsilon_{it-g}^{2}\right]+\frac{(T-g-1)E\left[\varepsilon_{it}^{4}\right]}{N(T-1)^{2}}-\left(\frac{T-g-1}{T-1}\right)^{2}\sigma^{4} \\
= & \frac{(N-1)(T-g-1)(T-g-2)}{N(T-1)^{2}}\sigma^{4}-\left(\frac{T-g-1}{T-1}\right)^{2}\sigma^{4}+\frac{(N-1)(T-g-1)}{N(T-1)^{2}}\sigma^{4} \\
& +\frac{(T-g-1)(T-g-2)}{N(T-1)^{2}}\sigma^{4}+\frac{(T-g-1)E\left[\varepsilon_{it}^{4}\right]}{N(T-1)^{2}} \\
= & \frac{(T-g-1)(T-g-2)}{(T-1)^{2}}\sigma^{4}-\frac{(T-g-1)(T-g-2)}{N(T-1)^{2}}\sigma^{4}-\frac{(T-1)^{2}-2g(T-1)+g^{2}}{(T-1)^{2}}\sigma^{4} \\
& +\frac{(T-g-1)}{(T-1)^{2}}\sigma^{4}-\frac{(T-g-1)}{N(T-1)^{2}}\sigma^{4}+\frac{(T-g-1)(T-g-2)}{N(T-1)^{2}}\sigma^{4} \\
& +\frac{E\left[\varepsilon_{it}^{4}\right]}{N(T-1)}-\frac{gE\left[\varepsilon_{it}^{4}\right]}{N(T-1)^{2}} \\
= & \frac{(T-g-1)(T-g-2)}{(T-1)^{2}}\sigma^{4}-\frac{(T-1)^{2}-2g(T-1)+g^{2}}{(T-1)^{2}}\sigma^{4}+\frac{(T-g-1)}{(T-1)^{2}}\sigma^{4}-\frac{(T-g-1)}{N(T-1)^{2}}\sigma^{4} \\
& +\frac{E\left[\varepsilon_{it}^{4}\right]}{N(T-1)}-\frac{gE\left[\varepsilon_{it}^{4}\right]}{N(T-1)^{2}} \\
= & \frac{(T-1)^{2}-g(T-1)-(g+1)(T-1)+g(g+1)}{(T-1)^{2}}\sigma^{4}-\frac{(T-1)^{2}-2g(T-1)+g^{2}}{(T-1)^{2}}\sigma^{4} \\
& +\frac{\sigma^{4}}{(T-1)}-\frac{g\sigma^{4}}{(T-1)^{2}}-\frac{\sigma^{4}}{N(T-1)}+\frac{g\sigma^{4}}{N(T-1)^{2}}+\frac{E\left[\varepsilon_{it}^{4}\right]}{N(T-1)}-\frac{gE\left[\varepsilon_{it}^{4}\right]}{N(T-1)^{2}} \\
= & -\frac{\sigma^{4}}{(T-1)}+\frac{g\sigma^{4}}{(T-1)^{2}}+\frac{\sigma^{4}}{(T-1)}-\frac{g\sigma^{4}}{(T-1)^{2}}-\frac{\sigma^{4}}{N(T-1)}+\frac{g\sigma^{4}}{N(T-1)^{2}} \\
& +\frac{E\left[\varepsilon_{it}^{4}\right]}{N(T-1)}-\frac{gE\left[\varepsilon_{it}^{4}\right]}{N(T-1)^{2}} \\
= & -\frac{\sigma^{4}}{N(T-1)}+\frac{E\left[\varepsilon_{it}^{4}\right]}{N(T-1)}+O\left(\frac{1}{NT^{2}}\right) \\
= & O\left(\frac{1}{NT}\right)
\end{aligned}
$$

It follows by Markov's inequality that as $N, T \to \infty$

$$\frac{1}{N(T-1)} \sum_{i=1}^{N} \sum_{t=g+2}^{T} \varepsilon_{it-g}^2 = \sigma^2 + O\left(\frac{1}{T}\right) + O_p\left(\frac{1}{\sqrt{NT}}\right) = \sigma^2 + O_p\left(\max\left\{\frac{1}{\sqrt{NT}}, \frac{1}{T}\right\}\right).$$

To show part (g), write

$$\sum_{i=1}^{N} \sum_{t=2}^{T} a_i \Delta\varepsilon_{it} = \sum_{i=1}^{N} a_i \sum_{t=2}^{T} \Delta\varepsilon_{it} = \sum_{i=1}^{N} a_i \left(\varepsilon_{iT} - \varepsilon_{i1}\right).$$

Note that

$$\begin{aligned} E\left[\sum_{i=1}^{N} \sum_{t=2}^{T} a_i \Delta\varepsilon_{it}\right]^2 &= \sum_{i=1}^{N} \sum_{j=1}^{N} E\left[a_i a_j\right] E\left[\left(\varepsilon_{iT} - \varepsilon_{i1}\right)\left(\varepsilon_{jT} - \varepsilon_{j1}\right)\right] \\ &= \sum_{i=1}^{N} E\left[a_i^2\right] E\left[\left(\varepsilon_{iT} - \varepsilon_{i1}\right)^2\right] = 2\sigma^2\left(\mu_a^2 + \sigma_a^2\right) N \end{aligned}$$

from which it follows by Markov's inequality that

$$\sum_{i=1}^{N} \sum_{t=2}^{T} a_i \Delta\varepsilon_{it} = O_p\left(\sqrt{N}\right).$$

To show part (h), let $Y_{i,T} = \frac{1}{\sqrt{T}} \sum_{t=g+3}^{T} \varepsilon_{it-g-1}\varepsilon_{it-g}$ and note that

$$E\left[Y_{i,T}\right] = \frac{1}{\sqrt{T}} \sum_{t=g+3}^{T} E\left[\varepsilon_{it-g-1}\right] E\left[\varepsilon_{it-g}\right] = 0.$$

Moreover,

$$\begin{aligned} \omega_{N,T}^2 &= \sum_{i=1}^{N} E\left[Y_{i,T}^2\right] = \frac{1}{T} \sum_{i=1}^{N} \sum_{t=g+3}^{T} \sum_{s=g+3}^{T} E\left[\varepsilon_{it-g-1}\varepsilon_{it-g}\varepsilon_{is-g-1}\varepsilon_{is-g}\right] \\ &= \sigma^4 \frac{N(T-g-2)}{T} = \sigma^4 N\left[1 + O\left(\frac{1}{T}\right)\right], \end{aligned}$$

so that

$$\frac{\omega_{N,T}}{\sqrt{N}} \to \sigma^2 \in (0, \infty) \quad \text{as } N, T \to \infty.$$

Hence, to show the asymptotic normality of

$$U_{N,T} = \frac{1}{\omega_{N,T}/\sqrt{N}} \frac{1}{\sqrt{N}} \sum_{i=1}^{N} Y_{i,T},$$

it suffices to verify a Liapounov-type condition

$$\lim_{N,T\to\infty} \frac{1}{N^2} \sum_{i=1}^{N} E\left[Y_{i,T}^4\right] = 0. \tag{14}$$

By direct calculation,

$$
\begin{aligned}
\frac{1}{N^2}\sum_{i=1}^{N} E\left[Y_{i,T}^4\right] &= \frac{1}{N^2T^2}\sum_{i=1}^{N} E\left[\left(\sum_{t=g+3}^{T}\varepsilon_{it-g-1}\varepsilon_{it-g}\right)^4\right] \\
&= \frac{1}{N^2T^2}\sum_{i=1}^{N}\sum_{g=4}^{T}\sum_{s=4}^{T}\sum_{t=4}^{T}\sum_{u=4}^{T} E\left[\varepsilon_{ig-2}\varepsilon_{is-2}\varepsilon_{it-2}\varepsilon_{iu-2}\varepsilon_{ig-1}\varepsilon_{is-1}\varepsilon_{it-1}\varepsilon_{iu-1}\right] \\
&= \frac{1}{N^2T^2}\sum_{i=1}^{N}\sum_{t=4}^{T} E\left[\varepsilon_{it-2}^4\right] E\left[\varepsilon_{it-1}^4\right] + \frac{6}{N^2T^2}\sum_{i=1}^{N}\sum_{s=6}^{T}\sum_{t=4}^{s-2} E\left[\varepsilon_{is-2}^2\right] E\left[\varepsilon_{is-1}^2\right] E\left[\varepsilon_{it-2}^2\right] E\left[\varepsilon_{it-1}^2\right] \\
&\quad + \frac{6}{N^2T^2}\sum_{i=1}^{N}\sum_{t=4}^{T-1} E\left[\varepsilon_{it}^2\right] E\left[\varepsilon_{it-1}^4\right] E\left[\varepsilon_{it-2}^2\right] \\
&= \left(E\left[\varepsilon_{it-1}^4\right]\right)^2\frac{(T-3)}{NT^2} + \frac{6\sigma^8}{NT^2}\sum_{s=6}^{T}(s-5) + 6E\left[\varepsilon_{it-1}^4\right]\sigma^4\frac{(T-4)}{NT^2} \\
&= O\left(\frac{1}{NT}\right).
\end{aligned}
$$

Since the Liapounov-type condition (14) implies Lindeberg-type condition (13) given in Lemma SE-10 above, it follows from Lemma SE-10 that

$$
\begin{aligned}
U_{N,T} &= \frac{1}{\omega_{N,T}}\sum_{i=1}^{N} Y_{i,T} = \frac{1}{\omega_{N,T}}\frac{1}{\sqrt{T}}\sum_{i=1}^{N}\sum_{t=g+3}^{T}\varepsilon_{it-g-1}\varepsilon_{it-g} \\
&= \frac{1}{\sigma^2\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=g+3}^{T}\varepsilon_{it-g-1}\varepsilon_{it-g} + o_p(1) \Rightarrow N(0,1).
\end{aligned}
$$

Futhermore, we deduce from the Cramér convergence theorem that

$$
\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=g+3}^{T}\varepsilon_{it-g-1}\varepsilon_{it-g} \Rightarrow N\left(0,\sigma^4\right),
$$

as required.

Finally, to show part (i), note that

$$
E\left[\sum_{i=1}^{N}\varepsilon_{i1}\varepsilon_{i2}\right]^2 = \sum_{i=1}^{N}\sum_{j=1}^{N} E\left[\varepsilon_{i1}\varepsilon_{i2}\varepsilon_{j1}\varepsilon_{j2}\right] = \sum_{i=1}^{N} E\left[\varepsilon_{i1}^2\right] E\left[\varepsilon_{i2}^2\right] = \sigma^4 N,
$$

so that

$$
\sum_{i=1}^{N}\varepsilon_{i1}\varepsilon_{i2} = O_p\left(\sqrt{N}\right),
$$

as desired for part (i). □

Lemma SE-12:

Suppose that Assumptions 1 and 4 hold. Then, for $T/q(T) \to 0$ and $r \in (0,1]$

$$\frac{1}{\sqrt{T}} w_{i,[Tr]} \Rightarrow \sigma W_i(r) \text{ for each } i,$$

as $T \to \infty$, where

$$w_{i,[Tr]} = \sum_{j=1}^{[Tr]} \exp\left(-\frac{([Tr]-j)}{q(T)}\right)\varepsilon_{ij} + \exp\left\{-\frac{[Tr]}{q(T)}\right\} w_{i0}.$$

Proof of Lemma SE-12:

To proceed, for $(j-1)/T \leq s < j/T$, set

$$\int_{(j-1)/T}^{j/T} dX_{i,T}(s) = \frac{1}{\sigma\sqrt{T}}\varepsilon_{ij}$$

and note that, since $w_{i0} = O_p(1)$ for all i in light of Assumption 4 and the Markov's inequality, we have that

$$\begin{aligned}
\frac{1}{\sqrt{T}} w_{i,[Tr]} &= \frac{1}{\sqrt{T}} \sum_{j=1}^{[Tr]} \exp\left(-\frac{([Tr]-j)}{q(T)}\right)\varepsilon_{ij} + \frac{1}{\sqrt{T}} \exp\left\{-\frac{[Tr]}{q(T)}\right\} w_{i0} \\
&= \sigma \sum_{j=1}^{[Tr]} \exp\left(-\frac{([Tr]-j)}{q(T)}\right) \int_{(j-1)/T}^{j/T} dX_{i,T}(s) + o_p(1) \\
&= \sigma \sum_{j=1}^{[Tr]} \exp\left\{-\frac{T}{q(T)}\frac{[Tr]-j}{T}\right\} \int_{(j-1)/T}^{j/T} dX_{i,T}(s) + o_p(1) \\
&= \sigma \sum_{j=1}^{[Tr]} \int_{(j-1)/T}^{j/T} \exp\left\{-\frac{T}{q(T)}(r-s)\right\} \exp\left\{-\frac{T}{q(T)}\left[\frac{[Tr]}{T} - r\right]\right\} \\
&\quad \times \exp\left\{-\frac{T}{q(T)}\left[s - \frac{j}{T}\right]\right\} dX_{i,T}(s) + o_p(1).
\end{aligned}$$

Now, for $(j-1)/T \leq s < j/T$ and $j = 1, ..., [Tr]$

$$\begin{aligned}
& \int_{(j-1)/T}^{j/T} \exp\left\{-\frac{T}{q(T)}(r-s)\right\} \mathbb{I}\{\varepsilon_{ij} \geq 0\} dX_{i,T}(s) \\
\leq\ & \int_{(j-1)/T}^{j/T} \exp\left\{-\frac{T}{q(T)}(r-s)\right\} \exp\left\{-\frac{T}{q(T)}\left[\frac{[Tr]}{T} - r\right]\right\} \\
& \times \exp\left\{-\frac{T}{q(T)}\left[s - \frac{j}{T}\right]\right\} \mathbb{I}\{\varepsilon_{ij} \geq 0\} dX_{i,T}(s) \\
\leq\ & \exp\left\{\frac{2}{q(T)}\right\} \int_{(j-1)/T}^{j/T} \exp\left\{-\frac{T}{q(T)}(r-s)\right\} \mathbb{I}\{\varepsilon_{ij} \geq 0\} dX_{i,T}(s) \\
=\ & \int_{(j-1)/T}^{j/T} \exp\left\{-\frac{T}{q(T)}(r-s)\right\} \mathbb{I}\{\varepsilon_{ij} \geq 0\} dX_{i,T}(s) \left[1 + O\left(\frac{1}{q(T)}\right)\right],
\end{aligned}$$

so that

$$\begin{aligned}
& \int_{(j-1)/T}^{j/T} \exp\left\{-\frac{T}{q(T)}(r-s)\right\} \exp\left\{-\frac{T}{q(T)}\left[\frac{[Tr]}{T}-r\right]\right\} \\
& \times \exp\left\{-\frac{T}{q(T)}\left[s-\frac{j}{T}\right]\right\} \mathbb{I}\{\varepsilon_{ij} \geq 0\} dX_{i,T}(s) \\
= & \int_{(j-1)/T}^{j/T} \exp\left\{-\frac{T}{q(T)}(r-s)\right\} \mathbb{I}\{\varepsilon_{ij} \geq 0\} dX_{i,T}(s) \left[1+O\left(\frac{1}{q(T)}\right)\right]
\end{aligned}$$

Moreover,

$$\begin{aligned}
& \int_{(j-1)/T}^{j/T} \exp\left\{-\frac{T}{q(T)}(r-s)\right\} \mathbb{I}\{\varepsilon_{ij} < 0\} dX_{i,T}(s) \\
\geq & \int_{(j-1)/T}^{j/T} \exp\left\{-\frac{T}{q(T)}(r-s)\right\} \exp\left\{-\frac{T}{q(T)}\left[\frac{[Tr]}{T}-r\right]\right\} \\
& \times \exp\left\{-\frac{T}{q(T)}\left[s-\frac{j}{T}\right]\right\} \mathbb{I}\{\varepsilon_{ij} < 0\} dX_{i,T}(s) \\
\geq & \exp\left\{\frac{2}{q(T)}\right\} \int_{(j-1)/T}^{j/T} \exp\left\{-\frac{T}{q(T)}(r-s)\right\} \mathbb{I}\{\varepsilon_{ij} < 0\} dX_{i,T}(s) \\
= & \int_{(j-1)/T}^{j/T} \exp\left\{-\frac{T}{q(T)}(r-s)\right\} \mathbb{I}\{\varepsilon_{ij} < 0\} dX_{i,T}(s) \left[1+O\left(\frac{1}{q(T)}\right)\right],
\end{aligned}$$

so that

$$\begin{aligned}
& \int_{(j-1)/T}^{j/T} \exp\left\{-\frac{T}{q(T)}(r-s)\right\} \exp\left\{-\frac{T}{q(T)}\left[\frac{[Tr]}{T}-r\right]\right\} \\
& \times \exp\left\{-\frac{T}{q(T)}\left[s-\frac{j}{T}\right]\right\} \mathbb{I}\{\varepsilon_{ij} < 0\} dX_{i,T}(s) \\
= & \int_{(j-1)/T}^{j/T} \exp\left\{-\frac{T}{q(T)}(r-s)\right\} \mathbb{I}\{\varepsilon_{ij} < 0\} dX_{i,T}(s) \left[1+O\left(\frac{1}{q(T)}\right)\right].
\end{aligned}$$

It follows that

$$
\begin{aligned}
& \int_{(j-1)/T}^{j/T} \exp\left\{-\frac{T}{q(T)}(r-s)\right\} \exp\left\{-\frac{T}{q(T)}\left[\frac{[Tr]}{T}-r\right]\right\} \\
& \times \exp\left\{-\frac{T}{q(T)}\left[s-\frac{j}{T}\right]\right\} dX_{i,T}(s) \\
= & \int_{(j-1)/T}^{j/T} \exp\left\{-\frac{T}{q(T)}(r-s)\right\} \exp\left\{-\frac{T}{q(T)}\left[\frac{[Tr]}{T}-r\right]\right\} \\
& \times \exp\left\{-\frac{T}{q(T)}\left[s-\frac{j}{T}\right]\right\} \mathbb{I}\{\varepsilon_{ij}<0\} dX_{i,T}(s) \\
& + \int_{(j-1)/T}^{j/T} \exp\left\{-\frac{T}{q(T)}(r-s)\right\} \exp\left\{-\frac{T}{q(T)}\left[\frac{[Tr]}{T}-r\right]\right\} \\
& \times \exp\left\{-\frac{T}{q(T)}\left[s-\frac{j}{T}\right]\right\} \mathbb{I}\{\varepsilon_{ij}\geq 0\} dX_{i,T}(s) \\
= & \int_{(j-1)/T}^{j/T} \exp\left\{-\frac{T}{q(T)}(r-s)\right\} \mathbb{I}\{\varepsilon_{ij}<0\} dX_{i,T}(s)\left[1+O\left(\frac{1}{q(T)}\right)\right] \\
& + \int_{(j-1)/T}^{j/T} \exp\left\{-\frac{T}{q(T)}(r-s)\right\} \mathbb{I}\{\varepsilon_{ij}\geq 0\} dX_{i,T}(s)\left[1+O\left(\frac{1}{q(T)}\right)\right] \\
= & \int_{(j-1)/T}^{j/T} \exp\left\{-\frac{T}{q(T)}(r-s)\right\} dX_{i,T}(s)\left[1+O\left(\frac{1}{q(T)}\right)\right].
\end{aligned}
$$

Hence,

$$
\begin{aligned}
& \frac{1}{\sqrt{T}} w_{i,[Tr]} \\
= & \sigma \sum_{j=1}^{[Tr]} \int_{(j-1)/T}^{j/T} \exp\left\{-\frac{T}{q(T)}(r-s)\right\} \exp\left\{-\frac{T}{q(T)}\left[\frac{[Tr]}{T}-r\right]\right\} \\
& \times \exp\left\{-\frac{T}{q(T)}\left[s-\frac{j}{T}\right]\right\} dX_{i,T}(s) + o_p(1) \\
= & \sigma \sum_{j=1}^{[Tr]} \int_{(j-1)/T}^{j/T} \exp\left\{-\frac{T}{q(T)}(r-s)\right\} dX_{i,T}(s)\left[1+O\left(\frac{1}{q(T)}\right)\right] + o_p(1) \\
= & \sigma \int_0^r \exp\left\{-\frac{T}{q(T)}(r-s)\right\} dX_{i,T}(s)\left[1+O\left(\frac{1}{q(T)}\right)\right] + o_p(1) \\
= & \sigma X_{i,T}(r) - \frac{T}{q(T)} \int_0^r \sigma \exp\left(-\frac{T}{q(T)}(r-s)\right) X_{i,T}(s)\, ds + o_p(1) \\
= & \sigma X_{i,T}(r) + O_p\left(\frac{T}{q(T)}\right) + o_p(1) \\
\Rightarrow & \sigma W_i(r),
\end{aligned}
$$

as $T \to \infty$, which is the required result. □

Lemma SE-13: (Phillips and Moon, 1999, Corollary 1): Suppose that $Y_{i,T} = C_i Q_{i,T}$, where $Q_{i,T}$ are *i.i.d.* across i for all T, and the C_i are $(m \times m)$ nonrandom matrices for all i. Assume that

$Q_{i,T}$ are integrable for all T and $Q_{i,T} \Rightarrow Q_i$ as $T \to \infty$. Assume that $C = \lim_N (1/N) \sum_{i=1}^N C_i$ exists. If $\|Q_{i,T}\|$ is uniformly integrable in T for all i, and if $\sup_i \|C_i\| < \infty$, then

$$\frac{1}{N} \sum_{i=1}^{N} Y_{i,T} \xrightarrow{p} CE\left[Q_i\right]$$

as $N, T \to \infty$.

In the subsequent lemmas, we find it useful to decompose w_{it} as

$$w_{it} = \underline{w}_{it} + \rho_T^t w_{i0}$$

where $\underline{w}_{it} = \sum_{j=1}^t \rho_T^{(t-j)} \varepsilon_{ij}$.

Lemma SE-14:

Suppose that Assumptions 1 and 4 hold. If $\rho_T = 1$ for all T sufficiently large, then, as $N, T \to \infty$,

$$\frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{it-g,T}^2 \xrightarrow{p} \frac{\sigma^2}{2} \text{ for } g \in \{1, 2\}.$$

Proof of Lemma SE-14:

To proceed, note first that, under the assumption here, there exists a positive integer I_ρ such that for all $T \geq I_\rho$, the triangular array process $\left\{\underline{w}_{it-g,T}\right\}$ has the partial sum representation $\underline{w}_{it-1,T} = \sum_{j=1}^{t-g} \varepsilon_{ij}$. Thus, for all T sufficiently large, we can write

$$\begin{aligned}
\frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{it-g,T}^2 &= \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=g+1}^{T} \underline{w}_{it-g,T}^2 + 2\frac{2}{NT^{3/2}} \sum_{i=1}^{N} \sum_{t=2}^{T} \underline{w}_{it-g,T} w_{i0} + \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{i0}^2 \\
&= \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=2}^{T} \left(\frac{1}{\sqrt{T}} \sum_{j=1}^{t-g} \varepsilon_{ij} \right)^2 + 2\frac{1}{NT^{3/2}} \sum_{i=1}^{N} \sum_{t=2}^{T} \left(\frac{1}{\sqrt{T}} \sum_{j=1}^{t-g} \varepsilon_{ij} \right) w_{i0} \\
&\quad + \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{i0}^2.
\end{aligned}$$

Consider first the case where $g = 1$. Define

$$Q_{i,T} = \frac{1}{T} \sum_{t=2}^{T} \left(\frac{1}{\sqrt{T}} \sum_{j=1}^{t-1} \varepsilon_{ij} \right)^2.$$

Note that we can apply arguments similar to that given in the proof of Theorem 3.1 part (a) in Phillips (1987) to obtain

$$Q_{i,T} \Rightarrow \sigma^2 \int_0^1 \left[W_i(r)\right]^2 dr = Q_i \quad (say).$$

Next, we verify the conditions of Corollary 1 of Phillips and Moon (1999), given here as Lemma SE-13. Note first that $Q_{i,T}$ is integrable in light of Assumption 1. Moreover, in this case, $C_i = 1$ for all i, so that trivially, $C = \lim_N (1/N) \sum_{i=1}^N C_i = 1 < \infty$ and $\sup_i \|C_i\| = 1 < \infty$. In addition, note that, in this case, for all $T \geq I_\rho$,

$$\begin{aligned} E\left[|Q_{i,T}|\right] &= E\left[Q_{i,T}\right] = \frac{1}{T^2} \sum_{t=2}^{T} \sum_{j=1}^{t-1} \sum_{k=1}^{t-1} E\left[\varepsilon_{ij}\varepsilon_{ik}\right] \\ &= \sigma^2 \frac{1}{T^2} \sum_{t=2}^{T} (t-1) = \sigma^2 \frac{1}{T^2} \frac{T(T-1)}{2} \to \frac{\sigma^2}{2}, \end{aligned}$$

so that

$$\lim_{T\to\infty} E\left[|Q_{i,T}|\right] = \frac{\sigma^2}{2} = \sigma^2 \int_0^1 E\left[W_i(g)\right]^2 dg = E\left[Q_i\right] \text{ for all } i.$$

It follows from Theorem 5.4 of Billingsley (1968) that $\{|Q_{i,T}|\}$ is uniformly integrable in T for all i. Hence, all the conditions of Lemma SE-13 are satisfied, and we deduce from this lemma that

$$\frac{1}{N} \sum_{i=1}^{N} Q_{i,T} = \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1}^2 \xrightarrow{p} CE\left[Q_i\right] = \frac{\sigma^2}{2},$$

as $N, T \to \infty$.

Now, by Assumption 4, there exists a positive constant C such that

$$E\left[\frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{i0}^2\right] = \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} E\left[w_{i0}^2\right] \leq \sup_i E\left[w_{i0}^2\right] \frac{(T-1)}{T^2} \leq C \frac{(T-1)}{T^2} = O\left(T^{-1}\right)$$

so that, applying Markov's inequality, we obtain

$$\frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{i0}^2 = O_p\left(\frac{1}{T}\right).$$

It follows from the Cauchy-Schwarz inequality that

$$\left|\frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} \underline{w}_{it-1,T} w_{i0}\right| \leq \sqrt{\frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} \underline{w}_{it-1,T}^2} \sqrt{\frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{i0}^2} = O_p(1) \times O_p\left(T^{-1/2}\right)$$

so that by the Cramér convergence theorem, we deduce that

$$\frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{it-1,T}^2 \xrightarrow{p} \frac{\sigma^2}{2}.$$

Next, consider the case $g = 2$. Here, note that, for T sufficiently large,

$$\begin{aligned} &\frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=3}^{T} w_{it-2,T}^2 = \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1,T}^2 - \frac{1}{NT^2} \sum_{i=1}^{N} w_{iT-1,T}^2 \\ = &\frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1,T}^2 - \frac{1}{NT^2} \sum_{i=1}^{N} \left(\sum_{j=1}^{T-1} \varepsilon_{ij}\right)^2 - 2\frac{1}{NT^2} \sum_{i=1}^{N} \left(\sum_{j=1}^{T-1} \varepsilon_{ij}\right) w_{i0} - \frac{1}{NT^2} \sum_{i=1}^{N} w_{i0}^2. \end{aligned}$$

Now, using Assumption 4

$$E\left(\frac{1}{NT^2}\sum_{i=1}^{N}w_{i0}^2\right)=\frac{1}{NT^2}\sum_{i=1}^{N}E\left[w_{i0}^2\right]\leq\frac{C}{T^2}=O_p\left(\frac{1}{T^2}\right).$$

Moreover,

$$E\left(\frac{1}{NT^2}\sum_{i=1}^{N}\left[\sum_{j=1}^{T-1}\varepsilon_{ij}\right]^2\right)=\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{j=1}^{T-1}\sum_{k=1}^{T-1}E\left[\varepsilon_{ij}\varepsilon_{ik}\right]=\sigma^2\frac{T-1}{T^2}=O\left(\frac{1}{T}\right),$$

so that by Markov's inequality, we deduce that

$$\frac{1}{NT^2}\sum_{i=1}^{N}w_{i0}^2=O_p\left(\frac{1}{T^2}\right),\ \frac{1}{NT^2}\sum_{i=1}^{N}\left[\sum_{j=1}^{T-1}\varepsilon_{ij}\right]^2=O_p\left(\frac{1}{T}\right).$$

The Cauchy-Schwarz inequality further implies that

$$\begin{aligned}\left|\frac{1}{NT^2}\sum_{i=1}^{N}\left(\sum_{j=1}^{T-1}\varepsilon_{ij}\right)w_{i0}\right| &\leq \sqrt{\frac{1}{NT^2}\sum_{i=1}^{N}\left(\sum_{j=1}^{T-1}\varepsilon_{ij}\right)^2}\sqrt{\frac{1}{NT^2}\sum_{i=1}^{N}w_{i0}^2}\\ &= O_p\left(\frac{1}{\sqrt{T}}\right)O_p\left(\frac{1}{T}\right)=O_p\left(\frac{1}{T^{3/2}}\right).\end{aligned}$$

It follows that

$$\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=3}^{T}w_{it-2,T}^2=\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T}w_{it-1,T}^2+O_p\left(\frac{1}{T}\right)=\frac{\sigma^2}{2}+o_p\left(1\right),$$

as required. □

Lemma SE-15:

Suppose that Assumptions 1 and 4 hold. If $\rho_T=\exp\left\{-1/q\left(T\right)\right\}$ such that $T/q\left(T\right)\to 0$, then, as $N,T\to\infty$,

$$\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=g+1}^{T}w_{it-g,T}^2\xrightarrow{p}\frac{\sigma^2}{2}\text{ for }g\in\{1,2\}.$$

Proof of Lemma SE-15:

To proceed, first write

$$\begin{aligned}w_{it-g,T}^2 &= \underline{w}_{it-g,T}^2+2\underline{w}_{it-g}\rho_T^{t-g}w_{i0}+\rho_T^{2(t-g)}w_{i0}^2\\ &= \left(\sum_{j=1}^{t-g}\exp\left\{-\frac{(t-g-j)}{q\left(T\right)}\right\}\varepsilon_{ij}\right)^2+2\left(\sum_{j=1}^{t-g}\exp\left\{-\frac{(t-g-j)}{q\left(T\right)}\right\}\varepsilon_{ij}\right)\exp\left\{-\frac{(t-g)}{q\left(T\right)}\right\}w_{i0}\\ &\quad+\exp\left\{-\frac{2\left(t-g\right)}{q\left(T\right)}\right\}w_{i0}^2\end{aligned}$$

Now, consider the case where $g = 1$. In this case, note that

$$\begin{aligned}
\frac{1}{T^2}\sum_{t=2}^{T}\underline{w}_{it-1,T}^2 &= \frac{1}{T^2}\sum_{t=1}^{T}\left(\sum_{j=1}^{t-1}\exp\left\{-\frac{(t-1-j)}{q(T)}\right\}\varepsilon_{ij}\right)^2 \\
&= \sigma^2\frac{1}{T}\sum_{t=1}^{T}\left[\sum_{j=1}^{t-1}\exp\left\{-\frac{T}{q(T)}\left(\frac{t-1}{T}-\frac{j}{T}\right)\right\}\frac{\varepsilon_{ij}}{\sigma\sqrt{T}}\right]^2 \\
&= \sigma^2\sum_{t=1}^{T}\exp\left\{-\frac{2T}{q(T)}\left(\frac{t-1}{T}\right)\right\}\frac{1}{T}\left[\sum_{j=1}^{t-1}\exp\left\{\frac{T}{q(T)}\frac{j}{T}\right\}\frac{\varepsilon_{ij}}{\sigma\sqrt{T}}\right]^2.
\end{aligned}$$

Now, for $\frac{t-1}{T} \leq r < \frac{t}{T}$, we define

$$X_{i,T}(r) = \frac{1}{\sigma\sqrt{T}}\sum_{j=1}^{[Tr]}\varepsilon_{ij} = \frac{1}{\sigma\sqrt{T}}\sum_{j=1}^{t-1}\varepsilon_{ij}, \tag{15}$$

$$\widetilde{X}_{i,T}(r) = \sum_{j=1}^{t-1}\exp\left\{\frac{T}{q(T)}\frac{j}{T}\right\}\frac{\varepsilon_{ij}}{\sigma\sqrt{T}}. \tag{16}$$

Next, write

$$\begin{aligned}
\frac{1}{T^2}\sum_{t=1}^{T}\underline{w}_{it-1,T}^2 &= \sigma^2\sum_{t=1}^{T}\exp\left\{-\frac{2T}{q(T)}\left(\frac{t-1}{T}\right)\right\}\int_{(t-1)/T}^{t/T}\widetilde{X}_{i,T}^2(r)\,dr \\
&= \sigma^2\sum_{t=1}^{T}\int_{(t-1)/T}^{t/T}\exp\left\{-\frac{2T}{q(T)}\left[\left(\frac{t-1}{T}\right)-r+r\right]\right\}\widetilde{X}_{i,T}^2(r)\,dr \\
&= \sigma^2\int_0^1\exp\left\{-\frac{2T}{q(T)}r\right\}\exp\left\{\frac{2T}{q(T)}\left[r-\left(\frac{t-1}{T}\right)\right]\right\}\widetilde{X}_{i,T}^2(r)\,dr,
\end{aligned}$$

and note that for $t = 1, ..., T$

$$\begin{aligned}
\int_{(t-1)/T}^{t/T}\exp\left\{-\frac{2Tr}{q(T)}\right\}\widetilde{X}_{i,T}^2(r)\,dr &\leq \int_{(t-1)/T}^{t/T}\exp\left\{-\frac{2Tr}{q(T)}\right\}\exp\left\{\frac{2T}{q(T)}\left[r-\left(\frac{t-1}{T}\right)\right]\right\}\widetilde{X}_{i,T}^2(r)\,dr \\
&\leq \exp\left\{\frac{2}{q(T)}\right\}\int_{(t-1)/T}^{t/T}\exp\left\{-\frac{2T}{q(T)}r\right\}\widetilde{X}_{i,T}^2(r)\,dr,
\end{aligned}$$

from which it follows that

$$\begin{aligned}
\sigma^2\int_0^1\exp\left\{-\frac{2Tr}{q(T)}\right\}\widetilde{X}_{i,T}^2(r)\,dr &\leq \sigma^2\int_0^1\exp\left\{-\frac{2Tr}{q(T)}\right\}\exp\left\{\frac{2T}{q(T)}\left[r-\left(\frac{t-1}{T}\right)\right]\right\}\widetilde{X}_{i,T}^2(r)\,dr \\
&\leq \sigma^2\exp\left\{\frac{2}{q(T)}\right\}\int_0^1\exp\left\{-\frac{2T}{q(T)}r\right\}\widetilde{X}_{i,T}^2(r)\,dr \\
&= \sigma^2\int_0^1\exp\left\{-\frac{2T}{q(T)}r\right\}\widetilde{X}_{i,T}^2(r)\,dr\left[1+O_p\left(\frac{1}{q(T)}\right)\right],
\end{aligned}$$

so that

$$\begin{aligned}
& \sigma^2 \int_0^1 \exp\left\{-\frac{2T}{q(T)}r\right\} \exp\left\{\frac{2T}{q(T)}\left[r-\left(\frac{t-1}{T}\right)\right]\right\} \widetilde{X}_{i,T}^2(r)\,dr \\
= \; & \sigma^2 \int_0^1 \exp\left\{-\frac{2T}{q(T)}r\right\} \widetilde{X}_{i,T}^2(r)\,dr \left[1+O_p\left(\frac{1}{q(T)}\right)\right].
\end{aligned}$$

By similar argument, we have

$$\begin{aligned}
& \int_{(j-1)/T}^{j/T} \exp\left\{\frac{T}{q(T)}s\right\} \exp\left\{\frac{T}{q(T)}\left[\frac{j}{T}-s\right]\right\} dX_{i,T}(s) \\
= \; & \int_{(j-1)/T}^{j/T} \exp\left\{\frac{T}{q(T)}s\right\} dX_{i,T}(s) \left[1+O_p\left(\frac{1}{q(T)}\right)\right], \qquad (17)
\end{aligned}$$

for $j=1,...,T$ and $(j-1)/T \leq s < j/T$.

Moreover, for $j=1,...,t$ we have

$$\begin{aligned}
\int_{(j-1)/T}^{j/T} dX_{i,T}(s) &= X_{i,T}\left(\frac{j}{T}\right) - X_{i,T}\left(\frac{j-1}{T}\right) = \frac{1}{\sigma\sqrt{T}}\left\{\sum_{k=1}^{j}\varepsilon_{ik} - \sum_{k=1}^{j-1}\varepsilon_{ik}\right\} \\
&= \frac{1}{\sigma\sqrt{T}}\varepsilon_{ij}. \qquad (18)
\end{aligned}$$

Using (18), we can define for $(t-1)/T \leq r < t/T$,

$$\begin{aligned}
& \widetilde{X}_T(r) \\
= \; & \sum_{j=1}^{t-1} \exp\left\{\frac{T}{q(T)}\frac{j}{T}\right\} \frac{\varepsilon_{ij}}{\sigma\sqrt{T}} \\
= \; & \sum_{j=1}^{t-1} \int_{(j-1)/T}^{j/T} \exp\left\{\frac{T}{q(T)}\frac{j}{T}\right\} dX_{i,T}(s) \\
= \; & \sum_{j=1}^{t-1} \int_{(j-1)/T}^{j/T} \exp\left\{\frac{T}{q(T)}s\right\} \exp\left\{\frac{T}{q(T)}\left[\frac{j}{T}-s\right]\right\} dX_{i,T}(s) \\
= \; & \sum_{j=1}^{t-1} \int_{(j-1)/T}^{j/T} \exp\left\{\frac{T}{q(T)}s\right\} dX_{i,T}(s) \left[1+O_p\left(\frac{1}{q(T)}\right)\right] \quad \text{(by (17))} \\
= \; & \int_0^{[Tr]/T} \exp\left\{\frac{T}{q(T)}s\right\} dX_{i,T}(s) \left[1+O_p\left(\frac{1}{q(T)}\right)\right] \\
= \; & \int_0^{r} \exp\left\{\frac{T}{q(T)}s\right\} dX_{i,T}(s) \left[1+O_p\left(\frac{1}{T}\right)\right] \\
= \; & \left(\exp\left\{\frac{T}{q(T)}s\right\} X_{i,T}(s)\Big|_0^r + \int_0^r -\frac{T}{q(T)} \exp\left(\frac{T}{q(T)}s\right) X_{i,T}(s)\,ds\right) \left[1+O_p\left(\frac{1}{q(T)}\right)\right] \\
= \; & \left[\exp\left\{\frac{T}{q(T)}r\right\} X_{i,T}(r) - \frac{T}{q(T)}\int_0^r \exp\left(\frac{T}{q(T)}s\right) X_{i,T}(s)\,ds\right] \left[1+O_p\left(\frac{1}{q(T)}\right)\right],
\end{aligned}$$

so that

$$
\begin{aligned}
& \frac{1}{T^2}\sum_{t=1}^{T}\underline{w}_{it-1,T}^2 \\
= \; & \sigma^2\int_0^1 \exp\left\{-\frac{2T}{q(T)}r\right\}\widetilde{X}_T^2(r)\,dr\left[1+O_p\left(\frac{1}{q(T)}\right)\right] \\
= \; & \sigma^2\int_0^1 \exp\left\{-\frac{2T}{q(T)}r\right\}\left[\exp\left\{\frac{T}{q(T)}r\right\}X_{i,T}(r)-\frac{T}{q(T)}\int_0^r \exp\left(\frac{T}{q(T)}s\right)X_{i,T}(s)\,ds\right]^2 dr \\
& \times\left[1+O_p\left(\frac{1}{q(T)}\right)\right] \\
= \; & \sigma^2\int_0^1 \exp\left\{-\frac{2T}{q(T)}r\right\}\exp\left\{\frac{2T}{q(T)}r\right\}\left[X_{i,T}(r)-\frac{T}{q(T)}\int_0^r \exp\left(\frac{T}{q(T)}[s-r]\right)X_{i,T}(s)\,ds\right]^2 dr \\
& \times\left[1+O_p\left(\frac{1}{q(T)}\right)\right] \\
= \; & \sigma^2\int_0^1 \left[X_{i,T}(r)-\frac{T}{q(T)}\int_0^r \exp\left(\frac{T}{q(T)}[s-r]\right)X_{i,T}(s)\,ds\right]^2 dr\left[1+O_p\left(\frac{1}{q(T)}\right)\right] \\
= \; & \sigma^2\int_0^1 [X_{i,T}(g)]^2\,dr\left[1+O_p\left(\frac{T}{q(T)}\right)+O_p\left(\frac{1}{q(T)}\right)\right] \\
= \; & \sigma^2\int_0^1 [X_{i,T}(g)]^2\,dr\left[1+O_p\left(\frac{T}{q(T)}\right)\right].
\end{aligned}
$$

Hence, by the continuous mapping theorem,

$$
\frac{1}{T^2}\sum_{t=1}^{T}\underline{w}_{it-1,T}^2 \Rightarrow \sigma^2\int_0^1 [W_i(r)]^2\,dr
$$

as $T\to\infty$.

Next, we verify the conditions of Corollary 1 of Phillips and Moon (1999), given here as Lemma SE-13. To proceed, define

$$
Q_{i,T}=\frac{1}{T^2}\sum_{t=2}^{T}\underline{w}_{it-1}^2.
$$

Note first that $Q_{i,T}$ is integrable in light of Assumption 1; and, by the argument given previously, we have that, as $T\to\infty$,

$$
Q_{i,T}\Rightarrow\sigma^2\int_0^1 [W_i(r)]^2\,dr=Q_i \quad (say).
$$

Moreover, in this case, $C_i=1$ for all i, so that trivially, $C=\lim_N (1/N)\sum_{i=1}^{N}C_i=1<\infty$ and

$\sup_i \|C_i\| = 1 < \infty$. In addition, applying the results of part (a) of Lemma SE-1, we get

$$\begin{aligned}
E\left[|Q_{i,T}|\right] &= E\left[Q_{i,T}\right] = \frac{1}{T^2}\sum_{t=2}^{T} E\left[w_{it-1,T}^2\right] \\
&= \frac{1}{T^2}\sum_{t=2}^{T}\sum_{j=1}^{t-1}\sum_{k=1}^{t-1}\exp\left\{-\frac{(t-1-j)}{q(T)}\right\}\exp\left\{-\frac{(t-1-k)}{q(T)}\right\}E\left[\varepsilon_{ij}\varepsilon_{ik}\right] \\
&= \sigma^2\frac{1}{T^2}\sum_{t=2}^{T}\sum_{j=1}^{t-1}\exp\left\{-2\frac{(t-1-j)}{q(T)}\right\} \\
&= \sigma^2\frac{1}{T^2}\frac{T^2}{2}\left[1+O\left(\frac{T}{q(T)}\right)+O\left(\frac{1}{T}\right)\right] \\
&= \frac{\sigma^2}{2}+O\left(\frac{T}{q(T)}\right)+O\left(\frac{1}{T}\right),
\end{aligned}$$

so that

$$\lim_{T\to\infty} E\left[|Q_{i,T}|\right] = \frac{\sigma^2}{2} = \sigma^2\int_0^1 E\left[W_i(g)\right]^2 dg = E\left[Q_i\right] \text{ for all } i.$$

It follows from Theorem 5.4 of Billingsley (1968) that $\{|Q_{i,T}|\}$ is uniformly integrable in T for all i. Hence, all the conditions of Lemma SE-13 are satisfied, and we deduce from this lemma that

$$\frac{1}{N}\sum_{i=1}^{N} Q_{i,T} = \frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T}\underline{w}_{it-1}^2 \xrightarrow{p} CE\left[Q_i\right] = \frac{\sigma^2}{2},$$

as $N, T \to \infty$.

In addition, note that, by Assumption 4, there exists a positive constant C such that

$$\begin{aligned}
& E\left[\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T}\rho_T^{2(t-1)}w_{i0}^2\right] \\
\leq\ & \sup_i E\left[w_{i0}^2\right]\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T}\rho_T^{2(t-1)} \\
\leq\ & \frac{C}{T^2}\exp\left\{-\frac{2}{q(T)}\right\}\left[1-\exp\left\{-\frac{2}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{2(T-1)}{q(T)}\right\}\right] \\
=\ & O\left(T^{-2}\right)O(1)\,O(q(T))\,O(T/q(T)) = O\left(T^{-1}\right),
\end{aligned}$$

so that, applying Markov's inequality, we obtain

$$\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T}\rho_T^{2(t-1)}w_{i0}^2 = O_p\left(\frac{1}{T}\right).$$

It follows from the Cauchy-Schwarz inequality that

$$\begin{aligned}
\left|\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T}\underline{w}_{it-1}\rho_T^{t-1}w_{i0}\right| &\leq \sqrt{\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T}\underline{w}_{it-1}^2}\sqrt{\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T}\rho_T^{2(t-1)}w_{i0}^2} \\
&= O_p(1)\,O_p\left(T^{-1/2}\right) = O_p\left(T^{-1/2}\right),
\end{aligned}$$

so that, using Slutsky's Theorem, we further deduce that

$$\begin{aligned}
&\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-1,T}^2 \\
= \;& \frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T} \underline{w}_{it-1,T}^2 + \frac{2}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T} \underline{w}_{it-1}\rho_T^{t-1} w_{i0} + \frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T} \rho_T^{2(t-1)} w_{i0}^2 \xrightarrow{p} \frac{\sigma^2}{2}.
\end{aligned}$$

Next, consider the case $g = 2$. Here, note that in this case

$$\begin{aligned}
&\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=3}^{T} \underline{w}_{it-2,T}^2 \\
= \;& \frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T} \underline{w}_{it-1,T}^2 - \frac{1}{NT^2}\sum_{i=1}^{N} \underline{w}_{iT-1,T}^2 \\
= \;& \frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T} \underline{w}_{it-1,T}^2 - \frac{1}{NT^2}\sum_{i=1}^{N}\left[\sum_{j=1}^{T-1}\exp\left(-\frac{(T-1-j)}{q(T)}\right)\varepsilon_{ij}\right]^2.
\end{aligned}$$

Now,

$$\begin{aligned}
&E\left(\frac{1}{T^2}\left[\sum_{j=1}^{T-1}\exp\left\{-\frac{(T-1-j)}{q(T)}\right\}\varepsilon_{ij}\right]^2\right) \\
= \;& \frac{1}{T^2}\sum_{j=1}^{T-1}\sum_{k=1}^{T-1}\exp\left\{-\frac{(T-1-j)}{q(T)}\right\}\exp\left\{-\frac{(T-1-k)}{q(T)}\right\}E\left[\varepsilon_{ij}\varepsilon_{ik}\right] \\
= \;& \sigma^2\frac{1}{T^2}\sum_{j=1}^{T-1}\exp\left\{-2\frac{(T-1-j)}{q(T)}\right\}.
\end{aligned}$$

Applying part (a) of Lemma SE-3 with $b = 2$ and $g = 1$, we obtain

$$\begin{aligned}
&\sigma^2\frac{1}{T^2}\sum_{j=1}^{T-1}\exp\left\{-2\left(\frac{T-1-j}{q(T)}\right)\right\} \\
= \;& \sigma^2\frac{1}{T}\left[1-\frac{1}{T}-\frac{T}{q(T)}+O\left(\max\left\{\frac{T^2}{q(T)^2},\frac{1}{q(T)}\right\}\right)\right] \\
= \;& O\left(T^{-1}\right).
\end{aligned}$$

It follows from Markov's inequality that

$$\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=3}^{T} \underline{w}_{it-2,T}^2 = \frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=2}^{T} \underline{w}_{it-1,T}^2 + O_p\left(\frac{1}{\sqrt{T}}\right) = \frac{\sigma^2}{2} + o_p(1).$$

Moreover, note that, again by Assumption 4, we obtain

$$\begin{aligned}
& E\left[\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=3}^{T}\rho_T^{2(t-2)}w_{i0}^2\right] \\
\leq\ & \frac{C}{T^2}\exp\left\{-\frac{2}{q(T)}\right\}\left[1-\exp\left\{-\frac{2}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{2(T-2)}{q(T)}\right\}\right] \\
=\ & O\left(T^{-2}\right)O(1)\,O(q(T))\,O(T/q(T)) = O\left(T^{-1}\right),
\end{aligned}$$

from which we deduce, using Markov's inequality, that

$$\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=3}^{T}\rho_T^{2(t-2)}w_{i0}^2 = O_p\left(\frac{1}{T}\right)$$

The Cauchy-Schwarz inequality then further implies that

$$\begin{aligned}
\left|\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=3}^{T}\underline{w}_{it-2}\rho_T^{t-1}w_{i0}\right| &\leq \sqrt{\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=3}^{T}\underline{w}_{it-2}^2}\sqrt{\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=3}^{T}\rho_T^{2(t-2)}w_{i0}^2} \\
&= O_p(1)\,O_p\left(T^{-1/2}\right) = O_p\left(T^{-1/2}\right),
\end{aligned}$$

so that, by making use of the Slutsky's Theorem, we get

$$\begin{aligned}
& \frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=3}^{T}w_{it-2,T}^2 \\
=\ & \frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=3}^{T}\underline{w}_{it-2,T}^2 + \frac{2}{NT^2}\sum_{i=1}^{N}\sum_{t=3}^{T}\underline{w}_{it-2}\rho_T^{t-2}w_{i0} + \frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=3}^{T}\rho_T^{2(t-2)}w_{i0}^2 \xrightarrow{p} \frac{\sigma^2}{2},
\end{aligned}$$

as required. □

Lemma SE-16:

Let g be a non-negative integer. Under Assumptions 1 and 4, the following results hold as $N, T \to \infty$.

(a) If $\rho_T = 1$ for all T sufficiently large, then

$$\begin{aligned}
\sum_{i=1}^{N}\sum_{t=g+1}^{T}E\left[\underline{w}_{it-g,T}^2\right] &= NT^2\frac{\sigma^2}{2}\left[1+O\left(\frac{1}{T}\right)\right], \\
\sum_{i=1}^{N}\sum_{t=g+2}^{T}\sum_{s=g+1}^{t-1}E\left[\underline{w}_{it-g,T}\underline{w}_{is-g,T}\right] &= NT^3\frac{\sigma^2}{6}\left[1+O\left(\frac{1}{T}\right)\right].
\end{aligned}$$

(b) If $\rho_T = \exp\{-1/q(T)\}$ such that $T/q(T) \to 0$, then

$$\begin{aligned}
\sum_{i=1}^{N}\sum_{t=g+1}^{T}E\left[\underline{w}_{it-g,T}^2\right] &= NT^2\frac{\sigma^2}{2}\left[1+O\left(\max\left\{\frac{T}{q(T)},\frac{1}{T}\right\}\right)\right], \\
\sum_{i=1}^{N}\sum_{t=g+2}^{T}\sum_{s=g+1}^{t-1}E\left[\underline{w}_{it-g,T}\underline{w}_{is-g,T}\right] &= NT^3\frac{\sigma^2}{6}\left[1+O\left(\max\left\{\frac{T}{q(T)},\frac{1}{T}\right\}\right)\right].
\end{aligned}$$

(c) If $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \sim T$, then

$$\sum_{i=1}^{N}\sum_{t=g+1}^{T} E\left[\underline{w}_{it-g,T}^2\right] = \sigma^2 N \frac{q(T)^2}{4}\left[\exp\left\{-\frac{2T}{q(T)}\right\} + \frac{2T}{q(T)} - 1\right]\left[1 + O\left(\frac{1}{T}\right)\right],$$

and

$$\begin{aligned}
&\sum_{i=1}^{N}\sum_{t=g+2}^{T}\sum_{s=g+1}^{t-1} E\left[\underline{w}_{it-g,T}\underline{w}_{is-g,T}\right] \\
= \;& NTq(T)^2 \frac{\sigma^2}{2}\left[1 - \frac{3}{2}\frac{q(T)}{T} + \frac{2q(T)}{T}\exp\left\{-\frac{T}{q(T)}\right\} - \frac{1}{2}\frac{q(T)}{T}\exp\left\{-\frac{2T}{q(T)}\right\}\right]\left[1 + O\left(\frac{1}{T}\right)\right]
\end{aligned}$$

(d) If $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \to \infty$ but $q(T)/T \to 0$, then

$$\sum_{i=1}^{N}\sum_{t=g+1}^{T} E\left[\underline{w}_{it-g,T}^2\right] = NTq(T)\frac{\sigma^2}{2}\left[1 + O\left(\frac{1}{q(T)}\right) + O\left(\frac{q(T)}{T}\right)\right],$$

and

$$\sum_{i=1}^{N}\sum_{t=g+2}^{T}\sum_{s=g+1}^{t-1} E\left[\underline{w}_{it-g,T}\underline{w}_{is-g,T}\right] = NTq(T)^2\frac{\sigma^2}{2}\left[1 + O\left(\frac{q(T)}{T}\right)\right].$$

Proof of Lemma SE-16:

First consider part (a). Note that, under the assumption here, there exists a positive integer I_ρ such that for all $T \geq \max\{I_\rho, g+2\}$, the triangular array process $\left\{\underline{w}_{it-g,T}\right\}$ has the partial sum representation $\underline{w}_{it-g,T} = \sum_{j=1}^{t-g}\varepsilon_{ij}$. Thus, by direct calculation, we have that, for all $T \geq \max\{I_\rho, g+2\}$,

$$\begin{aligned}
\sum_{i=1}^{N}\sum_{t=g+1}^{T} E\left[\underline{w}_{it-g,T}^2\right] &= \sum_{i=1}^{N}\sum_{t=g+1}^{T}\sum_{j=1}^{t-g}\sum_{k=1}^{t-g} E\left[\varepsilon_{ij}\varepsilon_{ik}\right] = \sigma^2 N\sum_{t=g+1}^{T}(t-g) \\
&= \frac{\sigma^2}{2}N(T-g)(T-g+1) = NT^2\frac{\sigma^2}{2}\left[1 + O\left(\frac{1}{T}\right)\right],
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{i=1}^{N}\sum_{t=g+2}^{T}\sum_{s=g+1}^{t-1} E\left[\underline{w}_{it-g,T}\underline{w}_{is-g,T}\right] = \sum_{i=1}^{N}\sum_{t=g+2}^{T}\sum_{s=g+1}^{t-1}\sum_{j=1}^{s-g}\sum_{k=1}^{t-g} E\left[\varepsilon_{ij}\varepsilon_{ik}\right] = \sigma^2 N\sum_{t=g+2}^{T}\sum_{s=g+1}^{t-1}(s-g) \\
= \;& N\frac{\sigma^2}{2}\sum_{t=g+2}^{T}(t-g)(t-g-1) = N\frac{\sigma^2}{2}\left[\frac{(T-g)(T-g+1)(2T-2g+1)}{6} - \frac{(T-g)(T-g+1)}{2}\right] \\
= \;& NT^3\frac{\sigma^2}{6}\left[1 + O\left(\frac{1}{T}\right)\right],
\end{aligned}$$

which completes the proof for part (a).

Now, to show parts (b)-(d), note first that, for all $T \geq g+2$, we have

$$\sum_{i=1}^{N}\sum_{t=g+1}^{T} E\left[\underline{w}_{it-g,T}^{2}\right] = \sum_{i=1}^{N}\sum_{t=g+1}^{T}\sum_{j=1}^{t-g}\sum_{k=1}^{t-g}\exp\left\{-\frac{(t-g-j)}{q(T)}\right\}\exp\left\{-\frac{(t-g-k)}{q(T)}\right\}E\left[\varepsilon_{ij}\varepsilon_{ik}\right]$$
$$= \sigma^{2}N\sum_{t=g+1}^{T}\sum_{j=1}^{t-g}\exp\left\{-2\frac{(t-g-j)}{q(T)}\right\},$$

and

$$\sum_{i=1}^{N}\sum_{t=g+2}^{T}\sum_{s=g+1}^{t-1} E\left[\underline{w}_{it-g,T}\underline{w}_{is-g,T}\right]$$
$$= \sum_{i=1}^{N}\sum_{t=g+2}^{T}\sum_{s=g+1}^{t-1}\sum_{j=1}^{s-g}\sum_{k=1}^{t-g}\exp\left\{-\left(\frac{t-g-j}{q(T)}\right)\right\}\exp\left\{-\left(\frac{s-g-k}{q(T)}\right)\right\}E\left[\varepsilon_{ij}\varepsilon_{ik}\right]$$
$$= \sigma^{2}N\sum_{t=g+2}^{T}\sum_{s=g+1}^{t-1}\sum_{k=1}^{s-g}\exp\left\{-\left(\frac{t-g-k}{q(T)}\right)\right\}\exp\left\{-\left(\frac{s-g-k}{q(T)}\right)\right\}.$$

Consider part (b), where we take $\rho_T = \exp\{-1/q(T)\}$ such that $T/q(T) \to 0$. Applying part (a) of Lemma SE-1 with $b = g+1$ and $d = 2$, we obtain

$$\sum_{i=1}^{N}\sum_{t=g+1}^{T} E\left[\underline{w}_{it-g,T}^{2}\right] = \sigma^{2}N\sum_{t=g+1}^{T}\sum_{j=1}^{t-g}\exp\left\{-2\frac{(t-g-j)}{q(T)}\right\}$$
$$= NT^{2}\frac{\sigma^{2}}{2}\left[1++O\left(\max\left\{\frac{T}{q(T)},\frac{1}{T}\right\}\right)O\left(\frac{1}{T}\right)+O\left(\frac{T}{q(T)}\right)\right],$$

while, applying part (a) of Lemma SE-7, we get

$$\sum_{i=1}^{N}\sum_{t=g+2}^{T}\sum_{s=g+1}^{t-1} E\left[\underline{w}_{it-g,T}\underline{w}_{is-g,T}\right] = \sigma^{2}N\sum_{t=g+2}^{T}\sum_{s=g+1}^{t-1}\sum_{k=1}^{s-g}\exp\left\{-\left(\frac{t-g-k}{q(T)}\right)\right\}\exp\left\{-\left(\frac{s-g-k}{q(T)}\right)\right\}$$
$$= NT^{3}\frac{\sigma^{2}}{6}\left[1+O\left(\max\left\{\frac{T}{q(T)},\frac{1}{T}\right\}\right)\right],$$

as required for part (b).

We now turn our attention to part (c), where we take $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \sim T$. Applying part (b) of Lemma SE-1 with $b = g+1$ and $d = 2$, we get in this case

$$\sum_{i=1}^{N}\sum_{t=g+1}^{T} E\left[\underline{w}_{it-g,T}^{2}\right] = \sigma^{2}N\sum_{t=g+1}^{T}\sum_{j=1}^{t-g}\exp\left\{-2\frac{(t-g-j)}{q(T)}\right\}$$
$$= \sigma^{2}N\frac{q(T)^{2}}{4}\left[\exp\left\{-\frac{2T}{q(T)}\right\}+\frac{2T}{q(T)}-1\right]\left[1+O\left(\frac{1}{T}\right)\right].$$

Moreover, applying part (b) of Lemma SE-7 with $b = 1$, we obtain

$$\begin{aligned}
& \sum_{i=1}^{N}\sum_{t=g+2}^{T}\sum_{s=g+1}^{t-1} E\left[\underline{w}_{it-g,T}\underline{w}_{is-g,T}\right] \\
= \; & \sigma^2 N \sum_{t=g+2}^{T}\sum_{s=g+1}^{t-1}\sum_{k=1}^{s-g} \exp\left\{-\left(\frac{t-g-k}{q(T)}\right)\right\}\exp\left\{-\left(\frac{s-g-k}{q(T)}\right)\right\} \\
= \; & NTq(T)^2\frac{\sigma^2}{2}\left[1-\frac{3}{2}\frac{q(T)}{T}+\frac{2q(T)}{T}\exp\left\{-\frac{T}{q(T)}\right\}-\frac{1}{2}\frac{q(T)}{T}\exp\left\{-\frac{2T}{q(T)}\right\}\right]\left[1+O\left(\frac{1}{T}\right)\right].
\end{aligned}$$

Together, these two results show part (c).

Finally, consider part (d), where we take $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \to \infty$ but $q(T)/T \to 0$. Applying part (c) of Lemma SE-1 with $b = g+1$ and $d = 2$, we get

$$\begin{aligned}
\sum_{i=1}^{N}\sum_{t=g+1}^{T} E\left[\underline{w}_{it-g,T}^2\right] &= \sigma^2 N \sum_{t=g+1}^{T}\sum_{j=1}^{t-g}\exp\left\{-2\frac{(t-g-j)}{q(T)}\right\} \\
&= NTq(T)\frac{\sigma^2}{2}\left[1+O\left(\frac{1}{q(T)}\right)+O\left(\frac{q(T)}{T}\right)\right].
\end{aligned}$$

Moreover, applying part (c) of Lemma SE-7 with $b = 1$, we obtain

$$\begin{aligned}
& \sum_{i=1}^{N}\sum_{t=g+2}^{T}\sum_{s=g+1}^{t-1} E\left[\underline{w}_{it-g,T}\underline{w}_{is-g,T}\right] \\
= \; & \sigma^2 N \sum_{t=g+2}^{T}\sum_{s=g+1}^{t-1}\sum_{k=1}^{s-g}\exp\left\{-\frac{(t-g-k)}{q(T)}\right\}\exp\left\{-b\frac{(s-g-k)}{q(T)}\right\} \\
= \; & NTq(T)^2\frac{\sigma^2}{2}\left[1+O\left(\frac{q(T)}{T}\right)\right].
\end{aligned}$$

Together, these two results show part (d). □

Lemma SE-17:

Let g be a positive integer, and suppose that Assumptions 1 and 4 hold. Then, the following statements are true as $N, T \to \infty$.

(a) If $q(T) \sim T$, then

$$\begin{aligned}
\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=g+1}^{T} w_{it-g,T}^2 &= \frac{q(T)^2}{T^2}\frac{\sigma^2}{4}\left[\exp\left\{-\frac{2T}{q(T)}\right\}+\frac{2T}{q(T)}-1\right] \\
&\quad +O_p\left(\frac{1}{\sqrt{N}}\right)+O_p\left(\frac{1}{\sqrt{T}}\right).
\end{aligned}$$

(b) If $q(T) \to \infty$ such that $q(T)/T \to 0$, then

$$\frac{1}{NTq(T)} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{it-g,T}^2 = \frac{\sigma^2}{2} + O_p\left(\max\left\{\frac{1}{q(T)}, \frac{q(T)}{T}, \sqrt{\frac{q(T)}{NT}}, \frac{1}{\sqrt{T}}\right\}\right).$$

(c) If $\rho_T \in \mathcal{G}_{\mathrm{St}} = \left\{|\rho_T| = \exp\left\{-\frac{1}{q(T)}\right\} : q(T) \geq 0 \text{ and } q(T) = O(1) \text{ as } T \to \infty\right\}$, then

$$\frac{1-\rho_T^2}{NT} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{it-g,T}^2 = \sigma^2 + O_p\left(T^{-1/2}\right).$$

Proof of Lemma SE-17:

To proceed, again write

$$\begin{aligned}\frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=g+1}^{T} w_{it-g,T}^2 &= \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=g+1}^{T} \underline{w}_{it-g,T}^2 + 2\frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=g+1}^{T} \underline{w}_{it-g} \rho_T^{t-g} w_{i0} \\ &\quad + \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=g+1}^{T} \rho_T^{2(t-g)} w_{i0}^2.\end{aligned}$$

To show part (a), note first that from part (c) of Lemma SE-16 above, we have that

$$\frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=g+1}^{T} E\left[\underline{w}_{it-g,T}^2\right] - \frac{q(T)^2}{T^2} \frac{\sigma^2}{4} \left[\exp\left\{-\frac{2T}{q(T)}\right\} + \frac{2T}{q(T)} - 1\right] = O\left(\frac{1}{T}\right),$$

as $N, T \to \infty$. Thus, to show the desired result, it suffices to show that

$$\frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=g+1}^{T} \underline{w}_{it-g,T}^2 - \frac{1}{NT^2} \sum_{i=1}^{N} \sum_{t=g+1}^{T} E\left[\underline{w}_{it-g,T}^2\right] = O_p\left(\frac{1}{\sqrt{N}}\right),$$

as $N, T \to \infty$. To proceed, note that for all $T \geq g+1$, we have

$$
\begin{aligned}
& E\left(\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\underline{w}_{it-g,T}^2 - \frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=g+1}^{T}E\left[\underline{w}_{it-g,T}^2\right]\right)^2 \\
= \; & E\left(\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\underline{w}_{it-g,T}^2 - \frac{1}{NT^2}\sigma^2\sum_{i=1}^{N}\sum_{t=g+1}^{T}\sum_{k=1}^{t-g}\exp\left\{-2\frac{(t-g-k)}{q(T)}\right\}\right)^2 \\
= \; & \frac{1}{N^2T^4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=g+1}^{T}\sum_{s=g+1}^{T}E\left[\left(\underline{w}_{it-g,T}^2 - \sigma^2\sum_{k=1}^{t-g}\exp\left\{-2\frac{(t-2-k)}{q(T)}\right\}\right)\right. \\
& \qquad\qquad \left.\times\left(\underline{w}_{js-g,T}^2 - \sigma^2\sum_{\ell=1}^{s-g}\exp\left\{-2\frac{(s-g-\ell)}{q(T)}\right\}\right)\right] \\
= \; & \frac{1}{N^2T^4}\sum_{i=1}^{N}\sum_{t=g+1}^{T}E\left[\underline{w}_{it-g,T}^2 - \sigma^2\sum_{j=1}^{t-g}\exp\left\{-2\frac{(t-g-j)}{q(T)}\right\}\right]^2 \\
& + \frac{2}{N^2T^4}\sum_{i=1}^{N}\sum_{t=g+2}^{T}\sum_{s=g+1}^{t-1}E\left\{\left[\underline{w}_{it-g,T}^2 - \sigma^2\sum_{j=1}^{t-g}\exp\left\{-2\frac{(t-g-j)}{q(T)}\right\}\right]\right. \\
& \qquad\qquad \left.\times\left[\underline{w}_{is-g,T}^2 - \sigma^2\sum_{k=1}^{s-g}\exp\left\{-2\frac{(s-g-k)}{q(T)}\right\}\right]\right\}. \qquad (19)
\end{aligned}
$$

Now, taking each of the two terms on the right-hand side of (19) in turn, we have upon applying part

(b) of Lemma SE-1 with $b = g+1$ and $d = 4$ and part (b) of Lemma SE-2 with $b = 2$

$$
\begin{aligned}
& \frac{1}{N^2T^4}\sum_{i=1}^{N}\sum_{t=g+1}^{T} E\left[\underline{w}_{it-g,T}^2 - \sigma^2\sum_{j=1}^{t-g}\exp\left\{-2\frac{(t-g-j)}{q(T)}\right\}\right]^2 \\
= & \frac{1}{N^2T^4}\sum_{i=1}^{N}\sum_{t=g+1}^{T} E\left[\underline{w}_{it-g,T}^4\right] - \frac{1}{N^2T^4}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\left(\sigma^2\sum_{j=1}^{t-g}\exp\left\{-2\frac{(t-g-j)}{q(T)}\right\}\right)^2 \\
= & \frac{1}{N^2T^4}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\sum_{h=1}^{t-g}\sum_{j=1}^{t-g}\sum_{k=1}^{t-g}\sum_{\ell=1}^{t-g}\left\{\exp\left\{-\frac{(t-g-h)}{q(T)}\right\}\exp\left\{-\frac{(t-g-j)}{q(T)}\right\}\right. \\
& \qquad\qquad \left.\times\exp\left\{-\frac{(t-g-k)}{q(T)}\right\}\exp\left\{-\frac{(t-g-\ell)}{q(T)}\right\}E\left[\varepsilon_{ih}\varepsilon_{ij}\varepsilon_{ik}\varepsilon_{i\ell}\right]\right\} \\
& -\frac{\sigma^4}{N^2T^4}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\left(\sum_{j=1}^{t-g}\exp\left\{-2\frac{(t-g-j)}{q(T)}\right\}\right)^2 \\
\leq & \frac{1}{N^2T^4}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\sum_{j=1}^{t-g}\exp\left\{-4\frac{(t-g-j)}{q(T)}\right\}E\left[\varepsilon_{ij}^4\right] \\
& +\frac{1}{N^2T^4}3\sigma^4\sum_{i=1}^{N}\sum_{t=g+1}^{T}\left[\sum_{j=1}^{t-g}\exp\left\{-2\frac{(t-g-j)}{q(T)}\right\}\right]^2 \\
& -\frac{\sigma^4}{N^2T^4}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\left(\sum_{j=1}^{t-g}\exp\left\{-2\frac{(t-g-j)}{q(T)}\right\}\right)^2 \\
= & \frac{1}{N^2T^4}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\sum_{j=1}^{t-g}\exp\left\{-4\frac{(t-g-j)}{q(T)}\right\}E\left[\varepsilon_{ij}^4\right] \\
& +2\frac{\sigma^4}{N^2T^4}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\left[\sum_{j=1}^{t-g}\exp\left\{-2\frac{(t-g-j)}{q(T)}\right\}\right]^2 \\
= & \frac{E\left[\varepsilon_{ij}^4\right]}{N^2T^4}\frac{Nq(T)^2}{16}\left[\exp\left\{-\frac{4T}{q(T)}\right\}+\frac{4T}{q(T)}-1\right]\left[1+O\left(\frac{1}{T}\right)\right] \\
& -2\frac{\sigma^4}{N^2T^4}\frac{Nq(T)^3}{16}\left[3-\frac{4T}{q(T)}-4\exp\left\{-\frac{2T}{q(T)}\right\}+\exp\left\{-\frac{4T}{q(T)}\right\}\right]\left[1+O\left(\frac{1}{T}\right)\right] \\
= & O\left(\frac{1}{NT^2}\right)+O\left(\frac{1}{NT}\right) \\
= & O\left(\frac{1}{NT}\right). \qquad (20)
\end{aligned}
$$

Moreover, note that

$$
\begin{aligned}
& \frac{1}{N^2T^4}\sum_{i=1}^{N}\sum_{t=g+2}^{T}\sum_{s=g+1}^{t-1} E\left\{\left[w_{it-g,T}^2-\sigma^2\sum_{j=1}^{t-g}\exp\left\{-2\frac{(t-g-j)}{q(T)}\right\}\right]\right. \\
& \qquad\qquad \left.\times\left[w_{is-g,T}^2-\sigma^2\sum_{k=1}^{s-g}\exp\left\{-2\frac{(s-g-k)}{q(T)}\right\}\right]\right\} \\
= \; & \frac{1}{N^2T^4}\sum_{i=1}^{N}\sum_{t=g+2}^{T}\sum_{s=g+1}^{t-1} E\left[w_{it-g,T}^2 w_{is-g,T}^2\right] \\
& -\frac{2\sigma^4}{N^2T^4}\sum_{i=1}^{N}\sum_{t=g+2}^{T}\sum_{s=g+1}^{t-1}\sum_{j=1}^{t-g}\exp\left\{-2\frac{(t-g-j)}{q(T)}\right\}\sum_{k=1}^{s-g}\exp\left\{-2\frac{(s-g-k)}{q(T)}\right\} \\
& +\frac{\sigma^4}{N^2T^4}\sum_{i=1}^{N}\sum_{t=g+2}^{T}\sum_{s=g+1}^{t-1}\sum_{j=1}^{t-g}\exp\left\{-2\frac{(t-g-j)}{q(T)}\right\}\sum_{k=1}^{s-2}\exp\left\{-2\frac{(s-g-k)}{q(T)}\right\} \\
= \; & \frac{1}{N^2T^4}\sum_{i=1}^{N}\sum_{t=g+2}^{T}\sum_{s=g+1}^{t-1}\left\{\sum_{h=1}^{t-g}\sum_{j=1}^{t-g}\exp\left(-\frac{(t-g-h)}{q(T)}\right)\exp\left(-\frac{(t-g-j)}{q(T)}\right)\right. \\
& \qquad \left.\times\sum_{k=1}^{s-g}\sum_{\ell=1}^{s-g}\exp\left(-\frac{(s-g-k)}{q(T)}\right)\exp\left(-\frac{(s-g-\ell)}{q(T)}\right)E\left[\varepsilon_{ih}\varepsilon_{ij}\varepsilon_{ik}\varepsilon_{i\ell}\right]\right\} \\
& -\sigma^4\frac{1}{N^2T^4}\sum_{i=1}^{N}\sum_{t=g+2}^{T}\sum_{s=g+1}^{t-1}\sum_{j=1}^{t-g}\exp\left\{-2\frac{(t-g-j)}{q(T)}\right\}\sum_{k=1}^{s-g}\exp\left\{-2\frac{(s-g-k)}{q(T)}\right\} \\
= \; & E\left[\varepsilon_{ij}^4\right]\frac{1}{N^2T^4}\sum_{i=1}^{N}\sum_{t=g+2}^{T}\sum_{s=g+1}^{t-1}\sum_{j=1}^{s-g}\exp\left(-2\frac{(t-g-j)}{q(T)}\right)\exp\left(-2\frac{(s-g-j)}{q(T)}\right) \\
& +\sigma^4\frac{1}{N^2T^4}\sum_{i=1}^{N}\sum_{t=g+2}^{T}\sum_{s=g+1}^{t-1}\sum_{j=1}^{t-g}\exp\left(-2\frac{(t-g-j)}{q(T)}\right)\sum_{k=1}^{s-g}\exp\left(-2\frac{(s-g-k)}{q(T)}\right) \\
& -\sigma^4\frac{1}{N^2T^4}\sum_{i=1}^{N}\sum_{t=g+2}^{T}\sum_{s=g+1}^{t-1}\sum_{j=1}^{s-g}\exp\left(-2\frac{(t-g-j)}{q(T)}\right)\exp\left(-2\frac{(s-g-j)}{q(T)}\right) \\
& +2\sigma^4\frac{1}{N^2T^4}\sum_{i=1}^{N}\sum_{t=g+2}^{T}\sum_{s=g+1}^{t-1}\sum_{j=1}^{s-g}\exp\left(-\frac{(t-g-j)}{q(T)}\right)\exp\left(-\frac{(s-g-j)}{q(T)}\right) \\
& \qquad\qquad \times\sum_{k=1}^{s-g}\exp\left(-\frac{(t-g-k)}{q(T)}\right)\exp\left(-\frac{(s-g-k)}{q(T)}\right) \\
& -2\sigma^4\frac{1}{N^2T^4}\sum_{i=1}^{N}\sum_{t=g+2}^{T}\sum_{s=g+1}^{t-1}\sum_{j=1}^{s-g}\exp\left(-2\frac{(t-g-j)}{q(T)}\right)\exp\left(-2\frac{(s-g-j)}{q(T)}\right) \\
& -\sigma^4\frac{1}{N^2T^4}\sum_{i=1}^{N}\sum_{t=g+2}^{T}\sum_{s=g+1}^{t-1}\sum_{j=1}^{t-g}\exp\left\{-2\frac{(t-g-j)}{q(T)}\right\}\sum_{k=1}^{s-g}\exp\left\{-2\frac{(s-g-k)}{q(T)}\right\}
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\left|\frac{1}{N^2T^4}\sum_{i=1}^{N}\sum_{t=g+2}^{T}\sum_{s=g+1}^{t-1}E\left\{\left[\underline{w}_{it-g,T}^2-\sigma^2\sum_{j=1}^{t-g}\exp\left\{-2\frac{(t-g-j)}{q(T)}\right\}\right]\right.\right.\\
&\qquad\qquad\left.\left.\times\left[\underline{w}_{is-g,T}^2-\sigma^2\sum_{k=1}^{s-g}\exp\left\{-2\frac{(s-g-k)}{q(T)}\right\}\right]\right\}\right|\\
\leq\ &\frac{\left(E\left[\varepsilon_{ij}^4\right]+3\sigma^4\right)}{N^2T^4}\sum_{i=1}^{N}\sum_{t=g+2}^{T}\sum_{s=g+1}^{t-1}\sum_{j=1}^{s-g}\exp\left(-2\frac{(t-g-j)}{q(T)}\right)\exp\left(-2\frac{(s-g-j)}{q(T)}\right)\\
&+\frac{2\sigma^4}{N^2T^4}\sum_{i=1}^{N}\sum_{t=g+2}^{T}\sum_{s=g+1}^{t-1}\sum_{j=1}^{s-g}\exp\left(-\frac{(t-g-j)}{q(T)}\right)\exp\left(-\frac{(s-g-j)}{q(T)}\right)\\
&\qquad\qquad\times\sum_{k=1}^{s-g}\exp\left(-\frac{(t-g-k)}{q(T)}\right)\exp\left(-\frac{(s-g-k)}{q(T)}\right)\\
=\ &\frac{\left(E\left[\varepsilon_{ij}^4\right]+3\sigma^4\right)}{NT^4}\frac{Tq(T)^2}{8}\\
&\times\left[1-\frac{3}{4}\frac{q(T)}{T}+\frac{q(T)}{T}\exp\left\{-\frac{2T}{q(T)}\right\}-\frac{1}{4}\frac{q(T)}{T}\exp\left\{-\frac{4T}{q(T)}\right\}\right]\left[1+O\left(\frac{1}{T}\right)\right]\\
&+\frac{2\sigma^4}{NT^4}\left(\frac{q(T)^4}{32}\left[\exp\left\{-\frac{4T}{q(T)}\right\}+4(g-1)\exp\left\{-\frac{2T}{q(T)}\right\}-5\right]\right.\\
&\qquad\left.+\frac{Tq(T)^3}{8}\left[2\exp\left\{-\frac{2T}{q(T)}\right\}+1\right]\right)\left[1+O\left(\frac{1}{T}\right)\right]\\
=\ &O\left(\frac{1}{NT}\right)+O\left(\frac{1}{N}\right)\\
=\ &O\left(\frac{1}{N}\right),
\end{aligned}
\tag{21}
$$

where the third-to-last equality is justified by applying part (b) of Lemma SE-7 with $b=2$ and part (b) of Lemma SE-6 with $b=1$.

It follows from (19), (20), and (21), that

$$
\begin{aligned}
& E\left(\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\underline{w}_{it-g,T}^2-\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=g+1}^{T}E\left[\underline{w}_{it-g}^2\right]\right)^2 \\
= & \frac{1}{N^2T^4}\sum_{i=1}^{N}\sum_{t=g+1}^{T}E\left[\underline{w}_{it-g,T}^2-\sigma^2\sum_{j=1}^{t-g}\exp\left\{-2\frac{(t-g-j)}{q(T)}\right\}\right]^2 \\
& +\frac{2}{N^2T^4}\sum_{i=1}^{N}\sum_{t=4}^{T}\sum_{s=3}^{t-1}E\left\{\left[\underline{w}_{it-g,T}^2-\sigma^2\sum_{j=1}^{t-2}\exp\left\{-2\frac{(t-2-j)}{q(T)}\right\}\right]\right. \\
& \qquad\qquad \left.\times\left[\underline{w}_{is-g,T}^2-\sigma^2\sum_{k=1}^{s-2}\exp\left\{-2\frac{(s-2-k)}{q(T)}\right\}\right]\right\} \\
= & O\left(\frac{1}{NT}\right)+O\left(\frac{1}{N}\right) \\
= & O\left(\frac{1}{N}\right)=o(1),
\end{aligned}
$$

from which it follows by Markov's inequality that

$$
\begin{aligned}
\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=3}^{T}\underline{w}_{it-g,T}^2 &= \frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=3}^{T}E\left[\underline{w}_{it-g,T}^2\right]+O_p\left(\frac{1}{\sqrt{N}}\right) \\
&= \frac{q(T)^2}{T^2}\frac{\sigma^2}{4}\left[\exp\left\{-\frac{2T}{q(T)}\right\}+\frac{2T}{q(T)}-1\right]+O_p\left(\frac{1}{T}\right)+O_p\left(\frac{1}{\sqrt{N}}\right) \\
&= \frac{q(T)^2}{T^2}\frac{\sigma^2}{4}\left[\exp\left\{-\frac{2T}{q(T)}\right\}+\frac{2T}{q(T)}-1\right]+O_p\left(\frac{1}{\sqrt{N}}\right),
\end{aligned}
$$

as $N,T\to\infty$.

Moreover, note that, by Assumption 4, we have

$$
\begin{aligned}
& E\left[\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\rho_T^{2(t-g)}w_{i0}^2\right] \\
\leq & \frac{\sup_i E\left[w_{i0}^2\right]}{T^2}\exp\left\{-\frac{2}{q(T)}\right\}\left[1-\exp\left\{-\frac{2}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{2(T-g)}{q(T)}\right\}\right] \\
= & O\left(T^{-2}\right)O(1)O(T)O(1)=O\left(T^{-1}\right),
\end{aligned}
$$

so that, applying Markov's inequality, we further deduce that

$$
\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\rho_T^{2(t-g)}w_{i0}^2=O_p\left(\frac{1}{T}\right).
$$

The Cauchy-Schwarz inequality then implies that

$$\begin{aligned}\left|\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\underline{w}_{it-g}\rho_T^{t-g}w_{i0}\right| &\leq \sqrt{\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\underline{w}_{it-g}^2}\sqrt{\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\rho_T^{2(t-g)}w_{i0}^2} \\ &= O_p(1)\,O_p\left(T^{-1/2}\right) = O_p\left(T^{-1/2}\right)\end{aligned}$$

from which it follows that

$$\begin{aligned}&\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=g+1}^{T}w_{it-g,T}^2 \\ =\ &\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\underline{w}_{it-g,T}^2 + \frac{2}{NT^2}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\underline{w}_{it-g}\rho_T^{t-g}w_{i0} + \frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\rho_T^{2(t-g)}w_{i0}^2 \\ =\ &\frac{q(T)^2}{T^2}\frac{\sigma^2}{4}\left[\exp\left\{-\frac{2T}{q(T)}\right\}+\frac{2T}{q(T)}-1\right] + O_p\left(\frac{1}{\sqrt{N}}\right) + O_p\left(\frac{1}{\sqrt{T}}\right),\end{aligned}$$

as required.

To show part (b), write

$$\begin{aligned}&\frac{1}{NTq(T)}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\underline{w}_{it-g,T}^2 - \frac{\sigma^2}{2} \\ =\ &\frac{1}{NTq(T)}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\underline{w}_{it-g,T}^2 - \frac{1}{NTq(T)}\sum_{i=1}^{N}\sum_{t=g+1}^{T}E\left[\underline{w}_{it-g,T}^2\right] + \frac{1}{NTq(T)}\sum_{i=1}^{N}\sum_{t=g+1}^{T}E\left[\underline{w}_{it-g,T}^2\right] - \frac{\sigma^2}{2}.\end{aligned}$$

Now, from part (d) of Lemma SE-16 above, we have that, as $N, T \to \infty$,

$$\frac{1}{NTq(T)}\sum_{i=1}^{N}\sum_{t=g+1}^{T}E\left[\underline{w}_{it-g,T}^2\right] = \frac{\sigma^2}{2} + O\left(\frac{1}{q(T)}\right) + O\left(\frac{q(T)}{T}\right).$$

Thus, to show part (b), it suffices to show that

$$\frac{1}{NTq(T)}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\underline{w}_{it-g,T}^2 = \frac{1}{NTq(T)}\sum_{i=1}^{N}\sum_{t=g+1}^{T}E\left[\underline{w}_{it-g}^2\right] + O_p\left(\sqrt{\frac{q(T)}{NT}}\right),$$

as $N, T \to \infty$. Similar to the proof of part (a) above, we have

$$
\begin{aligned}
& E\left(\frac{1}{NTq(T)}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\underline{w}_{it-g,T}^{2}-\frac{1}{NTq(T)}\sum_{i=1}^{N}\sum_{t=g+1}^{T}E\left[\underline{w}_{it-g}^{2}\right]\right)^{2} \\
\leq\ & \frac{1}{N^{2}T^{2}q(T)^{2}}\sum_{i=1}^{N}\sum_{t=g+1}^{T}E\left[\underline{w}_{it-g,T}^{2}-\sigma^{2}\sum_{j=1}^{t-g}\exp\left\{-2\frac{(t-g-j)}{q(T)}\right\}\right]^{2} \\
& +\frac{2}{N^{2}T^{2}q(T)^{2}}\left|\sum_{i=1}^{N}\sum_{t=g+2}^{T}\sum_{s=g+1}^{t-1}E\left\{\left[\underline{w}_{it-g,T}^{2}-\sigma^{2}\sum_{j=1}^{t-g}\exp\left\{-2\frac{(t-g-j)}{q(T)}\right\}\right]\right.\right. \\
& \qquad\qquad\left.\left.\times\left[\underline{w}_{is-g,T}^{2}-\sigma^{2}\sum_{k=1}^{s-g}\exp\left\{-2\frac{(s-g-k)}{q(T)}\right\}\right]\right\}\right| \\
\leq\ & \frac{1}{N^{2}T^{2}q(T)^{2}}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\sum_{j=1}^{t-g}\exp\left(-4\frac{(t-g-j)}{q(T)}\right)E\left[\varepsilon_{ij}^{4}\right] \\
& +2\frac{\sigma^{4}}{N^{2}T^{2}q(T)^{2}}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\left[\sum_{j=1}^{t-g}\exp\left(-2\frac{(t-g-j)}{q(T)}\right)\right]^{2} \\
& +\frac{\left(E\left[\varepsilon_{ij}^{4}\right]+3\sigma^{4}\right)}{N^{2}T^{2}q(T)^{2}}\sum_{i=1}^{N}\sum_{t=g+2}^{T}\sum_{s=g+1}^{t-1}\sum_{j=1}^{s-g}\exp\left(-2\frac{(t-g-j)}{q(T)}\right)\exp\left(-2\frac{(s-g-j)}{q(T)}\right) \\
& +\frac{2\sigma^{4}}{N^{2}T^{2}q(T)^{2}}\sum_{i=1}^{N}\sum_{t=g+2}^{T}\sum_{s=g+1}^{t-1}\sum_{j=1}^{s-g}\exp\left(-\frac{(t-g-j)}{q(T)}\right)\exp\left(-\frac{(s-g-j)}{q(T)}\right) \\
& \qquad\qquad\times\sum_{k=1}^{s-g}\exp\left(-\frac{(t-g-k)}{q(T)}\right)\exp\left(-\frac{(s-g-k)}{q(T)}\right) \\
=\ & \frac{E\left[\varepsilon_{ij}^{4}\right]}{N^{2}T^{2}q(T)^{2}}N\frac{Tq(T)}{4}\left[1+O\left(\frac{1}{q(T)}\right)+O\left(\frac{q(T)}{T}\right)\right] \\
& +2\frac{\sigma^{4}}{N^{2}T^{2}q(T)^{2}}\frac{NTq(T)^{2}}{4}\left[1+O\left(\frac{q(T)}{T}\right)\right] \\
& +\frac{\left(E\left[\varepsilon_{ij}^{4}\right]+3\sigma^{4}\right)}{N^{2}T^{2}q(T)^{2}}\frac{NTq(T)^{2}}{8}\left[1+O\left(\max\left\{\frac{q(T)}{T},\frac{1}{q(T)}\right\}\right)\right] \\
& +2\sigma^{4}\frac{1}{N^{2}T^{2}q(T)^{2}}\frac{NTq(T)^{3}}{8}\left[1+O\left(\frac{1}{q(T)}\right)+O\left(\frac{q(T)}{T}\right)\right] \\
=\ & O\left(\frac{1}{NTq(T)}\right)+O\left(\frac{1}{NT}\right)+O\left(\frac{1}{NT}\right)+O\left(\frac{q(T)}{NT}\right) \\
=\ & O\left(\frac{q(T)}{NT}\right),
\end{aligned}
$$

where we have applied part (c) of Lemma SE-1, part (c) of Lemma SE-2, part (c) of Lemma SE-7 with

$b=2$, and part (a) of Lemma SE-6 with $b=1$ in calculating the order of magnitudes given above. It follows by Markov's inequality that, as $N,T\to\infty$,

$$\begin{aligned}\frac{1}{NTq(T)}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\underline{w}_{it-g,T}^{2} &= \frac{1}{NTq(T)}\sum_{i=1}^{N}\sum_{t=g+1}^{T}E\left[\underline{w}_{it-g,T}^{2}\right]+O_p\left(\sqrt{\frac{q(T)}{NT}}\right)\\ &= \frac{\sigma^2}{2}+O_p\left(\max\left\{\frac{1}{q(T)},\frac{q(T)}{T},\sqrt{\frac{q(T)}{NT}}\right\}\right)\end{aligned}$$

as required.

Moreover, note that, using Assumption 4, we have

$$\begin{aligned}&E\left[\frac{1}{NTq(T)}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\rho_T^{2(t-g)}w_{i0}^{2}\right]\\ \leq\ &\frac{\sup_i E\left[w_{i0}^{2}\right]}{Tq(T)}\exp\left\{-\frac{2}{q(T)}\right\}\left[1-\exp\left\{-\frac{2}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{2(T-g)}{q(T)}\right\}\right]\\ =\ &O\left(T^{-1}q(T)^{-1}\right)O(1)\,O(q(T))\,O(1)=O\left(\frac{1}{T}\right),\end{aligned}$$

so that

$$\frac{1}{NTq(T)}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\rho_T^{2(t-g)}w_{i0}^{2}=O_p\left(\frac{1}{T}\right),$$

by Markov's inequality. The Cauchy-Schwarz inequality furhter implies

$$\begin{aligned}\left|\frac{1}{NTq(T)}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\underline{w}_{it-g}\rho_T^{t-g}w_{i0}\right| &\leq \sqrt{\frac{1}{NTq(T)}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\underline{w}_{it-g}^{2}}\sqrt{\frac{1}{NTq(T)}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\rho_T^{2(t-g)}w_{i0}^{2}}\\ &= O_p(1)\,O_p\left(T^{-1/2}\right)=O_p\left(T^{-1/2}\right),\end{aligned}$$

from which it follows that

$$\begin{aligned}&\frac{1}{NTq(T)}\sum_{i=1}^{N}\sum_{t=g+1}^{T}w_{it-g,T}^{2}\\ =\ &\frac{1}{NTq(T)}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\underline{w}_{it-g,T}^{2}+\frac{2}{NTq(T)}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\underline{w}_{it-g}\rho_T^{t-g}w_{i0}+\frac{1}{NTq(T)}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\rho_T^{2(t-g)}w_{i0}^{2}\\ =\ &\frac{\sigma^2}{2}+O_p\left(\max\left\{\frac{1}{q(T)},\frac{q(T)}{T},\sqrt{\frac{q(T)}{NT}}\right\}\right)+O_p\left(\frac{1}{T}\right)+O_p\left(\frac{1}{\sqrt{T}}\right)\\ =\ &\frac{\sigma^2}{2}+O_p\left(\max\left\{\frac{1}{q(T)},\frac{q(T)}{T},\sqrt{\frac{q(T)}{NT}},\frac{1}{\sqrt{T}}\right\}\right),\end{aligned}$$

which is the desired result for part (b).

To show part (c), write

$$\begin{aligned}
&\frac{1-\rho_T^2}{NT}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\underline{w}_{it-g,T}^2-\sigma^2\\
=\;&\frac{1-\rho_T^2}{NT}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\underline{w}_{it-g,T}^2-\frac{1-\rho_T^2}{NT}\sum_{i=1}^{N}\sum_{t=g+1}^{T}E\left[\underline{w}_{it-g,T}^2\right]+\frac{1-\rho_T^2}{NT}\sum_{i=1}^{N}\sum_{t=g+1}^{T}E\left[\underline{w}_{it-g,T}^2\right]-\sigma^2.
\end{aligned}$$

Note that for all $T\geq g+1$

$$\begin{aligned}
\frac{1-\rho_T^2}{NT}\sum_{i=1}^{N}\sum_{t=g+1}^{T}E\left[\underline{w}_{it-g,T}^2\right] &= \frac{1-\rho_T^2}{NT}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\sum_{j=1}^{t-g}\sum_{k=1}^{t-g}\rho_T^{2(t-g)-j-k}E\left[\varepsilon_{ij}\varepsilon_{ik}\right]\\
&= \sigma^2\frac{1-\rho_T^2}{T}\sum_{t=g+1}^{T}\sum_{j=1}^{t-g}\rho_T^{2(t-g-j)}\\
&= \sigma^2\frac{1-\rho_T^2}{T}\sum_{t=g+1}^{T}\frac{1-\rho_T^{2(t-g)}}{1-\rho_T^2}\\
&= \sigma^2\left[1-\frac{g}{T}-\frac{\rho_T^2\left(1-\rho_T^{2(T-g)}\right)}{T\left(1-\rho_T^2\right)}\right].
\end{aligned}$$

Since we assume here that $\rho_T^2=\exp\{-2/q(T)\}$ with $q(T)=O(1)$, it follows that there exist a positive constant C_q and a positive integer T^* such that for all $T\geq\max\{T^*,g+1\}$,

$$0\leq\frac{\rho_T^2\left(1-\rho_T^{2(T-g)}\right)}{T\left(1-\rho_T^2\right)}\leq\frac{1}{T\left(1-\exp\{-2/C_q\}\right)}=O\left(\frac{1}{T}\right),$$

so that, in this case,

$$\frac{1-\rho_T^2}{NT}\sum_{i=1}^{N}\sum_{t=g+1}^{T}E\left[\underline{w}_{it-g,T}^2\right]=\sigma^2+O\left(\frac{1}{T}\right).$$

Thus, to complete the proof of this part, it suffices to show that

$$\frac{1-\rho_T^2}{NT}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\underline{w}_{it-g,T}^2=\frac{1-\rho_T^2}{NT}\sum_{i=1}^{N}\sum_{t=g+1}^{T}E\left[\underline{w}_{it-g,T}^2\right]+O_p\left(\frac{1}{\sqrt{NT}}\right).$$

To proceed, write

$$\begin{aligned}
&E\left(\frac{1-\rho_T^2}{NT}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\underline{w}_{it-g,T}^2-\frac{1-\rho_T^2}{NT}\sum_{i=1}^{N}\sum_{t=g+1}^{T}E\left[\underline{w}_{it-g,T}^2\right]\right)^2\\
=\;&E\left(\frac{1-\rho_T^2}{NT}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\left[\underline{w}_{it-g,T}^2-\sigma^2\sum_{j=1}^{t-g}\rho_T^{2(t-g-j)}\right]\right)^2
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\left(1-\rho_T^2\right)^2}{N^2T^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=g+1}^{T}\sum_{s=g+1}^{T} E\left[\left(\underline{w}_{it-g,T}^2-\sigma^2\sum_{k=1}^{t-g}\rho_T^{2(t-g-k)}\right)\left(\underline{w}_{js-g,T}^2-\sigma^2\sum_{\ell=1}^{t-g}\rho_T^{2(t-g-\ell)}\right)\right] \\
&= \frac{\left(1-\rho_T^2\right)^2}{N^2T^2}\sum_{i=1}^{N}\sum_{t=g+1}^{T} E\left[\underline{w}_{it-g,T}^2-\sigma^2\sum_{k=1}^{t-g}\rho_T^{2(t-g-k)}\right]^2 \\
&\quad +\frac{2\left(1-\rho_T^2\right)^2}{N^2T^2}\sum_{i=1}^{N}\sum_{t=g+2}^{T}\sum_{s=g+1}^{t-1} E\left[\left(\underline{w}_{it-g,T}^2-\sigma^2\sum_{k=1}^{t-g}\rho_T^{2(t-g-k)}\right)\left(\underline{w}_{is-g,T}^2-\sigma^2\sum_{\ell=1}^{t-g}\rho_T^{2(t-g-\ell)}\right)\right].
\end{aligned}
$$

Next, note that there exist a positive constant C_q and a positive integer T^* such that for all $T \geq T^*$

$$
\begin{aligned}
&\frac{\left(1-\rho_T^2\right)^2}{N^2T^2}\sum_{i=1}^{N}\sum_{t=g+1}^{T} E\left(\underline{w}_{it-g,T}^2-\sigma^2\sum_{k=1}^{t-g}\rho_T^{2(t-g-k)}\right)^2 \\
&= \frac{\left(1-\rho_T^2\right)^2}{N^2T^2}\sum_{i=1}^{N}\sum_{t=g+1}^{T} E\left[\underline{w}_{it-g,T}^4\right]-\frac{\left(1-\rho_T^2\right)^2}{N^2T^2}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\left(\sigma^2\sum_{k=1}^{t-g}\rho_T^{2(t-g-k)}\right)^2 \\
&= \frac{\left(1-\rho_T^2\right)^2}{N^2T^2}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\sum_{h=1}^{t-g}\sum_{j=1}^{t-g}\sum_{k=1}^{t-g}\sum_{\ell=1}^{t-g}\rho_T^{(t-g-h)}\rho_T^{(t-g-j)}\rho_T^{(t-g-k)}\rho_T^{(t-g-\ell)}E\left[\varepsilon_{ih}\varepsilon_{ij}\varepsilon_{ik}\varepsilon_{i\ell}\right] \\
&\quad -\frac{\left(1-\rho_T^2\right)^2}{N^2T^2}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\left(\sigma^2\sum_{k=1}^{t-g}\rho_T^{2(t-g-k)}\right)^2 \\
&\leq \frac{E\left[\varepsilon_{ij}^4\right]\left(1-\rho_T^2\right)^2}{N^2T^2}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\sum_{k=1}^{t-g}\rho_T^{4(t-g-k)}+\frac{3\sigma^4\left(1-\rho_T^2\right)^2}{N^2T^2}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\left[\sum_{k=1}^{t-g}\rho_T^{2(t-g-k)}\right]^2 \\
&\quad -\frac{\sigma^4\left(1-\rho_T^2\right)^2}{N^2T^2}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\left(\sum_{k=1}^{t-g}\rho_T^{2(t-g-k)}\right)^2 \\
&= \frac{E\left[\varepsilon_{ij}^4\right]\left(1-\rho_T^2\right)^2}{N^2T^2}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\frac{1-\rho_T^{4(t-g)}}{1-\rho_T^4}+\frac{2\sigma^4\left(1-\rho_T^2\right)^2}{N^2T^2}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\frac{1-2\rho_T^{2(t-g)}+\rho_T^{4(t-g)}}{\left(1-\rho_T^2\right)^2} \\
&= \frac{E\left[\varepsilon_{ij}^4\right]}{NT^2}\frac{1-\rho_T^2}{\left(1+\rho_T^2\right)}\left[(T-g)-\frac{\rho_T^4\left(1-\rho_T^{4(T-g)}\right)}{1-\rho_T^4}\right] \\
&\quad +\frac{2\sigma^4}{NT^2}\left[(T-g)-2\frac{\rho_T^2\left(1-\rho_T^{2(T-g)}\right)}{1-\rho_T^2}+\frac{\rho_T^4\left(1-\rho_T^{4(T-g)}\right)}{1-\rho_T^4}\right] \\
&\leq \frac{E\left[\varepsilon_{ij}^4\right]}{NT}\left[1+\frac{g}{T}+\frac{1}{T\left(1-\exp\left\{-4/C_q\right\}\right)}\right] \\
&\quad +\frac{2\sigma^4}{NT}\left[1+\frac{g}{T}+\frac{2}{T\left(1-\exp\left\{-2/C_q\right\}\right)}+\frac{1}{T\left(1-\exp\left\{-4/C_q\right\}\right)}\right]=O\left(\frac{1}{NT}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
& \frac{\left(1-\rho_T^2\right)^2}{N^2T^2}\sum_{i=1}^{N}\sum_{t=g+2}^{T}\sum_{s=g+1}^{t-1} E\left[\left(\underline{w}_{it-g,T}^2-\sigma^2\sum_{k=1}^{t-g}\rho_T^{2(t-g-k)}\right)\left(\underline{w}_{is-g,T}^2-\sigma^2\sum_{\ell=1}^{t-g}\rho_T^{2(t-g-\ell)}\right)\right] \\
= & \frac{\left(1-\rho_T^2\right)^2}{N^2T^2}\sum_{i=1}^{N}\sum_{t=g+2}^{T}\sum_{s=g+1}^{t-1} E\left[\underline{w}_{it-g,T}^2\underline{w}_{is-g,T}^2\right] \\
& -2\sigma^4\frac{\left(1-\rho_T^2\right)^2}{N^2T^2}\sum_{i=1}^{N}\sum_{t=g+2}^{T}\sum_{s=g+1}^{t-1}\sum_{k=1}^{t-g}\rho_T^{2(t-g-k)}\sum_{\ell=1}^{s-g}\rho_T^{2(t-g-\ell)} \\
& +\frac{\left(1-\rho_T^2\right)^2}{N^2T^2}\sum_{i=1}^{N}\sum_{t=g+2}^{T}\sum_{s=g+1}^{t-1}\sum_{k=1}^{t-g}\rho_T^{2(t-g-k)}\sum_{\ell=1}^{s-2}\rho_T^{2(t-g-\ell)} \\
= & \frac{\left(1-\rho_T^2\right)^2}{N^2T^2}\sum_{i=1}^{N}\sum_{t=g+2}^{T}\sum_{s=g+1}^{t-1}\left\{\sum_{h=1}^{t-g}\sum_{j=1}^{t-g}\rho_T^{(t-g-h)}\rho_T^{(t-g-j)}\sum_{k=1}^{s-g}\sum_{\ell=1}^{s-g}\rho_T^{(s-g-k)}\rho_T^{(s-g-\ell)}E\left[\varepsilon_{ih}\varepsilon_{ij}\varepsilon_{ik}\varepsilon_{i\ell}\right]\right\} \\
& -\frac{\sigma^4\left(1-\rho_T^2\right)^2}{N^2T^2}\sum_{i=1}^{N}\sum_{t=g+2}^{T}\sum_{s=g+1}^{t-1}\sum_{k=1}^{t-g}\rho_T^{2(t-g-k)}\sum_{\ell=1}^{s-g}\rho_T^{2(t-g-\ell)}
\end{aligned}
$$

$$
\begin{aligned}
= \quad & E\left[\varepsilon_{ij}^4\right] \frac{\left(1-\rho_T^2\right)^2}{N^2T^2} \sum_{i=1}^{N} \sum_{t=g+2}^{T} \sum_{s=g+1}^{t-1} \sum_{j=1}^{s-g} \rho_T^{2(t-g-j)} \rho_T^{2(s-g-j)} \\
& +\sigma^4 \frac{\left(1-\rho_T^2\right)^2}{N^2T^2} \sum_{i=1}^{N} \sum_{t=g+2}^{T} \sum_{s=g+1}^{t-1} \sum_{k=1}^{t-g} \rho_T^{2(t-g-k)} \sum_{\ell=1}^{s-2} \rho_T^{2(s-g-\ell)} \\
& -\sigma^4 \frac{\left(1-\rho_T^2\right)^2}{N^2T^2} \sum_{i=1}^{N} \sum_{t=g+2}^{T} \sum_{s=g+1}^{t-1} \sum_{j=1}^{s-g} \rho_T^{2(t-g-j)} \rho_T^{2(s-g-j)} \\
& +2\sigma^4 \frac{\left(1-\rho_T^2\right)^2}{N^2T^2} \sum_{i=1}^{N} \sum_{t=g+2}^{T} \sum_{s=g+1}^{t-1} \sum_{j=1}^{s-g} \rho_T^{(t-g-j)} \rho_T^{(s-g-j)} \sum_{k=1}^{s-g} \rho_T^{(t-g-k)} \rho_T^{(s-g-k)} \\
& -2\sigma^4 \frac{\left(1-\rho_T^2\right)^2}{N^2T^2} \sum_{i=1}^{N} \sum_{t=g+2}^{T} \sum_{s=g+1}^{t-1} \sum_{j=1}^{s-g} \rho_T^{2(t-g-j)} \rho_T^{2(s-g-j)} \\
& -\sigma^4 \frac{\left(1-\rho_T^2\right)^2}{N^2T^2} \sum_{i=1}^{N} \sum_{t=g+2}^{T} \sum_{s=g+1}^{t-1} \sum_{k=1}^{t-g} \rho_T^{2(t-g-k)} \sum_{\ell=1}^{s-2} \rho_T^{2(s-g-\ell)} \\
= \quad & \left(E\left[\varepsilon_{ij}^4\right] - 3\sigma^4\right) \frac{\left(1-\rho_T^2\right)^2}{N^2T^2} \sum_{i=1}^{N} \sum_{t=g+2}^{T} \sum_{s=g+1}^{t-1} \rho_T^{2(t-s)} \frac{1-\rho_T^{4(s-g)}}{1-\rho_T^4} \\
& +2\sigma^4 \frac{\left(1-\rho_T^2\right)^2}{N^2T^2} \sum_{i=1}^{N} \sum_{t=g+2}^{T} \sum_{s=g+1}^{t-1} \rho_T^{2(t-s)} \frac{1-2\rho_T^{2(s-g)}+\rho_T^{4(s-g)}}{\left(1-\rho_T^2\right)^2} \\
= \quad & \frac{E\left[\varepsilon_{ij}^4\right] - 3\sigma^4}{NT^2} \frac{1-\rho_T^2}{1+\rho_T^2} \left[\sum_{t=g+2}^{T} \frac{\rho_T^2\left(1-\rho_T^{2(t-g-1)}\right)}{\left(1-\rho_T^2\right)} - \sum_{t=g+2}^{T} \frac{\rho_T^{2(t-g+1)}\left(1-\rho_T^{2(t-g-1)}\right)}{\left(1-\rho_T^2\right)} \right] \\
& +\frac{2\sigma^4}{NT^2} \sum_{t=g+2}^{T} \frac{\rho_T^2\left(1-\rho_T^{2(t-g-1)}\right)}{\left(1-\rho_T^2\right)} - \frac{4\sigma^4\rho_T^2}{NT^2} \sum_{t=g+2}^{T} \rho_T^{2(t-g-1)} (t-g-1) \\
& +\frac{2\sigma^4}{NT^2} \sum_{t=g+2}^{T} \frac{\rho_T^{2(t-g+1)}\left(1-\rho_T^{2(t-g-1)}\right)}{\left(1-\rho_T^2\right)}
\end{aligned}
$$

Applying part (b) of Lemma SE-5 and performing additional calculation, we get

$$
\begin{aligned}
& \frac{\left(1-\rho_T^2\right)^2}{N^2T^2}\sum_{i=1}^{N}\sum_{t=g+2}^{T}\sum_{s=g+1}^{t-1} E\left[\left(\underline{w}_{it-g,T}^2-\sigma^2\sum_{k=1}^{t-g}\rho_T^{2(t-g-k)}\right)\left(\underline{w}_{is-g,T}^2-\sigma^2\sum_{\ell=1}^{t-g}\rho_T^{2(t-g-\ell)}\right)\right] \\
= & \frac{\left(E\left[\varepsilon_{ij}^4\right]-3\sigma^4\right)}{NT^2}\frac{\rho_T^2}{1+\rho_T^2}\left[(T-g-1)-\frac{\rho_T^2\left(1-\rho_T^{2(T-g-1)}\right)}{\left(1-\rho_T^2\right)}\right] \\
& -\frac{\left(E\left[\varepsilon_{ij}^4\right]-3\sigma^4\right)}{NT^2}\frac{\rho_T^6}{1+\rho_T^2}\frac{\left(1-\rho_T^{2(T-g-1)}\right)}{\left(1-\rho_T^2\right)} \\
& +\frac{\left(E\left[\varepsilon_{ij}^4\right]-3\sigma^4\right)}{NT^2}\frac{\rho_T^8}{1+\rho_T^2}\frac{\left(1-\rho_T^{4(T-g-1)}\right)}{\left(1-\rho_T^4\right)} \\
& +\frac{2\sigma^4}{NT^2}\frac{\rho_T^2}{\left(1-\rho_T^2\right)}\left[(T-g-1)-\frac{\rho_T^2\left(1-\rho_T^{2(T-g-1)}\right)}{\left(1-\rho_T^2\right)}\right] \\
& -\frac{4\sigma^4\rho_T^2}{NT^2}\frac{\rho_T^2-(T-g)\,\rho_T^{2(T-g)}+(T-g-1)\,\rho_T^{2(T-g+1)}}{\left(1-\rho_T^2\right)^2} \\
& +\frac{2\sigma^4}{NT^2}\frac{\rho_T^6}{1-\rho_T^2}\frac{\left(1-\rho_T^{2(T-g-1)}\right)}{\left(1-\rho_T^2\right)}-\frac{2\sigma^4}{NT^2}\frac{\rho_T^8}{1-\rho_T^2}\frac{\left(1-\rho_T^{4(T-g-1)}\right)}{\left(1-\rho_T^4\right)} \\
\leq & \frac{\left(E\left[\varepsilon_{ij}^4\right]+3\sigma^4\right)}{NT}\left[1+\frac{g+1}{T}+\frac{1}{1-\exp\{-2/C_q\}}\right] \\
& +\frac{2\sigma^4}{NT}\left[1+\frac{g+1}{T}+\frac{1}{1-\exp\{-2/C_q\}}\right]+\frac{4\sigma^4}{NT}\frac{1}{1-\exp\{-2/C_q\}} \\
& +\frac{\left(E\left[\varepsilon_{ij}^4\right]+3\sigma^4\right)}{NT^2}\left[\frac{1}{1-\exp\{-2/C_q\}}+\frac{1}{1-\exp\{-4/C_q\}}\right]+\frac{8\sigma^4}{NT^2}\frac{(g+1)}{\left(1-\exp\{-2/C_q\}\right)^2} \\
& +\frac{2\sigma^4}{NT^2}\left[\frac{1}{\left(1-\exp\{-2/C_q\}\right)^2}+\frac{1}{\left(1-\exp\{-2/C_q\}\right)\left(1-\exp\{-4/C_q\}\right)}\right] \\
= & O\left(\frac{1}{NT}\right).
\end{aligned}
$$

It follows that

$$
E\left(\frac{1-\rho_T^2}{NT}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\underline{w}_{it-g,T}^2-\frac{1-\rho_T^2}{NT}\sum_{i=1}^{N}\sum_{t=g+1}^{T}E\left[\underline{w}_{it-g,T}^2\right]\right)^2=O_p\left(\frac{1}{NT}\right),
$$

and by Markov's inequality, we deduce that

$$\begin{aligned}\frac{1-\rho_T^2}{NT}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\underline{w}_{it-g,T}^2 &= \frac{1-\rho_T^2}{NT}\sum_{i=1}^{N}\sum_{t=g+1}^{T}E\left[\underline{w}_{it-g,T}^2\right]+O\left(\frac{1}{\sqrt{NT}}\right)\\ &= \sigma^2+O_p\left(\max\left\{\frac{1}{\sqrt{NT}},\frac{1}{T}\right\}\right).\end{aligned}$$

Moreover, note that, using Assumption 4, we have

$$E\left[\frac{1-\rho_T^2}{NT}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\rho_T^{2(t-g)}w_{i0}^2\right]\leq\frac{\left(1-\rho_T^2\right)}{T}\sup_i E\left[w_{i0}^2\right]\rho_T^2\frac{1-\rho_T^{2(T-g)}}{1-\rho_T^2}=O\left(T^{-1}\right),$$

so that

$$\frac{1-\rho_T^2}{NT}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\rho_T^{2(t-g)}w_{i0}^2=O_p\left(\frac{1}{T}\right).$$

The Cauchy-Schwarz inequality then implies that

$$\begin{aligned}\left|\frac{1-\rho_T^2}{NT}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\underline{w}_{it-g}\rho_T^{t-g}w_{i0}\right| &\leq \sqrt{\frac{1-\rho_T^2}{NT}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\underline{w}_{it-g}^2}\sqrt{\frac{1-\rho_T^2}{NT}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\rho_T^{2(t-g)}w_{i0}^2}\\ &= O_p\left(1\right)O_p\left(T^{-1/2}\right)=O_p\left(T^{-1/2}\right),\end{aligned}$$

from which it follows that

$$\begin{aligned}&\frac{1-\rho_T^2}{NT^2}\sum_{i=1}^{N}\sum_{t=g+1}^{T}w_{it-g,T}^2\\ =\;&\frac{1-\rho_T^2}{NT}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\underline{w}_{it-g,T}^2+\frac{2\left(1-\rho_T^2\right)}{NT}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\underline{w}_{it-g}\rho_T^{t-g}w_{i0}+\frac{1-\rho_T^2}{NT}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\rho_T^{2(t-g)}w_{i0}^2\\ =\;&\sigma^2+O_p\left(\max\left\{\frac{1}{\sqrt{NT}},\frac{1}{T}\right\}\right)+O_p\left(T^{-1}\right)+O_p\left(T^{-1/2}\right)\\ =\;&\sigma^2+O_p\left(T^{-1/2}\right),\end{aligned}$$

as required for part (c). □

Lemma SE-18:

Let g be a positive integer. Under Assumptions 1-4, the following results hold as $N,T\to\infty$

(a) If $\rho_T=1$ for all T sufficiently large, then

$$\sum_{i=1}^{N}\sum_{t=g+1}^{T}a_iw_{it-g,T}=\;O_p\left(\max\left\{\sqrt{N}T^{3/2},NT\right\}\right).$$

(b) If $\rho_T = \exp\{-1/q(T)\}$ such that $T/q(T) \to 0$, then

$$\sum_{i=1}^{N}\sum_{t=g+1}^{T} a_i w_{it-g,T} = O_p\left(\max\left\{\sqrt{N}T^{3/2}, NT\right\}\right).$$

(c) If $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \sim T$, then

$$\sum_{i=1}^{N}\sum_{t=g+1}^{T} a_i w_{it-g,T} = O_p\left(\max\left\{\sqrt{N}T^{3/2}, NT\right\}\right).$$

(d) If $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \to \infty$ but $q(T)/T \to 0$, then

$$\sum_{i=1}^{N}\sum_{t=g+1}^{T} a_i w_{it-g,T} = O_p\left(\max\left\{\sqrt{NT}q(T), Nq(T)\right\}\right).$$

(e) If $\rho_T \in \mathcal{G}_{\mathrm{St}} = \left\{|\rho_T| = \exp\left\{-\frac{1}{q(T)}\right\} : q(T) \geq 0 \text{ and } q(T) = O(1) \text{ as } T \to \infty\right\}$, then

$$\sum_{i=1}^{N}\sum_{t=g+1}^{T} a_i w_{it-g,T} = O_p\left(\max\left\{\sqrt{NT}, N\right\}\right).$$

Proof of Lemma SE-18:

To proceed, write

$$\sum_{i=1}^{N}\sum_{t=g+1}^{T} a_i w_{it-g,T} = \sum_{i=1}^{N}\sum_{t=g+1}^{T} a_i \underline{w}_{it-g,T} + \sum_{i=1}^{N}\sum_{t=g+1}^{T} a_i \rho_T^{t-g} w_{i0},$$

where $\underline{w}_{it-g} = \sum_{j=1}^{t-g} \rho_T^{(t-g-j)}\varepsilon_{ij}$. Note that for $T \geq g+1$

$$\begin{aligned}
E\left[\sum_{i=1}^{N}\sum_{t=g+1}^{T} a_i \underline{w}_{it-g,T}\right]^2 &= \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=g+1}^{T}\sum_{s=g+1}^{T} E\left[a_i a_j \underline{w}_{it-g,T}\underline{w}_{js-g,T}\right] \\
&= \left(\mu_a^2 + \sigma_a^2\right)\sum_{i=1}^{N}\sum_{t=g+1}^{T}\sum_{s=g+1}^{T} E\left[\underline{w}_{it-g,T}\underline{w}_{is-g,T}\right] \\
&= \left(\mu_a^2 + \sigma_a^2\right)\sum_{i=1}^{N}\sum_{t=g+1}^{T} E\left[\underline{w}_{it-g,T}^2\right] + 2\left(\mu_a^2 + \sigma_a^2\right)\sum_{i=1}^{N}\sum_{s=g+2}^{T}\sum_{t=g+1}^{s-1} E\left[\underline{w}_{it-g,T}\underline{w}_{is-g,T}\right].
\end{aligned}$$

Now, consider part (a), where we take $\rho_T = 1$ for all T sufficiently large. In this case, we can apply

the results of part (a) of Lemma SE-16 to get

$$
\begin{aligned}
& E\left[\sum_{i=1}^{N}\sum_{t=g+1}^{T} a_i \underline{w}_{it-g,T}\right]^2 \\
= & \left(\mu_a^2+\sigma_a^2\right)\sum_{i=1}^{N}\sum_{t=g+1}^{T} E\left[\underline{w}_{it-g,T}^2\right] + 2\left(\mu_a^2+\sigma_a^2\right)\sum_{i=1}^{N}\sum_{s=g+2}^{T}\sum_{t=g+1}^{s-1} E\left[\underline{w}_{it-g,T}\underline{w}_{is-g,T}\right] \\
= & \sigma^2\left(\mu_a^2+\sigma_a^2\right)N\frac{T^2}{2}\left[1+O\left(\frac{1}{T}\right)\right] + 2\sigma^2\left(\mu_a^2+\sigma_a^2\right)\frac{NT^3}{6}\left[1+O\left(\frac{1}{T}\right)\right] \\
= & NT^3\frac{\sigma^2\left(\mu_a^2+\sigma_a^2\right)}{3}\left[1+O\left(\frac{1}{T}\right)\right] = O\left(NT^3\right),
\end{aligned}
$$

so that by Markov's inequality, we obtain $\sum_{i=1}^{N}\sum_{t=g+1}^{T} a_i \underline{w}_{it-g,T} = O_p\left(\sqrt{N}T^{3/2}\right)$.

Moreover, using Assumptions 2 and 4, we have

$$
\begin{aligned}
\sum_{i=1}^{N} E\left[a_i^2\right] &= N\left(\mu_a^2+\sigma_a^2\right) = O\left(N\right), \\
\sum_{i=1}^{N} E\left(\sum_{t=g+1}^{T}\rho_T^{t-g} w_{i0}\right)^2 &= \sum_{i=1}^{N} E\left(\sum_{t=g+1}^{T} w_{i0}\right)^2 \quad \text{(for all } T \text{ sufficiently large)} \\
&\leq \sup_i E\left[w_{i0}^2\right] N\left(T-g\right)^2 = O\left(NT^2\right),
\end{aligned}
$$

from which it follows by Markov's inequality that

$$
\sum_{i=1}^{N} a_i^2 = O_p\left(N\right), \quad \sum_{i=1}^{N}\left(\sum_{t=g+1}^{T}\rho_T^{t-g} w_{i0}\right)^2 = O_p\left(NT^2\right).
$$

Applying the Cauchy-Schwarz inequality, we further obtain that

$$
\left|\sum_{i=1}^{N}\sum_{t=g+1}^{T} a_i \rho_T^{t-g} w_{i0}\right| \leq \sqrt{\sum_{i=1}^{N} a_i^2}\sqrt{\sum_{i=1}^{N}\left(\sum_{t=g+1}^{T}\rho_T^{t-g} w_{i0}\right)^2} = O_p\left(\sqrt{N}\right)O_p\left(\sqrt{N}T\right) = O_p\left(NT\right),
$$

so that

$$
\begin{aligned}
\sum_{i=1}^{N}\sum_{t=g+1}^{T} a_i w_{it-g,T} &= \sum_{i=1}^{N}\sum_{t=g+1}^{T} a_i \underline{w}_{it-g,T} + \sum_{i=1}^{N}\sum_{t=g+1}^{T} a_i \rho_T^{t-g} w_{i0} \\
&= O_p\left(\sqrt{N}T^{3/2}\right) + O_p\left(NT\right) \\
&= O_p\left(\max\left\{\sqrt{N}T^{3/2}, NT\right\}\right),
\end{aligned}
$$

as required for part (a).

Next, consider part (b), where we take $\rho_T = \exp\{-1/q(T)\}$ such that $T/q(T) \to 0$. In this case, we apply the results of parts (b) of Lemma SE-16 to get

$$\begin{aligned}
& E\left[\sum_{i=1}^{N}\sum_{t=g+1}^{T} a_i \underline{w}_{it-g,T}\right]^2 \\
= & \left(\mu_a^2+\sigma_a^2\right)\sum_{i=1}^{N}\sum_{t=g+1}^{T} E\left[\underline{w}_{it-g,T}^2\right] + 2\left(\mu_a^2+\sigma_a^2\right)\sum_{i=1}^{N}\sum_{s=g+2}^{T}\sum_{t=g+1}^{s-1} E\left[\underline{w}_{it-g,T}\underline{w}_{is-g,T}\right] \\
= & \sigma^2\left(\mu_a^2+\sigma_a^2\right)N\frac{T^2}{2}\left[1+O\left(\max\left\{\frac{T}{q(T)},\frac{1}{T}\right\}\right)\right] \\
& +2\sigma^2\left(\mu_a^2+\sigma_a^2\right)\frac{NT^3}{6}\left[1+O\left(\max\left\{\frac{T}{q(T)},\frac{1}{T}\right\}\right)\right] \\
= & NT^3\frac{\sigma^2\left(\mu_a^2+\sigma_a^2\right)}{3}\left[1+O\left(\max\left\{\frac{T}{q(T)},\frac{1}{T}\right\}\right)\right] = O\left(NT^3\right).
\end{aligned}$$

It follows from Markov's inequality that we obtain $\sum_{i=1}^{N}\sum_{t=g+1}^{T} a_i \underline{w}_{it-g,T} = O_p\left(\sqrt{N}T^{3/2}\right)$.

Moreover, using Assumptions 2 and 4, we have

$$\begin{aligned}
\sum_{i=1}^{N} E\left[a_i^2\right] &= N\left(\mu_a^2+\sigma_a^2\right) = O(N), \\
\sum_{i=1}^{N} E\left(\sum_{t=g+1}^{T}\rho_T^{t-g} w_{i0}\right)^2 &\leq \sup_i E\left[w_{i0}^2\right] N\exp\left\{-\frac{2}{q(T)}\right\}\left[1-\exp\left\{-\frac{1}{q(T)}\right\}\right]^{-2}\left[1-\exp\left\{-\frac{(T-g)}{q(T)}\right\}\right]^2 \\
&= O(N)\times O(1)\times O\left(q(T)^2\right)\times O\left(T^2/q(T)^2\right) = O\left(NT^2\right),
\end{aligned}$$

from which it follows again by Markov's inequality that

$$\sum_{i=1}^{N} a_i^2 = O_p(N), \quad \sum_{i=1}^{N}\left(\sum_{t=g+1}^{T}\rho_T^{t-g} w_{i0}\right)^2 = O_p\left(NT^2\right).$$

Applying the Cauchy-Schwarz inequality, we further obtain that

$$\left|\sum_{i=1}^{N}\sum_{t=g+1}^{T} a_i \rho_T^{t-g} w_{i0}\right| \leq \sqrt{\sum_{i=1}^{N} a_i^2}\sqrt{\sum_{i=1}^{N}\left(\sum_{t=g+1}^{T}\rho_T^{t-g} w_{i0}\right)^2} = O_p\left(\sqrt{N}\right)O_p\left(\sqrt{N}T\right) = O_p(NT),$$

so that

$$\begin{aligned}
\sum_{i=1}^{N}\sum_{t=g+1}^{T} a_i w_{it-g,T} &= \sum_{i=1}^{N}\sum_{t=g+1}^{T} a_i \underline{w}_{it-g,T} + \sum_{i=1}^{N}\sum_{t=g+1}^{T} a_i \rho_T^{t-g} w_{i0} \\
&= O_p\left(\sqrt{N}T^{3/2}\right) + O_p(NT) \\
&= O_p\left(\max\left\{\sqrt{N}T^{3/2}, NT\right\}\right),
\end{aligned}$$

as required for part (b).

Next, consider part (c), where we take $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \sim T$. Here, we apply part (c) of Lemma SE-16 to obtain

$$
\begin{aligned}
& E\left[\sum_{i=1}^{N}\sum_{t=g+1}^{T} a_i \underline{w}_{it-g,T}\right]^2 \\
= & \left(\mu_a^2+\sigma_a^2\right)\sum_{i=1}^{N}\sum_{t=g+1}^{T} E\left[\underline{w}_{it-g,T}^2\right] + 2\left(\mu_a^2+\sigma_a^2\right)\sum_{i=1}^{N}\sum_{s=g+2}^{T}\sum_{t=g+1}^{s-1} E\left[\underline{w}_{it-g,T}\underline{w}_{is-g,T}\right] \\
= & \sigma^2\left(\mu_a^2+\sigma_a^2\right)\frac{Nq(T)^2}{4}\left[\exp\left\{-\frac{2T}{q(T)}\right\}+\frac{2T}{q(T)}-1\right]\left[1+O\left(\frac{1}{T}\right)\right] \\
& +2\left(\mu_a^2+\sigma_a^2\right)NTq(T)^2\frac{\sigma^2}{2} \\
& \times\left[1-\frac{3}{2}\frac{q(T)}{T}+\frac{2q(T)}{T}\exp\left\{-\frac{T}{q(T)}\right\}-\frac{1}{2}\frac{q(T)}{T}\exp\left\{-\frac{2T}{q(T)}\right\}\right]\left[1+O\left(\frac{1}{T}\right)\right] \\
= & \sigma^2\left(\mu_a^2+\sigma_a^2\right)NTq(T)^2\left[1-\frac{3}{2}\frac{q(T)}{T}+\frac{2q(T)}{T}\exp\left\{-\frac{T}{q(T)}\right\}-\frac{1}{2}\frac{q(T)}{T}\exp\left\{-\frac{2T}{q(T)}\right\}\right] \\
& \times\left[1+O\left(\frac{1}{T}\right)\right] \\
= & O\left(NT^3\right).
\end{aligned}
$$

It follows from Markov's inequality that we obtain $\sum_{i=1}^{N}\sum_{t=g+1}^{T} a_i \underline{w}_{it-g,T} = O_p\left(\sqrt{N}T^{3/2}\right)$.

Moreover, using Assumptions 2 and 4, we obtain

$$
\sum_{i=1}^{N} E\left[a_i^2\right] = N\left(\mu_a^2+\sigma_a^2\right) = O(N),
$$

and

$$
\begin{aligned}
& \sum_{i=1}^{N} E\left(\sum_{t=g+1}^{T}\rho_T^{t-g} w_{i0}\right)^2 \\
\leq & \sup_i E\left[w_{i0}^2\right] N\exp\left\{-\frac{2}{q(T)}\right\}\left[1-\exp\left\{-\frac{1}{q(T)}\right\}\right]^{-2}\left[1-\exp\left\{-\frac{(T-g)}{q(T)}\right\}\right]^2 \\
= & O(N)\times O(1)\times O\left(T^2\right)\times O(1) = O\left(NT^2\right),
\end{aligned}
$$

so that similar to the proof of part (b) above, we have

$$
\left|\sum_{i=1}^{N}\sum_{t=g+1}^{T} a_i\rho_T^{t-g} w_{i0}\right| \leq \sqrt{\sum_{i=1}^{N} a_i^2}\sqrt{\sum_{i=1}^{N}\left(\sum_{t=g+1}^{T}\rho_T^{t-g} w_{i0}\right)^2} = O_p\left(\sqrt{N}\right)\times O_p\left(\sqrt{N}T\right) = O_p(NT)
$$

upon application of the Markov and the Cauchy-Schwarz inequalities. It follows that

$$\begin{aligned}\sum_{i=1}^{N}\sum_{t=g+1}^{T} a_i w_{it-g,T} &= \sum_{i=1}^{N}\sum_{t=g+1}^{T} a_i \underline{w}_{it-g,T} + \sum_{i=1}^{N}\sum_{t=g+1}^{T} a_i \rho_T^{t-g} w_{i0} \\ &= O_p\left(\sqrt{N}T^{3/2}\right) + O_p\left(NT\right) \\ &= O_p\left(\max\left\{\sqrt{N}T^{3/2}, NT\right\}\right),\end{aligned}$$

as required for part (c).

For part (d), we consider the case where $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \to \infty$ but $q(T)/T \to 0$. Here, we apply part (d) of Lemma SE-16 to obtain

$$\begin{aligned}& E\left[\sum_{i=1}^{N}\sum_{t=g+1}^{T} a_i \underline{w}_{it-g,T}\right]^2 \\ =\ & \left(\mu_a^2+\sigma_a^2\right)\sum_{i=1}^{N}\sum_{t=g+1}^{T} E\left[\underline{w}_{it-g,T}^2\right] + 2\left(\mu_a^2+\sigma_a^2\right)\sum_{i=1}^{N}\sum_{s=g+2}^{T}\sum_{t=g+1}^{s-1} E\left[\underline{w}_{it-g,T}\underline{w}_{is-g,T}\right] \\ =\ & \sigma^2\left(\mu_a^2+\sigma_a^2\right)\frac{NTq(T)}{2}\left[1+O\left(\frac{1}{q(T)}\right)+O\left(\frac{q(T)}{T}\right)\right] \\ & +2\sigma^2\left(\mu_a^2+\sigma_a^2\right)\frac{NTq(T)^2}{2}\left[1+O\left(\frac{q(T)}{T}\right)\right] \\ =\ & NTq(T)^2\sigma^2\left(\mu_a^2+\sigma_a^2\right)\left[1+O\left(\max\left\{\frac{q(T)}{T},\frac{1}{q(T)}\right\}\right)\right] \\ =\ & O\left(NTq(T)^2\right),\end{aligned}$$

from which it follows, by Markov's inequality, that $\sum_{i=1}^{N}\sum_{t=g+1}^{T} a_i \underline{w}_{it-g,T} = O_p\left(\sqrt{NT}q(T)\right)$.

Moreover, using Assumptions 2 and 4, we obtain

$$\sum_{i=1}^{N} E\left[a_i^2\right] = N\left(\mu_a^2+\sigma_a^2\right) = O(N),$$

and

$$\begin{aligned}& \sum_{i=1}^{N} E\left(\sum_{t=g+1}^{T}\rho_T^{t-g} w_{i0}\right)^2 \\ \leq\ & \sup_i E\left[w_{i0}^2\right] N\exp\left\{-\frac{2}{q(T)}\right\}\left[1-\exp\left\{-\frac{1}{q(T)}\right\}\right]^{-2}\left[1-\exp\left\{-\frac{(T-g)}{q(T)}\right\}\right]^2 \\ =\ & O(N)\times O(1)\times O\left(q(T)^2\right)\times O(1) = O\left(Nq(T)^2\right),\end{aligned}$$

from which it follows by Markov's inequality that

$$\sum_{i=1}^{N} a_i^2 = O_p(N),\ \ \sum_{i=1}^{N}\left(\sum_{t=g+1}^{T}\rho_T^{t-g} w_{i0}\right)^2 = O_p\left(Nq(T)^2\right).$$

Applying the Cauchy-Schwarz inequality, we further obtain that

$$\left|\sum_{i=1}^{N}\sum_{t=g+1}^{T} a_i \rho_T^{t-g} w_{i0}\right| \leq \sqrt{\sum_{i=1}^{N} a_i^2}\sqrt{\sum_{i=1}^{N}\left(\sum_{t=g+1}^{T} \rho_T^{t-g} w_{i0}\right)^2} = O_p\left(\sqrt{N}\right) \times O_p\left(\sqrt{N} q(T)\right) = O_p\left(N q(T)\right),$$

so that

$$\begin{aligned}
\sum_{i=1}^{N}\sum_{t=g+1}^{T} a_i w_{it-g,T} &= \sum_{i=1}^{N}\sum_{t=g+1}^{T} a_i \underline{w}_{it-g,T} + \sum_{i=1}^{N}\sum_{t=g+1}^{T} a_i \rho_T^{t-g} w_{i0} \\
&= O_p\left(\sqrt{NT} q(T)\right) + O_p\left(N q(T)\right) \\
&= O_p\left(\max\left\{\sqrt{NT} q(T), N q(T)\right\}\right),
\end{aligned}$$

as required for part (d).

Finally, to show part (e), note that in this case

$$\begin{aligned}
& E\left[\sum_{i=1}^{N}\sum_{t=g+1}^{T} a_i \underline{w}_{it-g,T}\right]^2 \\
= & \left(\mu_a^2+\sigma_a^2\right) N \sum_{t=g+1}^{T} E\left[\underline{w}_{it-g,T}^2\right] + 2\left(\mu_a^2+\sigma_a^2\right) N \sum_{s=g+2}^{T}\sum_{t=g+1}^{s-1} E\left[\underline{w}_{it-g,T}\underline{w}_{is-g,T}\right] \\
= & \left(\mu_a^2+\sigma_a^2\right) N \sum_{t=g+1}^{T}\sum_{j=1}^{t-g}\sum_{k=1}^{t-g} \rho_T^{2(t-g)-j-k} E\left[\varepsilon_{ij}\varepsilon_{ik}\right] \\
& +2\left(\mu_a^2+\sigma_a^2\right) N \sum_{t=g+2}^{T}\sum_{s=g+1}^{t-1}\sum_{j=1}^{t-g}\sum_{k=1}^{s-g} \rho_T^{(t-g-j)} \rho_T^{(s-g-k)} E\left[\varepsilon_{ij}\varepsilon_{ik}\right] \\
= & \left(\mu_a^2+\sigma_a^2\right)\sigma^2 N \sum_{t=g+1}^{T}\sum_{j=1}^{t-g} \rho_T^{2(t-g-j)} + 2\left(\mu_a^2+\sigma_a^2\right)\sigma^2 N \sum_{t=g+2}^{T}\sum_{s=g+1}^{t-1}\sum_{j=1}^{s-g} \rho_T^{(t+s-2g-2j)} \\
= & \left(\mu_a^2+\sigma_a^2\right)\sigma^2 N \sum_{t=g+1}^{T} \frac{1-\rho_T^{2(t-g)}}{1-\rho_T^2} + 2\left(\mu_a^2+\sigma_a^2\right)\sigma^2 N \sum_{t=g+2}^{T}\sum_{s=g+1}^{t-1} \rho_T^{(t-s)} \sum_{j=1}^{s-g} \rho_T^{2(s-g-j)} \\
= & \frac{\left(\mu_a^2+\sigma_a^2\right)\sigma^2}{1-\rho_T^2} N(T-g) - \frac{\left(\mu_a^2+\sigma_a^2\right)\sigma^2\rho_T^2}{\left(1-\rho_T^2\right)} N \sum_{t=g+1}^{T} \rho_T^{2(t-g-1)} \\
& + \frac{2\left(\mu_a^2+\sigma_a^2\right)\sigma^2}{1-\rho_T^2} N \sum_{t=g+2}^{T}\sum_{s=g+1}^{t-1} \rho_T^{(t-s)}\left(1-\rho_T^{2(s-g)}\right) \\
= & \frac{\left(\mu_a^2+\sigma_a^2\right)\sigma^2}{1-\rho_T^2} N(T-g) - \left(\mu_a^2+\sigma_a^2\right)\sigma^2 \frac{\rho_T^2\left(1-\rho_T^{2(T-g)}\right)}{\left(1-\rho_T^2\right)^2} N \\
& + \frac{2\left(\mu_a^2+\sigma_a^2\right)\sigma^2\rho_T}{\left(1-\rho_T^2\right)\left(1-\rho_T\right)} N \sum_{t=g+2}^{T}\left(1-\rho_T^{(t-g-1)}\right) - \frac{2\left(\mu_a^2+\sigma_a^2\right)\sigma^2}{1-\rho_T^2} N \sum_{t=g+2}^{T} \rho_T^{(t-g+1)} \sum_{s=g+1}^{t-1} \rho_T^{(s-g-1)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\left(\mu_a^2+\sigma_a^2\right)\sigma^2}{1-\rho_T^2}N\left(T-g\right)-\left(\mu_a^2+\sigma_a^2\right)\sigma^2\frac{\rho_T^2\left(1-\rho_T^{2(T-g)}\right)}{\left(1-\rho_T^2\right)^2}N \\
&\quad +\frac{2\left(\mu_a^2+\sigma_a^2\right)\sigma^2\rho_T}{\left(1-\rho_T^2\right)\left(1-\rho_T\right)}N\left(T-g-1\right)-\frac{2\left(\mu_a^2+\sigma_a^2\right)\sigma^2\rho_T^2\left(1-\rho_T^{(T-g-1)}\right)}{\left(1-\rho_T^2\right)\left(1-\rho_T\right)^2}N \\
&\quad -\frac{2\left(\mu_a^2+\sigma_a^2\right)\sigma^2}{\left(1-\rho_T^2\right)\left(1-\rho_T\right)}N\sum_{t=g+2}^{T}\rho_T^{(t-g+1)}\left(1-\rho_T^{(t-g-1)}\right) \\
&= \frac{\left(\mu_a^2+\sigma_a^2\right)\sigma^2}{1-\rho_T^2}N\left(T-g\right)-\left(\mu_a^2+\sigma_a^2\right)\sigma^2\frac{\rho_T^2\left(1-\rho_T^{2(T-g)}\right)}{\left(1-\rho_T^2\right)^2}N \\
&\quad +\frac{2\left(\mu_a^2+\sigma_a^2\right)\sigma^2\rho_T}{\left(1-\rho_T^2\right)\left(1-\rho_T\right)}N\left(T-g-1\right)-\frac{2\left(\mu_a^2+\sigma_a^2\right)\sigma^2\rho_T^2\left(1-\rho_T^{(T-g-1)}\right)}{\left(1-\rho_T^2\right)\left(1-\rho_T\right)^2}N \\
&\quad -2\left(\mu_a^2+\sigma_a^2\right)\sigma^2N\frac{\rho_T^3\left(1-\rho_T^{(T-g-1)}\right)}{\left(1-\rho_T^2\right)\left(1-\rho_T\right)^2}+2\left(\mu_a^2+\sigma_a^2\right)\sigma^2N\frac{\rho_T^4\left(1-\rho_T^{2(T-g-1)}\right)}{\left(1-\rho_T^2\right)^2\left(1-\rho_T\right)} \\
&= O\left(NT\right)
\end{aligned}
$$

Since we assume that $|\rho_T| = \exp\left\{-\frac{1}{q(T)}\right\}$ with $q\left(T\right) = O\left(1\right)$ in this case, it follows that there exist a positive constant C_q and a positive integer T^* such that for all $T \geq T^*$

$$
|\rho_T| \leq \exp\left\{-\frac{1}{C_q}\right\} < 1
$$

Applying this upper bound we obtain for all $T \geq T^*$

$$
\begin{aligned}
&E\left[\sum_{i=1}^{N}\sum_{t=g+1}^{T}a_i\underline{w}_{it-g,T}\right]^2 \\
\leq\ & \frac{\left(\mu_a^2+\sigma_a^2\right)\sigma^2}{1-\rho_T^2}N\left(T-g\right)+\left(\mu_a^2+\sigma_a^2\right)\sigma^2\frac{\rho_T^2\left(1-\rho_T^{2(T-g)}\right)}{\left(1-\rho_T^2\right)^2}N \\
&+\frac{2\left(\mu_a^2+\sigma_a^2\right)\sigma^2\left|\rho_T\right|}{\left(1-\rho_T^2\right)\left(1-\left|\rho_T\right|\right)}N\left(T-g-1\right)+\frac{2\left(\mu_a^2+\sigma_a^2\right)\sigma^2\rho_T^2\left(1-\rho_T^{(T-g-1)}\right)}{\left(1-\rho_T^2\right)\left(1-\left|\rho_T\right|\right)^2}N \\
&+4\left(\mu_a^2+\sigma_a^2\right)\sigma^2N\frac{\left|\rho_T\right|^3}{\left(1-\rho_T^2\right)\left(1-\rho_T\right)^2}+2\left(\mu_a^2+\sigma_a^2\right)\sigma^2N\frac{\rho_T^4\left(1-\rho_T^{2(T-g-1)}\right)}{\left(1-\rho_T^2\right)^2\left(1-\left|\rho_T\right|\right)}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \left(\mu_a^2+\sigma_a^2\right)\sigma^2\left[\frac{N(T-g)}{1-\exp\{-2/C_q\}}+\frac{N}{(1-\exp\{-2/C_q\})^2}+\frac{2N(T-g-1)}{(1-\exp\{-2/C_q\})(1-\exp\{-1/C_q\})}\right.\\
&\quad+\frac{4N}{(1-\exp\{-2/C_q\})(1-\exp\{-1/C_q\})^2}+\frac{4N}{(1-\exp\{-2/C_q\})(1-\exp\{-1/C_q\})^2}\\
&\quad\left.+\frac{2N}{(1-\exp\{-2/C_q\})^2(1-\exp\{-1/C_q\})}\right]\\
&= O(NT),
\end{aligned}
$$

so that, using Markov's inequality, we deduce that $\sum_{i=1}^{N}\sum_{t=g+1}^{T} a_i \underline{w}_{it-g,T} = O_p\left(\sqrt{NT}\right)$.

Moreover, using Assumptions 2 and 4, we obtain

$$
\begin{aligned}
\sum_{i=1}^{N} E\left[a_i^2\right] &= N\left(\mu_a^2+\sigma_a^2\right)=O(N),\\
\sum_{i=1}^{N} E\left(\sum_{t=g+1}^{T}\rho_T^{t-g}w_{i0}\right)^2 &\leq N\sup_i E\left[w_{i0}^2\right]\rho_T^2\frac{\left(1-\rho_T^{T-g}\right)^2}{(1-\rho_T)^2}=O(N)\,O(1)=O(N),
\end{aligned}
$$

from which it follows by Markov's inequality that

$$
\sum_{i=1}^{N} a_i^2 = O_p(N),\quad \sum_{i=1}^{N}\left(\sum_{t=g+1}^{T}\rho_T^{t-g}w_{i0}\right)^2 = O_p(N).
$$

Applying the Cauchy-Schwarz inequality, we further obtain that

$$
\left|\sum_{i=1}^{N}\sum_{t=g+1}^{T} a_i\rho_T^{t-g}w_{i0}\right| \leq \sqrt{\sum_{i=1}^{N} a_i^2}\sqrt{\sum_{i=1}^{N}\left(\sum_{t=g+1}^{T}\rho_T^{t-g}w_{i0}\right)^2} = O_p\left(\sqrt{N}\right)O_p\left(\sqrt{N}\right)=O_p(N),
$$

so that

$$
\begin{aligned}
\sum_{i=1}^{N}\sum_{t=g+1}^{T} a_i w_{it-g,T} &= \sum_{i=1}^{N}\sum_{t=g+1}^{T} a_i\underline{w}_{it-g,T}+\sum_{i=1}^{N}\sum_{t=g+1}^{T} a_i\rho_T^{t-g}w_{i0}\\
&= O_p\left(\sqrt{NT}\right)+O_p(N)=O_p\left(\max\left\{\sqrt{NT},N\right\}\right),
\end{aligned}
$$

as required for part (e). □

Lemma SE-19 (Phillips and Moon, 1999, Theorem 3): Suppose that $Y_{i,T}=C_iQ_{i,T}$, where the $(m\times 1)$ random vectors $Q_{i,T}$ are *i.i.d.* $(0,\Sigma_T)$ across i for all T and the C_i are $(m\times m)$ nonzero and nonrandom matrices. Assume the following conditions hold.

(i) Let $\sigma_T^2=\lambda_{\min}(\Sigma_T)$ and $\liminf_T \sigma_T^2>0$;

(ii) $\max_{1\leq i\leq n}\|C_i\|^2/\lambda_{\min}\left(\sum_{i=1}^{N}C_iC_i'\right)=O(1/N)$ as $N\to\infty$;

(iii) $\|Q_{i,T}\|^2$ are uniformly integrable in T;

(iv) $\lim_{N,T}(1/N)\sum_{i=1}^{N} C_i \Sigma_T C_i' = \Omega > 0$.

Then,

$$X_{N,T} = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} Y_{i,T} \Rightarrow N(0,\Omega), \text{ as } N,T \to \infty.$$

Lemma SE-20:

Let g be a non-negative integer. Under Assumptions 1 and 4, the following results hold as $N,T \to \infty$.

(a) If $\rho_T = \exp\{-1/q(T)\}$ such that $T/q(T) \to 0$, then

$$\frac{1}{\sqrt{N}T} \sum_{i=1}^{N} \sum_{t=g+2}^{T} w_{it-g-1,T}\varepsilon_{it-g} \Rightarrow N\left(0, \frac{\sigma^4}{2}\right).$$

(b) If $\rho_T = 1$ for all T sufficiently large, then

$$\frac{1}{\sqrt{N}T} \sum_{i=1}^{N} \sum_{t=g+2}^{T} w_{it-g-1,T}\varepsilon_{it-g} \Rightarrow N\left(0, \frac{\sigma^4}{2}\right).$$

Proof of Lemma SE-20:

Consider first part (a), where we take $\rho_T = \exp\{-1/q(T)\}$ such that $T/q(T) \to 0$. Write

$$\frac{1}{\sqrt{N}T} \sum_{i=1}^{N} \sum_{t=g+2}^{T} w_{it-g-1,T}\varepsilon_{it-g} = \frac{1}{\sqrt{N}T} \sum_{i=1}^{N} \sum_{t=g+2}^{T} \underline{w}_{it-g-1,T}\varepsilon_{it-g} + \frac{1}{\sqrt{N}T} \sum_{i=1}^{N} \sum_{t=g+2}^{T} \varepsilon_{it-g}\rho_T^{t-g-1} w_{i0},$$

where $\underline{w}_{it-g-1,T} = \sum_{j=1}^{t-g-1} \rho_T^{(t-g-1-j)}\varepsilon_{ij}$. Note that

$$\begin{aligned}
& \frac{1}{T}\sum_{t=2}^{T} \underline{w}_{it,T}^2 \\
= \; & \frac{1}{T}\sum_{t=2}^{T} \left(\exp\left(-\frac{1}{q(T)}\right)\underline{w}_{it-1,T} + \varepsilon_{it}\right)^2 \\
= \; & \frac{1}{T}\sum_{t=2}^{T} \varepsilon_{it}^2 + 2\exp\left\{-\frac{1}{q(T)}\right\}\frac{1}{T}\sum_{t=2}^{T} \underline{w}_{it-,T}\varepsilon_{it} + \exp\left\{-\frac{2}{q(T)}\right\}\frac{1}{T}\sum_{t=g+2}^{T} \underline{w}_{it-1,T}^2
\end{aligned}$$

so that

$$\begin{aligned}
& \frac{1}{T}\sum_{t=2}^{T} \underline{w}_{it-1,T}\varepsilon_{it} \\
= \; & \frac{1}{2}\exp\left\{\frac{1}{q(T)}\right\}\left[\frac{1}{T}\sum_{t=2}^{T} \underline{w}_{it,T}^2 - \exp\left\{-\frac{2}{q(T)}\right\}\frac{1}{T}\sum_{t=1}^{T} \underline{w}_{it-1,T}^2 - \frac{1}{T}\sum_{t=1}^{T} \varepsilon_{it}^2\right].
\end{aligned}$$

Now,

$$
\begin{aligned}
& \frac{1}{T}\sum_{t=2}^{T}\underline{w}_{it,T}^{2}-\exp\left\{-\frac{2}{q\left(T\right)}\right\}\frac{1}{T}\sum_{t=2}^{T}\underline{w}_{it-1,T}^{2} \\
= & \frac{1}{T}\sum_{t=2}^{T}\underline{w}_{it,T}^{2}-\left[1-\frac{2}{q\left(T\right)}+O\left(\frac{1}{q\left(T\right)^{2}}\right)\right]\frac{1}{T}\sum_{t=2}^{T}\underline{w}_{it-1,T}^{2} \\
= & \frac{1}{T}\sum_{t=2}^{T}\underline{w}_{it,T}^{2}-\frac{1}{T}\sum_{t=2}^{T}\underline{w}_{it-1,T}^{2}+\frac{2T}{q\left(T\right)}\left(\frac{1}{T^{2}}\sum_{t=2}^{T}\underline{w}_{it-1,T}^{2}\right)+O_{p}\left(\frac{T}{q\left(T\right)^{2}}\right) \\
= & \frac{1}{T}\underline{w}_{iT,T}^{2}-\frac{1}{T}\underline{w}_{i1,T}^{2}+\frac{2T}{q\left(T\right)}\left(\frac{1}{T^{2}}\sum_{t=1}^{T}\underline{w}_{it-1,T}^{2}\right)+O_{p}\left(\frac{T}{q\left(T\right)^{2}}\right) \\
= & \frac{1}{T}\underline{w}_{iT,T}^{2}-\frac{1}{T}\varepsilon_{i1}^{2}+\frac{2T}{q\left(T\right)}\left(\frac{1}{T^{2}}\sum_{t=2}^{T}\underline{w}_{it-1,T}^{2}\right)+O_{p}\left(\frac{T}{q\left(T\right)^{2}}\right) \\
= & \frac{1}{T}\underline{w}_{iT,T}^{2}+O_{p}\left(\max\left\{\frac{T}{q\left(T\right)},\frac{1}{T}\right\}\right).
\end{aligned}
$$

Next, note that

$$
\frac{1}{T}\sum_{t=g+2}^{T}\underline{w}_{it-g-1,T}\varepsilon_{it-g}=\frac{1}{T}\sum_{t=2}^{T}\underline{w}_{it-1,T}\varepsilon_{it}-\frac{1}{T}\sum_{s=1}^{g-1}\underline{w}_{iT-g+s,T}\varepsilon_{iT-g+1+s}
$$

Observe that

$$
\begin{aligned}
& E\left[\frac{1}{T}\sum_{s=1}^{g-1}\underline{w}_{iT-g+s,T}\varepsilon_{iT-g+1+s}\right]^2 \\
= & \frac{1}{T^2}\sum_{s=1}^{g-1}\sum_{v=1}^{g-1}\sum_{k=1}^{T-g+s}\sum_{\ell=1}^{T-g+v}\exp\left\{-\frac{T-g+s-k}{q(T)}\right\}\exp\left\{-\frac{T-g+v-\ell}{q(T)}\right\}E\left[\varepsilon_{ik}\varepsilon_{i\ell}\varepsilon_{iT-g+1+s}\varepsilon_{iT-g+1+v}\right] \\
= & \frac{\sigma^4}{T^2}\sum_{s=1}^{g-1}\sum_{k=1}^{T-g+s}\exp\left\{-2\frac{T-g+s-k}{q(T)}\right\} \\
= & \frac{\sigma^4}{T^2}\sum_{s=1}^{g-1}\left[1-\exp\left\{-\frac{2}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-2\frac{T-g+s}{q(T)}\right\}\right] \\
= & \frac{\sigma^4}{T^2}\left[1-\exp\left\{-\frac{2}{q(T)}\right\}\right]^{-1}(g-1) \\
& -\frac{\sigma^4}{T^2}\left[1-\exp\left\{-\frac{2}{q(T)}\right\}\right]^{-1}\exp\left\{-2\frac{T-g+1}{q(T)}\right\}\sum_{s=1}^{g-1}\exp\left\{-2\frac{s-1}{q(T)}\right\} \\
= & \frac{\sigma^4}{T^2}\left[1-\exp\left\{-\frac{2}{q(T)}\right\}\right]^{-1}(g-1) \\
& -\frac{\sigma^4}{T^2}\left[1-\exp\left\{-\frac{2}{q(T)}\right\}\right]^{-2}\exp\left\{-2\frac{T-g+1}{q(T)}\right\}\left[1-\exp\left\{-2\frac{g-1}{q(T)}\right\}\right] \\
= & \frac{\sigma^4}{T^2}(g-1)\frac{q(T)}{2}\left[1+O\left(\frac{1}{q(T)}\right)\right] \\
& -\frac{\sigma^4}{T^2}\frac{q(T)^2}{4}\left[1+O\left(\frac{1}{q(T)}\right)\right]\left[1-2\frac{T-g+1}{q(T)}+\frac{4(T-g+1)^2}{2!q(T)^2}+O\left(\frac{T^3}{q(T)^3}\right)\right] \\
& \times\left[2\frac{g-1}{q(T)}-\frac{4(g-1)^2}{2!q(T)^2}+O\left(\frac{1}{q(T)^3}\right)\right] \\
= & \frac{\sigma^4}{T^2}(g-1)\frac{q(T)}{2}\left[1+O\left(\frac{1}{q(T)}\right)\right] \\
& -\frac{\sigma^4}{T^2}\frac{q(T)^2}{4}\left[\frac{2(g-1)}{q(T)}-\frac{4T(g-1)}{q(T)^2}+\frac{4(g-1)^2}{q(T)^2}-\frac{2(g-1)^2}{q(T)^2}+O\left(\frac{T^3}{q(T)^3}\right)\right] \\
= & \frac{\sigma^4(g-1)}{2}\frac{q(T)}{T^2}-\frac{\sigma^4(g-1)}{2}\frac{q(T)}{T^2}+\frac{\sigma^4(g-1)}{T}+O\left(\frac{1}{T^2}\right)+O\left(\frac{T}{q(T)}\right) \\
= & \frac{\sigma^4(g-1)}{T}+O\left(\frac{1}{T^2}\right)+O\left(\frac{T}{q(T)}\right)=O\left(\frac{1}{T}\right)
\end{aligned}
$$

so that applying Markov's inequality, we obtain

$$
\frac{1}{T}\sum_{s=1}^{g-1}\underline{w}_{iT-g+s,T}\varepsilon_{iT-g+1+s}=O_p\left(\frac{1}{\sqrt{T}}\right)
$$

Hence,

$$
\begin{aligned}
& \frac{1}{T}\sum_{t=g+2}^{T}\underline{w}_{it-g-1,T}\varepsilon_{it-g} \\
= \; & \frac{1}{T}\sum_{t=2}^{T}\underline{w}_{it-1,T}\varepsilon_{it}-\frac{1}{T}\sum_{s=1}^{g-1}\underline{w}_{iT-g+s,T}\varepsilon_{iT-g+1+s} \\
= \; & \frac{1}{2}\exp\left\{\frac{1}{q(T)}\right\}\left[\frac{1}{T}\sum_{t=1}^{T}\underline{w}_{it}^{2}-\exp\left\{-\frac{2}{q(T)}\right\}\frac{1}{T}\sum_{t=1}^{T}\underline{w}_{it-1}^{2}-\frac{1}{T}\sum_{t=1}^{T}\varepsilon_{it}^{2}\right]+O_p\left(\frac{1}{\sqrt{T}}\right) \\
= \; & \left[\frac{1}{2T}\underline{w}_{iT}^{2}-\frac{1}{2T}\sum_{t=1}^{T}\varepsilon_{it}^{2}+O_p\left(\max\left\{\frac{T}{q(T)},\frac{1}{T}\right\}\right)\right]\left[1+O\left(\frac{1}{q(T)}\right)\right]+O_p\left(\frac{1}{\sqrt{T}}\right) \\
= \; & \frac{1}{2T}\underline{w}_{iT}^{2}-\frac{1}{2}\sigma^{2}+O_p\left(\frac{T}{q(T)}\right)+O_p\left(\frac{1}{\sqrt{T}}\right) \\
\Rightarrow \; & \frac{1}{2}\left[\sigma W_i(1)\right]^{2}-\frac{1}{2}\sigma^{2}\equiv\frac{\sigma^{2}}{2}\left(\left[W_i(1)\right]^{2}-1\right), \qquad (22)
\end{aligned}
$$

where the last line follows from applying Lemma SE-12 with $r=1$ and the Cramér convergence theorem. Moreover, note that

$$
\begin{aligned}
d\left[W_i(g)\right]^{2} &= \left(W_i(g)+dW_i(g)\right)^{2}-\left[W_i(g)\right]^{2} \\
&= \left[W_i(g)\right]^{2}+2W_i(g)\,dW_i(g)+\left[dW_i(g)\right]^{2}-\left[W_i(g)\right]^{2} \\
&= 2W_i(g)\,dW_i(g)+\left[dW_i(g)\right]^{2} \\
&= 2W_i(g)\,dW_i(g)+dg \quad a.s.
\end{aligned}
$$

It follows that

$$
\int_0^1 W_i(g)\,dW_i(g)=\frac{1}{2}\left(\left[W_i(1)\right]^{2}-1\right). \qquad (23)
$$

Substituting (23) into (22), we obtain

$$
\frac{1}{T}\sum_{t=g+2}^{T}\underline{w}_{it-g-1,T}\varepsilon_{it-g}\Rightarrow\sigma^{2}\int_0^1 W_i(g)\,dW_i(g),
$$

as $T\to\infty$.

Next, we verify the conditions of Theorem 3 of Phillips and Moon (1999), given above as Lemma SE-19. First define

$$
Q_{i,T}=\frac{1}{T}\sum_{t=g+2}^{T}\underline{w}_{it-g-1,T}\varepsilon_{it-g},
$$

and note that by direct calculation and by applying the results given parts (a) and (b) of Lemma SE-1

with $d = 2$ and $b = g + 2$, we have

$$\begin{aligned}
\liminf_{T\to\infty} \sigma_T^2 = \liminf_{T\to\infty} E\left[Q_{i,T}^2\right] &= \lim_{T\to\infty} E\left[\sum_{i=1}^{N}\sum_{t=g+2}^{T} \underline{w}_{it-g-1,T}\varepsilon_{it-g}\right]^2 \\
&= \lim_{T\to\infty} \frac{1}{T^2}\sum_{t=g+2}^{T}\sum_{s=g+2}^{T}\sum_{k=1}^{t-g-1}\sum_{\ell=1}^{s-g-1} \exp\left\{-\frac{t-g-1-k}{q(T)}\right\}\exp\left\{-\frac{s-g-1-\ell}{q(T)}\right\} E\left[\varepsilon_{ik}\varepsilon_{j\ell}\varepsilon_{it-g}\varepsilon_{js-g}\right] \\
&= \sigma^4 \lim_{T\to\infty} \frac{1}{T^2}\sum_{t=g+2}^{T}\sum_{k=1}^{t-g-1} \exp\left\{-2\frac{t-g-1-k}{q(T)}\right\} = \frac{\sigma^4}{2} > 0,
\end{aligned}$$

so that condition (i) of Lemma SE-19 is satisfied. Moreover, in this case, we have $C_i = 1$ for all i, so that $\max_{1\le i\le n} \|C_i\|^2 / \lambda_{\min}\left(\sum_{i=1}^{N} C_i C_i'\right) = 1/N = O(1/N)$ as required by condition (ii). Next, note that by (22) above,

$$Q_{i,T} = \frac{1}{T}\sum_{t=g+2}^{T} \underline{w}_{it-g-1,T}\varepsilon_{it-g} \Rightarrow \frac{\sigma^2}{2}\left[\chi_1^2 - 1\right] \equiv Q,$$

as $T \to \infty$, so that by the continuous mapping theorem

$$Q_{i,T}^2 \Rightarrow Q^2 \equiv \frac{\sigma^4}{4}\left[\chi_1^2 - 1\right]^2, \text{ as } T \to \infty.$$

In addition,

$$\begin{aligned}
E\left[Q^2\right] &= \frac{\sigma^4}{4}\left\{E\left[\chi_1^2\right]^2 - 2E\left[\chi_1^2\right] + 1\right\} = \frac{\sigma^4}{4}\{3 - 2 + 1\} \\
&= \frac{\sigma^4}{2},
\end{aligned}$$

and note that, as $T \to \infty$,

$$\lim_{T\to\infty} E\left[Q_{i,T}^2\right] = \sigma^4 \lim_{T\to\infty} \frac{1}{T^2}\sum_{t=g+2}^{T}\sum_{k=1}^{t-g-1} \exp\left\{-2\frac{t-g-1-k}{q(T)}\right\} = \frac{\sigma^4}{2} = E\left[Q^2\right].$$

It follows again from Theorem 5.4 of Billingsley (1968) that $\left\{Q_{i,T}^2\right\}$ is uniformly integrable, so that condition (iii) of Lemma SE-19 is satisfied. Finally, note that in this case

$$\lim_{N,T} (1/N)\sum_{i=1}^{N} C_i \Sigma_T C_i' = \lim_{N,T} (1/N)\sum_{i=1}^{N} E\left[Q_{i,T}^2\right] = \frac{\sigma^4}{2} > 0,$$

so that condition (iv) is satisfied as well. It follows then from Lemma SE-19 that

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N} Q_{i,T} = \frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=g+2}^{T} \underline{w}_{it-g-1,T}\varepsilon_{it-g} \Rightarrow N\left(0, \frac{\sigma^4}{2}\right),$$

as $N, T \to \infty$.

Moreover, by Assumptions 1 and 4,

$$
\begin{aligned}
& E\left[\left(\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=g+2}^{T}\varepsilon_{it-g}\rho_T^{t-g-1}w_{i0}\right)^2\right] \\
= & \frac{1}{NT^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=g+2}^{T}\sum_{s=g+2}^{T}E\left[w_{i0}w_{j0}\right]\rho_T^{t-g-1}\rho_T^{s-g-1}E\left[\varepsilon_{it-g}\varepsilon_{js-g}\right] \\
= & \frac{\sigma^2}{NT^2}\sum_{i=1}^{N}E\left[w_{i0}^2\right]\sum_{t=g+2}^{T}\rho_T^{2(t-g-1)} \\
= & \frac{\sigma^2}{T^2}\left(\sup_i E\left[w_{i0}^2\right]\right)\exp\left\{-\frac{2}{q(T)}\right\}\left[1-\exp\left\{-\frac{2}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{2(T-g-1)}{q(T)}\right\}\right] \\
= & O\left(T^{-2}\right)\times O(1)\times O(1)\times O(q(T))\times O(T/q(T)) = O\left(T^{-1}\right).
\end{aligned}
$$

It follows from Markov's inequality that

$$
\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=g+2}^{T}\varepsilon_{it-g}\rho_T^{t-g-1}w_{i0} = O_p\left(\frac{1}{\sqrt{T}}\right).
$$

Hence,

$$
\begin{aligned}
& \frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=g+2}^{T}w_{it-g-1,T}\varepsilon_{it-g} = \frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=g+2}^{T}\underline{w}_{it-g-1,T}\varepsilon_{it-g} + \frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=g+2}^{T}\varepsilon_{it-g}\rho_T^{t-g-1}w_{i0} \\
= & \frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=g+2}^{T}\underline{w}_{it-g-1,T}\varepsilon_{it-g} + O_p\left(\frac{1}{\sqrt{T}}\right) \Rightarrow N\left(0,\frac{\sigma^4}{2}\right),
\end{aligned}
$$

which is the desired result.

To show part (b), note that, in this case,

$$
w_{it,T} = \underline{w}_{it,T} + \rho_T^{t-g-1}w_{i0} = \sum_{j=1}^{t}\varepsilon_{ij} + w_{i0}
$$

where the second equality holds for all T sufficiently large. Now, by direct calculation,

$$
\begin{aligned}
& E\left[\frac{1}{T}\sum_{s=1}^{g-1}\underline{w}_{iT-g+s,T}\varepsilon_{iT-g+1+s,T}\right]^2 = \frac{1}{T^2}\sum_{s=1}^{g-1}\sum_{v=1}^{g-1}\sum_{k=1}^{T-g+s}\sum_{\ell=1}^{T-g+v}E\left[\varepsilon_{ik}\varepsilon_{i\ell}\varepsilon_{iT-g+1+s}\varepsilon_{iT-g+1+v}\right] \\
= & \frac{\sigma^4}{T^2}\sum_{s=1}^{g-1}(T-g+s) = \frac{\sigma^4}{T^2}\left[(T-g)(g-1)+\frac{g(g-1)}{2}\right] = O\left(\frac{1}{T}\right),
\end{aligned}
$$

so that we can apply Markov's inequality to obtain

$$
\frac{1}{T}\sum_{s=1}^{g-1}\underline{w}_{iT-g+s,T}\varepsilon_{iT-g+1+s,T} = O_p\left(\frac{1}{\sqrt{T}}\right).
$$

Using this result and following the arguments of part (b) of Theorem 3.1 in Phillips (1987), we further obtain

$$\frac{1}{T}\sum_{t=g+2}^{T}\underline{w}_{it-g-1,T}\varepsilon_{it-g}=\frac{1}{T}\sum_{t=2}^{T}\underline{w}_{it-1,T}\varepsilon_{it}-\frac{1}{T}\sum_{s=1}^{g-1}\underline{w}_{iT-g+s,T}\varepsilon_{iT-g+1+s}$$

$$=\frac{1}{T}\sum_{t=2}^{T}\underline{w}_{it-1,T}\varepsilon_{it}+O_p\left(\frac{1}{\sqrt{T}}\right)\Rightarrow\frac{\sigma^2}{2}\left([W_i(1)]^2-1\right).$$

Next, we verify the conditions of Theorem 3 of Phillips and Moon (1999), given above as Lemma SE-19. To proceed, define $Q_{i,T}=T^{-1}\sum_{t=g+2}^{T}\underline{w}_{it-g-1,T}\varepsilon_{it-g}$ as before, and note that by direct calculation, we have

$$\lim_{T\to\infty}\inf\sigma_T^2=\lim_{T\to\infty}\inf E\left[Q_{i,T}^2\right]=\lim_{T\to\infty}\frac{1}{T^2}\sum_{t=g+2}^{T}\sum_{s=g+2}^{T}\sum_{k=1}^{t-g-1}\sum_{\ell=1}^{s-g-1}E\left[\varepsilon_{ik}\varepsilon_{j\ell}\varepsilon_{it-g}\varepsilon_{js-g}\right]$$

$$=\sigma^4\lim_{T\to\infty}\frac{1}{T^2}\sum_{t=g+2}^{T}(t-g-1)=\sigma^4\lim_{T\to\infty}\frac{1}{T^2}\frac{(T-g)(T-g-1)}{2}=\frac{\sigma^4}{2}>0.$$

The rest of the steps for verifying the conditions of Lemma SE-19 follows in a manner similar to that given in part (a) above. Hence, by applying Lemma SE-19, we deduce that

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N}Q_{i,T}=\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=g+2}^{T}\underline{w}_{it-g-1,T}\varepsilon_{it-g}\Rightarrow N\left(0,\frac{\sigma^4}{2}\right),$$

as $N,T\to\infty$.

Moreover, by Assumptions 1 and 4,

$$E\left[\left(\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=g+2}^{T}\varepsilon_{it-g}w_{i0}\right)^2\right]=\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=g+2}^{T}\sum_{s=g+2}^{T}E\left[w_{i0}w_{j0}\right]E\left[\varepsilon_{it-g}\varepsilon_{js-g}\right]$$

$$=\frac{\sigma^2}{NT^2}(T-g-1)\sum_{i=1}^{N}E\left[w_{i0}^2\right]=\frac{\sigma^2}{T}\left(\sup_i E\left[w_{i0}^2\right]\right)\left[1+O\left(\frac{1}{T}\right)\right]=O\left(T^{-1}\right).$$

It follows from Markov's inequality that

$$\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=g+2}^{T}\varepsilon_{it-g}w_{i0}=O_p\left(\frac{1}{\sqrt{T}}\right).$$

Hence, in this case

$$\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=g+2}^{T}w_{it-g-1,T}\varepsilon_{it-g}=\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=g+2}^{T}\underline{w}_{it-g-1,T}\varepsilon_{it-g}+\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=g+2}^{T}\varepsilon_{it-g}w_{i0}$$

(for all T sufficiently large)

$$=\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=g+2}^{T}\underline{w}_{it-g-1,T}\varepsilon_{it-g}+O_p\left(\frac{1}{\sqrt{T}}\right)\Rightarrow N\left(0,\frac{\sigma^4}{2}\right),$$

which is the desired result. □

Lemma SE-21:

Let $\underline{w}_{iT-2} = \sum_{j=1}^{T-2} \rho_T^{(T-2-j)} \varepsilon_{ij}$. Under Assumptions 1 and 4, the following statements are true, as $N, T \to \infty$.

(a) If $\rho_T = 1$ for all T sufficiently large, then

$$\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} w_{iT-2,T} \varepsilon_{iT} = \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \underline{w}_{iT-2,T} \varepsilon_{iT} + O_p\left(\frac{1}{\sqrt{T}}\right) = O_p(1).$$

(b) If $\rho_T = \exp\{-1/q(T)\}$ such that $T/q(T) \to 0$, then

$$\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} w_{iT-2,T} \varepsilon_{iT} = \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \underline{w}_{iT-2,T} \varepsilon_{iT} + O_p\left(\frac{1}{\sqrt{T}}\right) = O_p(1).$$

(c) If $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \sim T$, then

$$\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} w_{iT-2,T} \varepsilon_{iT} = \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \underline{w}_{iT-2,T} \varepsilon_{iT} + O_p\left(\frac{1}{\sqrt{T}}\right) = O_p(1).$$

(d) If $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \to \infty$ but $q(T)/T \to 0$, then

$$\begin{aligned} \frac{1}{\sqrt{Nq(T)}} \sum_{i=1}^{N} w_{iT-2,T} \varepsilon_{iT} &= \frac{1}{\sqrt{Nq(T)}} \sum_{i=1}^{N} \underline{w}_{iT-2,T} \varepsilon_{iT} + O_p\left(\frac{1}{\sqrt{q(T)}} \exp\left\{-\frac{T}{q(T)}\right\}\right) \\ &= O_p(1). \end{aligned}$$

(e) If $\rho_T \in \mathcal{G}_{\mathrm{St}} = \left\{|\rho_T| = \exp\left\{-\frac{1}{q(T)}\right\} : q(T) \geq 0 \text{ and } q(T) = O(1) \text{ as } T \to \infty\right\}$, then

$$\frac{1}{\sqrt{N}} \sum_{i=1}^{N} w_{iT-2,T} \varepsilon_{iT} = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \underline{w}_{iT-2,T} \varepsilon_{iT} + O_p\left(\rho_T^{(T-2)}\right) = O_p(1).$$

Proof of Lemma SE-21:

To proceed, write

$$\sum_{i=1}^{N} w_{iT-2,T} \varepsilon_{iT} = \sum_{i=1}^{N} \underline{w}_{iT-2,T} \varepsilon_{iT} + \sum_{i=1}^{N} \varepsilon_{iT} \rho_T^{T-2} w_{i0},$$

where $\underline{w}_{iT-2} = \sum_{j=1}^{T-2} \rho_T^{(T-2-j)} \varepsilon_{ij}$.

Now, consider part (a) where, by assumption, there exists a positive integer I_ρ such that for all $T \geq I_\rho$, the sequence $\left\{\underline{w}_{iT-2,T}\right\}$ has the partial sum representation $\underline{w}_{iT-2,T} = \sum_{j=1}^{T-2} \varepsilon_{ij}$. Hence, for all $T \geq I_\rho$, we have by direct calculation,

$$\begin{aligned} E\left[\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\underline{w}_{iT-2,T}\varepsilon_{iT}\right]^2 &= \frac{1}{NT}\sum_{i=1}^{N}\sum_{h=1}^{N}\sum_{j=1}^{T-2}\sum_{k=1}^{T-2} E\left[\varepsilon_{ij}\varepsilon_{iT}\varepsilon_{hk}\varepsilon_{hT}\right] \\ = \quad \frac{1}{NT}\sum_{i=1}^{N}\sum_{j=1}^{T-2} E\left[\varepsilon_{ij}^2\right] E\left[\varepsilon_{iT}^2\right] &= \sigma^4\frac{N(T-2)}{NT} = O(1). \end{aligned}$$

It follows from Markov's inequality that $(NT)^{-1/2}\sum_{i=1}^{N}\underline{w}_{iT-2,T}\varepsilon_{iT} = O_p(1)$.

Moreover, by Assumptions 1 and 4,

$$\begin{aligned} E\left[\left(\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\varepsilon_{iT}w_{i0}\right)^2\right] &= \frac{1}{NT}\sum_{i=1}^{N}\sum_{j=1}^{N} E\left[w_{i0}w_{j0}\right] E\left[\varepsilon_{iT}\varepsilon_{jT}\right] \\ &\leq \sigma^2\frac{N}{NT}\left(\sup_i E\left[w_{i0}^2\right]\right) = O\left(T^{-1}\right), \end{aligned}$$

from which, it follows by Markov's inequality

$$\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\varepsilon_{iT}w_{i0} = O_p\left(\frac{1}{\sqrt{T}}\right). \tag{24}$$

Hence, in this case,

$$\begin{aligned} \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}w_{iT-2,T}\varepsilon_{iT} &= \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\underline{w}_{iT-2,T}\varepsilon_{iT} + \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\varepsilon_{iT}w_{i0} \\ &\quad \text{(for all } T \text{ sufficiently large)} \\ &= \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\underline{w}_{iT-2,T}\varepsilon_{iT} + O_p\left(\frac{1}{\sqrt{T}}\right) = O_p(1), \end{aligned}$$

as required for part (a).

Next, to show parts (b)-(d), note first that

$$\begin{aligned} & E\left[\sum_{i=1}^{N}\underline{w}_{iT-2,T}\varepsilon_{iT}\right]^2 \\ = \quad & \sum_{i=1}^{N}\sum_{h=1}^{N}\sum_{j=1}^{T-2}\sum_{k=1}^{T-2}\exp\left\{-\left(\frac{T-2-j}{q(T)}\right)\right\}\exp\left\{-\left(\frac{T-2-k}{q(T)}\right)\right\} E\left[\varepsilon_{ij}\varepsilon_{iT}\varepsilon_{hk}\varepsilon_{hT}\right] \\ = \quad & \sum_{i=1}^{N}\sum_{j=1}^{T-2}\exp\left\{-2\left(\frac{T-2-j}{q(T)}\right)\right\} E\left[\varepsilon_{ij}^2\right] E\left[\varepsilon_{iT}^2\right] \\ = \quad & \sigma^4 N\sum_{j=1}^{T-2}\exp\left\{-2\left(\frac{T-2-j}{q(T)}\right)\right\}. \end{aligned}$$

Now, consider part (b), where we take $\rho_T = \exp\{-1/q(T)\}$ such that $T/q(T) \to 0$. Making use of part (a) of Lemma SE-3 with $b = 2$ and $g = 2$, we have

$$\sum_{j=1}^{T-2} \exp\left\{-2\left(\frac{T-2-j}{q(T)}\right)\right\} = T\left[1 + O\left(\max\left\{\frac{T}{q(T)}, \frac{1}{T}\right\}\right)\right] = O(T).$$

It follows that in this case

$$E\left[\frac{1}{\sqrt{NT}}\sum_{i=1}^{N} \underline{w}_{iT-2,T}\varepsilon_{iT}\right]^2 = \sigma^4 \frac{N}{NT}\sum_{j=1}^{T-2} \exp\left\{-2\left(\frac{T-2-j}{q(T)}\right)\right\} = O(1),$$

from which we deduce, using Markov's inequality, that $(NT)^{-1/2}\sum_{i=1}^{N} \underline{w}_{iT-2,T}\varepsilon_{iT} = O_p(1)$.

Moreover, by Assumptions 1 and 4,

$$\begin{aligned} E\left[\left(\frac{1}{\sqrt{NT}}\sum_{i=1}^{N} \varepsilon_{iT}\rho_T^{T-2} w_{i0}\right)^2\right] &= \frac{1}{NT}\sum_{i=1}^{N}\sum_{j=1}^{N} \rho_T^{2(T-2)} E\left[w_{i0}w_{j0}\right] E\left[\varepsilon_{iT}\varepsilon_{jT}\right] \\ &\leq \sigma^2 \frac{N}{NT}\left(\sup_i E\left[w_{i0}^2\right]\right) \exp\left\{-\frac{2(T-2)}{q(T)}\right\} = O\left(T^{-1}\right), \end{aligned}$$

from which it follows by Markov's inequality that

$$\frac{1}{\sqrt{NT}}\sum_{i=1}^{N} \varepsilon_{iT}\rho_T^{T-2} w_{i0} = O_p\left(\frac{1}{\sqrt{T}}\right). \tag{25}$$

Hence,

$$\begin{aligned} \frac{1}{\sqrt{NT}}\sum_{i=1}^{N} w_{iT-2,T}\varepsilon_{iT} &= \frac{1}{\sqrt{NT}}\sum_{i=1}^{N} \underline{w}_{iT-2,T}\varepsilon_{iT} + \frac{1}{\sqrt{NT}}\sum_{i=1}^{N} \varepsilon_{iT}\rho_T^{T-2} w_{i0} \\ &= \frac{1}{\sqrt{NT}}\sum_{i=1}^{N} \underline{w}_{iT-2,T}\varepsilon_{iT} + O_p\left(\frac{1}{\sqrt{T}}\right) = O_p(1), \end{aligned}$$

as required for part (b).

We now turn our attention to part (c), where we take $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \sim T$. Using part (b) of Lemma SE-3 with $b = 2$ and $g = 2$, we have that

$$\begin{aligned} E\left[\frac{1}{\sqrt{NT}}\sum_{i=1}^{N} \underline{w}_{iT-2,T}\varepsilon_{iT}\right]^2 &= \sigma^4 \frac{N}{NT}\sum_{j=1}^{T-2} \exp\left\{-2\left(\frac{T-2-j}{q(T)}\right)\right\} \\ &= \frac{\sigma^4}{2} N \frac{q(T)}{NT}\left[1 - \exp\left\{-\frac{2T}{q(T)}\right\}\right]\left[1 + O\left(\frac{1}{T}\right)\right] \\ &= O(1), \end{aligned}$$

so that again we deduce that $(NT)^{-1/2}\sum_{i=1}^{N} \underline{w}_{iT-2,T}\varepsilon_{iT} = O_p(1)$.

Moreover, by Assumptions 1 and 4,

$$\begin{aligned} E\left[\left(\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\varepsilon_{iT}\rho_T^{T-2}w_{i0}\right)^2\right] &= \frac{1}{NT}\sum_{i=1}^{N}\sum_{j=1}^{N}\rho_T^{2(T-2)}E\left[w_{i0}w_{j0}\right]E\left[\varepsilon_{iT}\varepsilon_{jT}\right] \\ &\leq \frac{\sigma^2}{T}\left(\sup_i E\left[w_{i0}^2\right]\right)\exp\left\{-\frac{2\left(T-2\right)}{q\left(T\right)}\right\} = O\left(T^{-1}\right), \end{aligned}$$

from which it follows by Markov's inequality that

$$\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\varepsilon_{iT}\rho_T^{T-2}w_{i0} = O_p\left(\frac{1}{\sqrt{T}}\right). \tag{26}$$

Hence,

$$\begin{aligned} \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}w_{iT-2,T}\varepsilon_{iT} &= \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\underline{w}_{iT-2,T}\varepsilon_{iT} + \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\varepsilon_{iT}\rho_T^{T-2}w_{i0} \\ &= \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\underline{w}_{iT-2,T}\varepsilon_{iT} + O_p\left(\frac{1}{\sqrt{T}}\right) = O_p\left(1\right), \end{aligned}$$

which shows part (c).

For part (d), we consider the case where $\rho_T = \exp\left\{-1/q\left(T\right)\right\}$ such that $q\left(T\right)\to\infty$ but $q\left(T\right)/T\to 0$. In this case, we use part (c) of Lemma SE-3 with $b=2$ and $g=2$ to obtain

$$\begin{aligned} E\left[\frac{1}{\sqrt{Nq\left(T\right)}}\sum_{i=1}^{N}w_{iT-2,T}\varepsilon_{iT}\right]^2 &= \sigma^4\frac{N}{Nq\left(T\right)}\sum_{j=1}^{T-2}\exp\left\{-2\left(\frac{T-2-j}{q\left(T\right)}\right)\right\} \\ &= \frac{\sigma^4}{2}\frac{Nq\left(T\right)}{Nq\left(T\right)}\left[1+O\left(\frac{1}{q\left(T\right)}\right)\right] = O\left(1\right), \end{aligned}$$

from which it follows, by Markov's inequality, that $\left(Nq\left(T\right)\right)^{-1/2}\sum_{i=1}^{N}\underline{w}_{iT-2,T}\varepsilon_{iT} = O_p\left(1\right)$.

Moreover, by Assumptions 1 and 4,

$$\begin{aligned} E\left[\left(\frac{1}{\sqrt{Nq\left(T\right)}}\sum_{i=1}^{N}\varepsilon_{iT}\rho_T^{T-2}w_{i0}\right)^2\right] &= \frac{1}{Nq\left(T\right)}\sum_{i=1}^{N}\sum_{j=1}^{N}\rho_T^{2(T-2)}E\left[w_{i0}w_{j0}\right]E\left[\varepsilon_{iT}\varepsilon_{jT}\right] \\ &\leq \sigma^2\frac{N}{Nq\left(T\right)}\left(\sup_i E\left[w_{i0}^2\right]\right)\exp\left\{-\frac{2\left(T-2\right)}{q\left(T\right)}\right\} \\ &= O\left(\frac{1}{q\left(T\right)}\exp\left\{-\frac{2T}{q\left(T\right)}\right\}\right), \end{aligned}$$

from which it follows by Markov's inequality that

$$\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\varepsilon_{iT}\rho_T^{T-2}w_{i0} = O_p\left(\frac{1}{\sqrt{q\left(T\right)}}\exp\left\{-\frac{T}{q\left(T\right)}\right\}\right). \tag{27}$$

Hence,

$$\begin{aligned}
\frac{1}{\sqrt{Nq(T)}}\sum_{i=1}^{N} w_{iT-2,T}\varepsilon_{iT} &= \frac{1}{\sqrt{Nq(T)}}\sum_{i=1}^{N} \underline{w}_{iT-2,T}\varepsilon_{iT} + \frac{1}{\sqrt{Nq(T)}}\sum_{i=1}^{N} \varepsilon_{iT}\rho_T^{T-2} w_{i0} \\
&= \frac{1}{\sqrt{Nq(T)}}\sum_{i=1}^{N} \underline{w}_{iT-2,T}\varepsilon_{iT} + O_p\left(\frac{1}{\sqrt{q(T)}}\exp\left\{-\frac{T}{q(T)}\right\}\right) = O_p(1),
\end{aligned}$$

as required for part (d)

Finally, to show part (e), note that since, in this case, $q(T) = O(1)$, there is some positive constant C_q and some positive integer T^* such that for all $T \geq T^*$

$$0 \leq \rho_T^2 = \exp\left\{-\frac{2}{q(T)}\right\} \leq \exp\left\{-\frac{2}{C_q}\right\} < 1,$$

from which it follows that

$$\begin{aligned}
E\left[\frac{1}{\sqrt{N}}\sum_{i=1}^{N} \underline{w}_{iT-2,T}\varepsilon_{iT}\right]^2 &= \frac{1}{N}\sum_{i=1}^{N}\sum_{k=1}^{N}\sum_{j=1}^{T-2}\sum_{\ell=1}^{T-2} \rho_T^{(T-2-j)}\rho_T^{(T-2-\ell)} E\left[\varepsilon_{ij}\varepsilon_{iT}\varepsilon_{k\ell}\varepsilon_{kT}\right] \\
&= \sigma^4 \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{T-2} \rho_T^{2(T-2-j)} \\
&= \sigma^4 N \frac{1}{N}\frac{1-\rho_T^{2(T-2)}}{1-\rho_T^2} \\
&\leq \frac{\sigma^4}{1-\exp\{-2/C_q\}} \quad (\text{for all } T \geq T^*) \\
&= O(1).
\end{aligned}$$

It follows from applying Markov's inequality that $N^{-1/2}\sum_{i=1}^{N} w_{iT-2,T}\varepsilon_{iT} = O_p(1)$

Moreover, by Assumptions 1 and 4,

$$\begin{aligned}
E\left[\left(\frac{1}{\sqrt{N}}\sum_{i=1}^{N} \varepsilon_{iT}\rho_T^{T-2} w_{i0}\right)^2\right] &= \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N} E\left[w_{i0}w_{j0}\right]\rho_T^{2(T-2)} E\left[\varepsilon_{iT}\varepsilon_{jT}\right] \\
&= \frac{\sigma^2}{N}\sum_{i=1}^{N} E\left[w_{i0}^2\right]\rho_T^{2(T-2)} \\
&\leq \sigma^2\left(\sup_i E\left[w_{i0}^2\right]\right)\rho_T^{2(T-2)} = O\left(\rho_T^{2(T-2)}\right),
\end{aligned}$$

from which it follows by Markov's inequality that

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N} \varepsilon_{iT}\rho_T^{T-2} w_{i0} = O_p\left(\rho_T^{(T-2)}\right).$$

Hence,

$$\begin{aligned}\frac{1}{\sqrt{N}}\sum_{i=1}^{N} w_{iT-2,T}\varepsilon_{iT} &= \frac{1}{\sqrt{N}}\sum_{i=1}^{N} \underline{w}_{iT-2,T}\varepsilon_{iT} + \frac{1}{\sqrt{N}}\sum_{i=1}^{N} \varepsilon_{iT}\rho_T^{T-2} w_{i0} \\ &= \frac{1}{\sqrt{N}}\sum_{i=1}^{N} \underline{w}_{iT-2,T}\varepsilon_{iT} + O_p\left(\rho_T^{(T-2)}\right) = O_p(1),\end{aligned}$$

as required for part (e). □.

To facilitate stating the next lemma, we introduce the following notations

$$X_{i,T} = -\frac{1}{\sqrt{T}}\sum_{t=4}^{T} \varepsilon_{it-2}\varepsilon_{it-1}, \tag{28}$$

$$Y_{i,T} = \frac{1}{\sqrt{T}} w_{iT-2}\varepsilon_{iT}. \tag{29}$$

Lemma SE-22:

Under Assumptions 1 and 4, the following statements are true as $N, T \to \infty$.

(a) If $\rho_T = \exp\{-1/q(T)\}$ such that $T/q(T) \to 0$, then

$$\frac{1}{\sqrt{2}\sigma^2\sqrt{N}}\sum_{i=1}^{N}(X_{i,T} + Y_{i,T}) \Rightarrow N(0,1).$$

(b) If $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \sim T$, then

$$\frac{1}{\overline{\omega}_T\sqrt{N}}\sum_{i=1}^{N}(X_{i,T} + Y_{i,T}) \Rightarrow N(0,1),$$

where

$$\overline{\omega}_T = \sigma^2\sqrt{1 + \frac{q(T)}{T}\left[\frac{1-\exp\{-2T/q(T)\}}{2}\right]}.$$

(c) If $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \to \infty$ but $q(T)/T \to 0$, then

$$\frac{1}{\sigma^2\sqrt{N}}\sum_{i=1}^{N}(X_{i,T} + Y_{i,T}) = \frac{1}{\sigma^2\sqrt{N}}\sum_{i=1}^{N} X_{i,T} + O_p\left(\sqrt{\frac{q(T)}{T}}\right) \Rightarrow N(0,1).$$

Proof of Lemma SE-22:

To proceed, first decompose $Y_{i,T}$ as follows:

$$\begin{aligned}Y_{i,T} &= \frac{1}{\sqrt{T}} w_{iT-2,T}\varepsilon_{iT} = \frac{1}{\sqrt{T}}\underline{w}_{iT-2,T}\varepsilon_{iT} + \frac{1}{\sqrt{T}}\varepsilon_{iT}\rho_T^{T-2} w_{i0} \\ &= \underline{Y}_{i,T} + \frac{1}{\sqrt{T}}\varepsilon_{iT}\rho_T^{T-2} w_{i0},\end{aligned}$$

where $\underline{w}_{iT-2} = \sum_{j=1}^{T-2} \rho_T^{(T-2-j)} \varepsilon_{ij}$ and $\underline{Y}_{i,T} = \frac{1}{\sqrt{T}} \underline{w}_{iT-2,T} \varepsilon_{iT}$, and we perform some preliminary moment calculations. Let

$$\underline{U}_{N,T} = \sum_{i=1}^{N} \left(X_{i,T} + \underline{Y}_{i,T} \right).$$

Next, note that

$$\begin{aligned} E\left[X_{i,T}\right] &= -\frac{1}{\sqrt{T}} \sum_{t=4}^{T} E\left[\varepsilon_{it-2}\varepsilon_{it-1}\right] = 0 \\ E\left[\underline{Y}_{i,T}\right] &= \frac{1}{\sqrt{T}} \sum_{j=1}^{T-2} \exp\left\{-\frac{T-2-j}{q(T)}\right\} E\left[\varepsilon_{ij}\varepsilon_{iT}\right] = 0, \end{aligned}$$

and, thus,

$$E\left[\underline{U}_{N,T}\right] = \sum_{i=1}^{N} \left(E\left[X_{i,T}\right] + E\left[\underline{Y}_{i,T}\right] \right) = 0.$$

In addition, note that

$$E\left[X_{i,T}^2\right] = \frac{1}{T} \sum_{t=4}^{T} \sum_{s=4}^{T} E\left[\varepsilon_{it-2}\varepsilon_{it-1}\varepsilon_{is-2}\varepsilon_{is-1}\right] = \frac{1}{T} \sum_{t=4}^{T} E\left[\varepsilon_{it-2}^2\right] E\left[\varepsilon_{it-1}^2\right] = \sigma^4 \left(\frac{T-3}{T}\right) = \sigma^4 + O\left(\frac{1}{T}\right),$$

$$\begin{aligned} E\left[\underline{Y}_{i,T}^2\right] &= \frac{1}{T} \sum_{j=1}^{T-2} \sum_{k=1}^{T-2} \exp\left\{-\frac{T-2-j}{q(T)}\right\} \exp\left\{-\frac{T-2-k}{q(T)}\right\} E\left[\varepsilon_{ij}\varepsilon_{ik}\varepsilon_{iT}\varepsilon_{iT}\right] \\ &= \frac{1}{T} \sum_{j=1}^{T-2} \exp\left\{-2\frac{T-2-j}{q(T)}\right\} E\left[\varepsilon_{it}^2\right] E\left[\varepsilon_{iT}^2\right] \\ &= \sigma^4 \frac{1}{T} \sum_{j=1}^{T-2} \exp\left\{-2\frac{T-2-j}{q(T)}\right\}, \end{aligned}$$

$$\begin{aligned} E\left[X_{i,T}\underline{Y}_{i,T}\right] &= -\frac{1}{T} \sum_{t=4}^{T} \sum_{\ell=1}^{T-2} \exp\left\{-\frac{T-2-\ell}{q(T)}\right\} E\left[\varepsilon_{it-2}\varepsilon_{it-1}\varepsilon_{i\ell}\varepsilon_{iT}\right] \\ &= -\frac{1}{T} \sum_{t=4}^{T} \sum_{\ell=1}^{T-2} \exp\left\{-\frac{T-2-\ell}{q(T)}\right\} E\left[\varepsilon_{it-2}\varepsilon_{it-1}\varepsilon_{i\ell}\right] E\left[\varepsilon_{iT}\right] \\ &= 0. \end{aligned}$$

It follows that

$$\begin{aligned} \omega_{i,T}^2 &= E\left[\left(X_{i,T} + \underline{Y}_{i,T}\right)^2\right] \\ &= E\left[X_{i,T}^2\right] + E\left[\underline{Y}_{i,T}^2\right] + 2E\left[X_{i,T}\underline{Y}_{i,T}\right] \\ &= \sigma^4 \left(\frac{T-3}{T}\right) + \sigma^4 \frac{1}{T} \sum_{\ell=1}^{T-2} \exp\left\{-2\frac{T-2-\ell}{q(T)}\right\}, \end{aligned}$$

so that

$$\omega_{N,T}^2 = \sum_{i=1}^{N} \omega_{i,T}^2 = \sigma^4 N \left[\left(\frac{T-3}{T} \right) + \frac{1}{T} \sum_{\ell=1}^{T-2} \exp\left\{ -2 \frac{T-2-\ell}{q(T)} \right\} \right].$$

Now, consider part (a), where we take $T/q(T) \to 0$. In this case, we apply part (a) of Lemma SE-3 with $b = 2$ and $g = 2$ to obtain

$$\begin{aligned} \frac{\omega_{N,T}^2}{N} &= \frac{1}{N} \sum_{i=1}^{N} \omega_{i,T}^2 \\ &= \sigma^4 \frac{1}{N} N \left\{ \left(\frac{T-3}{T} \right) + \frac{1}{T} T \left[1 + O\left(\max\left\{ \frac{T}{q(T)}, \frac{1}{T} \right\} \right) \right] \right\} \\ &= 2\sigma^4 + O\left(\max\left\{ \frac{T}{q(T)}, \frac{1}{T} \right\} \right). \end{aligned} \tag{30}$$

Expression (30) and Assumption 1 then imply that there exists a positive constant C such that

$$0 < \frac{1}{C} \leq \frac{\omega_{N,T}}{\sqrt{N}} \leq C < \infty \text{ eventually as } N, T \to \infty.$$

Next, note that, by the result given in expression (25)

$$\begin{aligned} & \frac{1}{\omega_{N,T}/\sqrt{N}} \frac{1}{\sqrt{N}} \sum_{i=1}^{N} (X_{i,T} + Y_{i,T}) \\ = & \frac{1}{\omega_{N,T}/\sqrt{N}} \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left(X_{i,T} + \underline{Y}_{i,T} \right) + \frac{1}{\omega_{N,T}/\sqrt{N}} \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \varepsilon_{iT} \rho_T^{T-2} w_{i0} \\ = & \frac{1}{\omega_{N,T}/\sqrt{N}} \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left(X_{i,T} + \underline{Y}_{i,T} \right) + o_p(1). \end{aligned}$$

Hence, showing the desired result is equivalent to showing the asymptotic normality of

$$U_{N,T} = \frac{1}{\omega_{N,T}/\sqrt{N}} \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left(X_{i,T} + \underline{Y}_{i,T} \right).$$

To show the asymptotic normality of $U_{N,T}$, it suffices to verify a Liapounov-type condition of the form

$$\lim_{N,T \to \infty} \frac{1}{N^2} \sum_{i=1}^{N} E\left[\left(X_{i,T} + \underline{Y}_{i,T} \right)^4 \right] = 0. \tag{31}$$

To show this, note first that by Loève's c_r inequality, we have that

$$\begin{aligned} & \frac{1}{N^2} \sum_{i=1}^{N} E\left[\left(X_{i,T} + \underline{Y}_{i,T} \right)^4 \right] \\ \leq & \frac{1}{N^2} \sum_{i=1}^{N} 8 \left\{ E\left[\left(-\frac{1}{\sqrt{T}} \sum_{t=4}^{T} \varepsilon_{it-2} \varepsilon_{it-1} \right)^4 \right] + E\left[\left(\frac{1}{\sqrt{T}} \sum_{j=1}^{T-2} \exp\left\{ -\frac{T-2-j}{q(T)} \right\} \varepsilon_{ij} \varepsilon_{iT} \right)^4 \right] \right\} \\ = & 8 \frac{1}{N^2 T^2} \sum_{i=1}^{N} E\left[\left(\sum_{t=4}^{T} \varepsilon_{it-2} \varepsilon_{it-1} \right)^4 \right] + 8 \frac{1}{N^2 T^2} \sum_{i=1}^{N} E\left[\left(\sum_{j=1}^{T-2} \exp\left\{ -\frac{T-2-j}{q(T)} \right\} \varepsilon_{ij} \varepsilon_{iT} \right)^4 \right]. \end{aligned} \tag{32}$$

Next, note that

$$
\begin{aligned}
& E\left[\left(\sum_{t=4}^{T}\varepsilon_{it-2}\varepsilon_{it-1}\right)^4\right] = \sum_{g=4}^{T}\sum_{s=4}^{T}\sum_{t=4}^{T}\sum_{u=4}^{T} E\left[\varepsilon_{ig-2}\varepsilon_{is-2}\varepsilon_{it-2}\varepsilon_{iu-2}\varepsilon_{ig-1}\varepsilon_{is-1}\varepsilon_{it-1}\varepsilon_{iu-1}\right] \\
= \; & \sum_{t=4}^{T} E\left[\varepsilon_{it-2}^4\right] E\left[\varepsilon_{it-1}^4\right] + 6\sum_{s=6}^{T}\sum_{t=4}^{s-2} E\left[\varepsilon_{is-2}^2\right] E\left[\varepsilon_{is-1}^2\right] E\left[\varepsilon_{it-2}^2\right] E\left[\varepsilon_{it-1}^2\right] \\
& +6\sum_{t=4}^{T-1} E\left[\varepsilon_{it}^2\right] E\left[\varepsilon_{it-1}^4\right] E\left[\varepsilon_{it-2}^2\right] \\
= \; & \left(E\left[\varepsilon_{it-1}^4\right]\right)^2 (T-3) + 6\sigma^8\sum_{s=6}^{T}(s-5) + 6E\left[\varepsilon_{it-1}^4\right]\sigma^4(T-4) \\
= \; & 6\sigma^8\frac{(T-5)(T-4)}{2} + \left(E\left[\varepsilon_{it-1}^4\right]\right)^2 (T-3) + 6E\left[\varepsilon_{it-1}^4\right]\sigma^4(T-4) \\
= \; & 3\sigma^8 T^2\left[1+O\left(\frac{1}{T}\right)\right], \qquad (33)
\end{aligned}
$$

and

$$
\begin{aligned}
& E\left[\left(\sum_{j=1}^{T-2}\exp\left\{-\frac{T-2-j}{q(T)}\right\}\varepsilon_{ij}\varepsilon_{iT}\right)^4\right] \\
= \; & \sum_{g=1}^{T-2}\sum_{s=1}^{T-2}\sum_{t=1}^{T-2}\sum_{u=1}^{T-2}\left\{\exp\left\{-\frac{T-2-g}{q(T)}\right\}\exp\left\{-\frac{T-2-s}{q(T)}\right\}\exp\left\{-\frac{T-2-t}{q(T)}\right\}\right. \\
& \qquad \left.\times\exp\left\{-\frac{T-2-u}{q(T)}\right\} E\left[\varepsilon_{ig}\varepsilon_{is}\varepsilon_{it}\varepsilon_{iu}\right] E\left[\varepsilon_{iT}^4\right]\right\} \\
= \; & \sum_{s=1}^{T-2}\exp\left\{-4\frac{T-2-s}{q(T)}\right\} E\left[\varepsilon_{is}^4\right] E\left[\varepsilon_{iT}^4\right] \\
& +6\sum_{g=2}^{T-2}\sum_{s=1}^{g-1}\exp\left\{-2\frac{T-2-g}{q(T)}\right\}\exp\left\{-2\frac{T-2-s}{q(T)}\right\} E\left[\varepsilon_{ig}^2\right] E\left[\varepsilon_{is}^2\right] E\left[\varepsilon_{iT}^4\right] \\
= \; & E\left[\varepsilon_{is}^4\right] E\left[\varepsilon_{iT}^4\right]\sum_{s=1}^{T-2}\exp\left\{-4\frac{T-2-s}{q(T)}\right\} \\
& +6\sigma^4 E\left[\varepsilon_{iT}^4\right]\sum_{g=2}^{T-2}\sum_{s=1}^{g-1}\exp\left\{-2\frac{T-2-g}{q(T)}\right\}\exp\left\{-2\frac{T-2-s}{q(T)}\right\} \\
= \; & E\left[\varepsilon_{is}^4\right] E\left[\varepsilon_{iT}^4\right] T\left[1+O\left(\max\left\{\frac{T}{q(T)},\frac{1}{T}\right\}\right)\right] \\
& +6\sigma^4 E\left[\varepsilon_{iT}^4\right]\frac{T^2}{2}\left[1+O\left(\max\left\{\frac{T}{q(T)},\frac{1}{T}\right\}\right)\right], \qquad (34)
\end{aligned}
$$

where the last equality follows from applying part (a) of Lemma SE-3 with $b = 4$ and $g = 2$ and by applying part (a) of Lemma SE-4 with $d = 2$. Now, applying (33) and (34) to the upper bound in (32),

we get

$$
\begin{aligned}
& \frac{1}{N^2}\sum_{i=1}^{N} E\left[\left(X_{i,T}+\underline{Y}_{i,T}\right)^4\right] \leq 8\frac{1}{N^2T^2}\sum_{i=1}^{N} 3\sigma^8 T^2\left[1+O\left(\frac{1}{T}\right)\right] \\
& +8E\left[\varepsilon_{is}^4\right]E\left[\varepsilon_{iT}^4\right]\frac{1}{N^2T^2}\sum_{i=1}^{N} T\left[1+O\left(\max\left\{\frac{T}{q(T)},\frac{1}{T}\right\}\right)\right] \\
& +8\sigma^4 E\left[\varepsilon_{iT}^4\right]\frac{1}{N^2T^2}\sum_{i=1}^{N} 6\frac{T^2}{2}\left[1+O\left(\max\left\{\frac{T}{q(T)},\frac{1}{T}\right\}\right)\right] \\
= \; & 24\sigma^8\frac{1}{N}+8E\left[\varepsilon_{is}^4\right]E\left[\varepsilon_{iT}^4\right]\frac{1}{NT}+24\sigma^4 E\left[\varepsilon_{iT}^4\right]\frac{1}{N}+O\left(\max\left\{\frac{T}{Nq(T)},\frac{1}{NT}\right\}\right) \\
= \; & O\left(N^{-1}\right).
\end{aligned}
$$

Since the Liapounov-type condition (31) implies the Lindeberg-type condition (13) given in Lemma SE-10 above, it follows from Lemma SE-10 that

$$
\begin{aligned}
U_{N,T} &= \frac{1}{\omega_{N,T}}\sum_{i=1}^{N}\left(X_{i,T}+\underline{Y}_{i,T}\right) \\
&= \frac{1}{\sqrt{2}\sigma^2}\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(X_{i,T}+\underline{Y}_{i,T}\right)+O\left(\max\left\{\frac{T}{q(T)},\frac{1}{T}\right\}\right) \\
&\Rightarrow N(0,1).
\end{aligned}
$$

Next, consider part (b), where we take $q(T)\sim T$. Here, we apply part (b) of Lemma SE-3 with $b=2$ and $g=2$ to obtain

$$
\begin{aligned}
\frac{\omega_{N,T}^2}{N} &= \frac{1}{N}\sum_{i=1}^{N}\omega_{i,T}^2 \\
&= \sigma^4\frac{1}{N}N\left[\left(\frac{T-3}{T}\right)+\frac{1}{T}\sum_{\ell=1}^{T-2}\exp\left\{-2\frac{T-2-\ell}{q(T)}\right\}\right] \\
&= \sigma^4\left\{\left(\frac{T-3}{T}\right)+\frac{1}{T}\frac{q(T)}{2}\left[1-\exp\left\{-\frac{2T}{q(T)}\right\}\right]\left[1+O\left(\frac{1}{T}\right)\right]\right\} \\
&= \sigma^4\left[1+\frac{q(T)}{T}\frac{1}{2}\left[1-\exp\left\{-\frac{2T}{q(T)}\right\}\right]\right]+O\left(\frac{1}{T}\right). \qquad (35)
\end{aligned}
$$

Note that, in light of Assumption 1, there exists a positive constant C such that

$$
\begin{aligned}
0 &< \frac{1}{C}\leq\frac{\omega_{N,T}}{\sqrt{N}} \\
&= \sigma^2\sqrt{1+\frac{q(T)}{T}\left[\frac{1-\exp\{-2T/q(T)\}}{2}\right]}\left[1+O\left(\frac{1}{T}\right)\right] \\
&\leq C<\infty \text{ eventually as } N,T\to\infty.
\end{aligned}
$$

Next, note that, by the result given in expression (26)

$$\begin{aligned}\frac{1}{\omega_{N,T}/\sqrt{N}}\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(X_{i,T}+Y_{i,T}\right) &= \frac{1}{\omega_{N,T}/\sqrt{N}}\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(X_{i,T}+\underline{Y}_{i,T}\right)+\frac{1}{\omega_{N,T}/\sqrt{N}}\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\varepsilon_{iT}\rho_T^{T-2}w_{i0}\\ &= \frac{1}{\omega_{N,T}/\sqrt{N}}\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(X_{i,T}+\underline{Y}_{i,T}\right)+o_p\left(1\right).\end{aligned}$$

Hence, showing the desired result is equivalent to showing the asymptotic normality of

$$U_{N,T}=\frac{1}{\omega_{N,T}}\underline{U}_{N,T}=\frac{1}{\omega_{N,T}/\sqrt{N}}\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(X_{i,T}+\underline{Y}_{i,T}\right).$$

To show the asymptotic normality of $U_{N,T}$, it suffices to verify a Liapounov-type condition of the form

$$\lim_{N,T\to\infty}\sum_{i=1}^{N}E\left[\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(X_{i,T}+\underline{Y}_{i,T}\right)\right]^4=\lim_{N,T\to\infty}\frac{1}{N^2}\sum_{i=1}^{N}E\left[\left(X_{i,T}+\underline{Y}_{i,T}\right)^4\right]=0.$$

To proceed, note that using calculations similar to that given for (34), we have

$$\begin{aligned}&E\left[\left(\sum_{j=1}^{T-2}\exp\left\{-\frac{T-2-j}{q\left(T\right)}\right\}\varepsilon_{ij}\varepsilon_{iT}\right)^4\right]\\ =\;& E\left[\varepsilon_{is}^4\right]E\left[\varepsilon_{iT}^4\right]\sum_{s=1}^{T-2}\exp\left\{-4\frac{T-2-s}{q\left(T\right)}\right\}\\ &+6\sigma^4E\left[\varepsilon_{iT}^4\right]\sum_{g=2}^{T-2}\sum_{s=1}^{g-1}\exp\left\{-2\frac{T-2-g}{q\left(T\right)}\right\}\exp\left\{-2\frac{T-2-s}{q\left(T\right)}\right\}\\ =\;& E\left[\varepsilon_{is}^4\right]E\left[\varepsilon_{iT}^4\right]q\left(T\right)\left[\frac{1-\exp\left\{-4T/q\left(T\right)\right\}}{4}\right]\left[1+O\left(\frac{1}{T}\right)\right]\\ &+6\sigma^4E\left[\varepsilon_{iT}^4\right]q\left(T\right)^2\frac{\left[1-\exp\left\{-2T/q\left(T\right)\right\}\right]^2}{8}\left[1+O\left(\frac{1}{T}\right)\right],\end{aligned}\tag{36}$$

where the last equality follows from applying part (b) of Lemma SE-3 with $b=4$ and $g=2$ and by applying part (b) of Lemma SE-4 with $d=2$.. Now, applying (33) and (36) to the upper bound in (32),

we get

$$
\begin{aligned}
& \frac{1}{N^2}\sum_{i=1}^{N} E\left[\left(X_{i,T}+\underline{Y}_{i,T}\right)^4\right] \\
\leq\ & 8\frac{1}{N^2T^2}\sum_{i=1}^{N} 3\sigma^8 T^2\left[1+O\left(\frac{1}{T}\right)\right] \\
& +8E\left[\varepsilon_{is}^4\right]E\left[\varepsilon_{iT}^4\right]\frac{1}{N^2T^2}\sum_{i=1}^{N} q\left(T\right)\left[\frac{1-\exp\left\{-4T/q\left(T\right)\right\}}{4}\right]\left[1+O\left(\frac{1}{T}\right)\right] \\
& +8\sigma^4 E\left[\varepsilon_{iT}^4\right]\frac{1}{N^2T^2}\sum_{i=1}^{N} 6q\left(T\right)^2\frac{\left[1-\exp\left\{-2T/q\left(T\right)\right\}\right]^2}{8}\left[1+O\left(\frac{1}{T}\right)\right]\Bigg\} \\
=\ & 24\sigma^8\frac{1}{N}+2E\left[\varepsilon_{is}^4\right]E\left[\varepsilon_{iT}^4\right]\frac{q\left(T\right)}{NT}\left[1-\exp\left\{-\frac{4T}{q\left(T\right)}\right\}\right] \\
& +6\sigma^4 E\left[\varepsilon_{iT}^4\right]\frac{q\left(T\right)^2}{NT^2}\left[1-\exp\left\{-\frac{2T}{q\left(T\right)}\right\}\right]^2+O\left(\frac{1}{NT}\right) \\
=\ & O\left(N^{-1}\right),
\end{aligned}
$$

which verifies the required Liapounov-type condition. Since this condition implies the Lindeberg-type condition (13) stated in Lemma SE-10 above, it follows from Lemma SE-10 that

$$
\begin{aligned}
U_{N,T} &= \frac{1}{\omega_{N,T}}\sum_{i=1}^{N}\left(X_{i,T}+\underline{Y}_{i,T}\right) \\
&= \frac{1}{\omega_{N,T}/\sqrt{N}}\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(X_{i,T}+\underline{Y}_{i,T}\right) \\
&= \frac{1}{\overline{\omega}_T\sqrt{N}}\sum_{i=1}^{N}\left(X_{i,T}+\underline{Y}_{i,T}\right)+O\left(\frac{1}{T}\right) \\
&\Rightarrow N\left(0,1\right),
\end{aligned}
$$

where

$$
\overline{\omega}_T=\sigma^2\sqrt{1+\frac{q\left(T\right)}{T}\left[\frac{1-\exp\left\{-2T/q\left(T\right)\right\}}{2}\right]}.
$$

Finally, we turn our attention to part (c). Here, we consider the case $q\left(T\right)\to\infty$ such that $q\left(T\right)/T\to 0$. Applying part (c) of Lemma SE-3 with $b=2$ and $g=2$, we obtain

$$
\begin{aligned}
\frac{\omega_{N,T}^2}{N} &= \frac{1}{N}\sum_{i=1}^{N}\omega_{i,T}^2=\sigma^4\frac{1}{N}N\left\{\left(\frac{T-3}{T}\right)+\frac{1}{T}\frac{q\left(T\right)}{2}\left[1+O\left(\frac{1}{q\left(T\right)}\right)\right]\right\} \\
&= \sigma^4+O\left(\frac{q\left(T\right)}{T}\right). \qquad (37)
\end{aligned}
$$

Hence, in light of Assumption 1, there exists positive constant C such that

$$
0<\frac{1}{C}\leq\frac{\omega_{N,T}^2}{N}\leq C<\infty \text{ eventually as } N,T\to\infty. \qquad (38)
$$

Next, applying the result given in expression (27), part (d) of Lemma SE-21 and (38) above, we obtain

$$\begin{aligned}
U_{N,T} &= \frac{1}{\omega_{N,T}} \sum_{i=1}^{N} (X_{i,T} + Y_{i,T}) \\
&= \frac{1}{\omega_{N,T}/\sqrt{N}} \frac{1}{\sqrt{N}} \sum_{i=1}^{N} X_{i,T} + \frac{1}{\omega_{N,T}/\sqrt{N}} \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \underline{Y}_{i,T} \\
&\quad + \frac{1}{\omega_{N,T}/\sqrt{N}} \sqrt{\frac{q(T)}{T}} \frac{1}{\sqrt{Nq(T)}} \sum_{i=1}^{N} \varepsilon_{iT} \rho_T^{T-2} w_{i0} \\
&= \frac{1}{\omega_{N,T}} \sum_{i=1}^{N} X_{i,T} + O_p\left(\sqrt{\frac{q(T)}{T}}\right) + o_p\left(\sqrt{\frac{q(T)}{T}}\right).
\end{aligned}$$

Hence, to prove the result in part (c), we need to show the asymptotic normality of

$$U_{N,T}^{(1)} = \frac{1}{\omega_{N,T}} \sum_{i=1}^{N} X_{i,T} = \frac{1}{\omega_{N,T}/\sqrt{N}} \frac{1}{\sqrt{N}} \sum_{i=1}^{N} X_{i,T}.$$

To do so, we verify the Liapounov-type condition

$$\lim_{N,T\to\infty} \sum_{i=1}^{N} E\left[\frac{1}{\sqrt{N}} \sum_{i=1}^{N} X_{i,T}\right]^4 = \lim_{N,T\to\infty} \frac{1}{N^2} \sum_{i=1}^{N} E\left[X_{i,T}^4\right] = 0.$$

Applying (33), we get

$$\begin{aligned}
\frac{1}{N^2} \sum_{i=1}^{N} E\left[X_{i,T}^4\right] &\leq \frac{1}{N^2 T^2} \sum_{i=1}^{N} 3\sigma^8 T^2 \left[1 + O\left(\frac{1}{T}\right)\right] \\
&= 3\sigma^8 \frac{1}{N} + O\left(\frac{1}{NT}\right) \\
&= O\left(N^{-1}\right),
\end{aligned}$$

which verifies the required Liapounov-type condition. Since this condition implies the Lindeberg-type condition (13) stated in Lemma SE-10 above, it follows from Lemma SE-10 that

$$\begin{aligned}
U_{N,T}^{(1)} &= \frac{1}{\omega_{N,T}} \sum_{i=1}^{N} X_{i,T} \\
&= \frac{1}{\sigma^2} \frac{1}{\sqrt{N}} \sum_{i=1}^{N} X_{i,T} + O\left(\frac{q(T)}{T}\right) \\
&\Rightarrow N(0,1),
\end{aligned}$$

as required. □

Lemma SE-23: Suppose that Assumptions 1 and 4 hold. If

$\rho_T \in \mathcal{G}_{\mathrm{St}} = \left\{ |\rho_T| = \exp\left\{-\frac{1}{q(T)}\right\} : q(T) \geq 0 \text{ and } q(T) = O(1) \text{ as } T \to \infty \right\}$, then as $N, T \to \infty$

$$\frac{1}{\omega_{N,T}} \sum_{i=1}^{N} (X_{i,T} + Y_{i,T}) = \sqrt{\frac{1+\rho_T}{2\sigma^4}} \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left(X_{i,T} + \underline{Y}_{i,T}\right) + o_p(1) \Rightarrow N(0,1).$$

where in this case

$$\begin{aligned} X_{i,T} &= -\frac{1}{\sqrt{T}} \sum_{t=4}^{T} \varepsilon_{it-2}\varepsilon_{it-1}, \; Y_{i,T} = (1-\rho_T) \frac{1}{\sqrt{T}} \sum_{t=4}^{T} w_{it-3,T}\varepsilon_{it-1}, \\ \underline{Y}_{i,T} &= (1-\rho_T) \frac{1}{\sqrt{T}} \sum_{t=4}^{T} \underline{w}_{it-3,T}\varepsilon_{it-1}, \; \underline{w}_{it-3,T} = \sum_{j=1}^{t-3} \rho_T^{(t-3-j)} \varepsilon_{ij}, \end{aligned}$$

and

$$\begin{aligned} \omega_{i,T}^2 &= \sigma^4 \left\{ \left(1 - \frac{3}{T}\right) + \frac{(1-\rho_T)}{1+\rho_T} \left[1 - \frac{3}{T} - \frac{\rho_T^2 \left(1 - \rho_T^{2(T-3)}\right)}{T\left(1-\rho_T^2\right)} \right] \right\} \\ &= \frac{2\sigma^4}{1+\rho_T} + O\left(\frac{1}{T}\right). \end{aligned}$$

Proof of Lemma SE-23:

To proceed, first decompose $Y_{i,T}$ as follows:

$$\begin{aligned} Y_{i,T} &= (1-\rho_T) \frac{1}{\sqrt{T}} \sum_{t=4}^{T} w_{it-3,T}\varepsilon_{it-1} \\ &= (1-\rho_T) \frac{1}{\sqrt{T}} \sum_{t=4}^{T} \underline{w}_{it-3,T}\varepsilon_{it-1} + (1-\rho_T) \frac{1}{\sqrt{T}} \sum_{t=4}^{T} \varepsilon_{it-1}\rho_T^{t-3} w_{i0} \\ &= \underline{Y}_{i,T} + (1-\rho_T) \frac{1}{\sqrt{T}} \sum_{t=4}^{T} \varepsilon_{it-1}\rho_T^{t-3} w_{i0}, \end{aligned}$$

where $\underline{Y}_{i,T}$ and $\underline{w}_{it-3,T} = \sum_{j=1}^{t-3} \rho_T^{(t-3-j)} \varepsilon_{ij}$ are defined in the statement of the lemma. Note that under the assumption of this lemma, $q(T) = O(1)$, so that there exist some positive constant C_q and some positive integer T^* such that for all $T \geq T^*$

$$0 \leq |\rho_T| = \exp\left\{-\frac{1}{q(T)}\right\} \leq \exp\left\{-\frac{1}{C_q}\right\} < 1. \tag{39}$$

We should use this bound in various places in the argument given below. Now, let

$$\underline{U}_{N,T} = \sum_{i=1}^{N} \left(X_{i,T} + \underline{Y}_{i,T}\right),$$

where $X_{i,T} = -T^{-1/2}\sum_{t=4}^{T}\varepsilon_{it-2}\varepsilon_{it-1}$ and $\underline{Y}_{i,T}$ is as defined above. Next, note that

$$\begin{aligned}
E\left[X_{i,T}\right] &= -\frac{1}{\sqrt{T}}\sum_{t=4}^{T}E\left[\varepsilon_{it-2}\varepsilon_{it-1}\right]=0,\\
E\left[\underline{Y}_{i,T}\right] &= \left(1-\rho_T\right)\frac{1}{\sqrt{T}}\sum_{t=4}^{T}\sum_{j=1}^{t-3}\rho_T^{(t-3-j)}E\left[\varepsilon_{ij}\varepsilon_{it-1}\right]=0,
\end{aligned}$$

and, thus,

$$E\left[\underline{U}_{N,T}\right]=\sum_{i=1}^{N}\left(E\left[X_{i,T}\right]+E\left[\underline{Y}_{i,T}\right]\right)=0.$$

In addition, note that

$$E\left[X_{i,T}^2\right]=\frac{1}{T}\sum_{t=4}^{T}\sum_{s=4}^{T}E\left[\varepsilon_{it-2}\varepsilon_{it-1}\varepsilon_{is-2}\varepsilon_{is-1}\right]=\frac{1}{T}\sum_{t=4}^{T}E\left[\varepsilon_{it-2}^2\right]E\left[\varepsilon_{it-1}^2\right]=\sigma^4\left(\frac{T-3}{T}\right)=\sigma^4+O\left(\frac{1}{T}\right),$$

$$\begin{aligned}
E\left[\underline{Y}_{i,T}^2\right] &= \left(1-\rho_T\right)^2\frac{1}{T}\sum_{t=4}^{T}\sum_{s=4}^{T}\sum_{j=1}^{t-3}\sum_{k=1}^{s-3}\rho_T^{(t-3-j)}\rho_T^{(s-3-k)}E\left[\varepsilon_{ij}\varepsilon_{it-1}\varepsilon_{ik}\varepsilon_{is-1}\right]\\
&= \sigma^4\frac{\left(1-\rho_T\right)}{1+\rho_T}\frac{1}{T}\left[\left(T-3\right)-\frac{\rho_T^2\left(1-\rho_T^{2(T-3)}\right)}{1-\rho_T^2}\right]\\
&= \sigma^4\frac{\left(1-\rho_T\right)}{1+\rho_T}+O\left(\frac{1}{T}\right),
\end{aligned}$$

$$E\left[X_{i,T}\underline{Y}_{i,T}\right]=-\left(1-\rho_T\right)\frac{1}{T}\sum_{t=4}^{T}\sum_{s=4}^{T}\sum_{j=1}^{t-3}\rho_T^{(t-3-j)}E\left[\varepsilon_{ij}\varepsilon_{it-1}\varepsilon_{is-2}\varepsilon_{is-1}\right]=0.$$

It follows that

$$\begin{aligned}
&\omega_{i,T}^2\\
&= E\left[\left(X_{i,T}+\underline{Y}_{i,T}\right)^2\right]\\
&= E\left[X_{i,T}^2\right]+E\left[\underline{Y}_{i,T}^2\right]+2E\left[X_{i,T}\underline{Y}_{i,T}\right]\\
&= \sigma^4\left(\frac{T-3}{T}\right)+\sigma^4\frac{\left(1-\rho_T\right)}{1+\rho_T}\left[1-\frac{3}{T}-\frac{\rho_T^2\left(1-\rho_T^{2(T-3)}\right)}{T\left(1-\rho_T^2\right)}\right]\\
&= \frac{2\sigma^4}{1+\rho_T}-\frac{\sigma^4}{T}\left[\frac{6}{1+\rho_T}+\frac{\rho_T^2\left(1-\rho_T\right)\left(1-\rho_T^{2(T-3)}\right)}{\left(1-\rho_T^2\right)\left(1+\rho_T\right)}\right]\\
&= \frac{2\sigma^4}{1+\rho_T}-\frac{\sigma^4}{T}\left[\frac{6}{1+\rho_T}+\frac{\rho_T^2\left(1-\rho_T^{2(T-3)}\right)}{\left(1+\rho_T\right)^2}\right],
\end{aligned}$$

and, thus,

$$\begin{aligned}
\frac{\omega_{N,T}^2}{N} &= \frac{1}{N}\sum_{i=1}^{N}\omega_{i,T}^2 \\
&= \frac{1}{N}\left\{\frac{2N\sigma^4}{1+\rho_T} - \frac{\sigma^4 N}{T}\left[\frac{6}{1+\rho_T} + \frac{\rho_T^2\left(1-\rho_T^{2(T-3)}\right)}{(1+\rho_T)^2}\right]\right\} \\
&= \frac{2\sigma^4}{1+\rho_T} - \frac{\sigma^4}{T}\left[\frac{6}{1+\rho_T} + \frac{\rho_T^2\left(1-\rho_T^{2(T-3)}\right)}{(1+\rho_T)^2}\right].
\end{aligned}$$

Next, note that

$$\frac{\sigma^4}{T}\left[\frac{6}{1+\rho_T} + \frac{\rho_T^2\left(1-\rho_T^{2(T-3)}\right)}{(1+\rho_T)^2}\right] \leq \frac{1}{T}\frac{\sigma^4}{1-\exp\{-1/C_q\}}\left[6 + \frac{1}{(1-\exp\{-1/C_q\})}\right],$$

for all $T \geq T^*$ and for all N, so that, in light of Assumption 1,

$$\frac{\sigma^4}{T}\left[\frac{6}{1+\rho_T} + \frac{\rho_T^2\left(1-\rho_T^{2(T-3)}\right)}{(1+\rho_T)^2}\right] = O\left(\frac{1}{T}\right),$$

and

$$\frac{\omega_{N,T}^2}{N} = \frac{1}{N}\sum_{i=1}^{N}\omega_{i,T}^2 = \frac{2\sigma^4}{1+\rho_T} + O\left(\frac{1}{T}\right).$$

Moreover, $0 < \sigma^4 < \infty$ by Assumption 1, so that

$$0 < \sigma^4 \leq \frac{2\sigma^4}{1+\rho_T} \leq \frac{2\sigma^4}{(1-\exp\{-1/C_q\})} < \infty,$$

from which it follows that there exists a positive constant C such that

$$0 < \frac{1}{C} \leq \frac{\omega_{N,T}}{\sqrt{N}} \leq C < \infty \text{ eventually as } N, T \to \infty.$$

Moreover, by Assumptions 1 and 4,

$$\begin{aligned}
& E\left[\left((1-\rho_T)\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=4}^{T}\varepsilon_{it-1}\rho_T^{t-3}w_{i0}\right)^2\right] \\
= & \frac{(1-\rho_T)^2}{NT}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=4}^{T}\sum_{s=4}^{T}E\left[w_{i0}w_{j0}\right]\rho_T^{t-3}\rho_T^{s-3}E\left[\varepsilon_{it-1}\varepsilon_{js-1}\right] \\
= & \frac{\sigma^2(1-\rho_T)^2}{NT}\sum_{i=1}^{N}E\left[w_{i0}^2\right]\sum_{t=4}^{T}\rho_T^{2(t-3)} \\
= & \frac{\sigma^2}{T}\left(\sup_i E\left[w_{i0}^2\right]\right)\frac{\rho_T^2\left(1-\rho_T^{2(T-3)}\right)}{1-\rho_T^2} = O\left(T^{-1}\right),
\end{aligned}$$

from which it follows by Markov's inequality that

$$(1-\rho_T)\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=4}^{T}\varepsilon_{it-1}\rho_T^{t-3}w_{i0}=O_p\left(\frac{1}{\sqrt{T}}\right).$$

Applying this result, we have

$$\begin{aligned}\frac{1}{\omega_{N,T}/\sqrt{N}}\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(X_{i,T}+Y_{i,T}\right) &= \frac{1}{\omega_{N,T}/\sqrt{N}}\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(X_{i,T}+\underline{Y}_{i,T}\right)\\ &\quad+\frac{1}{\omega_{N,T}/\sqrt{N}}(1-\rho_T)\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=4}^{T}\varepsilon_{it-1}\rho_T^{t-3}w_{i0}\\ &= \frac{1}{\omega_{N,T}/\sqrt{N}}\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(X_{i,T}+\underline{Y}_{i,T}\right)+O_p\left(\frac{1}{\sqrt{T}}\right).\end{aligned}$$

Hence, showing the desired result is equivalent to showing the asymptotic normality of

$$U_{N,T}=\frac{1}{\omega_{N,T}/\sqrt{N}}\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(X_{i,T}+\underline{Y}_{i,T}\right).$$

To proceed, we again verify the Liapounov-type condition

$$\lim_{N,T\to\infty}\frac{1}{N^2}\sum_{i=1}^{N}E\left[\left(X_{i,T}+\underline{Y}_{i,T}\right)^4\right]=0. \tag{40}$$

To show this, note first that by Loève's c_r inequality, we have

$$\begin{aligned}&\frac{1}{N^2}\sum_{i=1}^{N}E\left[\left(X_{i,T}+\underline{Y}_{i,T}\right)^4\right]\\ \leq\ &\frac{1}{N^2}\sum_{i=1}^{N}8\left\{E\left[\left(-\frac{1}{\sqrt{T}}\sum_{t=4}^{T}\varepsilon_{it-2}\varepsilon_{it-1}\right)^4\right]+E\left[\left((1-\rho_T)\frac{1}{\sqrt{T}}\sum_{t=4}^{T}\sum_{j=1}^{t-3}\rho_T^{(t-3-j)}\varepsilon_{ij}\varepsilon_{it-1}\right)^4\right]\right\}\\ =\ &8\frac{1}{N^2T^2}\sum_{i=1}^{N}E\left[\left(\sum_{t=4}^{T}\varepsilon_{it-2}\varepsilon_{it-1}\right)^4\right]+8\frac{(1-\rho_T)^4}{N^2T^2}\sum_{i=1}^{N}E\left[\left(\sum_{t=4}^{T}w_{it-3}\varepsilon_{it-1}\right)^4\right].\end{aligned}$$

From previous calculations, we have that

$$\begin{aligned}&E\left[\left(\sum_{t=4}^{T}\varepsilon_{it-2}\varepsilon_{it-1}\right)^4\right]=6\sigma^8\frac{(T-5)(T-4)}{2}+\left(E\left[\varepsilon_{it-1}^4\right]\right)^2(T-3)+6E\left[\varepsilon_{it-1}^4\right]\sigma^4(T-4)\\ =\ &3\sigma^8T^2\left[1+O\left(\frac{1}{T}\right)\right].\end{aligned}$$

Moreover, in light of Assumption 1, there exists a positive constant C such that

$$
\begin{aligned}
& E\left[\left(\sum_{t=4}^{T}\underline{w}_{it-3,T}\varepsilon_{it-1}\right)^4\right] \\
= & \sum_{g=4}^{T}\sum_{s=4}^{T}\sum_{t=4}^{T}\sum_{u=4}^{T}E\left[\underline{w}_{ig-3,T}\underline{w}_{is-3,T}\underline{w}_{it-3,T}\underline{w}_{iu-3,T}\varepsilon_{ig-1}\varepsilon_{is-1}\varepsilon_{it-1}\varepsilon_{iu-1}\right] \\
\leq & C\left(\sum_{t=4}^{T}\sum_{j=1}^{t-3}\rho_T^{4(t-3-j)}+\sum_{t=6}^{T}\sum_{s=4}^{t-2}\sum_{j=1}^{s-3}\left|\rho_T\right|^{2(t-3-[s-1])}\left|\rho_T\right|^{(t-3-j)}\left|\rho_T\right|^{(s-3-j)}\right. \\
& +\sum_{t=6}^{T}\sum_{s=4}^{t-2}\sum_{j=1}^{s-3}\left|\rho_T\right|^{(t-3-[s-1])}\left|\rho_T\right|^{2(t-3-j)}\left|\rho_T\right|^{(s-3-j)}+\sum_{t=6}^{T}\sum_{s=4}^{t-2}\sum_{j=1}^{s-3}\rho_T^{2(t-3-[s-1])}\rho_T^{2(s-3-j)} \\
& +\sum_{t=5}^{T}\sum_{s=4}^{t-1}\sum_{j=1}^{t-3}\sum_{k=1}^{s-3}\rho_T^{2(t-3-j)}\rho_T^{2(s-3-k)}+\sum_{t=5}^{T}\sum_{s=4}^{t-1}\left[\sum_{j=1}^{s-3}\left|\rho_T\right|^{(t-3-j)}\left|\rho_T\right|^{(s-3-j)}\right]^2 \\
& \left.+\sum_{t=6}^{T}\sum_{s=4}^{t-2}\sum_{j=1}^{s-3}\left|\rho_T\right|^{(t-3-[s-1])}\left|\rho_T\right|^{(t-3-j)}\left|\rho_T\right|^{2(s-3-j)}\right) \\
\leq & C\left(\sum_{t=4}^{T}\sum_{j=1}^{t-3}\rho_T^{4(t-3-j)}+\sum_{t=6}^{T}\sum_{s=4}^{t-2}\sum_{j=1}^{s-3}\left|\rho_T\right|^{2(t-3-[s-1])}\left|\rho_T\right|^{(t-s)}\left|\rho_T\right|^{2(s-3-j)}\right. \\
& +\sum_{t=6}^{T}\sum_{s=4}^{t-2}\sum_{j=1}^{s-3}\rho_T^{2(t-3-[s-1])}\rho_T^{2(s-3-j)}+\sum_{t=6}^{T}\sum_{s=4}^{t-2}\sum_{j=1}^{s-3}\left|\rho_T\right|^{(t-3-[s-1])}\left|\rho_T\right|^{(t-3-j)}\left|\rho_T\right|^{(t-s)}\left|\rho_T\right|^{2(s-3-j)} \\
& +\sum_{t=6}^{T}\sum_{s=4}^{t-2}\sum_{j=1}^{s-3}\left|\rho_T\right|^{(t-3-[s-1])}\left|\rho_T\right|^{(t-3-j)}\left|\rho_T\right|^{2(s-3-j)} \\
& \left.+\sum_{t=5}^{T}\sum_{s=4}^{t-1}\left[\sum_{j=1}^{s-3}\left|\rho_T\right|^{2(t-3-j)}\right]\left[\sum_{j=1}^{s-3}\left|\rho_T\right|^{2(s-3-j)}\right]+\sum_{t=5}^{T}\sum_{s=4}^{t-1}\sum_{j=1}^{t-3}\sum_{k=1}^{s-3}\rho_T^{2(t-3-j)}\rho_T^{2(s-3-k)}\right) \\
\leq & C\left(\sum_{t=4}^{T}\sum_{j=1}^{t-3}\rho_T^{4(t-3-j)}+2\sum_{t=6}^{T}\sum_{s=4}^{t-2}\sum_{j=1}^{s-3}\rho_T^{2(t-3-[s-1])}\rho_T^{2(s-3-j)}\right. \\
& \left.+2\sum_{t=6}^{T}\sum_{s=4}^{t-2}\sum_{j=1}^{s-3}\left|\rho_T\right|^{2(t-3-[s-1])}\left|\rho_T\right|^{(s-1-j)}\left|\rho_T\right|^{2(s-3-j)}+2\sum_{t=5}^{T}\sum_{s=4}^{t-1}\sum_{j=1}^{t-3}\sum_{k=1}^{s-3}\rho_T^{2(t-3-j)}\rho_T^{2(s-3-k)}\right) \\
\leq & C\left(\sum_{t=4}^{T}\sum_{j=1}^{t-3}\rho_T^{4(t-3-j)}+4\sum_{t=6}^{T}\sum_{s=4}^{t-2}\sum_{j=1}^{s-3}\rho_T^{2(t-3-[s-1])}\rho_T^{2(s-3-j)}+2\sum_{t=5}^{T}\sum_{s=4}^{t-1}\sum_{j=1}^{t-3}\sum_{k=1}^{s-3}\rho_T^{2(t-3-j)}\rho_T^{2(s-3-k)}\right) \\
= & C\left(I+II+III\right),\ (say).
\end{aligned}
$$

Now, using the bound (39), we obtain for all $T \geq T^*$

(i)

$$
\begin{aligned}
I &= \sum_{t=4}^{T}\sum_{j=1}^{t-3}\rho_T^{4(t-3-j)} = \sum_{t=4}^{T}\frac{1-\rho_T^{4(t-3)}}{1-\rho_T^4} = \frac{1}{1-\rho_T^4}\left[T-3-\frac{\rho_T^4\left(1-\rho_T^{4(T-3)}\right)}{1-\rho_T^4}\right] \\
&\leq \frac{T}{1-\exp\{-4/C_q\}}\left[1+\frac{3}{T}+\frac{1}{T\left(1-\exp\{-4/C_q\}\right)}\right] = O\left(T\right).
\end{aligned}
$$

(ii)

$$
\begin{aligned}
II &= 4\sum_{t=6}^{T}\sum_{s=4}^{t-2}\sum_{j=1}^{s-3}\rho_T^{2(t-3-[s-1])}\rho_T^{2(s-3-j)} \\
&= 4\sum_{t=6}^{T}\sum_{s=4}^{t-2}\rho_T^{2(t-3-[s-1])}\frac{1-\rho_T^{2(s-3)}}{1-\rho_T^2} \\
&= 4\frac{1}{1-\rho_T^2}\sum_{t=6}^{T}\sum_{s=4}^{t-2}\rho_T^{2(t-2-s)} - 4\frac{\rho_T^2}{1-\rho_T^2}\sum_{t=6}^{T}\sum_{s=4}^{t-2}\rho_T^{2(t-6)} \\
&= 4\frac{\rho_T^2}{\left(1-\rho_T^2\right)^2}\sum_{t=6}^{T}\left(1-\rho_T^{2(t-6)}\right) - 4\frac{\rho_T^2}{1-\rho_T^2}\sum_{t=6}^{T}\rho_T^{2(t-6)}\left(t-5\right) \\
&= 4\frac{\rho_T^2\left(T-5\right)}{\left(1-\rho_T^2\right)^2} - 4\frac{\rho_T^2\left(1-\rho_T^{2(T-5)}\right)}{\left(1-\rho_T^2\right)^3} - 4\frac{\rho_T^2}{1-\rho_T^2}\sum_{t=6}^{T}\rho_T^{2(t-6)} - 4\frac{\rho_T^2}{1-\rho_T^2}\sum_{t=6}^{T}\rho_T^{2(t-6)}\left(t-6\right).
\end{aligned}
$$

Applying part (b) of Lemma SE-5 and performing additional calculation, we get

$$
\begin{aligned}
II &= 4\frac{\rho_T^2\left(T-5\right)}{\left(1-\rho_T^2\right)^2} - 4\frac{\rho_T^2\left(1-\rho_T^{2(T-5)}\right)}{\left(1-\rho_T^2\right)^3} + 4\frac{\rho_T^2\left(1-\rho_T^{2(T-5)}\right)}{\left(1-\rho_T^2\right)^2} \\
&\quad -4\frac{\rho_T^2\left(\rho_T^2-\left(T-5\right)\rho_T^{2(T-5)}+\left(T-6\right)\rho_T^{2(T-4)}\right)}{\left(1-\rho_T^2\right)^3} \\
&= 4\frac{\rho_T^2 T}{\left(1-\rho_T^2\right)^2}\left[1+\frac{\rho_T^{2(T-5)}}{\left(1-\rho_T^2\right)}-\frac{\rho_T^{2(T-4)}}{\left(1-\rho_T^2\right)}\right] - \frac{20\rho_T^2}{\left(1-\rho_T^2\right)^2} - 4\frac{\rho_T^2\left(1-\rho_T^{2(T-5)}\right)}{\left(1-\rho_T^2\right)^3} \\
&\quad +4\frac{\rho_T^2\left(1-\rho_T^{2(T-5)}\right)}{\left(1-\rho_T^2\right)^2} - 4\frac{\rho_T^4\left[1+5\rho_T^{2(T-6)}-6\rho_T^{2(T-5)}\right]}{\left(1-\rho_T^2\right)^3} \\
&\leq \frac{4T}{\left(1-\exp\{-2/C_q\}\right)^2}\left[1+\frac{2}{1-\exp\{-2/C_q\}}\right] + \frac{20}{\left(1-\exp\{-2/C_q\}\right)^2} \\
&\quad +\frac{4}{\left(1-\exp\{-2/C_q\}\right)^3} + \frac{4}{\left(1-\exp\{-2/C_q\}\right)^2} + \frac{48}{\left(1-\exp\{-2/C_q\}\right)^3} \\
&= O\left(T\right).
\end{aligned}
$$

(iii)

$$
\begin{aligned}
III &= 2\sum_{t=5}^{T}\sum_{s=4}^{t-1}\sum_{j=1}^{t-3}\sum_{k=1}^{s-3}\rho_T^{2(t-3-j)}\rho_T^{2(s-3-k)} \\
&= \frac{2}{\left(1-\rho_T^2\right)^2}\sum_{t=5}^{T}\left(1-\rho_T^{2(t-3)}\right)\sum_{s=4}^{t-1}\left(1-\rho_T^{2(s-3)}\right) \\
&= \frac{2}{\left(1-\rho_T^2\right)^2}\sum_{t=5}^{T}\left(1-\rho_T^{2(t-3)}\right)\left[(t-4)-\frac{\rho_T^2\left(1-\rho_T^{2(t-4)}\right)}{\left(1-\rho_T^2\right)}\right] \\
&= \frac{2}{\left(1-\rho_T^2\right)^2}\left[\sum_{t=5}^{T}(t-4)-\rho_T^4\sum_{t=5}^{T}\rho_T^{2(t-5)}(t-5)-\rho_T^4\sum_{t=5}^{T}\rho_T^{2(t-5)}-\frac{\rho_T^2}{\left(1-\rho_T^2\right)}(T-4)\right. \\
&\quad \left.+\frac{\rho_T^6}{\left(1-\rho_T^2\right)}\sum_{t=5}^{T}\rho_T^{2(t-5)}+\frac{\rho_T^4}{\left(1-\rho_T^2\right)}\sum_{t=5}^{T}\rho_T^{2(t-5)}-\frac{\rho_T^8}{\left(1-\rho_T^2\right)}\sum_{t=5}^{T}\rho_T^{4(t-5)}\right] \\
&= \frac{2}{\left(1-\rho_T^2\right)^2}\left[\frac{(T-4)(T-3)}{2}-\frac{\rho_T^4\left(\rho_T^2-(T-4)\rho_T^{2(T-4)}+(T-5)\rho_T^{2(T-3)}\right)}{\left(1-\rho_T^2\right)^2}\right. \\
&\quad -\frac{\rho_T^4\left(1-\rho_T^{2(T-4)}\right)}{\left(1-\rho_T^2\right)}-\frac{\rho_T^2(T-4)}{\left(1-\rho_T^2\right)}+\frac{\rho_T^6\left(1-\rho_T^{2(T-4)}\right)}{\left(1-\rho_T^2\right)^2}+\frac{\rho_T^4\left(1-\rho_T^{2(T-4)}\right)}{\left(1-\rho_T^2\right)^2} \\
&\quad \left.-\frac{\rho_T^8\left(1-\rho_T^{4(T-4)}\right)}{\left(1-\rho_T^2\right)\left(1-\rho_T^4\right)}\right] \\
&= \frac{2T^2}{\left(1-\rho_T^2\right)^2}\left[\frac{1}{2}-\frac{7}{2T}-\frac{\rho_T^2}{T\left(1-\rho_T^2\right)}+\frac{6}{T^2}-\frac{\rho_T^{2(T-2)}}{T}-\frac{\rho_T^6+4\rho_T^{2(T-2)}-5\rho_T^{2(T-1)}}{T^2\left(1-\rho_T^2\right)^2}\right. \\
&\quad -\frac{\rho_T^4\left(1-\rho_T^{2(T-4)}\right)}{T^2\left(1-\rho_T^2\right)}+\frac{4\rho_T^2}{T^2\left(1-\rho_T^2\right)}+\frac{\rho_T^6\left(1-\rho_T^{2(T-4)}\right)}{T^2\left(1-\rho_T^2\right)^2}+\frac{\rho_T^4\left(1-\rho_T^{2(T-4)}\right)}{T^2\left(1-\rho_T^2\right)^2} \\
&\quad \left.-\frac{\rho_T^8\left(1-\rho_T^{4(T-4)}\right)}{T^2\left(1-\rho_T^2\right)\left(1-\rho_T^4\right)}\right] \\
&\leq \frac{T^2}{\left(1-\exp\{-2/C_q\}\right)^2}\left[1+\frac{9}{T}+\frac{2}{T\left(1-\exp\{-2/C_q\}\right)}+\frac{12}{T^2}+\frac{10}{T^2\left(1-\exp\{-2/C_q\}\right)}\right. \\
&\quad \left.+\frac{24}{T^2\left(1-\exp\{-2/C_q\}\right)^2}+\frac{2}{T^2\left(1-\exp\{-2/C_q\}\right)\left(1-\exp\{-4/C_q\}\right)}\right] \\
&= O\left(T^2\right),
\end{aligned}
$$

where the fifth equality above follows in part from applying part (b) of Lemma SE-5.

It follows from the results given for expressions $I - III$ that

$$\sum_{i=1}^{N} E\left[\left(\sum_{t=4}^{T} \underline{w}_{it-3,T}\varepsilon_{it-1}\right)^4\right] = O\left(NT^2\right).$$

Hence,

$$\begin{aligned}
&\frac{1}{N^2}\sum_{i=1}^{N} E\left[\left(X_{i,T} + \underline{Y}_{i,T}\right)^4\right] \\
= \;& 8\frac{1}{N^2T^2}\sum_{i=1}^{N} E\left[\left(\sum_{t=4}^{T}\varepsilon_{it-2}\varepsilon_{it-1}\right)^4\right] + 8\frac{(1-\rho_T)^4}{N^2T^2}\sum_{i=1}^{N} E\left[\left(\sum_{t=4}^{T}\underline{w}_{it-3}\varepsilon_{it-1}\right)^4\right] \\
= \;& O\left(\frac{1}{N^2T^2}\right)O\left(NT^2\right) + O\left(\frac{1}{N^2T^2}\right)O\left(NT^2\right) \\
= \;& O\left(\frac{1}{N}\right),
\end{aligned}$$

so the the Liapunov-type condition (40) is satisfied. Now, since the Liapounov-type condition (40) implies the Lindeberg-type condition (13) given in Lemma SE-10 above, it follows from Lemma SE-10 that

$$U_{N,T} = \frac{1}{\omega_{N,T}}\sum_{i=1}^{N}\left(X_{i,T} + \underline{Y}_{i,T}\right) \Rightarrow N(0,1).$$

Moreover, note that

$$\frac{1}{\omega_{N,T}}\sum_{i=1}^{N}\left(X_{i,T} + \underline{Y}_{i,T}\right) = \sqrt{\frac{1+\rho_T}{2\sigma^4}}\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(X_{i,T} + \underline{Y}_{i,T}\right) + O\left(\frac{1}{T}\right).$$

This implies that

$$\sqrt{\frac{1+\rho_T}{2\sigma^4}}\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(X_{i,T} + \underline{Y}_{i,T}\right) = \frac{1}{\omega_{N,T}}\sum_{i=1}^{N}\left(X_{i,T} + \underline{Y}_{i,T}\right) + O\left(\frac{1}{T}\right) \Rightarrow N(0,1),$$

as required. □

Lemma SE-24:

Suppose that Assumptions 1 and 4 hold. If $\rho_T = 1$ for all T sufficiently large, then, as $N, T \to \infty$,

$$\frac{1}{\sqrt{2}\sigma^2\sqrt{N}}\sum_{i=1}^{N}\left(X_{i,T} + Y_{i,T}\right) \Rightarrow N(0,1),$$

where

$$X_{i,T} = -\frac{1}{\sqrt{T}}\sum_{t=4}^{T}\varepsilon_{it-2}\varepsilon_{it-1},\; Y_{i,T} = \frac{1}{\sqrt{T}}w_{iT-2,T}\varepsilon_{iT}.$$

Proof of Lemma SE-24:

To proceed, first decompose $Y_{i,T}$ as follows:

$$\begin{aligned} Y_{i,T} &= \frac{1}{\sqrt{T}} w_{iT-2,T}\varepsilon_{iT} = \frac{1}{\sqrt{T}} \underline{w}_{iT-2,T}\varepsilon_{iT} + \frac{1}{\sqrt{T}}\varepsilon_{iT}\rho_T^{T-2} w_{i0} \\ &= \underline{Y}_{i,T} + \frac{1}{\sqrt{T}}\varepsilon_{iT}\rho_T^{T-2} w_{i0}, \end{aligned}$$

where $\underline{w}_{iT-2} = \sum_{j=1}^{T-2} \rho_T^{(T-2-j)}\varepsilon_{ij}$ and $\underline{Y}_{i,T} = \frac{1}{\sqrt{T}}\underline{w}_{iT-2,T}\varepsilon_{iT}$. Let $\underline{U}_{N,T} = \sum_{i=1}^{N}\left(X_{i,T} + \underline{Y}_{i,T}\right)$, as before. Now, as shown in the proof of Lemma SE-22, $E\left[X_{i,T}\right] = 0$ and

$$E\left[X_{i,T}^2\right] = \sigma^4\left(\frac{T-3}{T}\right) = \sigma^4 + O\left(\frac{1}{T}\right).$$

Moreover, given the assumption here, there exists a positive integer I_ρ such that for all $T \geq I_\rho$, the triangular array process $\{w_{it,T}\}$ has the partial sum representation $w_{it,T} = \sum_{j=1}^{t}\varepsilon_{ij}$. Hence, for all $T \geq I_\rho$, we can make the following moment calculations:

$$E\left[\underline{Y}_{i,T}\right] = \frac{1}{\sqrt{T}}\sum_{j=1}^{T-2} E\left[\varepsilon_{ij}\varepsilon_{iT}\right] = 0,$$

$$E\left[\underline{Y}_{i,T}^2\right] = \frac{1}{T}\sum_{j=1}^{T-2}\sum_{k=1}^{T-2} E\left[\varepsilon_{ij}\varepsilon_{ik}\varepsilon_{iT}\varepsilon_{iT}\right] = \frac{1}{T}\sum_{j=1}^{T-2} E\left[\varepsilon_{it}^2\right] E\left[\varepsilon_{iT}^2\right] = \sigma^4\frac{T-2}{T},$$

and

$$E\left[X_{i,T}\underline{Y}_{i,T}\right] = -\frac{1}{T}\sum_{t=4}^{T}\sum_{\ell=1}^{T-2} E\left[\varepsilon_{it-2}\varepsilon_{it-1}\varepsilon_{i\ell}\varepsilon_{iT}\right] = -\frac{1}{T}\sum_{t=4}^{T}\sum_{\ell=1}^{T-2} E\left[\varepsilon_{it-2}\varepsilon_{it-1}\varepsilon_{i\ell}\right] E\left[\varepsilon_{iT}\right] = 0.$$

It follows from these calculations that

$$E\left[\underline{U}_{N,T}\right] = \sum_{i=1}^{N}\left(E\left[X_{i,T}\right] + E\left[\underline{Y}_{i,T}\right]\right) = 0,$$

and

$$\begin{aligned} \frac{\omega_{N,T}^2}{N} &= \frac{1}{N}\sum_{i=1}^{N}\omega_{i,T}^2 = \frac{1}{N}\sum_{i=1}^{N} E\left[\left(X_{i,T} + \underline{Y}_{i,T}\right)^2\right] \\ &= \frac{1}{N}\sum_{i=1}^{N} E\left[X_{i,T}^2\right] + \frac{1}{N}\sum_{i=1}^{N} E\left[\underline{Y}_{i,T}^2\right] + 2\frac{1}{N}\sum_{i=1}^{N} E\left[X_{i,T}\underline{Y}_{i,T}\right] \\ &= \frac{\sigma^4}{N} N\left[\left(\frac{T-3}{T}\right) + \frac{T-2}{T}\right] \\ &= 2\sigma^4 + O\left(\frac{1}{T}\right). \end{aligned}$$

Given Assumption 1, it follows that, in this case, there exists a positive constant C such that

$$0 < \frac{1}{C} \leq \frac{\omega_{N,T}}{\sqrt{N}} \leq C < \infty \text{ eventually as } N, T \to \infty.$$

Next, note that, for all T sufficiently large,

$$\begin{aligned}\frac{1}{\omega_{N,T}/\sqrt{N}}\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(X_{i,T}+Y_{i,T}\right) &= \frac{1}{\omega_{N,T}/\sqrt{N}}\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(X_{i,T}+\underline{Y}_{i,T}\right)+\frac{1}{\omega_{N,T}/\sqrt{N}}\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\varepsilon_{iT}w_{i0}\\ &= \frac{1}{\omega_{N,T}/\sqrt{N}}\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(X_{i,T}+\underline{Y}_{i,T}\right)+o_p\left(1\right).\end{aligned}$$

where the second equality follows from the result given in expression (24) above. Hence, showing the desired result is equivalent to showing the asymptotic normality of

$$U_{N,T}=\frac{1}{\omega_{N,T}/\sqrt{N}}\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(X_{i,T}+\underline{Y}_{i,T}\right).$$

To show the asymptotic normality of $U_{N,T}$, it suffices to verify a Liapounov-type condition of the form

$$\lim_{N,T\to\infty}\frac{1}{N^2}\sum_{i=1}^{N}E\left[\left(X_{i,T}+\underline{Y}_{i,T}\right)^4\right]=0. \tag{41}$$

To show this, note first that by Loève's c_r inequality, we have

$$\begin{aligned}&\frac{1}{N^2}\sum_{i=1}^{N}E\left[\left(X_{i,T}+\underline{Y}_{i,T}\right)^4\right]\\ \leq\ & 8\frac{1}{N^2T^2}\sum_{i=1}^{N}E\left[\left(\sum_{t=4}^{T}\varepsilon_{it-2}\varepsilon_{it-1}\right)^4\right]+8\frac{1}{N^2T^2}\sum_{i=1}^{N}E\left[\left(\sum_{j=1}^{T-2}\varepsilon_{ij}\varepsilon_{iT}\right)^4\right].\end{aligned} \tag{42}$$

From the proof of Lemma SE-22, we have

$$\begin{aligned}&E\left[\left(\sum_{t=4}^{T}\varepsilon_{it-2}\varepsilon_{it-1}\right)^4\right]\\ =\ & 6\sigma^8\frac{(T-5)(T-4)}{2}+\left(E\left[\varepsilon_{it-1}^4\right]\right)^2(T-3)+6E\left[\varepsilon_{it-1}^4\right]\sigma^4(T-4)\\ =\ & 3\sigma^8T^2\left[1+O\left(\frac{1}{T}\right)\right].\end{aligned} \tag{43}$$

In addition, note that, under Assumption 1,

$$\begin{aligned}E\left[\left(\sum_{j=1}^{T-2}\varepsilon_{ij}\varepsilon_{iT}\right)^4\right] &= \sum_{g=1}^{T-2}\sum_{s=1}^{T-2}\sum_{t=1}^{T-2}\sum_{u=1}^{T-2}E\left[\varepsilon_{ig}\varepsilon_{is}\varepsilon_{it}\varepsilon_{iu}\right]E\left[\varepsilon_{iT}^4\right]\\ &= E\left[\varepsilon_{is}^4\right]E\left[\varepsilon_{iT}^4\right](T-2)+6\sigma^4E\left[\varepsilon_{iT}^4\right]\sum_{g=2}^{T-2}(g-1)\\ &= E\left[\varepsilon_{is}^4\right]E\left[\varepsilon_{iT}^4\right](T-2)+3\sigma^4E\left[\varepsilon_{iT}^4\right](T-3)(T-2)\\ &= 3\sigma^4E\left[\varepsilon_{iT}^4\right]T^2\left[1+O\left(\frac{1}{T}\right)\right].\end{aligned} \tag{44}$$

Now, applying (43) and (44) to the upper bound in (42), we get

$$\begin{aligned}
\frac{1}{N^2}\sum_{i=1}^{N} E\left[\left(X_{i,T}+\underline{Y}_{i,T}\right)^4\right] &\leq 8\frac{1}{N^2T^2}\sum_{i=1}^{N} 3\sigma^8 T^2\left[1+O\left(\frac{1}{T}\right)\right] \\
&\quad +8\frac{1}{N^2T^2}\sum_{i=1}^{N} 3\sigma^4 E\left[\varepsilon_{iT}^4\right] T^2\left[1+O\left(\frac{1}{T}\right)\right] \\
&= 24\sigma^8\frac{1}{N}+24\sigma^4 E\left[\varepsilon_{iT}^4\right]\frac{1}{N}+O\left(\frac{1}{NT}\right). \\
&= O\left(N^{-1}\right).
\end{aligned}$$

Since the Liapounov-type condition (41) implies the Lindeberg-type condition (13) given in Lemma SE-10 above, it follows from Lemma SE-10 that

$$\begin{aligned}
U_{N,T} &= \frac{1}{\omega_{N,T}}\sum_{i=1}^{N}\left(X_{i,T}+\underline{Y}_{i,T}\right) \\
&= \frac{1}{\sqrt{2}\sigma^2}\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(X_{i,T}+\underline{Y}_{i,T}\right)+O\left(\frac{1}{T}\right) \\
&\Rightarrow N\left(0,1\right),
\end{aligned}$$

as required. □

Lemma SE-25:

Let d be a non-negative integer and let g be an integer such that $g \geq 2$. Suppose further that $g > d$. Under Assumptions 1 and 4, the following statements are true as as $N, T \to \infty$.

(a) If $\rho_T = 1$ for all T sufficiently large, then

$$\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=g+1}^{T} w_{it-g,T}\varepsilon_{it-d} = \frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=g+1}^{T} \underline{w}_{it-g,T}\varepsilon_{it-d} + O_p\left(\frac{1}{\sqrt{T}}\right) = O_p\left(1\right).$$

(b) If $\rho_T = \exp\left\{-1/q\left(T\right)\right\}$ such that $T/q\left(T\right) \to 0$, then

$$\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=g+1}^{T} w_{it-g,T}\varepsilon_{it-d} = \frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=g+1}^{T} \underline{w}_{it-g,T}\varepsilon_{it-d} + O_p\left(\frac{1}{\sqrt{T}}\right) = O_p\left(1\right).$$

and

$$\frac{\left(1-\rho_T\right)}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=g+1}^{T} w_{it-g,T}\varepsilon_{it-d} = O_p\left(\frac{1}{q\left(T\right)}\right).$$

(c) If $\rho_T = \exp\left\{-1/q\left(T\right)\right\}$ such that $q\left(T\right) \sim T$, then

$$\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=g+1}^{T} w_{it-g,T}\varepsilon_{it-d} = \frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=g+1}^{T} \underline{w}_{it-g,T}\varepsilon_{it-d} + O_p\left(\frac{1}{\sqrt{T}}\right) = O_p\left(1\right).$$

and

$$\frac{(1-\rho_T)}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=g+1}^{T} w_{it-g,T}\varepsilon_{it-d} = O_p\left(\frac{1}{T}\right).$$

(d) If $\rho_T = \exp\{-1/q(T)\}$ such that $q(T)\to\infty$ but $q(T)/T\to 0$, then

$$\frac{1}{\sqrt{NTq(T)}}\sum_{i=1}^{N}\sum_{t=g+1}^{T} w_{it-g,T}\varepsilon_{it-d} = \frac{1}{\sqrt{NTq(T)}}\sum_{i=1}^{N}\sum_{t=g+1}^{T} \underline{w}_{it-g,T}\varepsilon_{it-d} + O_p\left(\frac{1}{\sqrt{T}}\right) = O_p(1),$$

and

$$\frac{(1-\rho_T)}{\sqrt{NTq(T)}}\sum_{i=1}^{N}\sum_{t=g+1}^{T} w_{it-g,T}\varepsilon_{it-d} = O_p\left(\frac{1}{q(T)}\right).$$

(e) If $\rho_T \in \mathcal{G}_{\mathrm{St}} = \left\{|\rho_T| = \exp\left\{-\frac{1}{q(T)}\right\} : q(T)\geq 0 \text{ and } q(T) = O(1) \text{ as } T\to\infty\right\}$, then

$$\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=g+1}^{T} w_{it-g,T}\varepsilon_{it-d} = \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=g+1}^{T} \underline{w}_{it-g,T}\varepsilon_{it-d} + O_p\left(\frac{1}{\sqrt{T}}\right) = O_p(1),$$

and

$$\frac{(1-\rho_T)}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=g+1}^{T} w_{it-g,T}\varepsilon_{it-d} = O_p(1).$$

Proof of Lemma SE-25:

To proceed, first write

$$\sum_{i=1}^{N}\sum_{t=g+1}^{T} w_{it-g,T}\varepsilon_{it-d} = \sum_{i=1}^{N}\sum_{t=g+1}^{T} \underline{w}_{it-g,T}\varepsilon_{it-d} + \sum_{i=1}^{N}\sum_{t=g+1}^{T} \varepsilon_{it-d}\rho_T^{t-g} w_{i0},$$

where $\underline{w}_{it-g,T} = \sum_{j=1}^{t-g}\rho_T^{(t-g-j)}\varepsilon_{ij}$.

Consider first part (a). Note that, under the assumption here, there exists a positive integer I_ρ such that for all $T\geq I_\rho$, the triangular array process $\left\{\underline{w}_{it-g,T}\right\}$ has the partial sum representation $\underline{w}_{it-g,T} = \sum_{j=1}^{t}\varepsilon_{ij}$. Hence, for all $T\geq \max\{I_\rho, g+1\}$, we obtain by direct calculation

$$\begin{aligned}
E\left[\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\underline{w}_{it-g,T}\varepsilon_{it-d}\right]^2 &= \frac{1}{NT^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=g+1}^{T}\sum_{s=g+1}^{T}\sum_{k=1}^{t-g}\sum_{\ell=1}^{s-g} E\left[\varepsilon_{ik}\varepsilon_{j\ell}\varepsilon_{it-d}\varepsilon_{js-d}\right] \\
&= \frac{\sigma^4}{NT^2}\sum_{i=1}^{N}\sum_{t=g+1}^{T}(t-g) \\
&= \frac{\sigma^4}{2}\frac{N(T-g)(T-g+1)}{NT^2} \\
&= \frac{\sigma^4}{2}\frac{NT^2}{NT^2}\left[1+O\left(\frac{1}{T}\right)\right] = O(1),
\end{aligned}$$

so that by Markov's inequality $N^{-1/2}T^{-1}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\underline{w}_{it-g,T}\varepsilon_{it-d}=O_p(1)$.

Moreover, by Assumptions 1 and 4,

$$\begin{aligned}
E\left[\left(\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\varepsilon_{it-d}w_{i0}\right)^2\right] &= \frac{1}{NT^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=g+1}^{T}\sum_{s=g+1}^{T}E\left[w_{i0}w_{j0}\right]E\left[\varepsilon_{it-d}\varepsilon_{js-d}\right] \\
&= \frac{\sigma^2}{NT^2}\sum_{i=1}^{N}E\left[w_{i0}^2\right](T-g) \\
&\leq \frac{\sigma^2}{T}\left(\sup_i E\left[w_{i0}^2\right]\right)\frac{(T-g)}{T} \\
&= O\left(T^{-1}\right),
\end{aligned}$$

from which it follows, by Markov's inequality, that

$$\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\varepsilon_{it-d}w_{i0}=O_p\left(\frac{1}{\sqrt{T}}\right).$$

Hence,

$$\begin{aligned}
\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=g+1}^{T}w_{it-g,T}\varepsilon_{it-d} &= \frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\underline{w}_{it-g,T}\varepsilon_{it-d}+\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\varepsilon_{it-d}w_{i0} \\
&\quad \text{(for all } T \text{ sufficiently large)} \\
&= \frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\underline{w}_{it-g,T}\varepsilon_{it-d}+O_p\left(\frac{1}{\sqrt{T}}\right)=O_p(1).
\end{aligned}$$

Now, to show parts (b)-(d), we first write

$$\begin{aligned}
&E\left[\sum_{i=1}^{N}\sum_{t=g+1}^{T}\underline{w}_{it-g,T}\varepsilon_{it-d}\right]^2 \\
&= \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=g+1}^{T}\sum_{s=g+1}^{T}\sum_{k=1}^{t-g}\sum_{\ell=1}^{s-g}\exp\left\{-\frac{t-g-k}{q(T)}\right\}\exp\left\{-\frac{s-g-\ell}{q(T)}\right\}E\left[\varepsilon_{ik}\varepsilon_{j\ell}\varepsilon_{it-d}\varepsilon_{js-d}\right] \\
&= \sigma^4\sum_{i=1}^{N}\sum_{t=g+1}^{T}\sum_{k=1}^{t-g}\exp\left\{-2\frac{t-g-k}{q(T)}\right\}
\end{aligned}$$

for $T\geq g+1$.

Next, consider part (b), where we take $\rho_T=\exp\{-1/q(T)\}$ such that $T/q(T)\to 0$. In this case, we apply part (a) of Lemma SE-1 with $b=g+1$ and $d=2$ to obtain

$$\begin{aligned}
E\left[\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\underline{w}_{it-g,T}\varepsilon_{it-d}\right]^2 &= \sigma^4\frac{N}{NT^2}\sum_{t=g+1}^{T}\sum_{k=1}^{t-g}\exp\left\{-2\frac{t-g-k}{q(T)}\right\} \\
&= \frac{NT^2}{NT^2}\frac{\sigma^4}{2}\left[1+O_p\left(\max\left\{\frac{T}{q(T)},\frac{1}{T}\right\}\right)\right]=O(1),
\end{aligned}$$

so that, by Markov's inequality, we obtain $N^{-1/2}T^{-1}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\underline{w}_{it-g,T}\varepsilon_{it-d}=O_p(1)$.

Moreover, by Assumptions 1 and 4,

$$\begin{aligned}
& E\left[\left(\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\varepsilon_{it-d}\rho_T^{t-g}w_{i0}\right)^2\right] \\
= & \frac{1}{NT^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=g+1}^{T}\sum_{s=g+1}^{T}\rho_T^{t-g}\rho_T^{s-g}E\left[w_{i0}w_{j0}\right]E\left[\varepsilon_{it-d}\varepsilon_{js-d}\right] \\
= & \frac{\sigma^2}{NT^2}\sum_{i=1}^{N}E\left[w_{i0}^2\right]\sum_{t=g+1}^{T}\rho_T^{2(t-g)} \\
= & \frac{\sigma^2}{T^2}\sup_i E\left[w_{i0}^2\right]\exp\left\{-\frac{2}{q(T)}\right\}\left[1-\exp\left\{-\frac{2}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{2(T-g)}{q(T)}\right\}\right] \\
= & O\left(T^{-2}\right)\times O(1)\times O(1)\times O(q(T))\times O(T/q(T))=O\left(T^{-1}\right),
\end{aligned}$$

from which it follows by Markov's inequality that

$$\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\varepsilon_{it-d}\rho_T^{t-g}w_{i0}=O_p\left(\frac{1}{\sqrt{T}}\right).$$

Hence,

$$\begin{aligned}
\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=g+1}^{T}w_{it-g,T}\varepsilon_{it-d} &= \frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\underline{w}_{it-g,T}\varepsilon_{it-d}+\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\varepsilon_{it-d}\rho_T^{t-g}w_{i0} \\
&= \frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\underline{w}_{it-g,T}\varepsilon_{it-d}+O_p\left(\frac{1}{\sqrt{T}}\right)=O_p(1).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\frac{(1-\rho_T)}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=g+1}^{T}w_{it-g,T}\varepsilon_{it-d} &= \left(1-\exp\left\{-\frac{1}{q(T)}\right\}\right)\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=g+1}^{T}w_{it-g,T}\varepsilon_{it-d} \\
&= \frac{1}{q(T)}\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=g+1}^{T}w_{it-g,T}\varepsilon_{it-d}\left[1+O_p\left(\frac{1}{q(T)}\right)\right] \\
&= O_p\left(\frac{1}{q(T)}\right).
\end{aligned}$$

Next, consider part (c), where we take $\rho_T=\exp\{-1/q(T)\}$ such that $q(T)\sim T$. Here, applying part (b) of Lemma SE-1 with $b=g+1$ and $d=2$, we get

$$\begin{aligned}
E\left[\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\underline{w}_{it-g,T}\varepsilon_{it-d}\right]^2 &= \sigma^4 N\frac{1}{NT^2}\sum_{t=g+1}^{T}\sum_{k=1}^{t-g}\exp\left\{-2\frac{t-g-k}{q(T)}\right\} \\
&= \frac{Nq(T)^2}{NT^2}\frac{\sigma^4}{4}\left[\exp\left\{-\frac{2T}{q(T)}\right\}+\frac{2T}{q(T)}-1\right]\left[1+O_p\left(\frac{1}{T}\right)\right] \\
&= O(1),
\end{aligned}$$

so that, using Markov's inequality, we again obtain $N^{-1/2}T^{-1}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\underline{w}_{it-g,T}\varepsilon_{it-d}=O_p(1)$.

Moreover, by Assumptions 1 and 4,

$$\begin{aligned}
& E\left[\left(\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\varepsilon_{it-d}\rho_T^{t-g}w_{i0}\right)^2\right] \\
= & \frac{1}{NT^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=g+1}^{T}\sum_{s=g+1}^{T}\rho_T^{t-g}\rho_T^{s-g}E\left[w_{i0}w_{j0}\right]E\left[\varepsilon_{it-d}\varepsilon_{js-d}\right] \\
= & \frac{\sigma^2}{NT^2}\sum_{i=1}^{N}E\left[w_{i0}^2\right]\sum_{t=g+1}^{T}\rho_T^{2(t-g)} \\
= & \frac{\sigma^2}{T^2}\left(\sup_i E\left[w_{i0}^2\right]\right)\exp\left\{-\frac{2}{q(T)}\right\}\left[1-\exp\left\{-\frac{2}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{2(T-g)}{q(T)}\right\}\right] \\
= & O\left(T^{-2}\right)O(1)O(1)O(T)O(1)=O\left(T^{-1}\right),
\end{aligned}$$

from which it follows, by Markov's inequality, that

$$\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\varepsilon_{it-d}\rho_T^{t-g}w_{i0}=O_p\left(\frac{1}{\sqrt{T}}\right).$$

Hence,

$$\begin{aligned}
\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=g+1}^{T}w_{it-g,T}\varepsilon_{it-d} &= \frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\underline{w}_{it-g,T}\varepsilon_{it-d}+\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\varepsilon_{it-d}\rho_T^{t-g}w_{i0} \\
&= \frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\underline{w}_{it-g,T}\varepsilon_{it-d}+O_p\left(\frac{1}{\sqrt{T}}\right)=O_p(1).
\end{aligned}$$

Furthermore, note that

$$\begin{aligned}
\frac{(1-\rho_T)}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=g+1}^{T}w_{it-g,T}\varepsilon_{it-d} &= \left(1-\exp\left\{-\frac{1}{q(T)}\right\}\right)\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=g+1}^{T}w_{it-g,T}\varepsilon_{it-d} \\
&= \frac{1}{q(T)}\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=g+1}^{T}w_{it-g,T}\varepsilon_{it-d}\left[1+O_p\left(\frac{1}{q(T)}\right)\right] \\
&= O_p\left(\frac{1}{T}\right).
\end{aligned}$$

Now, consider part (d), where we take $\rho_T=\exp\{-1/q(T)\}$ such that $q(T)\to\infty$ but $q(T)/T\to 0$. Here, applying part (c) of Lemma SE-1 with $b=g+1$ and $d=2$, we get

$$\begin{aligned}
E\left[\frac{1}{\sqrt{NTq(T)}}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\underline{w}_{it-g,T}\varepsilon_{it-d}\right]^2 &= \sigma^4\frac{N}{NTq(T)}\sum_{t=g+1}^{T}\sum_{k=1}^{t-g}\exp\left\{-2\frac{t-g-k}{q(T)}\right\} \\
&= \frac{NTq(T)}{NTq(T)}\frac{\sigma^4}{2}\left[1+O_p\left(\frac{q(T)}{T}\right)+O_p\left(\frac{1}{q(T)}\right)\right] \\
&= O(1),
\end{aligned}$$

so that, applying Markov's inequality, we obtain $N^{-1/2}T^{-1/2}q(T)^{-1/2}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\underline{w}_{it-g,T}\varepsilon_{it-d} = O_p(1)$.

Moreover, by Assumptions 1 and 4,

$$\begin{aligned}
& E\left[\left(\frac{1}{\sqrt{NTq(T)}}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\varepsilon_{it-d}\rho_T^{t-g}w_{i0}\right)^2\right] \\
= & \frac{1}{NTq(T)}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=g+1}^{T}\sum_{s=g+1}^{T}\rho_T^{t-g}\rho_T^{s-g}E\left[w_{i0}w_{j0}\right]E\left[\varepsilon_{it-d}\varepsilon_{js-d}\right] \\
= & \frac{\sigma^2}{NTq(T)}\sum_{i=1}^{N}E\left[w_{i0}^2\right]\sum_{t=g+1}^{T}\rho_T^{2(t-g)} \\
= & \frac{\sigma^2}{Tq(T)}\sup_i E\left[w_{i0}^2\right]\exp\left\{-\frac{2}{q(T)}\right\}\left[1-\exp\left\{-\frac{2}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{2(T-g)}{q(T)}\right\}\right] \\
= & O\left(T^{-1}q(T)^{-1}\right)O(1)O(1)O(q(T))O(1) \\
= & O\left(T^{-1}\right),
\end{aligned}$$

from which it follows by Markov's inequality that

$$\frac{1}{\sqrt{NTq(T)}}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\varepsilon_{it-d}\rho_T^{t-g}w_{i0} = O_p\left(\frac{1}{\sqrt{T}}\right).$$

Hence,

$$\begin{aligned}
& \frac{1}{\sqrt{NTq(T)}}\sum_{i=1}^{N}\sum_{t=g+1}^{T}w_{it-g,T}\varepsilon_{it-d} \\
= & \frac{1}{\sqrt{NTq(T)}}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\underline{w}_{it-g,T}\varepsilon_{it-d} + \frac{1}{\sqrt{NTq(T)}}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\varepsilon_{it-d}\rho_T^{t-g}w_{i0} \\
= & \frac{1}{\sqrt{NTq(T)}}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\underline{w}_{it-g,T}\varepsilon_{it-d} + O_p\left(\frac{1}{\sqrt{T}}\right) = O_p(1)
\end{aligned}$$

Furthermore, note that

$$\begin{aligned}
\frac{(1-\rho_T)}{\sqrt{NTq(T)}}\sum_{i=1}^{N}\sum_{t=g+1}^{T}w_{it-g,T}\varepsilon_{it-d} &= \left(1-\exp\left\{-\frac{1}{q(T)}\right\}\right)\frac{1}{\sqrt{NTq(T)}}\sum_{i=1}^{N}\sum_{t=g+1}^{T}w_{it-g,T}\varepsilon_{it-d} \\
&= \frac{1}{q(T)}\frac{1}{\sqrt{NTq(T)}}\sum_{i=1}^{N}\sum_{t=g+1}^{T}w_{it-g,T}\varepsilon_{it-d}\left[1+O_p\left(\frac{1}{q(T)}\right)\right] \\
&= O_p\left(\frac{1}{q(T)}\right).
\end{aligned}$$

Finally, for part (e), note that for all $T \geq g+1$, we have

$$\begin{aligned}
& E\left[\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\underline{w}_{it-g,T}\varepsilon_{it-d}\right]^2 \\
= & \frac{1}{NT}\sum_{i=1}^{N}\sum_{h=1}^{N}\sum_{t=g+1}^{T}\sum_{s=g+1}^{T}\sum_{j=1}^{t-g}\sum_{k=1}^{s-g}\rho_T^{(t-g-j)}\rho_T^{(s-g-k)}E\left[\varepsilon_{ij}\varepsilon_{hk}\varepsilon_{it-d}\varepsilon_{hs-d}\right] \\
= & \frac{\sigma^4}{NT}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\sum_{j=1}^{t-g}\rho_T^{2(t-g-j)} \\
= & \frac{\sigma^4 N}{NT}\sum_{t=g+1}^{T}\frac{1-\rho_T^{2(t-g)}}{1-\rho_T^2} \\
= & \sigma^4\frac{1}{1-\rho_T^2}\frac{1}{T}\left[(T-g)-\frac{\rho_T^2\left(1-\rho_T^{2(T-g)}\right)}{1-\rho_T^2}\right].
\end{aligned}$$

Since here we assume that $|\rho_T| = \exp\left\{-\frac{1}{q(T)}\right\}$ with $q(T) = O(1)$, it follows that there exist a positive constant C_q and a positive integer T^* such that for all $T \geq T^*$, we have

$$|\rho_T| \leq \exp\left\{-\frac{1}{C_q}\right\} < 1.$$

Using this bound, we have $T \geq \max\{T^*, g+1\}$

$$\begin{aligned}
E\left[\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\underline{w}_{it-g,T}\varepsilon_{it-d}\right]^2 &= \sigma^4\frac{1}{1-\rho_T^2}\frac{1}{T}\left[(T-g)-\frac{\rho_T^2\left(1-\rho_T^{2(T-g)}\right)}{1-\rho_T^2}\right] \\
&\leq \frac{\sigma^4}{1-\exp\{-2/C_q\}}\left[1+\frac{1}{1-\exp\{-2/C_q\}}\right] = O(1).
\end{aligned}$$

Hence, by applying Markov's inequality, we further obtain $N^{-1/2}T^{-1/2}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\underline{w}_{it-g,T}\varepsilon_{it-d} = O_p(1)$.

Moreover, by Assumptions 1 and 4,

$$\begin{aligned}
& E\left[\left(\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\varepsilon_{it-d}\rho_T^{t-g}w_{i0}\right)^2\right] \\
= & \frac{1}{NT}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=g+1}^{T}\sum_{s=g+1}^{T}\rho_T^{t-g}\rho_T^{s-g}E\left[w_{i0}w_{j0}\right]E\left[\varepsilon_{it-d}\varepsilon_{js-d}\right] \\
= & \frac{\sigma^2}{NT}\sum_{i=1}^{N}E\left[w_{i0}^2\right]\sum_{t=g+1}^{T}\rho_T^{2(t-g)} \\
= & \frac{\sigma^2}{T}\left(\sup_i E\left[w_{i0}^2\right]\right)^2\frac{\rho_T^2\left(1-\rho_T^{2(T-g)}\right)}{1-\rho_T^2} \\
= & O\left(T^{-1}\right)O\left(1\right)O\left(1\right) \\
= & O\left(T^{-1}\right),
\end{aligned}$$

from which it follows, by Markov's inequality, that

$$\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\varepsilon_{it-d}\rho_T^{t-g}w_{i0} = O_p\left(\frac{1}{\sqrt{T}}\right).$$

Hence,

$$\begin{aligned}
\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=g+1}^{T}w_{it-g,T}\varepsilon_{it-d} &= \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\underline{w}_{it-g,T}\varepsilon_{it-d} + \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\varepsilon_{it-d}\rho_T^{t-g}w_{i0} \\
&= \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=g+1}^{T}\underline{w}_{it-g,T}\varepsilon_{it-d} + O_p\left(\frac{1}{\sqrt{T}}\right) = O_p\left(1\right).
\end{aligned}$$

Finally, note that

$$\left|\frac{(1-\rho_T)}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=g+1}^{T}w_{it-g,T}\varepsilon_{it-d}\right| \leq 2\left|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=g+1}^{T}w_{it-g,T}\varepsilon_{it-d}\right| = O_p\left(1\right),$$

which completes the proof for part (e). □

Lemma SE-26:

Suppose that Assumptions 1 and 4 hold. Then, the following statements are true as $N, T \to \infty$

(a) If $\rho_T = 1$ for all T sufficiently large, then

$$\overline{w}_{-1,N,T} = O_p\left(\max\left\{\sqrt{\frac{T}{N}}, 1\right\}\right).$$

(b) If $\rho_T = \exp\{-1/q(T)\}$ such that $T/q(T) \to 0$, then

$$\overline{w}_{-1,N,T} = O_p\left(\max\left\{\sqrt{\frac{T}{N}}, 1\right\}\right),$$

as $N, T \to \infty$.

(c) If $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \sim T$, then

$$\overline{w}_{-1,N,T} = O_p\left(\max\left\{\sqrt{\frac{T}{N}}, 1\right\}\right),$$

as $N, T \to \infty$.

(d) If $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \to \infty$ but $q(T)/T \to 0$, then

$$\overline{w}_{-1,N,T} = O_p\left(\max\left\{\frac{q(T)}{\sqrt{NT}}, \frac{q(T)}{T}\right\}\right),$$

as $N, T \to \infty$.

(e) If $\rho_T \in \mathcal{G}_{\mathrm{St}} = \left\{|\rho_T| = \exp\left\{-\frac{1}{q(T)}\right\} : q(T) \geq 0 \text{ and } q(T) = O(1) \text{ as } T \to \infty\right\}$, then

$$\overline{w}_{-1,N,T} = O_p\left(\max\left\{\frac{1}{\sqrt{NT}}, \frac{1}{T}\right\}\right),$$

as $N, T \to \infty$.

Proof of Lemma SE-26:

To proceed, note that

$$\overline{w}_{-1,N,T} = \frac{1}{N(T-1)}\sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-1} = \frac{1}{N(T-1)}\sum_{i=1}^{N}\sum_{t=2}^{T} \underline{w}_{it-1} + \frac{1}{N(T-1)}\sum_{i=1}^{N}\sum_{t=2}^{T} \rho_T^{t-1} w_{i0},$$

where $\underline{w}_{it-1,T} = \sum_{j=1}^{t-1} \rho_T^{(t-1-j)} \varepsilon_{ij}$.

Consider first part (a). Note that, under the assumption here, there exists a positive integer I_ρ such that for all $T \geq I_\rho$, the triangular array process $\left\{\underline{w}_{it,T}\right\}$ has the partial sum representation

$\underline{w}_{it,T} = \sum_{j=1}^{t} \varepsilon_{ij}$. Hence, for all $T \geq \max\{I_\rho, g+1\}$, we obtain by direct calculation

$$
\begin{aligned}
& E\left[\left(\frac{1}{N(T-1)}\sum_{i=1}^{N}\sum_{t=2}^{T}\underline{w}_{it-1}\right)^2\right] \\
= & \frac{1}{N^2(T-1)^2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{t=2}^{T}\sum_{s=2}^{T} E\left[\underline{w}_{it-1,T}\underline{w}_{\ell s-1,T}\right] \\
= & \frac{1}{N^2(T-1)^2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{t=2}^{T}\sum_{s=2}^{T}\sum_{j=1}^{t-1}\sum_{k=1}^{s-1} E\left[\varepsilon_{ij}\varepsilon_{\ell k}\right] \\
= & \frac{\sigma^2}{N^2(T-1)^2}\sum_{i=1}^{N}\sum_{t=2}^{T}(t-1) + \frac{2\sigma^2}{N^2(T-1)^2}\sum_{i=1}^{N}\sum_{t=3}^{T}\sum_{s=2}^{t-1}(s-1) \\
= & \frac{\sigma^2}{N(T-1)^2}\frac{T(T-1)}{2} + \frac{2\sigma^2}{N(T-1)^2}\sum_{t=3}^{T}\frac{(t-1)(t-2)}{2} \\
= & \frac{T}{N(T-1)}\frac{\sigma^2}{2} + \frac{\sigma^2}{N(T-1)^2}\frac{(T-2)(T-1)(2T-3)}{6} + \frac{\sigma^2}{N(T-1)^2}\frac{(T-2)(T-1)}{2} \\
= & \frac{\sigma^2}{3}\frac{T}{N}\left[1 + O\left(\frac{1}{T}\right)\right].
\end{aligned}
$$

It follows by Markov's inequality that

$$
\frac{1}{N(T-1)}\sum_{i=1}^{N}\sum_{t=2}^{T}\underline{w}_{it-1} = O_p\left(\sqrt{\frac{T}{N}}\right).
$$

Moreover, by Assumption 4 and Liapounov's inequality, we have

$$
E\left|\frac{1}{N(T-1)}\sum_{i=1}^{N}\sum_{t=2}^{T}w_{i0}\right| \leq \frac{1}{N(T-1)}\sum_{i=1}^{N}\sum_{t=2}^{T}E\left|w_{i0}\right| \leq \left(\sup_i E\left[w_{i0}^2\right]\right)^{1/2} \leq C < \infty,
$$

for some (positive) constant C. It follows, by Markov's inequality, that

$$
\frac{1}{N(T-1)}\sum_{i=1}^{N}\sum_{t=2}^{T}w_{i0} = O_p(1).
$$

Hence,

$$
\begin{aligned}
\overline{w}_{-1,N,T} &= \frac{1}{N(T-1)}\sum_{i=1}^{N}\sum_{t=2}^{T}\underline{w}_{it-1} + \frac{1}{N(T-1)}\sum_{i=1}^{N}\sum_{t=2}^{T}w_{i0} \quad \text{(for all } T \text{ sufficiently large)} \\
&= O_p\left(\sqrt{\frac{T}{N}}\right) + O_p(1) = O_p\left(\max\left\{\sqrt{\frac{T}{N}}, 1\right\}\right),
\end{aligned}
$$

which shows part (a).

Now, to show parts (b)-(d), first write

$$
\begin{aligned}
& E\left[\left(\frac{1}{N(T-1)}\sum_{i=1}^{N}\sum_{t=2}^{T}\underline{w}_{it-1}\right)^2\right] \\
= & \frac{1}{N^2(T-1)^2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{t=2}^{T}\sum_{s=2}^{T}E\left[\underline{w}_{it-1,T}\underline{w}_{\ell s-1,T}\right] \\
= & \frac{1}{N^2(T-1)^2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{t=2}^{T}\sum_{s=2}^{T}\sum_{j=1}^{t-1}\sum_{k=1}^{s-1}\exp\left\{-\frac{t-1-j}{q(T)}\right\}\exp\left\{-\frac{s-1-k}{q(T)}\right\}E\left[\varepsilon_{ij}\varepsilon_{\ell k}\right] \\
= & \sigma^2\frac{1}{N^2(T-1)^2}\sum_{i=1}^{N}\sum_{t=2}^{T}\sum_{j=1}^{t-1}\exp\left\{-2\frac{t-1-j}{q(T)}\right\} \\
& +2\sigma^2\frac{1}{N^2(T-1)^2}\sum_{i=1}^{N}\sum_{t=3}^{T}\sum_{s=2}^{t-1}\sum_{j=1}^{s-1}\exp\left\{-\frac{t-1-j}{q(T)}\right\}\exp\left\{-\frac{s-1-j}{q(T)}\right\} \\
= & \sigma^2\frac{1}{N(T-1)^2}\sum_{t=2}^{T}\sum_{j=1}^{t-1}\exp\left\{-2\frac{t-1-j}{q(T)}\right\} \\
& +2\sigma^2\frac{1}{N(T-1)^2}\sum_{t=3}^{T}\sum_{s=2}^{t-1}\sum_{j=1}^{s-1}\exp\left\{-\frac{t-1-j}{q(T)}\right\}\exp\left\{-\frac{s-1-j}{q(T)}\right\}.
\end{aligned}
$$

Next, consider part (b), where we assume that $\rho_T = \exp\{-1/q(T)\}$ such that $T/q(T) \to 0$. Applying part (a) of Lemma SE-1 with $b = 2$ and $d = 2$ and part (a) of Lemma SE-7 with $b = 1$ and $g = 1$, we see that in this case

$$
\begin{aligned}
& E\left[\left(\frac{1}{N(T-1)}\sum_{i=1}^{N}\sum_{t=2}^{T}\underline{w}_{it-1}\right)^2\right] \\
= & \sigma^2\frac{1}{N(T-1)^2}\sum_{t=2}^{T}\sum_{j=1}^{t-1}\exp\left\{-2\frac{t-1-j}{q(T)}\right\} \\
& +2\sigma^2\frac{1}{N(T-1)^2}\sum_{t=3}^{T}\sum_{s=2}^{t-1}\sum_{j=1}^{s-1}\exp\left\{-\frac{t-1-j}{q(T)}\right\}\exp\left\{-\frac{s-1-j}{q(T)}\right\} \\
= & \sigma^2\frac{1}{N(T-1)^2}\frac{T^2}{2}\left[1+O\left(\max\left\{\frac{T}{q(T)},\frac{1}{T}\right\}\right)\right] \\
& +\frac{2\sigma^2}{N(T-1)^2}\frac{1}{6}T^3\left[1+O\left(\max\left\{\frac{T}{q(T)},\frac{1}{T}\right\}\right)\right] \\
= & O\left(\frac{T}{N}\right).
\end{aligned}
$$

It follows by the Markov's inequality that

$$
\frac{1}{N(T-1)}\sum_{i=1}^{N}\sum_{t=2}^{T}\underline{w}_{it-1} = O_p\left(\sqrt{\frac{T}{N}}\right)
$$

Moreover, by Assumption 4 and Liapounov's inequality, we have

$$\begin{aligned}
& E\left|\frac{1}{N(T-1)}\sum_{i=1}^{N}\sum_{t=2}^{T}\rho_T^{t-1}w_{i0}\right| \\
\leq\ & \frac{1}{N(T-1)}\sum_{i=1}^{N}E\left|w_{i0}\right|\sum_{t=2}^{T}\rho_T^{t-1} \\
\leq\ & \frac{1}{T-1}\left(\sup_i E\left[w_{i0}^2\right]\right)^{1/2}\exp\left\{-\frac{1}{q(T)}\right\}\left[1-\exp\left\{-\frac{1}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{T-1}{q(T)}\right\}\right] \\
\leq\ & O\left(T^{-1}\right)O(1)\,O(1)\,O(q(T))\,O(T/q(T)) = O(1)
\end{aligned}$$

from which it follows, by Markov's inequality, that

$$\frac{1}{N(T-1)}\sum_{i=1}^{N}\sum_{t=2}^{T}\rho_T^{t-1}w_{i0} = O_p(1).$$

Hence,

$$\begin{aligned}
\overline{w}_{-1,N,T} &= \frac{1}{N(T-1)}\sum_{i=1}^{N}\sum_{t=2}^{T}\underline{w}_{it-1} + \frac{1}{N(T-1)}\sum_{i=1}^{N}\sum_{t=2}^{T}\rho_T^{t-1}w_{i0} \\
&= O_p\left(\sqrt{\frac{T}{N}}\right) + O_p(1) = O_p\left(\max\left\{\sqrt{\frac{T}{N}},1\right\}\right),
\end{aligned}$$

which shows part (b).

Consider part (c), where we assume that $\rho_T = \exp\{-1/q(T)\}$ such that $q(T)\sim T$. Applying part (b) of Lemma SE-1 with $b=2$ and $d=2$ and part (b) of Lemma SE-7 with $b=1$, we see that in this case

$$\begin{aligned}
& E\left[\left(\frac{1}{N(T-1)}\sum_{i=1}^{N}\sum_{t=2}^{T}\underline{w}_{it-1}\right)^2\right] \\
=\ & \sigma^2\frac{1}{N(T-1)^2}\sum_{t=2}^{T}\sum_{j=1}^{t-1}\exp\left\{-2\frac{t-1-j}{q(T)}\right\} \\
& +2\sigma^2\frac{1}{N(T-1)^2}\sum_{t=3}^{T}\sum_{s=2}^{t-1}\sum_{j=1}^{s-1}\exp\left\{-\frac{t-1-j}{q(T)}\right\}\exp\left\{-\frac{s-1-j}{q(T)}\right\} \\
=\ & \sigma^2\frac{1}{N(T-1)^2}\frac{q(T)^2}{4}\left[\exp\left\{-\frac{2T}{q(T)}\right\}+\frac{2T}{q(T)}-1\right]\left[1+O\left(\frac{1}{T}\right)\right] \\
& +\frac{2\sigma^2}{N(T-1)^2}\frac{Tq(T)^2}{2}\left[1-\frac{3}{2}\frac{q(T)}{T}+2\frac{q(T)}{T}\exp\left\{-\frac{T}{q(T)}\right\}-\frac{1}{2}\frac{q(T)}{T}\exp\left\{-\frac{2T}{q(T)}\right\}\right] \\
& \qquad\times\left[1+O\left(\frac{1}{T}\right)\right] \\
=\ & O\left(\frac{q(T)^2}{NT}\right) = O\left(\frac{q(T)}{N}\right).
\end{aligned}$$

It follows by the Markov's inequality that

$$\frac{1}{N(T-1)}\sum_{i=1}^{N}\sum_{t=2}^{T}\underline{w}_{it-1}=O_p\left(\sqrt{\frac{q(T)}{N}}\right)=O_p\left(\sqrt{\frac{T}{N}}\right).$$

Moreover, by Assumption 4 and Liapounov's inequality, we have

$$\begin{aligned}
& E\left|\frac{1}{N(T-1)}\sum_{i=1}^{N}\sum_{t=2}^{T}\rho_T^{t-1}w_{i0}\right| \\
\leq\ & \frac{1}{N(T-1)}\sum_{i=1}^{N}E\left|w_{i0}\right|\sum_{t=2}^{T}\rho_T^{t-1} \\
\leq\ & \frac{1}{T-1}\left(\sup_i E\left[w_{i0}^2\right]\right)^{1/2}\exp\left\{-\frac{1}{q(T)}\right\}\left[1-\exp\left\{-\frac{1}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{T-1}{q(T)}\right\}\right] \\
\leq\ & O\left(T^{-1}\right)\times O(1)\times O(1)\times O(T)\times O(1)=O(1),
\end{aligned}$$

from which it follows, by Markov's inequality, that

$$\frac{1}{N(T-1)}\sum_{i=1}^{N}\sum_{t=2}^{T}\rho_T^{t-1}w_{i0}=O_p(1).$$

Hence,

$$\begin{aligned}
\overline{w}_{-1,N,T} &= \frac{1}{N(T-1)}\sum_{i=1}^{N}\sum_{t=2}^{T}\underline{w}_{it-1}+\frac{1}{N(T-1)}\sum_{i=1}^{N}\sum_{t=2}^{T}\rho_T^{t-1}w_{i0} \\
&= O_p\left(\sqrt{\frac{T}{N}}\right)+O_p(1)=O_p\left(\max\left\{\sqrt{\frac{T}{N}},1\right\}\right),
\end{aligned}$$

as required for part (c).

We turn our attention now to part (d) where we assume that $\rho_T=\exp\{-1/q(T)\}$ such that $q(T)\to\infty$ but $q(T)/T\to 0$. In this case, applying part (c) of Lemma SE-1 with $b=2$ and $d=2$ as well as part (c) of Lemma SE-7 with $b=1$, we get

$$\begin{aligned}
& E\left[\left(\frac{1}{N(T-1)}\sum_{i=1}^{N}\sum_{t=2}^{T}\underline{w}_{it-1}\right)^2\right] \\
=\ & \sigma^2\frac{1}{N(T-1)^2}\sum_{t=2}^{T}\sum_{j=1}^{t-1}\exp\left\{-2\frac{t-1-j}{q(T)}\right\} \\
& +2\sigma^2\frac{1}{N(T-1)^2}\sum_{t=3}^{T}\sum_{s=2}^{t-1}\sum_{j=1}^{s-1}\exp\left\{-\frac{t-1-j}{q(T)}\right\}\exp\left\{-\frac{s-1-j}{q(T)}\right\} \\
=\ & \sigma^2\frac{1}{N(T-1)^2}\frac{Tq(T)}{2}\left[1+O\left(\frac{1}{q(T)}\right)+O\left(\frac{q(T)}{T}\right)\right] \\
& +2\sigma^2\frac{1}{N(T-1)^2}\frac{Tq(T)^2}{2}\left[1+O\left(\frac{1}{q(T)}\right)+O\left(\frac{q(T)}{T}\right)\right] \\
=\ & O\left(\frac{q(T)^2}{NT}\right).
\end{aligned}$$

It follows by the Markov's inequality that

$$\frac{1}{N(T-1)}\sum_{i=1}^{N}\sum_{t=2}^{T}\underline{w}_{it-1}=O_p\left(\frac{q(T)}{\sqrt{NT}}\right)=o_p(1).$$

Moreover, by Assumption 4 and Liapounov's inequality, we have

$$\begin{aligned}
& E\left|\frac{1}{N(T-1)}\sum_{i=1}^{N}\sum_{t=2}^{T}\rho_T^{t-1}w_{i0}\right| \\
\leq\ & \frac{1}{N(T-1)}\sum_{i=1}^{N}E\left|w_{i0}\right|\sum_{t=2}^{T}\rho_T^{t-1} \\
\leq\ & \frac{1}{T-1}\left(\sup_i E\left[w_{i0}^2\right]\right)^{1/2}\exp\left\{-\frac{1}{q(T)}\right\}\left[1-\exp\left\{-\frac{1}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{T-1}{q(T)}\right\}\right] \\
\leq\ & O\left(T^{-1}\right)\times O(1)\times O(1)\times O(q(T))\times O(1)=O(q(T)/T),
\end{aligned}$$

from which it follows, by Markov's inequality, that

$$\frac{1}{N(T-1)}\sum_{i=1}^{N}\sum_{t=2}^{T}\rho_T^{t-1}w_{i0}=O_p\left(\frac{q(T)}{T}\right).$$

Hence,

$$\begin{aligned}
\overline{w}_{-1,N,T} &= \frac{1}{N(T-1)}\sum_{i=1}^{N}\sum_{t=2}^{T}\underline{w}_{it-1}+\frac{1}{N(T-1)}\sum_{i=1}^{N}\sum_{t=2}^{T}\rho_T^{t-1}w_{i0} \\
&= O_p\left(\frac{q(T)}{\sqrt{NT}}\right)+O_p\left(\frac{q(T)}{T}\right)=O_p\left(\max\left\{\frac{q(T)}{\sqrt{NT}},\frac{q(T)}{T}\right\}\right),
\end{aligned}$$

which shows part (d).

Finally, to show part (e), write

$$
\begin{aligned}
& E\left[\left(\frac{1}{N(T-1)}\sum_{i=1}^{N}\sum_{t=2}^{T}\underline{w}_{it-1}\right)^2\right] \\
= & \frac{1}{N^2(T-1)^2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{t=2}^{T}\sum_{s=2}^{T}E\left[\underline{w}_{it-1,T}\underline{w}_{\ell s-1,T}\right] \\
= & \frac{1}{N^2(T-1)^2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{t=2}^{T}\sum_{s=2}^{T}\sum_{j=1}^{t-1}\sum_{k=1}^{s-1}\rho_T^{t-1-j}\rho_T^{s-1-k}E\left[\varepsilon_{ij}\varepsilon_{\ell k}\right] \\
= & \sigma^2\frac{1}{N(T-1)^2}\sum_{t=2}^{T}\sum_{j=1}^{t-1}\rho_T^{2(t-1-j)}+2\sigma^2\frac{1}{N^2(T-1)^2}\sum_{i=1}^{N}\sum_{t=3}^{T}\sum_{s=2}^{t-1}\rho_T^{t-s}\sum_{j=1}^{s-1}\rho_T^{2(s-1-j)} \\
= & \sigma^2\frac{1}{N(T-1)^2}\sum_{t=2}^{T}\frac{1-\rho_T^{2(t-1)}}{1-\rho_T^2}+2\sigma^2\frac{1}{N(T-1)^2}\sum_{t=3}^{T}\sum_{s=2}^{t-1}\rho_T^{t-s}\frac{1-\rho_T^{2(s-1)}}{1-\rho_T^2} \\
= & \frac{\sigma^2}{N(T-1)}\frac{1}{1-\rho_T^2}-\frac{\sigma^2}{N(T-1)^2}\frac{\rho_T^2\left(1-\rho_T^{2(T-1)}\right)}{\left(1-\rho_T^2\right)^2} \\
& +\frac{2\sigma^2}{N(T-1)^2}\frac{\rho_T}{1-\rho_T^2}\sum_{t=3}^{T}\sum_{s=2}^{t-1}\rho_T^{t-1-s}-\frac{2\sigma^2}{N(T-1)^2}\frac{1}{1-\rho_T^2}\sum_{t=3}^{T}\rho_T^{t}\sum_{s=2}^{t-1}\rho_T^{(s-2)} \\
= & \frac{\sigma^2}{N(T-1)}\frac{1}{1-\rho_T^2}-\frac{\sigma^2}{N(T-1)^2}\frac{\rho_T^2\left(1-\rho_T^{2(T-1)}\right)}{\left(1-\rho_T^2\right)^2} \\
& +\frac{2\sigma^2}{N(T-1)^2}\frac{\rho_T}{\left(1-\rho_T^2\right)(1-\rho_T)}\sum_{t=3}^{T}\left(1-\rho_T^{(t-2)}\right) \\
& -\frac{2\sigma^2}{N(T-1)^2}\frac{1}{1-\rho_T^2}\sum_{t=3}^{T}\rho_T^{t}\frac{1-\rho_T^{t-2}}{1-\rho_T} \\
= & \frac{\sigma^2}{N(T-1)}\frac{1}{1-\rho_T^2}-\frac{\sigma^2}{N(T-1)^2}\frac{\rho_T^2\left(1-\rho_T^{2(T-1)}\right)}{\left(1-\rho_T^2\right)^2} \\
& +\frac{2\sigma^2(T-2)}{N(T-1)^2}\frac{\rho_T}{\left(1-\rho_T^2\right)(1-\rho_T)}-\frac{2\sigma^2}{N(T-1)^2}\frac{\rho_T^2\left(1-\rho_T^{(T-2)}\right)}{\left(1-\rho_T^2\right)(1-\rho_T)^2} \\
& -\frac{2\sigma^2}{N(T-1)^2}\frac{\rho_T^3\left(1-\rho_T^{(T-2)}\right)}{\left(1-\rho_T^2\right)(1-\rho_T)^2}+\frac{2\sigma^2}{N(T-1)^2}\frac{\rho_T^4\left(1-\rho_T^{2(T-2)}\right)}{\left(1-\rho_T^2\right)^2(1-\rho_T)}.
\end{aligned}
$$

Now, since $q(T)=O(1)$ in this case, there exist some positive constant C_q and some positive integer T^* such that for all $T\geq T^*$

$$
0\leq|\rho_T|=\exp\left\{-\frac{1}{q(T)}\right\}\leq\exp\left\{-\frac{1}{C_q}\right\}<1
$$

so that

$$
\begin{aligned}
& E\left[\overline{w}_{-1,N,T}^{2}\right] \\
\leq\ & \frac{\sigma^2}{N(T-1)}\frac{1}{1-\rho_T^2}+\frac{\sigma^2}{N(T-1)^2}\frac{\rho_T^2\left(1-\rho_T^{2(T-1)}\right)}{\left(1-\rho_T^2\right)^2} \\
& +\frac{2\sigma^2}{N(T-1)}\frac{|\rho_T|}{\left(1-\rho_T^2\right)(1-\rho_T)}+\frac{2\sigma^2}{N(T-1)^2}\frac{\rho_T^2\left(1-\rho_T^{(T-2)}\right)}{\left(1-\rho_T^2\right)(1-\rho_T)^2} \\
& +\frac{2\sigma^2}{N(T-1)^2}\frac{\rho_T^3\left(1-\rho_T^{(T-2)}\right)}{\left(1-\rho_T^2\right)(1-\rho_T)^2}+\frac{2\sigma^2}{N(T-1)^2}\frac{\rho_T^4\left(1-\rho_T^{2(T-2)}\right)}{\left(1-\rho_T^2\right)^2(1-\rho_T)} \\
\leq\ & \frac{1}{N(T-1)}\frac{\sigma^2}{1-\exp\{-2/C_q\}}+\frac{1}{N(T-1)^2}\frac{\sigma^2}{(1-\exp\{-2/C_q\})^2} \\
& +\frac{1}{N(T-1)}\frac{4\sigma^2}{(1-\exp\{-2/C_q\})(1-\exp\{-1/C_q\})} \\
& +\frac{1}{N(T-1)^2}\frac{2\sigma^2}{(1-\exp\{-2/C_q\})^2(1-\exp\{-1/C_q\})} \\
=\ & O\left(\frac{1}{NT}\right)
\end{aligned}
$$

It follows by the Markov's inequality that

$$
\frac{1}{N(T-1)}\sum_{i=1}^{N}\sum_{t=2}^{T}\underline{w}_{it-1}=O_p\left(\frac{1}{\sqrt{NT}}\right).
$$

Moreover, by Assumption 4 and Liapounov's inequality, we have

$$
\begin{aligned}
E\left|\frac{1}{N(T-1)}\sum_{i=1}^{N}\sum_{t=2}^{T}\rho_T^{t-1}w_{i0}\right| &\leq \frac{1}{N(T-1)}\sum_{i=1}^{N}\sum_{t=2}^{T}|\rho_T|^{t-1}E|w_{i0}| \\
&\leq \frac{1}{T-1}\left(\sup_i E\left[w_{i0}^2\right]\right)^{1/2}\frac{|\rho_T|\left(1-|\rho_T|^{T-1}\right)}{1-|\rho_T|} \\
&= O\left(\frac{1}{T}\right),
\end{aligned}
$$

from which it follows, by Markov's inequality, that

$$
\frac{1}{N(T-1)}\sum_{i=1}^{N}\sum_{t=2}^{T}\rho_T^{t-1}w_{i0}=O_p\left(\frac{1}{T}\right).
$$

Hence,

$$
\begin{aligned}
\overline{w}_{-1,N,T} &= \frac{1}{N(T-1)}\sum_{i=1}^{N}\sum_{t=2}^{T}\underline{w}_{it-1}+\frac{1}{N(T-1)}\sum_{i=1}^{N}\sum_{t=2}^{T}\rho_T^{t-1}w_{i0} \\
&= O_p\left(\frac{1}{\sqrt{NT}}\right)+O_p\left(\frac{1}{T}\right)=O_p\left(\max\left\{\frac{1}{\sqrt{NT}},\frac{1}{T}\right\}\right),
\end{aligned}
$$

which completes the proof of part (e). □

Lemma SE-27:

Suppose that Assumptions 1 and 4 hold. Then, the following statements are true as $N, T \to \infty$

(a) If $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \sim T$, then

$$\frac{1}{\overline{\omega}_{Z,N,T}} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1,T} \varepsilon_{it} \Rightarrow \mathcal{Z} \equiv N(0,1),$$

as N, $T \to \infty$, where

$$\begin{aligned} \overline{\omega}^2_{Z,N,T} &= \sigma^4 \sum_{i=1}^{N} \sum_{t=2}^{T} \sum_{j=1}^{t-1} \exp\left\{-2\frac{t-1-j}{q(T)}\right\} \\ &= \frac{\sigma^4}{4} N q(T)^2 \left[\exp\left\{-\frac{2T}{q(T)}\right\} + \frac{2T}{q(T)} - 1\right]\left[1 + O\left(\frac{1}{T}\right)\right]. \end{aligned}$$

(b) If $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \to \infty$ but $q(T)/T \to 0$, then

$$\frac{1}{\sqrt{NTq(T)}} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1,T} \varepsilon_{it} \Rightarrow N\left(0, \frac{\sigma^4}{2}\right),$$

as N, $T \to \infty$.

(c) If $\rho_T \in \mathcal{G}_{\mathrm{St}} = \left\{|\rho_T| = \exp\left\{-\frac{1}{q(T)}\right\} : q(T) \geq 0 \text{ and } q(T) = O(1) \text{ as } T \to \infty\right\}$, then

$$\sqrt{\frac{1-\rho_T^2}{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1,T} \varepsilon_{it} \Rightarrow N\left(0, \sigma^4\right)$$

Proof of Lemma SE-27:

Before embarking on the proof of the indvidual parts of this lemma, we first introduce some additional notation and perform some preliminary moment calculations. To proceed, first write

$$\sum_{i=1}^{N} \sum_{t=2}^{T} w_{it-1,T} \varepsilon_{it} = \sum_{i=1}^{N} \sum_{t=2}^{T} \underline{w}_{it-1,T} \varepsilon_{it} + \sum_{i=1}^{N} \sum_{t=2}^{T} \varepsilon_{it} \rho_T^{t-1} w_{i0},$$

where $\underline{w}_{it-1,T} = \sum_{j=1}^{t-1} \rho_T^{(t-1-j)} \varepsilon_{ij}$. Let

$$Z_{i,T} = \frac{1}{T} \sum_{t=2}^{T} \underline{w}_{it-1,T} \varepsilon_{it},$$

and note that

$$\begin{aligned}
E\left[Z_{i,T}\right] &= \frac{1}{T}\sum_{t=2}^{T} E\left[\underline{w}_{it-1,T}\varepsilon_{it}\right] = \frac{1}{T}\sum_{t=2}^{T}\sum_{j=1}^{t-1}\exp\left\{-\frac{t-1-j}{q\left(T\right)}\right\} E\left[\varepsilon_{ij}\varepsilon_{it}\right] = 0, \\
E\left[Z_{i,T}^{2}\right] &= \frac{1}{T^{2}}\sum_{t=2}^{T}\sum_{s=2}^{T} E\left[\underline{w}_{it-1,T}\underline{w}_{is-1,T}\varepsilon_{it}\varepsilon_{is}\right] \\
&= \frac{1}{T^{2}}\sum_{t=2}^{T}\sum_{s=2}^{T}\sum_{j=1}^{t-1}\sum_{k=1}^{s-1}\exp\left\{-\frac{t-1-j}{q\left(T\right)}\right\}\exp\left\{-\frac{s-1-k}{q\left(T\right)}\right\} E\left[\varepsilon_{ij}\varepsilon_{ik}\varepsilon_{it}\varepsilon_{is}\right] \\
&= \sigma^{4}\frac{1}{T^{2}}\sum_{t=2}^{T}\sum_{j=1}^{t-1}\exp\left\{-2\frac{t-1-j}{q\left(T\right)}\right\}.
\end{aligned}$$

It follows that

$$\omega_{Z,N,T}^{2} = \sum_{i=1}^{N} E\left[Z_{i,T}^{2}\right] = \sigma^{4}N\frac{1}{T^{2}}\sum_{t=2}^{T}\sum_{j=1}^{t-1}\exp\left\{-2\frac{t-1-j}{q\left(T\right)}\right\}.$$

Now, consider part (a), where we assume that $q\left(T\right)\sim T$. Using part (b) of Lemma SE-1 with $b=2$ and $d=2$, we see that in this case

$$\begin{aligned}
\omega_{Z,N,T}^{2} &= \frac{\overline{\omega}_{Z,N,T}^{2}}{T^{2}} = \sigma^{4}N\frac{1}{T^{2}}\sum_{t=2}^{T}\sum_{j=1}^{t-1}\exp\left\{-2\frac{t-1-j}{q\left(T\right)}\right\} \\
&= \sigma^{4}N\frac{q\left(T\right)^{2}}{4T^{2}}\left[\exp\left\{-2\frac{T}{q\left(T\right)}\right\}+\frac{2T}{q\left(T\right)}-1\right]\left[1+O\left(\frac{1}{T}\right)\right]
\end{aligned}$$

so that

$$\frac{\omega_{Z,N,T}^{2}}{N} = \sigma^{4}\frac{q\left(T\right)^{2}}{4T^{2}}\left[\exp\left\{-2\frac{T}{q\left(T\right)}\right\}+\frac{2T}{q\left(T\right)}-1\right]\left[1+O\left(\frac{1}{T}\right)\right].$$

Moreover, the assumption $q\left(T\right)\sim T$ implies that there exists a constant $\overline{C}>1$ such that

$$0<\frac{1}{\overline{C}}\leq\frac{T}{q\left(T\right)}\leq\overline{C}<\infty \text{ eventually as } T\to\infty.$$

In addition, let $f\left(x\right)=\exp\left\{-2x\right\}+2x-1$. Since $f\left(0\right)=0$ and $f'\left(x\right)=-2\exp\left\{-2x\right\}+2>0$ for all $x>0$, it follows that by the mean value theorem that for all $a\in\left(0,\infty\right)$ there exists some $b\in\left(0,a\right)$ such that

$$\exp\left\{-2a\right\}+2a-1=f\left(a\right)=f'\left(b\right)a>0.$$

Using these facts, we deduce that

$$0<\exp\left\{-\frac{2}{\overline{C}}\right\}+\frac{2}{\overline{C}}-1<\exp\left\{-2\overline{C}\right\}+2\overline{C}-1<\infty.$$

It then follows from Assumption 1 that there exists a positive constant C such that

$$0<\frac{1}{C}\leq\frac{\omega_{Z,N,T}}{\sqrt{N}}\leq C<\infty \text{ eventually as } N,T\to\infty.$$

Moreover, by Assumptions 1 and 4,

$$
\begin{aligned}
& E\left[\left(\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=2}^{T}\varepsilon_{it}\rho_T^{t-1}w_{i0}\right)^2\right] \\
= & \frac{1}{NT^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=2}^{T}\sum_{s=2}^{T}\rho_T^{t-1}\rho_T^{s-1}E\left[w_{i0}w_{j0}\right]E\left[\varepsilon_{it}\varepsilon_{js}\right] \\
= & \frac{\sigma^2}{NT^2}\sum_{i=1}^{N}E\left[w_{i0}^2\right]\sum_{t=2}^{T}\rho_T^{2(t-1)} \\
= & \frac{\sigma^2}{T^2}\left(\sup_i E\left[w_{i0}^2\right]\right)\exp\left\{-\frac{2}{q(T)}\right\}\left[1-\exp\left\{-\frac{2}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{2(T-1)}{q(T)}\right\}\right] \\
= & O\left(T^{-2}\right)\times O(1)\times O(1)\times O(T)\times O(1)=O\left(T^{-1}\right),
\end{aligned}
$$

from which it follows, by Markov's inequality, that

$$
\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=2}^{T}\varepsilon_{it}\rho_T^{t-1}w_{i0}=O_P\left(\frac{1}{\sqrt{T}}\right).
$$

Hence,

$$
\begin{aligned}
& \frac{1}{\omega_{Z,N,T}/\sqrt{N}}\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=2}^{T}w_{it-1,T}\varepsilon_{it} \\
= & \frac{1}{\omega_{Z,N,T}/\sqrt{N}}\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=g+2}^{T}\underline{w}_{it-1,T}\varepsilon_{it}+\frac{1}{\omega_{Z,N,T}/\sqrt{N}}\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=g+2}^{T}\varepsilon_{it}\rho_T^{t-1}w_{i0} \\
= & \frac{1}{\omega_{Z,N,T}/\sqrt{N}}\frac{1}{\sqrt{N}}\sum_{i=1}^{N}Z_{i,T}+O_P\left(\frac{1}{\sqrt{T}}\right).
\end{aligned}
$$

Thus, showing the desired result is equivalent to showing the asymptotic normality of

$$
U_{Z,N,T}=\frac{1}{\omega_{Z,N,T}/\sqrt{N}}\frac{1}{\sqrt{N}}\sum_{i=1}^{N}Z_{i,T},
$$

To show the asymptotic normality of $U_{N,T}$, it suffices to verify a Liapounov-type condition of the form

$$
\lim_{N,T\to\infty}\frac{1}{N^2}\sum_{i=1}^{N}E\left[Z_{i,T}^4\right]=0.
$$

Now, given Assumption 1, there exists a positive constant C such that

$$
\begin{aligned}
& E\left[Z_{i,T}^{4}\right] \\
= \; & \frac{1}{T^{4}}\sum_{t=2}^{T}\sum_{s=2}^{T}\sum_{g=2}^{T}\sum_{u=2}^{T} E\left[\underline{w}_{it-1,T}\underline{w}_{is-1,T}\underline{w}_{ig-1,T}\underline{w}_{iu-1,T}\varepsilon_{it}\varepsilon_{is}\varepsilon_{ig}\varepsilon_{iu}\right] \\
= \; & \frac{1}{T^{4}}\sum_{t=2}^{T}\sum_{s=2}^{T}\sum_{g=2}^{T}\sum_{u=2}^{T}\sum_{j=1}^{t-1}\sum_{k=1}^{s-1}\sum_{\ell=1}^{g-1}\sum_{h=1}^{u-1}\exp\left\{-\frac{t-1-j}{q\left(T\right)}\right\}\exp\left\{-\frac{s-1-k}{q\left(T\right)}\right\} \\
& \exp\left\{-\frac{g-1-\ell}{q\left(T\right)}\right\}\exp\left\{-\frac{u-1-h}{q\left(T\right)}\right\} E\left[\varepsilon_{ij}\varepsilon_{ik}\varepsilon_{i\ell}\varepsilon_{ih}\varepsilon_{it}\varepsilon_{is}\varepsilon_{ig}\varepsilon_{iu}\right] \\
\leq \; & C\left(\frac{1}{T^{4}}\sum_{t=2}^{T}\sum_{j=1}^{t-1}\exp\left\{-4\frac{t-1-j}{q\left(T\right)}\right\}\right. \\
& +\frac{1}{T^{4}}\sum_{t=3}^{T}\sum_{s=2}^{t-1}\sum_{j=1}^{s-1}\exp\left\{-2\frac{t-1-s}{q\left(T\right)}\right\}\exp\left\{-\frac{t-1-j}{q\left(T\right)}\right\}\exp\left\{-\frac{s-1-j}{q\left(T\right)}\right\} \\
& +\frac{1}{T^{4}}\sum_{t=3}^{T}\sum_{s=2}^{t-1}\sum_{j=1}^{s-1}\exp\left\{-\frac{t-1-s}{q\left(T\right)}\right\}\exp\left\{-2\frac{t-1-j}{q\left(T\right)}\right\}\exp\left\{-\frac{s-1-j}{q\left(T\right)}\right\} \\
& +\frac{1}{T^{4}}\sum_{t=3}^{T}\sum_{s=2}^{t-1}\sum_{j=1}^{s-1}\exp\left\{-2\frac{t-1-s}{q\left(T\right)}\right\}\exp\left\{-2\frac{s-1-j}{q\left(T\right)}\right\} \\
& +\frac{1}{T^{4}}\sum_{t=3}^{T}\sum_{s=2}^{t-1}\sum_{j=1}^{t-1}\sum_{k=1}^{s-1}\exp\left\{-2\frac{t-1-j}{q\left(T\right)}\right\}\exp\left\{-2\frac{s-1-k}{q\left(T\right)}\right\} \\
& +\frac{1}{T^{4}}\sum_{t=3}^{T}\sum_{s=2}^{t-1}\left[\sum_{j=1}^{s-1}\exp\left\{-\frac{t-1-j}{q\left(T\right)}\right\}\exp\left\{-\frac{s-1-j}{q\left(T\right)}\right\}\right]^{2} \\
& \left.+\frac{1}{T^{4}}\sum_{t=3}^{T}\sum_{s=2}^{t-1}\sum_{j=1}^{s-1}\exp\left\{-\frac{t-1-s}{q\left(T\right)}\right\}\exp\left\{-\frac{t-1-j}{q\left(T\right)}\right\}\exp\left\{-2\frac{s-1-j}{q\left(T\right)}\right\}\right)
\end{aligned}
$$

$$
\begin{aligned}
\leq\ & C\left(\frac{1}{T^4}\sum_{t=2}^{T}\sum_{j=1}^{t-1}\exp\left\{-4\frac{t-1-j}{q(T)}\right\}\right.\\
&+\frac{1}{T^4}\sum_{t=3}^{T}\sum_{s=2}^{t-1}\sum_{j=1}^{s-1}\exp\left\{-2\frac{t-1-s}{q(T)}\right\}\exp\left\{-\frac{t-s}{q(T)}\right\}\exp\left\{-2\frac{(s-1-j)}{q(T)}\right\}\\
&+\frac{1}{T^4}\sum_{t=3}^{T}\sum_{s=2}^{t-1}\sum_{j=1}^{s-1}\exp\left\{-2\frac{t-1-s}{q(T)}\right\}\exp\left\{-2\frac{s-1-j}{q(T)}\right\}\\
&+\frac{1}{T^4}\sum_{t=3}^{T}\sum_{s=2}^{t-1}\sum_{j=1}^{s-1}\exp\left\{-\frac{t-1-s}{q(T)}\right\}\exp\left\{-\frac{t-1-j}{q(T)}\right\}\exp\left\{-\frac{t-s}{q(T)}\right\}\exp\left\{-2\frac{(s-1-j)}{q(T)}\right\}\\
&+\frac{1}{T^4}\sum_{t=3}^{T}\sum_{s=2}^{t-1}\sum_{j=1}^{s-1}\exp\left\{-\frac{t-1-s}{q(T)}\right\}\exp\left\{-\frac{t-1-j}{q(T)}\right\}\exp\left\{-2\frac{s-1-j}{q(T)}\right\}\\
&+\frac{1}{T^4}\sum_{t=3}^{T}\sum_{s=2}^{t-1}\sum_{j=1}^{t-1}\sum_{k=1}^{s-1}\exp\left\{-2\frac{t-1-j}{q(T)}\right\}\exp\left\{-2\frac{s-1-k}{q(T)}\right\}\\
&\left.+\frac{1}{T^4}\sum_{t=3}^{T}\sum_{s=2}^{t-1}\left[\sum_{j=1}^{s-1}\exp\left\{-2\frac{(t-1-j)}{q(T)}\right\}\right]\left[\sum_{k=1}^{s-1}\exp\left\{-2\frac{(t-1-k)}{q(T)}\right\}\right]\right)\\
\leq\ & C\left(\frac{1}{T^4}\sum_{t=2}^{T}\sum_{j=1}^{t-1}\exp\left\{-4\frac{t-1-j}{q(T)}\right\}+\frac{2}{T^4}\sum_{t=3}^{T}\sum_{s=2}^{t-1}\sum_{j=1}^{s-1}\exp\left\{-2\frac{t-1-s}{q(T)}\right\}\exp\left\{-2\frac{s-1-j}{q(T)}\right\}\right.\\
&+\frac{2}{T^4}\sum_{t=3}^{T}\sum_{s=2}^{t-1}\sum_{j=1}^{s-1}\exp\left\{-2\frac{t-1-s}{q(T)}\right\}\exp\left\{-\frac{s-j}{q(T)}\right\}\exp\left\{-2\frac{s-1-j}{q(T)}\right\}\\
&\left.+\frac{2}{T^4}\sum_{t=3}^{T}\sum_{s=2}^{t-1}\sum_{j=1}^{t-1}\sum_{k=1}^{s-1}\exp\left\{-2\frac{t-1-j}{q(T)}\right\}\exp\left\{-2\frac{s-1-k}{q(T)}\right\}\right)\\
\leq\ & C\left(\frac{1}{T^4}\sum_{t=2}^{T}\sum_{j=1}^{t-1}\exp\left\{-4\frac{t-1-j}{q(T)}\right\}+\frac{4}{T^4}\sum_{t=3}^{T}\sum_{s=2}^{t-1}\sum_{j=1}^{s-1}\exp\left\{-2\frac{t-1-s}{q(T)}\right\}\exp\left\{-2\frac{s-1-j}{q(T)}\right\}\right.\\
&\left.+\frac{2}{T^4}\sum_{t=3}^{T}\sum_{s=2}^{t-1}\sum_{j=1}^{t-1}\sum_{k=1}^{s-1}\exp\left\{-2\frac{t-1-j}{q(T)}\right\}\exp\left\{-2\frac{s-1-k}{q(T)}\right\}\right)
\end{aligned}
$$

It follows from applying part (b) of Lemma SE-1 with $b = 2$ and $d = 4$, part (a) of Lemma SE-8 with

$b=2$, and part (a) of Lemma SE-9 with $b=c=2$ that

$$
\begin{aligned}
& E\left[Z_{i,T}^{4}\right] \\
\leq\ & C\left\{\frac{1}{T^{4}}\frac{q(T)^{2}}{16}\left[\exp\left\{-\frac{4T}{q(T)}\right\}+\frac{4T}{q(T)}-1\right]\left[1+O\left(\frac{1}{T}\right)\right]\right. \\
& +\frac{4}{T^{4}}\left(\frac{Tq(T)^{2}}{4}\left[1+\exp\left\{-\frac{2T}{q(T)}\right\}\right]-\frac{2q(T)^{3}}{8}\left[1-\exp\left\{-\frac{2T}{q(T)}\right\}\right]\right)\left[1+O\left(\frac{1}{T}\right)\right] \\
& +\frac{2}{T^{4}}\left(\frac{T^{2}q(T)^{2}}{8}-\frac{q(T)^{4}}{16}\left[1-\exp\left\{-\frac{2T}{q(T)}\right\}\right]-\frac{Tq(T)^{3}}{8}\left[1-\exp\left\{-\frac{2T}{q(T)}\right\}\right]\right. \\
& +\frac{q(T)^{4}}{16}\left[1-\exp\left\{-\frac{2T}{q(T)}\right\}\right]+\frac{q(T)^{4}}{16}\left[1-\exp\left\{-\frac{2T}{q(T)}\right\}\right] \\
& \left.\left.-\frac{q(T)^{4}}{32}\left[1-\exp\left\{-\frac{4T}{q(T)}\right\}\right]\right)\left[1+O\left(\frac{1}{T}\right)\right] \\
=\ & C\left(\frac{1}{4}\frac{q(T)^{2}}{T^{2}}-\frac{1}{4}\frac{q(T)^{3}}{T^{3}}\left[1-\exp\left\{-\frac{2T}{q(T)}\right\}\right]\right. \\
& \left.+\frac{1}{16}\frac{q(T)^{4}}{T^{4}}\left[1-2\exp\left\{-\frac{2T}{q(T)}\right\}+\exp\left\{-\frac{4T}{q(T)}\right\}\right]\right)\left[1+O\left(\frac{1}{T}\right)\right],
\end{aligned}
$$

so that

$$
\begin{aligned}
& \frac{1}{N^{2}}\sum_{i=1}^{N}E\left[Z_{i,T}^{4}\right] \\
=\ & C\frac{1}{4N}\left(\frac{q(T)^{2}}{T^{2}}-\frac{q(T)^{3}}{T^{3}}\left[1-\exp\left\{-\frac{2T}{q(T)}\right\}\right]+\frac{1}{4}\frac{q(T)^{4}}{T^{4}}\left[1-\exp\left\{-\frac{2T}{q(T)}\right\}\right]^{2}\right) \\
& \times\left[1+O\left(\frac{1}{T}\right)\right] \\
=\ & O\left(\frac{1}{N}\right).
\end{aligned}
$$

Since the Liapounov condition implies the Lindeberg-type condition (13), we deduce from Lemma SE-10 that

$$
U_{Z,N,T}=\frac{1}{\omega_{Z,N,T}}\sum_{i=1}^{N}Z_{i,T}\Rightarrow\mathcal{Z}\equiv N(0,1),
$$

as $N,T\to\infty$.

Next, consider part (b), where we assume that $\rho_T=\exp\{-1/q(T)\}$ such that $q(T)\to\infty$ but

$q(T)/T \to 0$. In this case, using part (c) of Lemma SE-1 with $b = 2$ and $d = 2$, we see that

$$\begin{aligned}
\omega_{Z,N,T}^2 &= \sigma^4 N \frac{1}{T^2} \sum_{t=2}^{T} \sum_{j=1}^{t-1} \exp\left\{-2\frac{t-1-j}{q(T)}\right\} \\
&= \sigma^4 N \frac{Tq(T)}{2T^2}\left[1 + O\left(\frac{1}{q(T)}\right) + O\left(\frac{q(T)}{T}\right)\right] \\
&= \frac{\sigma^4}{2}\frac{Nq(T)}{T}\left[1 + O\left(\frac{1}{q(T)}\right) + O\left(\frac{q(T)}{T}\right)\right],
\end{aligned}$$

so that

$$\frac{T}{Nq(T)}\omega_{Z,N,T}^2 = \frac{\sigma^4}{2}\left[1 + O\left(\frac{1}{q(T)}\right) + O\left(\frac{q(T)}{T}\right)\right].$$

It then follows from Assumption 1 that there exists a positive constant C such that

$$0 < \frac{1}{C} \leq \sqrt{\frac{T}{Nq(T)}}\omega_{Z,N,T} \leq C < \infty \text{ eventually as } N, T \to \infty.$$

Moreover, by Assumptions 1 and 4,

$$\begin{aligned}
&E\left[\left(\frac{1}{\sqrt{NTq(T)}}\sum_{i=1}^{N}\sum_{t=2}^{T}\varepsilon_{it}\rho_T^{t-1}w_{i0}\right)^2\right] = \frac{1}{NTq(T)}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=2}^{T}\sum_{s=2}^{T}\rho_T^{t-1}\rho_T^{s-1}E\left[w_{i0}w_{j0}\right]E\left[\varepsilon_{it}\varepsilon_{js}\right] \\
= &\frac{\sigma^2}{NTq(T)}\sum_{i=1}^{N}E\left[w_{i0}^2\right]\sum_{t=2}^{T}\rho_T^{2(t-1)} \\
= &\frac{\sigma^2}{q(T)T}\left(\sup_i E\left[w_{i0}^2\right]\right)\exp\left\{-\frac{2}{q(T)}\right\}\left[1-\exp\left\{-\frac{2}{q(T)}\right\}\right]^{-1}\left[1-\exp\left\{-\frac{2(T-1)}{q(T)}\right\}\right] \\
= &O\left(q(T)^{-1}T^{-1}\right) \times O(1) \times O(1) \times O(q(T)) \times O(1) = O\left(T^{-1}\right),
\end{aligned}$$

from which it follows, by Markov's inequality, that

$$\frac{1}{\sqrt{NTq(T)}}\sum_{i=1}^{N}\sum_{t=2}^{T}\varepsilon_{it}\rho_T^{t-1}w_{i0} = O_p\left(\frac{1}{\sqrt{T}}\right).$$

Hence,

$$\begin{aligned}
&\frac{1}{\omega_{Z,N,T}/\sqrt{Nq(T)/T}}\frac{1}{\sqrt{NTq(T)}}\sum_{i=1}^{N}\sum_{t=2}^{T}w_{it-1,T}\varepsilon_{it} \\
= &\frac{1}{\omega_{Z,N,T}/\sqrt{Nq(T)/T}}\frac{1}{\sqrt{NTq(T)}}\sum_{i=1}^{N}\sum_{t=g+2}^{T}\underline{w}_{it-1,T}\varepsilon_{it} \\
&+\frac{1}{\omega_{Z,N,T}/\sqrt{Nq(T)/T}}\frac{1}{\sqrt{NTq(T)}}\sum_{i=1}^{N}\sum_{t=g+2}^{T}\varepsilon_{it}\rho_T^{t-1}w_{i0} \\
= &\frac{1}{\omega_{Z,N,T}/\sqrt{Nq(T)/T}}\sqrt{\frac{T}{Nq(T)}}\sum_{i=1}^{N}Z_{i,T} + O_p\left(\frac{1}{\sqrt{T}}\right).
\end{aligned}$$

Thus, showing the desired result is equivalent to showing the asymptotic normality of

$$U_{Z,N,T} = \frac{1}{\omega_{Z,N,T}/\sqrt{Nq(T)/T}} \sqrt{\frac{T}{Nq(T)}} \sum_{i=1}^{N} Z_{i,T},$$

To show the asymptotic normality of $U_{N,T}$, it suffices to verify a Liapounov-type condition of the form

$$\lim_{N,T\to\infty} \left(\frac{T}{Nq(T)}\right)^2 \sum_{i=1}^{N} E\left[Z_{i,T}^4\right] = 0.$$

Now, similar to part (a) above, we have that, given Assumption 1, there exists a positive constant C such that

$$\begin{aligned}
& E\left[Z_{i,T}^4\right] \\
\leq\ & C\left(\frac{1}{T^4}\sum_{t=2}^{T}\sum_{j=1}^{t-1}\exp\left\{-4\frac{t-1-j}{q(T)}\right\} + \frac{4}{T^4}\sum_{t=3}^{T}\sum_{s=2}^{t-1}\sum_{j=1}^{s-1}\exp\left\{-2\frac{t-1-s}{q(T)}\right\}\exp\left\{-2\frac{s-1-j}{q(T)}\right\}\right. \\
& \left. +\frac{2}{T^4}\sum_{t=3}^{T}\sum_{s=2}^{t-1}\sum_{j=1}^{t-1}\sum_{k=1}^{s-1}\exp\left\{-2\frac{t-1-j}{q(T)}\right\}\exp\left\{-2\frac{s-1-k}{q(T)}\right\}\right),
\end{aligned}$$

from applying part (c) of Lemma SE-1 with $b = 2$ and $d = 4$, part (b) of Lemma SE-8 with $b = 2$, and part (a) of Lemma SE-9 with $b = c = 2$ that

$$\begin{aligned}
& E\left[Z_{i,T}^4\right] \\
\leq\ & C\left\{\frac{1}{T^4}\frac{Tq(T)}{4}\left[1+O\left(\frac{1}{q(T)}\right)+O\left(\frac{q(T)}{T}\right)\right] + \frac{4}{T^4}\frac{Tq(T)^2}{4}\left[1+O\left(\frac{1}{q(T)}\right)+O\left(\frac{q(T)}{T}\right)\right]\right. \\
& \left. +\frac{2}{T^4}\frac{T^2q(T)^2}{8}\left[1+O\left(\frac{1}{q(T)}\right)+O\left(\frac{q(T)}{T}\right)\right]\right\} \\
=\ & C\frac{1}{4}\frac{q(T)^2}{T^2}\left[1+\frac{4}{T}+\frac{1}{Tq(T)}\right]\left[1+O\left(\frac{1}{q(T)}\right)+O\left(\frac{q(T)}{T}\right)\right] \\
=\ & C\frac{1}{4}\left(\frac{q(T)}{T}\right)^2\left[1+O\left(\frac{1}{q(T)}\right)+O\left(\frac{q(T)}{T}\right)\right].
\end{aligned}$$

Hence,

$$\left(\frac{T}{Nq(T)}\right)^2 \sum_{i=1}^{N} E\left[Z_{i,T}^4\right] \leq C\frac{1}{4N}\left[1+O\left(\frac{1}{q(T)}\right)+O\left(\frac{q(T)}{T}\right)\right] = O\left(\frac{1}{N}\right).$$

Since the Liapounov condition implies the Lindeberg-type condition (13), we deduce from Lemma SE-10 that

$$U_{Z,N,T} = \frac{1}{\omega_{Z,N,T}/\sqrt{Nq(T)/T}} \sqrt{\frac{T}{Nq(T)}} \sum_{i=1}^{N} Z_{i,T} \Rightarrow \mathcal{Z} \equiv N(0,1),$$

as $N, T \to \infty$. By the Cramér convergence theorem, we then have that

$$\sqrt{\frac{T}{Nq(T)}} \sum_{i=1}^{N} Z_{i,T} = \frac{1}{\sqrt{NTq(T)}} \sum_{i=1}^{N}\sum_{t=2}^{T} w_{it-1,T}\varepsilon_{it} \Rightarrow N\left(0, \frac{\sigma^4}{2}\right),$$

as N, $T \to \infty$.

To show part (d), we first define

$$Q_{i,T} = \sqrt{\frac{1-\rho_T^2}{T}} \sum_{t=2}^{T} \underline{w}_{it-1,T} \varepsilon_{it},$$

and note that

$$\begin{aligned}
E\left[Q_{i,T}\right] &= \sqrt{\frac{1-\rho_T^2}{T}} \sum_{t=2}^{T} E\left[\underline{w}_{it-1,T}\varepsilon_{it}\right] = \sqrt{\frac{1-\rho_T^2}{T}} \sum_{t=2}^{T}\sum_{j=1}^{t-1} \rho_T^{(t-1-j)} E\left[\varepsilon_{ij}\varepsilon_{it}\right] = 0, \\
E\left[Q_{i,T}^2\right] &= \frac{1-\rho_T^2}{T} \sum_{t=2}^{T}\sum_{s=2}^{T} E\left[\underline{w}_{it-1,T}\underline{w}_{is-1,T}\varepsilon_{it}\varepsilon_{is}\right] \\
&= \frac{1-\rho_T^2}{T} \sum_{t=2}^{T}\sum_{s=2}^{T}\sum_{j=1}^{t-1}\sum_{k=1}^{s-1} \rho_T^{(t-1-j)}\rho_T^{(s-1-k)} E\left[\varepsilon_{ij}\varepsilon_{ik}\varepsilon_{it}\varepsilon_{is}\right] \\
&= \sigma^4 \frac{1-\rho_T^2}{T} \sum_{t=2}^{T}\sum_{j=1}^{t-1} \rho_T^{2(t-1-j)}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\omega_{Q,N,T}^2 &= \sum_{i=1}^{N} E\left[Q_{i,T}^2\right] = \sigma^4 N \frac{1-\rho_T^2}{T} \sum_{t=2}^{T}\sum_{j=1}^{t-1} \rho_T^{2(t-1-j)} \\
&= \sigma^4 N \frac{1-\rho_T^2}{T} \sum_{t=2}^{T} \left[\frac{1-\rho_T^{2(t-1)}}{1-\rho_T^2}\right] \\
&= \sigma^4 N \left[1 - \frac{1}{T} - \frac{\rho_T^2\left(1-\rho_T^{2(T-1)}\right)}{T\left(1-\rho_T^2\right)}\right].
\end{aligned}$$

Since, in this case, we take $\rho_T^2 = \exp\{-2/q(T)\}$ with $q(T) = O(1)$, it follows that there exist a positive constant C_q and a positive integer T^* such that for all $T \geq T^*$

$$\rho_T^2 \leq \exp\left\{-\frac{2}{C_q}\right\} < 1.$$

Using this bound, we further deduce that all $T \geq T^*$

$$0 \leq \frac{\rho_T^2\left(1-\rho_T^{2(T-1)}\right)}{T\left(1-\rho_T^2\right)} \leq \frac{1}{T\left(1-\exp\{-2/C_q\}\right)} = O\left(\frac{1}{T}\right),$$

so that

$$\frac{\omega_{Q,N,T}^2}{N} = \sigma^4\left[1 + O\left(\frac{1}{T}\right)\right].$$

Thus, it follows from Assumption 1 that there exists a positive constant C such that

$$0 < \frac{1}{C} \leq \frac{\omega_{Q,N,T}}{\sqrt{N}} \leq C < \infty \text{ eventually as } N, T \to \infty.$$

Moreover, by Assumptions 1 and 4,

$$
\begin{aligned}
& E\left[\left(\sqrt{\frac{1-\rho_T^2}{NT}}\sum_{i=1}^{N}\sum_{t=2}^{T}\varepsilon_{it}\rho_T^{t-1}w_{i0}\right)^2\right] \\
= & \frac{1-\rho_T^2}{NT}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=2}^{T}\sum_{s=2}^{T}\rho_T^{t-1}\rho_T^{s-1}E\left[w_{i0}w_{j0}\right]E\left[\varepsilon_{it}\varepsilon_{js}\right] \\
= & \frac{\sigma^2\left(1-\rho_T^2\right)}{NT}\sum_{i=1}^{N}E\left[w_{i0}^2\right]\sum_{t=2}^{T}\rho_T^{2(t-1)} \\
= & \frac{\sigma^2\left(1-\rho_T^2\right)}{T}\left(\sup_i E\left[w_{i0}^2\right]\right)\frac{\rho_T^2\left(1-\rho_T^{2(T-1)}\right)}{1-\rho_T^2} \\
= & O\left(T^{-1}\right)\times O\left(1\right)\times O\left(1\right)=O\left(T^{-1}\right),
\end{aligned}
$$

from which it follows, by Markov's inequality, that

$$
\sqrt{\frac{1-\rho_T^2}{NT}}\sum_{i=1}^{N}\sum_{t=2}^{T}\varepsilon_{it}\rho_T^{t-1}w_{i0}=O_p\left(\frac{1}{\sqrt{T}}\right).
$$

Hence,

$$
\begin{aligned}
& \frac{1}{\omega_{Q,N,T}/\sqrt{N}}\sqrt{\frac{1-\rho_T^2}{NT}}\sum_{i=1}^{N}\sum_{t=2}^{T}w_{it-1,T}\varepsilon_{it} \\
= & \frac{1}{\omega_{Q,N,T}/\sqrt{N}}\sqrt{\frac{1-\rho_T^2}{NT}}\sum_{i=1}^{N}\sum_{t=g+2}^{T}\underline{w}_{it-1,T}\varepsilon_{it}+\frac{1}{\omega_{Q,N,T}/\sqrt{N}}\sqrt{\frac{1-\rho_T^2}{NT}}\sum_{i=1}^{N}\sum_{t=g+2}^{T}\varepsilon_{it}\rho_T^{t-1}w_{i0} \\
= & \frac{1}{\omega_{Q,N,T}/\sqrt{N}}\frac{1}{\sqrt{N}}\sum_{i=1}^{N}Q_{i,T}+O_p\left(\frac{1}{\sqrt{T}}\right).
\end{aligned}
$$

Thus, showing the desired result is equivalent to showing the asymptotic normality of

$$
U_{Q,N,T}=\frac{1}{\omega_{Q,N,T}/\sqrt{N}}\frac{1}{\sqrt{N}}\sum_{i=1}^{N}Q_{i,T}=\frac{1}{\omega_{Q,N,T}/\sqrt{N}}\sqrt{\frac{1-\rho_T^2}{NT}}\sum_{i=1}^{N}\sum_{t=2}^{T}w_{it-1}\varepsilon_{it}.
$$

To show the asymptotic normality of $U_{N,T}$, it suffices to verify a Liapounov-type condition of the form

$$
\lim_{N,T\to\infty}\frac{1}{N^2}\sum_{i=1}^{N}E\left[Q_{i,T}^4\right]=0.
$$

To proceed, note that following calculations similar to that given in the proof of Lemma SE-23, we have

that, in light of Assumption 1, there exists a positive constant C such that

$$\begin{aligned}
& E\left[Q_{i,T}^4\right] \\
= \; & E\left[\left(\sqrt{\frac{1-\rho_T^2}{T}}\sum_{t=2}^{T}\underline{w}_{it-1,T}\varepsilon_{it}\right)^4\right] \\
= \; & \frac{\left(1-\rho_T^2\right)^2}{T^2}\sum_{g=2}^{T}\sum_{s=2}^{T}\sum_{t=2}^{T}\sum_{u=2}^{T}E\left[\underline{w}_{ig-1,T}\underline{w}_{is-1,T}\underline{w}_{it-1,T}\underline{w}_{iu-1,T}\varepsilon_{ig}\varepsilon_{is}\varepsilon_{it}\varepsilon_{iu}\right] \\
\leq \; & C\left[\frac{\left(1-\rho_T^2\right)^2}{T^2}\sum_{t=2}^{T}\sum_{j=1}^{t-1}\rho_T^{4(t-1-j)}+4\frac{\left(1-\rho_T^2\right)^2}{T^2}\sum_{t=3}^{T}\sum_{s=2}^{t-1}\sum_{j=1}^{s-1}\rho_T^{2(t-1-s)}\rho_T^{2(s-1-j)}\right. \\
& \left.+2\frac{\left(1-\rho_T^2\right)^2}{T^2}\sum_{t=3}^{T}\sum_{s=2}^{t-1}\sum_{j=1}^{t-1}\sum_{k=1}^{s-1}\rho_T^{2(t-1-j)}\rho_T^{2(s-1-k)}\right] \\
= \; & C\left(I+II+III\right), \, (say).
\end{aligned}$$

Moreover, using the upper bound $\rho_T^2 \leq \exp\left\{-2/C_q\right\} < 1$ for all $T \geq T^*$ as above, we have

$$\begin{aligned}
I \; = \; & \frac{\left(1-\rho_T^2\right)^2}{T^2}\sum_{t=2}^{T}\sum_{j=1}^{t-1}\rho_T^{4(t-1-j)} = \frac{\left(1-\rho_T^2\right)^2}{T^2}\sum_{t=2}^{T}\frac{1-\rho_T^{4(t-1)}}{1-\rho_T^4} \\
= \; & \frac{\left(1-\rho_T^2\right)^2}{T^2}\frac{1}{1-\rho_T^4}\left[T-1-\frac{\rho_T^4\left(1-\rho_T^{4(T-1)}\right)}{1-\rho_T^4}\right] \\
= \; & \frac{1}{T^2}\frac{1}{1+\rho_T^2}\left[\left(1-\rho_T^2\right)\left(T-1\right)-\frac{\rho_T^4\left(1-\rho_T^{4(T-1)}\right)}{1+\rho_T^2}\right] \\
\leq \; & \frac{1}{T}\left[1+\frac{1}{2T}\right] = O\left(\frac{1}{T}\right),
\end{aligned}$$

$$
\begin{aligned}
II &= 4\frac{\left(1-\rho_T^2\right)^2}{T^2}\sum_{t=3}^{T}\sum_{s=2}^{t-1}\sum_{j=1}^{s-1}\rho_T^{2(t-1-s)}\rho_T^{2(s-1-j)} \\
&= 4\frac{\left(1-\rho_T^2\right)^2}{T^2}\sum_{t=3}^{T}\sum_{s=2}^{t-1}\rho_T^{2(t-1-s)}\frac{1-\rho_T^{2(s-1)}}{1-\rho_T^2} \\
&= 4\frac{\left(1-\rho_T^2\right)^2}{T^2}\frac{1}{1-\rho_T^2}\sum_{t=3}^{T}\sum_{s=2}^{t-1}\rho_T^{2(t-1-s)} - 4\frac{\left(1-\rho_T^2\right)^2}{T^2}\frac{1}{1-\rho_T^2}\sum_{t=3}^{T}\sum_{s=2}^{t-1}\rho_T^{2(t-2)} \\
&= 4\frac{\left(1-\rho_T^2\right)^2}{T^2}\frac{1}{\left(1-\rho_T^2\right)^2}\sum_{t=3}^{T}\left(1-\rho_T^{2(t-2)}\right) - 4\frac{\left(1-\rho_T^2\right)^2}{T^2}\frac{1}{1-\rho_T^2}\sum_{t=3}^{T}\rho_T^{2(t-2)}\left(t-2\right) \\
&= 4\frac{T-2}{T^2} - 4\frac{1}{T^2}\frac{\rho_T^2\left(1-\rho_T^{2(T-2)}\right)}{\left(1-\rho_T^2\right)} - 4\frac{\left(1-\rho_T^2\right)}{T^2}\frac{\rho_T^2-\left(T-1\right)\rho_T^{2(T-1)}+\left(T-2\right)\rho_T^{2T}}{\left(1-\rho_T^2\right)^2} \\
&\quad \text{(by part(b) of Lemma SE-5)} \\
&= \frac{4}{T} - \frac{8}{T^2} - 4\frac{1}{T^2}\frac{\rho_T^2\left(1-\rho_T^{2(T-2)}\right)}{\left(1-\rho_T^2\right)} - \frac{4\rho_T^2}{T^2\left(1-\rho_T^2\right)} + \frac{4\rho_T^{2(T-1)}}{T\left(1-\rho_T^2\right)} - \frac{4\rho_T^{2T}}{T\left(1-\rho_T^2\right)} \\
&\quad -\frac{4\rho_T^{2(T-1)}}{T^2\left(1-\rho_T^2\right)} + \frac{8\rho_T^{2T}}{T^2\left(1-\rho_T^2\right)} \\
&\leq \frac{4}{T}\left[1+\frac{2}{\left(1-\exp\left\{-2/C_q\right\}\right)}\right] + \frac{4}{T^2}\left[2+\frac{5}{\left(1-\exp\left\{-2/C_q\right\}\right)}\right] \\
&= O\left(\frac{1}{T}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
III &= 2\frac{\left(1-\rho_T^2\right)^2}{T^2}\sum_{t=3}^{T}\sum_{s=2}^{t-1}\sum_{j=1}^{t-1}\sum_{k=1}^{s-1}\rho_T^{2(t-1-j)}\rho_T^{2(s-1-k)} \\
&= \frac{\left(1-\rho_T^2\right)^2}{T^2}\frac{2}{\left(1-\rho_T^2\right)^2}\sum_{t=3}^{T}\left(1-\rho_T^{2(t-1)}\right)\sum_{s=2}^{t-1}\left(1-\rho_T^{2(s-1)}\right) \\
&= \frac{2}{T^2}\sum_{t=3}^{T}\left(1-\rho_T^{2(t-1)}\right)\left[(t-2)-\frac{\rho_T^2\left(1-\rho_T^{2(t-2)}\right)}{\left(1-\rho_T^2\right)}\right] \\
&= \frac{2}{T^2}\sum_{t=3}^{T}(t-2)-\frac{2\rho_T^2}{T^2}\sum_{t=3}^{T}\rho_T^{2(t-2)}(t-2)-2\frac{(T-2)}{T^2}\frac{\rho_T^2}{\left(1-\rho_T^2\right)}+\frac{2}{T^2}\frac{\rho_T^6\left(1-\rho_T^{2(T-2)}\right)}{\left(1-\rho_T^2\right)^2} \\
&\quad +\frac{2}{T^2}\frac{\rho_T^4\left(1-\rho_T^{2(T-2)}\right)}{\left(1-\rho_T^2\right)^2}-\frac{2}{T^2}\frac{\rho_T^8\left(1-\rho_T^{4(T-2)}\right)}{\left(1-\rho_T^2\right)\left(1-\rho_T^4\right)} \\
&= \frac{(T-2)(T-1)}{T^2}-\frac{2\rho_T^2}{T^2}\frac{\rho_T^2-(T-1)\rho_T^{2(T-1)}+(T-2)\rho_T^{2T}}{\left(1-\rho_T^2\right)^2}\ \text{(by part(b) of Lemma SE-5)} \\
&\quad -\frac{2\rho_T^2}{T\left(1-\rho_T^2\right)}+\frac{4\rho_T^2}{T^2\left(1-\rho_T^2\right)}+\frac{2}{T^2}\left[\frac{\rho_T^6\left(1-\rho_T^{2(T-2)}\right)}{\left(1-\rho_T^2\right)^2}+\frac{\rho_T^4\left(1-\rho_T^{2(T-2)}\right)}{\left(1-\rho_T^2\right)^2}-\frac{\rho_T^8\left(1-\rho_T^{4(T-2)}\right)}{\left(1-\rho_T^2\right)\left(1-\rho_T^4\right)}\right] \\
&= 1-\frac{3}{T}-\frac{2\rho_T^2}{T\left(1-\rho_T^2\right)}+\frac{2\rho_T^{2T}}{T\left(1-\rho_T^2\right)^2}-\frac{2\rho_T^{2(T+1)}}{T\left(1-\rho_T^2\right)^2} \\
&\quad +\frac{2}{T^2}-\frac{2\rho_T^4}{T^2\left(1-\rho_T^2\right)^2}-\frac{2\rho_T^{2T}}{T^2\left(1-\rho_T^2\right)^2}+\frac{4\rho_T^{2(T+1)}}{T^2\left(1-\rho_T^2\right)^2}+\frac{4\rho_T^2}{T^2\left(1-\rho_T^2\right)} \\
&\quad +\frac{2}{T^2}\frac{\rho_T^6\left(1-\rho_T^{2(T-2)}\right)}{\left(1-\rho_T^2\right)^2}+\frac{2}{T^2}\frac{\rho_T^4\left(1-\rho_T^{2(T-2)}\right)}{\left(1-\rho_T^2\right)^2}-\frac{2}{T^2}\frac{\rho_T^8\left(1-\rho_T^{4(T-2)}\right)}{\left(1-\rho_T^2\right)\left(1-\rho_T^4\right)} \\
&\le 1+\frac{1}{T}\left[3+\frac{2}{\left(1-\exp\{-2/C_q\}\right)}+\frac{4}{\left(1-\exp\{-2/C_q\}\right)^2}\right] \\
&\quad +\frac{2}{T^2}\left[1+\frac{2}{\left(1-\exp\{-2/C_q\}\right)}+\frac{6}{\left(1-\exp\{-2/C_q\}\right)^2}+\frac{2}{\left(1-\exp\{-2/C_q\}\right)\left(1-\exp\{-4/C_q\}\right)}\right] \\
&= O(1),
\end{aligned}
$$

from which we deduce that

$$
\frac{1}{N^2}\sum_{i=1}^{N}E\left[Q_{i,T}^4\right]=O\left(\frac{1}{N}\right)=o(1).
$$

Since the Liapounov condition implies the Lindeberg-type condition (13), we deduce from Lemma SE-10 that

$$
U_{Q,N,T}=\frac{1}{\omega_{Q,N,T}/\sqrt{N}}\sqrt{\frac{1-\rho_T^2}{NT}}\sum_{i=1}^{N}\sum_{t=2}^{T}\underline{w}_{it-1}\varepsilon_{it}\Rightarrow\mathcal{Z}\equiv N(0,1)
$$

as $N, T \to \infty$. By the Cramér convergence theorem, we then have that

$$\sqrt{\frac{1-\rho_T^2}{NT}} \sum_{i=1}^{N} \sum_{t=2}^{T} \underline{w}_{it-1} \varepsilon_{it} = \frac{\omega_{Q,N,T}}{\sqrt{N}} U_{Q,N,T} = \sigma^2 U_{Q,N,T} \left[1 + O\left(\frac{1}{T}\right) \right] \Rightarrow N\left(0, \sigma^4\right)$$

as required. □

Lemma SE-28:

Under Assumptions 1 and 4, the following statements are hold as $N, T \to \infty$.

(a) If $\rho_T = 1$ for all T sufficiently large, then

$$\sum_{i=1}^{N} \overline{w}_{i,-1}^2 = O_p\left(NT\right).$$

(b) If $\rho_T = \exp\left\{-1/q\left(T\right)\right\}$ such that $T/q\left(T\right) \to 0$, then

$$\sum_{i=1}^{N} \overline{w}_{i,-1}^2 = O_p\left(NT\right).$$

(c) If $\rho_T = \exp\left\{-1/q\left(T\right)\right\}$ such that $q\left(T\right) \sim T$, then

$$\sum_{i=1}^{N} \overline{w}_{i,-1}^2 = O_p\left(Nq\left(T\right)\right) = O_p\left(NT\right).$$

(d) If $\rho_T = \exp\left\{-1/q\left(T\right)\right\}$ such that $q\left(T\right) \to \infty$ but $q\left(T\right)/T \to 0$, then

$$\sum_{i=1}^{N} \overline{w}_{i,-1}^2 = O_p\left(\frac{Nq\left(T\right)^2}{T}\right).$$

(e) If $\rho_T \in \mathcal{G}_{\mathrm{St}} = \left\{ |\rho_T| = \exp\left\{-\frac{1}{q(T)}\right\} : q\left(T\right) \geq 0 \text{ and } q\left(T\right) = O\left(1\right) \text{ as } T \to \infty \right\}$, then

$$\sum_{i=1}^{N} \overline{w}_{i,-1}^2 = O_p\left(\frac{N}{T}\right).$$

Proof of Lemma SE-28:

To proceed, note that

$$\begin{aligned}
\sum_{i=1}^{N} \overline{w}_{i,-1}^2 &= \sum_{i=1}^{N} \left(\frac{1}{T-1} \sum_{t=2}^{T} w_{it-1,T} \right)^2 \\
&= \sum_{i=1}^{N} \left(\frac{1}{T-1} \sum_{t=2}^{T} \underline{w}_{it-1,T} + \frac{1}{T-1} \sum_{t=2}^{T} \rho_T^{t-1} w_{i0} \right)^2 \\
&= \sum_{i=1}^{N} \left(\frac{1}{T-1} \sum_{t=2}^{T} \underline{w}_{it-1,T} \right)^2 + 2 \sum_{i=1}^{N} \left(\frac{1}{T-1} \sum_{t=2}^{T} \underline{w}_{it-1,T} \right) \left(\frac{1}{T-1} \sum_{t=2}^{T} \rho_T^{t-1} w_{i0} \right) \\
&\quad + \sum_{i=1}^{N} \left(\frac{1}{T-1} \sum_{t=2}^{T} \rho_T^{t-1} w_{i0} \right)^2,
\end{aligned}$$

where $\underline{w}_{it-1,T} = \sum_{j=1}^{t-1} \rho_T^{(t-1-j)} \varepsilon_{ij}$.

Consider first part (a). Note that, under the assumption here, there exists a positive integer I_ρ such that for all $T \geq I_\rho$, the triangular array process $\left\{\underline{w}_{it,T}\right\}$ has the partial sum representation $\underline{w}_{it,T} = \sum_{j=1}^{t} \varepsilon_{ij}$. Hence, for all $T \geq \max\{I_\rho, 3\}$, we obtain by direct calculation

$$\begin{aligned}
& E\left[\sum_{i=1}^{N}\left(\frac{1}{T-1}\sum_{t=2}^{T}\underline{w}_{it-1,T}\right)^2\right] \\
= & \frac{1}{(T-1)^2}\sum_{i=1}^{N}\sum_{s=2}^{T}\sum_{t=2}^{T}\sum_{g=1}^{s-1}\sum_{v=1}^{t-1} E\left[\varepsilon_{ig}\varepsilon_{iv}\right] \\
= & \frac{\sigma^2 N}{(T-1)^2}\sum_{t=2}^{T}(t-1) + \frac{2\sigma^2 N}{(T-1)^2}\sum_{t=3}^{T}\sum_{s=2}^{t-1}(s-1) \\
= & \frac{\sigma^2 N}{(T-1)^2}\frac{T(T-1)}{2} + \frac{2\sigma^2 N}{(T-1)^2}\sum_{t=3}^{T}\frac{(t-1)(t-2)}{2} \\
= & \frac{\sigma^2}{2}\frac{NT}{(T-1)} + \frac{\sigma^2 N}{(T-1)^2}\left[\frac{(T-2)(T-1)(2T-3)}{6} + \frac{(T-2)(T-1)}{2}\right] \\
= & \frac{\sigma^2}{3}NT\left[1 + O\left(\frac{1}{T}\right)\right].
\end{aligned}$$

Applying Markov's inequality, we deduce that

$$\sum_{i=1}^{N}\left(\frac{1}{T-1}\sum_{t=2}^{T}\underline{w}_{it-1,T}\right)^2 = O_p(NT).$$

Moreover, by Assumption 4,

$$E\left[\sum_{i=1}^{N}\left(\frac{1}{T-1}\sum_{t=2}^{T}w_{i0}\right)^2\right] \leq N\left(\sup_i E\left[w_{i0}^2\right]\right) = O(N)$$

from which it follows, by Markov's inequality, that

$$\sum_{i=1}^{N}\left(\frac{1}{T-1}\sum_{t=2}^{T}w_{i0}\right)^2 = O_p(N).$$

Also, by the Cauchy-Schwarz inequality,

$$\begin{aligned}
& \left|\sum_{i=1}^{N}\left(\frac{1}{T-1}\sum_{t=2}^{T}\underline{w}_{it-1,T}\right)\left(\frac{1}{T-1}\sum_{t=2}^{T}w_{i0}\right)\right| \\
\leq & \sqrt{\sum_{i=1}^{N}\left(\frac{1}{T-1}\sum_{t=2}^{T}\underline{w}_{it-1,T}\right)^2}\sqrt{\sum_{i=1}^{N}\left(\frac{1}{T-1}\sum_{t=2}^{T}w_{i0}\right)^2} = O_p\left(\sqrt{NT}\right)O_p\left(\sqrt{N}\right) = O_p\left(N\sqrt{T}\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
\sum_{i=1}^{N}\overline{w}_{i,-1}^{2} &= \sum_{i=1}^{N}\left(\frac{1}{T-1}\sum_{t=2}^{T}\underline{w}_{it-1,T}\right)^{2}+2\sum_{i=1}^{N}\left(\frac{1}{T-1}\sum_{t=2}^{T}\underline{w}_{it-1,T}\right)\left(\frac{1}{T-1}\sum_{t=2}^{T}w_{i0}\right)\\
&\quad+\sum_{i=1}^{N}\left(\frac{1}{T-1}\sum_{t=2}^{T}w_{i0}\right)^{2}\quad\text{(for all } T \text{ sufficiently large)}\\
&= O_p\left(NT\right)+O_p\left(N\sqrt{T}\right)+O_p\left(N\right)=O_p\left(NT\right)
\end{aligned}$$

as required for part (a).

Now, to show parts (b)-(d), we first make some preliminary moment calculations. Note that

$$\begin{aligned}
&E\left[\sum_{i=1}^{N}\left(\frac{1}{T-1}\sum_{s=2}^{T}\underline{w}_{is-1,T}\right)^{2}\right]\\
&= \frac{1}{(T-1)^{2}}\sum_{i=1}^{N}\sum_{s=2}^{T}\sum_{t=2}^{T}\sum_{g=1}^{s-1}\sum_{\upsilon=1}^{t-1}\exp\left\{-\frac{s-1-g}{q(T)}\right\}\exp\left\{-\frac{t-1-\upsilon}{q(T)}\right\}E\left[\varepsilon_{ig}\varepsilon_{i\upsilon}\right]\\
&= \frac{\sigma^{2}}{(T-1)^{2}}\sum_{i=1}^{N}\sum_{t=2}^{T}\sum_{\upsilon=1}^{t-1}\exp\left\{-2\frac{t-1-\upsilon}{q(T)}\right\}\\
&\quad+2\frac{\sigma^{2}}{(T-1)^{2}}\sum_{i=1}^{N}\sum_{t=3}^{T}\sum_{s=2}^{t-1}\sum_{g=1}^{s-1}\exp\left\{-\frac{s-1-g}{q(T)}\right\}\exp\left\{-\frac{t-1-g}{q(T)}\right\}\\
&= \frac{\sigma^{2}N}{(T-1)^{2}}\sum_{t=2}^{T}\sum_{\upsilon=1}^{t-1}\exp\left\{-2\frac{t-1-\upsilon}{q(T)}\right\}\\
&\quad+2\frac{\sigma^{2}N}{(T-1)^{2}}\sum_{t=3}^{T}\sum_{s=2}^{t-1}\sum_{g=1}^{s-1}\exp\left\{-\frac{s-1-g}{q(T)}\right\}\exp\left\{-\frac{t-1-g}{q(T)}\right\}.
\end{aligned}$$

Next, consider part (b), where we take $\rho_T=\exp\left\{-1/q(T)\right\}$ such that $T/q(T)\to 0$. Applying part (a) of Lemma SE-1 with $b=2$ and $d=2$ and part (a) of Lemma SE-7 with $b=1$ and $g=1$, we obtain

$$\begin{aligned}
&E\left[\sum_{i=1}^{N}\left(\frac{1}{T-1}\sum_{s=2}^{T}\underline{w}_{is-1,T}\right)^{2}\right]\\
&= \frac{\sigma^{2}N}{(T-1)^{2}}\sum_{t=2}^{T}\sum_{\upsilon=1}^{t-1}\exp\left\{-2\frac{t-1-\upsilon}{q(T)}\right\}\\
&\quad+2\frac{\sigma^{2}N}{(T-1)^{2}}\sum_{t=3}^{T}\sum_{s=2}^{t-1}\sum_{g=1}^{s-1}\exp\left\{-\frac{s-1-g}{q(T)}\right\}\exp\left\{-\frac{t-1-g}{q(T)}\right\}\\
&= \frac{\sigma^{2}N}{(T-1)^{2}}\frac{T^{2}}{2}\left[1+O\left(\frac{1}{T}\right)\right]+2\frac{\sigma^{2}N}{(T-1)^{2}}\frac{T^{3}}{6}\left[1+O\left(\frac{T}{q(T)}\right)+O\left(\frac{1}{T}\right)\right]\\
&= \frac{\sigma^{2}}{3}NT\left[1+O\left(\frac{T}{q(T)}\right)+O\left(\frac{1}{T}\right)\right]=O\left(NT\right).
\end{aligned}$$

Using Markov's inequality, we deduce that

$$\sum_{i=1}^{N}\left(\frac{1}{T-1}\sum_{s=2}^{T}\underline{w}_{is-1,T}\right)^{2}=O_{p}\left(NT\right).$$

Moreover, by Assumption 4,

$$\begin{aligned}
& E\left[\sum_{i=1}^{N}\left(\frac{1}{T-1}\sum_{t=2}^{T}\rho_{T}^{t-1}w_{i0}\right)^{2}\right] \\
\leq\ & \frac{N}{\left(T-1\right)^{2}}\left(\sup_{i}E\left[w_{i0}^{2}\right]\right)\exp\left\{-\frac{2}{q\left(T\right)}\right\}\left[1-\exp\left\{-\frac{1}{q\left(T\right)}\right\}\right]^{-2}\left[1-\exp\left\{-\frac{T-1}{q\left(T\right)}\right\}\right]^{2} \\
=\ & O\left(NT^{-2}\right)O\left(1\right)O\left(1\right)O\left(q\left(T\right)^{2}\right)O\left(T^{2}/q\left(T\right)^{2}\right)=O\left(N\right)
\end{aligned}$$

from which it follows by Markov's inequality that

$$\sum_{i=1}^{N}\left(\frac{1}{T-1}\sum_{t=2}^{T}\rho_{T}^{t-1}w_{i0}\right)^{2}=O_{p}\left(N\right).$$

Also, by the Cauchy-Schwarz inequality,

$$\begin{aligned}
& \left|\sum_{i=1}^{N}\left(\frac{1}{T-1}\sum_{t=2}^{T}\underline{w}_{it-1,T}\right)\left(\frac{1}{T-1}\sum_{t=2}^{T}\rho_{T}^{t-1}w_{i0}\right)\right| \\
\leq\ & \sqrt{\sum_{i=1}^{N}\left(\frac{1}{T-1}\sum_{s=2}^{T}\underline{w}_{is-1,T}\right)^{2}}\sqrt{\sum_{i=1}^{N}\left(\frac{1}{T-1}\sum_{t=2}^{T}\rho_{T}^{t-1}w_{i0}\right)^{2}} \\
=\ & O_{p}\left(\sqrt{NT}\right)O_{p}\left(\sqrt{N}\right)=O_{p}\left(N\sqrt{T}\right).
\end{aligned}$$

so that

$$\begin{aligned}
\sum_{i=1}^{N}\overline{w}_{i,-1}^{2} &= \sum_{i=1}^{N}\left(\frac{1}{T-1}\sum_{t=2}^{T}\underline{w}_{it-1,T}\right)^{2}+2\sum_{i=1}^{N}\left(\frac{1}{T-1}\sum_{t=2}^{T}\underline{w}_{it-1,T}\right)\left(\frac{1}{T-1}\sum_{t=2}^{T}\rho_{T}^{t-1}w_{i0}\right) \\
&\quad +\sum_{i=1}^{N}\left(\frac{1}{T-1}\sum_{t=2}^{T}\rho_{T}^{t-1}w_{i0}\right)^{2} \\
&= O_{p}\left(NT\right)+O_{p}\left(N\sqrt{T}\right)+O_{p}\left(N\right)=O_{p}\left(NT\right),
\end{aligned}$$

as required for part (b).

To show part (c), where we take $\rho_{T}=\exp\left\{-1/q\left(T\right)\right\}$ such that $q\left(T\right)\sim T$, we apply part (b) of

Lemma SE-1 with $b=2$ and $d=2$ and part (b) of Lemma SE-7 with $b=1$ to obtain

$$\begin{aligned}
& E\left[\sum_{i=1}^{N}\left(\frac{1}{T-1}\sum_{s=2}^{T}\underline{w}_{is-1,T}\right)^{2}\right] \\
= & \frac{\sigma^{2}N}{(T-1)^{2}}\sum_{t=2}^{T}\sum_{\upsilon=1}^{t-1}\exp\left\{-2\frac{t-1-\upsilon}{q(T)}\right\} \\
& +2\frac{\sigma^{2}N}{(T-1)^{2}}\sum_{t=3}^{T}\sum_{s=2}^{t-1}\sum_{g=1}^{s-1}\exp\left\{-\frac{s-1-g}{q(T)}\right\}\exp\left\{-\frac{t-1-g}{q(T)}\right\} \\
= & \frac{\sigma^{2}N}{(T-1)^{2}}\frac{q(T)^{2}}{4}\left[\exp\left\{-\frac{2T}{q(T)}\right\}+\frac{2T}{q(T)}-1\right]\left[1+O\left(\frac{1}{T}\right)\right] \\
& +\frac{2\sigma^{2}N}{(T-1)^{2}}\frac{Tq(T)^{2}}{2}\left[1-\frac{3}{2}\frac{q(T)}{T}+2\frac{q(T)}{T}\exp\left\{-\frac{T}{q(T)}\right\}-\frac{1}{2}\frac{q(T)}{T}\exp\left\{-\frac{2T}{q(T)}\right\}\right]\left[1+O\left(\frac{1}{T}\right)\right] \\
= & \sigma^{2}Nq(T)\frac{q(T)}{T}\left[1-\frac{3}{2}\frac{q(T)}{T}+2\frac{q(T)}{T}\exp\left\{-\frac{T}{q(T)}\right\}-\frac{1}{2}\frac{q(T)}{T}\exp\left\{-\frac{2T}{q(T)}\right\}\right]\left[1+O\left(\frac{1}{T}\right)\right] \\
= & O(Nq(T)).
\end{aligned}$$

Using Markov's inequality, we deduce that

$$\sum_{i=1}^{N}\left(\frac{1}{T-1}\sum_{s=2}^{T}\underline{w}_{is-1,T}\right)^{2}=O_{p}(Nq(T))=O_{p}(NT).$$

Moreover, by Assumption 4,

$$\begin{aligned}
& E\left[\sum_{i=1}^{N}\left(\frac{1}{T-1}\sum_{t=2}^{T}\rho_{T}^{t-1}w_{i0}\right)^{2}\right] \\
\leq & \frac{N}{(T-1)^{2}}\left(\sup_{i}E\left[w_{i0}^{2}\right]\right)\exp\left\{-\frac{2}{q(T)}\right\}\left[1-\exp\left\{-\frac{1}{q(T)}\right\}\right]^{-2}\left[1-\exp\left\{-\frac{T-1}{q(T)}\right\}\right]^{2} \\
= & O\left(NT^{-2}\right)\times O(1)\times O(1)\times O\left(T^{2}\right)\times O(1)=O(N),
\end{aligned}$$

from which it follows, by Markov's inequality, that

$$\sum_{i=1}^{N}\left(\frac{1}{T-1}\sum_{t=2}^{T}\rho_{T}^{t-1}w_{i0}\right)^{2}=O_{p}(N).$$

Also, by the Cauchy-Schwarz inequality,

$$\begin{aligned}
& \left|\sum_{i=1}^{N}\left(\frac{1}{T-1}\sum_{t=2}^{T}\underline{w}_{it-1,T}\right)\left(\frac{1}{T-1}\sum_{t=2}^{T}\rho_{T}^{t-1}w_{i0}\right)\right| \\
\leq & \sqrt{\sum_{i=1}^{N}\left(\frac{1}{T-1}\sum_{s=2}^{T}\underline{w}_{is-1,T}\right)^{2}}\sqrt{\sum_{i=1}^{N}\left(\frac{1}{T-1}\sum_{t=2}^{T}\rho_{T}^{t-1}w_{i0}\right)^{2}} \\
= & O_{p}\left(\sqrt{NT}\right)O_{p}\left(\sqrt{N}\right)=O_{p}\left(N\sqrt{T}\right),
\end{aligned}$$

so that

$$\begin{aligned}\sum_{i=1}^{N}\overline{w}_{i,-1}^{2} &= \sum_{i=1}^{N}\left(\frac{1}{T-1}\sum_{t=2}^{T}\underline{w}_{it-1,T}\right)^{2}+2\sum_{i=1}^{N}\left(\frac{1}{T-1}\sum_{t=2}^{T}\underline{w}_{it-1,T}\right)\left(\frac{1}{T-1}\sum_{t=2}^{T}\rho_{T}^{t-1}w_{i0}\right)\\ &\quad+\sum_{i=1}^{N}\left(\frac{1}{T-1}\sum_{t=2}^{T}\rho_{T}^{t-1}w_{i0}\right)^{2}\\ &= O_{p}\left(NT\right)+O_{p}\left(N\sqrt{T}\right)+O_{p}\left(N\right)=O_{p}\left(NT\right).\end{aligned}$$

We turn our attention now to part (d), where we take $\rho_T = \exp\{-1/q(T)\}$ such that $q(T)\to\infty$ but $q(T)/T\to 0$. Applying part (c) of Lemma SE-1 with $b=2$ and $d=2$ and part (c) of Lemma SE-7 with $b=1$, we obtain

$$\begin{aligned}&E\left[\sum_{i=1}^{N}\left(\frac{1}{T-1}\sum_{s=2}^{T}\underline{w}_{is-1,T}\right)^{2}\right]\\ &= \frac{\sigma^{2}N}{(T-1)^{2}}\sum_{t=2}^{T}\sum_{\upsilon=1}^{t-1}\exp\left\{-2\frac{t-1-\upsilon}{q(T)}\right\}\\ &\quad+2\frac{\sigma^{2}N}{(T-1)^{2}}\sum_{t=3}^{T}\sum_{s=2}^{t-1}\sum_{g=1}^{s-1}\exp\left\{-\frac{s-1-g}{q(T)}\right\}\exp\left\{-\frac{t-1-g}{q(T)}\right\}\\ &= \frac{\sigma^{2}N}{(T-1)^{2}}\frac{Tq(T)}{2}\left[1+O\left(\frac{1}{q(T)}\right)+O\left(\frac{q(T)}{T}\right)\right]\\ &\quad+2\frac{\sigma^{2}N}{(T-1)^{2}}\frac{Tq(T)^{2}}{2}\left[1+O\left(\max\left\{\frac{q(T)}{T},\frac{1}{q(T)}\right\}\right)\right]\\ &= O\left(\frac{Nq(T)^{2}}{T}\right).\end{aligned}$$

Using Markov's inequality, we deduce that

$$\sum_{i=1}^{N}\left(\frac{1}{T-1}\sum_{s=2}^{T}\underline{w}_{is-1,T}\right)^{2}=O_{p}\left(\frac{Nq(T)^{2}}{T}\right).$$

Moreover, note that, using Assumption 4, we obtain

$$\begin{aligned}&E\left[\sum_{i=1}^{N}\left(\frac{1}{T-1}\sum_{t=2}^{T}\rho_{T}^{t-1}w_{i0}\right)^{2}\right]\\ &\leq \frac{N}{(T-1)^{2}}\left(\sup_{i}E\left[w_{i0}^{2}\right]\right)\exp\left\{-\frac{2}{q(T)}\right\}\left[1-\exp\left\{-\frac{1}{q(T)}\right\}\right]^{-2}\left[1-\exp\left\{-\frac{T-1}{q(T)}\right\}\right]^{2}\\ &= O\left(NT^{-2}\right)\times O(1)\times O(1)\times O\left(q(T)^{2}\right)\times O(1)=O\left(Nq(T)^{2}/T^{2}\right),\end{aligned}$$

so that it follows, by Markov's inequality, that

$$\sum_{i=1}^{N}\left(\frac{1}{T-1}\sum_{t=2}^{T}\rho_{T}^{t-1}w_{i0}\right)^{2}=O_{p}\left(\frac{Nq(T)^{2}}{T^{2}}\right).$$

Also, by the Cauchy-Schwarz inequality,

$$\begin{aligned}
&\left|\sum_{i=1}^{N}\left(\frac{1}{T-1}\sum_{t=2}^{T}\underline{w}_{it-1,T}\right)\left(\frac{1}{T-1}\sum_{t=2}^{T}\rho_T^{t-1}w_{i0}\right)\right| \\
\leq\ & \sqrt{\sum_{i=1}^{N}\left(\frac{1}{T-1}\sum_{s=2}^{T}\underline{w}_{is-1,T}\right)^2}\sqrt{\sum_{i=1}^{N}\left(\frac{1}{T-1}\sum_{t=2}^{T}\rho_T^{t-1}w_{i0}\right)^2} \\
=\ & O_p\left(\sqrt{\frac{Nq(T)^2}{T}}\right)O_p\left(\frac{\sqrt{N}q(T)}{T}\right)=O_p\left(\frac{Nq(T)^2}{T^{3/2}}\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
\sum_{i=1}^{N}\overline{w}_{i,-1}^2 &= \sum_{i=1}^{N}\left(\frac{1}{T-1}\sum_{t=2}^{T}\underline{w}_{it-1,T}\right)^2+2\sum_{i=1}^{N}\left(\frac{1}{T-1}\sum_{t=2}^{T}\underline{w}_{it-1,T}\right)\left(\frac{1}{T-1}\sum_{t=2}^{T}\rho_T^{t-1}w_{i0}\right) \\
&\quad+\sum_{i=1}^{N}\left(\frac{1}{T-1}\sum_{t=2}^{T}\rho_T^{t-1}w_{i0}\right)^2 \\
&= O_p\left(\frac{Nq(T)^2}{T}\right)+O_p\left(\frac{Nq(T)^2}{T^{3/2}}\right)+O_{a.s.}\left(\frac{Nq(T)^2}{T^2}\right)=O_p\left(\frac{Nq(T)^2}{T}\right).
\end{aligned}$$

Finally, we consider part (e), where we take $\rho_T \in \mathcal{G}_{\mathrm{St}}$. Since we assume here that $q(T)=O(1)$, there exist some positive constant C_q and some positive integer T^* such that for all $T \geq T^*$

$$0 \leq |\rho_T| = \exp\left\{-\frac{1}{q(T)}\right\} \leq \exp\left\{-\frac{1}{C_q}\right\} < 1.$$

Using this bound, we obtain for all $T \geq T^*$

$$
\begin{aligned}
& E\left[\sum_{i=1}^{N}\left(\frac{1}{T-1}\sum_{s=2}^{T}\underline{w}_{is-1,T}\right)^2\right] \\
= & \frac{\sigma^2 N}{(T-1)^2}\sum_{t=2}^{T}\sum_{\upsilon=1}^{t-1}\rho_T^{2(t-1-\upsilon)} + 2\frac{\sigma^2 N}{(T-1)^2}\sum_{t=3}^{T}\sum_{s=2}^{t-1}\sum_{g=1}^{s-1}\rho_T^{(s-1-g)}\rho_T^{(t-1-g)} \\
= & \frac{\sigma^2 N}{(T-1)^2}\sum_{t=2}^{T}\frac{1-\rho_T^{2(t-1)}}{1-\rho_T^2} + 2\frac{\sigma^2 N}{(T-1)^2}\sum_{t=3}^{T}\sum_{s=2}^{t-1}\rho_T^{t-1-(s-1)}\sum_{g=1}^{s-1}\rho_T^{2(s-1-g)} \\
= & \frac{N}{(T-1)}\frac{\sigma^2}{1-\rho_T^2} - \frac{N\sigma^2}{(T-1)^2}\frac{\rho_T^2\left(1-\rho_T^{2(T-1)}\right)}{\left(1-\rho_T^2\right)^2} + 2\frac{\sigma^2 N}{(T-1)^2}\sum_{t=3}^{T}\sum_{s=2}^{t-1}\rho_T^{(t-s)}\frac{1-\rho_T^{2(s-1)}}{1-\rho_T^2} \\
= & \frac{N}{(T-1)}\frac{\sigma^2}{1-\rho_T^2} - \frac{N\sigma^2}{(T-1)^2}\frac{\rho_T^2\left(1-\rho_T^{2(T-1)}\right)}{\left(1-\rho_T^2\right)^2} \\
& +2\frac{N}{(T-1)^2}\frac{\sigma^2}{1-\rho_T^2}\sum_{t=3}^{T}\frac{\rho_T\left(1-\rho_T^{2(t-2)}\right)}{1-\rho_T^2} - 2\frac{N}{(T-1)^2}\frac{\sigma^2}{1-\rho_T^2}\sum_{t=3}^{T}\rho_T^t\frac{1-\rho_T^{(t-2)}}{1-\rho_T} \\
= & \frac{N}{(T-1)}\frac{\sigma^2}{1-\rho_T^2} - \frac{N\sigma^2}{(T-1)^2}\frac{\rho_T^2\left(1-\rho_T^{2(T-1)}\right)}{\left(1-\rho_T^2\right)^2} \\
& +2\frac{N(T-2)}{(T-1)^2}\frac{\sigma^2\rho_T}{\left(1-\rho_T^2\right)^2} - 2\frac{N}{(T-1)^2}\frac{\sigma^2\rho_T^3\left(1-\rho_T^{2(T-2)}\right)}{\left(1-\rho_T^2\right)^3} \\
& -2\frac{N}{(T-1)^2}\frac{\sigma^2\rho_T^3\left(1-\rho_T^{(T-2)}\right)}{\left(1-\rho_T^2\right)(1-\rho_T)^2} + 2\frac{N}{(T-1)^2}\frac{\sigma^2\rho_T^4\left(1-\rho_T^{2(T-2)}\right)}{\left(1-\rho_T^2\right)^2(1-\rho_T)} \\
\leq & \frac{N}{(T-1)}\frac{\sigma^2}{1-\exp\{-2/C_q\}} + \frac{N}{(T-1)}\frac{2\sigma^2}{(1-\exp\{-2/C_q\})^2} + \frac{N}{(T-1)^2}\frac{2\sigma^2}{(1-\exp\{-2/C_q\})^2} \\
& +\frac{N}{(T-1)^2}\frac{\sigma^2}{(1-\exp\{-2/C_q\})^2} + 2\frac{N}{(T-1)^2}\frac{\sigma^2}{(1-\exp\{-2/C_q\})^3} \\
& +2\frac{N}{(T-1)^2}\frac{\sigma^2}{(1-\exp\{-2/C_q\})(1-\exp\{-1/C_q\})^2} + \frac{N}{(T-1)^2}\frac{2\sigma^2}{(1-\exp\{-2/C_q\})^2(1-\exp\{-1/C_q\})} \\
= & O\left(\frac{N}{T}\right).
\end{aligned}
$$

Using Markov's inequality, we deduce that

$$
\sum_{i=1}^{N}\left(\frac{1}{T-1}\sum_{s=2}^{T}\underline{w}_{is-1,T}\right)^2 = O_p\left(\frac{N}{T}\right).
$$

Moreover, note that, by Assumption 4,

$$\begin{aligned} E\left[\sum_{i=1}^{N}\left(\frac{1}{T-1}\sum_{t=2}^{T}\rho_T^{t-1}w_{i0}\right)^2\right] &\leq \frac{N}{(T-1)^2}\left(\sup_i E\left[w_{i0}^2\right]\right)\left(\sum_{t=2}^{T}|\rho_T|^{t-1}\right)^2 \\ &\leq \frac{N}{(T-1)^2}\left(\sup_i E\left[w_{i0}^2\right]\right)^2\frac{|\rho_T|^2\left(1-|\rho_T|^{T-1}\right)^2}{(1-|\rho_T|)^2} \\ &= O\left(NT^{-2}\right)\times O(1)\times O(1) = O\left(N/T^2\right), \end{aligned}$$

so that it follows, by Markov's inequality, that

$$\sum_{i=1}^{N}\left(\frac{1}{T-1}\sum_{t=2}^{T}\rho_T^{t-1}w_{i0}\right)^2 = O_p\left(\frac{N}{T^2}\right).$$

Also, by the Cauchy-Schwarz inequality,

$$\begin{aligned} &\left|\sum_{i=1}^{N}\left(\frac{1}{T-1}\sum_{t=2}^{T}\underline{w}_{it-1,T}\right)\left(\frac{1}{T-1}\sum_{t=2}^{T}\rho_T^{t-1}w_{i0}\right)\right| \\ \leq\ & \sqrt{\sum_{i=1}^{N}\left(\frac{1}{T-1}\sum_{s=2}^{T}\underline{w}_{is-1,T}\right)^2}\sqrt{\sum_{i=1}^{N}\left(\frac{1}{T-1}\sum_{t=2}^{T}\rho_T^{t-1}w_{i0}\right)^2} \\ =\ & O_p\left(\sqrt{\frac{N}{T}}\right)O_p\left(\frac{\sqrt{N}}{T}\right) = O_p\left(\frac{N}{T^{3/2}}\right), \end{aligned}$$

so that

$$\begin{aligned} \sum_{i=1}^{N}\overline{w}_{i,-1}^2 &= \sum_{i=1}^{N}\left(\frac{1}{T-1}\sum_{t=2}^{T}\underline{w}_{it-1,T}\right)^2 + 2\sum_{i=1}^{N}\left(\frac{1}{T-1}\sum_{t=2}^{T}\underline{w}_{it-1,T}\right)\left(\frac{1}{T-1}\sum_{t=2}^{T}\rho_T^{t-1}w_{i0}\right) \\ &\quad + \sum_{i=1}^{N}\left(\frac{1}{T-1}\sum_{t=2}^{T}\rho_T^{t-1}w_{i0}\right)^2 \\ &= O_p\left(\frac{N}{T}\right) + O_p\left(\frac{N}{T^{3/2}}\right) + O_p\left(\frac{N}{T^2}\right) = O_p\left(\frac{N}{T}\right), \end{aligned}$$

as required for part (e). □

Lemma SE-29:

Under Assumptions 1 and 4, the following statements hold as $N, T \to \infty$.

(a) If $\rho_T = 1$ for all T sufficiently large, then

$$\frac{1}{N}\sum_{i=1}^{N}\overline{\varepsilon}_i\overline{w}_{i,-1} = O_p(1).$$

(b) If $\rho_T = \exp\{-1/q(T)\}$ such that $T/q(T) \to 0$, then

$$\frac{1}{N}\sum_{i=1}^{N} \overline{\varepsilon}_i \overline{w}_{i,-1} = O_p(1).$$

(c) If $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \sim T$, then

$$\frac{1}{N}\sum_{i=1}^{N} \overline{\varepsilon}_i \overline{w}_{i,-1} = O_p(1).$$

(d) If $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \to \infty$ but $q(T)/T \to 0$, then

$$\frac{1}{N}\sum_{i=1}^{N} \overline{\varepsilon}_i \overline{w}_{i,-1} = O_p\left(\frac{q(T)}{T}\right).$$

(e) If $\rho_T \in \mathcal{G}_{\mathrm{St}} = \left\{|\rho_T| = \exp\left\{-\frac{1}{q(T)}\right\} : q(T) \geq 0 \text{ and } q(T) = O(1) \text{ as } T \to \infty\right\}$, then

$$\frac{1}{N}\sum_{i=1}^{N} \overline{\varepsilon}_i \overline{w}_{i,-1} = O_p\left(\frac{1}{T}\right).$$

Proof of Lemma SE-29:

By the Cauchy-Schwarz inequality, we have

$$\left|\frac{1}{N}\sum_{i=1}^{N} \overline{\varepsilon}_i \overline{w}_{i,-1}\right| \leq \frac{1}{N}\sum_{i=1}^{N} |\overline{\varepsilon}_i \overline{w}_{i,-1}| \leq \sqrt{\frac{1}{N}\sum_{i=1}^{N} \overline{\varepsilon}_i^2}\sqrt{\frac{1}{N}\sum_{i=1}^{N} \overline{w}_{i,-1}^2}.$$

The results for parts (a)-(d) then follow immediately from applying, respectively, the results from parts (a)-(d) of Lemma SE-28 as well as part (e) of Lemma SE-11. □

Lemma SE-30:

Under Assumptions 1-4, the following statements hold as $N, T \to \infty$.

(a) If $\rho_T = 1$ for all T sufficiently large, then

$$\sum_{i=1}^{N} a_i w_{iT-2,T} = O_p\left(\max\left\{\sqrt{NT}, N\right\}\right).$$

(b) If $\rho_T = \exp\{-1/q(T)\}$ such that $T/q(T) \to 0$, then

$$\sum_{i=1}^{N} a_i w_{iT-2,T} = O_p\left(\max\left\{\sqrt{NT}, N\right\}\right).$$

(c) If $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \sim T$, then

$$\sum_{i=1}^{N} a_i w_{iT-2,T} = O_p\left(\max\left\{\sqrt{NT}, N\right\}\right).$$

(d) If $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \to \infty$ but $q(T)/T \to 0$, then

$$\sum_{i=1}^{N} a_i w_{iT-2,T} = O_p\left(\sqrt{Nq(T)}\right).$$

(e) If $\rho_T \in \mathcal{G}_{\mathrm{St}} = \left\{|\rho_T| = \exp\left\{-\frac{1}{q(T)}\right\} : q(T) \geq 0 \text{ and } q(T) = O(1) \text{ as } T \to \infty\right\}$, then

$$\sum_{i=1}^{N} a_i w_{iT-2,T} = O_p\left(\sqrt{N}\right).$$

Proof of Lemma SE-30:

To proceed, first write

$$\sum_{i=1}^{N} a_i w_{iT-2,T} = \sum_{i=1}^{N} a_i \underline{w}_{iT-2,T} + \sum_{i=1}^{N} a_i \rho_T^{T-2} w_{i0},$$

where $\underline{w}_{iT-2} = \sum_{j=1}^{T-2} \rho_T^{(T-2-j)} \varepsilon_{ij}$.

Consider part (a). Note that, under the assumption here, there exists a positive integer I_ρ such that for all $T \geq I_\rho$, $\left\{\underline{w}_{iT-2,T}\right\}$ has the partial sum representation $\underline{w}_{iT-2,T} = \sum_{j=1}^{T-2} \varepsilon_{ij}$. Hence, for all $T \geq I_\rho$, we obtain by direct calculation

$$\begin{aligned}
E\left(\sum_{i=1}^{N} a_i \underline{w}_{iT-2,T}\right)^2 &= \sum_{i=1}^{N}\sum_{j=1}^{N} E\left[a_i a_j \underline{w}_{iT-2,T} \underline{w}_{jT-2,T}\right] \\
&= \sum_{i=1}^{N} E\left[a_i^2\right] E\left[\underline{w}_{iT-2,T}^2\right] + \sum_{i \neq j} E\left[a_i\right] E\left[a_j\right] E\left[\underline{w}_{iT-2,T}\right] E\left[\underline{w}_{jT-2,T}\right] \\
&= \left(\mu_a^2 + \sigma_a^2\right) \sum_{i=1}^{N}\sum_{g=1}^{T-2}\sum_{s=1}^{T-2} E\left[\varepsilon_{ig}\varepsilon_{is}\right] \\
&= \left(\mu_a^2 + \sigma_a^2\right) \sigma^2 N (T-2) = O(NT).
\end{aligned}$$

It follows by Markov's inequality that $\sum_{i=1}^{N} a_i \underline{w}_{iT-2,T} = O_p\left(\sqrt{NT}\right)$.

Moreover, by Assumptions 2 and 4, we have

$$E\left[\sum_{i=1}^{N} a_i^2\right] = N\left(\mu_a^2 + \sigma_a^2\right) = O(N), \; E\left[\sum_{i=1}^{N} w_{i0}^2\right] \leq \sup_i E\left[w_{i0}^2\right] N = O(N),$$

from which it follows, by Markov's inequality, that

$$\sum_{i=1}^{N} a_i^2 = O_p(N), \quad \sum_{i=1}^{N} w_{i0}^2 = O_p(N).$$

Applying tge Cauchy-Schwarz inequality, we further obtain

$$\left|\sum_{i=1}^{N} a_i w_{i0}\right| \leq \sqrt{\sum_{i=1}^{N} a_i^2}\sqrt{\sum_{i=1}^{N} w_{i0}^2} = O_p\left(\sqrt{N}\right) O_p\left(\sqrt{N}\right) = O_p(N).$$

Hence,

$$\begin{aligned}
\sum_{i=1}^{N} a_i w_{iT-2,T} &= \sum_{i=1}^{N} a_i \underline{w}_{iT-2,T} + \sum_{i=1}^{N} a_i w_{i0} \\
&\quad (\text{ for all } T \text{ sufficiently large}) \\
&= O_p\left(\sqrt{NT}\right) + O_p(N) = O_p\left(\max\left\{\sqrt{NT}, N\right\}\right),
\end{aligned}$$

as required for part (a).

Now, to show parts (b)-(d), note first that

$$\begin{aligned}
E\left(\sum_{i=1}^{N} a_i \underline{w}_{iT-2,T}\right)^2 &= \sum_{i=1}^{N}\sum_{j=1}^{N} E\left[a_i a_j \underline{w}_{iT-2,T} \underline{w}_{jT-2,T}\right] \\
&= \sum_{i=1}^{N} E\left[a_i^2\right] E\left[\underline{w}_{iT-2,T}^2\right] + \sum_{i\neq j} E[a_i] E[a_j] E\left[\underline{w}_{iT-2,T}\right] E\left[\underline{w}_{jT-2,T}\right] \\
&= \left(\mu_a^2 + \sigma_a^2\right) \sum_{i=1}^{N}\sum_{g=1}^{T-2}\sum_{s=1}^{T-2} \exp\left\{-\frac{(T-2-g)}{q(T)}\right\} \exp\left\{-\frac{(T-2-s)}{q(T)}\right\} E\left[\varepsilon_{ig}\varepsilon_{is}\right] \\
&= \left(\mu_a^2 + \sigma_a^2\right) \sigma^2 \sum_{i=1}^{N}\sum_{s=1}^{T-2} \exp\left\{-2\frac{(T-2-s)}{q(T)}\right\} \\
&= \left(\mu_a^2 + \sigma_a^2\right) \sigma^2 N \sum_{s=1}^{T-2} \exp\left\{-2\frac{(T-2-s)}{q(T)}\right\}.
\end{aligned}$$

Next, consider part (b), where we take $\rho_T = \exp\{-1/q(T)\}$ such that $T/q(T) \to 0$. In this case, we apply Lemma SE-3 part (a) to obtain

$$E\left(\sum_{i=1}^{N} a_i \underline{w}_{iT-2,T}\right)^2 = \left(\mu_a^2 + \sigma_a^2\right) \sigma^2 NT \left[1 + O\left(\frac{1}{T}\right) + O\left(\frac{T}{q(T)}\right)\right],$$

so that we deduce via Markov's inequality $\sum_{i=1}^{N} a_i \underline{w}_{iT-2,T} = O_p\left(\sqrt{NT}\right)$.

Moreover, by Assumptions 2 and 4, we have

$$\begin{aligned}
E\left[\sum_{i=1}^{N} a_i^2\right] &= N\left(\mu_a^2 + \sigma_a^2\right) = O(N), \\
E\left[\sum_{i=1}^{N} \rho_T^{2(T-2)} w_{i0}^2\right] &\leq \sup_i E\left[w_{i0}^2\right] N \exp\left\{-\frac{2(T-2)}{q(T)}\right\} = O(N),
\end{aligned}$$

from which it follows, by Markov's inequality, that

$$\sum_{i=1}^{N} a_i^2 = O_p(N), \quad \sum_{i=1}^{N} \rho_T^{2(T-2)} w_{i0}^2 = O_p(N).$$

Applying tge Cauchy-Schwarz inequality, we further obtain

$$\left|\sum_{i=1}^{N} a_i \rho_T^{T-2} w_{i0}\right| \leq \sqrt{\sum_{i=1}^{N} a_i^2} \sqrt{\sum_{i=1}^{N} \rho_T^{2(T-2)} w_{i0}^2} = O_p\left(\sqrt{N}\right) O_p\left(\sqrt{N}\right) = O_p(N).$$

Hence,

$$\begin{aligned}
\sum_{i=1}^{N} a_i w_{iT-2,T} &= \sum_{i=1}^{N} a_i \underline{w}_{iT-2,T} + \sum_{i=1}^{N} a_i \rho_T^{T-2} w_{i0} \\
&= O_p\left(\sqrt{NT}\right) + O_p(N) \\
&= O_p\left(\sqrt{NT}\right) + O_p(N) = O_p\left(\max\left\{\sqrt{NT}, N\right\}\right)
\end{aligned}$$

as required for part (b).

Consider part (c), where we take $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \sim T$. Here, we apply Lemma SE-3 part (b) to obtain

$$E\left(\sum_{i=1}^{N} a_i \underline{w}_{iT-2,T}\right)^2 = \left(\mu_a^2 + \sigma_a^2\right) \sigma^2 N \frac{q(T)}{2} \left[1 - \exp\left\{-\frac{2T}{q(T)}\right\}\right] \left[1 + O\left(\frac{1}{T}\right)\right],$$

so again we deduce via Markov's inequality $\sum_{i=1}^{N} a_i \underline{w}_{iT-2,T} = O_p\left(\sqrt{NT}\right)$.

Moreover, by Assumptions 2 and 4, we have

$$\begin{aligned}
E\left[\sum_{i=1}^{N} a_i^2\right] &= N\left(\mu_a^2 + \sigma_a^2\right) = O(N), \\
E\left[\sum_{i=1}^{N} \rho_T^{2(T-2)} w_{i0}^2\right] &\leq \sup_i E\left[w_{i0}^2\right] N \exp\left\{-\frac{2(T-2)}{q(T)}\right\} = O(N),
\end{aligned}$$

from which it follows, by Markov's inequality, that

$$\sum_{i=1}^{N} a_i^2 = O_p(N), \quad \sum_{i=1}^{N} \rho_T^{2(T-2)} w_{i0}^2 = O_p(N).$$

Applying tge Cauchy-Schwarz inequality, we further obtain

$$\left|\sum_{i=1}^{N} a_i \rho_T^{T-2} w_{i0}\right| \leq \sqrt{\sum_{i=1}^{N} a_i^2} \sqrt{\sum_{i=1}^{N} \rho_T^{2(T-2)} w_{i0}^2} = O_p\left(\sqrt{N}\right) O_p\left(\sqrt{N}\right) = O_p(N).$$

Hence,

$$\begin{aligned}\sum_{i=1}^{N} a_i w_{iT-2,T} &= \sum_{i=1}^{N} a_i \underline{w}_{iT-2,T} + \sum_{i=1}^{N} a_i \rho_T^{T-2} w_{i0} \\ &= O_p\left(\sqrt{NT}\right) + O_p(N) \\ &= O_p\left(\sqrt{NT}\right) + O_p(N) = O_p\left(\max\left\{\sqrt{NT}, N\right\}\right),\end{aligned}$$

as required for part (c).

Turning our attention to part (d), where we take $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \to \infty$ but $q(T)/T \to 0$. Applying part (c) of Lemma SE-3, we obtain

$$E\left(\sum_{i=1}^{N} a_i \underline{w}_{iT-2,T}\right)^2 = \left(\mu_a^2 + \sigma_a^2\right)\sigma^2 N \frac{q(T)}{2}\left[1 + O\left(\frac{1}{q(T)}\right)\right],$$

so that the use of Markov's inequality yields $\sum_{i=1}^{N} a_i \underline{w}_{iT-2,T} = O_p\left(\sqrt{Nq(T)}\right)$.

Moreover, by Assumptions 2 and 4, we have

$$\begin{aligned}E\left[\sum_{i=1}^{N} a_i^2\right] &= N\left(\mu_a^2 + \sigma_a^2\right) = O(N), \\ E\left[\sum_{i=1}^{N} \rho_T^{2(T-2)} w_{i0}^2\right] &\leq \sup_i E\left[w_{i0}^2\right] N \exp\left\{-\frac{2(T-2)}{q(T)}\right\} = O\left(N \exp\left\{-\frac{2T}{q(T)}\right\}\right),\end{aligned}$$

from which it follows, by Markov's inequality, that

$$\sum_{i=1}^{N} a_i^2 = O_p(N), \quad \sum_{i=1}^{N} \rho_T^{2(T-2)} w_{i0}^2 = O_p\left(N \exp\left\{-\frac{2T}{q(T)}\right\}\right).$$

Applying the Cauchy-Schwarz inequality, we further obtain

$$\begin{aligned}\left|\sum_{i=1}^{N} a_i \rho_T^{T-2} w_{i0}\right| &\leq \sqrt{\sum_{i=1}^{N} a_i^2}\sqrt{\sum_{i=1}^{N} \rho_T^{2(T-2)} w_{i0}^2} \\ &= O_p\left(\sqrt{N}\right) O_p\left(\sqrt{N} \exp\left\{-\frac{T}{q(T)}\right\}\right) = O_p\left(N \exp\left\{-\frac{T}{q(T)}\right\}\right).\end{aligned}$$

Hence,

$$\begin{aligned}&\sum_{i=1}^{N} a_i w_{iT-2,T} = \sum_{i=1}^{N} a_i \underline{w}_{iT-2,T} + \sum_{i=1}^{N} a_i \rho_T^{T-2} w_{i0} \\ &= O_p\left(\sqrt{Nq(T)}\right) + O_p\left(N \exp\left\{-\frac{T}{q(T)}\right\}\right) = O_p\left(\sqrt{Nq(T)}\right),\end{aligned}$$

as required for part (d).

Finally, consider part (e), where we take $\rho_T \in \mathcal{G}_{\mathrm{St}}$. In this case,

$$\begin{aligned} E\left(\sum_{i=1}^{N} a_i \underline{w}_{iT-2,T}\right)^2 &= \sum_{i=1}^{N}\sum_{j=1}^{N} E\left[a_i a_j \underline{w}_{iT-2,T}\underline{w}_{jT-2,T}\right] \\ &= \left(\mu_a^2+\sigma_a^2\right)\sigma^2 N \sum_{s=1}^{T-2} \rho_T^{2(T-2-s)} = \left(\mu_a^2+\sigma_a^2\right)\sigma^2 N \frac{1-\rho_T^{2(T-2)}}{1-\rho_T^2}. \end{aligned}$$

Since we assume that $\rho_T^2 = \exp\left\{-\frac{2}{q(T)}\right\}$ with $q(T) = O(1)$ here, it follows that there exist a positive constant C_q and a positive integer T^* such that for all $T \geq T^*$

$$\rho_T^2 = \exp\left\{-\frac{2}{q(T)}\right\} \leq \exp\left\{-\frac{2}{C_q}\right\} < 1,$$

from which we further deduce, in light of Assumptions 1 and 2, that

$$E\left(\sum_{i=1}^{N} a_i \underline{w}_{iT-2,T}\right)^2 = \left(\mu_a^2+\sigma_a^2\right)\sigma^2 N \frac{1-\rho_T^{2(T-2)}}{1-\rho_T^2} \leq N \frac{\left(\mu_a^2+\sigma_a^2\right)\sigma^2}{1-\exp\{-2/C_q\}} = O(N).$$

Hence, by applying Markov's inequality, we obtain $\sum_{i=1}^{N} a_i \underline{w}_{iT-2,T} = O_p\left(\sqrt{N}\right)$.

Moreover, by Assumptions 2 and 4, we have

$$\begin{aligned} E\left[\sum_{i=1}^{N} a_i^2\right] &= N\left(\mu_a^2+\sigma_a^2\right) = O(N), \\ E\left[\sum_{i=1}^{N} \rho_T^{2(T-2)} w_{i0}^2\right] &\leq \sup_i E\left[w_{i0}^2\right] N \rho_T^{2(T-2)} = O\left(N\rho_T^{2T}\right), \end{aligned}$$

from which it follows, by Markov's inequality, that

$$\sum_{i=1}^{N} a_i^2 = O_p(N), \quad \sum_{i=1}^{N} \rho_T^{2(T-2)} w_{i0}^2 = O_p\left(N\rho_T^{2T}\right).$$

Applying tge Cauchy-Schwarz inequality, we further obtain

$$\left|\sum_{i=1}^{N} a_i \rho_T^{T-2} w_{i0}\right| \leq \sqrt{\sum_{i=1}^{N} a_i^2}\sqrt{\sum_{i=1}^{N} \rho_T^{2(T-2)} w_{i0}^2} = O_p\left(\sqrt{N}\right) O_p\left(\sqrt{N}\rho_T^T\right) = O_p\left(N\rho_T^T\right).$$

Hence,

$$\begin{aligned} \sum_{i=1}^{N} a_i w_{iT-2,T} &= \sum_{i=1}^{N} a_i \underline{w}_{iT-2,T} + \sum_{i=1}^{N} a_i \rho_T^{T-2} w_{i0} \\ &= O_p\left(\sqrt{N}\right) + O_p\left(N\rho_T^T\right) = O_p\left(\sqrt{N}\right), \end{aligned}$$

as required for part (e). □

Lemma SE-31:

Let $\underline{w}_{iT-2} = \sum_{j=1}^{T-2} \rho_T^{(T-2-j)} \varepsilon_{ij}$. Under Assumptions 1 and 4, the following statements hold as $N, T \to \infty$.

(a) If $\rho_T = 1$ for all T sufficiently large, then

$$\frac{1}{NT} \sum_{i=1}^{N} E\left[\underline{w}_{iT-2,T}^2\right] = \sigma^2 + O\left(\frac{1}{T}\right).$$

(b) If $\rho_T = \exp\{-1/q(T)\}$ such that $T/q(T) \to 0$,

$$\frac{1}{NT} \sum_{i=1}^{N} E\left[\underline{w}_{iT-2,T}^2\right] = \sigma^2 + O\left(\frac{1}{T}\right) + O\left(\frac{T}{q(T)}\right).$$

(c) If $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \sim T$,

$$\frac{1}{NT} \sum_{i=1}^{N} E\left[\underline{w}_{iT-2,T}^2\right] = \frac{\sigma^2}{2} \frac{q(T)}{T} \left[1 - \exp\left\{-\frac{2T}{q(T)}\right\}\right] + O\left(\frac{1}{T}\right).$$

(c) If $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \to \infty$ but $q(T)/T \to 0$,

$$\frac{1}{NT} \sum_{i=1}^{N} E\left[\underline{w}_{iT-2,T}^2\right] = O\left(\frac{q(T)}{T}\right) = o(1).$$

(e) If $\rho_T \in \mathcal{G}_{\mathrm{St}} = \left\{|\rho_T| = \exp\left\{-\frac{1}{q(T)}\right\} : q(T) \geq 0 \text{ and } q(T) = O(1) \text{ as } T \to \infty\right\}$, then

$$\frac{1}{NT} \sum_{i=1}^{N} E\left[\underline{w}_{iT-2,T}^2\right] = O\left(\frac{1}{T}\right) = o(1).$$

Proof of Lemma SE-31:

Consider first part (a). Note that, under the assumption here, there exists a positive integer I_ρ such that, for all $T \geq I_\rho$, $\left\{\underline{w}_{iT-2,T}\right\}$ has the partial sum representation $\underline{w}_{iT-2,T} = \sum_{j=1}^{T-2} \varepsilon_{ij}$. Hence, for all $T \geq I_\rho$, we obtain by direct calculation

$$\frac{1}{NT} \sum_{i=1}^{N} E\left[\underline{w}_{iT-2,T}^2\right] = \frac{1}{NT} \sum_{i=1}^{N} \sum_{g=1}^{T-2} \sum_{s=1}^{T-2} E\left[\varepsilon_{ig}\varepsilon_{is}\right] = \sigma^2 \frac{N(T-2)}{NT} = \sigma^2 + O\left(\frac{1}{T}\right).$$

Now, to show parts (b)-(d), note first that

$$\begin{aligned}
& \sum_{i=1}^{N} E\left[\underline{w}_{iT-2,T}^{2}\right] \\
= & \sum_{i=1}^{N}\sum_{g=1}^{T-2}\sum_{s=1}^{T-2} \exp\left\{-\frac{(T-2-g)}{q(T)}\right\} \exp\left\{-\frac{(T-2-s)}{q(T)}\right\} E\left[\varepsilon_{ig}\varepsilon_{is}\right] \\
= & \sigma^2 N \sum_{s=1}^{T-2} \exp\left\{-2\frac{(T-2-s)}{q(T)}\right\}.
\end{aligned}$$

Next, consider part (b), where we take $\rho_T = \exp\{-1/q(T)\}$ such that $T/q(T) \to 0$. In this case, we apply Lemma SE-3 part (a) to obtain

$$E\left(\sum_{i=1}^{N} \underline{w}_{iT-2,T}^{2}\right) = \sigma^2 NT\left[1 + O\left(\frac{1}{T}\right) + O\left(\frac{T}{q(T)}\right)\right],$$

so that

$$\frac{1}{NT}\sum_{i=1}^{N} E\left[\underline{w}_{iT-2,T}^{2}\right] = \sigma^2 + O\left(\frac{1}{T}\right) + O\left(\frac{T}{q(T)}\right).$$

Consider part (c), where we take $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \sim T$. Here, we apply Lemma SE-3 part (b) to obtain

$$E\left(\sum_{i=1}^{N} \underline{w}_{iT-2,T}^{2}\right) = \sigma^2 N\frac{q(T)}{2}\left[1 - \exp\left\{-\frac{2T}{q(T)}\right\}\right]\left[1 + O\left(\frac{1}{T}\right)\right],$$

so that

$$\frac{1}{NT}\sum_{i=1}^{N} E\left[\underline{w}_{iT-2,T}^{2}\right] = \frac{\sigma^2}{2}\frac{q(T)}{T}\left[1 - \exp\left\{-\frac{2T}{q(T)}\right\}\right] + O\left(\frac{1}{T}\right).$$

We turn our attention now to part (d), where we take $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \to \infty$ but $q(T)/T \to 0$. Applying part (c) of Lemma SE-3, we obtain

$$E\left(\sum_{i=1}^{N} \underline{w}_{iT-2,T}^{2}\right) = \sigma^2 N\frac{q(T)}{2}\left[1 + O\left(\frac{1}{q(T)}\right)\right],$$

so that

$$\frac{1}{NT}\sum_{i=1}^{N} E\left[\underline{w}_{iT-2,T}^{2}\right] = \frac{\sigma^2}{2}\frac{q(T)}{T} + O\left(\frac{1}{T}\right) = O\left(\frac{q(T)}{T}\right).$$

Finally, to show part (e),

$$\sum_{i=1}^{N} E\left[\underline{w}_{iT-2,T}^{2}\right] = \sigma^2 N\sum_{s=1}^{T-2}\rho_T^{2(T-2-s)} = \sigma^2 N\frac{1-\rho_T^{2(T-2)}}{1-\rho_T^2}$$

Here, we take $\rho_T^2 = \exp\left\{-\frac{2}{q(T)}\right\}$ with $q(T) = O(1)$, so that there exist a positive constant C_q and a positive integer T^* such that for all $T \geq T^*$

$$\rho_T^2 = \exp\left\{-\frac{2}{q(T)}\right\} \leq \exp\left\{-\frac{2}{C_q}\right\} < 1,$$

from which we deduce that

$$\frac{1}{NT}\sum_{i=1}^{N} E\left[\underline{w}_{iT-2,T}^2\right] \leq \frac{\sigma^2}{T}\frac{1}{1-\exp\{-2/C_q\}} = O\left(\frac{1}{T}\right) = o(1). \quad \square$$

Lemma SE-32:

Under Assumptions 1 and 4, the following statements hold as $N, T \to \infty$.

(a) If $\rho_T = 1$ for all T sufficiently large, then

$$\frac{1}{NT}\sum_{i=1}^{N} w_{iT-2,T}^2 = \sigma^2 + O_p\left(\max\left\{\frac{1}{\sqrt{N}}, \frac{1}{\sqrt{T}}\right\}\right).$$

(b) If $\rho_T = \exp\{-1/q(T)\}$ such that $T/q(T) \to 0$, then

$$\frac{1}{NT}\sum_{i=1}^{N} w_{iT-2,T}^2 = \sigma^2 + O_p\left(\max\left\{\frac{1}{\sqrt{N}}, \frac{1}{\sqrt{T}}, \frac{T}{q(T)}\right\}\right).$$

(c) If $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \sim T$, then

$$\frac{1}{NT}\sum_{i=1}^{N} w_{iT-2,T}^2 = \frac{\sigma^2}{2}\frac{q(T)}{T}\left[1-\exp\left\{-\frac{2T}{q(T)}\right\}\right] + O_p\left(\max\left\{\frac{1}{\sqrt{N}}, \frac{1}{\sqrt{T}}\right\}\right).$$

(d) If $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \to \infty$ but $q(T)/T \to 0$, then

$$\frac{1}{NT}\sum_{i=1}^{N} w_{iT-2,T}^2 = O_p\left(\frac{q(T)}{T}\right).$$

(e) If $\rho_T \in \mathcal{G}_{\mathrm{St}} = \left\{|\rho_T| = \exp\left\{-\frac{1}{q(T)}\right\} : q(T) \geq 0 \text{ and } q(T) = O(1) \text{ as } T \to \infty\right\}$, then

$$\frac{1}{NT}\sum_{i=1}^{N} w_{iT-2,T}^2 = O_p\left(\frac{1}{T}\right).$$

Proof of Lemma SE-32:

To proceed, first write

$$\frac{1}{NT}\sum_{i=1}^{N} w_{iT-2,T}^2 = \frac{1}{NT}\sum_{i=1}^{N} \underline{w}_{iT-2,T}^2 + 2\frac{1}{NT}\sum_{i=1}^{N} \underline{w}_{iT-2}\rho_T^{T-2} w_{i0} + \frac{1}{NT}\sum_{i=1}^{N} \rho_T^{2(T-2)} w_{i0}^2,$$

$\underline{w}_{iT-2} = \sum_{j=1}^{T-2} \rho_T^{(T-2-j)} \varepsilon_{ij}$.

Consider first part (a). Note that, under the assumption here, there exists a positive integer I_ρ such that for all $T \geq I_\rho$, $\left\{\underline{w}_{iT-2,T}\right\}$ has the partial sum representation $\underline{w}_{iT-2,T} = \sum_{j=1}^{T-2} \varepsilon_{ij}$. Hence, for all $T \geq I_\rho$, we obtain by direct calculation

$$\begin{aligned}
& E\left(\frac{1}{NT}\sum_{i=1}^{N}\left(\underline{w}_{iT-2,T}^2 - E\left[\underline{w}_{iT-2,T}^2\right]\right)\right)^2 \\
= & \frac{1}{N^2T^2}\sum_{i=1}^{N}\sum_{j=1}^{N} E\left\{\left(\underline{w}_{iT-2,T}^2 - E\left[\underline{w}_{iT-2,T}^2\right]\right)\left(\underline{w}_{jT-2,T}^2 - E\left[\underline{w}_{jT-2,T}^2\right]\right)\right\} \\
= & \frac{1}{N^2T^2}\sum_{i=1}^{N} E\left(\underline{w}_{iT-2,T}^2 - E\left[\underline{w}_{iT-2,T}^2\right]\right)^2 \\
= & \frac{1}{N^2T^2}\sum_{i=1}^{N}\left\{E\left[\underline{w}_{iT-2,T}^4\right] - \left(E\left[\underline{w}_{iT-2,T}^2\right]\right)^2\right\} \\
= & \frac{1}{NT^2}\sum_{g=1}^{T-2}\sum_{s=1}^{T-2}\sum_{t=1}^{T-2}\sum_{v=1}^{T-2} E\left[\varepsilon_{ig}\varepsilon_{is}\varepsilon_{it}\varepsilon_{iv}\right] - \frac{1}{NT^2}\left(\sum_{g=1}^{T-2}\sum_{s=1}^{T-2} E\left[\varepsilon_{ig}\varepsilon_{is}\right]\right)^2 \\
\leq & E\left[\varepsilon_{it}^4\right]\frac{T-2}{NT^2} + 3\sigma^4\frac{(T-2)^2}{NT^2} - \sigma^4\frac{(T-2)^2}{NT^2} \\
= & E\left[\varepsilon_{it}^4\right]\frac{T-2}{NT^2} + 2\sigma^4\frac{(T-2)^2}{NT^2} = O\left(N^{-1}\right).
\end{aligned}$$

It follows from Markov's inequality and part (a) of Lemma SE-31 that

$$\begin{aligned}
\frac{1}{NT}\sum_{i=1}^{N}\underline{w}_{iT-2,T}^2 &= \frac{1}{NT}\sum_{i=1}^{N} E\left[\underline{w}_{iT-2,T}^2\right] + O_p\left(\frac{1}{\sqrt{N}}\right) = \sigma^2 + O_p\left(\frac{1}{\sqrt{N}}\right) + O_p\left(\frac{1}{T}\right) \\
&= \sigma^2 + O_p\left(\frac{1}{\sqrt{N}}\right).
\end{aligned}$$

Moreover, using Assumption 4, we have

$$E\left[\frac{1}{NT}\sum_{i=1}^{N} w_{i0}^2\right] \leq \frac{1}{T}\sup_i E\left[w_{i0}^2\right] = O\left(T^{-1}\right),$$

from which, it follows, by Markov's inequality, that

$$\frac{1}{NT}\sum_{i=1}^{N} w_{i0}^2 = O_p\left(\frac{1}{\sqrt{T}}\right).$$

Also, by the Cauchy-Schwarz inequality,

$$\left|\frac{1}{NT}\sum_{i=1}^{N}\underline{w}_{iT-2}w_{i0}\right| \leq \sqrt{\frac{1}{NT}\sum_{i=1}^{N}\underline{w}_{iT-2}^{2}}\sqrt{\frac{1}{NT}\sum_{i=1}^{N}w_{i0}^{2}} = O_p(1)\,O_p\left(T^{-1/2}\right) = O_p\left(T^{-1/2}\right).$$

Hence,

$$\begin{aligned}
\frac{1}{NT}\sum_{i=1}^{N}w_{iT-2,T}^{2} &= \frac{1}{NT}\sum_{i=1}^{N}\underline{w}_{iT-2,T}^{2} + 2\frac{1}{NT}\sum_{i=1}^{N}\underline{w}_{iT-2}w_{i0} + \frac{1}{NT}\sum_{i=1}^{N}w_{i0}^{2} \\
&\quad \text{(for all } T \text{ sufficiently large)} \\
&= \sigma^2 + O_p\left(\max\left\{\frac{1}{\sqrt{N}},\frac{1}{\sqrt{T}}\right\}\right),
\end{aligned}$$

as required for part (a).

Now, to show parts (b)-(d), note first that

$$\begin{aligned}
&E\left(\frac{1}{NT}\sum_{i=1}^{N}\left(\underline{w}_{iT-2,T}^{2} - E\left[\underline{w}_{iT-2,T}^{2}\right]\right)\right)^2 = \frac{1}{N^2T^2}\sum_{i=1}^{N}\left\{E\left[\underline{w}_{iT-2,T}^{4}\right] - \left(E\left[\underline{w}_{iT-2,T}^{2}\right]\right)^2\right\} \\
= \;& \frac{1}{NT^2}\sum_{g=1}^{T-2}\sum_{s=1}^{T-2}\sum_{t=1}^{T-2}\sum_{\upsilon=1}^{T-2}\exp\left\{-\frac{(T-2-g)}{q(T)}\right\}\exp\left\{-\frac{(T-2-s)}{q(T)}\right\}\exp\left\{-\frac{(T-2-t)}{q(T)}\right\} \\
&\qquad \times\exp\left\{-\frac{(T-2-\upsilon)}{q(T)}\right\}E\left[\varepsilon_{ig}\varepsilon_{is}\varepsilon_{it}\varepsilon_{i\upsilon}\right] \\
&-\frac{1}{NT^2}\left(\sum_{g=1}^{T-2}\sum_{s=1}^{T-2}\exp\left\{-\frac{(T-2-g)}{q(T)}\right\}\exp\left\{-\frac{(T-2-s)}{q(T)}\right\}E\left[\varepsilon_{ig}\varepsilon_{is}\right]\right)^2 \\
\leq \;& \frac{1}{NT^2}\sum_{s=1}^{T-2}\exp\left\{-4\frac{(T-2-s)}{q(T)}\right\}E\left[\varepsilon_{is}^4\right] + \frac{3\sigma^4}{N}\left(\frac{1}{T}\sum_{s=1}^{T-2}\exp\left\{-2\frac{(T-2-s)}{q(T)}\right\}\right)^2 \\
&-\frac{\sigma^4}{N}\left(\frac{1}{T}\sum_{s=1}^{T-2}\exp\left\{-2\frac{(T-2-s)}{q(T)}\right\}\right)^2 \\
= \;& \frac{1}{NT^2}\sum_{s=1}^{T-2}\exp\left\{-4\frac{(T-2-s)}{q(T)}\right\}E\left[\varepsilon_{is}^4\right] + \frac{2\sigma^4}{N}\left(\frac{1}{T}\sum_{s=1}^{T-2}\exp\left\{-2\frac{(T-2-s)}{q(T)}\right\}\right)^2.
\end{aligned}$$

Next, consider part (b), where we take $\rho_T = \exp\{-1/q(T)\}$ such that $T/q(T) \to 0$. In this case,

we apply Lemma SE-3 part (a) to obtain

$$\begin{aligned}
& E\left(\frac{1}{NT}\sum_{i=1}^{N}\left(\underline{w}_{iT-2,T}^{2}-E\left[\underline{w}_{iT-2,T}^{2}\right]\right)\right)^{2} \\
= & \frac{1}{NT^{2}}\sum_{s=1}^{T-2}\exp\left\{-4\frac{(T-2-s)}{q(T)}\right\}E\left[\varepsilon_{is}^{4}\right]+\frac{2\sigma^{4}}{N}\left(\frac{1}{T}\sum_{s=1}^{T-2}\exp\left\{-2\frac{(T-2-s)}{q(T)}\right\}\right)^{2} \\
= & \frac{E\left[\varepsilon_{is}^{4}\right]}{NT^{2}}T\left[1+O\left(\frac{1}{T}\right)+O\left(\frac{T}{q(T)}\right)\right]+\frac{2\sigma^{4}}{NT^{2}}T^{2}\left[1+O\left(\frac{1}{T}\right)+O\left(\frac{T}{q(T)}\right)\right] \\
= & O\left(\frac{1}{N}\right).
\end{aligned}$$

It follows from Markov's inequality and part (b) of Lemma SE-31 above that

$$\begin{aligned}
\frac{1}{NT}\sum_{i=1}^{N}\underline{w}_{iT-2,T}^{2} &= \frac{1}{NT}\sum_{i=1}^{N}E\left[\underline{w}_{iT-2,T}^{2}\right]+O_{p}\left(\frac{1}{\sqrt{N}}\right) \\
&= \sigma^{2}+O_{p}\left(\frac{1}{\sqrt{N}}\right)+O_{p}\left(\frac{T}{q(T)}\right).
\end{aligned}$$

Moreover, using Assumption 4, we have

$$E\left[\frac{1}{NT}\sum_{i=1}^{N}\rho_{T}^{2(T-2)}w_{i0}^{2}\right]\leq\frac{1}{T}\left(\sup_{i}E\left[w_{i0}^{2}\right]\right)\exp\left\{-\frac{2(T-2)}{q(T)}\right\}=O\left(\frac{1}{T}\right)O(1)O(1)=O\left(\frac{1}{T}\right)$$

from which it follows, by Markov's inequality, that

$$\frac{1}{NT}\sum_{i=1}^{N}\rho_{T}^{2(T-2)}w_{i0}^{2}=O_{p}\left(\frac{1}{T}\right).$$

Also, by the Cauchy-Schwarz inequality,

$$\left|\frac{1}{NT}\sum_{i=1}^{N}\underline{w}_{iT-2}\rho_{T}^{T-2}w_{i0}\right|\leq\sqrt{\frac{1}{NT}\sum_{i=1}^{N}\underline{w}_{iT-2,T}^{2}}\sqrt{\frac{1}{NT}\sum_{i=1}^{N}\rho_{T}^{2(T-2)}w_{i0}^{2}}=O_{p}(1)O_{p}\left(\frac{1}{\sqrt{T}}\right)=O_{p}\left(\frac{1}{\sqrt{T}}\right).$$

Hence,

$$\begin{aligned}
\frac{1}{NT}\sum_{i=1}^{N}w_{iT-2,T}^{2} &= \frac{1}{NT}\sum_{i=1}^{N}\underline{w}_{iT-2,T}^{2}+2\frac{1}{NT}\sum_{i=1}^{N}\underline{w}_{iT-2}\rho_{T}^{T-2}w_{i0}+\frac{1}{NT}\sum_{i=1}^{N}\rho_{T}^{2(T-2)}w_{i0}^{2} \\
&= \sigma^{2}+O_{p}\left(\max\left\{\frac{1}{\sqrt{N}},\frac{1}{\sqrt{T}},\frac{T}{q(T)}\right\}\right),
\end{aligned}$$

as required for part (b).

Consider part (c), where we take $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \sim T$. Here, we apply Lemma SE-3 part (b) to obtain

$$\begin{aligned}
& E\left(\frac{1}{NT}\sum_{i=1}^{N}\left(\underline{w}_{iT-2,T}^2 - E\left[\underline{w}_{iT-2,T}^2\right]\right)\right)^2 \\
= & \frac{1}{NT^2}\sum_{s=1}^{T-2}\exp\left\{-4\frac{(T-2-s)}{q(T)}\right\}E\left[\varepsilon_{is}^4\right] + \frac{2\sigma^4}{N}\left(\frac{1}{T}\sum_{s=1}^{T-2}\exp\left\{-2\frac{(T-2-s)}{q(T)}\right\}\right)^2 \\
= & \frac{E\left[\varepsilon_{is}^4\right]}{NT^2}\frac{q(T)}{4}\left[1-\exp\left\{-\frac{4T}{q(T)}\right\}\right]\left[1+O\left(\frac{1}{T}\right)\right] \\
& + \frac{2\sigma^4}{NT^2}\frac{q(T)^2}{4}\left[1-\exp\left\{-\frac{2T}{q(T)}\right\}\right]^2\left[1+O\left(\frac{1}{T}\right)\right] \\
= & O\left(\frac{1}{N}\right).
\end{aligned}$$

It follows from Markov's inequality and part (c) of Lemma SE-31 that

$$\begin{aligned}
\frac{1}{NT}\sum_{i=1}^{N}\underline{w}_{iT-2,T}^2 &= \frac{1}{NT}\sum_{i=1}^{N}E\left[\underline{w}_{iT-2,T}^2\right] + O_p\left(\frac{1}{\sqrt{N}}\right) \\
&= \frac{\sigma^2}{2}\frac{q(T)}{T}\left[1-\exp\left\{-\frac{2T}{q(T)}\right\}\right] + O_p\left(\frac{1}{\sqrt{N}}\right) + O_p\left(\frac{1}{T}\right) \\
&= \frac{\sigma^2}{2}\frac{q(T)}{T}\left[1-\exp\left\{-\frac{2T}{q(T)}\right\}\right] + O_p\left(\frac{1}{\sqrt{N}}\right).
\end{aligned}$$

Moreover, using Assumption 4, we have

$$E\left[\frac{1}{NT}\sum_{i=1}^{N}\rho_T^{2(T-2)}w_{i0}^2\right] \leq \frac{1}{T}\left(\sup_i E\left[w_{i0}^2\right]\right)\exp\left\{-\frac{2(T-2)}{q(T)}\right\} = O\left(\frac{1}{T}\right)O(1)O(1) = O\left(\frac{1}{T}\right),$$

from which it follows, by Markov's inequality, that

$$\frac{1}{NT}\sum_{i=1}^{N}\rho_T^{2(T-2)}w_{i0}^2 = O_p\left(\frac{1}{T}\right).$$

Also, by the Cauchy-Schwarz inequality,

$$\left|\frac{1}{NT}\sum_{i=1}^{N}\underline{w}_{iT-2}\rho_T^{T-2}w_{i0}\right| \leq \sqrt{\frac{1}{NT}\sum_{i=1}^{N}\underline{w}_{iT-2,T}^2}\sqrt{\frac{1}{NT}\sum_{i=1}^{N}\rho_T^{2(T-2)}w_{i0}^2} = O_p(1)O_p\left(\frac{1}{\sqrt{T}}\right) = O_p\left(\frac{1}{\sqrt{T}}\right).$$

Hence,

$$\begin{aligned}
\frac{1}{NT}\sum_{i=1}^{N}w_{iT-2,T}^2 &= \frac{1}{NT}\sum_{i=1}^{N}\underline{w}_{iT-2,T}^2 + 2\frac{1}{NT}\sum_{i=1}^{N}\underline{w}_{iT-2}\rho_T^{T-2}w_{i0} + \frac{1}{NT}\sum_{i=1}^{N}\rho_T^{2(T-2)}w_{i0}^2 \\
&= \frac{\sigma^2}{2}\frac{q(T)}{T}\left[1-\exp\left\{-\frac{2T}{q(T)}\right\}\right] + O_p\left(\max\left\{\frac{1}{\sqrt{N}},\frac{1}{\sqrt{T}}\right\}\right),
\end{aligned}$$

as required for part (c).

We now turn our attention to part (d), where we take $\rho_T = \exp\{-1/q(T)\}$ such that $q(T) \to \infty$ but $q(T)/T \to 0$. Applying part (c) of Lemma SE-3, we obtain

$$\begin{aligned}
& E\left(\frac{1}{NT}\sum_{i=1}^{N}\left(\underline{w}_{iT-2,T}^2 - E\left[\underline{w}_{iT-2,T}^2\right]\right)\right)^2 \\
= & \frac{1}{NT^2}\sum_{s=1}^{T-2}\exp\left\{-4\frac{(T-2-s)}{q(T)}\right\}E\left[\varepsilon_{is}^4\right] + \frac{2\sigma^4}{N}\left(\frac{1}{T}\sum_{s=1}^{T-2}\exp\left\{-2\frac{(T-2-s)}{q(T)}\right\}\right)^2 \\
= & \frac{E\left[\varepsilon_{is}^4\right]}{NT^2}\frac{q(T)}{4}\left[1+O\left(\frac{1}{q(T)}\right)\right] + \frac{2\sigma^4}{NT^2}\frac{q(T)^2}{4}\left[1+O\left(\frac{1}{q(T)}\right)\right] \\
= & O\left(\frac{q(T)^2}{NT^2}\right).
\end{aligned}$$

It follows from Markov's inequality and part (d) of Lemma SE-31 that

$$\begin{aligned}
\frac{1}{NT}\sum_{i=1}^{N}\underline{w}_{iT-2,T}^2 &= \frac{1}{NT}\sum_{i=1}^{N}E\left[\underline{w}_{iT-2,T}^2\right] + O_p\left(\frac{q(T)}{\sqrt{N}T}\right) \\
&= O\left(\frac{q(T)}{T}\right) + O_p\left(\frac{q(T)}{\sqrt{N}T}\right) = O_p\left(\frac{q(T)}{T}\right).
\end{aligned}$$

Moreover, using Assumption 4, we have

$$\begin{aligned}
E\left[\frac{1}{NT}\sum_{i=1}^{N}\rho_T^{2(T-2)}w_{i0}^2\right] &\leq \frac{1}{T}\left(\sup_i E\left[w_{i0}^2\right]\right)\exp\left\{-\frac{2(T-2)}{q(T)}\right\} \\
&= O\left(\frac{1}{T}\right)O(1)\,O\left(\exp\left\{-\frac{2T}{q(T)}\right\}\right) = O\left(\frac{1}{T}\exp\left\{-\frac{2T}{q(T)}\right\}\right),
\end{aligned}$$

from which it follows, by Markov's inequality, that

$$\frac{1}{NT}\sum_{i=1}^{N}\rho_T^{2(T-2)}w_{i0}^2 = O_p\left(\frac{1}{T}\exp\left\{-\frac{2T}{q(T)}\right\}\right).$$

Also, by the Cauchy-Schwarz inequality,

$$\begin{aligned}
& \left|\frac{1}{NT}\sum_{i=1}^{N}\underline{w}_{iT-2}\rho_T^{T-2}w_{i0}\right| \leq \sqrt{\frac{1}{NT}\sum_{i=1}^{N}\underline{w}_{iT-2,T}^2}\sqrt{\frac{1}{NT}\sum_{i=1}^{N}\rho_T^{2(T-2)}w_{i0}^2} \\
= & O_p\left(\sqrt{\frac{q(T)}{T}}\right)O_p\left(\frac{1}{\sqrt{T}}\exp\left\{-\frac{T}{q(T)}\right\}\right) = O_p\left(\frac{\sqrt{q(T)}}{T}\exp\left\{-\frac{T}{q(T)}\right\}\right).
\end{aligned}$$

Hence,

$$
\begin{aligned}
\frac{1}{NT}\sum_{i=1}^{N} w_{iT-2,T}^{2} &= \frac{1}{NT}\sum_{i=1}^{N} \underline{w}_{iT-2,T}^{2} + 2\frac{1}{NT}\sum_{i=1}^{N} \underline{w}_{iT-2}\rho_T^{T-2} w_{i0} + \frac{1}{NT}\sum_{i=1}^{N} \rho_T^{2(T-2)} w_{i0}^{2} \\
&= O_p\left(\frac{q(T)}{T}\right) + O_p\left(\frac{\sqrt{q(T)}}{T}\exp\left\{-\frac{T}{q(T)}\right\}\right) + O_p\left(\frac{1}{T}\exp\left\{-\frac{2T}{q(T)}\right\}\right) \\
&= O_p\left(\frac{q(T)}{T}\right),
\end{aligned}
$$

as required for part (d).

Finally, to show part (e), note that

$$
\begin{aligned}
& E\left(\frac{1}{NT}\sum_{i=1}^{N}\left(\underline{w}_{iT-2,T}^{2} - E\left[\underline{w}_{iT-2,T}^{2}\right]\right)\right)^{2} \\
= & \frac{1}{NT^2}\sum_{s=1}^{T-2}\rho_T^{4(T-2-s)} E\left[\varepsilon_{is}^{4}\right] + \frac{2\sigma^4}{N}\left(\frac{1}{T}\sum_{s=1}^{T-2}\rho_T^{2(T-2-s)}\right)^{2} \\
= & \frac{E\left[\varepsilon_{is}^{4}\right]}{NT^2}\frac{\left(1-\rho_T^{4(T-2)}\right)}{1-\rho_T^{4}} + \frac{2\sigma^4}{NT^2}\left(\frac{1-\rho_T^{2(T-2)}}{1-\rho_T^{2}}\right)^{2}.
\end{aligned}
$$

Here, we take $q(T) = O(1)$, so that there exist a positive constant C_q and a positive integer T^* such that for all $T \geq T^*$

$$
\rho_T^{g} = \exp\left\{-\frac{g}{q(T)}\right\} \leq \exp\left\{-\frac{g}{C_q}\right\} < 1 \quad \text{with } g \in \{2,4\},
$$

from which we further deduce that

$$
\begin{aligned}
& E\left(\frac{1}{NT}\sum_{i=1}^{N}\left(\underline{w}_{iT-2,T}^{2} - E\left[\underline{w}_{iT-2,T}^{2}\right]\right)\right)^{2} \\
\leq & \frac{1}{NT^2}\left[\frac{E\left[\varepsilon_{is}^{4}\right]}{1-\exp\{-4/C_q\}} + \frac{2\sigma^4}{\left(1-\exp\{-2/C_q\}\right)^2}\right] = O\left(\frac{1}{NT^2}\right).
\end{aligned}
$$

It follows from Markov's inequality and part (e) of Lemma SE-31 that

$$
\begin{aligned}
\frac{1}{NT}\sum_{i=1}^{N}\underline{w}_{iT-2,T}^{2} &= \frac{1}{NT}\sum_{i=1}^{N} E\left[\underline{w}_{iT-2,T}^{2}\right] + O_p\left(\frac{1}{\sqrt{N}T}\right) \\
&= O_p\left(\frac{1}{T}\right) + O_p\left(\frac{1}{\sqrt{N}T}\right) = O_p\left(\frac{1}{T}\right).
\end{aligned}
$$

Moreover, using Assumption 4, we have

$$
E\left[\frac{1}{NT}\sum_{i=1}^{N}\rho_T^{2(T-2)} w_{i0}^{2}\right] \leq \frac{1}{T}\left(\sup_i E\left[w_{i0}^{2}\right]\right)\rho_T^{2(T-2)} = O\left(\frac{1}{T}\right) O(1)\, O\left(\rho_T^{2T}\right) = O\left(\frac{\rho_T^{2T}}{T}\right),
$$

from which it follows, by Markov's inequality, that

$$\frac{1}{NT}\sum_{i=1}^{N}\rho_T^{2(T-2)}w_{i0}^2 = O_p\left(\frac{\rho_T^{2T}}{T}\right).$$

Also, by the Cauchy-Schwarz inequality,

$$\begin{aligned}\left|\frac{1}{NT}\sum_{i=1}^{N}\underline{w}_{iT-2}\rho_T^{T-2}w_{i0}\right| &\leq \sqrt{\frac{1}{NT}\sum_{i=1}^{N}\underline{w}_{iT-2,T}^2}\sqrt{\frac{1}{NT}\sum_{i=1}^{N}\rho_T^{2(T-2)}w_{i0}^2} \\ &= O_p\left(\frac{1}{\sqrt{T}}\right)O_p\left(\frac{\rho_T^T}{\sqrt{T}}\right) = O_p\left(\frac{\rho_T^T}{T}\right).\end{aligned}$$

Hence,

$$\begin{aligned}\frac{1}{NT}\sum_{i=1}^{N}w_{iT-2,T}^2 &= \frac{1}{NT}\sum_{i=1}^{N}\underline{w}_{iT-2,T}^2 + 2\frac{1}{NT}\sum_{i=1}^{N}\underline{w}_{iT-2}\rho_T^{T-2}w_{i0} + \frac{1}{NT}\sum_{i=1}^{N}\rho_T^{2(T-2)}w_{i0}^2 \\ &= O_p\left(\frac{1}{T}\right) + O_p\left(T^{-1}\rho_T^T\right) + O_p\left(T^{-1}\rho_T^{2T}\right) = O_p\left(\frac{1}{T}\right),\end{aligned}$$

as required for part (e). □

Lemma SE-33 below is a well-known lemma concerning strictly increasing functions which characterize subsequences.

Lemma SE-33: Let $f:\mathbb{N}\to\mathbb{N}$ be a strictly increasing function in its argument, where $\mathbb{N}$ denotes the set of natural numbers, i.e., $\{1,2,....\}$. Then, $f(T)\geq T$ for all $T\in\mathbb{N}$.

Proof:

The proof follows trivially by mathematical induction after first noting that $f(1)\geq 1$ since $f(1)\in\mathbb{N}$. □

Lemma SE-34:

Let $\varphi(x) = 1-\frac{1}{4x}\left[\exp\{-2x\}+2x-1\right]$. Then, $\varphi(x)\geq 1/2$, for $0<x<\infty$.

Proof:

Note that

$$\begin{aligned}\varphi'(x) &= \frac{1}{4x^2}\left[\exp\{-2x\}+2x-1\right] - \frac{1}{4x}\left[-2\exp\{-2x\}+2\right] \\ &= \frac{1}{4x^2}\left[\exp\{-2x\}+2x-1+2x\exp\{-2x\}-2x\right] \\ &= \frac{1}{4x^2}\left[(1+2x)\exp\{-2x\}-1\right] \\ &< 0, \text{ for all } x \text{ such that } 0<x<\infty,\end{aligned}$$

where the last inequality follows from the inequality $1+2x < \exp\{2x\}$ for all $x \in (0,\infty)$. Next, observe that

$$\lim_{x\to\infty} \varphi(x) = \lim_{x\to\infty} \left[1 + \frac{1}{4x} + \frac{1}{4xe^{2x}} - \frac{1}{2}\right] = \frac{1}{2}.$$

Also, let $f(x) = \frac{1}{4x}\left[\exp\{-2x\} + 2x - 1\right]$, and note that, by L'Hôpital's rule,

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{-2\exp\{-2x\} + 2}{4} = 0.$$

From these calculations, we see that $\varphi(x)$ is a function which approaches 1 as $x \to 0$, which approaches $1/2$ as $x \to \infty$, and which is monotonically decreasing in between. The required result, thus, follows. □

Lemma SE-35:

Let

$$\begin{aligned}
\mathcal{G}_1^0 &= \{\rho_T : \ \rho_T = 1 \text{ for all } T \text{ sufficiently large}\}, \\
\mathcal{G}_2^0 &= \left\{\rho_T = \exp\left\{-\frac{1}{q(T)}\right\} : T/q(T) \to 0\right\}, \\
\mathcal{G}_3^0 &= \left\{\rho_T = \exp\left\{-\frac{1}{q(T)}\right\} : q(T) \sim T\right\}, \\
\mathcal{G}_4^0 &= \left\{\rho_T = \exp\left\{-\frac{1}{q(T)}\right\} : q(T) \to \infty \text{ but } q(T)/T \to 0\right\} \\
\mathcal{G}_5^0 &= \mathcal{G}_{\mathrm{St}} = \left\{|\rho_T| = \exp\left\{-\frac{1}{q(T)}\right\} : q(T) \geq 0 \text{ and } q(T) = O(1) \text{ as } T \to \infty\right\}.
\end{aligned}$$

Suppose that $\rho_T \in \bigcup_{j=1}^5 \mathcal{G}_j^0$. Then, under Assumptions 1 and 4,

$$\frac{\rho_T}{\sqrt{NT}} \sum_{i=1}^N w_{i0}\varepsilon_{i2} = O_p\left(\frac{1}{\sqrt{T}}\right),$$

as $N, T \to \infty$.

Proof of Lemma SE-35:

Note that, by Assumptions 1 and 4,

$$\begin{aligned}
E\left(\frac{1}{\sqrt{NT}} \sum_{i=1}^N w_{i0}\varepsilon_{i2}\right)^2 &= \frac{1}{NT} \sum_{i=1}^N \sum_{j=1}^N E\left[w_{i0}w_{j0}\right] E\left[\varepsilon_{i2}\varepsilon_{j2}\right] \\
&= \frac{\sigma^2}{NT} \sum_{i=1}^N E\left[w_{i0}^2\right] \\
&\leq \frac{\sigma^2}{T} \sup_i E\left[w_{i0}^2\right] = O\left(\frac{1}{T}\right),
\end{aligned}$$

from which we deduce, using Markov's inequality, that

$$\frac{1}{\sqrt{NT}} \sum_{i=1}^N w_{i0}\varepsilon_{i2} = O_p\left(\frac{1}{\sqrt{T}}\right).$$

Moreover, since $|\rho_T| \leq 1$ for all T sufficiently large for $\rho_T \in \bigcup_{j=1}^{5} \mathcal{G}_j^0$, it further follows that

$$\frac{\rho_T}{\sqrt{NT}} \sum_{i=1}^{N} w_{i0}\varepsilon_{i2} = O_p\left(\frac{1}{\sqrt{T}}\right),$$

as required. □

Appendix SF: Additional Monte Carlo Results

This section reports additional Monte Carlo results focusing on comparing finite sample performance of alternative point estimators. The simulation results reported here therefore complement those reported in the main text of the paper, which focused on comparing finite sample performance of confidence procedures. The data generating processes we consider for the purpose of this study are similar to those considered in the paper. In particular, data are generated by the process

$$\begin{aligned} y_{it} &= a_i + w_{it}, \\ w_{it} &= \rho w_{it-1} + \varepsilon_{it}, \text{ for } i = 1, ...N \text{ and } t = 1, ..., T; \end{aligned}$$

where $\{\varepsilon_{it}\} \equiv i.i.d.N(0,1)$ and $\{a_i\} \equiv i.i.d.N(2,1)$. We vary $w_{i0} = 0, 2$ and $\rho = 1.00$, 0.99, 0.95, 0.90, 0.80, and 0.60. In addition, we let $N = 100, 200$. When $N = 100$, we take $T = 50$, 100; and when $N = 200$, we consider $T = 100, 200$.

We compare the point estimation properties of the AIP estimator

$$\hat{\rho}_{\text{AIP}} = w_{IC}\hat{\rho}_{\text{IVD}} + (1 - w_{IC})\,\hat{\rho}_{\text{pols}},$$

where

$$w_{IC} = \left[1 + \exp\left\{\frac{1}{2}\left(\mathbb{T}_{NT} + \sqrt{N}\left(1 + \ln\left(\ln T\right)^{1/2}\right)\right)\right\}\right]^{-1},$$

with that of the the bias-corrected within-group (BCWG) estimator of Hahn and Kuersteiner (2002), the POLS estimator, the Anderson-Hsiao IV estimator, the X-differencing estimator of Han, Phillips, and Sul (2014), and the Arellano-Bover IV estimator proposed in Arellano and Bover (1995) and further analyzed in Blundell and Bond (1998). Here, we differentiate between the Anderson-Hsiao IV estimator and the Arellano-Bover IV estimator by denoting the former as $\widehat{\rho}_{\text{IVD}}$, since it is based on performing IV on the first-differenced equation, and denoting the Arellano-Bover IV estimator as $\widehat{\rho}_{\text{IVL}}$, since it is based on performing IV on the equation in levels. Tables SF-1 through SF-4 report median bias of the estimators included in the comparison for different configurations of N, T, ρ_0, and w_{i0}, whereas Tables SF-5 through SF-8 report results for the range between the 0.05 and the 0.95 quantiles for the same set of experiments[1]. Looking at the results reported in these tables, it seems that the only general conclusion we can make is that no particular estimator dominates all others either in terms of median bias or in terms of 0.05-0.95 quantile range. Different estimators perform better or worse for different subclasses of experiments characterized by N, T, ρ_0, and w_{i0}. However, there are some patterns in the simulation data which we will summarize below.

[1] We have chosen to use median bias and 0.05-0.95 quantile range to measure the centrality and the dispersion of the point estimators because these measures are robust to the possible non-existence of finite sample moments of estimators.

First, the results of the first four tables show that, in the cases where $w_{i0} = 0$ which is consistent with the assumption of mean stationarity of the initial condition, $\widehat{\rho}_{\text{IVL}}$ tends to be the best performer in terms of median bias overall. On the other hand, $\widehat{\rho}_{\text{IVL}}$ is also the estimator whose performance is most sensitive to the specification of the initial condition, so that, under the specification that $w_{i0} = 2$ which violates the assumption of mean stationarity, the performance of $\widehat{\rho}_{\text{IVL}}$ deteriorates considerably, particularly in the cases where the underlying process is stable. More specifically, for experiments where $w_{i0} = 2$ and $\rho_0 \leq 0.9$, the performance of $\widehat{\rho}_{\text{IVL}}$ in terms of median bias is only fifth amongst the six estimators. On the other hand, relative to the other estimators, the performance of the AIP estimator $\widehat{\rho}_{\text{AIP}}$ is the most robust across different specifications of N, T, ρ_0, and w_{i0} in the sense that, across all experiments, it is the only estimator which never ranked in the bottom two in terms of median bias. Moreover, compared to the other five estimators, $\widehat{\rho}_{\text{AIP}}$ seems to perform particularly well in terms of median bias in the experiments reported in Table SF-4, where the sample sizes are relatively large ($N = 200$ and $T = 100$ or 200) and where mean stationarity of the initial condition is not assumed, i.e., $w_{i0} = 2$. In this set of experiments, $\widehat{\rho}_{\text{AIP}}$ ranks first or second in ten of the twelve experiments.

With respect to the other estimators, note that the results on the median bias for the Anderson-Hsiao IV estimator $\widehat{\rho}_{\text{IVD}}$ and the POLS estimator $\widehat{\rho}_{\text{POLS}}$ are very much in agreement with what is predicted by our large sample theory. POLS does poorly in terms of median bias when the underlying process is stable but much better when $\rho_0 = 1$, while IV is just the opposite. The performance of the X-differencing estimator of Han, Phillips, and Sul (2014) is similar to that of POLS in the sense that it is better when ρ_0 is in the "more persistent" region of the parameter space than when it is in the "more stable" region; but, overall, the median bias of the X-differencing estimator is smaller than that of POLS, and its performance across the parameter space is more uniform than that of POLS. Finally, the median biases of the BCWG estimator are also smaller when ρ_0 is in the $0.6 - 0.9$ range than in cases where $0.95 \leq \rho_0 \leq 1$. These results are consistent with results given in Hahn and Kuersteiner (2002), as their bias-correction procedure is specifically designed to remove second-order biases in the stable case, not the unit root case.

Next, we turn our attention to Tables SF-5 through SF-8, which report results on the dispersion of various estimators as measured by the 0.05-0.95 quantile range. Here, note that the POLS estimator tends to be the best performer, particularly in the unit root and near unit root cases. On the other hand, AIP does well in the persistent region of the parameter space (e.g., when $\rho_0 = 1$ or $\rho_0 = 0.99$) but tends to do less well relative to all the "OLS-type" estimators (i.e., $\widehat{\rho}_{\text{BCWG}}$, $\widehat{\rho}_{\text{POLS}}$, and $\widehat{\rho}_{\text{XD}}$) in terms of dispersion when the underlying process is stable. The degree of dispersion exhibited by AIP very much reflects that of the constituent estimators, $\widehat{\rho}_{\text{IVD}}$ and $\widehat{\rho}_{\text{POLS}}$, from which it is constructed. Hence, when the underlying process is very persistent, the weights on AIP shift toward POLS, thus, taking advantage of the efficiency of the latter estimator when the true autoregressive coefficient is unity or very close to unity. On the other hand, when the underlying process is stable, the weights on AIP shift toward the Anderson-Hsiao IV estimator, which, although well-centered in terms of having a small median bias, tends to be less efficient relative to the "OLS-type" estimators; and our simulation results show that the AIP estimator inherits both the virtue and the vice of this estimator.

Table SF-1: Median Bias
$N = 100$, $w_{i0} = 0$

ρ_0	T	$\widehat{\rho}_{\text{BCWG}}$	$\widehat{\rho}_{\text{POLS}}$	$\widehat{\rho}_{\text{IVD}}$	$\widehat{\rho}_{\text{XD}}$	$\widehat{\rho}_{\text{IVL}}$	$\widehat{\rho}_{\text{AIP}}$
1.00	50	-2.17×10^{-2}	-2.66×10^{-4}	-4.23×10^{-2}	-1.52×10^{-4}	-1.85×10^{-5}	-3.50×10^{-4}
1.00	100	-1.05×10^{-2}	-1.64×10^{-4}	-2.33×10^{-2}	-1.56×10^{-4}	-4.25×10^{-5}	-2.07×10^{-4}
0.99	50	-2.45×10^{-2}	1.81×10^{-4}	-5.31×10^{-4}	-8.57×10^{-3}	-2.21×10^{-5}	1.70×10^{-4}
0.99	100	-1.22×10^{-2}	1.17×10^{-4}	-6.02×10^{-4}	-7.21×10^{-3}	-8.61×10^{-5}	1.21×10^{-4}
0.95	50	-1.97×10^{-2}	4.93×10^{-3}	-6.27×10^{-4}	-1.89×10^{-2}	-2.87×10^{-4}	4.90×10^{-3}
0.95	100	-5.99×10^{-3}	4.59×10^{-3}	5.74×10^{-4}	-9.84×10^{-3}	5.29×10^{-4}	4.29×10^{-3}
0.90	50	-1.20×10^{-2}	1.66×10^{-2}	2.95×10^{-4}	-1.68×10^{-2}	2.99×10^{-4}	1.40×10^{-2}
0.90	100	-3.21×10^{-3}	1.62×10^{-2}	-2.61×10^{-5}	-8.15×10^{-3}	2.55×10^{-4}	3.50×10^{-3}
0.80	50	-6.05×10^{-3}	5.34×10^{-2}	4.82×10^{-4}	-1.12×10^{-2}	-1.48×10^{-4}	1.10×10^{-2}
0.80	100	-1.57×10^{-3}	5.30×10^{-2}	3.62×10^{-6}	-5.52×10^{-3}	-8.80×10^{-5}	2.82×10^{-4}
0.60	50	-3.08×10^{-3}	1.56×10^{-1}	1.82×10^{-4}	-4.77×10^{-3}	-1.64×10^{-4}	1.07×10^{-3}
0.60	100	-7.23×10^{-4}	1.55×10^{-1}	-4.56×10^{-4}	-2.20×10^{-3}	-3.21×10^{-4}	-4.51×10^{-4}

Results are based on 10,000 simulations.

Table SF-2: Median Bias
$N = 100$, $w_{i0} = 2$

ρ_0	T	$\widehat{\rho}_{\text{BCWG}}$	$\widehat{\rho}_{\text{POLS}}$	$\widehat{\rho}_{\text{IVD}}$	$\widehat{\rho}_{\text{XD}}$	$\widehat{\rho}_{\text{IVL}}$	$\widehat{\rho}_{\text{AIP}}$
1.00	50	-2.17×10^{-2}	-2.66×10^{-4}	-3.20×10^{-2}	-1.52×10^{-4}	-1.86×10^{-5}	-3.01×10^{-4}
1.00	100	-1.05×10^{-2}	-1.64×10^{-4}	-1.95×10^{-2}	-1.56×10^{-4}	-4.13×10^{-5}	-1.77×10^{-4}
0.99	50	-2.41×10^{-2}	1.76×10^{-4}	6.63×10^{-4}	-7.91×10^{-3}	-4.70×10^{-4}	1.65×10^{-4}
0.99	100	-1.18×10^{-2}	1.20×10^{-4}	-5.14×10^{-4}	-6.56×10^{-3}	-4.58×10^{-4}	1.30×10^{-4}
0.95	50	-1.67×10^{-2}	4.81×10^{-3}	-2.37×10^{-4}	-1.11×10^{-2}	-7.63×10^{-3}	4.76×10^{-3}
0.95	100	-5.12×10^{-3}	4.48×10^{-3}	5.62×10^{-4}	-5.85×10^{-3}	-3.16×10^{-3}	4.17×10^{-3}
0.90	50	-9.05×10^{-3}	1.60×10^{-2}	1.42×10^{-4}	-3.96×10^{-3}	-1.58×10^{-2}	1.29×10^{-2}
0.90	100	-2.51×10^{-3}	1.58×10^{-2}	1.06×10^{-4}	-2.05×10^{-3}	-7.15×10^{-3}	2.97×10^{-3}
0.80	50	-4.03×10^{-3}	5.18×10^{-2}	6.04×10^{-4}	4.61×10^{-3}	-2.76×10^{-2}	8.58×10^{-3}
0.80	100	-1.13×10^{-3}	5.21×10^{-2}	4.20×10^{-5}	2.13×10^{-3}	-1.27×10^{-2}	2.46×10^{-4}
0.60	50	-2.02×10^{-3}	1.53×10^{-1}	3.55×10^{-4}	6.63×10^{-3}	-3.51×10^{-2}	9.86×10^{-4}
0.60	100	-4.25×10^{-4}	1.53×10^{-1}	-4.95×10^{-4}	3.46×10^{-3}	-1.67×10^{-2}	-4.93×10^{-4}

Results are based on 10,000 simulations.

Table SF-3: Median Bias
$N = 200$, $w_{i0} = 0$

ρ_0	T	$\widehat{\rho}_{\text{BCWG}}$	$\widehat{\rho}_{\text{POLS}}$	$\widehat{\rho}_{\text{IVD}}$	$\widehat{\rho}_{\text{XD}}$	$\widehat{\rho}_{\text{IVL}}$	$\widehat{\rho}_{\text{AIP}}$
1.00	100	-1.04×10^{-2}	-9.41×10^{-5}	-1.73×10^{-2}	-5.88×10^{-5}	-5.75×10^{-5}	-9.30×10^{-5}
1.00	200	-5.07×10^{-3}	-3.77×10^{-5}	-5.71×10^{-3}	-2.82×10^{-5}	-9.46×10^{-5}	-3.85×10^{-5}
0.99	100	-1.21×10^{-2}	2.25×10^{-4}	-5.58×10^{-4}	-7.14×10^{-3}	1.68×10^{-5}	2.23×10^{-4}
0.99	200	-5.25×10^{-3}	1.95×10^{-4}	-2.66×10^{-5}	-4.97×10^{-3}	2.82×10^{-5}	1.93×10^{-4}
0.95	100	-6.00×10^{-3}	4.72×10^{-3}	2.26×10^{-4}	-9.85×10^{-3}	1.41×10^{-4}	4.45×10^{-3}
0.95	200	-1.57×10^{-3}	4.53×10^{-3}	-6.39×10^{-5}	-4.71×10^{-3}	-7.69×10^{-5}	6.10×10^{-4}
0.90	100	-3.07×10^{-3}	1.64×10^{-2}	9.40×10^{-5}	-7.96×10^{-3}	-8.47×10^{-5}	2.13×10^{-3}
0.90	200	-7.77×10^{-4}	1.61×10^{-2}	-1.09×10^{-4}	-3.84×10^{-3}	1.99×10^{-6}	-1.01×10^{-4}
0.80	100	-1.57×10^{-3}	5.33×10^{-2}	-1.31×10^{-4}	-5.42×10^{-3}	-8.25×10^{-5}	-9.95×10^{-5}
0.80	200	-4.05×10^{-4}	5.29×10^{-2}	-2.52×10^{-4}	-2.68×10^{-3}	-2.23×10^{-4}	-2.52×10^{-4}
0.60	100	-7.10×10^{-4}	1.56×10^{-1}	8.92×10^{-5}	-2.27×10^{-3}	1.04×10^{-4}	8.92×10^{-5}
0.60	200	-1.60×10^{-4}	1.55×10^{-1}	3.93×10^{-5}	-1.10×10^{-3}	5.69×10^{-5}	3.93×10^{-5}

Results are based on 10,000 simulations.

Table SF-4: Median Bias
$N = 200$, $w_{i0} = 2$

ρ_0	T	$\widehat{\rho}_{\text{BCWG}}$	$\widehat{\rho}_{\text{POLS}}$	$\widehat{\rho}_{\text{IVD}}$	$\widehat{\rho}_{\text{XD}}$	$\widehat{\rho}_{\text{IVL}}$	$\widehat{\rho}_{\text{AIP}}$
1.00	100	-1.04×10^{-2}	-9.41×10^{-5}	-1.65×10^{-2}	-5.88×10^{-5}	-5.78×10^{-5}	-9.57×10^{-5}
1.00	200	-5.07×10^{-3}	-3.77×10^{-5}	-5.31×10^{-3}	-2.82×10^{-5}	-9.53×10^{-5}	-3.68×10^{-5}
0.99	100	-1.18×10^{-2}	2.26×10^{-4}	-2.50×10^{-4}	-6.51×10^{-3}	-3.55×10^{-4}	2.26×10^{-4}
0.99	200	-5.07×10^{-3}	1.95×10^{-4}	-8.64×10^{-5}	-4.54×10^{-3}	-2.70×10^{-4}	1.92×10^{-4}
0.95	100	-5.14×10^{-3}	4.61×10^{-3}	1.16×10^{-4}	-5.88×10^{-3}	-3.60×10^{-3}	4.31×10^{-3}
0.95	200	-1.36×10^{-3}	4.46×10^{-3}	-7.90×10^{-6}	-2.86×10^{-3}	-1.99×10^{-3}	5.69×10^{-4}
0.90	100	-2.32×10^{-3}	1.60×10^{-2}	1.97×10^{-4}	-1.90×10^{-3}	-7.56×10^{-3}	1.83×10^{-3}
0.90	200	-5.96×10^{-4}	1.59×10^{-2}	-8.86×10^{-5}	-9.11×10^{-4}	-3.59×10^{-3}	-8.39×10^{-5}
0.80	100	-1.04×10^{-3}	5.24×10^{-2}	-4.79×10^{-5}	2.24×10^{-3}	-1.27×10^{-2}	-1.84×10^{-5}
0.80	200	-2.60×10^{-4}	5.25×10^{-2}	-2.40×10^{-4}	1.13×10^{-3}	-6.28×10^{-3}	-2.40×10^{-4}
0.60	100	-4.36×10^{-4}	1.54×10^{-1}	5.13×10^{-5}	3.39×10^{-3}	-1.63×10^{-2}	5.13×10^{-5}
0.60	200	-7.32×10^{-5}	1.55×10^{-1}	-2.60×10^{-6}	1.73×10^{-3}	-7.88×10^{-3}	-2.60×10^{-6}

Results are based on 10,000 simulations.

Table SF-5: Nine Decile Range 0.05 to 0.95 $N = 100$, $w_{i0} = 0$							
ρ_0	T	$\widehat{\rho}_{\text{BCWG}}$	$\widehat{\rho}_{\text{POLS}}$	$\widehat{\rho}_{\text{IVD}}$	$\widehat{\rho}_{\text{XD}}$	$\widehat{\rho}_{\text{IVL}}$	$\widehat{\rho}_{\text{AIP}}$
1.00	50	0.0211	0.0093	3.6701	0.0206	0.0471	0.0095
1.00	100	0.0106	0.0046	2.5061	0.0101	0.0334	0.0048
0.99	50	0.0222	0.0108	0.4070	0.0212	0.0579	0.0108
0.99	100	0.0117	0.0062	0.1479	0.0108	0.0461	0.0062
0.95	50	0.0259	0.0157	0.1380	0.0245	0.0797	0.0162
0.95	100	0.0149	0.0108	0.0793	0.0142	0.0594	0.0176
0.90	50	0.0292	0.0212	0.1161	0.0282	0.0855	0.0329
0.90	100	0.0173	0.0150	0.0712	0.0169	0.0616	0.0585
0.80	50	0.0336	0.0303	0.1034	0.0335	0.0878	0.0877
0.80	100	0.0220	0.0251	0.0685	0.0217	0.0624	0.0682
0.60	50	0.0407	0.0533	0.0944	0.0429	0.0860	0.0941
0.60	100	0.0273	0.0483	0.0629	0.0274	0.0594	0.0629

Results are based on 10,000 simulations.

Table SF-6: Nine Decile Range 0.05 to 0.95 $N = 100$, $w_{i0} = 2$							
ρ_0	T	$\widehat{\rho}_{\text{BCWG}}$	$\widehat{\rho}_{\text{POLS}}$	$\widehat{\rho}_{\text{IVD}}$	$\widehat{\rho}_{\text{XD}}$	$\widehat{\rho}_{\text{IVL}}$	$\widehat{\rho}_{\text{AIP}}$
1.00	50	0.0211	0.0093	3.3846	0.0206	0.0471	0.0096
1.00	100	0.0106	0.0046	2.3667	0.0101	0.0335	0.0048
0.99	50	0.0223	0.0107	0.3283	0.0211	0.0625	0.0108
0.99	100	0.0116	0.0062	0.1354	0.0107	0.0492	0.0062
0.95	50	0.0252	0.0155	0.1175	0.0235	0.0969	0.0160
0.95	100	0.0145	0.0106	0.0731	0.0136	0.0661	0.0174
0.90	50	0.0282	0.0208	0.1020	0.0270	0.1058	0.0326
0.90	100	0.0171	0.0149	0.0674	0.0163	0.0682	0.0570
0.80	50	0.0323	0.0299	0.0940	0.0323	0.1052	0.0828
0.80	100	0.0216	0.0249	0.0653	0.0212	0.0680	0.0651
0.60	50	0.0400	0.0530	0.0887	0.0427	0.0960	0.0884
0.60	100	0.0271	0.0481	0.0607	0.0273	0.0625	0.0607

Results are based on 10,000 simulations.

Table SF-7: Nine Decile Range 0.05 to 0.95 $N = 200$, $w_{i0} = 0$							
ρ_0	T	$\widehat{\rho}_{\text{BCWG}}$	$\widehat{\rho}_{\text{POLS}}$	$\widehat{\rho}_{\text{IVD}}$	$\widehat{\rho}_{\text{XD}}$	$\widehat{\rho}_{\text{IVL}}$	$\widehat{\rho}_{\text{AIP}}$
1.00	100	0.0074	0.0033	2.4315	0.0071	0.0234	0.0033
1.00	200	0.0037	0.0017	1.8462	0.0035	0.0166	0.0017
0.99	100	0.0084	0.0043	0.1022	0.0077	0.0324	0.0043
0.99	200	0.0045	0.0026	0.0492	0.0041	0.0262	0.0026
0.95	100	0.0105	0.0076	0.0557	0.0100	0.0426	0.0097
0.95	200	0.0063	0.0053	0.0358	0.0061	0.0313	0.0313
0.90	100	0.0123	0.0106	0.0513	0.0122	0.0441	0.0456
0.90	200	0.0079	0.0081	0.0342	0.0078	0.0317	0.0342
0.80	100	0.0155	0.0178	0.0474	0.0155	0.0435	0.0474
0.80	200	0.0104	0.0154	0.0328	0.0103	0.0311	0.0328
0.60	100	0.0195	0.0346	0.0442	0.0201	0.0420	0.0442
0.60	200	0.0136	0.0327	0.0302	0.0136	0.0294	0.0302

Results are based on 10,000 simulations.

Table SF-8: Nine Decile Range 0.05 to 0.95 $N = 200$, $w_{i0} = 2$							
ρ_0	T	$\widehat{\rho}_{\text{BCWG}}$	$\widehat{\rho}_{\text{POLS}}$	$\widehat{\rho}_{\text{IVD}}$	$\widehat{\rho}_{\text{XD}}$	$\widehat{\rho}_{\text{IVL}}$	$\widehat{\rho}_{\text{AIP}}$
1.00	100	0.0074	0.0033	2.3574	0.0071	0.0234	0.0033
1.00	200	0.0037	0.0017	1.8048	0.0035	0.0166	0.0017
0.99	100	0.0083	0.0043	0.0934	0.0077	0.0345	0.0043
0.99	200	0.0045	0.0026	0.0471	0.0041	0.0274	0.0026
0.95	100	0.0103	0.0075	0.0524	0.0098	0.0474	0.0098
0.95	200	0.0063	0.0053	0.0347	0.0060	0.0330	0.0308
0.90	100	0.0122	0.0105	0.0484	0.0117	0.0488	0.0442
0.90	200	0.0078	0.0080	0.0333	0.0076	0.0334	0.0333
0.80	100	0.0153	0.0176	0.0451	0.0151	0.0474	0.0451
0.80	200	0.0103	0.0153	0.0319	0.0102	0.0324	0.0319
0.60	100	0.0195	0.0345	0.0430	0.0200	0.0443	0.0430
0.60	200	0.0135	0.0327	0.0296	0.0135	0.0302	0.0296

Results are based on 10,000 simulations.

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