Achieving Efficiency in Dynamic Contribution Games*

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Abstract

We analyze a dynamic contribution game in which a group of agents exert costly effort over time to make progress on a project. The state of the project progresses gradually at a rate that depends on the agents' cumulative efforts, and it is completed once it teaches a pre-specified threshold, at which point it generates a lump sum payoff. We characterize a budget balanced mechanism which at every moment, induces each agent to exert the first best effort level as the outcome of a Markov Perfect Equilibrium, thus eliminating the free-rider problem. This mechanism specifies for each agent flow payments that are due while the project is in progress, and rewards that are disbursed upon completion of the project. The payments are placed in a savings account that accumulates interest, and their magnitude is chosen to balance the budget. Our mechanism resembles the incentives structure in early-stage entrepreneurial ventures and it can shed light on why the free-rider problem is not as prevalent as traditional economic models suggest.

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1 Introduction

Team problems are ubiquitous in modern economies as individuals and firms often need to collaborate in production and service provision. Moreover, the collaboration is often geared towards a defined deliverable; *i.e.*, towards completing a project. For example, in early-stage entrepreneurial ventures (*e.g.*, startups), individuals collaborate over time to create value, and the firm generates a payoff predominantly when it goes public or it is acquired by another corporation. Joint R&D and new product development projects also share many of these characteristics: corporations work together to achieve a common goal, progress is gradual, and the payoff from the collaboration is harvested primarily after said goal is achieved; for example, once a patent is secured or the product under development is released to the market. It is well known that such environments are susceptible to the free-rider problem (Olson (1965) and Alchian and Demsetz (1972)).

We study a game of dynamic contributions to a joint project, wherein at every moment, each agent in the group chooses his (costly) effort level to progressively bring the project closer to completion, at which point it will generate a lump-sum payoff. An important feature of this game is that efforts are strategic complements across time (Kessing (2007)). That is because effort increases with progress, and so by raising his effort, an agent induces the other agents to raise their future efforts. By comparing the equilibrium strategies to the efficient ones, we identify two kinds of inefficiencies. First, because each agent is incentivized by his share of the project's payoff (rather than the entire payoff), in equilibrium, he shirks by exerting inefficiently low effort. Second, because efforts are strategic complements, in equilibrium, each agent front-loads his effort in order to induce others to raise their future efforts, which in turn renders him better off. The former has been extensively discussed in the literature, but the latter, while subtle, is also intuitive. For instance, startups often use vesting schemes to disincentivize entrepreneurs from front-loading effort; *i.e.*, working hard early on and then walking away while retaining their stake in the firm.

We propose a mechanism which at every moment, induces each agent to exert the efficient level of effort as the outcome of a Markov Perfect equilibrium. This mechanism specifies for each agent flow payments that are due while the project is in progress, and a reward that is disbursed upon completion of the project. These payments are placed in a savings account that accumulates interest, and their magnitude is chosen to balance the budget. The mechanism effectively rewards each agent with the entire payoff that the project generates, thus making him the full residual claimant, while the flow payments increase with progress so that the marginal benefit from front-loading effort is exactly offset by the marginal cost associated with larger future flow payments. This mechanism is implementable provide that the agents have sufficient cash reserves at the outset of the game to make the specified flow payments.[1] This is a reasonable assumption in most relevant applications. In startups for example, the founders need to have some cash in hand to finance their project until they can raise capital. Similarly, the firms that partner in R&D or new product development ventures typically do have substantial cash reserves or access to credit.

Our mechanism resembles many of the features in the incentives structure in startups. In particular, entrepreneurs usually receive a salary that is below the market rate (if any). Therefore, the flow payments in our mechanism can be interpreted as the differential between what an individual earns in the startup and his outside option; *i.e.*, what he would earn if he were to seek employment in the market. Moreover, it is reasonable to assume that his outside option becomes more valuable as progress is made in the venture, which is consistent with the flow payments increasing with progress in our mechanism. Finally, entrepreneurs typically own shares, and so they are residual claimants.

To facilitate the intuition, in our base model, the project progresses deterministically, and the mechanism depends on the state of the project, but not on time. We extend our model to incorporate uncertainty in the evolution of the project, and we characterize the efficient time-dependent mechanism. In this case, flow payments are specified as a function of both time and the state of the project, and terminal rewards as a function of the completion time. This mechanism specifies a terminal reward that decreases in the completion time of the project, and at every moment, the flow payments are chosen so as to penalize the agents if the project is on expectation closer to completion than it should if all agents were applying the first best strategies. Intuitively, the former acts as a lever to mitigate the agents' incentives to shirk, while the latter as a lever to eliminate their incentives to front-load effort. An important difference relative to the deterministic mechanism is that with uncertainty, it is impossible to simultaneously achieve efficiency and ex-post budget balance; instead, the efficient mechanism can only be budget balanced ex-ante (*i.e.*, on expectation). Finally, in the special case in which the project progresses deterministically, the efficient time-dependent mechanism specifies that the agents make zero flow payments on the equilibrium path (but positive off path), and as

[1] For the case in which the efficient mechanism is not implementable, we characterize the mechanism that maximizes the agents' ex-ante discounted payoff as a function of the agents' cash reserves.

such, it is implementable even if the agents are credit constrained.

Naturally, our paper is related to the static moral hazard in teams literature (Olson (1965), Holmström (1982), Ma, Moore and Turnbull (1988), Bagnoli and Lipman (1989), Legros and Matthews (1993), and others). These papers focus on the free-rider problem that arises when each agent must share the output of his effort with the other members of the team but he alone bears its cost, and they explore ways to restore efficiency. Holmström (1982) shows that in a setting wherein a group of agents jointly produce output, there exists no budget balanced sharing rule that induces the agents to choose efficient actions. Moreover, he shows how efficiency is attainable by introducing a third party who makes each agent the full residual claimant of output and breaks the budget using transfers that are independent of output. Legros and Matthews (1993) provide a necessary and sufficient condition for the existence of a sharing rule such that the team sustains efficiency. Their construction uses mixed strategies where all agents but one choose the efficient actions, and the last agent chooses the efficient action with probability *close* to 1. However, a necessary condition for this mechanism to sustain efficiency is that the agents have unlimited liability. The idea behind our mechanism can be viewed as a dynamic counterpart of Holmström's budget breaker, except that the savings account plays the role of the budget breaker in our setting.

Closer related to this paper is the literature on dynamic contribution games. Admati and Perry (1991) characterizes a MPE with two players, and they show that efforts are inefficiently low relative to the efficient outcome. Marx and Matthews (2000) generalize this result to games with n players and also characterize non-Markov subgame perfect equilibria. More recently, Yildirim (2006) and Kessing (2007) show that if the project generates a payoff only upon completion, then efforts become strategic complements across time. Georgiadis (2015) studies how the size of the group influences incentives, and he analyzes the problem faced by a principal who chooses the group size and the agents' incentive contracts. Our model is a generalized version of Georgiadis et. al. (2014), who focus on the problem faced by a project manager who must determine the optimal project size. In contrast to these papers, which highlight the inefficiencies that arise in these games and explore the differences to their static counterparts, our objective in this paper is to construct a mechanism that restores efficiency.

Finally, our work is also related to the dynamic mechanisms of Bergemann and Valimaki (2010) and Athey and Segal (2013). Our mechanism is similar in that it induces each agent to internalize his externality on the other agents, and as a result, behave efficiently in every period. A key difference however is that in our model, each agent faces a moral hazard problem, whereas in these two papers, the agents have private information and so they face an adverse selection problem.

The remainder of this paper is organized as follows. We present the model in Section 2, and in Section 3, we characterize the MPE when the agents are credit constrained, as well as the efficient outcome of this game. In Section 4, we present out main result: a budget balanced mechanism that achieves efficiency as the outcome of a MPE. In Section 5, we extend our model to incorporate uncertainty and we characterize the efficient mechanism, and in Section 6 we conclude. In Appendix A, we characterize the mechanism that maximizes the agents' total ex-ante discounted payoff when the agents don't have sufficient cash reserves and so the efficient (time-independent) mechanism characterized in Section 4 cannot be implemented. In Appendix B, we consider two extensions to our model, and in Appendix C, we present an example with quadratic effort costs and symmetric agents, which enables us to characterize the mechanism in closed-form. All proofs are provided in Appendix D.

2 Model

A group of n agents collaborate to complete a project. Time $t \in [0, \infty)$ is continuous. The project starts at initial state q_0, its state q_t evolves at a rate that depends on the agents' instantaneous effort levels, and it is completed at the first time τ such that q_t hits the completion state which is denoted by Q.[2] Each agent is risk neutral, discounts time at rate $r > 0$, and receives a pre-specified reward $V_i = \alpha_i V > 0$ upon completing the project, where $\sum_{i=1}^{n} \alpha_i = 1$.[3] We assume that each agent i has cash reserves $w_i \geq 0$ at the outset of the game, and outside option $\bar{u}_i \geq 0$. An incomplete project has zero value. At every moment t, each agent i observes the state of the project q_t, and privately chooses his effort level $a_{i,t} \geq 0$ at flow cost $c_i(a_{i,t})$ to influence the process

$$dq_t = \left(\sum_{i=1}^{n} a_{i,t} \right) dt \,.$$

[4] We assume that each agent's effort choice is not observable to the other agents, and $c_i(\cdot)$ is a strictly increasing, strictly convex, differentiable function satisfying $c_i'''(a) \geq 0$ for all a, $c_i(0) = c_i'(0) = 0$ and $\lim_{a \to \infty} c_i'(a) = \infty$ for all i. Moreover, we shall assume that the agents

[2] For simplicity, we assume that the completion state is specified exogenously. In Appendix B.2, we also consider the case in which the project size is endogenous, and we show that all results carry over.

[3] Note that the project's reward can random, and V can be interpreted as its expected reward. We assume here that the project generates a payoff only upon completion. In Appendix B.1, we consider the case in which it also generates flow payoffs while it is in progress.

[4] For simplicity, we assume that the project progresses deterministically in the base model. We extend our model to incorporate stochastic progress in Section 5.

are ranked from most to least efficient so that $c_i'(a) \leq c_j'(a)$ for all $i < j$ and $a \geq 0$. Finally, we shall restrict attention to Markov Perfect equilibria (hereafter MPE), where at every moment, each agent chooses his effort level as a function of the current state of the project q_t (but not its entire evolution path $\{q_s\}_{s \leq t}$).[5]

3 Equilibrium Analysis

For benchmarking purposes, in this section we consider the case in which the agents have no cash reserves at the outset of the game (*i.e.*, $w_i = 0$ for all i). First, we characterize the MPE of this game, wherein at every moment, each agent chooses his strategy (as a function of q_t) to maximize his discounted payoff while accounting for the effort strategies of the other team members. Then, we determine the first best outcome of the game; this is the problem faced by a social planner who at every moment chooses the agents' strategies to maximize their total discounted payoff. Finally, we compare the MPE and the first best outcome of the game to pinpoint the inefficiencies that arise when the agents do not take into account the (positive) externality of their actions on the other agents.

3.1 Markov Perfect Equilibrium

In a MPE, at every moment t, each agent i observes the state of the project q_t, and chooses his effort $a_{i,t}$ as a function of q_t to maximize his discounted payoff while accounting for the effort strategies of the other team members. Thus, for a given set of strategies and a given initial value $q_t = q$, agent i's value function satisfies

$$J_i(q) = e^{-r(\tau - t)} \alpha_i V - \int_t^\tau e^{-r(s-t)} c_i(a_{i,s})\, ds\,, \tag{1}$$

where τ is the completion time of the project. Using standard arguments, we expect that the function $J_i(\cdot)$ satisfies the Hamilton-Jacobi-Bellman (hereafter HJB) equation:

$$rJ_i(q) = \max_{a_i} \left\{ -c_i(a_i) + \left(\sum_{j=1}^n a_j \right) J_i'(q) \right\} \tag{2}$$

[5]We focus on the MPE, because (i) it requires minimal coordination among the agents, and (ii) it constitutes the worst case scenario in terms of the severity of the free-rider problem. As shown by Georgiadis et. al. (2014), this game has a continuum of non-Markovian equilibria that yield the agents a higher discounted payoff than the MPE.

defined on $[q_0, Q]$, subject to the boundary condition

$$J_i(Q) = \alpha_i V\,. \tag{3}$$

The boundary condition states that upon completing the project, each agent receives his reward, and the game ends.

It is noteworthy that the MPE need not be unique in this game. Unless the project is sufficiently short (*i.e.*, $Q - q_0$ is sufficiently small) such that at least one agent is willing to undertake the project single-handedly, there exists another (Markovian) equilibrium in which no agent ever exerts any effort, and the project is never completed. In that case, the boundary condition (3) is not satisfied. This leads us to search for a sufficiently large value $\underline{q}$ such that for all $q > \underline{q}$, there exists a non-trivial MPE.

Assuming that (2) holds for each agent i's value function, it follows that the corresponding first order condition is $c_i'(a_i) = J_i'(q)$: at every moment, the agent chooses his effort level such that the marginal cost of effort is equal to the marginal benefit associated with bringing the project closer to completion. By noting that the second order condition is satisfied, $c'(0) = 0$ and $c(\cdot)$ is strictly convex, it follows that for any q, agent i's optimal effort level $a_i(q) = f_i(J_i'(q))$, where $f_i(\cdot) = \max\left\{0, c_i'^{-1}(\cdot)\right\}$. By substituting this into (2), the discounted payoff function of agent i satisfies

$$rJ_i(q) = -c_i\left(f_i\left(J_i'(q)\right)\right) + \left[\sum_{j=1}^{n} f_j\left(J_j'(q)\right)\right] J_i'(q) \tag{4}$$

subject to the boundary condition (3).

Thus, a MPE is characterized by the system of ODE's defined by (4) subject to the boundary condition (3) for all $i \in \{1,..,n\}$, provided that the system has a solution. The following proposition characterizes the unique project-completing MPE of this game, and establishes its main properties.

Proposition 1. *There exists a $\underline{q} < Q$ such that the game defined by (2) subject to the boundary condition (3) for all $i \in \{1,..,n\}$ has a unique MPE on $\left(\underline{q}, Q\right]$. We have $\underline{q} = \inf\{q : J_i(q) > 0$ for all $i\}$. On $\left(\underline{q}, Q\right]$, each agent's discounted payoff is strictly positive, strictly increasing, and, if twice differentiable, strictly convex in q; i.e., $J_i(q) > 0$, $J_i'(q) > 0$, and $J_i''(q) > 0$. The latter also implies that each agent's effort is strictly increasing in q; i.e., $a_i'(q) > 0$.*

The first part of this proposition asserts that if the project is not too long (*i.e.*, $Q - q_0$ is not too large), then the agents find it optimal to exert positive effort, in which case the project is completed. It is intuitive that $J_i(\cdot)$ is strictly increasing: each agent is better off, the closer the project is to completion. That each agent's effort level increases with progress is also intuitive: since he incurs the cost of effort at the time effort is exerted but is only compensated upon completion of the project, his incentives are stronger the closer the project is to completion. An implication of this last result is that efforts are strategic complements in this game. That is because by raising his effort, an agent brings the project closer to completion, thus incentivizing agents to raise their future efforts (Kessing (2007)).

3.2 First Best Outcome

In this section, we consider the problem faced by a social planner, who at every moment chooses each agent's effort level to maximize the group's total discounted payoff. Denoting the planner's discounted payoff function by $\bar{S}(q)$, we expect that it satisfies the HJB equation

$$r\bar{S}(q) = \max_{a_1,..,a_n} \left\{ -\sum_{i=1}^{n} c_i(a_i) + \left(\sum_{i=1}^{n} a_i\right) \bar{S}'(q) \right\} \tag{5}$$

defined on some interval $\left(\underline{q}^s, Q\right]$, subject to the boundary condition

$$\bar{S}(Q) = V \,. \tag{6}$$

The first order conditions for the planner's problem are $c_i'(a_i) = \bar{S}'(q)$ for every i: at every moment, each agent's effort is chosen such that the marginal cost of effort is equal to the group's total marginal benefit of bringing the project closer to completion. Note the difference between the MPE and the first best outcome: here, each agent's effort level trades off his marginal cost of effort and the marginal benefit of progress to the entire group, whereas in the MPE, he trades off his marginal cost of effort and his marginal benefit of progress, while ignoring the (positive) externality of his effort on the other agents.

Denoting the first best effort level of agent i by $\bar{a}_i(q)$, we have that $\bar{a}_i(q) = f_i\left(\bar{S}'(q)\right)$, and by substituting the first order condition into (5), it follows that the planner's discounted payoff function satisfies

$$r\bar{S}(q) = -\sum_{i=1}^{n} c_i\left(f_i\left(\bar{S}'(q)\right)\right) + \left[\sum_{i=1}^{n} f_i\left(\bar{S}'(q)\right)\right] \bar{S}'(q) \tag{7}$$

subject to the boundary condition (6).

The following proposition characterizes the solution to the planner's problem.

Proposition 2. *There exists a* $\underline{q}^s < Q$ *such that the ODE defined by (5) subject to the boundary condition (6) has a unique solution on* $\left(\underline{q}^s, Q\right]$. *We have* $\underline{q}^s = \inf\left\{q : \bar{S}(q) > 0\right\}$. *On this interval, the planner's discounted payoff is strictly positive, strictly increasing and, if twice differentiable, strictly convex in q; i.e.,* $\bar{S}(q) > 0$, $\bar{S}'(q) > 0$, *and* $\bar{S}''(q) > 0$. *The latter also implies that the efficient level of effort for each agent is strictly increasing in q; i.e.,* $\bar{a}_i'(q) > 0$. *In contrast,* $\bar{S}(q) = \bar{a}_i(q) = 0$ *for all* i *and* $q \leq \underline{q}^s$.

Provided that the project is not too long (*i.e.*, Q is not too large), it is socially efficient to undertake and complete the project. The intuition for why each agent's effort level increases with progress is similar to that in Proposition 1.

Note that it is efficient to undertake the project if and only if $\sum_{i=1}^n \bar{u}_i \leq \bar{S}(0)$, because otherwise, the agents are better off collecting their outside options instead. We shall assume that this inequality is satisfied throughout the remainder of this paper.

Next, because the project progresses deterministically, it is possible to study the completion time of the project, which is also deterministic. A comparative static that will be useful for the analysis that follows concerns how the completion time $\bar{\tau}$ depends on the group size n. We suppose that $\underline{q}^s < q_0$.

Remark 1. *Suppose that the agents are identical; i.e.,* $c_i(a) = c_j(a)$ *for all* i, j, *and* a. *Then the completion time of the project in the first best outcome* $\bar{\tau}$ *decreases in the group size* n.

Intuitively, because effort costs are convex, the larger the number of agents in the group, the faster they will complete the project in the first best outcome. As shown in Georgiadis (2015), this is generally not true in the MPE of this game, where the completion time of the project decreases in n if and only if the project is sufficiently long (*i.e.*, Q is sufficiently large).

3.3 Comparing the MPE and the First Best Outcome

Before we propose our mechanism, it is useful to pinpoint the inefficiencies that arise in this game when each agent chooses his effort at every moment to maximize his discounted payoff. To do so, we compare the solution to the planner's problem to the MPE in the following proposition.

Proposition 3. *In the MPE characterized in Proposition 1, at every* $q > \underline{q}$, *each agent exerts less effort relative to the planner's solution characterized in Proposition 1, i.e.,* $a_i(q) < \bar{a}_i(q)$ *for all* i *and* q.

Intuitively, because each agent's incentives are driven by the payoff that he receives upon completion of the project $\alpha_i V < V$, whereas the social planner's incentives are driven by the entire payoff that the project generates, in equilibrium, each agent exerts less effort relative to the first best outcome. This result resembles the earlier results on free-riding in partnerships (*e.g.*, Holmström (1982), Admati and Perry (1991), and others).

It turns out that in this dynamic game, there is a second, more subtle source of inefficiency. Because this game exhibits positive externalities and efforts are strategic complements across time, each agent has an incentive to front-load his effort in order to induce the other agents to raise their effort, which in turn renders him better off.[6]

To see this formally, we use the stochastic maximum principle of optimal control and write the agents' effort strategies as a function of time t instead of the state of the project q.[7] To begin, let us write down the planner's Hamiltonian:

$$\bar{H}_t = -\sum_{i=1}^{n} e^{-rt} c_i\left(a_{i,t}\right) + \lambda_t^{fb} \left(\sum_{i=1}^{n} a_{i,t}\right)$$

where λ_t^{fb} is his co-state variable. The optimality and adjoint equations are

$$\frac{d\bar{H}_t}{da_{i,t}} = 0 \quad \text{and} \quad \dot{\lambda}_t^{fb} = -\frac{dH_t}{dq} \,,$$

respectively. These can be re-written as

$$e^{-rt} c_i'\left(\bar{a}_{i,t}\right) = \lambda_t^{fb} \quad \text{and} \quad \dot{\lambda}_t^{fb} = 0 \,.$$

Therefore, each agent's discounted marginal cost of effort (*i.e.*, $e^{-rt} c_i'\left(\bar{a}_{i,t}\right)$) must be constant over time. This is intuitive: because effort costs are convex, efficiency requires that the agents smooth their effort over time. Next, let us consider the game in which each agent chooses his effort to maximize his own discounted payoff. The Hamiltonian of agent i is

$$H_{i,t} = -e^{-rt} c_i\left(a_{i,t}\right) + \lambda_{i,t} \left(\sum_{j=1}^{n} a_{j,t}\right) \,,$$

where $\lambda_{i,t}$ is his co-state variable. The optimality equation implies that $e^{-rt} c_i'\left(a_{i,t}\right) = \lambda_{i,t}$,

[6]Front-loading here refers to an agent's discounted marginal cost of effort decreasing over time (and with progress), whereas efficiency requires that it is constant across time.

[7]Note that because the project progresses deterministically, there is a 1-1 mapping between t and q, so that the two approaches are equivalent.

whereas the adjoint equation can be written as

$$\dot{\lambda}_{i,t} = -\sum_{j \neq i} \frac{dH_{i,t}}{da_{j,t}} \frac{da_{j,t}}{dt} \frac{dt}{dq} = -\sum_{j \neq i} \frac{\lambda_{i,t} \dot{a}_{j,t}}{\sum_{l=1}^{n} \lambda_{l,t}} .$$

Because efforts are non-negative and increasing in q (and hence increasing in t), we have $\lambda_{j,t} \geq 0$ and $\dot{a}_{j,t} \geq 0$ for all j and t. Therefore, $\dot{\lambda}_{i,t} \leq 0$, which in turn implies that $e^{-rt} c_i' (a_{i,t})$ decreases in t; *i.e.*, in equilibrium, each agent's discounted marginal cost of effort decreases over time. Intuitively, because (i) at every moment t, each agent chooses his effort upon observing the state of the project q_t, (ii) his equilibrium effort level increases in q, and (iii) he is better off the harder others work, each agent has incentives to front-load his effort in order to induce others to raise their future efforts.

4 An Efficient Mechanism

In this section, we establish our main result: We construct a mechanism that induces each self-interested agent to always exert the efficient level of effort as the outcome of a MPE. This mechanism specifies flow payments for each agent while the project is in progress, which are a function of q and they are placed in a savings account that accumulates interest at rate r, and the reward that each agent receives upon completion of the project. We show that on the equilibrium path, the mechanism is budget balanced, whereas off the equilibrium path, there may be a budget surplus in which case money must be burned. However, the mechanism never results in a budget deficit.

The timing of the game proceeds as follows: First, a mechanism is proposed, which specifies (i) an upfront payment $P_{i,0}$ for each agent i to be made before work commences (which turns out to be 0 in the optimal mechanism), and (ii) a schedule of flow payments $h_i (q_t) \, dt$ to be made during every interval $(t, \, t + dt)$ while the project is in progress, and are a function of the state q_t.[8] Then, each agent decides whether to accept or reject the mechanism. If all agents accept the mechanism, then each agent makes the upfront payment specified by the mechanism, and work commences. The payments are then placed in a savings account and they accumulate interest at rate r. Otherwise, the group dissolves, and each agent receives his outside option.

An important remark is that for this mechanism to be implementable, the agents must have

[8]We abstract from the bargaining process that leads to a particular mechanism being proposed. However, since this is a game with complete and symmetric information, multilateral bargaining will result in the agents finding it optimal to agree to a mechanism that maximizes the ex-ante total surplus.

sufficient cash reserves at the outset of the game (*i.e.*, w_i's must be sufficiently large). As such, our analysis is organized as follows. We begin in Section 4.1 by assuming that $w_i = \infty$ for all i (i.e., that the agents have unlimited cash reserves), and we characterize a mechanism that implements the efficient outcome. In Section 4.2, we establish a necessary and sufficient condition for the mechanism to be implementable when agents' cash reserves are unlimited.[9] Then in Section 4.3, we consider the more general environment in which the mechanism may depend on both t and q. In this case, there exists an efficient mechanism which specifies zero flow payments on the equilibrium path and positive flow payments only off the equilibrium path. As a result, this mechanism can be implemented even if the agents are cash constrained.

4.1 Unlimited Cash Reserves

Incentivizing Efficient Actions

Given a set of flow payment functions $\{h_i(q)\}_{i=1}^n$, agent i's discounted payoff function, which we denote by $\hat{J}_i(q)$, satisfies the HJB equation

$$r\hat{J}_i(q) = \max_{a_i} \left\{ -c_i(a_i) + \left(\sum_{j=1}^n a_j \right) \hat{J}_i'(q) - h_i(q) \right\} \tag{8}$$

on $[q_0, Q]$, subject to a boundary condition that remains to be determined. His first order condition is $c_i'(a_i) = \hat{J}_i'(q)$, and because we want to induce each agent to exert the efficient effort level, we must have $c_i'(a_i) = \bar{S}'(q)$, where $\bar{S}(\cdot)$ is characterized in Proposition 2. Therefore,

$$\hat{J}_i(q) = \bar{S}(q) - p_i \, ,$$

where p_i is a constant to be determined. Upon completion of the project, agent i must receive

$$\hat{J}_i(Q) = \bar{S}(Q) - p_i = V - p_i \, ,$$

which gives us the boundary condition. Note that neither the upfront payments $\{P_{i,0}\}_{i=1}^n$, nor the constants $\{p_i\}_{i=1}^n$ affect incentives directly; their role will be to ensure that the mechanism is budget balanced. Using (7) and (8), we can solve for each agent's flow payment function that induces first best effort:

$$h_i(q) \;=\; \sum_{j \neq i} c_j\left(f_j\left(\bar{S}'(q)\right)\right) + rp_i \, . \tag{9}$$

[9] In Appendix A we characterize the mechanism that maximizes the agents' discounted payoff when this condition is not satisfied, and hence the efficient mechanism cannot be implemented.

Based on this analysis, we can establish the following lemma.

Lemma 1. *Suppose that all agents make flow payments that satisfy (9) and receive $V - p_i$ upon completion of the project. Then:*
(i) There exists a MPE in which at every moment, each agent exerts the efficient level of effort $\bar{a}_i(q)$ as characterized in Proposition 2.
(ii) The efficient flow payment function satisfies $h_i(q) \geq 0$ and $h_i'(q) \geq 0$ for all i and q; i.e., the flow payments are always non-negative and non-decreasing in the state of the project.

To understand the intuition behind this construction, recall that the MPE characterized in Section 3.1 is plagued by two sources of inefficiency: first, the agents have incentives to shirk by exerting inefficiently low effort, and second, they front-load their effort, *i.e.*, their discounted marginal cost of effort decreases over time, whereas it is constant in the efficient outcome. Therefore, to induce agents to always exert the first best effort, this mechanism must neutralize both sources of inefficiency. To eradicate the former, each agent must effectively be made the full residual claimant of the project.[10] To neutralize the latter, the flow payments must increase with progress at a rate such that each agent's benefit from front-loading effort is exactly offset by the cost associated with having to make larger flow payments in the future.

By observing (9), we make the following remark regarding the total flow cost that each agent incurs while the project is in progress.

Remark 2. *The flow cost that agent i incurs given the state of the project q is equal to*

$$c_i\left(f_i\left(\bar{S}'(q)\right)\right) + h_i(q) = \sum_{j=1}^{n} c_j\left(f_j\left(\bar{S}'(q)\right)\right) + r\,p_i\,;$$

i.e., it is the same across all agents and equal to flow cost faced by the social planner (up to a constant) .

Intuitively, to induce efficient incentives, the flow payment function for each agent i must be chosen such that he faces the same flow cost (and reward upon completion) that the social planner faces in the original problem (up to a constant).

Budget Balance

In this section, we characterize the payments $\{P_{i,0}\}_{i=1}^n$ and $\{p_i\}_{i=1}^n$ such that the mechanism is budget balanced, while at the same time minimizing the total discounted cost of the upfront

[10]Notice that ignoring the state-dependent component in (9), the *ad infinitum* discounted cost of making the flow payments is equal to $\int_0^\infty e^{-rt} r p_i dt = p_i$, so the net benefit to each agent from completing the project is equal to $(V - p_i) - (-p_i) = V$.

and flow payments along the efficient effort path.

Before the agents begin working on the project, each agent i's discounted payoff is equal to $\hat{J}_i(0) - P_{i,0} = \bar{S}(0) - p_i - P_{i,0}$. Because each agent exerts the efficient effort level along the equilibrium path, the sum of the agents' ex-ante discounted payoffs (before any upfront payments) must equal $\bar{S}(0)$. Therefore, budget balance requires that $n\bar{S}(0) - \sum_{i=1}^{n}(p_i + P_{i,0}) = \bar{S}(0)$, or equivalently, $\sum_{i=1}^{n}(P_{i,0} + p_i) = (n-1)\bar{S}(0)$. Notice that the payments $\{P_{i,0}\}_{i=1}^{n}$ and $\{p_i\}_{i=1}^{n}$ are not determined uniquely.

The total discounted cost of the payments along the equilibrium path are equal to

$$\sum_{i=1}^{n}\left[P_{i,0} + \int_0^{\bar{\tau}} e^{-rs}\sum_{j\neq i} c_j\left(f_j\left(\bar{S}'(q_s)\right)\right) + rp_i\,ds\right] = e^{-r\bar{\tau}}\left[(n-1)V - \sum_{i=1}^{n} p_i\right]. \tag{10}$$

Because we require that the mechanism is never in deficit, it must be the case that $\sum_{i=1}^{n} P_{i,0} \geq 0$, and so $\sum_{i=1}^{n} p_i \leq (n-1)\bar{S}(0)$. Therefore, (10) is minimized when

$$\sum_{i=1}^{n} p_i = (n-1)\bar{S}(0) \quad \text{and} \quad P_{i,0} = 0 \;\; \forall i\,.$$

Observe that the budget balance constraint only pins down the sum $\sum_{i=1}^{n} p_i$, while the individual p_i's will depend on the agents' relative bargaining powers.

So far, we have characterized a mechanism that induces each agent to exert the first best effort at every moment while the project is in progress, and it is budget balanced on the equilibrium path. A potential concern however, is that after a deviation from the equilibrium path, the total amount in the savings account upon completion of the project may be greater or less than $(n-1)\left[V - \bar{S}(0)\right]$, in which case the mechanism will have a budget surplus or deficit, respectively. A desirable attribute for such a mechanism is to be always budget balanced, so that the rewards that the agents receive upon completion of the project sum up to the balance in the savings account, both on and off the equilibrium path. To analyze this case, we let

$$H_t = \sum_{i=1}^{n}\int_0^t e^{r(t-s)} h_i(q_s)\,ds \tag{11}$$

denote the balance in the savings account at time t. We say that the mechanism is *strictly budget balanced* if the sum of the agents' rewards upon completion of the project (at time τ) is equal to $V + H_\tau$. We assume that each agent i is promised reward $\beta_i(V + H_\tau)$ upon completion of the project, where $\beta_i \in [0,1]$ for all i and $\sum_{i=1}^{n} \beta_i = 1$. The following lemma shows that there exists no efficient, strictly budged balanced mechanism.

Lemma 2. *The maximal social welfare of the strictly budget balanced game is equal to the maximal social welfare of the game in which each player i is promised reward $\beta_i V$ upon completion of the project, and the flow payments add up to zero; i.e., $\sum_{i=1}^{n} h_i(q) = 0$ for all q. In particular, with strict budget balance, there exist no non-negative flow payment functions $h_i(\cdot)$ that lead to the efficient outcome. Moreover, in the symmetric case, the maximal social welfare in the strictly budget balanced game is obtained if and only if $h_i(q) = 0$ for all i and q.*

An implication of this result is that it is impossible to construct a mechanism that is simultaneously efficient and strictly budget balanced. Moreover, if the agents are symmetric (*i.e.*, $c_i(a) = c_j(a)$ for all $i \neq j$ and a, and $\beta_i = \frac{1}{n}$ for all i), then there exists no strictly budget balanced mechanism that yields a greater total discounted payoff as an outcome of a MPE than the payoff corresponding to the cash constrained case characterized in Proposition 1. This result can be viewed as a dynamic counterpart to Theorem 1 in Holmström (1982).

While this is a negative result, as shown in the following proposition, it is possible to construct a mechanism that is efficient and budget balanced on the equilibrium path, and it will never have a budget deficit.

Proposition 4. *Suppose that each agent commits to make flow payments as given by (9) for all q, where $\sum_{i=1}^{n} p_i = (n-1)\bar{S}(0)$, and he receives lump-sum reward* $\min\{V - p_i, \beta_i(V + H_\tau)\}$ *upon completion of the project, where $\beta_i = \frac{V - p_i}{V + (n-1)\left[V - \bar{S}(0)\right]}$. Then:*
(i) There exists a Markov Perfect equilibrium in which at every moment, each agent exerts the efficient level of effort $\bar{a}_i(q)$ as characterized in Proposition 2.
(ii) The mechanism is budget balanced on the equilibrium path, and it will never result in a budget deficit (off equilibrium).
(iii) The agents' total ex-ante discounted payoff is equal to the first-best discounted payoff; i.e., $\sum_{i=1}^{n} \hat{J}_i(0) = \bar{S}(0)$.

The intuition for why there exists no efficient, strictly budget balanced mechanism (as shown in Lemma 2) is as follows: To attain efficiency, the mechanism must eliminate the agents' incentives to shirk. However, with strict budget balance, by shirking, an agent delays the completion of the project, lets the balance in the savings account grow, and collects a larger reward upon completion. By capping each agent's reward, the agents no longer have incentives to procrastinate since they will receive at most $V - p_i$ upon completion. As a result, these perverse incentives are eliminated, and efficiency becomes attainable.

This modification raises the following concern: because the mechanism may burn money off the equilibrium path, it requires that the agents commit to not renegotiate the mechanism should that contingency arise.[11] As such, for the mechanism to implement the efficient outcome, it is necessary that the agents can commit to not renegotiate it or that the costs associated with renegotiating the mechanism are sufficiently large.

4.2 Limited (but Sufficient) Cash Reserves

In this section, we consider the case in which the agents have limited cash reserves. Because the mechanism requires that the agents make flow payments while the project is in progress, two complications arise. First, the mechanism must incorporate the contingency in which one or more agents run out of cash before the project is completed. Second, for the efficient mechanism to be implementable, the agents must have sufficient cash in hand. This section is concerned with addressing these issues.

The following proposition establishes a necessary and sufficient condition for the mechanism characterized in Proposition 4 to be implementable; that is each agent has enough cash w_i to make the flow payments $h_i(\cdot)$ on the equilibrium path.

Proposition 5. *The mechanism characterized in Proposition 4 is implementable if and only if there exist* $\{p_i\}_{i=1}^n$ *satisfying* $\sum_{i=1}^n p_i = (n-1)\bar{S}(0)$ *such that*

$$p_i\left(1-e^{-r\bar{\tau}}\right) < w_i - \int_0^{\bar{\tau}} e^{-rt} \sum_{j \neq i} c_j\left(f_j\left(\bar{S}'(q_t)\right)\right) dt \quad \text{and} \quad p_i \leq \bar{S}(0) - \bar{u}_i$$

for all i*, where* $\bar{S}(0)$ *and* $\bar{\tau}$ *are characterized in Proposition 2.*
If the agents are symmetric (i.e., $c_i(\cdot) \equiv c_j(\cdot)$*,* $w_i = w_j$*, and* $\bar{u}_i = \bar{u}_j$ *for all* i *and* j*), then the mechanism is implementable if and only if* $w_i > e^{-r\bar{\tau}}\left(\frac{n-1}{n}\right)\left[V - \bar{S}(0)\right]$ *for all* i*.*

The first condition asserts that each agent must have sufficient cash to make the flow payments $h_i(\cdot)$ on the equilibrium path. The second condition is each agent's individual rationality constraint; *i.e.*, each agent must prefer to participate in the mechanism rather than collect his outside option $\bar{u}_i$. In general, these conditions are satisfied if each agent's cash reserves are

[11] This contingency will arise if the agents exert less than first best effort, which will result in the project being completed at some time $\tau > \bar{\tau}$ and the balance in the savings account H_τ to exceed $(n-1)V$. In this case, the mechanism specifies that each agent receives reward V upon completion and the surplus $H_\tau - (n-1)V$ is burned.

sufficiently large and his outside option is sufficiently small.[12,13]

Focusing on the symmetric case, the following remark examines how the minimum amount of cash that each agent must have in hand at time 0 for the efficient mechanism to be implementable (*i.e.*, $e^{-r\bar{\tau}}\left(\frac{n-1}{n}\right)\left[V-\bar{S}(0)\right]$) depends on the number of agents in the group n.

Remark 3. *The minimum amount of initial wealth* $\underline{w}=e^{-r\bar{\tau}}\left(\frac{n-1}{n}\right)\left[V-\bar{S}(0)\right]$ *that each agent must have for the mechanism characterized in Proposition 4 to be implementable increases in the group size* n.[14]

Intuitively, the inefficiencies due to shirking and front-loading become more severe the larger the group. As a result, the mechanism must specify larger payments to neutralize those inefficiencies, which in turn implies that the agents' initial wealth must be larger.

Even if the conditions in Proposition 5 are satisfied, it is still possible that off equilibrium, one or more agents runs out of cash before the project is completed, in which case they will be unable to make the flow payments specified by the mechanism. Thus, the mechanism must specify the flow payments as a function not only of the state q, but also of the agents' cash reserves.

To analyze this case, we denote by $I_i(q)$ the indicator random variable that is equal to one if agent i still has cash reserves given the state of the project q, and is equal to zero if he has run out of cash at state q. The following proposition shows that the efficiency properties established in Proposition 4 continue to hold if an agent "loses" his share if he runs out of cash before the project is completed.

Proposition 6. *Suppose that each agent commits to make flow payments* $h_i(q)\,I_i(q)$ *for all* q *(where* $\sum_{i=1}^{n}p_i=(n-1)\,\bar{S}(0)$*), and he receives lump-sum reward* $\min\left\{V-p_i,\ \beta_i\left(V+H_\tau\right)\right\}I_i(Q)$ *upon completion of the project, where* $\beta_i=\frac{V-p_i}{V+(n-1)\left[V-\bar{S}(0)\right]}$. *Moreover, assume that the conditions of Proposition 5 are satisfied. Then the properties of Proposition 4 continue to hold.*

For the mechanism to induce efficient incentives, it must punish an agent who runs out of cash, for this should not occur on the equilibrium path. In addition, it must deter others from shirking to run an agent out of cash. Thus, if an agent runs out of cash, then he must "lose" his share, and this share must be burned; *i.e.*, it cannot be shared among the other agents.

[12]Note that in the symmetric case, we have $p_i=\left(\frac{n-1}{n}\right)\bar{S}(0)$ and $\int_0^{\bar{\tau}}e^{-rt}\sum_{j\neq i}c_j\left(f_j\left(\bar{S}'(q_t)\right)\right)=\frac{n-1}{n}\left[e^{-r\bar{\tau}}V-\bar{S}(0)\right]$ for all i, and by assumption, $\frac{\bar{S}(0)}{n}\geq\bar{u}_i$ for all i. As such, the first condition reduces to $w_i>e^{-r\bar{\tau}}\left(\frac{n-1}{n}\right)\left[V-\bar{S}(0)\right]$ and the second condition is automatically satisfied.

[13]In Appendix A, we characterize the socially optimal mechanism when the conditions in Proposition 5 are not satisfied.

[14]*This follows from Remark 1, which shows that the first best completion time* $\bar{\tau}$ *decreases in* n. *Therefore* $e^{-r\bar{\tau}}\left(\frac{n-1}{n}\right)\left[V-\bar{S}(0)\right]$ *increases in* n.

4.3 Time-Dependent Mechanism

So far, we have focused on mechanisms that condition flow payments and terminal rewards on the state of the project q, but not on time t. In this section, we consider the more general environment in which the mechanism may depend on both t and q, and we characterize optimal mechanisms. In particular, we consider the set of mechanisms that specify for each agent i a flow payment function $h_i(t,q)$, and a reward $g_i(\tau)$, where τ denotes the completion time of the project.[15]

Given a set of flow payment functions $\{h_i(t,q)\}_{i=1}^n$ and reward functions $\{g_i(t)\}_{i=1}^n$, and assuming that every agent other than i applies effort of the form $a_j(t,q)$, agent i's discounted payoff, which we denote by $J_i(t,q)$, satisfies the HJB equation

$$rJ_i(t,q) - J_{i,t}(t,q) = \max_{a_i}\left\{-c_i(a_i) + \left(\sum_{j=1}^n a_j(t,q)\right) J_{i,q}(t,q) - h_i(t,q)\right\} \tag{12}$$

subject to the boundary condition $J_i(t,Q) = g_i(t)$.[16] Each agent's first order condition is $c_i'(a_i) = J_{i,q}(t,q)$, and because we want to induce him to exert the efficient effort level, we must have $c_i'(a_i) = \bar{S}'(q)$, where $\bar{S}(\cdot)$ is the social planner's payoff function characterized in Proposition 2. Therefore,

$$J_i(t,q) = \bar{S}(q) - p_i(t) \ ,$$

where $p_i(\cdot)$ is a function of t that remains to be determined. From the boundary conditions for $J_i(t,q)$ and $\bar{S}(q)$, we have $g_i(t) = V - p_i(t)$ for all t, and using (7) and (12), we can solve for each agent's flow payment function that induces first best effort:

$$h_i(t,q) = \sum_{j\neq i} c_j\left(f_j\left(\bar{S}'(q)\right)\right) + r\left(V - g_i(t)\right) + g_i'(t) \ . \tag{13}$$

We aim to have zero flow payments on the equilibrium path, and noting that given any fixed set of strategies, there is a 1-1 correspondence between q and t, we let $\bar{q}(t)$ denote the state of the project on the efficient path at time t, starting at q_0. To pin down the functions $\{g_i(t)\}_{i=1}^n$ so that flow payments are zero along the equilibrium path, we solve the ODE

$$g_i'(t) - rg_i(t) + \sum_{j\neq i} c_j\left(f_j\left(\bar{S}'(\bar{q}(t))\right)\right) + rV = 0 \tag{14}$$

subject to the boundary condition $g_i(\bar{\tau}) = \alpha_i V$ for each i, where $\bar{\tau}$ denotes the first-best

[15] For example, $g_i(\tau) = \alpha_i V \mathbf{1}_{\{\tau\leq T\}}$ corresponds to the case with deadline T.

[16] $J_{i,t}(t,q)$ and $J_{i,q}(t,q)$ denote the derivative of $J_i(t,q)$ with respect to t and q, respectively.

completion time of the project. This is a linear non-homogeneous ODE, and a unique solution exists. Using (14), we can re-write each agent's flow payment function as

$$h_i\left(t,q\right)=\sum_{j\neq i}\left[c_j\left(f_j\left(\bar{S}'\left(q\right)\right)\right)-c_j\left(f_j\left(\bar{S}'\left(\bar{q}\left(t\right)\right)\right)\right]\ . \tag{15}$$

Noting that $c_j\left(f_j\left(x\right)\right)$ increases in x and $\bar{S}''\left(q\right)>0$ for all q, it follows that $h_i\left(t,q\right)>0$ if and only if $q>\bar{q}\left(t\right)$; *i.e.*, by time t, the project has progressed farther than is efficient.

If we want a mechanism that never has a budget deficit, then, the flow payments must be nonnegative, and if we want to ensure that the agents always have incentives to complete the project, then, the reward functions $\{g_i\left(t\right)\}_{i=1}^n$ must also be nonnegative. The following proposition characterizes a so modified mechanism that implements the first-best outcome.

Proposition 7. *Suppose that each agent makes flow payments $\hat{h}_i\left(t,q\right)=\left[h_i\left(t,q\right)\right]^+$ and receives $\hat{g}_i\left(\tau\right)=\left[\min\left\{\alpha_iV,\,g_i\left(\tau\right)\right\}\right]^+$ upon completion of the project, where $h_i\left(t,q\right)$ and $g_i\left(\tau\right)$ satisfy (15) and (14), respectively, and $\hat{g}_i\left(\tau\right)$ decreases in τ.[17] Then there exists a MPE in which at every moment, each agent exerts the first-best effort level, he makes 0 flow payments along the evolution path of the project, and he receives reward α_iV upon completion. In this equilibrium, the sum of the agents' discounted payoffs is equal to the first-best discounted payoff. Moreover, this mechanism is budget balanced on the equilibrium path and never has a budget deficit.*

Recall that the mechanism must neutralize the two inefficiencies described in Section 3.3. To eliminate the agents' incentives to shirk due to receiving only a share of the project's payoff (*i.e.*, α_iV instead of V), the mechanism specifies that each agent's reward decreases in τ at a sufficiently rapid rate, such that the agents find it optimal to complete the project no later than the first-best completion time $\bar{\tau}$.[18] To neutralize the front-loading effect, the mechanism specifies that as long as the state q at time t coincides with $\bar{q}\left(t\right)$, then flow payments are 0, but they are strictly positive whenever $q>\bar{q}\left(t\right)$, which will occur if one or more agents work harder than first-best.

In Proposition 7, we characterize a time-dependent mechanism in which along the equilibrium path, each agent exerts the efficient effort level, makes 0 flow payments, and receives α_iV upon completion. Therefore, in contrast to the time-independent mechanism characterized in Section 4.2, this mechanism is implementable even if the agents are cash constrained.

[17] Note that $[\cdot]^+=\max\{\cdot,0\}$.

[18] This is reminiscent of the optimal contract in Mason and Valimaki (2015) who study "Poisson projects".

5 Uncertainty

An important restriction of the model is that the project progresses deterministically (rather than stochastically). To evaluate the robustness of this assumption, in this section, we consider the case in which the project progresses stochastically over time. We incorporate uncertainty by assuming that its state evolves according to

$$dq_t = \left(\sum_{i=1}^{n} a_{i,t} \right) dt + \sigma dW_t \,,$$

where $\sigma > 0$ captures the degree of uncertainty associated with the evolution of the project, and W_t is a standard Brownian motion. It is straightforward to show that the planner's problem satisfies the ODE

$$r\bar{S}(q) = -\sum_{i=1}^{n} c_i \left(f_i \left(\bar{S}'(q) \right) \right) + \left[\sum_{i=1}^{n} f_i \left(\bar{S}'(q) \right) \right] \bar{S}'(q) + \frac{\sigma^2}{2} \bar{S}''(q) \tag{16}$$

subject to the boundary conditions

$$\lim_{q \to -\infty} \bar{S}(q) = 0 \quad \text{and} \quad \bar{S}(Q) = V \,. \tag{17}$$

Georgiadis (2015) provides conditions under which (16) subject to (17) admits a unique, smooth solution. Note that each agent's first-best level of effort satisfies $\bar{a}_i(q) = f_i \left(\bar{S}'(q) \right)$, and similarly to the deterministic case, $\bar{a}_i'(q) > 0$ for all i and q.

Next consider each agent's problem facing an arbitrary flow payments function $h_i(t,q)$ and a reward function $g_i(t)$. Using standard arguments, we expect agent i's discounted payoff function to satisfy the HJB equation

$$rJ_i(t,q) - J_{t,i}(t,q) = \max_{a_i} \left\{ -c_i(a_i) + \left(\sum_{j=1}^{n} a_j \right) J_{q,i}(t,q) + \frac{\sigma^2}{2} J_{qq,i}(t,q) - h_i(t,q) \right\} \tag{18}$$

subject to $\lim_{q \to -\infty} \hat{J}_i(q) = 0$ and $J_i(t,Q) = g_i(t)$. His first order condition is $c_i'(a_i) = J_{q,i}(t,q)$, and because we want to induce each agent to exert the efficient effort level, we must have $c_i'(a_i) = \bar{S}'(q)$. Therefore, it needs to be the case that each agent's discounted payoff

$$J_i(t,q) = \bar{S}(q) - p_i(t) \ ,$$

where $p_i(\cdot)$ is a function of time that remains to be determined. From the boundary conditions for $J_i(\cdot,\cdot)$ and $\bar{S}(\cdot)$, it follows that $p_i(t) = V - g_i(t)$. Using this, (18) and (16), we can solve for each agent's flow payment function that induces first best effort:

$$h_i(t,q) = \sum_{j\neq i} c_j\left(f_j\left(\bar{S}'(q)\right)\right) + r\left(V - g_i(t)\right) + g_i'(t). \tag{19}$$

Similar to the deterministic case, we choose the set of functions $\{g_i(t)\}_{i=1}^n$ such that along the equilibrium (or equivalently, first best) path $\bar{q}_t$, each agent i's expected flow payment $\mathbb{E}\left[h(t,\bar{q}_t)\right]$ is equal to zero; *i.e.*,

$$0 = \sum_{j\neq i} \mathbb{E}\left[c_j\left(f_j\left(\bar{S}'(\bar{q}_t)\right)\right)\right] + r\left(V - g_i(t)\right) + g_i'(t) \tag{20}$$

This is an ODE that can be solved for any given initial condition $g_i(0)$. To ensure that the sum of the agents' discounted payoffs is equal to the first best payoff, the set of initial conditions $\{g_i(0)\}_{i=1}^n$ must satisfy $\sum_{i=1}^n J_i(0,0) = \bar{S}(0)$, or equivalently

$$\sum_{i=1}^n g_i(0) = V + (n-1)\left[V - \bar{S}(0)\right]. \tag{21}$$

This condition also ensures that the mechanism is ex-ante budget balanced (in expectation).[19] Therefore, we have established the following proposition:

Proposition 8. *Suppose that each agent makes flow payments $h_i(t,q)$ and receives $g_i(\tau)$ upon completion of the project, which satisfy (19) and (20), respectively. Then there exists a MPE in which at every moment, each agent exerts the first-best effort level. In this equilibrium, the sum of the agents' expected discounted payoffs is equal to the first-best expected discounted payoff, and each agent's local flow payment has expected value equal to* 0*. Moreover, this mechanism is ex-ante budget balanced (*i.e., *in expectation).*

The intuition behind this mechanism is similar to the deterministic case: the flow payment functions and the terminal reward functions are chosen so as to eliminate the inefficiencies described in Section 3.3. However, there are two important differences relative to the deterministic case. First, in the stochastic mechanism, it is impossible to simultaneously achieve efficiency and ex-post budget balance. Instead, the mechanism is budget balanced in expectation, and as a result, for the mechanism to be implementable, there needs to be a third party who collects the surplus if the balance in the savings account upon completion of the project exceeds $\sum_{i=1}^n g_i(\tau) - V$,

[19] This follows by noting that the ex-ante budget balance condition is $\sum_{i=1}^n \mathbb{E}\left[\int_0^\tau e^{-rt} h_{i,t}\,dt - e^{-r\tau} g_i(\tau) + e^{-r\tau}\frac{V}{n}\right] = 0$, and by using (1) and (19), one can show that it is equivalent to (21).

and she pays the difference otherwise. Second, flow payments may be positive or negative, and because the path of the Brownian motion until the project is completed can be arbitrarily long, for the mechanism to be implementable, it is necessary that the agents as well as the third party are cash unconstrained.

Finally, the following remark characterizes the set of time-independent mechanisms that implement the efficient outcome in the game with uncertainty.

Remark 4. Let $g_i(t) = G_i$, where $G_i \geq 0$ and $\sum_{i=1}^n G_i = \sum_{i=1}^n V + (n-1)\left[V - \bar{S}(0)\right]$. Suppose that each agent makes (time-independent) flow payments that satisfy (15), and he receives G_i upon completion of the project. Then there exists a MPE in which at every moment, each agent exerts the first-best effort level. Moreover, this mechanism is ex-ante budget balanced (*i.e.*, in expectation).

6 Discussion

We study a simple model of dynamic contributions to a discrete public good. In our model, at every moment, each of n agents chooses his contribution to a group project, and the project generates a lump sum payoff once the cumulative contributions reach a pre-specified threshold. A standard result in such models is that effort is under-provided relative to the efficient outcome, due to the free-rider problem. We propose a mechanism that induces each agent to always exert the efficient level of effort as the outcome of a Markov Perfect equilibrium (MPE). The mechanism specifies for each agent flow payments that are due while the project is in progress, and a lump sum reward that is disbursed upon completion of the project.

Our mechanism has two key limitations: First, the agents must have sufficient cash in hand at the outset of the game in order to implement the mechanism. In particular, if the project progresses stochastically, then each agent must have unlimited liability. Second, for the mechanism to achieve the efficient outcome, the agents must be able to commit to burn money off the equilibrium path.

This paper opens several opportunities for future research, both theoretical and applied. On the theory side, an open question is to characterize the mechanism that maximizes the agents' total ex-ante discounted payoff when the project progresses stochastically and the agents' cash reserves are limited. In addition, our framework can be used to restore efficiency in other multi-agent environments that are susceptible to free-riding such as the dynamic extraction of a common resource or experimentation in teams. The key ingredients for our mechanism to be applicable are that the game is dynamic and it exhibits positive or negative externalities. From an applied perspective, it would be interesting to examine how such a mechanism performs in practice. This could be accomplished by conducting laboratory experiments as a first step, and then empirically by analyzing data on project teams.

A Cash Constraints

While the time-dependent mechanism characterized in Section 4.3 implements the efficient outcome even if the agents are cash constrained, a disadvantage of this mechanism is that this result is very sensitive to the assumption that the project progresses deterministically. In particular, with Brownian uncertainty (as analyzed in Section 5), efficiency is achievable as the outcome of a MPE only of the agents have unlimited liability. In addition, the strategies in the time-dependent mechanism are more complex, in that they must depend both on time t and the state q. In this section, we consider the case in which the agents' cash reserves are insufficient to implement the efficient mechanism characterized in Proposition 4, and we characterize the time-independent mechanism that maximizes the agents' total discounted payoff as a function of their cash reserves.

From a technical perspective, this problem is more challenging than the one analyzed in Section 4, because one needs to determine the set of upfront payments and flow payment functions that induce each agent to choose the strategy that maximizes the total surplus subject to a set of budget and budget balance constraints, which involves solving a partial differential equation and a fixed point problem.[20] To obtain tractable results, we assume that the agents are symmetric (*i.e.*, $c_i(a) = c_j(a)$ and $\bar{u}_i = \bar{u}_j = 0$ for all a and $i \neq j$), each has $w \in \left(0, e^{-r\bar{\tau}} \left(\frac{n-1}{n}\right) V\right)$ in cash at time 0, and we restrict attention to mechanisms that treat the agents symmetrically (*i.e.*, $P_{i,0} = P_{j,0}$ and $h_i(q) = h_j(q)$ for all q and $i \neq j$).[21] We also restrict the set of possible flow payment functions $h_i(q)$ to those for which the agent's value function uniquely solves the corresponding HJB.

Before we proceed with characterizing the optimal mechanism, it is necessary to introduce some notation. First, we let $t_\emptyset$ denote the time at which the agents run out of cash (and $q_\emptyset$ the corresponding state). Observe that on $[q_\emptyset, Q]$, the agents have no cash left in hand, and so they have no choice but to play the MPE characterized in Section 3.1, albeit with a different final reward, which we denote by K. Our approach comprises of four steps:

1. We start with an initial guess for the triplet $\{t_\emptyset, q_\emptyset, K\}$, which will be pinned down later by solving a fixed point problem, and we characterize the MPE of the game when the agents have no cash left in hand and each agent receives reward K upon completion of the project.

[20] In contrast, in Section 4, we chose the flow payment functions such that the agents always exert effort efficiently. Because this mechanism yields the efficient payoffs to the agents, it immediately follows that it maximizes their total discounted payoff.

[21] For notational simplicity, we drop the subscript i that denotes agent i.

2. We characterize the game while the agents still have cash in hand and they make flow payments according to some arbitrary flow payment function, which is defined on $[0, q_\emptyset]$.
3. We consider the planner's problem who pursues to maximize the agents' total discounted payoff, and we determine the optimal upfront payment P_0 and flow payment function $h(\cdot)$.
4. We solve a fixed point problem to pin down $t_\emptyset$, $q_\emptyset$, and K to complete the characterization of the optimal mechanism.

Step 1: First, we fix the reward that each agent receives upon completion of the project K, and the time and state at which the agents run out of cash ($i.e.$, $t_\emptyset$ and $q_\emptyset$, respectively). For a given set of strategies, each agent's discounted payoff, after he has run out of cash, is given by

$$J_\emptyset(q) = e^{-r(\tau-t)} K - \int_t^\tau e^{-r(s-t)} c(a_s)\, ds\,. \tag{22}$$

Writing the HJB equation for each agent's payoff function and using his first order condition yields the ODE

$$rJ_\emptyset(q) = -c\left(f\left(J'_\emptyset(q)\right)\right) + nf\left(J'_\emptyset(q)\right) J'_\emptyset(q) \tag{23}$$

subject to the boundary condition

$$J_\emptyset(Q) = K\,, \tag{24}$$

Moreover, note that each agent's effort level is $a_\emptyset(q) = f\left(J'_\emptyset(q)\right)$. We know from Proposition 1 that (23) subject to (24) has a unique, well-defined solution.

Step 2: We now characterize the agents' problem when they still have cash in hand ($i.e.$, for $q \leq q_\emptyset$) given an arbitrary smooth flow payment function $h(\cdot)$. In this case, using the same approach as before, it follows that each agent's payoff function satisfies

$$rJ_h(q) = -c\left(f\left(J'_h(q)\right)\right) + nf\left(J'_h(q)\right) J'_h(q) - h(q)\,, \tag{25}$$

subject to the boundary condition

$$J_h(q_\emptyset) = J_\emptyset(q_\emptyset)\,. \tag{26}$$

Condition (26) is a value matching condition: it ensures that each agent's discounted payoff is continuous at $q_\emptyset$, and we denote each agent's effort function $a_h(q) = f\left(J'_h(q)\right)$. By assumption on $h(q)$, we know that the agent's value function is the solution to (25) subject to (26).

Step 3: Next, we turn to the planner's problem who chooses the agents' upfront and flow payments to maximize their total discounted payoff. First, we define the process

$$dP_t = e^{-rt} h\left(q_t\right) dt \text{ , or equivalently, } P_t = P_0 + \int_0^t e^{-rs} h\left(q_s\right) ds$$

to denote the per-agent balance in the savings account at time t discounted to time 0, where P_0 denotes each agent's upfront payment. Note that $t_\emptyset$ is the first time such that P_t hits w, and $q_\emptyset$ is the corresponding state at $t_\emptyset$. Letting $L\left(t, J_t, q_t, P_t\right)$ denote the sum of the agents' discounted payoffs at time t, given each agent's continuation payoff J_t, the state of the project q_t, and the discounted balance in the savings account P_t, the planner solves

$$\begin{aligned} L\left(t, J_t, q_t, P_t\right) = \max_h & \left\{ e^{-r(\tau - t)} V - \int_t^\tau e^{-r(s-t)} n\, c\left(a_s\right) ds \right\} \\ \text{s.t.} \quad & P_t \leq w \end{aligned} \tag{27}$$

where a_s denotes each agent's equilibrium effort level, and τ is the completion time of the project, which is determined in equilibrium. The constraint asserts that each agent's flow payments cannot exceed the amount of cash that he has in hand. Note that in the definition of $L\left(t, J_t, q_t, P_t\right)$, we took into account the budget balance constraint $nK = nP_\tau e^{r\tau} + V$: the total rewards disbursed to the agents is equal to the total balance in the savings account when the project is completed (*i.e.*, at time τ) plus the payoff that the project generates.

Using (22) and (27), observe that after the agents have run out of cash (*i.e.*, for $P = w$, or equivalently, for $q \geq q_\emptyset$ and $t \geq t_\emptyset$), their total discounted payoff function satisfies

$$L\left(t, J_t, q_t, w\right) = nJ_\emptyset\left(q_t\right) - e^{-r(\tau - t)}\left(nK - V\right) , \tag{28}$$

where $J_\emptyset\left(q_t\right)$ is characterized in Step 1. Focusing on the case in which the agents still have cash in hand and re-writing (27) in differential form, it follows that for all $P < w$, L satisfies the HJB equation[22]

$$rL - \frac{\partial L}{\partial t} = \max_h \left\{ -nc\left(f\left(J_h'\left(q\right)\right)\right) + nf\left(J_h'\left(q\right)\right) \frac{\partial L}{\partial q} + ne^{-rt} h\left(q\right) \frac{\partial L}{\partial P} + nf\left(J_h'\left(q\right)\right) J_h'\left(q\right) \frac{\partial L}{\partial J} \right\} . \tag{29}$$

On the set $P \leq w$, we conjecture (and verify in the proof that follows) that $L\left(t, J, q, P\right) = L\left(q\right)$,

[22]For notational brevity, we drop the arguments of the function $L\left(t, J_t, q_t, P_t\right)$.

and so (29) can be re-written as

$$rL(q) = \max_{h} \{ nf(J_h'(q)) L'(q) - nc(f(J_h'(q))) \} \ , \tag{30}$$

subject to the boundary condition

$$L(q_\emptyset) = nJ_\emptyset(q_\emptyset) - e^{-r(\tau - t_\emptyset)}(nK - V) \ . \tag{31}$$

This condition follows from (28), and it ensures that $L(q)$ is continuous at $q = q_\emptyset$. Observe that we can express $h(\cdot)$ in terms of $J_h(\cdot)$ and $J_h'(q)$, and then maximize (30) with respect to $J_h'(q)$. By definition, $c'(f(J_h'(q))) = J_h'(q)$, and so the first order condition in (30) with respect to $J_h'(q)$ is $L'(q) = J_h'(q)$. Substituting this into (30) yields the ODE

$$rL(q) = nL'(q) f(L'(q)) - nc(f(L'(q))) \tag{32}$$

subject to the boundary condition (31). It is straightforward to verify using Proposition 1 that there exists a unique solution to (32) subject to (31). By using the first order condition and (26), we can pin down each agent's discounted payoff function for $q \leq q_\emptyset$:

$$J_h(q) = L(q) - L(q_\emptyset) + J_\emptyset(q_\emptyset) \ . \tag{33}$$

From (25) and (32), it follows that each agent's flow payment function satisfies

$$h(q) = (n-1) c(f(L'(q))) - r[J_\emptyset(q_\emptyset) - L(q_\emptyset)] \ . \tag{34}$$

Because each agent's ex-ante discounted payoff (*i.e.*, $J_h(0) - P_0$) must be equal to $\frac{L(0)}{n}$, each agent's upfront payment must be equal to

$$P_0 = \frac{n-1}{n} L(0) + [J_\emptyset(q_\emptyset) - L(q_\emptyset)] \ . \tag{35}$$

Step 4: So far, we have characterized the upfront and flow payments that maximize the agents' total discounted payoff, and the corresponding payoffs and strategies given an arbitrary triplet $\{t_\emptyset, q_\emptyset, K\}$; *i.e.*, the time and state at which the agents run out of cash, and the reward that each agent receives upon completion of the project. These parameters are pinned down in equilibrium from the budget balance constraint and the agents' budget constraint (27). In the optimal mechanism, these constraints will bind at completion[23], and so the triplet $\{t_\emptyset, q_\emptyset, K\}$

[23]If the constraint $P_\tau \leq w$ does not bind, then the agents still have cash in hand upon completion of the project, and yet, by Proposition 4, they cannot attain efficiency. By raising their upfront payments and by

solves the fixed point problem

$$P_0 + \int_0^{t_\emptyset} e^{-rt} h(q_t)\, dt = w \quad \text{and} \quad n\, w\, e^{r\tau} + V = nK\,, \tag{36}$$

where P_0 and $h(\cdot)$ are given in (35) and (34), respectively, $J_\emptyset(q)$ satisfies (23) subject to (24), and $L(q)$ satisfies (32) subject to (31).

This is a complex fixed point problem, and it does not seem feasible to show that a solution exists or that it is unique. As such, we make the following assumption:

Assumption 1. *There exists a triplet* $\left\{t_\emptyset^*,\, q_\emptyset^*,\, K^*\right\}$ *such that on the equilibrium path, (36) is satisfied.*

We summarize our analysis in the following proposition, characterizing the optimal mechanism when the agents' cash reserves are insufficient to implement the efficient outcome.

Proposition 9. *Suppose that the agents are symmetric (i.e.,* $c_i(a) = c_j(a)$ *and* $\bar{u}_i = \bar{u}_j$ *for all* q *and* $i \neq j$*), each has* $w \in [0, \underline{w}]$ *in cash at the outset of the game and Assumption 1 is satisfied. Then the optimal symmetric budget-balanced mechanism specifies that each agent makes an upfront payment as given by (35), flow payments as given by (34) on* $\left[0, q_\emptyset^*\right)$*, he makes no flow payments on* $\left[q_\emptyset^*, Q\right]$*, and upon completion of the project, he receives reward* K^**.*

Intuitively, the optimal mechanism specifies that while the agents still have cash in hand, they make flow payments to induce them to exert the efficient effort level that corresponds to the boundary condition (31). After their cash reserves are exhausted, they make no further payments and they exert the effort levels that correspond to the MPE in the case with zero flow payments. The following remark establishes how the agents' total ex-ante discounted payoff depends on their initial cash reserves.

Remark 5. The agents' total ex-ante discounted payoff under the optimal mechanism characterized in Proposition 9 (weakly) increases in his initial wealth w.

This result is intuitive: If the agents' initial wealth increases, then the mechanism corresponding to the lower wealth is still feasible. Therefore, the optimal mechanism corresponding to the higher wealth must give each agent an ex-ante discounted payoff that is at least as large as the payoff corresponding to the lower wealth.

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consequence the reward K that they receive upon completion, they can strengthen their incentives and increase their joint payoff.

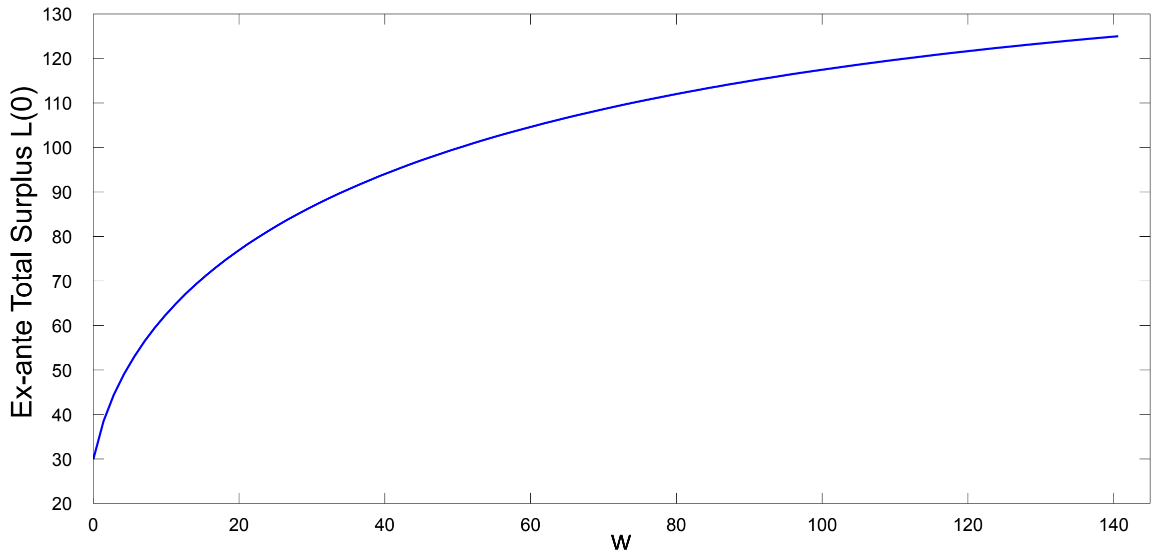

Figure 1: An example with $c_i(a) = \frac{a^2}{2}$ for all i, $Q = 100$, $V = 500$, $r = 0.1$, and $n = 4$.

Figure 1 uses the example analyzed in Appendix B with quadratic effort costs to illustrate how the optimal mechanism depends on the agents' initial cash reserves w. First, as shown in Remark 5, the total surplus generated by the optimal mechanism ($i.e.$, $L(0)$) increases in the agents' wealth w. Moreover, observe that the relationship is concave in w, which implies that even small cash reserves can have a significant effect in mitigating the free-rider problem and increasing total surplus. Moreover, in the optimal mechanism, each agent makes no upfront payment, and both the state $q_\emptyset^*$ at which the agents run out of cash, and each agent's reward upon completion of the project K^* increase steadily in the wealth w.

B Extensions

In this section, we extend our work in two directions to illustrate the versatility of our mechanism. To simplify the exposition, we shall assume throughout this section that each agent has *large* cash reserves at the outset of the game and outside option 0.

B.1 Flow Payoffs

First, we consider the case in which the project generates flow payoffs while in it is progress, in addition to a lump-sum payoff upon completion. To model such flow payoffs, we assume that during every $(t, t+dt)$ interval while the project is in progress, it generates a payoff $K(q_t)\,dt$, plus a lump-sum $V \geq \frac{K(Q)}{r}$ upon completion, where $K(\cdot)$ is a well-behaved function so as to ensure that the planner's problem has a well-defined solution. As in the base model, each agent i is entitled a share α_i of those payoffs, where $\sum_{i=1}^{n} \alpha_i = 1$.

Using the same approach and notation as in Sections 3.2 and 4, we shall construct a mechanism that induces each agent to exert the efficient effort level as an outcome of the MPE of the game. To begin, we characterize the efficient outcome of this game. The planner's problem satisfies the ODE

$$r\bar{S}(q) = K(q_t) - \sum_{i=1}^{n} c_i\left(f_i\left(\bar{S}'(q)\right)\right) + \left[\sum_{i=1}^{n} f_i\left(\bar{S}'(q)\right)\right]\bar{S}'(q) \tag{37}$$

defined on some interval $\left(\underline{q}^s, Q\right]$, subject to the boundary condition

$$\bar{S}(Q) = V\,, \tag{38}$$

and each agent i's first best effort level is given by $\bar{a}_i(q) = f_i\left(\bar{S}'(q)\right)$. It follows from Proposition 2 that the ODE defined by (37) subject to (38) has a unique solution on $\left(\underline{q}^s, Q\right]$.

Next, consider the problem faced by each agent given a mechanism that specifies arbitrary flow payments $\{h_i(q)\}_{i=1}^{n}$. Using standard arguments, we expect agent i's discounted payoff function to satisfy the HJB equation

$$r\hat{J}_i(q) = \max_{a_i}\left\{\alpha_i K(q) - c_i(a_i) + \left(\sum_{j=1}^{n} a_j\right)\hat{J}_i'(q) - h_i(q)\right\}$$

subject to a boundary condition that remains to be determined. As in the base model, his first order condition is $c_i'(a_i) = \hat{J}_i'(q)$. To induce each agent to exert the efficient effort level, we require that $\hat{J}_i'(q) = \bar{S}'(q)$ for all q, which also implies that $\hat{J}_i(q) = \bar{S}(q)$ for all q. It then follows that each agent's flow payments function must satisfy

$$h_i(q) = \sum_{j\neq i} c_j\left(f_j\left(\bar{S}'(q)\right)\right) - (1-\alpha_i)K(q)\ .$$

To ensure that the budget is balanced (on the equilibrium path), the agents' upfront payments must be chosen such that $\sum_{i=1}^{n} P_{i,0} = (n-1)\bar{S}(0)$. Finally, it follows that Proposition 6 that as long as each agent receives $\min\left\{V, \frac{V+H_\tau}{n}\right\}$ upon completion, where H_τ denotes the balance in the savings account upon completion of the project, the mechanism characterized above implements the efficient outcome, it is budget balanced on the equilibrium path, and it will never result in a budget deficit.

B.2 Endogenous Project Size

In this base model, the project size was given exogenously. Motivated by the fact that in many natural applications, the size of the project Q to be undertaken is part of the decision making process, in this section, we endogenize the project size. In particular, we assume that if the agents choose to undertake a project of size Q, then it generates a lump-sum payoff equal to $g(Q)$ upon completion, where $g(\cdot)$ is an arbitrary function.

To begin, let Q_q^* denote the project size that maximizes the social planner's ex-ante payoff given the current state q; *i.e.*, $Q_q^* \in \arg\max_Q \left\{\bar{S}(q;Q)\right\}$, where $\bar{S}(q;Q)$ denotes the planner's discounted payoff given project size Q as characterized in Proposition 2. Recall from Section 4.1 that each agent's ex-ante discounted payoff is equal to $\bar{S}(q_0;Q) - P_{i,0}$, and observe that this is maximized at $Q_{q_0}^*$ for all i. Therefore, all agents will be in agreement with respect to the project size that maximizes their ex-ante payoff, and this project size coincides with the socially optimal one. Moreover, Georgiadis et. al. (2014) shows hat Q_q^* is independent of q, which implies that the agents will not have any incentives to later renegotiate the project size chosen at $t = 0$.

C An Example

In this section, we use an example to illustrate the mechanism proposed in Sections 3 and 4. In particular, we assume that the agents are symmetric, have quadratic effort costs, outside option 0, and cash reserves $w \geq 0$ at the outset of the game (*i.e.*, $c_i(a) = \frac{a^2}{2}$, $\bar{u}_i = 0$ and $w_i = w$ for all i). These simplifications enable us to compute the optimal mechanism analytically and obtain closed form formulae, which are amenable for conducting comparative statics. We first characterize the the MPE of this game when the agents are cash constrained (*i.e.*, $w = 0$), and the first best outcome in Sections C.1 and C.2, respectively. Then in Section C.3, we characterize the mechanism that implements the efficient outcome, per the analysis in Section 4. Finally, in Section C.4, we characterize the optimal mechanism when the agents' cash reserves are not large enough to implement the efficient outcome.

C.1 Markov Perfect Equilibrium

Using the same approach as in Section 3.1, it follows that in any MPE, each agent i's discounted payoff function satisfies

$$rJ_i(q) = -\frac{1}{2}\left[J_i'(q)\right]^2 + \left[\sum_{j=1}^{n} J_j'(q)\right] J_i'(q)$$

subject to the boundary condition $J_i(Q) = \frac{V}{n}$. Georgiadis et. al. (2014) shows that this game has a unique project-completing MPE that is symmetric, and each agent's discounted payoff and effort level satisfies

$$J(q) = \frac{r}{2}\frac{\left([q-C]^+\right)^2}{2n-1} \quad \text{and} \quad a(q) = \frac{r}{2n-1}[q-C]^+ ,$$

respectively, where $C = Q - \sqrt{\frac{2V}{r}\frac{2n-1}{n}}$. Provided that $C < 0$, in this equilibrium, the project is completed at $\tau = \frac{2n-1}{rn}\ln\left(1 - \frac{Q}{C}\right)$.[24]

Note that this game may have another MPE in which no agent ever exerts any effort, the project is never completed, and each agent's discounted payoff is 0. A non-completing equilibrium exists if the project is sufficiently large such that no agent is willing to undertake the project single-handedly; *i.e.*, if $C|_{n=1} < 0$. In this case, if all other agents exert no effort, then each agent's best response is to also exert no effort, and consequently, the project is never completed.

C.2 First Best Outcome

We now turn to analyze the social planner's problem who at every moment, chooses the agents' efforts to maximize the group's total discounted payoff. From Section 3.2 and using the results of Georgiadis et. al. (2014), it follows that the planner's discounted payoff is given by

$$\bar{S}(q) = \frac{r}{2n}\left(\left[q - \bar{C}\right]^+\right)^2 , \tag{39}$$

where $\bar{C} = Q - \sqrt{\frac{2nV}{r}}$, and the first best effort level of each agent is $\bar{a}(q) = \frac{r}{n}\left[q - \bar{C}\right]^+$. If $\bar{C} < 0$, then the project is completed at $\bar{\tau} = \frac{1}{r}\ln\left(1 - \frac{Q}{\bar{C}}\right)$, whereas otherwise, no agent ever exerts any effort and the project is not completed. Intuitively, if the project's payoff or the group size (*i.e.*, V or n) is too small, or the agents are too impatient (*i.e.*, r is too large), or the project is too long (*i.e.*, Q is too large), then efficiency dictates that the project should not be

[24] Because the equilibrium is symmetric, we henceforth drop the subscript i.

undertaken.

First, it is straightforward to verify that $\bar{C} < C$ for all $n \geq 2$. Therefore, some projects which are efficient to undertake, are not completed in the MPE. Moreover, both the first best payoffs and the corresponding efforts are always greater than their MPE counterparts; *i.e.*, $\bar{S}(q) \geq nJ(q)$ and $\bar{a}(q) \geq a(q)$ for all q.

C.3 An Efficient Mechanism

In this section, we compute the mechanism that implements efficiency as the unique outcome of a MPE. Using (39), it follows from Proposition 4 that assuming $\bar{C} < 0$, then the efficiency-inducing, budget balanced mechanism specifies that each agent's flow payment function satisfies

$$h(q) = \frac{r^2(n-1)}{2n^2}\left[\left(q-\bar{C}\right)^2 + \bar{C}^2\right],$$

and upon completion of the project, he receives the smaller of V and one-n^{th} of the total balance in the savings account. Moreover, this mechanism is implementable if and only if each agent's cash reserves

$$w \geq \underline{w} = \left(\frac{n-1}{n}\right)\left(\frac{\bar{C}}{\bar{C}-Q}\right)\left[V - \frac{r\bar{C}^2}{2n}\right].$$

Time-Dependent Mechanism

At this point, it is useful to also characterize the efficient time-dependent mechanism as analyzed in Section A.3. First, note that along the efficient path, $\bar{q}(t) = -\bar{C}\left(e^{rt}-1\right)$. From Proposition 7, and using (14) and (15), we know that there exists a MPE in which agents always exert the efficient effort level when each agent's flow payment and reward function satisfies

$$h(t,q) = \left[\frac{r^2(n-1)}{2n^2}\left(\left(q-\bar{C}\right)^2 - \bar{C}^2 e^{2rt}\right)\right]^+ \quad \text{and} \quad g(\tau) = \left[V - \frac{r(n-1)\bar{C}^2}{2n^2}e^{r\tau}\right]^+,$$

respectively. Observe that each agent's reward decreases in the completion time at an exponential rate, so as to provide incentives to complete the project no later than the first-best completion time $\bar{\tau}$. As a result, any deviation will be towards higher effort (i.e., they will front-load effort) to induce others to raise their future efforts. On the equilibrium path, $q = \bar{q}(t)$ and so $h(t,q) = 0$. However, if $q > \bar{q}(t)$ for some t, then the mechanism punishes the agents by specifying a strictly positive flow payment.

C.4 Cash Constraints

We now consider the case in which each agent's cash reserves satisfy $w < \underline{w}$. To analyze this case, we use the four-step approach proposed in Appendix A. To begin, let us fix the state $q_\emptyset$ at which the agents run out of cash, and the reward that each agent receives upon completion of the project K, which will be pinned down in the final step by solving a fixed point problem.

Step 1: First, we characterize the game after the agents have run out of cash. The agents will play the MPE that corresponds to the cash constrained case with boundary condition $J_\emptyset(Q) = K$. It follows from Sections 5.1 and B.1 that each agent's discounted payoff and effort level satisfies

$$J_\emptyset(q) = \frac{r}{2}\frac{\left([q - C_\emptyset]^+\right)^2}{2n-1} \quad \text{and} \quad a_\emptyset(q) = \frac{r}{2n-1}\left[q_t - C_\emptyset\right]^+ ,$$

respectively, where $C_\emptyset = Q - \sqrt{\frac{2(2n-1)K}{r}}$. To compute the amount of time it takes for the project to progress from $q_\emptyset$ to Q, we solve the ODE $\dot{q}_t = n\, a_\emptyset(q_t)$ subject to the boundary condition $q_{t_\emptyset} = q_\emptyset$, which assuming that $C_\emptyset < q_\emptyset$, yields

$$\tau - t_\emptyset = \frac{2n-1}{r\,n}\ln\left(\frac{Q - C_\emptyset}{q_\emptyset - C_\emptyset}\right) .$$

Steps 2 and 3: Turning to the planner's problem and considering the game while the agents still have cash in hand, it follows from (31) and (32) that the planner's discounted payoff function satisfies $rL(q) = \frac{n}{2}\left[L'(q)\right]^2$ subject to the boundary condition $L(q_\emptyset) = nJ_\emptyset(q_\emptyset) - e^{-r(\tau - t_\emptyset)}(nK - V)$. This ODE has unique solution

$$L(q) = \frac{r}{2n}\left([q - C_h]^+\right)^2 ,$$

where $C_h = q_\emptyset - \sqrt{\frac{n^2}{2n-1}(q_\emptyset - C_\emptyset)^2 - \frac{2n(nK-V)}{r}\left(\frac{q_\emptyset - C_\emptyset}{Q - C_\emptyset}\right)^{\frac{2n-1}{n}}}$. It follows that on $[0, q_\emptyset]$, each agent's discounted payoff and effort level satisfies

$$J_h(q) = L(q) - L(q_\emptyset) + J_\emptyset(q_\emptyset) \quad \text{and} \quad a_h(q) = \frac{r}{n}\left[q - C_h\right]^+ ,$$

respectively. Next, we must compute the amount of time it takes for the project to progress from $q_0 = 0$ to $q_\emptyset$. By solving the ODE $\dot{q}_t = r\left[q - C_h\right]^+$ subject to the boundary condition $q_0 = 0$, we obtain $q_t = -C_h\left(e^{rt} - 1\right)$. Then solving for $q_{t_\emptyset} = q_\emptyset$, we find that, provided $C_h < 0$, the agents run out of cash at $t_\emptyset = \frac{1}{r}\ln\left(\frac{C_h - q_\emptyset}{C_h}\right)$.

Using (34), we can compute each agent's flow payment function

$$h(q) = \frac{r^2(n-1)}{2n^2}\left([q - C_h]^+\right)^2 - r\left[J_\emptyset(q_\emptyset) - L(q_\emptyset)\right], \tag{40}$$

and his upfront payment

$$P_0 = \frac{r(n-1)}{2n^2}C_h^2 + \left[J_\emptyset(q_\emptyset) - L(q_\emptyset)\right], \tag{41}$$

where $J_\emptyset(q_\emptyset) - L(q_\emptyset) = (nK - V)\left(\frac{q_\emptyset - C_\emptyset}{Q - C_\emptyset}\right)^{\frac{2n-1}{n}} - \frac{r(n-1)}{2(2n-1)}(q_\emptyset - C_\emptyset)^2$. So far, we have characterized the upfront payment and flow payment function that maximizes the agents' total discounted payoff given an arbitrary pair $\{q_\emptyset, K\}$. To complete the characterization, we must ensure that the agents indeed run out of cash at $q_\emptyset$, and upon completion of the project, the total balance in the savings account is $nK - V$. We do this by solving a fixed point problem in the next step.

Step 4: To write the fixed point problem, it is useful to express $h(q)$ as a function of time. Using (40) that $q_t = -C_h(e^{rt} - 1)$ for $t \leq t_\emptyset$, we have $h(t) = \frac{r^2(n-1)}{2n^2}C_h^2 e^{2rt} - r\left[J_\emptyset(q_\emptyset) - L(q_\emptyset)\right]$. The total payments that each agent makes along the equilibrium path, discounted to time 0 is equal to $P_0 + \int_0^{t_\emptyset} e^{-rt}h(q_t)\,dt = -\frac{r(n-1)}{2n^2}q_\emptyset C_h + \frac{q_\emptyset}{C_h - q_\emptyset}\left[J_\emptyset(q_\emptyset) - L(q_\emptyset)\right]$, and so each agent's budget constraint must satisfy

$$\frac{r(n-1)}{2n^2}C_h(C_h - q_\emptyset) + \frac{C_h}{C_h - q_\emptyset}\left[J_\emptyset(q_\emptyset) - L(q_\emptyset)\right] = w\,. \tag{42}$$

Budget balance requires that $nwe^{r\tau} + V = nK$, or equivalently

$$nw\left(\frac{Q - C_\emptyset}{q_\emptyset - C_\emptyset}\right)^{\frac{2n-1}{n}}\frac{C_h - q_\emptyset}{C_h} = nK - V\,. \tag{43}$$

Therefore, in the optimal mechanism is pinned down by a pair $\{q_\emptyset, K\}$ that simultaneously satisfies (42) and (43). Unfortunately, it is not feasible to show that a solution to this fixed point problem exists, or that it is unique[25]. As such, we assume that this is the case to establish the following result.

Result 1. *Suppose that the agents are identical, and each has initial cash reserves $w \in [0, \underline{w}]$. Moreover, assume that there exists a pair $\left\{q_\emptyset^*, K^*\right\}$ that satisfies (42) and (43). Then the optimal, symmetric, budget-balanced mechanism specifies that each agent makes an upfront pay-*

[25] In the numerical example mentioned at the end of Appendix A, our procedure does find a fixed point.

ment as given in (41), flow payments as given in (40) for all $q \leq q_{\emptyset}^{*}$, and upon completion of the project, he receives reward K^{*}.

D Proofs

Proof of Proposition 1.

To study the existence of a solution, we proceed as follows. First, we note that the value function of each agent is bounded below by 0, since he can exert no effort and guarantee himself a nonnegative payoff. Next, we write the ODE system (4) in the form

$$J_i(q) = G_i\left(J_1'(q), \ldots, J_n'(q)\right)$$

where $G_i\left(J_1'(q), \ldots, J_n'(q)\right) = \frac{1}{r}\left\{-c_i\left(f_i\left(J_i'\right)\right) + \left[\sum_{j=1}^{n} f_j\left(J_j'\right)\right] J_i'\right\}$. Because the value function of each agent is nonnegative, we are only interested in solutions for which $J_i(q) \geq 0$, and from (4), it follows then that $J_i'(q) \geq 0$, too. Thus, we consider G_i only on the domain $[0,\infty)^n$. Note that

$$\frac{\partial G_i}{\partial J_i'} = \frac{1}{r}\sum_{j=1}^{n} f\left(J_j'\right) \quad \text{and} \quad \frac{\partial G_i}{\partial J_j'} = \frac{1}{r} f_j'\left(J_j'\right) J_i' \quad \text{for } j \neq i\,.$$

Thus, G_i is strictly increasing in J_i' on the domain $\left\{\left(J_1', \ldots, J_n'\right) : \max_j J_j' > 0\ ,\ \min_j J_j' \geq 0\right\}$. Moreover, if $J_j(q_t)$ and $J_j'(q_t)$ are zero for some q_t and all j at time t , then the project cannot move away from q, hence, it cannot be completed, and $J_j(q_s) = J_j'(q_s) = 0$ for all j and $s > t$. This is the degenerate equilibrium that does not satisfy the boundary condition of the ODE system. Thus, we consider G_i only on the domain $\mathcal{R}_+^n = (0,\infty)^n$. We first prove the following lemma.

Lemma 3. *The mapping $G = (G_1, \ldots, G_n)$ from $\mathcal{R}_+^n$ to $\mathcal{R}_+^n$ is invertible and the inverse mapping F is continuously differentiable. Moreover, $\frac{\partial F_i}{\partial x_i} > 0$, that is, the i-th component of the inverse mapping is strictly increasing in the i^{th} variable.*

Proof of Lemma 3.

By the Gale-Nikaido generalization of Inverse Function Theorem (see, for example, Theorem 20.4 in Nikaido (1968)), for the first part of the lemma it is sufficient to show that all the principal minors of the Jacobian matrix of the mapping G are strictly positive for all $(J_1', .., J_n') \in \mathcal{R}_+^n$. We prove this by induction. The 1×1 principal minors, that is, the diagonal entries $\frac{1}{r}\sum_{j=1}^{n} f_j\left(J_j'\right)$ of the Jacobian are positive (for any n). For $n = 2$, from the expressions for the partial derivatives of G_i, we see that the determinant of the Jacobian is proportional to

$$\left(\sum_{j=1}^{2} f_j\left(J'_j\right)\right)^2 - f'_1\left(J'_1\right) f'_2\left(J'_2\right) J'_1 J'_2 > 0$$

where the inequality holds on the domain $\mathcal{R}_+^2$, by noting that $f_i\left(x\right) > x f'_i\left(x\right)$ for all i and $x > 0$.[26] The same argument shows that 2×2 principal minors of the Jacobian are positive, for any n. For $n = 3$, we can compute the determinant of the Jacobian by summing the diagonal terms multiplied with the 2×2 principal minors, that we have just shown to be strictly positive. Since the diagonal terms are strictly positive, the determinant of the Jacobian is also strictly positive. And so on, for $n = 4, 5, \ldots$

For the last statement, by the Inverse Function Theorem, we need to show that the diagonal entries of the inverse of the Jacobian matrix are strictly positive. By Cramer's rule, those entries are proportional to the diagonal entries of the adjugate of the Jacobian, thus proportional to the diagonal entries of the cofactor matrix of the Jacobian. Those entries are equal to the corresponding $(n-1) \times (n-1)$ principal minors, which, by the argument above, are positive. $\square$

This lemma asserts that the values J_i and the marginal values J'_i, and hence the optimal actions a_i, are in one-to-one smooth strictly increasing correspondence, outside of the trivial equilibrium in which all the values are zero. That is, increasing an agent's action also increases his value function, and vice-versa (holding everything else constant).

Next, we show that the set of solutions to the system of ODE's is not empty.

Lemma 4. *For every $\epsilon \in (0, \min_i \{V_i\})$, there exists some $q_\epsilon < Q$ such that there exists a unique solution $(J_1, \ldots, J_n)$ to the ODE system on interval $[q_\epsilon, Q]$ that satisfies $J_i \geq \epsilon$ on that interval, for all i.*

Proof of Lemma 4.
The ODE system (4) can be written as

$$J'_i\left(q\right) = F_i\left(J_1\left(q\right), \ldots, J_n\left(q\right)\right) \tag{44}$$

[26] To see why, first note that for all $x > 0$ we have $f'_i\left(c'_i\left(x\right)\right) = \frac{1}{c''_i(x)}$ and so $f''_i\left(c'_i\left(x\right)\right) = -\frac{c'''_i(x)}{\left[c''_i(x)\right]^3} \leq 0$. Next, let $k_i\left(x\right) = f_i\left(x\right) - x f'_i\left(x\right)$, and observe that $k_i\left(0\right) = 0$ by assumption and $k'_i\left(x\right) = -x f''_i\left(x\right) \geq 0$ for all $x > 0$. Therefore, $f_i\left(x\right) > x f'_i\left(x\right)$ for all $x > 0$.

for those values of $J_j(q)$ for which $\max_j J_j(q) > 0$. For given $\epsilon > 0$, denote

$$M_F = \max_i \max_{\epsilon \le x_i \le V_i} F_i(x_1, \ldots, x_n)$$

Pick $q_\epsilon < Q$ so close to Q such that, for all i,

$$V_i - (Q - q_\epsilon) M_F \ge \epsilon$$

Then, define $\Delta q = \frac{Q - q_\epsilon}{N}$ and functions J_i^N by Picard iterations, going backwards from Q,

$$J_i^N(Q) = V_i$$

$$J_i^N(Q - \Delta q) = V_i - \Delta q F_i(V_1, \ldots, V_n)$$

$$\begin{aligned} J_i^N(Q - 2\Delta q) &= J_i^N(Q - \Delta q) - \Delta q F_i\left(J_1^N(Q - \Delta q), \ldots, J_n^N(Q - \Delta q)\right) \\ &= V_i - \Delta q F_i(V_1, \ldots, V_n) - \Delta q F_i\left(J_1^N(Q - \Delta q), \ldots, J_n^N(Q - \Delta q)\right) \end{aligned}$$

and so on, until $J_i^N(Q - N\Delta q) = J_i(q_\epsilon)$. Then, we complete the definition of function J_i^N by making it piecewise linear between the points $Q - k\Delta q$, $k = 1, \ldots, N$. Notice, from the assumption on $Q - q_\epsilon$ that $J_i^N(Q - k\Delta q) \ge \epsilon$, for all $k = 1, \ldots, N$. Since F_i are continuously differentiable, they are Lipschitz on the $n-$dimensional bounded domain $\Pi_{j=1}^n [\epsilon, V_i]$. Thus, by the standard ODE argument, $\left\{J_i^N\right\}$ converge to a unique solution $\{J_i\}$ of the ODE system on $q \in [q_\epsilon, Q]$, and we have $J_i \ge \epsilon$. □

We can define

$$\underline{q} = \inf_{\epsilon > 0} q_\epsilon$$

with $\underline{q} \in (-\infty, Q)$, and Lemma 4 shows that the the system of ODE's has a unique solution on $[q_\epsilon, Q]$ for every $\epsilon > 0$. Thus, there exists a unique solution on $\left(\underline{q}, Q\right]$. Then, by standard optimal control arguments, $J_i(q)$ is the value function of agent i, for every initial project value $q > \underline{q}$.

We now want to show that the ODE solution $J_i(q)$ is the value function of agent i, for every initial project value $q_0 > \underline{q}$, if every other agent plays $a_j(q) = J_j'(q)$. We have, for any action

$a_i\left(q_t\right)$,

$$\begin{aligned} e^{-r\tau} J_i(Q) &= J_i\left(q_0\right)+\int_0^\tau d\left(e^{-rt} J_i\left(q_t\right)\right) \\ &= J_i\left(q_0\right)+\int_0^\tau e^{-rt}\left[-r J_i\left(q_t\right)+J_i^{\prime}\left(q_t\right) \sum_{j=1}^n a_j\left(q_t\right)\right] dt \\ &\leq J_i\left(q_0\right)+\int_0^\tau e^{-rt} c_i\left(a_i\left(q_t\right)\right) dt \end{aligned}$$

Since $J_i(Q)=\alpha_i V$, this implies that the agent's value function, denoted J_i^*, satisfies $J_i^*\left(q_0\right) \leq J_i\left(q_0\right)$. Moreover, the upper bound is attained if the agent plays $a_i(q)=J_i^{\prime}(q)$. A similar argument works for $J(q)$ for any $q=q_t \in\left(\underline{q}, Q\right]$.

To establish convexity, we differentiate the ODE for J_i to obtain

$$r J_i^{\prime}=-J_i^{\prime} f_i^{\prime} J_i^{\prime\prime}$$

We can view this as a linear system for the vector $J^{\prime\prime}$ with entries $J_i^{\prime\prime}$, which can be written as $M J^{\prime\prime}=r J^{\prime}$, where $J^{\prime}$ denotes the vector with entries $J_i^{\prime}$, and the i-th row of matrix M is

$$\left(J_i^{\prime} f_1\left(J_1^{\prime}\right), \ldots, J_i^{\prime} f_{i-1}\left(J_{i-1}^{\prime}\right), \sum_j f_j\left(J_j^{\prime}\right), J_i^{\prime} f_{i+1}\left(J_{i+1}^{\prime}\right), \ldots, J_i^{\prime} f_n\left(J_n^{\prime}\right)\right)$$

Using the main result of Kaykobad (1985), a sufficient condition for $J^{\prime\prime}$ to have all strictly positive entries is that $J^{\prime}$ has all strictly positive entries and

$$\sum_{j=1}^n f_j\left(J_j^{\prime}\right)>\sum_{j \neq i} J_j^{\prime} f_j^{\prime}\left(J_j^{\prime}\right),$$

which holds if $f_j(x) \geq x f^{\prime}{}_j(x)$ for all $x \geq 0$ and j. By noting that this condition is always satisfied (see footnote 26 for details), the proof is complete. □

Proof of Proposition 2.
To study the existence of a solution, we proceed as in Proposition 1. First, we write the system of ODE's (7) in the form

$$r \bar{S}(q)=G\left(\bar{S}^{\prime}(q)\right)$$

for an appropriate function G. As in the MPE case, we can show that $G(x)$ has a continuously differentiable and strictly increasing inverse function $F(y)$, for $y > 0$. Except for convexity, the proof is similar to the proof of Proposition 1 (and so we omit the details). To show that $\bar{S}''(q) > 0$ for all $q > \underline{q}^S$, note that differentiating (7) yields

$$r\bar{S}'(q) = \left[\sum_{j=1}^{n} f_j\left(\bar{S}'(q)\right)\right]\bar{S}''(q) .$$

Thus, $\bar{S}''(q) > 0$ for all q for which $\bar{S}'(q) > 0$, which completes the proof. □

Remark 1.

Let $\bar{S}_n(q)$ denote the social planner's value function when the group comprises of n symmetric agents, and observe that $\bar{S}_n(q)$ is non-decreasing in n. That is because the social planner can always instruct the additional agent(s) to exert zero effort without decreasing her payoff. Then it follows from (7) that $G\left(\bar{S}_n'(q)\right)$ also increases in n, where

$$G(x) = n\left[f(x)\,x - c(f(x))\right] .$$

Now consider the change of variables

$$y = nf(x) \quad \text{so that} \quad x = c'\left(\frac{y}{n}\right) ,$$

and the function

$$\tilde{G}(y) = yc'\left(\frac{y}{n}\right) - nc\left(\frac{y}{n}\right) .$$

Observe that $\tilde{G}'(y) = c'\left(\frac{y}{n}\right) + \frac{y}{n}c''\left(\frac{y}{n}\right) - c'\left(\frac{y}{n}\right) = \frac{y}{n}c''\left(\frac{y}{n}\right) > 0$, so $\tilde{G}(y)$ is increasing in $y = nf(x)$. To show that $n\left(f\left(\bar{S}_n'(q)\right)\right)$ increases in n for all q, suppose that the contrary is true; *i.e.*, that there exists some q for which $n\left(f\left(\bar{S}_n'(q)\right)\right)$ decreases in n. Then $\tilde{G}\left(n\left(f\left(\bar{S}_n'(q)\right)\right)\right)$ must also decrease in n. However, this is a contradiction , because $\tilde{G}\left(n\left(f\left(\bar{S}_n'(q)\right)\right)\right) = \bar{S}_n(q)$. Therefore, $q_t = q_0 + \int_0^t n\left(f\left(\bar{S}_n'(q_s)\right)\right) ds$ also increases in n, which in turn implies that the completion time $\bar{\tau}$ decreases in n. □

Proof of Proposition 3.

Let us define $D(q) = \bar{S}(q) - \sum_{i=1}^{n} J_i(q)$. By definitions, $D(q) \geq 0$. This implies that $\underline{q} \geq \underline{q}^S$. First, we claim that $D'(Q) \geq 0$. Suppose not. Then, there exists a value $q^* \in (\underline{q}^*, Q)$, such that for $q \in [q^*, Q]$, $D(q)$ is decreasing. Since $D(Q) = 0$, this would mean that $D(q) < 0$, for those values of q, which is a contradiction. Next, let $\Delta_i(q) = \bar{S}(q) - J_i(q)$. Note that $\Delta_i'(q) > D'(q)$.

We then have $\Delta_i'(Q) > 0$. Therefore, either $\Delta_i'(q) > 0$ for all $q \in (\underline{q}, Q]$, or there exists some z such that $\Delta_i'(z) = 0$. Suppose that the latter is true. Then, using (4) and (7), we have

$$r\Delta_i(z) = -\sum_{j \neq i} c_j\left(f_j\left(\bar{S}'(z)\right)\right) < 0$$

However, this is a contradiction, because $\Delta_i(q) \geq D(q) \geq 0$ for all q. Therefore, we conclude that $\Delta_i'(\cdot) > 0$, that is, $\bar{S}'(q) > J_i'(q)$, which in turn implies that $\bar{a}_i(q) > a_i(q)$ for all i and q. □

Proof of Lemma 1.

To establish existence of a MPE that implements the efficient outcome, substitute (9) into (8) and take agent i's the first order condition. This yields the system of ODE

$$\begin{aligned} r\hat{J}_i(q) &= -c_i\left(f_i\left(\hat{J}_i'(q)\right)\right) + \left[\sum_{j=1}^n f_j\left(\hat{J}_i'(q)\right)\right]\hat{J}_i'(q) - \sum_{j\neq i} c_j\left(f_j\left(\bar{S}'(q)\right)\right) \\ &= \left[\sum_{j=1}^n f_j\left(\hat{J}_i'(q)\right)\right]\hat{J}_i'(q) - \sum_{j=1}^n c_j\left(f_j\left(\bar{S}'(q)\right)\right) \end{aligned}$$

subject to $\hat{J}_i(Q) = V$. It follows from (7) that $\hat{J}_i(q) = S(q)$ satisfies this ODE for all i and $q > \underline{q}^S$. Because the planner's problem has a unique solution (by Proposition 1), it follows that this is the unique project-completing MPE. Budget balance on the equilibrium path follows by constructions of the upfront payments given in (**??**).

The first assertion for part (iii) is straightforward. The second follows by noting that $f_i(\cdot)$ is increasing for all i, $\bar{S}'(q) \geq 0$ and $\bar{S}''(q) \geq 0$ for all q, so that $h_i'(q) = \bar{S}'(q)\bar{S}''(q)\sum_{j\neq i} f_j'\left(\bar{S}'(q)\right) \geq 0$. □

Proof of Lemma 2.

Noting that the balance in the savings account evolves according to $dH_t = \left[rH_t + \sum_{j=1}^n h_j(q_t)\right]dt$, each agent i's problem is

$$rJ_i(q,H) = \max_{a_i}\left\{-c_i(a_i) + \left(\sum_{j=1}^n a_j\right)\frac{J_i(q,H)}{\partial q} + \left(rH + \sum_{j=1}^n h_j\right)\frac{J_i(q,H)}{\partial H} - h_i(q)\right\}$$

subject to the boundary condition

$$J_i(Q,H) = \beta_i (V+H)$$

Note that here, each agent's discounted payoff function J_i is a function of of both the state of the project q and the balance in the savings account H. We conjecture that J_i is of the form

$$J_i(q,H) = \beta_i H + \tilde{J}_i(q)$$

in which case the HJB equation for $\tilde{J}_i(q)$ can be re-written as

$$r\tilde{J}_i(q) = \max_{a_i}\left\{-c_i(a_i) + \left(\sum_{j=1}^n a_j\right)\tilde{J}_i'(q) - \left[h_i(q) - \beta_i \sum_{j=1}^n h_j(q)\right]\right\}$$

subject to

$$\tilde{J}_i(Q) = \beta_i V$$

The first order condition gives $a_i(q) = f_i\left(\tilde{J}_i'(q)\right)$ so that

$$r\tilde{J}_i(q) = -c_i\left(f_i\left(\tilde{J}_i'(q)\right)\right) + \left[\sum_{j=1}^n f_j\left(\tilde{J}_j'(q)\right)\right]\tilde{J}_i'(q) - \left[h_i(q) - \beta_i \sum_{j=1}^n h_j(q)\right]$$

Thus, we see that $\tilde{J}_i(q)$ is equal to the value function of agent i who receives $\beta_i V$ upon completion of the project and makes flow payments $\tilde{h}_i(q) = h_i(q) - \beta_i \sum_{j=1}^n h_j(q)$. These flow payments have to add up to zero, and the proposition is proved. □

Proof of Proposition 4.

Let $\bar{J}_i(q)$ denote agent i's discounted payoff function for the problem in which each agent i plays the efficient strategy $\bar{a}_i(q)$, he makes flow payments $h_i(q) = \sum_{j\neq i} c_j\left(f_j\left(\bar{S}'(q)\right)\right) + rp_i$ and receives $V - p_i$ upon completion, where $\sum_{i=1}^n p_i = (n-1)\bar{S}(0)$. Fix an agent i, and assume that every agent $j \neq i$ plays $\bar{a}_j(q) = f_j(S'(q))$. Moreover, let $\beta_i = \frac{V-p_i}{V+(n-1)\left[V-\bar{S}(0)\right]}$. We have, for any effort $a_i(q)$ process of agent i, with τ^a denoting the corresponding completion time,

$$\begin{aligned}
& e^{-r\tau^a} \min\left\{V - p_i,\, \beta_i\left(V + H_\tau\right)\right\} - \int_u^{\tau^a} e^{-rs}\left\{h_i\left(q_s\right) + c_i\left(a_i\left(q_s\right)\right)\right\} ds \\
\leq & e^{-r\tau^a}\left(V - p_i\right) - \int_u^{\tau^a} e^{-rs}\left\{h_i\left(q_s\right) + c_i\left(a_i\left(q_s\right)\right)\right\} ds \\
\leq & e^{-ru} \bar{J}_i\left(q_u\right)
\end{aligned}$$

where the last equality holds because $\bar{J}_i$ is the value function of the problem. In particular, with $u = t$, we see that $\bar{J}_i(q_t)$ is an upper bound for agent i's problem. On the other hand, with $a_i(q) = \bar{a}_i(q)$ for all q, all the inequalities above become equalities, so it is optimal for agent i to also choose strategy $\bar{a}_i(q)$. We see that this is also true starting from any point in time $t \leq u \leq \tau$, thus all playing $\bar{a}_i(q)$ is an MPE.

So far, we have shown that this game has a unique project-completing MPE. To complete the proof for part (i), we must show that there exists no equilibrium in which the project is not completed. Clearly, there cannot exist an equilibrium in which the agents make the upfront payments and then the equilibrium is not completed; at least one agent would be better off rejecting the mechanism to guarantee himself a zero rather than a negative payoff. Moreover, observe that each agent has a (weakly) dominant strategy to accept the mechanism, since he will have to make the upfront payments only if all agents accept the mechanism, in which case he obtains a nonnegative payoff, whereas rejecting the mechanism yields him payoff 0. Therefore, this game has a unique MPE.

Since each agent's reward is capped by $\beta_i(V + H_\tau)$, the mechanism will never result in a budget deficit. From Lemma 1, it follows that on the equilibrium path, the total balance in the savings account is $H_\tau = (n-1)\left[V - \bar{S}(0)\right]$, which implies that the budget is balanced, and hence proves part (ii). Parts (iii) follows directly from the construction in Section 4.1. $\square$

Proof of Proposition 6. Suppose first that each agent i receives zero if he runs out of cash before the project is completed, and otherwise he gets exactly V upon completion. Then, the system of ODE looks the same as before, except $h_i(q)$ in the ODE is multiplied by $I_i(q)$ and V in the terminal condition is multiplied by $I_i(Q)$. The efficient outcome is still an MPE, because with the corresponding choice of effort, $I_i(q) = 1$ for all i and all $q \leq Q$, and the system of ODE is the same as before.

Suppose now that each agent i is promised $\min\left\{V, \frac{V+H_\tau}{n}\right\}$ instead of V at completion, and zero if he runs out of cash. Then, the agent's payoff is equal to

$$
\begin{aligned}
&I_i(Q)\, e^{-r\tau} \min\left\{V - p_i,\ \beta_i (V + H_\tau)\right\} - \int_0^\tau e^{-rs} \left\{ I_i(q_s)\, h_i(q_s) + c_i(a_i(q_s)) \right\} ds \\
\leq &I_i(Q)\, e^{-r\tau} (V - p_i) - \int_0^\tau e^{-rs} \left\{ I_i(q_s)\, h_i(q_s) + c_i(a_i(q_s)) \right\} ds
\end{aligned}
$$

Consider first the deviations in which the agent does not run out of cash by completion. Then, as before, with $\bar{J}_i(q)$ denoting agent i's value function in the efficient MPE, the last term is no greater than $\bar{J}_i(q_0)$, and hence the agent would not deviate if no one else does. If the agent applies a deviation in which he runs out of cash, then his value function is negative, and the same conclusion holds. □

Proof of Proposition 7. We first show that $\hat{g}_i(\tau)$ is non-increasing in τ. For all τ such that $g_i(\tau) \leq \alpha_i V \leq V$, because $\sum_{j \neq i} c_j\left(f_j\left(\bar{S}'(Q)\right)\right) > 0$ for all i, it follows from (14) that $g_i'(\tau) < 0$, and so $\hat{g}_i'(\tau) \leq 0$. On the other hand, for τ such that $g_i(\tau) > \alpha_i V$, we have $\hat{g}_i(\tau) = \alpha_i V$ and so it is independent of τ. From this it follows that for $\tau < \bar{\tau}$, we have $g_i(\tau) > \alpha_i V$. Because, if we had $g_i(\tau) \leq \alpha_i V$ for some $\tau < \bar{\tau}$, then, $g_i(\cdot)$ would be strictly decreasing on $s > \tau$, and we could not have $g_i(\bar{\tau}) = \alpha_i V$.

Next, let $\bar{J}_i(t, q)$ denote agent i's discounted payoff function corresponding to the game in which each agent i pays $h_i(t, q)$ and receives $g_i(\tau)$ at completion time τ, and agents $j \neq i$ play the first best strategies. Then, by the arguments in Section 4.3, agent i also plays first best strategy $a_i(t, q) = \bar{S}'(q)$.

We aim to show that if each agent makes flow payments $\hat{h}_i(t, q) = [h_i(t, q)]^+$ and receives $\hat{g}_i(\tau) = [\min\{\alpha_i V,\ g_i(\tau)\}]^+$ upon completion of the project, and all agents except for i play first best strategies, then agent i cannot do better then playing first best himself. There are three cases to consider:

(i) If the effort process $a_i(t, q)$ of agent i does not lead to completion, then he has to pay non-negative payments forever but will never receive a lump-sum reward, which is worse than playing first best.

(ii) Similarly, playing $a_i(t, q)$ such that the corresponding completion time τ^a is larger than or equal to τ_i^0, where $\tau_i^0 = \min\{\tau : g_i(\tau) = 0\}$, is worse than first best, because he gets zero reward upon completion.

(iii) Suppose now he plays $a_i(t,q)$ such that the corresponding completion time τ^a is less than τ_i^0. Then, for $\tau < \tau_i^0$, we have $\hat{g}_i(\tau) = \min\{\alpha_i V,\, g_i(\tau)\}$ and the discounted payoff satisfies

$$
\begin{aligned}
& e^{-r\tau^a}\hat{g}_i(\tau^a) - \int_t^{\tau^a} e^{-rs}\left\{h^+(s,q_s) + c_i(a_i(s,q_s))\right\} ds \\
\leq & e^{-r\tau^a} g_i(\tau^a) - \int_t^{\tau^a} e^{-rs}\left\{h_i(s,q_s) + c_i(a_i(q_s))\right\} ds \\
\leq & e^{-rt}\bar{J}_i(t,q_t)
\end{aligned}
$$

where the last equality holds because $\bar{J}_i$ is the value function of the problem of maximizing the expression in the second line. In particular, with $t=0$, we see that $\bar{J}_i(0,q_0)$ is an upper bound for agent i's payoff. Moreover, with all other agents playing the first best strategy, if agent i also plays first best, we have $\tau^a = \bar{\tau}$, $g_i(\bar{\tau}) = \hat{g}_i(\bar{\tau}) = \alpha_i V$, $h_i \equiv \hat{h}_i \equiv 0$, and all the inequalities above become equalities, so it is optimal for agent i to also choose first best strategy $\bar{S}'(q)$.

Altogether, we see that under the mechanism $\left\{\hat{g},\, \hat{h}\right\}$ playing first best is a MPE, in which case the budget is balanced on the equilibrium path, and there is no budget deficit even off equilibrium path. □

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