Generalized Dynamic Factor Models and Volatilities

Recovering the Market Volatility Shocks

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Abstract

Decomposing volatilities into a *common* market-driven component and an *idiosyncratic* item-specific one is an important issue in financial econometrics. This, however, requires the statistical analysis of large panels of time series, hence faces the usual challenges associated with high-dimensional data. Factor model methods in such a context are an ideal tool, but they do not readily apply to the analysis of volatilities. Focusing on the reconstruction of the unobserved market shocks and the way they are loaded by the various items (stocks) in the panel, we propose an entirely non-parametric and model-free two-step general dynamic factor approach to the problem, which avoids the usual curse of dimensionality. Applied to the S&P100 asset return dataset, the method provides evidence that a non-negligible proportion of the market-driven volatility of returns originates in the volatilities of the idiosyncratic components of returns.

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1 Introduction

Decomposing asset returns and risks or volatilities into a *common*, market-driven component and an individual, *idiosyncratic* one, is one of the main issues in financial econometrics, risk management, and portfolio optimization. Market-driven risks indeed cannot be diversified away, while individual ones can be eliminated through clever portfolio diversification. Some of the first examples are in Connor and Korajczyk (1986) for returns and in Engle and Marcucci (2006) for volatilities.

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Since market features are involved, achieving such decompositions unavoidably implies the analysis of large portfolios, hence the usual challenges associated with high-dimensional datasets–here, moreover, in a time-series context. This problem lately has attracted much interest, in conjunction with the surge of activity in the estimation of high-dimensional covariance matrices and the analysis of large panels of time series data. A number of methods have been proposed; see the references below or the recent monograph by Ghysels (2014) for a review of the literature.

Among the latest and most effective contributions is Fan et al. (2013), where the analysis is based on a decomposition of the covariance matrix $\mathbf{\Gamma}$ of the observed high-dimensional process $\{\mathbf{Y}_t\}$–in the context of portfolio optimization, the time series of returns, or levels, of the collection of stocks under study–into a sum of the "low rank plus sparse" type (as in Fan et al., 2013), which also can be interpreted as a factor model decomposition of $\{\mathbf{Y}_t\}$ into a *common* component plus an *idiosyncratic* one.[1]

That approach, which is exclusively based on the marginal covariance matrix $\mathbf{\Gamma}$ of the series under study (in practice, an estimator thereof), is entirely static: it does not take into account, hence fails to exploit, the time-series nature of the problem. If the same dataset is considered from a dynamic point of view–if, for instance, *dynamic* portfolio management, that is, minimization of the conditional risk at specific time t, is the objective–that static approach can be improved on several counts.

(i) Rather than unconditional variances and covariances, conditional volatilities (that depend on past values) at specific time t should serve as the cornerstone of the analysis.

(ii) The "low rank plus sparse" type decomposition of $\mathbf{\Gamma}$ corresponds to a strictly static factor model decomposition of $\{\mathbf{Y}_t\}$ of the type considered by Chamberlain and Rothschild (1983), while the econometric literature has established the superiority, in the presence of serial dependence, of the various forms of *dynamic* factor models over the strictly static ones (see Forni et al., 2000; Stock and Watson, 2005; Bai and Ng, 2007; Forni et al., 2009; Forni and Lippi, 2011; Hallin and Lippi, 2013; Forni et al., 2015, to quote only a few). Dynamic factor models here are likely to be the most appropriate tool.

(iii) The same decomposition moreover exclusively relies on the common/idiosyncratic features of the process $\{\mathbf{Y}_t\}$ of levels or returns, surmising that the "common components of return volatilities" coincide with "the volatilities of the common components of returns", and the "idiosyncratic components of return volatilities" with "the volatilities of the idiosyncratic components of returns", an assumption which is unlikely to hold true.

In this paper, we propose a two-step dynamic factor approach taking care of those three points, then apply that method to the problem of reconstructing the unobserved market volatility shocks.

The first step yields a common plus idiosyncratic *general dynamic factor model* decomposition of the levels, $\{\mathbf{Y}_t\}$. We call the two components of this factor model decomposition *level-common* and *level-idiosyncratic* in order to distinguish them from those obtained from the factor decomposition of volatilities, which we define below. Based on recent results by Forni et al. (2015), the decomposition of the levels is one-sided, i.e. only involves one-sided filters, hence past observations. The same results also yield a (reduced rank) fundamental VAR representation of the level-common components, hence residuals that provide a consistent reconstruction of the level-common, market-driven, fundamental shocks. As for the level-idiosyncratic components, which by definition are only mildly

[1] With a definition of "idiosyncratic" which is not the same as the one we are using here, though.

cross-correlated, residuals can be obtained via univariate AR fitting, providing a reconstruction of the level-idiosyncratic fundamental shocks.

Those shocks in turn serve as the basis of a dynamic factor analysis of volatilities. After adequate non-linear transformation, they constitute a panel of volatility proxies. Actually, if the original dataset consisted of n stocks observed over a time period T, they constitute a panel of $2n \times T$ observations, subdivided into two $n \times T$ blocks or subpanels of volatility proxies: the *level-common* and the *level-idiosyncratic* one, respectively. The dynamic factor analysis of such panels with block structures has been studied by Hallin and Liška (2011), who show how to extract mutually orthogonal *strongly common* components (common to both subpanels), *strongly idiosyncratic* ones (idiosyncratic to both), as opposed to *weakly common* (common to one block but not to the other) and *weakly idiosyncratic* components. That approach, which perfectly applies here, allows us to take into account the presence of *volatility-common* shocks (market volatility shocks) both in the level-common residuals as in the level-idiosyncratic ones–only the strongly idiosyncratic components are free of market volatility impacts.

The method is applied to a panel of stock returns of the S&P100 index over a period spanning the last ten years. Results confirm the impact of market volatility shocks both on level-common as on level-idiosyncratic components. In particular, we find evidence of one market volatility shock accounting for about 60% of the total variation of logged level-common volatilities and about 13% of the total variation of logged level-idiosyncratic volatilities.

Multivariate models of conditional variance and covariance matrices are not new in the literature. Among the first proposed are the multivariate stochastic volatility models by Harvey et al. (1994) and the GARCH-DCC model by Engle (2002). We refer to the surveys by Bauwens et al. (2006), Asai et al. (2006), and Silvennoinen and Teräsvirta (2009) for recent reviews of the subject. However, being parametric, those models all suffer of the "curse of dimensionality": when considering high-dimensional panels, estimation rapidly becomes unfeasible. In order to solve this problem and in agreement with the idea of a market volatility common to all components of a financial index, factor structures in volatilities have been developed by Engle and Marcucci (2006), Engle et al. (2008), Rangel and Engle (2012), Luciani and Veredas (2014), and Ghysels (2014), among others, while a semi-parametric approach is proposed by Barigozzi et al. (2014). Recently, Fan et al. (2013) improved this model by relaxing the assumptions and allowing for the presence of idiosyncratic variances which are modelled as a sparse matrix. Finally, the papers most related to our work are those proposing a factor structure on the returns and then assuming a GARCH model for the latent factors, as, for example, Ng et al. (1992), Harvey et al. (1992), Diebold and Nerlove (1989), Van der Weide (2002), Connor et al. (2006), and Sentana et al. (2008), among others; see also Jurado et al. (2013) for an application to macroeconomic data. All those factor models, however, are static, and of the exact type (strictly no idiosyncratic cross-correlations); thus, they neither exploit the serial correlation in the data nor are able to account for idiosyncratic cross-sectional dependencies which are very likely to exist in large datasets.

In contrast with most of that literature, our analysis is purely non-parametric, and essentially model-free (see Hallin and Lippi (2013) for a discussion of this latter fact); it allows for mild cross-sectional correlation among idiosyncratic components, and avoids the dimensionality problems inherent to multivariate volatility models. Our main contribution with respect to the existing literature is then the introduction of a two-step generalized dynamic factor model: one for the returns, and another one for volatilities, with a focus not only on level-common but also on level-idiosyncratic volatility shocks.

The rest of the paper is organized as follows. Section 2.1 presents the general dynamic factor model for returns, Sections 2.2-2.4 a block dynamic factor model for volatilities based on the level-common and level-idiosyncratic shocks resulting from the previous decomposition. Section 3 describes estimation. Section 4 investigates the small sample properties of the proposed estimators by means of a Monte Carlo simulation study. Section 5 provides empirical results for the S&P100 market volatility. Finally, in Section 6, we conclude and discuss possible extensions of the present framework.

2 A two-stage general dynamic factor model for volatilities

2.1 A general dynamic factor model for returns

We throughout assume that all stochastic variables in this paper belong to the Hilbert space $L_2(\Omega, \mathcal{F}, \mathrm{P})$, where $(\Omega, \mathcal{F}, \mathrm{P})$ is some given probability space. The observation we are dealing with is an $n \times T$ panel of stock returns or levels, that is, a finite realization

$$\begin{array}{cccc} Y_{11}, & Y_{12}, & \ldots, & Y_{1T} \\ \vdots & \vdots & & \vdots \\ Y_{n1}, & Y_{n2}, & \ldots, & Y_{nT} \end{array}$$

of a double-indexed stochastic process, of the form

$$\mathbf{Y} := \{Y_{it} | i \in \mathbb{N},\ t \in \mathbb{Z}\},$$

where t stands for time and i for the cross-sectional index; equivalently, the $n \times T$ panel can be considered a collection of n observed time series (length T), or a unique observed time series in dimension n. We assume that $\{Y_{it}\}$, as a process, is centered and strictly stationary. As both n and T are "large", (n, T)-asymptotics, where both n and T tend to infinity, are considered throughout.

Let $\mathbf{Y}_n := \{\mathbf{Y}_{n,t} = (Y_{1t}, Y_{2t}, \ldots, Y_{nt})' | \, t \in \mathbb{Z}\}$ be the n-dimensional subprocess of $\mathbf{Y}$ and consider the following assumptions.

ASSUMPTION (A1). *For all $n \in \mathbb{N}$, the vector process $\mathbf{Y}_n$ is strictly stationary, with mean* $\mathbf{0}$ *and finite variances.*

ASSUMPTION (A2). *For all $n \in \mathbb{N}$, the spectral measure of $\mathbf{Y}_n$ is absolutely continuous with respect to the Lebesgue measure on $[-\pi, \pi]$, that is, $\mathbf{Y}_n$ has a spectral density matrix $\mathbf{\Sigma}_{\mathbf{Y};n}(\theta)$, $\theta \in [-\pi, \pi]$.*

For any $\theta \in [-\pi, \pi]$, denote by $\lambda_{\mathbf{Y};n,1}(\theta), \ldots, \lambda_{\mathbf{Y};n,n}(\theta)$ the eigenvalues (in decreasing order of magnitude) of $\mathbf{\Sigma}_{\mathbf{Y};n}(\theta)$; the mapping $\theta \mapsto \lambda_{\mathbf{Y};n,i}(\theta)$ is also called $\mathbf{\Sigma}_{\mathbf{Y};n}(\theta)$'s ith *dynamic eigenvalue*.

The n observed series $\mathbf{Y}_n$ are exposed, in general, to the influence of the same environment of unrecorded covariates, inducing complex interrelations that are not statistically tractable, or would involve prohibitively many parameters. Parametric methods thus, as a rule, are helpless or unrealistic. Factor model methods in this context are the ideal tool–arguably, the only successful ones. We say that $\mathbf{Y}$ admits a *dynamic factor representation* with q factors if Y_{it} for all i and t decomposes into

$$\begin{aligned} Y_{it} &= \text{“common”}_{it} + \text{“idiosyncratic”}_{it} \\ &=: X_{it} + Z_{it} =: \sum_{k=1}^{q} b_{ik}(L) u_{kt} + Z_{it}, \quad i \in \mathbb{N}, \quad t \in \mathbb{Z}, \end{aligned} \tag{2.1}$$

(L, as usual, stands for the lag operator) where

(i) the q-dimensional vector process $\mathbf{u} := \{\mathbf{u}_t = (u_{1t}u_{2t}\dots u_{qt})' |\, t \in \mathbb{Z}\}$ is orthonormal zero-mean white noise;

(ii) the idiosyncratic n-dimensional processes $\mathbf{Z}_n := \{\mathbf{Z}_{n,t} = (Z_{1t}Z_{2t}\dots Z_{nt})' |\, t \in \mathbb{Z}\}$ are zero-mean second-order stationary for any n, with θ-a.e. bounded (as $n \to \infty$) dynamic eigenvalues;

(iii) Z_{kt_1} and u_{ht_2} are mutually orthogonal for any k, h, t_1 and t_2;

(iv) the filters $b_{ik}(L)$ are one-sided and square-summable: $\sum_{m=1}^{\infty} b_{ikm}^2 < \infty$ for all $i \in \mathbb{N}$ and $k = 1, \dots, q$;

(v) q is minimal with respect to (i)-(iv).

In particular, the common and idiosyncratic components are identified by means of the following assumption.

ASSUMPTION (A3). *For some $q \in \mathbb{N}$, the qth dynamic eigenvalue of $\mathbf{\Sigma}_{\mathbf{Y};n}(\theta)$, $\lambda_{\mathbf{Y};n,q}(\theta)$, diverges as $n \to \infty$, θ-a.e. in $[-\pi, \pi]$, while the $(q+1)$th one, $\lambda_{\mathbf{Y};n,q+1}(\theta)$, is θ-a.e. bounded. Moreover, the divergence is at least linear in n, i.e.*

$$\liminf_{n\to\infty} \left(\inf_{\theta} \frac{\lambda_{\mathbf{Y};n,q}(\theta)}{n} \right) > 0.$$

We know from Forni et al. (2000) and Forni and Lippi (2001) that, given Assumptions (A1) and (A2), Assumption (A3) is necessary and sufficient for the process $\mathbf{Y}$ to admit the dynamic factor representation (2.1)[2]. Hallin and Lippi (2013) moreover provide very weak primitive conditions under which (2.1), hence Assumption (A3), holds for some $q < \infty$.

For any n, we can write (2.1) in vector notation as

$$\mathbf{Y}_{n,t} = \mathbf{X}_{n,t} + \mathbf{Z}_{n,t} = \mathbf{B}_n(L)\mathbf{u}_t + \mathbf{Z}_{n,t}, \quad n \in \mathbb{N}, \quad t \in \mathbb{Z}. \tag{2.2}$$

The decomposition (2.2) of $\mathbf{Y}_n$ induces (with obvious notation) a decomposition

$$\mathbf{\Gamma}_{\mathbf{Y};n,k} = \mathbf{\Gamma}_{\mathbf{X};n,k} + \mathbf{\Gamma}_{\mathbf{Z};n,k}$$

of the cross-covariance matrices $\mathbf{\Gamma}_{\mathbf{Y};n,k} := \mathsf{E}[\mathbf{Y}_{n,t}\mathbf{Y}'_{n,t-k}]$ of the $\mathbf{Y}_n$'s, and a decomposition

$$\mathbf{\Sigma}_{\mathbf{Y};n}(\theta) = \mathbf{\Sigma}_{\mathbf{X};n}(\theta) + \mathbf{\Sigma}_{\mathbf{Z};n}(\theta)$$

of their spectral density matrices $\mathbf{\Sigma}_{\mathbf{Y};n}(\theta)$.

The statistical treatment of (2.1) comprises

(i) a consistent (as both n and T tend to infinity) reconstruction of $\mathbf{Y}_n$'s decomposition into common and idiosyncratic components based on Brillinger's concept of *dynamic principal components* (Forni et al., 2000);

(ii) a consistent data-driven method for the identification of q (Hallin and Liška, 2007);

(iii) a one-sided version of (i) (Forni et al., 2015) exploiting properties of the so-called *tall processes* (Anderson and Deistler, 2008).

[2]Those references in Assumption (A1) only assume second-order stationarity, though. We are assuming strict stationarity in order to apply factor model methods to non-linear transformations of the Y_{it}'s (Sections 2.2-2.4).

Since $\mathbf{Y}_n$ decomposes into two components $\mathbf{X}_n$ and $\mathbf{Z}_n$, where $\mathbf{X}_n$ is driven by "common", that is, "market" shocks, and $\mathbf{Z}_n$ is orthogonal to the same, two distinct sources of volatility are to be expected: the volatility originating in the shocks driving the level-common components $\mathbf{X}_n$ (volatility of level-common components), and the volatility originating in the shocks driving the level-idiosyncratic components $\mathbf{Z}_n$ (volatility of level-idiosyncratic components). It is tempting to call "market volatility" the volatility of the level-common components, and "idiosyncratic" the volatility of the level-idiosyncratic ones.

"Natural" as it is, that idea is likely to be over-simplistic. The decomposition (2.2) between common and idiosyncratic indeed has been based on level autocovariances only, which do not carry any information on volatilities–a fact we emphasize by calling $\mathbf{X}_n$ and $\mathbf{Z}_n$ *level*-common and *level*-idiosyncratic, respectively. There is no reason for volatilities to exhibit the same common/idiosyncratic pattern as the levels. For instance, the volatilities of level-idiosyncratic components are quite likely to be affected by market-wide volatility shocks and there is no reason for level-common volatilities to be driven by market volatility shocks only: both are likely to present market-driven *and* item-specific features. The very concept of an identifiable *market-driven volatility shock* therefore requires a common/idiosyncratic analysis that cannot be based, as the one that has been performed so far, on the autocovariances of returns.

The analysis of volatility, typically, is based on the autocovariance structure of some non-linear transform of innovation processes–something the factor model decomposition (2.1) does not readily provide. For the common component $\mathbf{X}_n$, however, such residuals can be obtained (see Section 2.2) from results by Forni and Lippi (2011) and Forni et al. (2015). As for the idiosyncratic components $\mathbf{Z}_n$, since they are only mildly cross-correlated, componentwise residuals can be obtained (see Section 2.3) via univariate AR fitting.

2.2 The volatility of the level-common component

Assume, without loss of generality and for the simplicity of notation, that n is an integer multiple of $(q+1)$, that is, $n = m(q+1)$ for some $m \in \mathbb{N}$. Forni and Lippi (2011) and Forni et al. (2015) show that, under Assumptions (A1)-(A3) and the mild additional condition of a rational spectrum, there exist

(*i*) an $m(q+1) \times m(q+1)$ block-diagonal matrix of one-sided filters

$$\mathbf{A}_n(L) = \begin{pmatrix} \mathbf{A}^{(1)}(L) & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{A}^{(2)}(L) & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{A}^{(m)}(L) \end{pmatrix} \tag{2.3}$$

with $(q+1)\times(q+1)$ blocks $\mathbf{A}^{(i)}(L)$ such that the VAR operators $\mathbf{I}_{q+1} - \mathbf{A}^{(i)}(L)$ are square-summable and *fundamental*,

(*ii*) a full-rank $n \times q$ matrix of constants $\mathbf{H}_n$,

such that $\mathbf{Y}_n$ admits a VAR representation of the form

$$(\mathbf{I}_n - \mathbf{A}_n(L))\,\mathbf{Y}_{n,t} = \mathbf{H}_n\mathbf{u}_t + (\mathbf{I}_n - \mathbf{A}_n(L))\,\mathbf{Z}_{n,t} =: \mathbf{H}_n\mathbf{u}_t + \widetilde{\mathbf{Z}}_{n,t}, \quad n \in \mathbb{N}, \quad t \in \mathbb{Z}, \tag{2.4}$$

where $\widetilde{\mathbf{Z}}_n := (\mathbf{I}_n - \mathbf{A}_n(L))\,\mathbf{Z}_{n,t}$ is idiosyncratic, i.e. only has θ-a.e. bounded (as $n \to \infty$) dynamic eigenvalues.

The form of the extreme-right-hand side of (2.4) is of particular importance. It shows, indeed, that the filtered panel $(\mathbf{I}_n - \mathbf{A}_n(L))\,\mathbf{Y}_{n,t}$, where the AR filters in $\mathbf{A}_n(L)$ can be estimated via low-dimensional AR fitting, admit a *static* factor model representation: the (unlagged) common shocks $\mathbf{u}_t$ indeed are loaded via the matrix loadings $\mathbf{H}_n$. Those shocks, their loadings, and the $\widetilde{\mathbf{Z}}_{n,t}$,'s therefore can be recovered from the observations by means of traditional static factor methods–as described, for instance, in Stock and Watson (2005) or Bai and Ng (2007)–applied to the filtered panel $(\mathbf{I}_n - \mathbf{A}_n(L))\,\mathbf{Y}_{n,t}$.

Contrary to a widespread opinion, general dynamic model methods thus are not technically more involved than the apparently simpler static ones, as the difference essentially consists in the additional $m = n/(q+1)$ AR fittings, each of dimension $(q+1)$, required in the estimation of $\mathbf{A}_n(L)$.

Denote by $\mathbf{e} := \{e_{it} := (\mathbf{H}_n\mathbf{u}_t)_i |\, i \in \mathbb{N},\, t \in \mathbb{Z}\}$, the double-indexed process of those level-common residuals. The n-dimensional singular subprocess $\mathbf{e}_n := \mathbf{H}_n\mathbf{u}$ of $\mathbf{e}$ is the innovation process of $\mathbf{Y}_n$'s common component $\mathbf{X}_n$, hence is zero-mean second-order white noise.

For any fixed $i \in \mathbb{N}$, classical volatility analyses are based on the autocovariance structure of some non-linear transform s_{it}, called *volatility proxy*, of the residual e_{it} resulting from some second-order fit. Standard volatility proxies, in that context, are squared residuals ($s_{it} := e_{it}^2$), or absolute values thereof ($s_{it} := |e_{it}|$); but any monotone increasing function of e_{it}^2, in principle, could serve as well, and many other choices have been considered in the literature. In particular, a reasonable candidate, as proposed by Engle and Marcucci (2006), is

$$s_{it} := \log(e_{it}^2) = 2\log|e_{it}|, \quad i \in \mathbb{N}, \quad t \in \mathbb{Z}. \tag{2.5}$$

The advantage of a logarithmic proxy as s_{it} over the squared residuals e_{it}^2 lies in the fact that it can be analyzed via an additive factor model, while a similar analysis of the e_{it}^2's would require imposing intricate positivity constraints when estimating the model. Just as the original observations, the s_{it}'s constitute a double-indexed process $\mathbf{s}$, hence, for any finite n and T, an $n \times T$ panel of level-common volatility proxies; the notation $\mathbf{s}_n := \{\mathbf{s}_{n,t} = (s_{1t}, s_{2t}, \ldots, s_{nt})' |\, t \in \mathbb{Z}\}$ will be used for the n-dimensional subprocess of $\mathbf{s}$.

If the panel of volatilities is to be analyzed via general dynamic factor model techniques, we need the existence of spectral densities.

ASSUMPTION (B1). *The second-order moments* $\mathsf{E}[s_{it}^2]$ *exist for all* $i \in \mathbb{N}$ *and, for all* $n \in \mathbb{N}$*, the spectral density of* $\mathbf{s}_n$ *is absolutely continuous with respect to the Lebesgue measure over* $[-\pi, \pi]$*, that is,* $\mathbf{s}_n$ *has a spectral density matrix* $\mathbf{\Sigma}_{\mathbf{s};n}(\theta)$ *for* $\theta \in [-\pi, \pi]$.

We now make the following assumption on the dynamic eigenvalues of the level-common volatility panel.

ASSUMPTION (B2). *There exists a* $q_s \in \mathbb{N}$ *such that the* q_s*th eigenvalue* $\lambda_{\mathbf{s};n,q_s}(\theta)$ *of* $\mathbf{\Sigma}_{\mathbf{s};n}(\theta)$ *diverges as* $n \to \infty$*,* θ*-a.e. in* $[-\pi, \pi]$*, while the* (q_s+1)*th one,* $\lambda_{\mathbf{s};n,q_s+1}(\theta)$*, is* θ*-a.e. bounded.*

Assumption (B2) implies that the panel $\mathbf{s}_n$ of level-common volatility proxies admits a dynamic factor representation with q_s common factors, with common and idiosyncratic components $\chi_{\mathbf{s};it}$ and $\xi_{\mathbf{s};it}$, respectively. Namely, writing $\mathring{s}_{it}$ for $s_{it} - \mathsf{E}[s_{it}]$, Assumption (B2) entails the existence of a decomposition

$$\mathring{s}_{it} = \chi_{\mathbf{s};it} + \xi_{\mathbf{s};it} = \sum_{k=1}^{q_s} d_{\mathbf{s};ik}(L)\varepsilon_{\mathbf{s};kt} + \xi_{\mathbf{s};it} \quad i \in \mathbb{N}, \quad t \in \mathbb{Z}, \tag{2.6}$$

or, with obvious vector notation,

$$\mathring{\mathbf{s}}_{n,t} = \boldsymbol{\chi}_{\mathbf{s};n,t} + \boldsymbol{\xi}_{\mathbf{s};n,t}, \quad n \in \mathbb{N}, \quad t \in \mathbb{Z}, \tag{2.7}$$

such that (*i*)-(*v*) of Section 2.1 hold. Intuitively, the existence of such a factor structure for the level-common volatility panel is motivated by the fact that $\mathbf{e}_n$ is a reduced-rank process; it is reasonable thus to assume a similar structure for the volatility proxy which is a function of $\mathbf{e}_n$. Moreover, Hallin and Lippi (2013) show that the existence of a finite q_s is a very natural assumption, which is also justified empirically in Section 5.

2.3 The volatility of the level-idiosyncratic component

A general dynamic factor analysis of the volatilities of the level-idiosyncratic components $\widetilde{Z}_{it}$'s similarly requires a volatility proxy, hence, to begin with, a definition of residuals. Since, being idiosyncratic, those $\widetilde{Z}_{it}$'s are only mildly cross-correlated, a componentwise residual analysis only overlooks negligible information, and we therefore assume, for each $\{\widetilde{Z}_{it}|\, t \in \mathbb{Z}\}$, a univariate AR representation, of the form

$$\big(1 - c_i(L)\big)\widetilde{Z}_{it} = v_{it}, \quad i \in \mathbb{N}, \quad t \in \mathbb{Z}, \tag{2.8}$$

where the AR filters $c_i(L)$ are one-sided, square-summable, and such that the roots of $c(z) = 0$ all lie outside the unit disc. Denote by $\mathbf{v} := \{v_{it}|\, i \in \mathbb{N},\, t \in \mathbb{Z}\}$ the corresponding double-indexed process of residuals: the $\mathbf{v}_{it}$'s are zero-mean second-order white noise, and constitute the univariate innovations of the level-idiosyncratic components $\widetilde{Z}_{it}$'s. In general, they are not mutually orthogonal and some possible mild cross-correlation remains among them. The corresponding n-dimensional subprocess is denoted as $\mathbf{v}_n := \{\mathbf{v}_{nt} = (v_{1t}, v_{2t}, \dots, v_{nt})'|\, t \in \mathbb{Z}\}$.

Analogously to (2.5), we consider, for the level-idiosyncratic component of $\mathbf{Y}_n$, the volatility proxy

$$w_{it} := \log(v_{it}^2), \quad i \in \mathbb{N}, \quad t \in \mathbb{Z}. \tag{2.9}$$

The w_{it}'s constitute a double-indexed process $\mathbf{w}$, hence, for any finite n and T, an $n \times T$ panel of level-idiosyncratic volatility proxies; the notation $\mathbf{w}_n := \{\mathbf{w}_{n,t} = (w_{1t}, w_{2t}, \dots, w_{nt})'|\, t \in \mathbb{Z}\}$ will be used for the n-dimensional subprocess of $\mathbf{w}$. In order to analyze also this panel by means of general dynamic factor models, we have to assume the existence of spectral densities.

ASSUMPTION (C1). *The second-order moments* $\mathsf{E}[w_{it}^2]$ *exist for all* $i \in \mathbb{N}$*, and For all* $n \in \mathbb{N}$*, the spectral density of* $\mathbf{w}_n$ *is absolutely continuous with respect to the Lebesgue measure on* $[-\pi, \pi]$*, that is,* $\mathbf{w}_n$ *has a spectral density matrix,* $\mathbf{\Sigma}_{\mathbf{w};n}(\theta)$ *for* $\theta \in [-\pi, \pi]$.

Finally, we make the following assumption on the dynamic eigenvalues of those level-idiosyncratic volatility proxies.

ASSUMPTION (C2). *There exists a* $q_w \in \mathbb{N}$ *such that the* q_w*th eigenvalue* $\lambda_{\mathbf{w};n,q_w}(\theta)$ *of* $\mathbf{\Sigma}_{\mathbf{w};n}(\theta)$ *diverges as* $n \to \infty$*,* θ*-a.e. in* $[-\pi, \pi]$*, while the* $(q_w + 1)$*th one,* $\lambda_{\mathbf{w};n,q_w+1}(\theta)$*, is* θ*-a.e. bounded.*

As for the case of level-common volatilities, the existence of a finite q_w, as shown by Hallin and Lippi (2013), is a very natural assumption, and is justified empirically in Section 5. As a consequence, we have a dynamic factor model representation for $\mathbf{w}_n$ with q_w factors, of the form

$$\mathring{w}_{it} = \chi_{\mathbf{w};it} + \xi_{\mathbf{w};it} = \sum_{k=1}^{q_w} d_{\mathbf{w};ik}(L)\varepsilon_{\mathbf{w};kt} + \xi_{\mathbf{w};it}, \quad i \in \mathbb{N}, \quad t \in \mathbb{Z}, \tag{2.10}$$

with common and idiosyncratic components $\chi_{\mathbf{w};it}$ and $\xi_{\mathbf{w};it}$, respectively, or, in obvious vector notation,

$$\mathring{\mathbf{w}}_{n,t} = \boldsymbol{\chi}_{\mathbf{w};n,t} + \boldsymbol{\xi}_{\mathbf{w};n,t}, \quad n \in \mathbb{N}, \quad t \in \mathbb{Z}, \tag{2.11}$$

such that (*i*)-(*v*) hold.

2.4 A block structure for volatilities

Limiting the analysis to one of the two factor model decompositions (2.6) and (2.10), however, means throwing away a lot of information about market volatility shocks. Two separate analyses, on the other hand, are, in general, not adequate: indeed, while $\boldsymbol{\xi}_{\mathbf{w};n}$, by definition, is orthogonal to $\boldsymbol{\chi}_{\mathbf{w};n}$, it is not orthogonal, in general, to $\boldsymbol{\chi}_{\mathbf{s};n}$; nor is $\boldsymbol{\xi}_{\mathbf{s};n}$ orthogonal to $\boldsymbol{\chi}_{\mathbf{w};n}$. Both $\boldsymbol{\xi}_{\mathbf{s};n}$ and $\boldsymbol{\xi}_{\mathbf{w};n}$ thus may yield a market-driven component (in the terminology below, a *weakly idiosyncratic component*). A joint analysis of (2.6) and (2.10) is thus in order, leading to a *two-block general dynamic factor* analysis of the type studied in Hallin and Liška (2011).

When put together, the two n-dimensional panels $\{\mathring{s}_{it}\}$ and $\{\mathring{w}_{it}\}$ indeed constitute the two subpanels or *blocks* of a $2n$-dimensional panel of level-common and level-idiosyncratic volatility proxies with *block structure*. Consider the joint process $\boldsymbol{\eta} := \{\eta_{it}|\, i \in \mathbb{N},\, t \in \mathbb{Z}\}$, where η_{it} is either $\mathring{s}_{jt}$ or $\mathring{w}_{jt}$ for some $j \in \mathbb{N}$. It follows from Lemma 1 in Hallin and Liška (2011) that, given the sets of Assumptions B and C, there exists $Q \in \mathbb{N}$, with $\max(q_s, q_w) \le Q \le q_s + q_w$, such that the Qth eigenvalue, $\lambda_{\boldsymbol{\eta};n,Q}(\theta)$, of the spectral density matrix, $\boldsymbol{\Sigma}_{\boldsymbol{\eta};n}(\theta)$, diverges ($\theta$-a.e. in $[-\pi, \pi]$) as $n \to \infty$, while the $(Q+1)$th one, $\lambda_{\boldsymbol{\eta};n,Q+1}(\theta)$, is θ-a.e. bounded. Therefore, $\boldsymbol{\eta}$ also admits a dynamic factor representation with Q factors, of the form

$$\eta_{it} = \begin{cases} \mathring{s}_{it} = \chi^{\mathbf{s}}_{\boldsymbol{\eta};it} + \xi^{\mathbf{s}}_{\boldsymbol{\eta};it} = \sum_{k=1}^{Q} d^{\mathbf{s}}_{\boldsymbol{\eta};ik}(L)\varepsilon_{kt} + \xi^{\mathbf{s}}_{\boldsymbol{\eta};it}, & i \in \mathbb{N}, \quad t \in \mathbb{Z} \\ \mathring{w}_{it} = \chi^{\mathbf{w}}_{\boldsymbol{\eta};it} + \xi^{\mathbf{w}}_{\boldsymbol{\eta};it} = \sum_{k=1}^{Q} d^{\mathbf{w}}_{\boldsymbol{\eta};ik}(L)\varepsilon_{kt} + \xi^{\mathbf{w}}_{\boldsymbol{\eta};it}, & i \in \mathbb{N}, \quad t \in \mathbb{Z}, \end{cases} \tag{2.12}$$

such that (*i*) to (*v*) hold for $\boldsymbol{\varepsilon}$, $\boldsymbol{\xi}^{\mathbf{s}}_{\boldsymbol{\eta};n}$, $\boldsymbol{\xi}^{\mathbf{w}}_{\boldsymbol{\eta};n}$, $d^{\mathbf{s}}_{\boldsymbol{\eta};ik}(L)$ and $d^{\mathbf{w}}_{\boldsymbol{\eta};ik}(L)$.

Combining (2.6), (2.10), and (2.12) yields

$$\begin{aligned} \mathring{s}_{it} &= \overbrace{\underbrace{\phi_{\mathbf{s};it} + \psi_{\mathbf{s};it}}_{\chi_{\mathbf{s};it}} + \zeta_{\mathbf{s};it}}^{\chi^{\mathbf{s}}_{\boldsymbol{\eta};it}} \!\!\!\!\!\!\!\!\underbrace{\phantom{\zeta_{\mathbf{s};it}} + \xi^{\mathbf{s}}_{\boldsymbol{\eta};it}}_{\xi_{\mathbf{s};it}}, \quad i \in \mathbb{N}, \quad t \in \mathbb{Z}, \\ \mathring{w}_{it} &= \overbrace{\underbrace{\phi_{\mathbf{w};it} + \psi_{\mathbf{w};it}}_{\chi_{\mathbf{w};it}} + \zeta_{\mathbf{w};it}}^{\chi^{\mathbf{w}}_{\boldsymbol{\eta};it}} \!\!\!\!\!\!\!\!\underbrace{\phantom{\zeta_{\mathbf{w};it}} + \xi^{\mathbf{w}}_{\boldsymbol{\eta};it}}_{\xi_{\mathbf{w};it}}, \quad i \in \mathbb{N}, \quad t \in \mathbb{Z}. \end{aligned} \tag{2.13}$$

The $\phi_{\mathbf{s};it}$ and $\phi_{\mathbf{w};it}$ components are called *strongly common*, as they are driven by shocks which are common both to the volatilities of the level-common and the volatilities of the level-idiosyncratic components. The components $\psi_{\mathbf{s};it}$ and $\psi_{\mathbf{w};it}$ are called *weakly common*; they are indeed common either to the $\mathbf{s}$ or to the $\mathbf{w}$ block, but not to both. Being idiosyncratic to one block but not to the other, $\zeta_{\mathbf{s};it}$ and $\zeta_{\mathbf{w};it}$ are called *weakly idiosyncratic*. Finally, $\xi^{\mathbf{s}}_{\boldsymbol{\eta};it}$ and $\xi^{\mathbf{w}}_{\boldsymbol{\eta};it}$ are called *strongly idiosyncratic*. We refer to Hallin and Liška (2011) for details.

All those components, except for the strongly idiosyncratic ones, are market-driven, i.e. they are driven by the *market volatility shocks* $\boldsymbol{\varepsilon} := \{\boldsymbol{\varepsilon}_t = (\varepsilon_{1t}, \dots, \varepsilon_{Qt})' |\, t \in \mathbb{Z}\}$. The decompositions (2.13), thus provide an insight into the impacts of the shocks and the way they are loaded through the ϕ_{it}'s, the ψ_{it}'s, and the ζ_{it}'s. Those various loadings could be obtained from an application of the Forni et al. (2015) one-sided method (as described in *(i)-(vi)* of Section 3.1) to the $6n$-dimensional panel consisting of all ϕ_{it}'s, ψ_{it}'s, and ζ_{it}'s. If the objective is limited to the estimation of market volatility shocks and the way they are loaded by level-common and level-idiosyncratic returns, the same analysis can be limited to the $2n$-dimensional panel of the $\mathring{s}_{it}$'s and $\mathring{w}_{it}$'s, yielding an estimation of the decomposition (2.12). Indeed, since $\phi_{\mathbf{s};it}$, $\psi_{\mathbf{s};it}$ and $\zeta_{\mathbf{s};it}$ are mutually orthogonal, the loadings for $\mathring{s}_{it}$ are the sum of the loadings for those three components; the same holds, of course, for $\mathring{w}_{it}$.

When, however, $Q = q_s = q_w$, implying that all market volatility shocks are common to both the level-common and level-idiosyncratic blocks, the weakly common and weakly idiosyncratic components are vanishing. Hence, (2.13) reduces to

$$\begin{cases} \mathring{s}_{it} &= \phi_{\mathbf{s};it} + \xi^{\mathbf{s}}_{\boldsymbol{\eta};it} = \chi^{\mathbf{s}}_{\boldsymbol{\eta};it} + \xi^{\mathbf{s}}_{\boldsymbol{\eta};it}, \quad i \in \mathbb{N}, \quad t \in \mathbb{Z}, \\ \mathring{w}_{it} &= \phi_{\mathbf{w};it} + \xi^{\mathbf{w}}_{\boldsymbol{\eta};it} = \chi^{\mathbf{w}}_{\boldsymbol{\eta};it} + \xi^{\mathbf{w}}_{\boldsymbol{\eta};it}, \quad i \in \mathbb{N}, \quad t \in \mathbb{Z}, \end{cases} \tag{2.14}$$

where $(\phi_{\mathbf{s};it}, \phi_{\mathbf{w};it})$ and $(\xi^{\mathbf{s}}_{\boldsymbol{\eta};it}, \xi^{\mathbf{w}}_{\boldsymbol{\eta};it})$ now are mutually orthogonal at all leads and lags,[3] and the above-mentioned $6n$-dimensional panel reduces to a $2n$-dimensional one. The analysis then can be conducted along the same ways as for the Y_{it}'s in Section 3.1, applying the Forni et al. (2015) methodology to the $2n$-dimensional panel (2.14). This is what we are doing in Section 3.2 below, where $Q = q_s = q_w = 1$. A reconstruction of the market volatility shocks $\boldsymbol{\varepsilon}$ follows, which involves only one-sided filters (i.e., based on present and past observables only), along with an estimation of their loadings by the level-common volatility proxies s_{it} and the level-idiosyncratic ones w_{it}, respectively. Details are provided in Section 3.

The above factor decompositions correspond to a multiplicative factor model for the squared innovations of the level-common and level-idiosyncratic components. Thus, from (2.14) we have

$$e^2_{it} = \exp\left(\phi_{\mathbf{s};it} + \xi^{\mathbf{s}}_{\boldsymbol{\eta};it} + \mathsf{E}[s_{it}]\right), \quad i \in \mathbb{N}, \quad t \in \mathbb{Z}, \tag{2.15}$$

and

$$v^2_{it} = \exp\left(\phi_{\mathbf{w};it} + \xi^{\mathbf{w}}_{\boldsymbol{\eta};it} + \mathsf{E}[w_{it}]\right), \quad i \in \mathbb{N}, \quad t \in \mathbb{Z}, \tag{2.16}$$

respectively.

3 Estimation

The main objects of interest here are the market volatility shocks and the way they are loaded, via the corresponding proxies, by the level-common and level-idiosyncratic components of returns. Their estimation, based on a finite $n \times T$ panel of Y_{it}'s, proceeds in two steps, which we now describe. A superscript T is used for estimated quantities, as opposed to population ones.

3.1 Step 1: estimating the level-common and level-idiosyncratic shocks

Estimation of the level-common and level-idiosyncratic innovations is in seven steps.

(*i*) Start with a consistent estimator $\boldsymbol{\Sigma}^T_{\mathbf{Y};n}(\theta)$ of the spectral density matrix of the returns. Use the Hallin and Liška (2007) information criterion to select the number q^T of level-common shocks, and compute the eigenvectors $\mathbf{p}^T_{\mathbf{Y};n,1}(\theta), \ldots, \mathbf{p}^T_{\mathbf{Y};n,q^T}(\theta)$ corresponding to $\boldsymbol{\Sigma}^T_{\mathbf{Y};n}(\theta)$'s q^T largest dynamic eigenvalues $\lambda^T_{\mathbf{Y};n,1}(\theta), \ldots, \lambda^T_{\mathbf{Y};n,q^T}(\theta)$.

[3] Unlike, for instance, $(\psi_{\mathbf{s};it}, \psi_{\mathbf{w};it})$ and $(\zeta_{\mathbf{s};it}, \zeta_{\mathbf{w};it})$, in case $Q = q_s = q_w$ does not hold.

(ii) Decompose the spectral density matrix $\mathbf{\Sigma}^T_{\mathbf{Y};n}(\theta)$ into the contributions

$$\sum_{k=1}^{q^T} \lambda^T_{\mathbf{Y};n,k}(\theta)\mathbf{p}^T_{\mathbf{Y};n,k}(\theta)\mathbf{p}^{T*}_{\mathbf{Y};n,k}(\theta) =: \mathbf{\Sigma}^T_{\mathbf{X};n}(\theta)$$

of those q^T largest eigenvalues and its complement

$$\mathbf{\Sigma}^T_{\mathbf{Y};n}(\theta) - \mathbf{\Sigma}^T_{\mathbf{X};n}(\theta) =: \mathbf{\Sigma}^T_{\mathbf{Z};n}(\theta)$$

($\mathbf{p}^*$ stands for the transposed complex conjugate of $\mathbf{p}$). In line with the notation, $\mathbf{\Sigma}^T_{\mathbf{X};n}(\theta)$ and $\mathbf{\Sigma}^T_{\mathbf{Z};n}(\theta)$ are our estimates for the spectral density matrices of the common component process $\mathbf{X}_n$ and the idiosyncratic one $\mathbf{Z}_n$, respectively.

(iii) By classical inverse Fourier transform of $\mathbf{\Sigma}^T_{\mathbf{X};n}(\theta)$, estimate the autocovariances $\mathbf{\Gamma}^T_{\mathbf{X};n,k}$, $k \in \mathbb{Z}$ of the level-common components.

(iv) Assuming, for simplicity, that $n = m(q+1)$ for some $m \in \mathbb{N}$, consider the m $(q^T+1) \times (q^T+1)$ diagonal blocks of the $\mathbf{\Gamma}^T_{\mathbf{X};n,k}$'s. From each of them, estimate (via standard AIC or BIC methods) the order, and, via a Yule-Walker method, the coefficients, of a (q^T+1)-dimensional VAR model. This yields, for the ith diagonal blocks, an estimator $\mathbf{A}^{(i)T}(L)$ of the autoregressive filter $\mathbf{A}^{(i)}(L)$ appearing in (2.4), hence an estimator of the block-diagonal operator in (2.3), which we denote by $\mathbf{A}^T_n(L)$. Let $\widetilde{\mathbf{Y}}^T_n := \left(\mathbf{I}_n - \mathbf{A}^T_n(L)\right)\mathbf{Y}_n$.

(v) An estimator $\mathbf{\Gamma}^T_{\tilde{\mathbf{Y}};n,0}$ of the covariance matrix $\mathbf{\Gamma}_{\tilde{\mathbf{Y}};n,0}$ of $\widetilde{\mathbf{Y}}_n := (\mathbf{I}_n - \mathbf{A}_n(L))\mathbf{Y}_n$ can be obtained either in the time domain as

$$\mathbf{\Gamma}^T_{\tilde{\mathbf{Y}};n,0} = \frac{1}{T}\sum_{t=1}^{T} \widetilde{\mathbf{Y}}^T_n \widetilde{\mathbf{Y}}^{T\prime}_n,$$

or in the frequency domain as

$$\mathbf{\Gamma}^T_{\tilde{\mathbf{Y}};n,0} = \frac{1}{H}\sum_{h=1}^{H} \left(\mathbf{I}_n - \mathbf{A}^T_n(e^{-i\theta_h})\right)\mathbf{\Sigma}^T_{\mathbf{Y};n}(\theta_h)\left(\mathbf{I}_n - \mathbf{A}^T_n(e^{i\theta_h})\right)',$$

where $\theta_h = 2h\pi/H$.

(vi) Projecting the $\widetilde{Y}^T_{it}$'s onto their q^T largest static principal components (computed from the eigenvectors of $\mathbf{\Gamma}^T_{\tilde{\mathbf{Y}};n,0}$) provides an estimate $\mathbf{e}^T_n = \mathbf{H}^T_n\mathbf{u}^T$ of the level-common innovation process $\mathbf{e}_n$.[4]

(vii) The estimator of the idiosyncratic component $\widetilde{\mathbf{Z}}_n$ is then $\widetilde{\mathbf{Z}}^T_n := \left(\mathbf{I}_n - \mathbf{A}^T_n(L)\right)\mathbf{Y}^T_n - \mathbf{e}^T_n$. Fitting a univariate AR model (the order of which, again, is identified via standard AIC or BIC methods) to each of the n components of $\widetilde{\mathbf{Z}}^T_n$, denote by $\mathbf{v}^T_n$ the resulting $n \times 1$ vector of residuals.

The results of Forni et al. (2015) establish the consistency, as $n, T \to \infty$, of all those estimators.

[4]If, furthermore, the identification constraint $\mathbf{H}'_n\mathbf{H}_n = \mathbf{I}_q$ is imposed, estimators $\mathbf{H}^T_n$ of the loadings $\mathbf{H}_n$ can be disentangled from those, $\mathbf{u}^T_n$, of the shocks $\mathbf{u}_n$ by enforcing $\mathbf{H}^{T\prime}_n\mathbf{H}^T_n = \mathbf{I}_{q^T}$.

3.2 Step 2: estimating the market volatility shocks

The estimated innovations $\mathbf{e}_n^T$ and $\mathbf{v}_n^T$ obtained in Step 1 *(vi)-(vii)* are the starting point of the block-factor analysis of Step 2, which leads to the estimation of the market volatility shocks.

(*viii*) From the components of $\mathbf{e}_n^T$ and $\mathbf{v}_n^T$, compute the (estimated) volatility proxies $\mathbf{s}_n^T$ and $\mathbf{w}_n^T$ as in (2.5) and (2.9).

(*ix*) Apply the Hallin and Liška (2007) method to identify the number of factors in the two subpanels and the pooled panel; this yields Q^T, q_s^T and q_w^T, respectively. As already mentioned, we obtain, for the S&P100 dataset in Section 5, $Q^T = q_s^T = q_w^T = 1$, and we suspect this is a general feature of financial data; the method described in steps (x) below applies to this case.

(*x*) Repeat steps *(i)-(vi)* of Section 3.1, on the $2n$-dimensional joint panel of centered volatility measures $\mathring{s}_{it}$ and $\mathring{w}_{it}$, with (using obvious notation) an estimator

$$\boldsymbol{\Sigma}_{\boldsymbol{\eta};n}^T(\theta) := \begin{pmatrix} \boldsymbol{\Sigma}_{\mathbf{s};n}^T(\theta) & \boldsymbol{\Sigma}_{\mathbf{sw};n}^T(\theta) \\ \boldsymbol{\Sigma}_{\mathbf{ws};n}^T(\theta) & \boldsymbol{\Sigma}_{\mathbf{w};n}^T(\theta) \end{pmatrix}$$

of their joint spectral density matrix $\boldsymbol{\Sigma}_{\boldsymbol{\eta};n}(\theta)$.[5] Step *(iv)* produces a $2n$-dimensional block-diagonal VAR operator (with n two-dimensional diagonal blocks) of the form $(\mathbf{I}_{2n} - \mathbf{B}_{2n;\boldsymbol{\eta}}^T(L))$. Step *(vi)* eventually yields estimated innovations

$$\begin{pmatrix} \mathbf{H}_{\mathbf{s};n}^T \\ \mathbf{H}_{\mathbf{w};n}^T \end{pmatrix} \varepsilon_t^T, \qquad t = 1, \dots, T, \tag{3.1}$$

hence transfer or impulse-response functions of the form

$$\mathbf{C}_{2n}^T(L) := (\mathbf{I}_{2n} - \mathbf{B}_{2n;\boldsymbol{\eta}}^T(L))^{-1} \begin{pmatrix} \mathbf{H}_{\mathbf{s};n}^T \\ \mathbf{H}_{\mathbf{w};n}^T \end{pmatrix}, \tag{3.2}$$

where $\mathbf{H}_{\mathbf{s};n}^T$ and $\mathbf{H}_{\mathbf{w};n}^T$ are $n \times 1$, while ε_t^T is scalar; $\mathbf{C}_{2n}^T(L)$, typically, is a $2n \times 1$ vector of one-sided filters describing the dynamic loading, by the volatility proxies s_{it}^T and w_{it}^T, of the market volatility shocks.

The estimated shocks ε^T in (3.1) still are not fully identified. Their scale can be fixed by enforcing the identification constraint $\mathbf{H}_{\mathbf{s};n}^{T\prime}\mathbf{H}_{\mathbf{s};n}^T + \mathbf{H}_{\mathbf{w};n}^{T\prime}\mathbf{H}_{\mathbf{w};n}^T = \mathbf{I}_{Q^T}$ (here, 1), and their sign (as well as that of $\mathbf{H}_{\mathbf{s};n}^T$ and $\mathbf{H}_{\mathbf{w};n}^T$) can be chosen so that the empirical covariance $(nT)^{-1}\sum_{t=1}^T \varepsilon^T \sum_{i=1}^n (s_{it}^T + w_{it}^T)$ be positive.

This estimator fully exploits the fact that the market shock is present both in $\mathbf{e}_n^T$ and $\mathbf{v}_n^T$. Combining the loadings $\mathbf{H}_{\mathbf{s};n}^T$ and $\mathbf{H}_{\mathbf{w};n}^T$ with the inverse[6] of the VAR operator $(\mathbf{I}_{2n} - \mathbf{B}_{2n;\boldsymbol{\eta}}^T(L))$, one easily computes, for any given stock i, the volatility impulse-response functions of the market-driven components of the level-common proxies $\mathring{s}_{it}$, and that of the idiosyncratic proxies $\mathring{w}_{it}$, respectively.

As already mentioned, the consistency, as $n, T \to \infty$, of all estimators derived in this section is established in Hallin and Liška (2011) and Forni et al. (2015) in case they are computed from observed data. Here, however, they are based on the estimated volatility proxies $\mathbf{s}_n^T$ and $\mathbf{w}_n^T$ obtained in Section 3.1. A formal consistency proof thus is needed, which, with consistency rates and results on volatility forecasting, is the subject of a companion paper.

[5] Note that step *(vii)* here is not required.

[6] Due to block-diagonality, computing $(\mathbf{I}_{2n} - \mathbf{B}_{2n;\boldsymbol{\eta}}^T(L))^{-1}$ only requires the inversion of $(Q^T + 1)$-dimensional VARs, that is, for $Q^T = 1$, the inversion of bivariate autoregressive operators.

4 Monte Carlo simulation study

In order to address the finite-sample properties of the estimation method described in the previous section, we conduct here a small Monte Carlo study. We simulate a model with $n = 100$ time series, representing stock returns, and $T = 1000$ according to the description of Section 2. The choice of the parameters range reflects the empirical findings of Section 5.

In particular, following the model in (2.14), we generate two blocks of volatilities of size n, driven by a single strongly common shock ($Q = q_s = q_w = 1$) as

$$\begin{aligned} \mathring{s}_{it} &= \phi_{\mathbf{s};it} + \xi^{\mathbf{s}}_{\boldsymbol{\eta};it} = \frac{d^{\mathbf{s}}_i}{1-\alpha L}\varepsilon_t + \frac{\theta}{1-a^{\mathbf{s}}_i L}\nu^{\mathbf{s}}_{it}, \\ \mathring{w}_{it} &= \phi_{\mathbf{w};it} + \xi^{\mathbf{w}}_{\boldsymbol{\eta};it} = \frac{d^{\mathbf{w}}_i}{1-\beta_i L}\varepsilon_t + \frac{1}{1-a^{\mathbf{w}}_i L}\nu^{\mathbf{w}}_{it}, \end{aligned}$$

where $\alpha \sim U[.8, .9]$, $\beta_i \overset{i.i.d.}{\sim} U[.5, .9]$, $d^{\mathbf{s}}_i, d^{\mathbf{w}}_{\boldsymbol{\eta};i} \overset{i.i.d.}{\sim} U[.5, 1.5]$, and $a^{\mathbf{s}}_i, a^{\mathbf{w}}_i \overset{i.i.d.}{\sim} U[-.9, .9]$. The innovations ε_t, $\nu^{\mathbf{s}}_{it}$, and $\nu^{\mathbf{w}}_{it}$ are Gaussian white noise processes with zero mean and unit variance. Moreover, the idiosyncratic innovations have cross-covariances given by matrices $\boldsymbol{\Gamma}_{\boldsymbol{\nu}^{\mathbf{s}};n,0}$ and $\boldsymbol{\Gamma}_{\boldsymbol{\nu}^{\mathbf{w}};n,0}$ with ones on the main diagonal and entries on the kth diagonal equal to 0.2^k and 0.3^k respectively, for $k = 1, \dots, 10$. No cross-sectional dependence is imposed at other than contemporaneously.

Since the simulated series have a dimension of log-volatilities, the above parametrization implies that the common components of level-common volatility are driven by a single factor following a stochastic volatility process with parameter α. Finally, we set $\theta = .5$. These choices reflect the high collinearity of the level-common innovation panel as assumed from the factor structure in the levels and as also suggested by the S&P500 panel analyzed below. We allow for more heterogeneity in the common components of the level-idiosyncratic volatilities.

We then create the level-common and level-idiosyncratic innovation panels as follows:

$$\begin{aligned} e_{it} &= \left(\exp\left(\phi_{\mathbf{s};it} + \xi^{\mathbf{s}}_{\boldsymbol{\eta};it}\right)\right)^{1/2} p^{\mathbf{s}}_i, \\ v_{it} &= \left(\exp\left(\phi_{\mathbf{w};it} + \xi^{\mathbf{w}}_{\boldsymbol{\eta};it}\right)\right)^{1/2} p^{\mathbf{w}}_i, \end{aligned}$$

where $p^{\mathbf{s}}_i$ and $p^{\mathbf{w}}_i$ are Bernoulli random variables taking values ± 1 with equal probabilities. For small θ, the level-common innovations are then approximately $e_{it} \simeq h_i u_t$, with $u_t = (\exp((1 - \alpha L)^{-1}\varepsilon_t))^{1/2} p^{\mathbf{s}}_i$ and $h_i = (\exp(d^{\mathbf{s}}_i))^{1/2}$ as it is the ith entry of the n-dimensional vector $\mathbf{H}_n$ such that $\mathbf{e}_n \simeq \mathbf{H}_n u_t$. The model for levels is then simulated as

$$Y_{it} = X_{it} + Z_{it} = \frac{h_i}{1-\gamma_i L}u_t + \frac{1}{1-a^{\mathbf{Z}}_i L}\nu^{\mathbf{Z}}_{it},$$

where h_i is defined above, $\gamma_i \overset{i.i.d.}{\sim} U[-.5, .5]$, and $a^{\mathbf{Z}}_i \overset{i.i.d.}{\sim} U[-.9, .9]$. The innovations $\nu^{\mathbf{Z}}_{it}$ are Gaussian white noise processes with zero mean and unit variance and, as before, have cross-covariance given by a matrix $\boldsymbol{\Gamma}_{\boldsymbol{\nu}^{\mathbf{Z}};n,0}$ with ones on the main diagonal and entries on the kth diagonal equal to 0.5^k, for $k = 1, \dots, 10$. No cross-sectional dependence is imposed at other than contemporaneously. Finally, given the non-linear transformations involved, we cannot directly control for the autocorrelation of u_t but from the simulated series we observe that it is almost not correlated thus resembling a white noise process.

Estimation is carried out according as in Section 3.1-3.2, by applying twice the estimation method by Forni et al. (2015).[7] We simulate and estimate the model described above 1000 times. At each

[7]When estimating the level-common innovations $\mathbf{e}_n$ we do not estimate $\mathbf{H}_n$ but we take it as given in order to identify the innovations' vector. Moreover, all centering steps required by the model are taken into account when estimating.

replication, we compute the following standardized mean squared errors (MSE) (see also Forni et al., 2000)

$$
\begin{aligned}
R^2(\mathbf{X}_n^T, \mathbf{X}_n) &= \frac{\sum_{i=1}^n \sum_{t=1}^T (X_{it} - X_{it}^T)^2}{\sum_{i=1}^n \sum_{t=1}^T X_{it}^2}, \\
R^2(\boldsymbol{\phi}_{\mathbf{s};n}^T, \boldsymbol{\phi}_{\mathbf{s};n}) &= \frac{\sum_{i=1}^n \sum_{t=1}^T (\phi_{\mathbf{s};it} - \phi_{\mathbf{s};it}^T)^2}{\sum_{i=1}^n \sum_{t=1}^T \phi_{\mathbf{s};it}^2}, \\
R^2(\boldsymbol{\phi}_{\mathbf{w};n}^T, \boldsymbol{\phi}_{\mathbf{w};n}) &= \frac{\sum_{i=1}^n \sum_{t=1}^T (\phi_{\mathbf{w};it} - \phi_{\mathbf{w};it}^T)^2}{\sum_{i=1}^n \sum_{t=1}^T \phi_{\mathbf{w};it}^2}.
\end{aligned} \tag{4.1}
$$

The first part of Table 1 reports the mean and the 10th and 90th percentiles of the distribution of these measures over the 1000 Monte Carlo simulations. The MSEs in the second step, $R^2(\boldsymbol{\phi}_{\mathbf{s};n}^T, \boldsymbol{\phi}_{\mathbf{s};n})$ and $R^2(\boldsymbol{\phi}_{\mathbf{w};n}^T, \boldsymbol{\phi}_{\mathbf{w};n})$, are comparable to those in the first step, $R^2(\mathbf{X}_n^T, \mathbf{X}_n)$, thus showing that the second step is not too much affected by the first step estimation error.

Lastly, we compare our methodology with a static approach based on principal components as in Stock and Watson (2005) or Forni et al. (2009). In this case, we first have to determine the number of static factors driving both the levels and the volatility panels. Given the dynamic structure of the model a static approach is likely to require more than one factor to explain the same amount of variance as explained by the common shocks u_t and ε_t. Therefore, in the second part of Table 1 we report the MSEs defined in (4.1) for the static approach jointly with the average (over all replications) number of factors chosen by the Alessi et al. (2010) criterion and defined as r for the levels and r_s and r_w for the volatility panels.[8] On average, the dynamic approach is more parsimonious, and it seems to outperform the static one.

TABLE 1: *Monte Carlo simulation results.*

		Levels $R^2(\mathbf{X}_n^T, \mathbf{X}_n)$	Volatility level-common $R^2(\boldsymbol{\phi}_{\mathbf{s};n}^T, \boldsymbol{\phi}_{\mathbf{s};n})$	Volatility level-idiosyncratic $R^2(\boldsymbol{\phi}_{\mathbf{w};n}^T, \boldsymbol{\phi}_{\mathbf{w};n})$
Dynamic	mean	0.1225	0.1383	0.2217
	10th percentile	0.1104	0.0767	0.1886
	90th percentile	0.1352	0.2137	0.2652
num. factors		$q = 1$	$q_s = 1$	$q_w = 1$
Static	mean	0.1756	0.4141	0.3062
	10th percentile	0.1115	0.2382	0.1926
	90th percentile	0.2507	0.5895	0.4730
num. factors		$r = 6.5$	$r_s = 1$	$r_w = 2.5$

[8]Results using Bai and Ng (2002) criterion are similar and not reported.

5 The S&P100 panel

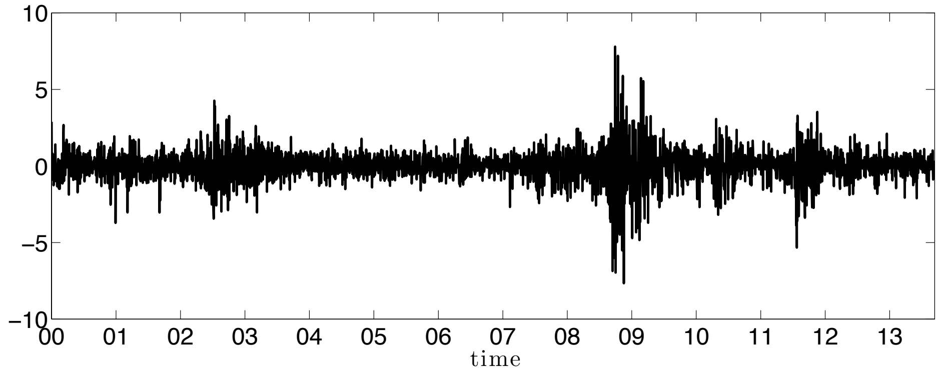
As an application, we consider the panel of stocks used in the construction of the Standard &Poor's 100 (S&P100) index and, based on daily adjusted closing prices, we compute daily log-returns from January 3rd 2000 to September 30th 2013. We have thus an observation period of $T = 3457$ days. Since not all 100 constituents of the index were traded during the observation period, we end up with a panel of $n = 90$ time series.[9]

5.1 Extracting the market volatility shocks

We first run Step 1 of the method to estimate model (2.4) on the centered log-returns Y_{it}. Applying the Hallin and Liška (2007) criterion, we obtain $q^T = 1$, that is, a one-dimensional common shock. Proceeding as described in Section 3.1, we compute the estimated shocks u^T, the estimated residuals $\mathbf{e}_n^T = \mathbf{H}_n^T u^T$, and the the n-dimensional vector of idiosyncratic shocks $\mathbf{v}_n^T$ to be used in Step 2 of the procedure. Autocorrelation in the level-idiosyncratic components apparently is weak, as AIC mostly returns an autoregressive fit of order zero of the $\tilde{Z}_{it}$'s: those components thus can be treated as white noise without the need of further filtering.

Figure 1 shows a plot of the estimated market shock u_t^T on the returns. From that plot, one easily can spot some well-identified periods of high volatility: the 2001-2003 series of crises, related to the dot-com bubble, the Enron (late 2001) and Worldcom (mid-2002) scandals; the 2003 Iraq war; the Great 2008-2009 Financial crisis starting with Lehman Brothers bankruptcy (September 2008); the 2010-2012 euro sovereign bond crisis. The largest shocks over the period, by far, are those related with the 2008-2009 financial crisis.

FIGURE 1: *The market shocks u_t^T on returns, period 2000-2013.*



Still from Step 1, we can quantify the contribution of market shocks to the total variation of returns, as the ratio between the sum of the (empirical) variances of the estimated common components $\mathbf{X}_n^T$ to the sum of the (empirical) variances of the observed returns:

$$R^2_{Y.\text{market}} := \frac{\sum_{i=1}^n \sum_{t=1}^T (X_{it}^T)^2}{\sum_{i=1}^n \sum_{t=1}^T (Y_{it})^2}.$$

Those common components can be obtained at the end of part (vi) of Step 1, as

$$\mathbf{X}_{n,t}^T = (\mathbf{I}_n - \mathbf{A}^T(L))^{-1}\mathbf{H}_n^T u_t^T = (\mathbf{I}_n - \mathbf{A}^T(L))^{-1}\mathbf{e}_{n,t}^T, \qquad t = 1, \dots, T.$$

[9]The dataset is downloadable from Yahoo Finance and a list of the series used is provided in the Appendix.

The same quantity equivalently can be estimated in the spectral domain, as the ratio of (an approximate value of) the integral over all frequencies of $\mathbf{Y}_n$'s first dynamic eigenvalue and (an approximate value of) the integral over all frequencies of the sum of $\mathbf{Y}_n$'s first n dynamic eigenvalue, that is,

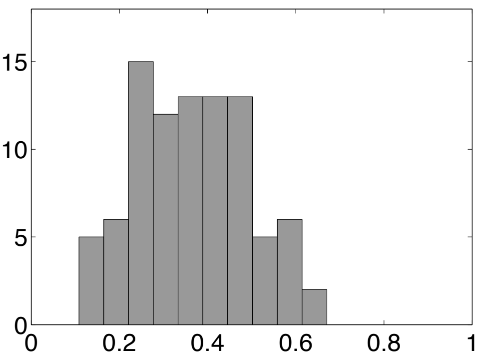
$$R^2_{Y.\text{market}} := \frac{\sum_{h=1}^{H} \lambda^T_{\mathbf{Y};n,1}(2h\pi/H)}{\sum_{h=1}^{H}\sum_{j=1}^{n} \lambda^T_{\mathbf{Y};n,j}(2h\pi/H)}.$$

In both cases, we obtain $R^2_{Y.\text{market}} \approx 0.36$: the market-driven level-common component accounts for about 36% of the total variance of returns. The same quantity can be evaluated for each individual stock, or each time point, by computing

$$R^2_{Y_i.\text{market}} := \frac{\sum_{t=1}^{T}(X^T_{it})^2}{\sum_{t=1}^{T}(Y_{it})^2},\ \ i = 1,\ldots,n, \quad \text{and} \quad R^2_{Y_t.\text{market}} := \frac{\sum_{i=1}^{n}(X^T_{it})^2}{\sum_{i=1}^{n}(Y_{it})^2},\ \ t = 1,\ldots,T,$$

respectively. A histogram of the $R^2_{Y_i.\text{market}}$'s ($i = 1,\ldots,n$) and a plot of the $R^2_{Y_t.\text{market}}$'s ($t = 1,\ldots,T$) are shown in Figures 2 and 3, respectively. Concerning the distribution of explained variances we notice that market shocks explain up to 60% of the total variation of returns, but in general this percentage is lower, pointing towards an important role for idiosyncratic returns. On the other hand, when looking at the series of $R^2_{Y_t.\text{market}}$ values, the periods of crisis are clearly captured by the market shocks to returns.

FIGURE 2: *Distribution of market-driven variances of returns, period 2000-2013.*

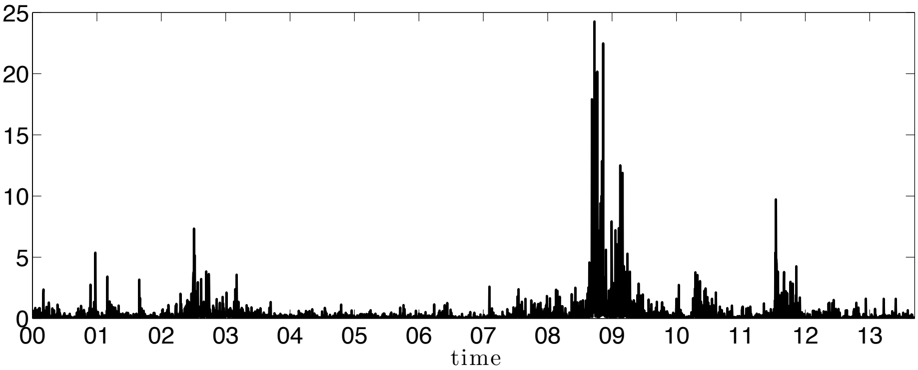


Histogram for the proportions $R^2_{Y_i.\text{market}}$ of variance explained by the market shocks to returns across the panel.

Both this analysis of the impact of market shocks and the fact that $q^T = 1$ are in close agreement with most of the empirical literature on financial returns, and with classical asset pricing theory models like the CAPM or APT, which imply that the unexpected return of risky assets can be expressed as a linear function of a systematic common factor representing the market and an idiosyncratic component.[10] Moreover, the single common shock u^T has a correlation of 0.95 with the total daily return $\sum_{i=1}^{n} Y_{it}$ of the panel, which is consistent with the interpretation of u^T as the market return shock.

[10]Some exceptions in the financial econometrics literature are for example the Nelson and Siegel (1987) level-slope-curvature 3-factor model for yield curves, the Fama and French (1993) 3-factor model for equity returns, and the Fung and Hsieh (2004) 7-factor model for hedge fund returns. However, the "factors" in these models are not always latent dynamic shocks, as in the present paper, but covariables (sometimes observed) explaining the contemporaneous (and not dynamic) variation of the data.

FIGURE 3: *Evolution of market-driven variances of returns, period 2000-2013.*



Time series of the proportion $R^2_{Y_t.\text{market}}$ of variance explained by the market shocks to returns at time t.

Turning to Step 2, as described in Section 3.2, we first compute the volatility proxies $\mathbf{s}_n^T$ and $\mathbf{w}_n^T$ from the logs of squared estimated residuals (equations (2.5) and (2.9)), and center them about their empirical means, obtaining the centered proxies $\mathring{\mathbf{s}}_n^T$ and $\mathring{\mathbf{w}}_n^T$ which constitute the two blocks of the joint panel $\boldsymbol{\eta}_n^T$.

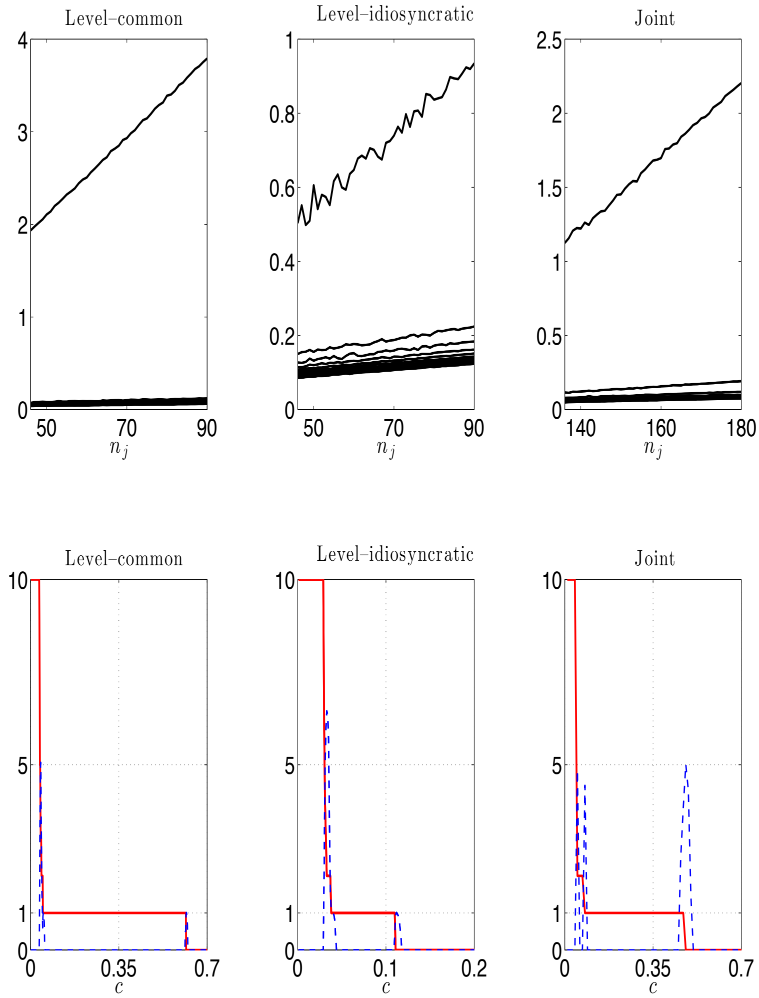
Before estimating a dynamic factor model from the $\boldsymbol{\eta}_n^T$'s, however, it is wise to check for possible departures from stationarity. It is well known that realized volatilities typically exhibit long-memory dependence (see for example Andersen et al., 2003). Therefore, we first estimate an ARFIMA$(1, d, 0)$ on each component of $\mathbf{s}_n^T$ and $\mathbf{w}_n^T$. Following Beran (1995), we obtain estimators d^T for the fractional differencing parameters d which are compatible with the assumption of stationarity. Indeed, the maximum values of these estimators indicate that d^T never reaches 0.25, which is significantly less than 0.50.

A general dynamic factor structure for volatility proxies is justified by looking at the behavior of their dynamic eigenvalues. In particular, in Figure 4 (a)-(c), we show, for subpanels of increasing sizes $n_j \uparrow n = 90$, the dynamic eigenvalues, averaged over frequencies, of the level-common volatility and the level-idiosyncratic volatility panels, and for subpanels of increasing sizes $n_j \uparrow 2n = 180$ for the joint volatility panel. For all three of them, we clearly see one eigenvalue dominating over all others, and diverging faster as n_j increases. This finding is the empirical justification for Assumptions (B2) and (C2) on the factor structure of the two volatility subpanels, and supports the idea that a unique common shock is driving both subpanels. The Hallin-Liška identification method confirms (see Figure 4 (d)-(f)) that fact–recall that the method identifies the number of common shocks as the value shown by the red curve at the second stability interval[11] of the blue one. We thus proceed with estimation and $Q^T = q_s^T = q_w^T = 1$.

The block decomposition (2.13) in this case reduces, in each block, to a sum of two terms: a strongly common component, and a strongly idiosyncratic one. Those strongly common components are estimated by $\boldsymbol{\phi}_{\mathbf{s};n}^T = \boldsymbol{\chi}_{\boldsymbol{\eta};n}^{\mathbf{s}T}$ and $\boldsymbol{\phi}_{\mathbf{w};n}^T = \boldsymbol{\chi}_{\boldsymbol{\eta};n}^{\mathbf{w}T}$, respectively. Since $Q^T = 1$, a single volatility market shock ε_t^T driving both the level-common and level-idiosyncratic volatilities of the S&P100 is identified, in line with the remark made after (3.1). Rather than a plot of ε_t^T itself, Figure 5 is providing a plot of the multiplicative shocks $\exp(\varepsilon_t^T)$, which have the same scale as the squared level-residuals e_{it}^2 and v_{it}^2; that series of course is intrinsically non-negative. It has to be noticed that this

[11]A stability interval is an interval over which the blue curve coincides with the horizontal axis; see Hallin and Liška (2007) for details.

FIGURE 4: *Evidence of factor structure in the volatility proxy panels.*



Top: the ten largest dynamic eigenvalues, averaged over frequencies, computed for panels of increasing sizes: $45 \leq n_j \leq n = 90$ for the level-common and level-idiosyncratic volatility panels, and $135 \leq n_j \leq 2n = 180$ for the joint volatility panel. Bottom: the plots associated with the Hallin-Liška identification method for the same panels.

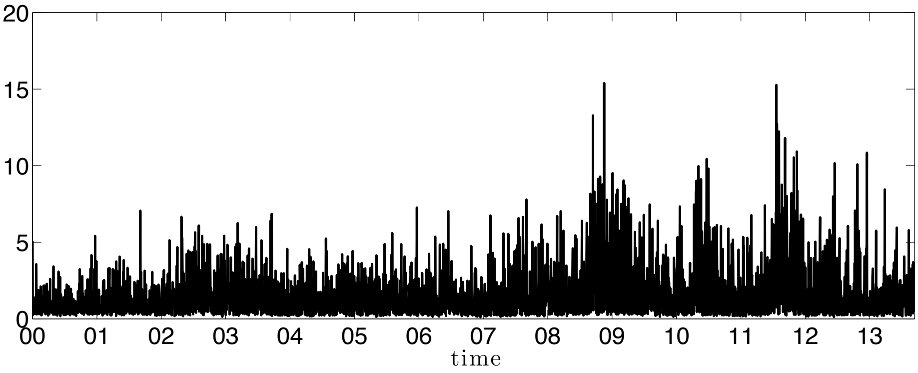
shock does not represent the volatility of the market shock on returns u^T. Compared to the plot of u^T in Figure 1, the market volatility shock ε^T shows periods of high volatility mainly in correspondence of the 2008-2009 and 2010-2012 crises while lower volatility is related to the 2002-2003 crisis.

5.2 Analyzing the volatility shocks and their impact

The method described in the previous sections does not just yield an estimation of the market volatility shocks (as plotted in Figure 5), it also provides insightful information on the way those shocks are loaded by the various stocks in the panel.

The overall contribution of market shocks to the variances of the volatility proxies s_{it} and w_{it} can

FIGURE 5: *The market shock* $\exp(\varepsilon_t^T)$ *on volatilities, period 2000-2013.*



be evaluated by means of the ratios

$$R^2_{s.\text{market}} := \frac{\sum_{t=1}^T \sum_{i=1}^n (\phi^T_{\mathbf{s};it})^2}{\sum_{t=1}^T \sum_{i=1}^n (s^T_{it})^2} \quad \text{and} \quad R^2_{w.\text{market}} := \frac{\sum_{t=1}^T \sum_{i=1}^n (\phi^T_{\mathbf{w};it})^2}{\sum_{t=1}^T \sum_{i=1}^n (s^T_{it})^2}. \tag{5.1}$$

For each individual stock i, a measure of the same impact is

$$R^2_{s_i.\text{market}} := \frac{\sum_{t=1}^T (\phi^T_{\mathbf{s};it})^2}{\sum_{t=1}^T (s^T_{it})^2} \quad \text{and} \quad R^2_{w_i.\text{market}} := \frac{\sum_{t=1}^T (\phi^T_{\mathbf{w};it})^2}{\sum_{t=1}^T (s^T_{it})^2}, \quad i = 1, \dots, n; \tag{5.2}$$

while their evolution through time is captured by

$$R^2_{s_t.\text{market}} := \frac{\sum_{i=1}^n (\phi^T_{\mathbf{s};it})^2}{\sum_{i=1}^n (s^T_{it})^2} \quad \text{and} \quad R^2_{w_t.\text{market}} := \frac{\sum_{i=1}^n (\phi^T_{\mathbf{w};it})^2}{\sum_{i=1}^n (s^T_{it})^2}, \quad t = 1, \dots, T. \tag{5.3}$$

The values of $R^2_{\boldsymbol{s}.\text{market}}$ and $R^2_{\boldsymbol{w}.\text{market}}$ in (5.1) are displayed in Table 2: the market-driven components $\boldsymbol{\phi}^T_{\mathbf{s};n}$ of the volatility of level-common components account for about 60% of the total variance of the $\mathbf{s}^T_n$ panel. The same measure, for the market-driven components $\boldsymbol{\phi}^T_{\mathbf{w};n}$ of the $\mathbf{w}^T_n$ panel, is still about 13%, which is highly non-negligible.

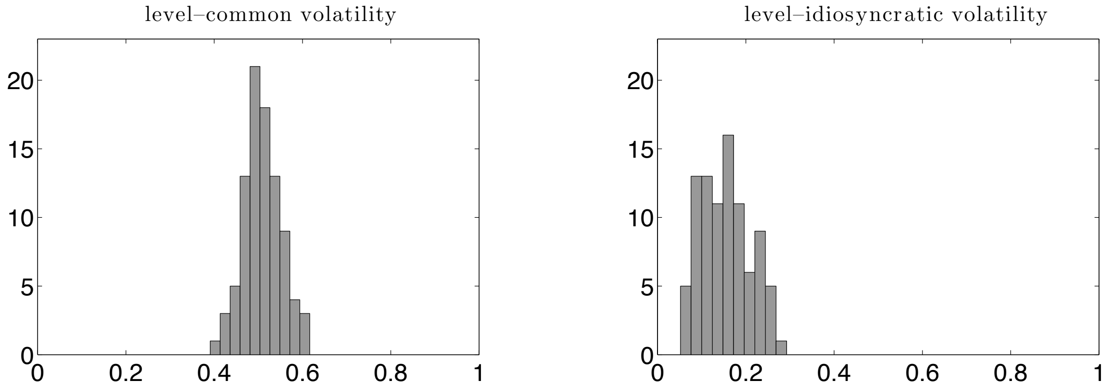
Figures 6 and 7 show histograms of the ratios $R^2_{s_i.\text{market}}$ and $R^2_{w_i.\text{market}}$ in (5.2) and evolution through time of the market impact, i.e., plots of $R^2_{s_t.\text{market}}$ and $R^2_{s_t.\text{market}}$ against time. The distribution of explained variances is quite homogeneous inside each block. As for the evolution through time of proportions of explained variances, most of the contribution of market volatility shocks to level-idiosyncratic volatility is observed during the recent Great Financial crisis (2008-2009); during that period, that contribution is comparable to the one observed for level-common volatility.

The transfer or impulse-response functions (3.2), which describe how market shocks are loaded dynamically by the volatility proxies, is another most informative byproduct of our method. For each stock i, those functions take the form of scalar filters (one for s_{it}, another one for w_{it}), hence a sequence of coefficients associated with the various lags. Those coefficients are shown in Figure 9 for a selection of ten stocks; median, maximum and minimum values are provided in Figure 8. Two findings emerge from inspection of Figures 8 and Figures 9. First, the reaction to market shocks of level-common volatilities (the s_{it}'s) and level-idiosyncratic volatilities (the w_{it}'s) are markedly different. That reaction, for the s_{it}'s, is extremely homogenous across the panel, with a strong loading

TABLE 2: *Explained variances of market volatility proxies, period 2000-2013.*

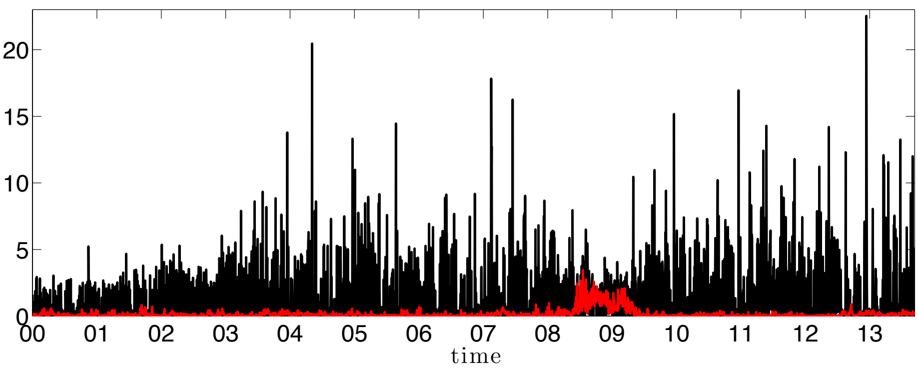
Volatility	Strongly common	Strongly idiosyncratic
Level-common $\mathbf{s}_n^T$	$R^2_{s.\text{market}} = 0.5997$ $\boldsymbol{\phi}^T_{\mathbf{s};n}$	$1 - R^2_{s.\text{market}} = 0.4003$ $\boldsymbol{\xi}^{\mathbf{s}T}_{\boldsymbol{\eta};n}$
Level-idiosyncratic $\mathbf{w}_n^T$	$R^2_{w.\text{market}} = 0.1740$ $\boldsymbol{\phi}^T_{\mathbf{w};n}$	$1 - R^2_{w.\text{market}} = 0.8260$ $\boldsymbol{\xi}^{\mathbf{w}T}_{\boldsymbol{\eta};n}$

FIGURE 6: *Distribution of market-driven variances of volatility proxies, period 2000-2013.*



Histograms for the proportions of variances explained by the market volatility shocks across the panel: $R^2_{s_i.\text{market}}$ (left) and $R^2_{w_i.\text{market}}$ (right).

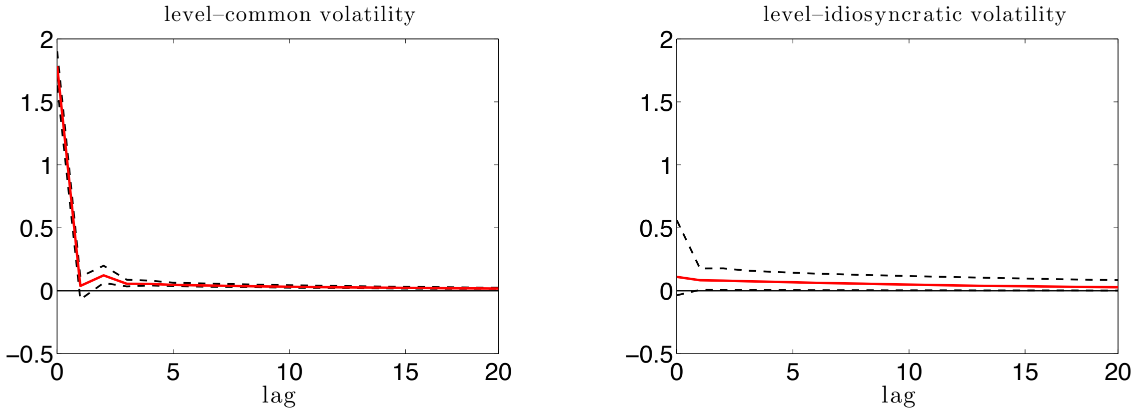
FIGURE 7: *Evolution of market-driven variances of volatility proxies, period 2000-2013.*



Time series of the proportions of variances explained by the market volatility shocks: $R^2_{s_t.\text{market}}$ (black) and $R^2_{w_t.\text{market}}$ (red).

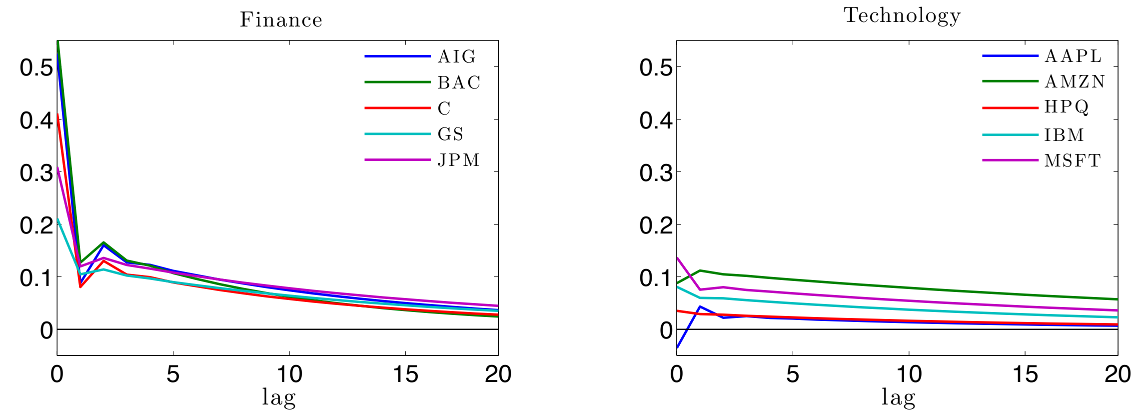
coefficient at lag zero and very short persistence: the coefficients rapidly decrease with the lag, and essentially vanish within one week time (5 lags). Quite on the contrary, the same reaction, for the w_{it}'s, varies considerably across the panel, and is more persistent, sometimes lasting over one month (20 lags). Finally, except for a few level-idiosyncratic volatilities (such as Apple Inc.), the instant impact of a market shock is always positive.

FIGURE 8: *Impulse-response functions for market volatility shocks.*



Median, maximum, and minimum of the distribution, over the 90 stocks in the panel, of impulse-response functions of volatilities to one-standard-deviation market volatility shock, that is, the sequence of loading coefficients divided by the standard error of the shocks, for level-common (left) and level-idiosyncratic (right) volatilities, respectively.

FIGURE 9: *Impulse-response functions for market volatility shocks to level-idiosyncratic volatilities.*



Impulse-response functions of volatilities to one-standard-deviation market volatility shock, that is, the sequence of loading coefficients divided by the standard error of the shocks, for level-idiosyncratic volatilities of selected stocks from the Financial (left) and Technology (right) sectors, respectively; see Appendix for tickers' definitions.

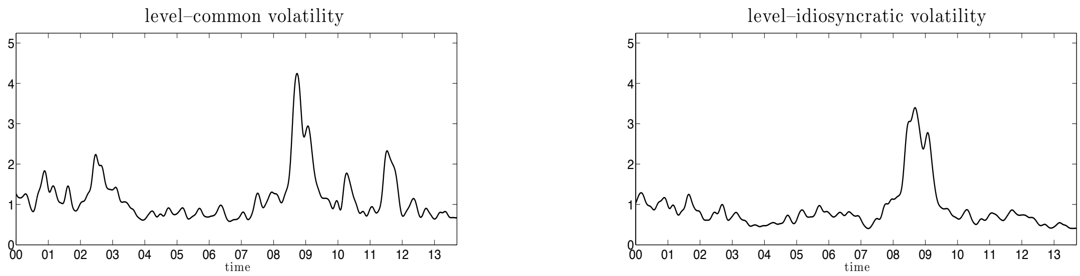
To conclude, we turn to the analysis, for a few selected stocks, of the market-driven volatilities, which, referring to (2.15) and (2.16), we define as

$$\chi^T_{\mathbf{e}^2;it} := \exp(\phi^T_{\mathbf{s};it} + \bar{s}^T_{it}), \quad \chi^T_{\mathbf{v}^2;it} := \exp(\phi^T_{\mathbf{w};it} + \bar{w}^T_{it}), \quad i = 1, \dots, n, \quad t = 1, \dots, T,$$

where $\bar{s}^T_{it}$ and $\bar{w}^T_{it}$ stand for empirical means. In Figure 10, we show kernel-smoothed cross-sectional averages of these quantities, while in Figures 11 and 12 we show the market volatilities (level-common and level-idiosyncratic), for selected stocks from the Financial and Technology sectors respectively, together with their smoothed versions.

The time span under study is known to display at least three periods of high volatility: *(i)* the Great Financial crisis of 2008-2009, *(ii)* the European Sovereign debt crisis of 2011-2012, and *(iii)* a period in the early 2000s including the dot-com bubble, the Enron and Worldcom scandals, and the Second Iraq war. The market volatilities of the level-common and level-idiosyncratic components exhibit somewhat distinct reactions to those events. Level-idiosyncratic volatility is affected mainly by the Great Financial crisis of 2008-2009, the effect of which is particularly significant for the Financial sector. The more recent European Sovereign debt crisis of 2011-2012 is contributing to the market volatility of level-common components only, in agreement with the fact that this crisis represents an external shock to the US market , so that no idiosyncratic return is likely to be seriously affected. Finally, the dot-com bubble of the early 2000s affects the Technology sector through level-common market volatilities only. Overall, these figures confirm the heterogeneity in the contributions of market volatility shocks to level-idiosyncratic volatilities, with a significant impact on the Financial sector but a more limited one on the Technology sector.

FIGURE 10: *Cross-sectional averages of market volatilities.*



The figure shows kernel-smoothed cross-sectional averages of market volatilities. The bandwidth used corresponds to three weeks of trading (15 days).

6 Conclusion

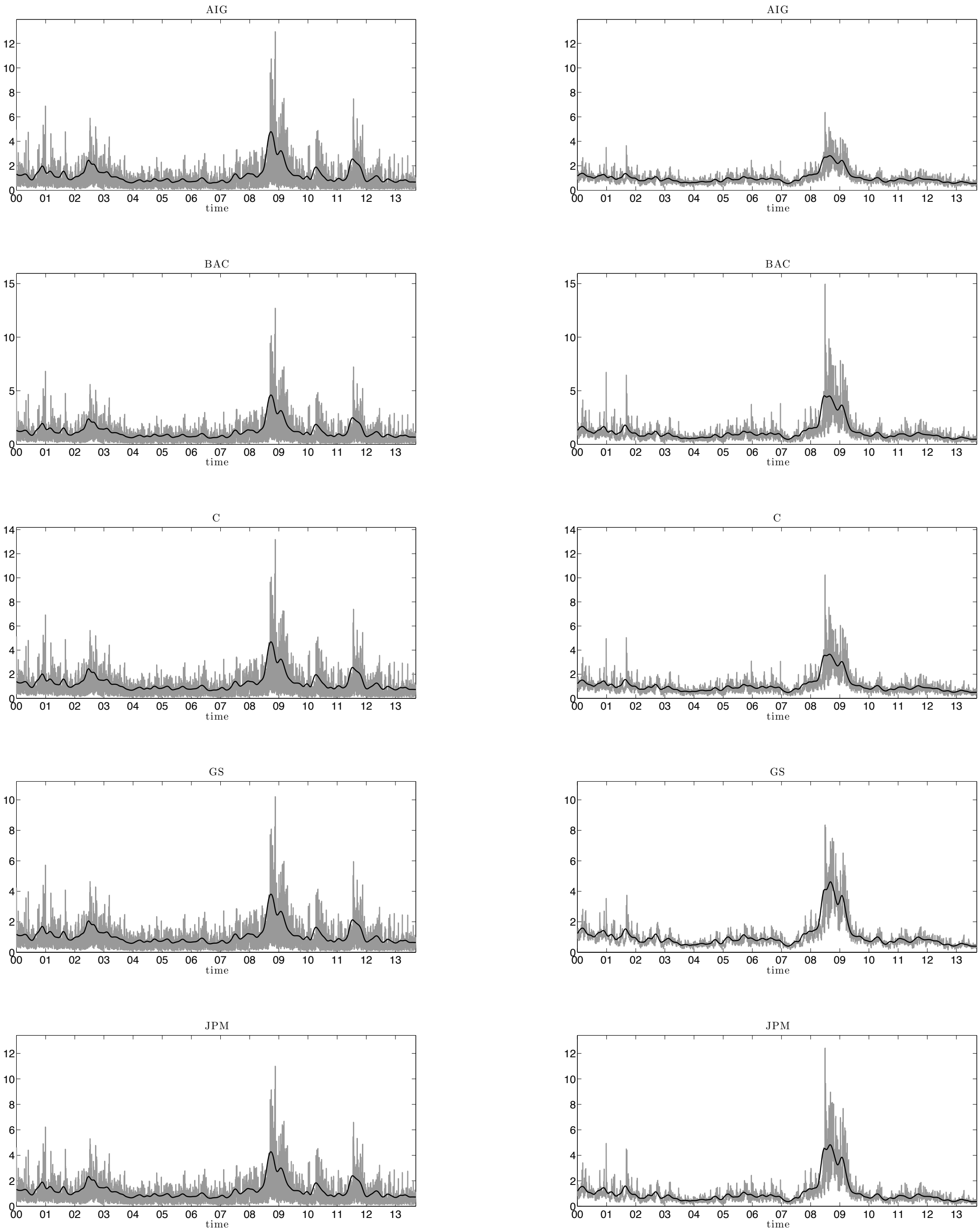
In this paper, we propose a two-step general dynamic factor method for the analysis of financial volatilities in large panels of stock returns. Our focus throughout is on identifying and recovering the market volatility shocks. We show that the decomposition into "common" or "market-driven" and "idiosyncratic" component of the returns does not necessarily coincide with the corresponding decomposition for volatilities, in the sense that level-idiosyncratic components, just as much as the the level-common ones, are affected by market volatility shocks. The empirical study of Section 5 actually suggests that the market shocks are univariate quantities, a finding most practitioners seem to agree with. Whether this is a general feature of financial data is quite plausible, but should be checked against other financial datasets.

A Monte Carlo simulation study assesses the validity of the two-step estimation approach in finite samples. Results from this study also show the superiority in this setting of dynamic factor analysis over static factor models.

The present framework can be extended in many directions of potential interest in financial econometrics and risk management. Our approach is a first step towards a general model-free and nonparametric analysis of covolatilities, with obvious applications in risk management and the optimization of financial portfolios. Moreover, the various variance decompositions and impulse-response

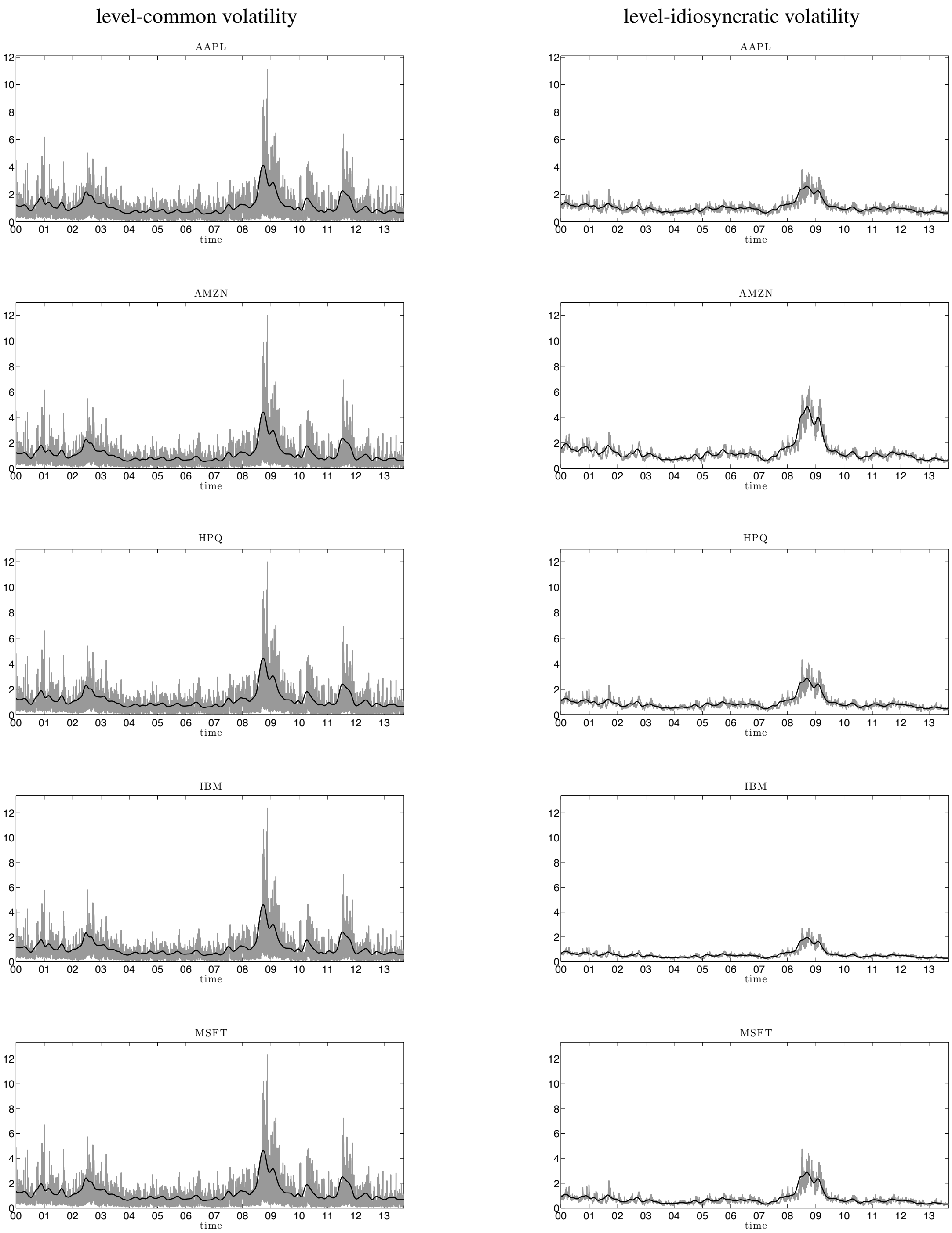
FIGURE 11: *Market volatilities - Financial sector.*

level-common volatility | level-idiosyncratic volatility



Estimated market volatilities for five selected stocks from the Financial sector, along with their smoothed versions (black solid line); see Appendix for tickers' definitions.

FIGURE 12: *Market volatilities - Technology sector.*



The figure shows the estimated market volatilities for five selected stocks from the Technology sector, along with their smoothed versions (black solid line); see Appendix for tickers' definitions.

functions following as by-products of our approach open the way to model-free and non-parametric simultaneous forecasting of large numbers of volatilities.

References

Alessi, L., M. Barigozzi, and M. Capasso (2010). Improved penalization for determining the number of factors in approximate static factor models. *Statistics and Probability Letters 80*, 1806–1813.

Andersen, T. G., T. Bollerslev, F. X. Diebold, and P. Labys (2003). Modeling and forecasting realized volatility. *Econometrica 71*, 579–625.

Anderson, B. D. and M. Deistler (2008). Generalized linear dynamic factor models. A structure theory. *IEE Conference on Decision and Control*.

Asai, M., M. McAleer, and J. Yu (2006). Multivariate stochastic volatility: A review. *Econometric Reviews 25*, 145–175.

Bai, J. and S. Ng (2002). Determining the number of factors in approximate factor models. *Econometrica 70*, 191–221.

Bai, J. and S. Ng (2007). Determining the number of primitive shocks in factor models. *Journal of Business and Economic Statistics 25*, 52–60.

Barigozzi, M., C. T. Brownlees, G. M. Gallo, and D. Veredas (2014). Disentangling systematic and idiosyncratic dynamics in panels of volatility measures. *Journal of Econometrics 182*, 364–384.

Bauwens, L., S. Laurent, and J. V. K. Rombouts (2006). Multivariate GARCH models: A survey. *Journal of the Applied Econometrics 21*, 79–109.

Beran, J. (1995). Maximum likelihood estimation of the differencing parameter for invertible short and long memory autoregressive integrated moving average models. *Journal of the Royal Statistical Society. Series B 57*, 659–672.

Chamberlain, G. and M. Rothschild (1983). Arbitrage, factor structure, and mean–variance analysis on large asset markets. *Econometrica 51*, 1281–304.

Connor, G. and R. A. Korajczyk (1986). Performance measurement with the arbitrage pricing theorem. a new framework for analysis. *Journal of Financial Economics 15*, 373–394.

Connor, G., R. A. Korajczyk, and O. Linton (2006). The common and specific components of dynamic volatility. *Journal of Econometrics 132*, 231–255.

Diebold, F. X. and M. Nerlove (1989). The dynamics of exchange rate volatility: a multivariate latent factor ARCH model. *Journal of Applied Econometrics 4*, 1–21.

Diebold, F. X. and K. Yılmaz (2014). On the network topology of variance decompositions: Measuring the connectedness of financial firms. *Journal of Econometrics 182*, 119–134.

Engle, R. F. (2002). Dynamic conditional correlation: A simple class of multivariate generalized autoregressive conditional heteroskedasticity models. *Journal of Business & Economic Statistics 20*, 339–350.

Engle, R. F. and J. Marcucci (2006). A long–run pure variance common features model for the common volatilities of the Dow Jones. *Journal of Econometrics 132*, 7–42.

Engle, R. F., N. Shephard, and K. Sheppard (2008). Fitting vast dimensional time–varying covariance models. mimeo.

Fama, E. F. and K. R. French (1993). Common risk factors in the returns on stocks and bonds. *Journal of Financial Economics 33*, 3–56.

Fan, J., Y. Liao, and M. Mincheva (2013). Large covariance estimation by thresholding principal orthogonal complements. *Journal of the Royal Statistical Society, Series B 75*, 603–680.

Fan, J., Y. Liao, and X. Shi (2013). Risk of large portfolios. arXiv 1302.0926.

Forni, M., D. Giannone, M. Lippi, and L. Reichlin (2009). Opening the Black Box: Structural Factor Models versus Structural VARs. *Econometric Theory 25*, 1319–1347.

Forni, M., M. Hallin, M. Lippi, and L. Reichlin (2000). The Generalized Dynamic Factor Model: Identification and Estimation. *The Review of Economics and Statistics 82*, 540–554.

Forni, M., M. Hallin, M. Lippi, and P. Zaffaroni (2015). Dynamic factor models with infinite dimensional factor spaces: one–sided representations. *Journal of Econometrics*. in press.

Forni, M. and M. Lippi (2001). The Generalized Dynamic Factor Model: Representation Theory. *Econometric Theory 17*, 1113–1141.

Forni, M. and M. Lippi (2011). The Unrestricted Dynamic Factor Model: One–sided Representation Results. *Journal of Econometrics 163*, 23–28.

Fung, W. and D. A. Hsieh (2004). Hedge fund benchmarks: A risk-based approach. *Financial Analysts Journal*, 65–80.

Ghysels, E. (2014). Factor analysis with large panels of volatility proxies. mimeo.

Hallin, M. and M. Lippi (2013). Factor models in high–dimensional time series. A time-domain approach. *Stochastic Processes and their Applications 123*, 2678–2695.

Hallin, M. and R. Liška (2007). Determining the number of factors in the general dynamic factor model. *Journal of the American Statistical Association 102*, 603–617.

Hallin, M. and R. Liška (2011). Dynamic factors in the presence of blocks. *Journal of Econometrics 163*, 29–41.

Harvey, A., E. Ruiz, and E. Sentana (1992). Unobserved component time series models with arch disturbances. *Journal of Econometrics 52*, 129–157.

Harvey, A., E. Ruiz, and N. Shephard (1994). Multivariate stochastic variance models. *The Review of Economic Studies 61*, 247–264.

Jurado, K., S. C. Ludvigson, and S. Ng (2013). Measuring uncertainty. Technical report, National Bureau of Economic Research.

Luciani, M. and D. Veredas (2014). Estimating and forecasting large panels of volatilities with approximate dynamic factor models. mimeo.

Nelson, C. R. and A. F. Siegel (1987). Parsimonious modeling of yield curves. *Journal of Business*, 473–489.

Ng, V., R. F. Engle, and M. Rothschild (1992). A multi-dynamic-factor model for stock returns. *Journal of Econometrics 52*, 245–266.

Rangel, J. G. and R. F. Engle (2012). The Factor–Spline–GARCH model for high and low frequency correlations. *Journal of Business & Economic Statistics 30*(1), 109–124.

Sentana, E., G. Calzolari, and G. Fiorentini (2008). Indirect estimation of large conditionally heteroskedastic factor models, with an application to the Dow 30 stocks. *Journal of Econometrics 146*, 10–25.

Silvennoinen, A. and T. Teräsvirta (2009). Multivariate garch models. In *Handbook of Financial Time Series*, pp. 201–229. Springer.

Stock, J. H. and M. W. Watson (2005). Implications of dynamic factor models for VAR analysis. Working Paper 11467, NBER.

Van der Weide, R. (2002). GO–GARCH: A multivariate generalized orthogonal GARCH model. *Journal of Applied Econometrics 17*, 549–564.

A Data

TABLE 3: *S&P100 consituents.*

Ticker	Name		
AAPL	Apple Inc.	HPQ	Hewlett Packard Co.
ABT	Abbott Laboratories	IBM	International Business Machines
AEP	American Electric Power Co.	INTC	Intel Corporation
AIG	American International Group Inc.	JNJ	Johnson & Johnson Inc.
ALL	Allstate Corp.	JPM	JP Morgan Chase & Co.
AMGN	Amgen Inc.	KO	The Coca-Cola Company
AMZN	Amazon.com	LLY	Eli Lilly and Company
APA	Apache Corp.	LMT	Lockheed-Martin
APC	Anadarko Petroleum Corp.	LOW	Lowe's
AXP	American Express Inc.	MCD	McDonald's Corp.
BA	Boeing Co.	MDT	Medtronic Inc.
BAC	Bank of America Corp.	MMM	3M Company
BAX	Baxter International Inc.	MO	Altria Group
BK	Bank of New York	MRK	Merck & Co.
BMY	Bristol-Myers Squibb	MS	Morgan Stanley
BRK.B	Berkshire Hathaway	MSFT	Microsoft
C	Citigroup Inc.	NKE	Nike
CAT	Caterpillar Inc.	NOV	National Oilwell Varco
CL	Colgate-Palmolive Co.	NSC	Norfolk Southern Corp.
CMCSA	Comcast Corp.	ORCL	Oracle Corporation
COF	Capital One Financial Corp.	OXY	Occidental Petroleum Corp.
COP	ConocoPhillips	PEP	Pepsico Inc.
COST	Costco	PFE	Pfizer Inc.
CSCO	Cisco Systems	PG	Procter & Gamble Co.
CVS	CVS Caremark	QCOM	Qualcomm Inc.
CVX	Chevron	RTN	Raytheon Co.
DD	DuPont	SBUX	Starbucks Corporation
DELL	Dell	SLB	Schlumberger
DIS	The Walt Disney Company	SO	Southern Company
DOW	Dow Chemical	SPG	Simon Property Group, Inc.
DVN	Devon Energy	T	AT&T Inc.
EBAY	eBay Inc.	TGT	Target Corp.
EMC	EMC Corporation	TWX	Time Warner Inc.
EMR	Emerson Electric Co.	TXN	Texas Instruments
EXC	Exelon	UNH	UnitedHealth Group Inc.
F	Ford Motor	UNP	Union Pacific Corp.
FCX	Freeport-McMoran	UPS	United Parcel Service Inc.
FDX	FedEx	USB	US Bancorp
GD	General Dynamics	UTX	United Technologies Corp.
GE	General Electric Co.	VZ	Verizon Communications Inc.
GILD	Gilead Sciences	WAG	Walgreens
GS	Goldman Sachs	WFC	Wells Fargo
HAL	Halliburton	WMB	Williams Companies
HD	Home Depot	WMT	Wal-Mart
HON	Honeywell	XOM	Exxon Mobil Corp.